01 Indefinite Integration

If f & F are function of x such that $F'(x) = f(x)$ then the function F is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of $f(x)$ w.r.t. x and is written symbolically as

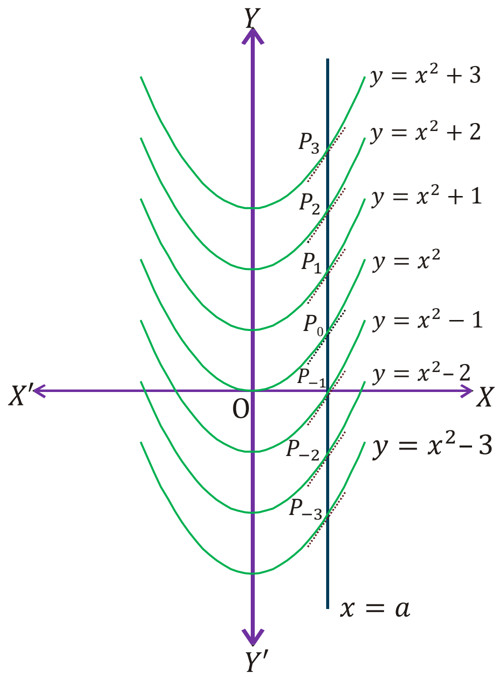
$\int f(x)dx = F(x) + C \Leftrightarrow \frac{d}{dx}\{F(x)+C\} = f(x)$, where C is called the **constant of integration.**

1. Geometrical Interpretation of Indefinite Integral :

$\int f(x)dx = F(x) + C = y$ (say), represents a family of curves. The different values of c will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

Let $f(x) = 2x$. Then $\int f(x)dx = x^2 + C$. For different values of C, we get different integrals. But these integrals are very similar geometrically.

Thus, $y = x^2 + C$, where C is arbitrary constant, represents a family of integrals. By assigning different values to C, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.



If the line $x = a$ intersects the parabolas $y = x^2, y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1, y = x^2 - 2$ at $P_0, P_1, P_2, P_{-1}, P_{-2}$ etc., respectively, then $\frac{dy}{dx}$ at these points equals $2a$. This indicates that the tangents to the curves at these points are parallel. Thus, $\int 2x dx = x^2 + C = f(x) + C$ **(say), implies that the tangents to all the curves $f(x) + C$, $C \in R$, at the points of intersection of the curves by the line $x = a$, $(a \in R)$, are parallel.**

2. Standard Formulae :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C;\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n|ax+b| + C$

(iii) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$

(iv) $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{\ell na} + C, (a > 0)$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+C$ **(vi)** $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+C$

(vii) $\int \tan(ax+b)dx = \frac{1}{a}\ell n|\sec(ax+b)|+C$ **(viii)** $\int \cot(ax+b)dx = \frac{1}{a}\ell n|\sin(ax+b)|+C$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+C$ **(x)** $\int \text{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+C$

(xi) $\int \text{cosec}(ax+b).\cot(ax+b)dx = -\frac{1}{a}\text{cosec}(ax+b)+C$

(xii) $\int \sec(ax+b).\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+C$

(xiii) $\int \sec x\,dx = \ell n|\sec x+\tan x|+C = \ell n\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|+C$

(xiv) $\int \text{cosec}\,x\,dx = \ell n|\text{cosec}\,x-\cot x|+C = \ell n\left|\tan\frac{x}{2}\right|+C = -\ell n|\text{cosec}\,x+\cot x|+C$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C$ **(xvi)** $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$

(xvii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+C$ **(xviii)** $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x+\sqrt{x^2+a^2}\right]+C$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x+\sqrt{x^2-a^2}\right]+C$ **(xx)** $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ell n\left|\frac{a+x}{a-x}\right|+C$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right|+C$ **(xxii)** $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

(xxiii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ell n\left(x+\sqrt{x^2+a^2}\right)+C$

(xxiv) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ell n\left(x+\sqrt{x^2-a^2}\right)+C$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)+C$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+C$

Elementary Integrals (Loving Integrals)
(Direct formulae Based or Converting into Known formulae)

Illustration 1:

$\int (x+5)^2 dx$

Solution:

M-1 : $= \int (x^2+25+10x)dx = \frac{x^3}{3}+\frac{10x^2}{2}+25x+C$

M-2 : $= \frac{(x+5)^3}{3}+C = \frac{x^3+15x^2+75x+125}{3}+C$

Illustration 2 :

$$\int \frac{x^4+1}{x^3}dx$$

Solution:

$$\int x^1+\frac{1}{x^3}\,dx=\frac{x^2}{2}+\frac{x^{-2}}{(-2)}+C=\frac{x^2}{2}-\frac{1}{2x^2}+C$$

Extension of Standard Formulae :

If x is replaced by $(Ax+B)$ then also the standard result remain true provided the result is divided by co-efficient of x

$$\int (Ax+B)^n\,dx=\frac{(Ax+B)^{n+1}}{(n+1)A}+C\,,\,n\neq -1$$

Similarly if $\int f(x)dx=g(x)+C$ then $\int f(Ax+B)dx=\frac{g(Ax+B)}{A}+C$

Illustration 3 :

Evaluate (i) $\int \frac{dx}{8x+3}$ (ii) $\int e^{8x+9}dx$

Solution:

(i) $\frac{1}{8}\ell n|8x+9|+C$ (ii) $\frac{e^{8x+9}}{8}+C$

Illustration 4 :

Evaluate (i) $\int \cos(6+2x)dx$ (ii) $\int \sec^2(6-5x)dx$

Solution:

(i) $\frac{\sin(5+4x)}{4}+C$ (ii) $\frac{\tan(3-4x)}{-4}+C$

Illustration 5 :

$$\int \cos^2 x\,dx$$

Solution:

$$\int \frac{1+\cos 2x}{2}dx=\frac{x}{2}+\frac{\sin 2x}{2\cdot 2}+C$$

Illustration 6 :

$$\int \cos^3 x\,dx$$

Solution:

$$\int \frac{3\cos x+\cos 3x}{4}dx \qquad [\cos 3x = 4\cos^3 x - 3\cos x]$$

$$\Rightarrow \int \frac{3\cos x}{4}dx+\int \frac{\cos 3x}{4}\,dx$$

$$\Rightarrow \frac{3\sin x}{4}+\frac{\sin 3x}{12}+C$$

Illustration 7:

$\int \sin^4 x\, dx$

Solution:

$\int (\sin^2 x)^2\, dx$

$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$

$= \int \frac{1+\cos^2 2x - 2\cos 2x}{4} dx$

$= \int \frac{1}{4} dx + \int \frac{\cos^2 2x}{4} dx - \frac{2}{4}\int \cos 2x\, dx$

$= \int \frac{1}{4} dx + \int \frac{1+\cos 4x}{2\cdot 4} dx - \int \frac{2\cos 2x}{4} dx$

$= \frac{x}{4} + \frac{1}{8}x + \frac{1}{8}\cdot\frac{\sin 4x}{4} - \frac{1}{2}\cdot\frac{\sin 2x}{2} + C$

$= \frac{3}{8}x + \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + C$

Illustration 8:

$\int \tan 3x \tan 2x \tan x\, dx$

Solution :

$3x = x + 2x$

$\Rightarrow \tan 3x = \tan(x + 2x)$

$\Rightarrow \tan 3x = \frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$

$\Rightarrow \tan 3x - \tan x.\tan 2x.\tan 3x = \tan x + \tan 2x$

$\Rightarrow \tan x \tan 2x \tan 3x = \tan 3x - \tan x - \tan 2x$

$\Rightarrow \int (\tan 3x - \tan 2x - \tan x)\, dx$

$\Rightarrow \int \tan 3x\, dx - \int \tan 2x\, dx - \int \tan x\, dx$

$\Rightarrow \frac{\ell n|\sec 3x|}{3} - \frac{\ell n|\sec 2x|}{2} - \ell n|\sec x| + C$

3. Methods of Integration:

(a) Substitution or change of independent variable :

If $\phi(x)$ is a continuous differentiable function, then to evaluate integrals of the form $\int f(\phi(x))\phi'(x)dx$, we substitute $\phi(x) = t$ and $\phi'(x)dx = dt$.

Hence $I = \int f(\phi(x))\phi'(x)dx$ reduces to $\int f(t)dt$.

(i) Fundamental deductions of method of substitution :

$\int [f(x)]^n f'(x)dx$ OR $\int \frac{f'(x)}{[f(x)]^n}dx$ put $f(x) = t$ & proceed.

