Indefinite Integration

SOLUTIONS

EXERCISE - O

1. **Ans. (B)**

$\because \int \frac{3x^4-1}{(x^4+x+1)^2}\cdot dx$

$= \int \frac{3x^4-1}{x^2\left(x^3+1+\frac{1}{x}\right)^2}\cdot dx$

$= \int \frac{3x^2-x^{-2}}{(x^3+1+x^{-1})^2}\cdot dx$ {Put $x^3+1+x^{-1}=t \Rightarrow (3x^2-x^{-2}).dx = dt$}

$= \int \frac{1}{t^2}\cdot dt$

$= -\frac{1}{t}+c$

$= \frac{-1}{x^3+1+x^{-1}}+c$

$= \boxed{\frac{-x}{x^4+x+1}+c}$

2. **Ans. (D)**

$I = \int \frac{x(x+1)}{e^{2x}\left(1+\frac{x+1}{e^x}\right)^2}dx = \int \frac{\frac{x}{e^x}\cdot\frac{x+1}{e^x}}{\left(1+\frac{x+1}{e^x}\right)^2}dx$

Put $1+\frac{x+1}{e^x}=t \Rightarrow \frac{e^x(1)-(x+1)e^x}{e^{2x}}dx = dt$

$\Rightarrow \frac{-x}{e^x}dx = dt$

So , $I = -\int \frac{(t-1)dt}{t^2} = \int \frac{dt}{t^2} - \int \frac{dt}{t}$

$= \frac{-1}{t} - \ell n|t| + C$

$= \frac{-1}{1+\frac{x+1}{e^x}} - \ell n\left|1+\frac{x+1}{e^x}\right| + C$

3. **Ans. (A)**

Let $I = \int \frac{x\cdot\ell n[\sqrt{1+x^2}+x]}{\sqrt{1+x^2}}\cdot dx$

Put $\ell n[\sqrt{1+x^2}+x] = t$

$\Rightarrow \sqrt{1+x^2}+x = e^t$

$\Rightarrow 1 + x^2 = (e^t - x)^2$

$\Rightarrow 1 = e^{2t} - 2xe^t$

$$\Rightarrow x = \frac{e^t - e^{-t}}{2}$$

$$\Rightarrow \boxed{dx = \frac{e^t + e^{-t}}{2} \cdot dt}$$

Now,

$\because \sqrt{1+x^2} + x = e^t$ ...(1)

& $\sqrt{1+x^2} - x = e^{-t}$ ...(2)

$\therefore$ (1) + (2) $\Rightarrow \boxed{\sqrt{1+x^2} = \frac{e^t + e^{-t}}{2}}$

(1) – (2) $\Rightarrow \boxed{x = \frac{e^t - e^{-t}}{2}}$

$$\therefore I = \int \frac{\left(\frac{e^t - e^{-t}}{2}\right) \cdot t \cdot \left(\frac{e^t + e^{-t}}{2}\right) dt}{\left(\frac{e^t + e^{-t}}{2}\right)}$$

$$\Rightarrow I = \int \underset{I}{t} \cdot \underset{II}{\left(\frac{e^t - e^{-t}}{2}\right)} \cdot \underset{\text{use by part}}{dt}$$

$$\Rightarrow I = t \cdot \left(\frac{e^t + e^{-t}}{2}\right) - \int 1 \cdot \left(\frac{e^t + e^{-t}}{2}\right) dt$$

$$\Rightarrow I = t\left(\frac{e^t + e^{-t}}{2}\right) - \left(\frac{e^t - e^{-t}}{2}\right) + C$$

$$\Rightarrow I = t\sqrt{1+x^2} - x + C$$

$$\Rightarrow \boxed{I = \sqrt{1+x^2} . ln[\sqrt{1+x^2} + x] - x + C}$$

4. **Ans. (B)**

$$I = 2\int \frac{e^{\sqrt{\sin x}} \cos x}{2\sqrt{\sin x}} dx$$

put $\sqrt{\sin x} = t$

$$\Rightarrow \frac{1}{2\sqrt{\sin x}} \times \cos x \, dx = dt$$

$$I = 2\int e^t dt = 2e^{\sqrt{\sin x}} + c$$

5. **Ans. (A)**

$\sin(100x + x) \sin^{99}x$

$= \sin(100x)\cdot\cos x\cdot\sin^{99}x + \cos(100x) \cdot (\sin x)^{100}$

$$I = \int \underbrace{\sin(100x)}_{I} \cdot \underbrace{\cos x \cdot \sin^{99} x}_{II} dx + \int \cos(100x) \cdot (\sin x)^{100} dx$$

use I.B.P. on first integration

$$I = \frac{(\sin x)^{100}}{100}\cdot\sin(100x) - \int(\sin x)^{100}\cos(100)x dx + \int\cos(100x)(\sin x)^{100}dx$$

$$= \frac{\sin(100x)(\sin x)^{100}}{100} + C$$

6. **Ans. (C)**

$$\frac{1}{2}\int\left\{2\sin\alpha\sin(x-\alpha)+2\sin^2\left(\frac{x}{2}-\alpha\right)\right\}dx$$

$$= \frac{1}{2}\{\{\cos(2\alpha-x)-\cos(x)+1-\cos(x-2\alpha)\}dx$$

$$= \frac{1}{2}\int(1-\cos x)dx = \frac{1}{2}(x-\sin x)+C$$

7. **Ans. (B)**

$$\int\frac{dx}{(x-\alpha)^2\sqrt{\frac{x-\beta}{x-\alpha}}}$$

put $\frac{x-\beta}{x-\alpha}=t^2 \Rightarrow \frac{(x-\alpha)-(x-\beta)}{(x-\alpha)^2}dx = 2tdt$

$$\Rightarrow \frac{dx}{(x-\alpha)^2}=\frac{2}{(\beta-\alpha)}tdt$$

$$I = \frac{2}{(\beta-\alpha)}\int\frac{tdt}{t}=\frac{2}{(\beta-\alpha)}\int dt$$

$$= \frac{2}{(\beta-\alpha)}t+C=\frac{2}{(\beta-\alpha)}\sqrt{\frac{x-\beta}{x-\alpha}}+C$$

8. **Ans. (D)**

$$= \int\frac{4x(x^2+1)^2-7(x^2+1)^2+12x^2}{x^2(x^2+1)^2}dx$$

$$= \int\frac{4}{x}dx-\int\frac{7}{x^2}dx+\int\frac{12}{(x^2+1)^2}dx$$

$$I = 4\,\ell nx + \frac{7}{x}+12\int\frac{dx}{(x^2+1)^2} \qquad ...(1)$$

Let $I_1 = \int\frac{dx}{(x^2+1)^2}$ put $x = \tan\theta$

$$I_1 = \int\frac{\sec^2\theta d\theta}{\sec^4\theta}=\int\cos^2\theta d\theta=\int\frac{1+\cos2\theta}{2}d\theta$$

$$= \frac{1}{2}\left(\theta+\frac{\sin2\theta}{2}\right)=\frac{1}{2}\tan^{-1}x+\frac{1}{4}\left\{\frac{2x}{1+x^2}\right\}$$

$$= \frac{1}{2}\tan^{-1}x+\frac{x}{2(1+x^2)}$$

So, $I = 4lnx + \frac{7}{x} + 6\tan^{-1}x + \frac{6x}{1+x^2}+C$

9. Ans. (C)

$(x+1)^3 - (x-1)^3 = 6x^2 + 2 = 2(3x^2+1)$

So $I = \frac{1}{2}\int \frac{(x+1)^3-(x-1)^3}{(x^2-1)^3}dx = \frac{1}{2}\int \frac{(x+1)^3-(x-1)^3}{(x+1)^3(x-1)^3}dx$

$= \frac{1}{2}\int \frac{dx}{(x-1)^3} - \frac{1}{2}\int \frac{dx}{(x+1)^3}$

$= -\frac{1}{4\cdot(x-1)^2} + \frac{1}{4\cdot(x+1)^2} + C$

10. Ans. (A)

$\frac{t+3}{t^3(t+1)} = \frac{A}{t+1} + \frac{B}{t} + \frac{C}{t^2} + \frac{D}{t^3}$

$\Rightarrow t + 3 = At^3 + Bt^2(t+1) + ct(t+1) + D(t+1)$

$t = 0 \Rightarrow \boxed{3=D}$, $t = -1 \Rightarrow 2 = -A \Rightarrow \boxed{A=-2}$

coef. of $t^3 = 0 = A + B \Rightarrow \boxed{B=2}$

coef. of $t^2 = 0 = B + C \Rightarrow \boxed{C=-2}$

So $\frac{t+3}{t^3(t+1)} = -\frac{2}{t+1} + \frac{2}{t} - \frac{2}{t^2} + \frac{3}{t^3}$

$\Rightarrow \int \frac{x^2+3}{x^6(x^2+1)}dx = \int \frac{-2}{x^2+1}dx + \int \frac{2}{x^2}dx - \int \frac{2}{x^4}dx + \int \frac{3}{x^6}dx$

$= -2\tan^{-1}(x) - \frac{2}{x} + \frac{2}{3x^3} - \frac{3}{5x^5} + C$

11. Ans. (B,D)

$I = \int \frac{\ell n\left(\frac{x-1}{x+1}\right)}{x^2-1}dx$ put $\ell n\left(\frac{x-1}{x+1}\right) = t \Rightarrow \frac{2}{x^2-1}dx = dt$

$\Rightarrow I = \int t\frac{dt}{2} = \frac{t^2}{4} + C$

$= \frac{1}{4}\ell n^2\left(\frac{x-1}{x+1}\right) + C = \frac{1}{4}\ell n^2\left(\frac{x+1}{x-1}\right) + C$

12. Ans. (A,C)

$I = \int \frac{(x-1)\ dx}{x^2\sqrt{2x^2-2x+1}} = \int \frac{(x-1)\ dx}{x^3\sqrt{2-\frac{2}{x}+\frac{1}{x^2}}}$

Put $2 - \frac{2}{x} + \frac{1}{x^2} = t^2$, then $\left(\frac{x-1}{x^3}\right)dx = t\ dt$

$\Rightarrow I = \int \frac{t\ dt}{t} = t + C = \sqrt{2-\frac{2}{x}+\frac{1}{x^2}} + C = \frac{\sqrt{2x^2-2x+1}}{x} + C$

So $f(x) = \sqrt{2x^2-2x+1}$ and $g(x) = x$

13. Ans. (A,B)

$\int \frac{4e^{x}+6e^{-x}}{9e^{x}-4e^{-x}}.dx = Ax + B\ell n\,|9e^{2x} - 4| + C$

put $4e^{x} + 6e^{-x} = P\left(9e^{x}-4e^{-x}\right) + Q(9e^{x} + 4e^{-x})$

$\Rightarrow 4 = 9P + 9Q$ and $6 = 4Q - 4P$

comparing, $P = -\frac{19}{36}, Q = \frac{35}{36}$

$I = -\frac{19}{36}\int dx + \frac{35}{36}\int \frac{9e^{x}+4e^{-x}}{9e^{x}-4e^{-x}}dx = -\frac{19}{36}.x + \frac{35}{36}\ell n\left|\left(9e^{x}-4e^{-x}\right)\right| + C$

$= -\frac{19}{36}x + \frac{35}{36}\ell n\left|\left(9e^{2x}-4\right)\right| - \frac{35}{36}x + C = \frac{35}{36}\ell n\left|\left(9e^{2x}-4\right)\right| - \frac{54}{36}x + C$

$= \frac{35}{36}\ell n\left|\left(9e^{2x}-4\right)\right| - \frac{3}{2}x + C$

So $A = -\frac{3}{2}, B = \frac{35}{36}, C \in R$

14. Ans. (A,C)

Put $x = t^{6},\ dx = 6t^{5}dt$

$I = 6\int \frac{t^{6}+t^{4}+t}{t^{6}(1+t^{2})}(t^{5})dt = 6\int \frac{t^{5}+t^{3}+1}{1+t^{2}}dt = 6\int\left(t^{3}+\frac{1}{1+t^{2}}\right)dt = 6\left(\frac{t^{4}}{4}+\tan^{-1}t\right)+c$

$= \frac{3}{2}x^{2/3} + 6\tan^{-1}(x^{1/6}) + c$

$\Rightarrow P = \frac{3}{2}$ & $q = 6$

15. Ans. (A→Q, B→P, C→P,R,S ,D→P,R)

(A) $\frac{d}{dx}(x^{\sin x}) = x^{\sin x}\left(\cos x\ \ell nx + \frac{\sin x}{x}\right)$

$f(x) = \int x^{\sin x}dx + \int x^{\sin x}\left(\cos x.\ell nx + \frac{\sin x}{x}\right)xdx$

$= x.x^{\sin x} - \int x.x^{\sin x}\left(\cos x.\ell nx + \frac{\sin x}{x}\right)dx + \int x.x^{\sin x}\left(\cos x.\ell nx + \frac{\sin x}{x}\right)dx$

$\Rightarrow f(x) = x \cdot x^{\sin x} + c$

As $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^{2}}{4} \Rightarrow c = 0 \Rightarrow f(x) = x \cdot x^{\sin x}$

$\Rightarrow f(\pi) = \pi \cdot \pi^{0} = \pi$ (Q)

(B) $g(x) = \int \frac{1+2\cos x}{(\cos x+2)^{2}}dx$

divide numerator and denominator by $\sin^{2} x$

$= \int \frac{\text{cosec}^{2} + 2\cot x\cdot\text{cosec}x}{(\cot x + 2\text{cosec}x)^{2}}dx$

put $2\text{cosec}x + \cot x = t$

$(-2\ \text{cosec}x.\cot x - \text{cosec}^{2}x)dx = dt$

$I = -\int \frac{dt}{t^2} = \frac{1}{t} + C$

$= \frac{1}{\cot x + 2\text{cosec}x} + C$

$= \frac{\sin x}{\cos x + 2} + C$

$\Rightarrow g(x) = \frac{\sin x}{\cos x + 2}$, as $g(0) = 0$

$\Rightarrow g\left(\frac{\pi}{2}\right) = \frac{1}{2}$ (P)

(C) $k(x) = \int \frac{(x^2+1)dx}{\sqrt[3]{x^3+3x+6}}$

put $x^3 + 3x + 6 = t^3$

$(3x^2 + 3)dx = 3t^2 dt$

$I = \int \frac{t^2 dt}{t} = \frac{t^2}{2} + C$

$= \frac{(x^3+3x+6)^{2/3}}{2} + C$

$\Rightarrow k(x) = \frac{1}{2}(x^3+3x+6)^{2/3}$ $\left(\because\ k(-1) = \frac{1}{\sqrt[3]{2}}\right)$