Substitution is said to be appropriate if the integrand in (i) is a loving one.

If $\int f(x)dx = \phi(x)$

$\int f(g(x))g'(x)dx = ?$ ...(i)

Let $g(x)=t \Rightarrow g'(x)dx=dt$

$\int f(t)dt = \phi(t)+c$

$=\phi(t)+c$

When substitution is directly observed :

Illustration 9 :

$$\int \frac{\ell n x^2}{x}dx$$

Solution :

Let $\ell n x^2 = t$

$$\frac{1}{x^2}.2xdx = dt$$

$$\frac{2dx}{x} = dt$$

$$\Rightarrow \int \frac{t}{2}dt = \frac{t^2}{4}+C = \frac{(\ell n x^2)^2}{4}+C$$

Illustration 10 :

$$\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$$

Solution :

Let $\sin^{-1}x = t$

$$\int \frac{1}{\sqrt{1-x^2}}dx = dt$$

$$\Rightarrow \int t\,dt = \frac{t^2}{2}+C \Rightarrow (\sin^{-1}x)^2 + C$$

Illustration 11 :

$$\int \frac{\sin(\tan^{-1}x)}{1+x^2}dx$$

Solution :

Put, $\tan^{-1}x = t$

$$\frac{1}{1+x^2}dx = dt$$

$$\Rightarrow \int \sin t\,dt = -\cos t + C = -\cos(\tan^{-1}x) + C$$

Illustration 12 :

$$\int (4x+6)\sqrt{2x^2+6x+5}\,dx$$

Solution :

Let $2x^2 + 6x + 5 = t$

$(4x + 6)dx = dt$

$$\Rightarrow \int \sqrt{t}\,dt = \frac{t^{3/2}}{3/2} + C$$

$$= \frac{2}{3}(2x^2+6x+5)^{\frac{3}{2}} + C$$

Converting the given integral form to make it suitable for substitution or to get directly loving form :

Illustration 13 :

$$\int \frac{\sqrt{\tan x}}{\sin 2x}dx$$

Solution :

$$\int \frac{\sqrt{\tan x}}{2\tan x}(1+\tan^2 x)dx$$

$$\Rightarrow \int \frac{\sqrt{\tan x}}{2\tan x}\sec^2 x\,dx$$

Put $\tan x = t$

$\sec^2 x\,dx = dt$

$$\frac{1}{2}\int (t)^{-1/2}dt = \frac{t^{1/2}\times 2}{2\times 1} + C$$

$$= \sqrt{t} + C$$

$$= \sqrt{\tan x} + C$$

Illustration 14 :

$$\int \frac{2x^3 dx}{1+x^2}$$

Solution :

Put $\quad 1 + x^2 = t$

$2xdx = dt$

$$\Rightarrow \int \frac{2x\cdot x^2\,dx}{1+x^2} = \int \frac{(t-1)}{t}dt$$

$$= \int \left(1 - \frac{1}{t}\right)dt$$

$= t - \ell n t + C$

$= (1 + x^2) - \ell n(1 + x^2) + C$

Illustration 15 :

$$\int \frac{9x^2+2}{3x^3+2x+10}dx$$

Solution :

Put $3x^3 + 2x + 10 = t$

$(9x^2 + 2)dx = dt$

$$= \int \frac{dt}{t} = \ell n\, t + C$$

$$= \ell n(3x^2+2x+10)+C$$

Illustration 16 :

Evaluate $\displaystyle\int \frac{(x^2-1)dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}$

Solution :

The given integral can be written as

$$I = \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left[\left(x+\frac{1}{x}\right)^2+1\right]\tan^{-1}\left(x+\frac{1}{x}\right)}$$

Let $\left(x+\frac{1}{x}\right) = t$. Differentiating we get $\left(1-\frac{1}{x^2}\right)dx = dt$

Hence $I = \displaystyle\int \frac{dt}{(t^2+1)\tan^{-1} t}$

Now make one more substitution $\tan^{-1}t = u$. Then $\dfrac{dt}{t^2+1} = du$ and $I = \displaystyle\int \frac{du}{u} = \ell n|u|+C$

Returning to t, and then to x, we have

$$I = \ell n|\tan^{-1} t|+C = \ell n\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right|+C$$ **Ans.**

Substitutions involving trigonometric functions :

- $\displaystyle\int \cot x dx = \int \frac{\cos x}{\sin x}dx = \ell n|\sin x|+C$
- $\displaystyle\int \tan x dx = -\int \frac{-\sin x}{\cos x}dx = -\ell n\,|\cos x| + C$ or $\ell n\,|\sec x| + C$
- $\displaystyle\int \sec x dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}dx = \ell n|\sec x + \tan x| + C$ or $\ell n \tan\left(\frac{x}{2}+\frac{\pi}{4}\right)+C$
- $\displaystyle\int \text{cosec}\, x\, dx = \int \frac{\text{cosec}\, x(\cot x-\text{cosec}\, x)}{\cot x-\text{cosec}\, x}dx = \ell n|(\cot x-\text{cosec}\, x)|+C$ or $\ell n\left|\tan\frac{x}{2}\right|+C$

Illustration 17:

Evaluate $\int \frac{\cos^3 x}{\sin^2 x + \sin x} dx$

Solution :

$$I = \int \frac{(1-\sin^2 x)\cos x}{\sin x(1+\sin x)} dx = \int \frac{1-\sin x}{\sin x} \cos x \, dx$$

Put $\sin x = t \Rightarrow \cos x \, dx = dt$

$$\Rightarrow I = \int \frac{1-t}{t} dt = \ell n|t| - t + C = \ell n|\sin x| - \sin x + C$$ **Ans.**

Illustration 18 :

$$\int \frac{e^{2x}(1+2x)}{\sin(xe^{2x})} dx$$

Solution :

Put, $xe^{2x} = t$

$e^{2x} + 2xe^{2x} dx = dt$

$e^{2x}(1 + 2x)dx = dt$

$$\Rightarrow \int \frac{dt}{\sin t} = \int \text{cosec}\, t \, dt$$

$$= \ell n\left|\tan \frac{t}{2}\right| + C$$

$$= \ell n\left|\tan \frac{xe^{2x}}{2}\right| + C$$

Illustration 19 :

$$\int \frac{\cos 2x}{\cos x} dx$$

Solution :

$$\int \frac{2\cos^2 x - 1}{\cos x} dx$$

$$\Rightarrow \int (2\cos x - \sec x) dx$$

$\Rightarrow 2\sin x - \ell n|\sec x + \tan x| + C$

Illustration 20 :

$$\int \text{cosec} x \, \ell n|\cot x - \text{cosec} x| \, dx$$

Solution :

Put $\ell n|\cot x - \text{cosec} x| = t$

$\Rightarrow \text{cosec}\, x \, dx = dt$

$$\int t dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{(\ell n|\cot x - \text{cosec} x|)^2}{2} + C$$

Illustration 21:

$$\int \frac{dx}{\sin(x+5)\sin(x+8)}$$

Solution :

Multiplying & dividing by sin(8 – 5)

$$\Rightarrow \frac{1}{\sin(8-5)}\int \frac{\sin(8-5)}{\sin(x+5)\sin(x+8)}dx$$

$$\Rightarrow \frac{1}{\sin 3}\int \frac{\sin((x+8)-(x+5))dx}{\sin(x+5)\sin(x+8)}$$

$$\Rightarrow \frac{1}{\sin 3}\int \frac{\sin(x+8)\cos(x+5)-\cos(x+8)\sin(x+5)}{\sin(x+5)\sin(x+8)}dx$$

$$\Rightarrow \frac{1}{\sin 3}\int \left(\cot(x+5)-\cot(x+8)\right)dx$$

$$\Rightarrow \frac{1}{\sin 3}\left(\ell n\left(\sin\left(x+5\right)\right)-\ell n\left(\sin(x+8)\right)\right)+C$$

(ii) Standard substitutions :

$\int \frac{dx}{\sqrt{a^2+x^2}}$ or $\int \sqrt{a^2+x^2}dx$; put $x = a\tan\theta$ or $x = a\cot\theta$

$\int \frac{dx}{\sqrt{a^2-x^2}}$ or $\int \sqrt{a^2-x^2}dx$; put $x = a\sin\theta$ or $x = a\cos\theta$

$\int \frac{dx}{\sqrt{x^2-a^2}}$ or $\int \sqrt{x^2-a^2}dx$; put $x = a\sec\theta$ or $x = a\,\mathrm{cosec}\theta$

$\int \sqrt{\frac{a-x}{a+x}}dx$; put $x = a\cos 2\theta$

$\int \sqrt{\frac{x-\alpha}{\beta-x}}dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

Formulae :

1. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

Sol. $\int \frac{dx}{a^2+x^2}$ Put $x = a\tan\theta$ $dx = a\sec^2\theta\, d\theta$

$$\Rightarrow \int \frac{a\sec^2\theta\, d\theta}{a^2+a^2\tan^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$$

2. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x+\sqrt{x^2+a^2}\right] + c$

Sol. $\int \frac{dx}{\sqrt{x^2+a^2}}$ Put $x = a\tan\theta$ $dx = a\sec^2\theta\, d\theta$

$$\Rightarrow \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\tan^2\theta+a^2}} = \int \sec\theta\, d\theta = \ell n[\sec\theta+\tan\theta] = \ell n\left[\sqrt{1+\frac{x^2}{a^2}}+\frac{x}{a}\right] = \ell n\left[\sqrt{x^2+a^2}+x\right]+C$$

3. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ell n\left|\frac{x+a}{x-a}\right|+C$

Sol. $\int \frac{1}{(a-x)(a+x)}dx = \frac{1}{2a}\int\left(\frac{1}{a-x}+\frac{1}{a+x}\right)dx = \frac{1}{2a}[-\ell n|x-a|+\ell n|x+a|]+C$

4. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c$

5. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x+\sqrt{x^2-a^2}\right] + c$

6. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ell n\left|\frac{x-a}{x+a}\right|+c$

Illustration 22 :

Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}.\frac{1}{x}dx$

Solution :

Put $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta\, d\theta$

$$\Rightarrow I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.\frac{1}{\cos^2\theta}(-2\sin\theta\cos\theta)d\theta = -\int 2\tan\frac{\theta}{2}\tan\theta\, d\theta$$

$$= -4\int \frac{\sin^2(\theta/2)}{\cos\theta}d\theta = -2\int \frac{1-\cos\theta}{\cos\theta}d\theta = -2\ell n|\sec\theta+\tan\theta|+2\theta+C$$

$$= -2\ell n\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right|+2\cos^{-1}\sqrt{x}+C$$

Integral of the form :

(ii) $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}$

Express ax^2+bx+c in the form of perfect square & then apply the standard results.

(iii) $\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Express $px+q = \ell$ (differential coefficient of denominator) $+ m$.

Illustration 23 :

$$\int \frac{\sin x\,dx}{\cos^2 x - 36}$$

Solution :

put $\cos x = t \Rightarrow -\sin x\,dx = dt$

$$\int \frac{-dt}{t^2 - (6)^2} = -\frac{1}{2.6}\ell n\left|\frac{t-6}{t+6}\right| + C$$

$$= -\frac{1}{12}\ell n\left|\frac{\cos x - 6}{\cos x + 6}\right| + C$$

Illustration 24 :

$$\int \frac{dx}{3x^2 + 6x + 15}$$

Solution :

$$\Rightarrow \frac{1}{3}\int \frac{dx}{x^2 + 2x + 5}$$

$$\Rightarrow \frac{1}{3}\int \frac{dx}{(x+1)^2 + 4} = \frac{1}{3}\int \frac{dx}{(x+1)^2 + (2)^2}$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

Illustration 25 :

$$\int \frac{dx}{\sqrt{x^2 - 5x}}$$

Solution :

$$\int \frac{dx}{\sqrt{\left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2}}$$

$$= \ell n\left[\left(x - \frac{5}{2}\right) + \sqrt{\left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2}\right] + C$$

Illustration 26 :

$$\int \frac{2x}{\sqrt{x^4 + 2x^2 + 4}}dx$$

Solution:

put $x^2 = t \;\Rightarrow\; 2x\,dx = t$

$$\int \frac{dt}{\sqrt{t^2 + 2t + 4}} = \int \frac{dt}{\sqrt{(t+1)^2 + \left(\sqrt{3}\right)^2}}$$

$$= \ell n\left|(t+1) + \sqrt{(t+1)^2 + \left(\sqrt{3}\right)^2}\right| + C$$

$t = x^2$

Illustration 27 :

Evaluate $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$

Solution:

Put $x = a\cos^2\theta + b\sin^2\theta$, the given integral becomes

$$I = \int \frac{2(b-a)\sin\theta\cos\theta d\theta}{\left\{(a\cos^2\theta + b\sin^2\theta - a)(b - a\cos^2\theta - b\sin^2\theta\right\}^{\frac{1}{2}}}$$

$$= \int \frac{2(b-a)\sin\theta\cos\theta d\theta}{(b-a)\sin\theta\cos\theta} = \left(\frac{b-a}{b-a}\right)\int 2d\theta = 2\theta + C = 2\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)} + C$$ **Ans.**

Illustration 28 :

Evaluate $\int \frac{dx}{2x^2+x-1}$

Solution:

$$I = \int \frac{dx}{2x^2+x-1} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}-\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}+\frac{1}{16}-\frac{1}{16}-\frac{1}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{(x+1/4)^2 - 9/16} = \frac{1}{2}\int \frac{dx}{(x+1/4)^2-(3/4)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right| + C \qquad \left\{\text{using, } \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right\}$$

$$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right| + C = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + C$$ **Ans.**

Illustration 29 :

Evaluate $\int \frac{3x+2}{4x^2+4x+5}dx$

Solution:

Express $3x + 2 = \ell(\text{d.c. of } 4x^2 + 4x + 5) + m$

or, $3x + 2 = \ell(8x + 4) + m$

Comparing the coefficients, we get

$8\ell = 3$ and $4\ell + m = 2 \Rightarrow \ell = 3/8$ and $m = 2 - 4\ell = 1/2$

$$\Rightarrow I = \frac{3}{8}\int \frac{8x+4}{4x^2+4x+5}dx + \frac{1}{2}\int \frac{dx}{4x^2+4x+5}$$

$$= \frac{3}{8}\log\left|4x^2+4x+5\right| + \frac{1}{8}\int \frac{dx}{x^2+x+\frac{5}{4}}$$

$$= \frac{3}{8}\log\left|4x^2+4x+5\right| + \frac{1}{8}\tan^{-1}\left(x+\frac{1}{2}\right) + C$$ **Ans.**

(b) Integration by part : $\int u.v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}.\int v\,dx\right]dx$ where u & v are differentiable functions and are commonly designated as first & second function respectively.

Note : While using integration by parts, choose u & v such that

(i) $\int v dx$ & (ii) $\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ are simple to integrate.

This is generally obtained by choosing first function as the function which comes first in the word **ILATE**, where; I-Inverse function, L-Logarithmic function, A-Algebraic function, T-Trigonometric function & E-Exponential function.

Let $\mathrm{I} = \int \underset{\mathrm{I}}{f(x)}.\underset{\mathrm{II}}{g(x)}dx = f(x)\cdot\int g(x)dx - \int\left(f'(x)\right)\left(\int g(x)dx\right)dx$

= 1^{st} function × integral of 2^{nd} – $\int(\text{diff. coeff. of } \mathrm{I}^{st}) \times (\text{integral of } 2^{nd})dx$

Proof : $\frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g(x).f'(x)$

$\therefore \int \underset{I}{f(x)}.\underset{II}{g'(x)}dx = f(x).g(x) - \int g(x).f'(x)dx$

Note : In applying the above rule care has to be taken in the selection of the first function(I) and the second function (II). Normally we use the following methods :

(i) If in the product of the two functions, one of the functions is not directly integrable (e.g. ℓnx, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.) then we take it as the first function and the remaining function is taken as the second function. e.g. in the integration of $\int x\tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and x as the second function.

(ii) If there is no other function, then unity is taken as the second function e.g. in the integration of $\int\tan^{-1}x\,dx$, $\tan^{-1}x$ is taken as the first function and 1 as the second function.

(iii) If both of the functions are directly integrable then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable. Usually we use the following preference order for the function (Inverse, Logarithmic, Algebraic, Trigonometric, Exponent).

In the above stated order, the function on the left is always chosen as the first function. This rule is called as ILATE e.g. in the integration of $\int x\sin x\,dx$, x is taken as the first function and $\sin x$ is taken as the second function.