$k(-2) = \frac{1}{2}(-8-6+6)^{2/3}$

$= \frac{1}{2}(-2)^2 = 2$

(P), (R), (S)

(D) $\because \int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x}\,dx$

$= \int \frac{e^{-x}\cos x - e^{-x}\sin x + e^{-x} - e^{-x}x}{1 + e^{-x}\sin x + e^{-x}x}\,dx$

Put $1 + e^{-x}\sin x + e^{-x}x = t$

$\Rightarrow \left(e^{-x}\cos x - e^{-x}\sin x + e^{-x} - e^{-x}x\right)dx = dt$

$= \int \frac{1}{t}\,dt$

$= \ell n(t) + C$

$= \ell n\left(\frac{e^x + \sin x + x}{e^x}\right) + C$

$= \ell n\left(e^x + \sin x + x\right) - x + C$

$\therefore\ f(x) = e^x + \sin x + x\ \&\ g(x) = -x$

So, $f(x) + g(x) = e^x + \sin(x)$

$f(0) + g(0) = 1$ ....(P,R)

EXERCISE - S

1. **Ans. (0)**

$\because\ f''(x) = -\dfrac{1}{x^2} - \pi^2 \sin(\pi x)$

$\Rightarrow\ f'(x) = \int -\dfrac{1}{x^2} - \pi^2 \sin(\pi x)\cdot dx$

$\Rightarrow\ f'(x) = \dfrac{1}{x} + \pi^2 \cdot \dfrac{\cos(\pi x)}{\pi} + c$

$$\left\{\begin{array}{l} \because\ f'(2) = \pi + \dfrac{1}{2} \\ \dfrac{1}{2} + \pi\cos(2x) + c = \pi + \dfrac{1}{2} \\ \Rightarrow c = 0 \end{array}\right\}$$

$\Rightarrow f'(x) = \dfrac{1}{x} + \pi\cos(\pi x)$

$\Rightarrow f(x) = \int \dfrac{1}{x} + \pi\cos(\pi x)\cdot dx$

$\Rightarrow f(x) = ln(x) + \dfrac{\pi\sin(\pi x)}{\pi} + C_1$

$$\left\{\begin{array}{l} \because\ f(1) = 0 \\ \Rightarrow \sin\pi + c_1 = 0 \\ \Rightarrow c_1 = 0 \end{array}\right\}$$

$\Rightarrow f(x) = ln(x) + \sin(\pi x)$

$\therefore\ f\left(\dfrac{1}{2}\right) = \ln\left(\dfrac{1}{2}\right) + \sin\left(\dfrac{\pi}{2}\right) = \boxed{1 - \ell n(2)}$

2. **Ans. (5)**

$\because\ \int \dfrac{2x+3}{x(x+3)(x+1)(x+2)+1}\cdot dx$

$= \int \dfrac{2x+3}{(x^2+3x)(x^2+3x+2)+1}\cdot dx$

$$\left\{\begin{array}{l} \text{Put } x^2+3x = t \\ \Rightarrow (2x+3)\cdot dx = dt \end{array}\right\}$$

$= \int \dfrac{dt}{t(t+2)+1} = \int \dfrac{dt}{(t+1)^2}$

$= -\left(\dfrac{1}{t+1}\right) + c$

$= c - \dfrac{1}{x^2+3x+1}$

$\therefore f(x) = x^2 + 3x + 1$

through compare

$$\therefore \left.\begin{matrix} a=1 \\ b=3 \\ c=1 \end{matrix}\right\} \qquad \therefore a+b+c=5 \textbf{ Ans.}$$

3. **Ans. (6)**

$$\because \int \frac{dx}{\sqrt[3]{x^{5/2}\cdot(x+1)^{7/2}}} = \int \frac{dx}{\sqrt[3]{x^{5/2}\cdot x^{7/2}\cdot\left(1+\frac{1}{x}\right)^{7/2}}}$$

$$= \int \frac{x^{-2}}{\left(1+\frac{1}{x}\right)^{7/6}}\cdot dx \qquad \left\{\begin{matrix}\text{Put } 1+\frac{1}{x}=t \\ -x^{-2}.\,dx=dt\end{matrix}\right\}$$

$$= -\int \frac{dt}{t^{7/6}}$$

$$= 6\cdot\frac{1}{t^{1/6}}+c$$

$$= \boxed{6\left(\frac{x+1}{x}\right)^{-\frac{1}{6}}+c}$$

4. **Ans. (2)**

$$\because \int \frac{e^{\sqrt{x}}}{\sqrt{x}}(x+\sqrt{x}).dx \qquad \{\text{Put } x=t^2 \Rightarrow dx=2t\,.dt\}$$

$$= \int \frac{e^t}{t}(t^2+t).2t\ dt$$

$$= \int e^t(2t^2+2t).dt = \int e^t[(2t^2+4t)-2t]\,.dt$$

$$= \int e^t(\underset{\substack{\Downarrow \\ f(x)}}{2t^2}+\underset{\substack{\Downarrow \\ f'(x)}}{4t})dt - 2\int \underset{I}{t}\cdot \underset{II}{e^t}\,.dt$$

$$= e^t\cdot 2t^2 - 2[t\cdot e^t - \int 1\cdot e^t\cdot dt]$$

$$= 2\cdot e^t\cdot t^2 - 2[e^t\cdot t - e^t]+c$$

$$= 2e^t\cdot t^2 - 2e^t\cdot t + 2e^t + c$$

$$= e^2\,[2t^2 - 2t + 2] + c$$

$$= \boxed{2e^{\sqrt{x}}[x-\sqrt{x}+1]+c}$$

5. **Ans. (2010)**

$$I = \int \frac{(\sin x)^{2008}-(\cos x)^{2008}}{(\sin x)^{2010}+(\cos x)^{2010}}\times(\cos x)(\sin x)dx$$

$$I = \int \frac{\cos x(\sin x)^{2009}-\sin x(\cos x)^{2009}}{(\sin x)^{2010}+(\cos x)^{2010}}dx$$

put $(\sin x)^{2010} + (\cos x)^{2010} = t$

$\Rightarrow (\cos x\ (\sin x)^{2009} - \sin x\ (\cos x)^{2009})dx = \dfrac{dt}{2010}$

$I = \frac{1}{2010}\int \frac{dt}{t} = \frac{1}{2010}\ell n|t| + C$

$= \frac{1}{2010}\ell n|\sin^{2010} x + \cos^{2010} x| + C$ **Ans.**

6. **Ans. (3)**

Let $f(x) = ax^2 + bx + c$

To find: $f'(0) = b$

$f(0) = 1 \Rightarrow c = 1$

$\int \frac{ax^2 + bx + 1}{x^2[x+1]^3} dx$ is a rational function

$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$

become log fun $\Rightarrow A = C = O$ (one of the posibility)

$\Rightarrow ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$

$x = 0 \Rightarrow \boxed{B = 1}$

compare coff. of $x \Rightarrow$

$b = 3B \Rightarrow \boxed{b = 3}$

$f'(0) = 3$

7. **Ans. (2)**

$\because \int \frac{5\tan(x)}{\tan(x) - 2} dx$

$= \int \frac{5\sin(x)}{\sin(x) - 2\cos(x)} \cdot dx$

$= \int \frac{(\sin(x) - 2\cos(x)) + 2(\cos(x) + 2\sin(x))}{\sin(x) - 2\cos x} dx$

$= \int 1 \cdot dx + 2\int \frac{\cos(x) + 2\sin(x)}{\sin(x) - 2\cos(x)} \cdot dx$

$= x + 2\ell n|\sin(x) - 2\cos(x)| + C$

8. **Ans. (5)**

Let $x(1 + x^{-2})^2 + 1 = t^2$

$x(1 + x^{-4} + 2x^{-2}) + 1 = t^2$

$x + x^{-3} + 2x^{-1} + 1 = t^2$

$(1 - 3x^{-4} - 2x^{-2})dx = 2tdt$

$\frac{(x^4 - 2x^2 - 3)}{x^4} dx = 2tdt$

$I = \int \frac{2tdt}{t} = 2t + C$

$= 2\sqrt{x + \frac{1}{x^3} + \frac{2}{x} + 1} + C$

$a = 1, b = 1, c = 2, d = 1$

$a + b + c + d = 5$

9. Ans. (46)

$\int f(x)dx = F(x)$

$\int 5(x^4 - 1)dx = F(x)$

$\Rightarrow F(x) = x^5 - 5x + C$

$\because F(1) = 20 \Rightarrow C = 24$

$\therefore F(x) = x^5 - 5x + 24$

$F(2) = 46$

10. Ans. (0.19 or 0.20)

$$\int \frac{dx}{16+x^2} = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$$

given $f(4) = \frac{\pi}{16} \Rightarrow f(x) = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)$

$$f(2) + f\left(\frac{4}{3}\right) = \frac{1}{4}\left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right) = \frac{\pi}{16}$$

EXERCISE - JEE (Main) PYQ

1. Ans. (1)

$$\int \frac{\cos x\, dx}{\sin^3 x(1+\sin^6 x)^{2/3}} = \int \frac{\cos x\, dx}{\sin^7 x\left(\frac{1}{\sin^6 x}+1\right)^{2/3}}$$

Put $\frac{1}{\sin^6 x} + 1 = t$

$$\frac{-6\cos x}{\sin^7 x}dx = dt$$

$$-\frac{1}{6}\int \frac{dt}{t^{\frac{2}{3}}}$$

$$= -\frac{1}{6}\times 3\left(\frac{1}{\sin^6 x}+1\right)^{1/3} + c$$

$$= -\frac{1}{2}\frac{(1+\sin^6 x)^{1/3}}{\sin^2 x} + c$$

Hence, $\lambda = 3$ and $f(x) = -\frac{1}{2\sin^2 x}$

so, $\lambda f\left(\frac{\pi}{3}\right) = -2$

REMARK : Technically, this question should be marked as bonus. Because $f(x)$ and λ cannot be found uniquely.

From example, another such $f(x)$ and λ can be $-\frac{(1+\sin^6 x)^{\frac{1}{6}}}{2\sin^2 x}$ and 6 respectively.

2. Ans. (1)

$$I=\int \frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)}$$

$$I=\int \frac{\sec^2\theta\, d\theta}{\frac{2\tan\theta}{1-\tan^2\theta}+\frac{1+\tan^2\theta}{1-\tan^2\theta}}=\int \frac{(1-\tan^2\theta)\sec^2\theta\, d\theta}{(1+\tan\theta)^2}$$

$\tan\theta = t \Rightarrow \sec^2\theta\ d\theta = dt$

$$I=\int \frac{1-t^2}{(1+t)^2}dt=\int \frac{(1-t)(1+t)}{(1+t)^2}dt$$

$$=\int \frac{1}{1+t}-\frac{t}{1+t}dt$$

$$=\ell n|1+t|-\int\left(\frac{1+t}{1+t}-\frac{1}{1+t}\right)dt$$

$= \ell n\ |1+t| - t + \ell n|1+t| + C$

$= 2\ell n\ |1+t| - t + C$

$= 2\ell n\ |1+\tan\theta| - \tan\theta + C$

$\lambda = -1, f(\theta) = 1+\tan\theta$

3. Ans. (1)

$$I=\int \frac{dx}{(x+4)^{\frac{8}{7}}(x-3)^{\frac{6}{7}}}=\int \frac{dx}{\left(\frac{x+4}{x-3}\right)^{\frac{8}{7}}(x-3)^2}$$

Let $\frac{x+4}{x-3}=t \Rightarrow \frac{dx}{(x-3)^2}=-\frac{1}{7}dt$

$$\Rightarrow I=-\frac{1}{7}\int \frac{dt}{t^{8/7}}=-\frac{1}{7}\int t^{-8/7}dt$$

$$=t^{-1/7}+C=\left(\frac{x+4}{x-3}\right)^{-1/7}+C=\left(\frac{x-3}{x+4}\right)^{1/7}+C$$

4. Ans. (3)

$$\int \frac{\cos x-\sin x}{\sqrt{8-\sin 2x}}dx$$

$$=\int \frac{\cos x-\sin x}{\sqrt{9-(\sin x+\cos x)^2}}dx$$

Let $\sin x + \cos x = t$

$$\int \frac{dt}{\sqrt{9-t^2}}=\sin^{-1}\frac{t}{3}+c$$

$$=\sin^{-1}\left(\frac{\sin x+\cos x}{3}\right)+c$$

So, $a = 1, b = 3$.

5. Ans. (2)

$$\int\left(\frac{x^2+1}{(x+1)^2}\right)e^x\,dx$$

$$= \int\left(\frac{x^2-1+2}{(x+1)^2}\right)e^x\,dx$$

$$= \int\left(\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right)e^x\,dx$$

$$= \int(f(x)+f'(x))e^x\,dx$$

$$= f(x)e^x+c$$

Where $f(x)=\frac{x-1}{x+1}$

$$f'(x)=\frac{2}{(x+1)^2}$$

$$f''(x)=\frac{-4}{(x+1)^3}$$

$$f'''(x)=\frac{12}{(x+1)^4}$$

$$f'''(1)=\frac{12}{16}$$

$$=\frac{3}{4}$$

6. Ans. (1)

$$I(x)=\int\underset{\text{II}}{\sec^2 x}.\underset{\text{I}}{\sin^{-2022}x}\,dx-2022\int\sin^{-2022}x\,dx$$

$$=\tan x.(\sin x)^{-2022}+\int(2022)\tan x.(\sin x)^{-2023}\cos x\,dx-2022\int(\sin x)^{-2022}\,dx$$

$$I(x) = (\tan x)(\sin x)^{-2022} + C$$

At $X = \pi/4$, $\quad 2^{1011} = \left(\frac{1}{\sqrt{2}}\right)^{-2022} + C \quad \therefore\ C = 0$