Illustration 30:

$\int e^{\ln x + x}.dx$

Solution:

$x\int e^x dx - \int(1)\left(\int e^x dx\right)dx$

$xe^x - \int e^x dx$

$xe^x - e^x + C$

Illustration 31:

$$\int \underset{II}{x^4}\ \underset{I}{\ell n x}\, dx$$

Solution:

$$\Rightarrow \ell n x \int x^4 dx - \int \left(\frac{d(\ell n x)}{dx} \int x^4 dx \right) dx$$

$$\Rightarrow \ell n x \cdot \frac{x^5}{5} - \int \frac{1}{x} \times \frac{x^5}{5} dx$$

$$\Rightarrow \frac{x^5}{5} \ell n x - \frac{x^5}{25} + C$$

Illustration 32:

$$\int x \cos 2x\, dx$$

Solution:

$$\Rightarrow x \int \cos 2x\, dx - \int (1) \left(\int \cos 2x\, dx \right) dx$$

$$\Rightarrow \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} dx$$

$$\Rightarrow \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C$$

Illustration 33:

$$\int \ell n (2x+3)^{(2x+3)} dx$$

Solution:

$$\Rightarrow \int \underbrace{(2x+3)}_{II} \underbrace{\ell n(2x+3)}_{I} dx$$

$$\Rightarrow \ell n(2x+3) \int (2x+3) dx - \int \frac{1}{(2x+3)} \left(\int (2x+3) dx \right) dx$$

$$\Rightarrow \ell n(2x+3) \frac{(2x+3)^2}{4} - \int \frac{1}{2x+3} \times \frac{(2x+3)^2}{2} dx$$

$$\Rightarrow \ell n(2x+3) \frac{(2x+3)^2}{4} - \frac{(2x+3)^2}{4} + C$$

Illustration 34:

$$\int \underset{(I)}{x^2}\ \underset{(II)}{e^x}\ dx$$

Solution:

$$x^2(e^x) - \int \underset{I}{2x}\, \underset{II}{e^x} dx$$

$$x^2(e^x) - \left[2x e^x - \int 2e^x dx \right] = x^2 e^x - 2x e^x + 2e^x + C$$

Illustration 35:

$\int \sin^{-1} x \, dx$

Solution:

M-1 :

Let 1 is algebraic function

$\int \underset{II}{1} \cdot \underbrace{\sin^{-1} x}_{I} \, dx$

$\sin^{-1} x(x) - \int \left(\frac{1}{\sqrt{1-x^2}} \right)(x) \, dx$

$1 - x^2 = t$

$-2xdx = dt$

$x \sin^{-1}(x) + \frac{1}{2}\int \frac{dt}{\sqrt{t}}$

$x\sin^{-1} x + \frac{1}{2}\frac{\sqrt{t}}{\left(\frac{1}{2}\right)} + C$

$x \sin^{-1} x + \sqrt{1-x^2} + C$

M-2 :

$\sin^{-1} x = t$

$x = \sin t$

$dx = \cos t \, dt$

$\int \underset{I}{t} \; \underset{II}{\cos t} \; dt$

$t(\sin t) - \int (1)\sin t \, dt$

$t \sin t + \cos t + C$

$\sin^{-1} x \, (x) + \sqrt{1-x^2} + C$

Illustration 36:

$\int \ell nx \, dx$

Solution:

M-1 :

$\int \underset{II}{1} \cdot \underset{I}{\ell nx} \; dx$

$\ell n(x) \cdot (x) - \int \frac{1}{x} \cdot x \, dx$

$x\ell nx - x + C$

M-2 :

$\int \ell nx\, dx$

Put $\ell nx = t$

$x = e^t$

$dx = e^t dt$

$\int \underset{I}{t} \cdot \underset{II}{e^t}\ dt$

$\Rightarrow te^t - \int e^t dt$

$\Rightarrow te^t - e^t + C$

$\Rightarrow x\ell nx - x + C$

Illustration 37:

Evaluate : $\int \cos\sqrt{x}\, dx$

Solution:

Consider $\quad I = \int \cos\sqrt{x} dx$

Let $\quad \sqrt{x} = t \qquad$ then $\quad \frac{1}{2\sqrt{x}} dx = dt$

i.e. $\quad dx = 2\sqrt{x} dt \qquad$ or $\quad dx = 2t\ dt$

so $\quad I = \int \cos t.2tdt$

Taking t as first function, integrate it by part

$\Rightarrow I = 2\left[t\int \cos tdt - \int\left\{\frac{dt}{dt}\int \cos tdt\right\}dt\right]$

$I = 2\left[t\sin t - \int 1.\sin tdt\right] = 2[t\sin t + \cos t] + C$

$I = 2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + C$ **Ans.**

Illustration 38:

Evaluate : $\int \frac{x}{1+\sin x} dx$

Solution:

Let $I = \int \frac{x}{1+\sin x} dx = \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$

$= \int \frac{x(1-\sin x)}{1-\sin^2 x} dx = \int \frac{x(1-\sin x)}{\cos^2 x} dx = \int x\sec^2 x dx - \int x\sec x\tan x dx$

$= \left[x\int \sec^2 x dx - \int\left\{\frac{dx}{dx}\int \sec^2 x dx\right\}dx\right] - \left[x\int \sec x\tan x dx - \int\left\{\frac{dx}{dx}\int \sec x\tan x dx\right\}dx\right]$

$= \left[x\tan x - \int \tan x dx\right] - \left[x\sec x - \int \sec x dx\right]$

$= [x\tan x - \ell n|\sec x|] - [x\sec x - \ell n|\sec x + \tan x|] + C$

$= x(\tan x - \sec x) + \ell n\left|\frac{(\sec x + \tan x)}{\sec x}\right| + C$

$= \frac{-x(1-\sin x)}{\cos x} + \ell n|1 + \sin x| + C$ **Ans.**

Two Classic Integrands :

(a) $\int e^x (f(x) + f'(x))dx = e^x f(x) + c$

Proof:

Let $I = \int e^x [f(x) + f'(x)]dx$

$\Rightarrow I = \int e^x f(x)dx + \int e^x f'(x)dx$

Integrate the first integral on the RHS by parts taking e^x as the second function, we get

$I = e^x f(x) - \int e^x f'(x)dx + \int e^x f'(x)dx$

$\Rightarrow I = e^x f(x)$

Thus to evaluate the integrals of the type $\int e^x g(x)dx$, we first try to express $g(x)$ as the sum of the function and its derivative i.e., $g(x) = f(x) + f'(x)$ and then we use the result derived above.

Illustration 39:

$\int e^x (\sin x + \cos x)dx$

Solution:

$\int e^x (\underbrace{\sin x}_{f(x)} + \underbrace{\cos x}_{f'(x)})dx$

$\Rightarrow e^x \sin x + C$

Illustration 40:

$\int e^x \left(\frac{2x^2 + x^3}{x}\right)dx$

Solution:

$\int e^x \left(\underbrace{2x}_{f'(x)} + \underbrace{x^2}_{f(x)}\right)dx$

$\Rightarrow e^x . x^2 + C$

Illustration 41 :

$\int \frac{xe^x}{(1+x)^2}dx$

Solution:

$\int e^x \left(\frac{x+1}{(x+1)^2} + \frac{(-1)}{(x+1)^2}\right)dx$

$\Rightarrow \int e^x \left(\underbrace{\frac{1}{x+1}}_{f(x)} + \underbrace{\frac{(-1)}{(x+1)^2}}_{f'(x)}\right)dx$

$= e^x \cdot \frac{1}{1+x} + C$

Illustration 42 :

$$\int e^x\left(\frac{\sin 2x-2}{1-\cos 2x}\right)dx$$

Solution:

$$\int e^x\left(\frac{2\sin x\cos x-2}{2\sin^2 x}\right)$$

$$\int e^x\left(\frac{2\sin x\cos x}{2\sin^2 x}-\frac{2}{2\sin^2 x}\right)dx$$

$$\int e^x(\underbrace{\cot x}_{f(x)}+\underbrace{(-\text{cosec}^2 x)}_{f'(x)})dx$$

$\Rightarrow e^x\cot x + C$

Illustration 43 :

Evaluate $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

Solution:

$$\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx=\int e^x\frac{(1-2x+x^2)}{(1+x^2)^2}dx$$

$$=\int e^x\left(\frac{1}{(1+x^2)}-\frac{2x}{(1+x^2)^2}\right)dx=\frac{e^x}{1+x^2}+C$$ **Ans.**

Illustration 44 :

The value of $\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx$ is equal to -

(A) $\frac{e^x(x+1)}{(1+x^2)^{3/2}}+C$ (B) $\frac{e^x(1-x+x^2)}{(1+x^2)^{3/2}}+C$ (C) $\frac{e^x(1-x)}{(1+x^2)^{3/2}}+C$ (D) none of these

Ans. (D)

Solution:

Let $I=\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx=\int e^x\left(\frac{1}{(1+x^2)^{1/2}}+\frac{1-2x^2}{(1+x^2)^{5/2}}\right).dx$

$$=\int e^x\left(\frac{1}{(1+x^2)^{1/2}}-\frac{x}{(1+x^2)^{3/2}}+\frac{x}{(1+x^2)^{3/2}}+\frac{1-2x^2}{(1+x^2)^{5/2}}\right)dx$$

$$=\frac{e^x}{(1+x^2)^{1/2}}+\frac{xe^x}{(1+x^2)^{3/2}}+C=\frac{e^x\{1+x^2+x\}}{(1+x^2)^{3/2}}+C$$

Illustration 45 :

Prove that $\int e^{g(x)}(g'(x).f(x)+f'(x))dx=e^{g(x)}.f(x)$.