Hence $I(x)=\frac{\tan x}{(\sin x)^{2022}}$

$$I(\pi/6)=\frac{1}{\sqrt{3}\left(\frac{1}{2}\right)^{2022}}=\frac{2^{2022}}{\sqrt{3}}$$

$$I(\pi/3)=\frac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)^{2022}}=\frac{2^{2022}}{(\sqrt{3})^{2021}}=\frac{1}{3^{1010}}I\left(\frac{\pi}{6}\right)$$

$$3^{1010}I\left(\frac{\pi}{3}\right)=I\left(\frac{\pi}{6}\right)$$

7. Ans. (4)

$\left(x=e^{\ln x}\right) \Rightarrow \int\left(\left(\frac{x}{e}\right)^{2x}+\left(\frac{e}{x}\right)^{2x}\right)\log_e x\,dx=\int\left[e^{2(x\ln x-x)}+e^{-2(x\ln x-x)}\right]\ln x dx$

sx $\ln x - x = t$

$\ln x . dx = dt$

$\int\left(e^{2t}+e^{-2t}\right)dt$

$\frac{e^{2t}}{2}-\frac{e^{-2t}}{2}+C$

$=\frac{1}{2}\left(\frac{x}{e}\right)^{2x}-\frac{1}{2}\left(\frac{e}{x}\right)^{2x}+C$

$\alpha = \beta = \gamma = \delta = 2$

$\alpha + 2\beta + 3\gamma - 4\delta = 4$

8. Ans. (1)

Put $x^2 = t$

$\int\frac{dt}{(t+1)(t+3)}=\frac{1}{2}\int\left(\frac{1}{t+1}-\frac{1}{t+3}\right)dt$

$f(x)=\frac{1}{2}\ln\left(\frac{x^2+1}{x^2+3}\right)+C$

$f(3)=\frac{1}{2}(\ln 10-\ln 12)+C$

$\Rightarrow C=0$

$f(4)=\frac{1}{2}\ln\left(\frac{17}{19}\right)$

9. Ans. (64)

$\int\sqrt{\frac{x+7}{x}}\ dx$

Put $x = t^2$

$dx = 2tdt$

$\int 2\sqrt{t2+7}\ \ dt = 2\int\sqrt{t2+\sqrt{7}2}\ dt$

$I(t)=2\left[\frac{t}{2}\sqrt{t^2+7}+\frac{7}{2}\ln\left|t+\sqrt{t^2+7}\right|\right]+C$

$I(x) = \sqrt{x}\ + 7\ ln\ \left|\sqrt{x}+\sqrt{1+7}\right|+C$

$I(9) = 12 + 7\ ln7 = 12 + 7\ (ln\ (3 + 4)) + C$

$\Rightarrow C = 0$

$I(x) = \sqrt{x}\ \ \sqrt{x+7}\ + 7\ ln\ \left(\sqrt{x}+\sqrt{x+7}\right)$

$I(1) = 1\ \sqrt{8}\ + 7\ ln\left(1+\sqrt{8}\right)$

$I(1) = \sqrt{8}\ + 7\ ln\ (1 + 2\sqrt{2}\,)$

$\alpha = \sqrt{8}$

$\alpha^4 = (8^{1/2})^4$

$\alpha^4 = 8^2 = 64$

10. **Ans. (1)**

$$\int \sqrt{\sec 2x - 1}\,dx = \int \sqrt{\frac{1-\cos 2x}{\cos 2x}}\,dx$$

$$= \sqrt{2}\int \frac{\sin x}{\sqrt{2\cos^2 x - 1}}\,dx$$

put $\cos x = t \Rightarrow -\sin x\,dx = dt$

$$= -\sqrt{2}\int \frac{dt}{\sqrt{2t^2 - 1}}$$

$$= -\ln\left|\sqrt{2}\cos x + \sqrt{\cos 2x}\right| + c$$

$$= -\frac{1}{2}\ln\left|2\cos^2 x + \cos 2x + 2\sqrt{\cos 2x}\cdot\sqrt{2}\cos x\right| + c \quad = -\frac{1}{2}\ln\left|\cos 2x + \frac{1}{2} + \sqrt{\cos 2x}\cdot\sqrt{1+\cos 2x}\right| + c$$

$$\because \beta = \frac{1}{2}, \alpha = -\frac{1}{2} \Rightarrow \beta - \alpha = 1$$

11. **Ans. (3)**

$$I(x) = \int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2}dx$$

Let $x\tan x + 1 = t$

$$I = x^2\left(\frac{-1}{x\tan x + 1}\right) + \int \frac{2x}{x\tan x + 1}dx$$

$$I = x^2\left(\frac{-1}{x\tan x + 1}\right) + 2\int \frac{x\cos x}{x\sin x + \cos x}dx$$

$$I = x^2\left(\frac{-1}{x\tan x + 1}\right) + 2\ln|x\sin x + \cos x| + C$$

As $I(0) = 0 \Rightarrow C = 0$

$$I\left(\frac{\pi}{4}\right) = \ln\left(\frac{(\pi+4)^2}{32}\right) - \frac{\pi^2}{4(\pi+4)}$$

EXERCISE - JEE (Advanced) PYQ

1. **Ans. (D)**

$$\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}.dx$$

$$=\int \frac{x^2-1}{x^5\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}dx$$

$$=\int \frac{\left(\frac{1}{x^3}-\frac{1}{x^5}\right)dx}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}$$

$$\left\{\begin{array}{l}\text{Put } 2-\frac{2}{x^2}+\frac{1}{x^4}=t^2\\ \Rightarrow 4\left(\frac{1}{x^3}-\frac{1}{x^5}\right)dx=2t.dt\end{array}\right\}$$

$$=\frac{1}{2}\int \frac{t}{\sqrt{t^2}}.dt$$

$$=\frac{t}{2}+C$$

$$=\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+C$$

2. **Ans. (A)**

$$f(x)=\frac{x}{(1+x^n)^{1/n}}$$

$$f(f(x))=\frac{f(x)}{[1+(f(x))^n]^{1/n}}$$

$$\text{Put } f(x)=\frac{x}{(1+x^n)^{1/n}}$$

$$f(f(x))=\frac{x}{(1+2.x^n)^{1/n}}$$

$$\text{again } f(f(f(x)))=\frac{f(x)}{[1+2(f(x))^n]^{1/n}}$$

$$=\frac{x}{(1+3x^n)^{1/n}}$$

$$(fo\,fo....of)(x)=\frac{x}{(1+n.x^n)^{1/n}}=g(x)$$

$$\therefore \int x^{n-2}.g(x)dx$$

$$=\int x^{n-2}\frac{x}{(1+nx^n)}.dx$$

$$=\int \frac{(x)^{n-1}}{(1+nx^n)^{1/n}}.dx$$

$$\left\{\begin{array}{l}\text{Put } 1+n.x^n=t\\ n^2(x)^{n-1}.dx=dt\end{array}\right\}$$

$$=\frac{1}{n^2}\int (t)^{-1/n}.dt$$

$$=\frac{1}{n^2}.\frac{(t)^{1-\frac{1}{n}}}{1-\frac{1}{n}}+K$$

$$=\frac{1}{n(n-1)}.(1+nx^n)^{1-\frac{1}{n}}+K$$

3. Ans. (D)

$F(x)=\int \sin^2(x).dx$

$\Rightarrow F(x)=\frac{1}{4}[2x-\sin 2x]+C$

So, $F(x+\pi)\neq F(x)$

$\therefore$ Statement 1 is FALSE

But statement 2 is TRUE, because period of $\sin^2(x)$ is π.

4. Ans. (C)

$J=\int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}.dx$

$=\int \frac{e^{3x}}{1+e^{2x}+e^{4x}}dx$

$I=\int \frac{e^{x}}{e^{4x}+e^{2x}+1}.dx$

$\therefore J-I=\int \frac{e^{x}(e^{2x}-1)}{e^{4x}+e^{2x}+1}dx$

$\begin{Bmatrix}\text{Put } e^x=t\\ e^x.dx=dt\end{Bmatrix}$

$=\int \frac{t^2-1}{t^4+t^2+1}.dt$

$=\int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}.dt$

$=\int \frac{\left(1-\frac{1}{t^2}\right).dt}{\left(t+\frac{1}{t}\right)^2-1}$

$\begin{Bmatrix}\text{Put } t+\frac{1}{t}=z\\ \Rightarrow\left(1-\frac{1}{t^2}\right)dt=dz\end{Bmatrix}$

$=\int \frac{dz}{z^2-1}$

$=\frac{1}{2\times 1}\ell n\left(\frac{z-1}{z+1}\right)+C$

$=\frac{1}{2}\ell n\left(\frac{t^2-t+1}{t^2+t+1}\right)+C$

$=\frac{1}{2}\ell n\left(\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right)+C$

5. Ans. (C)

Let $I=\int\frac{\sec(x).\sec(x).dx}{(\sec x+\tan x)^{9/2}}$

Put $\sec x + \tan x = t$

$\Rightarrow (\sec(x)\ \tan(x) + \sec^2(x)).\ dx = dt$

$\Rightarrow \sec(x)\ [\sec(x) + \tan(x)]\ .dx = dt$

$\sec(x)\ .\ t.\ dx = dt$

$\Rightarrow \sec(x).dx=\frac{1}{t}.dt$

& $\because \sec x + \tan x = t$

$\Rightarrow \sec x - \tan x = \frac{1}{t}$

$\Rightarrow 2\sec(x) = t+\frac{1}{t}$

$\Rightarrow \boxed{\sec(x)=\frac{1}{2}\left(t+\frac{1}{t}\right)}$

$\therefore I=\frac{1}{2}\int\frac{\left(t+\frac{1}{t}\right).\frac{dt}{t}}{(t)^{9/2}}$

$=\frac{1}{2}\left[\int t^{-9/2}.dt+\int t^{-13/2}.dt\right]$

$=\left[\frac{t^{-7/2}}{-7}+\frac{t^{-11/2}}{-11}\right]+C$

$=-t^{-11/2}\left[\frac{t^2}{7}+\frac{1}{11}\right]+C$

$=\frac{-1}{(\sec(x)+\tan(x))^{11/2}}\left[\frac{1}{11}+\frac{(\sec x+\tan x)^2}{7}\right]+C$

JEE (Main) Practice Paper

SECTION–A

1. Ans. (2)

$I=\int\frac{\ell n\left(\frac{x-1}{x+1}\right)}{x^2-1}dx$

put $\ell n\left(\frac{x-1}{x+1}\right) = t \Rightarrow \frac{2}{x^2-1}dx = dt$

$\Rightarrow I = \int t\frac{dt}{2} = \frac{t^2}{4} + C = \frac{1}{4}\log^2\left(\frac{x-1}{x+1}\right)+C$

2. Ans. (1)

$I = \int \frac{\ell n\,(\tan x)}{\sin x\ \cos x} dx$

put ℓn tan$x$ = $t \Rightarrow \frac{1}{\sin x \cos x} dx = dt$

$I = \int t\ \ dt = \frac{t^2}{2} + C = \frac{1}{2}(\ell n \tan x)^2 + C$

3. Ans. (2)

$f(x) = \int \frac{2\sin x - \sin 2x}{x^3} dx$

$f'(x) = \frac{2\sin x - \sin 2x}{x^3} = \frac{2\sin x}{x} \cdot \frac{1-\cos x}{x^2} = 2\left(\frac{\sin x}{x}\right) \cdot \frac{2\sin^2 \frac{x}{2}}{\frac{x^2}{4}\times 4} = \frac{2\times 2}{4}\left(\frac{\sin x}{x}\right) \cdot \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2$

$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right) \cdot \lim_{x\to 0}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = 1$

4. Ans. (3)

$I = \int x^3(x^6 + x^3 + 1)(2x^6 + 3x^3 + 6)^{1/3}\ dx$

$= \int x^2(x^6 + x^3 + 1)(2x^9 + 3x^6 + 6x^3)^{1/3}\ dx$

Let $2x^9 + 3x^6 + 6x^3 = t$

$18(x^8 + x^5 + x^2)\ dx = dt$

$I = \frac{1}{18}\int t^{1/3} dt = \frac{1}{18} \cdot \frac{t^{4/3}}{4/3} + c = \frac{1}{24}\ (2x^9 + 3x^6 + 6x^3)^{4/3} + c$

5. Ans. (1)

$\int \frac{dx}{\sin x . \sin(x+\alpha)}$

$= \frac{1}{\sin\alpha} \int \frac{\sin(\alpha + x - x)}{\sin x\ \sin(x+\alpha)} dx$

$= \text{cosec}\ \alpha \int \frac{\sin(x+\alpha)\cos x - \cos(x+\alpha)\sin x}{\sin\ x \sin(x+\alpha)} = \text{cosec}\ \alpha \left[\int \cot x\, dx - \int \cot(x+\alpha)\right] + C$

$= \text{cosec}\ \alpha\ [\log|\sin x| - \log|\sin(x+\alpha)|] + C = \text{cosec}\ \alpha \log\left|\frac{\sin x}{\sin(x+\alpha)}\right| + C$

6. Ans. (3)

$\int (1 + \tan x\ \tan(x+\alpha))dx$

$= \int \frac{\sin x \sin(x+\alpha) + \cos x \cos(x+\alpha)}{\cos x \cos(x+\alpha)} = \int \frac{\cos(x+\alpha-x)}{\cos x \cos(x+\alpha)} dx$

$= \cot\alpha \int \frac{\sin(x+\alpha-x)}{\cos x \cos(x+\alpha)} dx = \cot\alpha \left[\int \frac{\sin(x+\alpha)\cos x}{\cos x \cos(x+\alpha)} - \frac{\cos(x+\alpha)\sin x}{\cos x \cos(x+\alpha)} dx\right]$

$= \cot\alpha\left[\int \tan(x+\alpha)\,dx - \int \tan x\,dx\right] = \cot\alpha\,[\ell n|\sec(x+\alpha)| - \ell n|\sec x|]$

$= \cot\alpha\ \ell n\left|\dfrac{\cos x}{\cos(x+\alpha)}\right| + C = \cot\alpha\ \ell n\left(\left|\dfrac{\sec(x+\alpha)}{\sec x}\right|\right) + C$

7. **Ans. (3)**

$\displaystyle\int \frac{x^2+\cos^2 x}{1+x^2}\operatorname{cosec}^2 x\,dx = \int \operatorname{cosec}^2 x\,dx - \int \frac{1}{1+x^2}\,dx$

$= -\cot x - \tan^{-1}x + c = -\tan^{-1}x - \dfrac{\operatorname{cosec} x}{\sec x} + c$

8. **Ans. (3)**

$I = \displaystyle\int \sqrt{\frac{x-1}{x+1}} \times \frac{1}{x^2}\,dx$

Put $\dfrac{1}{x} = \cos 2\theta \Rightarrow -\dfrac{dx}{x^2} = -2\sin 2\theta\,d\theta$

$I = \displaystyle\int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\ 2\sin 2\theta\,d\theta$

$= \displaystyle\int 4\sin^2\theta\,d\theta = 2\int (1-\cos 2\theta)\,d\theta = 2\theta - \sin 2\theta + C = \cos^{-1}\left(\frac{1}{x}\right) - \sqrt{1-\frac{1}{x^2}} + C$