Solution:

$$I=\int e^{g(x)}(g'(x)f(x)+f'(x))dx$$

$$=\int e^{g(x)}g'(x).f(x)dx+\int e^{g(x)}f'(x)dx$$

Integrate the first integral on the R.H.S. by parts taking $e^{g(x)}.g'(x)$ as the second function, we get

$$I=e^{g(x)}f(x)-\int f'(x)e^{g(x)}dx+\int e^{g(x)}.f'(x)dx$$

$= e^{g(x)}.f(x)$

(b) $\int (f(x)+xf'(x))dx = x\,f(x)+c$

Proof : = $\int f(x)\cdot 1dx + \int xf'(x)dx$

$=\left[f(x)\cdot x - \int f'(x)\cdot x\right] + \int xf'(x)dx$

$= x\,f(x) + c$

Illustration 46:

$\int (\sin x + x\cos x)dx$

Solution:

$\Rightarrow \int (\underset{f(x)}{\sin x} + x\underset{f'(x)}{\cos x})dx$

$\Rightarrow x\ \sin x + C$

Illustration 47:

$\int \left(\ell n(\ell nx^2) + \frac{2}{\ell nx^2}\right)dx$

Solution:

$\int (\underbrace{\ell n(\ell nx^2)}_{f(x)} + x\underbrace{\left(\frac{1}{\ell nx^2}\times\frac{1}{x^2}\times 2x\right)}_{f'(x)}dx$

$\Rightarrow x\ \ell n\ (\ell nx^2) + C$

Illustration 48:

$\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

Solution:

$\int e^{\tan^{-1}x}\left(\frac{1+x^2}{1+x^2}+\frac{x}{1+x^2}\right)dx = \int e^{\tan^{-1}x}\left(1+\frac{x}{1+x^2}\right)dx$

$= \int \underbrace{e^{\tan^{-1}x}}_{f(x)} + \underbrace{x\cdot\frac{e^{\tan^{-1}x}}{1+x^2}}_{x\,f'(x)}\ dx = xe^{\tan^{-1}x} + C$

Illustration 49:

Evaluate $\int \frac{x+\sin x}{1+\cos x}dx$

Solution:

$I = \int \frac{x+\sin x}{1+\cos x}dx = \int \left(\frac{x+\sin x}{2\cos^2\frac{x}{2}}\right)dx = \int \left(x\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx = x\tan\frac{x}{2} + C$ **Ans.**

Integrals of the Type :

$\int e^{ax}\sin bx\,dx,\ \int e^{ax}\cos bx\,dx$

Results :

(i) $\int e^{ax}.\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+c$

(ii) $\int e^{ax}.\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+c$

Proof :

Method-I Let $I=\int e^{ax}\sin bx\,dx$

Integrate by parts taking e^{ax} as the first part, we get

$$I=\frac{-e^{ax}\cos bx}{b}-\int ae^{ax}\left(\frac{-\cos bx}{b}\right)dx$$

On integrating the second term by parts again, we get

$$I=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b}\left[\frac{1}{b}e^{ax}\sin bx-\int\frac{a}{b}e^{ax}\sin bx\,dx\right]$$

$$\Rightarrow\ I=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx-\frac{a^2}{b^2}I$$

$$\Rightarrow\ \left(1+\frac{a^2}{b^2}\right)I=\frac{e^{ax}}{b^2}\ (a\sin bx-b\cos bx)+C$$

$$\Rightarrow\ I=\frac{e^{ax}}{a^2+b^2}\ (a\sin bx-b\cos bx)+C$$

Similarly we can show that $\int e^{ax}\cos bx=\frac{e^{ax}}{a^2+b^2}\ (a\cos bx+b\sin bx)+C$

Illustration 50:

Evaluate $I=\int e^x.\sin x\,dx=\sin x.e^x-\int\cos x.e^x dx$

Solution:

$=\ \sin x e^x-\left[\cos x e^x-\int(-\sin x)e^x dx\right]$

$=\sin x\, e^x-\cos x\, e^x-\ I$

$\Rightarrow 2I=\sin x\, e^x-\cos x\, e^x$

$\Rightarrow I=\frac{1}{2}(\sin x-\cos x)e^x+c$

Method-II Assume $\int e^{ax}\cos bx\,dx=e^{ax}(A\cos bx+B\sin bx)$ and then differentiate both sides

Illustration 51:

$\int e^{2x}\cos 3x dx=e^{2x}\{A\cos 3x+B\sin 3x\}+C$ find A & B.

Solution:

differentiate both sides

$e^{2x}\cos 3x=e^{2x}(-3A\sin 3x+3B\cos 3x)+2e^{2x}(A\cos 3x+B\sin 3x)$

$=e^{2x}[(2A+3B)\cos 3x+(2B-3A)\sin 3x]$

$\Rightarrow 2A+3B=1$

$2B-3A=0$

$\Rightarrow\ A=\frac{2}{13}$ and $B=\frac{3}{13}$ **Ans.**

(c) Integration of trigonometric functions :

(i) $\int \sin^m x \cos^n x dx$

Case-I : When m & $n \in$ natural numbers.

* If one of them is odd, then substitute for the term of even power.
* If both are odd, substitute either of the term.
* If both are even, use trigonometric identities to convert integrand into cosines of multiple angles.

Case-II : $m + n$ is a negative even integer.

* In this case the best substitution is $\tan x = t$.

Illustration 52:

Evaluate $\int \sin^3 x \cos^5 x\, dx$

Solution:

Put $\cos x = t$; $-\sin x\, dx = dt$.

so that $I = -\int (1-t^2).t^5 dt$

$= \int (t^7 - t^5)dt = \frac{t^8}{8} - \frac{t^6}{6} = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C$

Alternate :

Put $\sin x = t$; $\cos x\, dx = dt$

so that $I = \int t^3(1-t^2)^2 dt = \int \left(t^3 - 2t^5 + t^7\right)dt$

$= \frac{\sin^4 x}{4} - \frac{2\sin^6 x}{6} + \frac{\sin^8 x}{8} + C$

Note : This problem can also be handled by successive reduction or by trigonometric identities.

Illustration 53:

Evaluate $\int \sin^2 x \cos^4 x\, dx$

Solution:

$$\int \sin^2 x \cos^4 x dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{\cos 2x + 1}{2}\right)^2 dx$$

$$= \int \frac{1}{8}(1-\cos 2x)\left(\cos^2 2x + 2\cos 2x + 1\right)dx$$

$$= \frac{1}{8}\int \left(\cos^2 2x + 2\cos 2x + 1 - \cos^3 2x - 2\cos^2 2x - \cos 2x\right)dx$$

$$= \frac{1}{8}\int \left(-\cos^3 2x - \cos^2 2x + \cos 2x + 1\right)dx$$

$$= -\frac{1}{8}\int \left(\frac{\cos 6x + 3\cos 2x}{4} + \frac{1+\cos 4x}{2} - \cos 2x - 1\right)dx$$

$$= -\frac{1}{32}\left[\frac{\sin 6x}{6} + \frac{3\sin 2x}{2}\right] - \frac{1}{16}x - \frac{\sin 4x}{64} + \frac{\sin 2x}{16} + \frac{x}{8} + C$$

$$= -\frac{\sin 6x}{192} - \frac{\sin 4x}{64} + \frac{1}{64}\sin 2x + \frac{x}{16} + C$$

Illustration 54:

Evaluate $\int \frac{\sqrt{\sin x}}{\cos^{9/2} x} dx$

Solution:

Let $I = \int \frac{\sin^{1/2} x}{\cos^{9/2} x} dx = \int \frac{dx}{\sin^{-1/2} x \cos^{9/2} x}$

Here $m + n = \frac{1}{2} - \frac{9}{2} = -4$ (negative even integer).