9. **Ans. (1)**

$\displaystyle\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}.\ dx = Ax + B\ell n\,|9e^{2x}-4| + C$

put $4e^x + 6e^{-x} = P\left(9e^x - 4e^{-x}\right) + Q\,(9e^x + 4e^{-x})$

$\Rightarrow 4 = 9P + 9Q$ and $6 = 4Q - 4P$

comparing, $P = -\dfrac{19}{36}, Q = \dfrac{35}{36}$

$I = -\dfrac{19}{36}\displaystyle\int dx + \frac{35}{36}\int \frac{9e^x+4e^{-x}}{9e^x-4e^{-x}}\,dx = -\frac{19}{36}.x + \frac{35}{36}\ \ell n\left|\left(9e^x-4e^{-x}\right)\right| + C$

$= -\dfrac{19}{36}x + \dfrac{35}{36}\ \ell n\left|\left(9e^{2x}-4\right)\right| - \dfrac{35}{36}x + C = \dfrac{35}{36}\ \ell n\left|\left(9e^{2x}-4\right)\right| - \dfrac{54}{36}x + C$

$= \dfrac{35}{36}\ \ell n\left|\left(9e^{2x}-4\right)\right| - \dfrac{3}{2}x + C$

So $A = -\dfrac{3}{2}, B = \dfrac{35}{36}, C \in R$

10. **Ans. (2)**

$\displaystyle\int \frac{dx}{\sqrt{\sin^3 x(\sin x\cos\alpha - \cos x\sin\alpha)}} = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha - \sin\alpha\cot x}}\,dx = \frac{2}{\sin\alpha}\sqrt{\cos\alpha - \sin\alpha\cot x} + C$

11. **Ans. (2)**

$\displaystyle\int \frac{\cos 4x + 1}{\cot x - \tan x}\,dx = A\cos 4x + B$

$= \displaystyle\int \frac{2\cos^2 2x\sin x\cos x}{\cos 2x}\,dx = \int 2\cos 2x\sin x\cos x\,dx = \frac{1}{2}\int \sin 4x\,dx = \frac{1}{2}\frac{(-\cos 4x)}{4} + B$

12. Ans. (4)

We known that the reduction formula of cos$x$ is

$I_n = \int \cos^n x dx = \frac{n-1}{n} I_{n-2} + \frac{1}{n} cos^{n-1} x sin^x$

$\therefore I_6 = \frac{5}{6} I_4 + \frac{1}{6} \cos^5 x \sin x \Rightarrow I_6 = \frac{5}{6}\left[\frac{3}{4} I_2 + \frac{1}{4}\cos^3 x \sin x\right] \frac{1}{6}\cos^5 x \sin x$

$\Rightarrow I_6 = \frac{5}{8} I_2 + \frac{5}{24} \cos^3 x \sin x + \frac{1}{6} \cos^5 \sin x$

$\Rightarrow I_6 = \frac{5}{8}\left[\frac{1}{2}x + \frac{1}{2}\cos x \sin x\right] + \frac{5}{24} x \cos^3 x \sin x + \frac{1}{6} \cos^5 x \sin x + C$

$\Rightarrow I_6 = \frac{5}{16}[x + \cos x \sin x] + \frac{5}{24}\cos^3 x \sin x + \frac{1}{6}\cos^5 x \sin x + C$

13. Ans. (2)

$I = \int \sqrt{\frac{1}{(\tan x)^{11}(\cos^8 x)}} dx = \int (\tan x)^{-11/2} \sec^4 x dx$

$= \int (\tan x)^{-11/2}(1 + \tan^2 x)\sec^2 x dx$

put tan$x$ = t $\sec^2 x dx = dt$

$I = \int (t^{-11/2} + t^{-7/2}) dt$

$= \frac{-2}{9} t^{-9/2} + \frac{-2}{5} t^{-5/2} + C = \frac{-2}{9}(\tan x)^{-9/2} + \frac{-2}{5}(\tan x)^{-5/2} + C \Rightarrow A = \frac{1}{9}, B = \frac{1}{5}$

14. Ans. (3)

$\int \frac{dx}{(x+1)\sqrt{x-2}}$

let $x - 2 = t^2$

$dx = 2tdt$

$\int \frac{2t\,dt}{(t^2+3)t} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} + C$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\sqrt{\frac{x-2}{3}}\right) + C.$

15. Ans. (4)

$\int \frac{1}{x^5(1+\frac{1}{x^4})^{3/4}} dx$

Let $1 + \frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5} dx = dt \quad = -\frac{1}{4}\int \frac{dt}{t^{3/4}} = -\frac{1}{4} \cdot \frac{t^{1/4}}{1/4} + c = -\left(1 + \frac{1}{x^4}\right)^{1/4} + c$

16. Ans. (3)

$\int \frac{1 - x^7}{x(1 + x^7)} dx = \int \frac{(1 + x^7) - 2x^7}{x(1 + x^7)} dx = \int \frac{1}{x} dx - 2\int \frac{x^6}{(1 + x^7)} dx = \ell n |x| - \frac{2}{7} \ell n |1 + x^7| + c$

17. Ans. (1)

Put $1 - x^3 = t^2 \Rightarrow -3x^2\,dx = 2t\,dt$

$$I = \int \frac{dx}{x\sqrt{1-x^3}} = \int \frac{x^2 dx}{x^3\sqrt{1-x^3}} = -\frac{1}{3}\int \frac{2tdt}{(1-t^2).t} = -\frac{2}{3}\int \frac{dt}{1-t^2} = \frac{2}{3}\int \frac{dt}{t^2-1}$$

$$= \frac{2.1}{3.2}\ell n\left|\frac{1-t}{1+t}\right| = \frac{1}{3}\ell n\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + c$$

18. Ans. (1)

$$I = \int \frac{e^x - 1}{\sqrt{e^{2x}-1}}\,dx$$

$$= \int \frac{e^x}{\sqrt{e^{2x}-1}}\,dx - \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}dx$$

put $e^x = t \Rightarrow e^x\,dx = dt$

$$\Rightarrow I = \int \frac{dt}{\sqrt{t^2-1}} - \int \frac{dt}{t\sqrt{t^2-1}}$$

$$= \ell n\left|t + \sqrt{t^2-1}\right| - \sec^{-1}(t) + C,\quad \text{where } t = e^x$$

19. Ans. (3)

$$I = \int \frac{e^{\sqrt{x}}(x+\sqrt{x})}{\sqrt{x}}dx$$

put $\sqrt{x} = t \Rightarrow \frac{1}{\sqrt{x}}dx = 2\,dt$

$$\Rightarrow I = 2\int e^t\left(t^2+t\right) dt = 2\left[e^t.t^2 - \int 2t.e^t\,dt + \int e^t.t\,dt\right] = 2\left[e^t.t^2 - \int t.e^t\,dt\right] = 2\left[e^t.t^2 - t.e^t + e^t\right] + C$$

$$= 2e^t\left[t^2 - t + 1\right] + C = 2\,e^{\sqrt{x}}\,[x - \sqrt{x} + 1] + C$$

20. Ans. (4)

$$F(x) = \int e^{\sin^{-1}x}\left(1 - \frac{x}{\sqrt{1-x^2}}\right)dx$$

$$= \int e^{\sin^{-1}x}\left(\frac{1}{\sqrt{1-x^2}}\quad\sqrt{1-x^2} + \frac{-x}{\sqrt{1-x^2}}\right)dx$$

put $\sin^{-1}x = t \Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$

$$\Rightarrow F(x) = \int e^t\left(\sqrt{1-\sin^2 t} - \sin t\right)dt = e^t\cos t + C = e^{\sin^{-1}x}\sqrt{1-x^2} + C$$