Divide Numerator & Denominator by $\cos^4 x$.

$I = \int \sqrt{\tan x} \sec^4 x dx = \int \sqrt{\tan x}(1+\tan^2 x)\sec^2 x dx$

$= \int \sqrt{t}(1+t^2)dt$ (using $\tan x = t$)

$= \frac{2}{3}t^{3/2} + \frac{2}{7}t^{7/2} + C = \frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + C$

Integral of the form :

(ii) $\int \frac{dx}{a + b\sin^2 x}$ OR $\int \frac{dx}{a + b\cos^2 x}$ OR $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Divide N^r & D^r by $\cos^2 x$ & put $\tan x = t$.

Illustration 55:

Evaluate : $\int \frac{dx}{2 + \sin^2 x}$

Solution:

Divide numerator and denominator by $\cos^2 x$

$I = \int \frac{\sec^2 x dx}{2\sec^2 x + \tan^2 x} = \int \frac{\sec^2 x dx}{2 + 3\tan^2 x}$

Let $\sqrt{3}\tan x = t$ $\therefore$ $\sqrt{3}\sec^2 x dx = dt$

So $I = \frac{1}{\sqrt{3}}\int \frac{dt}{2+t^2} = \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right) + C$ **Ans.**

Illustration 56:

Evaluate : $\int \frac{dx}{(2\sin x + 3\cos x)^2}$

Solution:

$\int \frac{dx}{4\sin^2 x + 12\sin x\cos x + 9\cos^2 x}$

Divide numerator and denominator by $\cos^2 x$

$\therefore$ $I = \int \frac{\sec^2 x dx}{4\tan^2 x + 12\tan x + 9} = \int \frac{\sec^2 x dx}{(2\tan x + 3)^2}$

Let $2\tan x + 3 = t$, $\therefore$ $2\sec^2 x\, dx = dt$

$I = \frac{1}{2}\int \frac{dt}{t^2} = -\frac{1}{2t} + C = -\frac{1}{2(2\tan x + 3)} + C$ **Ans.**

Integral of the form :

(iii) $\int \frac{dx}{a+b\sin x}$ OR $\int \frac{dx}{a+b\cos x}$ OR $\int \frac{dx}{a+b\sin x+c\cos x}$

Convert sines & cosines into their respective tangents of half the angles

& put $\tan\frac{x}{2}=t$

In this case $\sin x = \frac{2t}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}, x = 2\tan^{-1} t; dx = \frac{2dt}{1+t^2}$

Illustration 57 :

$\int \frac{dx}{3+4\sin 2x}$

Solution:

$\int \frac{dx}{3+\frac{4.2\tan x}{1+\tan^2 x}}$

$=\int \frac{(1+\tan^2 x)dx}{3+3\tan^2 x+8\tan x}$

Let $\tan x = t$

$\sec^2 x\, dx = dt$

$I=\int \frac{dt}{3t^2+8t+3}$

$=\frac{1}{3}\int \frac{dt}{\left(t+\frac{4}{3}\right)^2-\frac{7}{9}}$

$=\frac{1}{3}\cdot\frac{3}{2\sqrt{7}}\ell n\left|\frac{t+\frac{4}{3}-\frac{\sqrt{7}}{3}}{t+\frac{4}{3}+\frac{\sqrt{7}}{3}}\right|+c$

$=\frac{1}{2\sqrt{7}}\ell n\left|\frac{\tan x+\frac{4}{3}-\frac{\sqrt{7}}{3}}{\tan x+\frac{4}{3}+\frac{\sqrt{7}}{3}}\right|+c$

Illustration 58:

Evaluate : $\int \frac{dx}{3\sin x+4\cos x}$

Solution:

$$I=\int \frac{dx}{3\sin x+4\cos x}=\int \frac{dx}{3\left\{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}+4\left\{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}}=\int \frac{\sec^2\frac{x}{2}dx}{4+6\tan\frac{x}{2}-4\tan^2\frac{x}{2}}$$

let $\tan\frac{x}{2}=t$, $\therefore\ \frac{1}{2}\sec^2\frac{x}{2}dx=dt$

so $I = \int \frac{2dt}{4+6t-4t^2} = \frac{1}{2}\int \frac{dt}{1-\left(t^2-\frac{3}{2}t\right)} = \frac{1}{2}\int \frac{dt}{\frac{25}{16}-\left(t-\frac{3}{4}\right)^2}$

$$= \frac{1}{2}\cdot\frac{1}{2\left(\frac{5}{4}\right)}\ell n\left|\frac{\frac{5}{4}+\left(t-\frac{3}{4}\right)}{\frac{5}{4}-\left(t-\frac{3}{4}\right)}\right| + C = \frac{1}{5}\ell n\left|\frac{1+2\tan\frac{x}{2}}{4-2\tan\frac{x}{2}}\right| + C \textbf{ Ans.}$$

Integral of the form :

(iv) $\int \frac{a\cos x + b\sin x + c}{p\cos x + q\sin x + r}dx$

Express Numerator $(N^r) = \ell(D^r) + m\frac{d}{dx}(D^r) + n$ & proceed.

Illustration 59:

Evaluate : $\int \frac{2+3\cos\theta}{\sin\theta + 2\cos\theta + 3}d\theta$

Solution:

Write the Numerator = ℓ(denominator) + m(d.c. of denominator) + n

$\Rightarrow 2 + 3\cos\theta = \ell(\sin\theta + 2\cos\theta + 3) + m(\cos\theta - 2\sin\theta) + n$.

Comparing the coefficients of $\sin\theta$, $\cos\theta$ and constant terms,

we get $3\ell + n = 2,\ 2\ell + m = 3,\ \ell - 2m = 0 \Rightarrow \ell = 6/5, m = 3/5$ and $n = -8/5$

Hence $\quad I = \int \frac{6}{5}d\theta + \frac{3}{5}\int \frac{\cos\theta - 2\sin\theta}{\sin\theta + 2\cos\theta + 3}d\theta - \frac{8}{5}\int \frac{d\theta}{\sin\theta + 2\cos\theta + 3}$

$= \frac{6}{5}\theta + \frac{3}{5}\ell n|\sin\theta + 2\cos\theta + 3| - \frac{8}{5}I_3$ where $I_3 = \int \frac{d\theta}{\sin\theta + 2\cos\theta + 3}$

In I_3, put $\tan\frac{\theta}{2} = t \Rightarrow \sec^2\frac{\theta}{2}d\theta = 2dt$

$I_3 = 2\int \frac{dt}{t^2+2t+5} = 2\int \frac{dt}{(t+1)^2+2^2} = 2.\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right)$

Hence $I = \frac{6\theta}{5} + \frac{3}{5}\ell n|\sin\theta + 2\cos\theta + 3| - \frac{8}{5}\tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right) + C$ **Ans.**

(d) Integration of rational function :

(i) Rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. If the degree of $P(x)$ is less than the degree of $Q(x)$, then the rational function is called proper, otherwise, it is called improper. The improper rational function can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where $T(x)$ is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods.

S. No.	Form of the rational function	Form of the partial fraction
1.	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
2.	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
3.	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$

where x^2+bx+c cannot be factorised further

Illustration 60:

Evaluate : $\int \dfrac{x}{(x-2)(x+5)}dx$

Solution:

$$\frac{x}{(x-2)(x+5)}=\frac{A}{x-2}+\frac{B}{x+5}$$

or $x = A(x+5)+B(x-2)$.

by comparing the coefficients, we get

$A = 2/7$ and $B = 5/7$ so that

$$\int \frac{x}{(x-2)(x+5)}dx=\frac{2}{7}\int\frac{dx}{x-2}+\frac{5}{7}\int\frac{dx}{x+5}=\frac{2}{7}\ell n|(x-2)|+\frac{5}{7}\ell n|(x+5)|+C$$ **Ans.**

Illustration 61:

Evaluate $\int \dfrac{x^4}{(x+2)(x^2+1)}dx$

Solution:

$$\frac{x^4}{(x+2)(x^2+1)}=(x-2)+\frac{3x^2+4}{(x+2)(x^2+1)}$$

Now, $\dfrac{3x^2+4}{(x+2)(x^2+1)}=\dfrac{16}{5(x+2)}+\dfrac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}$

So, $\dfrac{x^4}{(x+2)(x^2+1)}=x-2+\dfrac{16}{5(x+2)}+\dfrac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}$

Now, $\int\left((x-2)+\dfrac{16}{5(x+2)}+\dfrac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}\right)dx$

$$=\frac{x^2}{2}-2x+\frac{2}{5}\tan^{-1}x+\frac{16}{5}\ell n|x+2|-\frac{1}{10}\ell n(x^2+1)+C$$ **Ans.**

(iv) Integrals of the form $\int\dfrac{x^2+1}{x^4+Kx^2+1}dx$ OR $\int\dfrac{x^2-1}{x^4+Kx^2+1}dx$, where K is any constant.

Divide N^r & D^r by x^2 & proceed.

Note : Sometimes it is useful to write the integral as a sum of two related integrals, which can be evaluated by making suitable substitutions e.g.

* $\int \frac{2x^2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx + \int \frac{x^2-1}{x^4+1}dx$

* $\int \frac{2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx - \int \frac{x^2-1}{x^4+1}dx$

These integrals can be called as **Algebric Twins**.

Illustration 62:

Evaluate : $\int \frac{4}{\sin^4 x + \cos^4 x}dx$

Solution:

$I = 4\int \frac{1}{\sin^4 x + \cos^4 x}dx = 4\int \frac{\sec^4 x}{1+\tan^4 x}dx$

$= 4\int \frac{(\tan^2 x + 1)\sec^2 x}{(\tan^4 x + 1)}dx$

Now, put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\Rightarrow I = 4\int \frac{1+t^2}{1+t^4}dt = 4\int \frac{1/t^2+1}{t^2+1/t^2}dt$

Now, put $t - 1/t = z \Rightarrow \left(1+\frac{1}{t^2}\right)dt = dz$

$\Rightarrow I = 4\int \frac{dz}{z^2+2} = \frac{4}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} + C = 2\sqrt{2}\tan^{-1}\frac{t-1/t}{\sqrt{2}} + C$

$= 2\sqrt{2}\tan^{-1}\left(\frac{\tan x - 1/\tan x}{\sqrt{2}}\right) + C$ **Ans.**

Illustration 63 :

Evaluate : $\int \frac{1}{x^4+5x^2+1}dx$

Solution:

$I = \frac{1}{2}\int \frac{2}{x^4+5x^2+1}dx$

$\Rightarrow I = \frac{1}{2}\int \frac{1+x^2}{x^4+5x^2+1}dx + \frac{1}{2}\int \frac{1-x^2}{x^4+5x^2+1}dx$

$= \frac{1}{2}\int \frac{1+1/x^2}{x^2+5+1/x^2}dx - \frac{1}{2}\int \frac{1-1/x^2}{x^2+5+1/x^2}dx$

{dividing N^r and D^r by x^2}

$= \frac{1}{2}\int \frac{(1+1/x^2)}{(x-1/x)^2+7}dx - \frac{1}{2}\int \frac{(1-1/x^2)dx}{(x+1/x)^2+3}$

$= \frac{1}{2}\int \frac{dt}{t^2+(\sqrt{7})^2} - \frac{1}{2}\int \frac{du}{u^2+(\sqrt{3})^2}$

where $t = x - \frac{1}{x}$ and $u = x + \frac{1}{x}$

$I = \frac{1}{2}.\frac{1}{\sqrt{7}}\left(\tan^{-1}\frac{t}{\sqrt{7}}\right) - \frac{1}{2}.\frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{u}{\sqrt{3}}\right) + C$

$= \frac{1}{2}\left[\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{7}}\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/x}{\sqrt{3}}\right)\right] + C$ **Ans.**

(e) Integration of Irrational functions :

(i) $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ & $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$;put $px+q=t^2$

(ii) $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$,put $ax+b=\frac{1}{t}$; $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$, put $x = \frac{1}{t}$

Illustration 64:

Evaluate $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}.dx$

Solution:

Let, $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}.dx$ Put $x + 1 = t^2 \Rightarrow dx = 2tdt$

$$\therefore I = \int \frac{(t^2-1)+2}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}.(2t)dt = 2\int \frac{t^2+1}{t^4+t^2+1}dt = 2\int \frac{1+1/t^2}{t^2+1+1/t^2}dt$$

$$= 2\int \frac{1+1/t^2}{(t-1/t)^2+(\sqrt{3})^2}.dt = 2\int \frac{du}{u^2+(\sqrt{3})^2} \qquad \left\{\text{where } u = t-\frac{1}{t}\right\}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right)+C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)+C \quad \textbf{Ans.}$$

Illustration 65:

Evaluate $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

Solution:

Let, $I = \int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$ put $x - 1 = \frac{1}{t} \Rightarrow dx = -1/t^2 dt$

$$I = \int \frac{-1/t^2 dt}{1/t\sqrt{\left(\frac{1}{t}+1\right)^2+\left(\frac{1}{t}+1\right)+1}} = -\int \frac{dt}{\sqrt{3t^2+3t+1}}$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+1/12}} = -\frac{1}{\sqrt{3}}\log\left|(t+1/2)+\sqrt{(t+1/2)^2+1/12}\right|+C$$

$$= -\frac{1}{\sqrt{3}}\log\left|\left(\frac{1}{x-1}+\frac{1}{2}\right)+\sqrt{\frac{12\left(\frac{1}{x-1}+\frac{1}{2}\right)^2+1}{12}}\right|+C \quad \textbf{Ans.}$$

Illustration 66:

Evaluate : $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Solution:

Let, $I = \int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ Put $x = \frac{1}{t}$, So that $dx = \frac{-1}{t^2}dt$

$$\therefore\ I = \int \frac{-1/t^2 dt}{(1+1/t^2)\sqrt{1-1/t^2}} = -\int \frac{tdt}{(t^2+1)\sqrt{t^2-1}}$$

Again let, $t^2 = u$. So that $2t\ dt = du$.

$= \frac{-1}{2}\int \frac{du}{(u+1)\sqrt{u-1}}$ which reduces to the form $\int \frac{dx}{P\sqrt{Q}}$ where both P and Q are linear so that we put $u - 1 = z^2$ so that $du = 2z\, dz$

$$\therefore\ I = -\frac{1}{2}\int \frac{2zdz}{(z^2+1+1)\sqrt{z^2}} = -\int \frac{dz}{(z^2+2)}$$

$$I = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + C$$

$$I = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{u-1}}{\sqrt{2}}\right) + C = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right) + C$$

$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + C$ **Ans.**

Integrals of Some More Types

Here, we discuss some special types of standard integrals based on the technique of integration by parts :

(i) $\int \sqrt{x^2-a^2}\,dx$ (ii) $\int \sqrt{x^2+a^2}\,dx$ (iii) $\int \sqrt{a^2-x^2}\,dx$

(i) Let $I = \int \sqrt{x^2-a^2}\,dx$

Taking constant function 1 as the second function and itnegratin by parts, we have

$$I = x\sqrt{x^2-a^2} - \int \frac{1}{2}\frac{2x}{\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$= x\sqrt{x^2-a^2} - I - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

or $2I = x\sqrt{x^2-a^2} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$

or $I=\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

Similarly, integrating other two integrals, by parts, taking constant function 1 as the second function, we get

(ii) $\int\sqrt{x^2+a^2}\,dx=\frac{1}{2}x\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

(iii) $\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

Alternatively, integrals (i), (ii) and (iii) can also be found by making trigonometric substitution $x = a\sec\theta$ in (i), $x = a\tan\theta$ in (ii) and $x = a\sin\theta$ in (iii) respectively.

Illustration 67:

Find $\int\sqrt{x^2+2x+5}\,dx$

Solution:

Note that

$\int\sqrt{x^2+2x+5}\,dx=\int\sqrt{(x+1)^2+4}\,dx$

Put $x + 1 = y$, so that $dx = dy$. Then s

$\int\sqrt{x^2+2x+5}\,dx=\int\sqrt{y^2+2^2}\,dy$

$=\frac{1}{2}y\sqrt{y^2+4}+\frac{4}{2}\log\left|y+\sqrt{y^2+4}\right|+C$

$=\frac{1}{2}(x+1)\sqrt{x^2+2x+5}+2\log\left|x+1+\sqrt{x^2+2x+5}\right|+C$

Illustration 68:

Find $\int\sqrt{3-2x-x^2}\,dx$

Solution:

Note that $\int\sqrt{3-2x-x^2}\,dx=\int\sqrt{4-(x+1)^2}\,dx$

Put $x + 1 = y$ so that $dx = dy$

Thus $\int\sqrt{3-2x-x^2}\,dx=\int\sqrt{4-y^2}\,dy$

$=\frac{1}{2}y\sqrt{4-y^2}+\frac{4}{2}\sin^{-1}\frac{y}{2}+C$

$=\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+2\sin^{-1}\left(\frac{x+1}{2}\right)+C$

(f) Manipulating integrands :

(i) $\int\frac{dx}{x(x^n+1)}$, $n\in N$, take x^n common & put $1 + x^{-n} = t$.

(ii) $\int\frac{dx}{x^2(x^n+1)^{(n-1)/n}}$, $n\in N$, take x^n common & put $1 + x^{-n} = t^n$

(iii) $\int\frac{dx}{x^n(1+x^n)^{1/n}}$, take x^n common and put $1 + x^{-n} = t^n$.

Illustration 69:

Evaluate : $\int \frac{dx}{x^n(1+x^n)^{1/n}}$

Solution:

Let $I = \int \frac{dx}{x^n(1+x^n)^{1/n}} = \int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)^{1/n}}$

Put $1+\frac{1}{x^n} = t^n$, then $\frac{1}{x^{n+1}}dx = -t^{n-1}dt$

$I = -\int \frac{t^{n-1}dt}{t} = -\int t^{n-2}dt = -\frac{t^{n-1}}{n-1}+C = \frac{-1}{n-1}\left(1+\frac{1}{x^n}\right)^{\frac{n-1}{n}} + C$ **Ans.**

Miscellaneous Illustrations :

Illustration 70:

Evaluate $\int \frac{\cos^4 x dx}{\sin^3 x\{\sin^5 x+\cos^5 x\}^{\frac{3}{5}}}$

Solution:

$I = \int \frac{\cos^4 x}{\sin^3 x\{\sin^5 x+\cos^5 x\}^{\frac{3}{5}}}dx = \int \frac{\cos^4 x}{\sin^6 x\{1+\cot^5 x\}^{\frac{3}{5}}}dx = \int \frac{\cot^4 x \operatorname{cosec}^2 x dx}{(1+\cot^5 x)^{3/5}}$

Put $1 + \cot^5 x = t$

$5\cot^4 x \operatorname{cosec}^2 x\, dx = -dt$

$= -\frac{1}{5}\int \frac{dt}{t^{3/5}} = -\frac{1}{2}t^{2/5} + C = -\frac{1}{2}(1+\cot^5 x)^{2/5} + C$ **Ans.**

Illustration 71:

$\int \frac{dx}{\cos^6 x+\sin^6 x}$ is equal to -

(A) $\ell n|\tan x - \cot x| + C$

(B) $\ell n|\cot x - \tan x| + C$

(C) $\tan^{-1}(\tan x - \cot x) + C$

(D) $\tan^{-1}(-2\cot 2x) + C$

Ans. (C,D)

Solution:

Let $I = \int \frac{dx}{\cos^6 x+\sin^6 x} = \int \frac{\sec^6 x}{1+\tan^6 x}dx = \int \frac{(1+\tan^2 x)^2 \sec^2 x dx}{1+\tan^6 x}$

If $\tan x = p$, then $\sec^2 x\, dx = dp$

$\Rightarrow I = \int \frac{(1+p^2)^2 dp}{1+p^6} = \int \frac{(1+p^2)}{p^4-p^2+1}dp = \int \frac{p^2\left(1+\frac{1}{p^2}\right)}{p^2\left(p^2+\frac{1}{p^2}-1\right)}dp$

$= \int \frac{dk}{k^2+1} = \tan^{-1}(k)+C \qquad \left(\text{where } p-\frac{1}{p} = k, \left(1+\frac{1}{p^2}\right)dp = dk\right)$

$= \tan^{-1}\left(p-\frac{1}{p}\right)+C = \tan^{-1}(\tan x - \cot x)+C = \tan^{-1}(-2\cot 2x) + C$

Illustration 72:

Evaluate : $\int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x}dx$

Solution:

$I = \int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4\sin x}dx = \int \frac{(4\sin x - 1)\cos x}{6 - (1 - \sin^2 x) - 4\sin x}dx = \int \frac{(4\sin x - 1)\cos x}{\sin^2 x - 4\sin x + 5}dx$

Put $\sin x = t$, so that $\cos x\, dx = dt$.

$\therefore I = \int \frac{(4t-1)dt}{(t^2 - 4t + 5)}$ ...(i)

Now, let $(4t - 1) = \lambda(2t - 4) + \mu$

Comparing coefficients of like powers of t, we get

$2\lambda = 4, -4\lambda + \mu = -1$ ...(ii)

$\lambda = 2, \mu = 7$

$\therefore I = \int \frac{2(2t-4)+7}{t^2 - 4t + 5}dt$ {using (i) and (ii)}

$= 2\int \frac{2t-4}{t^2-4t+5}dt + 7\int \frac{dt}{t^2-4t+5} = 2\log|t^2 - 4t + 5| + 7\int \frac{dt}{t^2 - 4t + 4 - 4 + 5}$

$= 2\log|t^2 - 4t + 5| + 7\int \frac{dt}{(t-2)^2 + (1)^2} = 2\log|t^2 - 4t + 5| + 7.\tan^{-1}(t - 2) + C$

$= 2\log|\sin^2 x - 4\sin x + 5| + 7\tan^{-1}(\sin x - 2) + C$. **Ans.**

Illustration 73:

The value of $\int \sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx$, is equal to -

(A) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$

(B) $\frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\sin^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$

(C) $\frac{1}{4}\left\{-3\left(\sin^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\sin^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$

(D) none of these

Ans. (A)

Solution:

Here, $I = \int \sqrt{\frac{3-x}{3+x}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-x}\right)dx$

Put $x = 3\cos 2\theta \Rightarrow dx = -6\sin 2\theta d\theta$

$= \int \sqrt{\frac{3-3\cos 2\theta}{3+3\cos 2\theta}}.\sin^{-1}\left(\frac{1}{\sqrt{6}}\sqrt{3-3\cos 2\theta}\right)(-6\sin 2\theta)d\theta$

$= \int \frac{\sin\theta}{\cos\theta}.\sin^{-1}(\sin\theta).(-6\sin 2\theta)d\theta = -6\int \theta.(2\sin^2\theta)d\theta$

$$= -6\int \theta(1-\cos 2\theta)d\theta = -6\left\{\frac{\theta^2}{2} - \int \theta\cos 2\theta\, d\theta\right\} + C$$

$$= -6\left\{\frac{\theta^2}{2} - \left(\theta\frac{\sin 2\theta}{2} - \int 1.\left(\frac{\sin 2\theta}{2}\right)d\theta\right)\right\} + C = -3\theta^2 + 6\left\{\theta\frac{\sin 2\theta}{2} + \frac{\cos 2\theta}{4}\right\} + C$$

$$= \frac{1}{4}\left\{-3\left(\cos^{-1}\left(\frac{x}{3}\right)\right)^2 + 2\sqrt{9-x^2}.\cos^{-1}\left(\frac{x}{3}\right) + 2x\right\} + C$$

Illustration 74:

Evaluate : $\displaystyle\int \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}}dx$

Solution:

$$I = \int \frac{\tan\left(\frac{\pi}{4}-x\right)}{\cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}}dx = \int \frac{(1-\tan^2 x)dx}{(1+\tan x)^2\cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}}$$

$$I = \int \frac{-\left(1-\frac{1}{\tan^2 x}\right)\sec^2 x dx}{\left(\tan x + 2 + \frac{1}{\tan x}\right)\sqrt{\tan x + 1 + \frac{1}{\tan x}}}$$

let, $y = \sqrt{\tan x + 1 + \frac{1}{\tan x}} \Rightarrow 2y\, dy = \left(\sec^2 x - \frac{1}{\tan^2 x}.\sec^2 x\right)dx$

$$\therefore I = \int \frac{-2ydy}{(y^2+1).y} = -2\int \frac{dy}{1+y^2}$$

$= -2\tan^{-1} y + c = -2\tan^{-1}\left(\sqrt{\tan x + 1 + \frac{1}{\tan x}}\right) + C$ **Ans.**