$\because\ 1 = F(0) \Rightarrow C = 0$

Hence, $F(1/2) = e^{\pi/6}.\ \frac{\sqrt{3}}{2} = \frac{k\sqrt{3}}{\pi}\ e^{\pi/6}$ (given) $\therefore\ k = \frac{\pi}{2}$

SECTION-B

1. Ans. (1)

$$f(x) = \int \frac{2\sin x - \sin 2x}{x^3}\, dx$$

$$f'(x) = \frac{2\sin x - \sin 2x}{x^3} = \frac{2\sin x}{x} \cdot \frac{1-\cos x}{x^2} = 2\left(\frac{\sin x}{x}\right) \cdot \frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}\times 4} = \frac{2\times 2}{4}\left(\frac{\sin x}{x}\right) \cdot \left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right) \cdot \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = 1$$

2. Ans. (3)

$$\int \sin^4 x \cos^4 x\, dx = \frac{1}{16}\int \sin^4 2x\, dx = \frac{1}{16}\int\left(\frac{1-\cos 4x}{2}\right)^2 dx = \frac{1}{64}\int\left(1+\cos^2 4x - 2\cos 4x\right) dx$$

$$= \frac{1}{64}\int dx - \frac{1}{32}\int \cos 4x\, dx + \frac{1}{64}\int\left(\frac{1+\cos 8x}{2}\right)dx$$

$$= \frac{x}{64} - \frac{\sin 4x}{128} + \frac{1}{64}\times\frac{x}{2} + \frac{1}{128}\frac{\sin 8x}{8} + C$$

$$= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{\sin 8x}{128\times 8} + C$$

$$= \frac{1}{128}\left[3x - \sin 4x + \frac{\sin 8x}{8}\right] + C$$

3. Ans. (12)

$$f(x) = \int \frac{3x+2}{\sqrt{x-9}}dx$$

put $x - 9 = t^2$

$$= 2\int (29+3t^2)dt$$

$f(x) = 2(29\sqrt{x-9} + (x-9)^{3/2}) + C$

$\therefore f(10) = 60 \Rightarrow C = 0$

$f(x) = 2(29\sqrt{x-9} + (x-9)^{3/2})$

$f(13) = 132 \Rightarrow$ sum of digits = 6

2(sum of digits) = 12

4. Ans. (10)

$$I = \int \frac{\sqrt{4+x^2}}{x^6}\, dx$$

Put $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$

$$\Rightarrow I = \int \frac{2\sec\theta . 2\sec^2\theta\, d\theta}{2^6\tan^6\theta} = \frac{1}{2^4}\int \frac{\cos^3\theta\, d\theta}{\sin^6\theta}$$

$$= \frac{1}{2^4}\int\left(\frac{1-\sin^2\theta}{\sin^6\theta}\right)\cos\theta\, d\theta$$

put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$$= \frac{1}{2^4}\int \frac{dt}{t^6} - \int \frac{1}{t^4}dt = \frac{1}{16}\left[\frac{1}{3t^3} - \frac{1}{5t^5}\right] + C = \frac{1}{16}\left(\frac{5\sin^2\theta - 3}{15\sin^5\theta}\right) + C$$

but $\tan\theta = \frac{x}{2}$

So $\sin\theta = \frac{x}{\sqrt{4+x^2}}$

$$= \frac{1}{16}\left(\frac{\frac{5x^2}{x^2+4} - 3}{\frac{15x^5}{(x^2+4)^{5/2}}}\right) + C = \frac{1}{16}\left(\frac{(2x^2-12)\,(x^2+4)^{3/2}}{15x^5}\right) + C = \frac{1}{120}\left(\frac{(x^2+4)^{3/2}\,(x^2-6)}{x^5}\right) + C$$

5. Ans. (5)

$$I = \int \sqrt{\frac{x}{a^3 - x^3}}\, dx$$

put $x^3 = a^3 \sin^2\theta$

$$\Rightarrow x = a\sin^{2/3}\theta \Rightarrow dx = a \times \frac{2}{3}\frac{\cos\theta}{\sin^{1/3}\theta}\, d\theta$$

$$\Rightarrow I = \int \sqrt{\frac{a\sin^{2/3}\theta}{a^3(1-\sin^2\theta)}} \times \frac{2a}{3}\left(\frac{\cos\theta}{\sin^{1/3}\theta}\right) d\theta = \int \frac{\sin^{1/3}\theta}{a\cos\theta} \times \frac{2a}{3}\frac{\cos\theta}{\sin^{1/3}\theta}\, d\theta = \frac{2}{3}\theta + C$$

$$= \frac{2}{3}\sin^{-1}\left(\left(\frac{x}{a}\right)^{3/2}\right) + C$$

6. Ans. (2)

$$I = \int \frac{x\ dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}$$

Put $1 + x^2 = t^2$, then $2x\,dx = 2t\,dt$

$$\Rightarrow I = \int \frac{t\ dt}{\sqrt{t^2+t^3}} = \int \frac{dt}{\sqrt{1+t}} = 2\sqrt{1+t} = 2\sqrt{1+\sqrt{1+x^2}} + C$$

7. Ans. (2)

$$\int e^{\sin x}.\frac{x\cos^3 x - \sin x}{\cos^2 x}\, dx = \int e^{\sin x}(x\cos x - \tan x\sec x)\ dx = \int e^{\sin x}.\ x\cos x\ dx - \int e^{\sin x}\tan x\sec x dx$$

(by parts integration)

$$= xe^{\sin x} - \int 1.\ e^{\sin x}dx - [e^{\sin x}\sec x - \int e^{\sin x}\cos x.\sec x dx] + C$$

$$= e^{\sin x}(x - \sec x) + C$$

$\therefore f(x) = x - \sec x$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2$$

8. Ans. (16)

$g(x) = \int \frac{1+2\cos x}{(\cos x+2)^2}dx$; $g(x) = \int \frac{\cos x(\cos x+2)+\sin^2 x}{(\cos x+2)^2}dx$

$g(x) = \int \cos x.\frac{1}{(\cos x+2)}dx + \int \frac{\sin^2 x}{(\cos x+2)^2}dx$

$= \frac{1}{(\cos x+2)}.\sin x - \int -\frac{(-\sin x)}{(\cos x+2)^2}.\sin x dx + \int \frac{\sin^2 x}{(\cos x+2)^2}dx$

$g(x) = \frac{\sin x}{(\cos x+2)} + C$

$g(0) = 0 \Rightarrow C = 0$

$g(x) = \frac{\sin x}{\cos x+2}$

$g\left(\frac{\pi}{2}\right) = \frac{1}{2}$; $32g\left(\frac{\pi}{2}\right) = 16$

9. Ans. (12)

$fog(x) = \sqrt{e^x - 1}$

$\int fog(x)dx = \int \sqrt{e^x-1}\, dx$ put $e^x - 1 = t^2$

$= \int \frac{2t^2}{1+t^2}dt = 2t - 2\tan^{-1}t + C = 2\sqrt{e^x-1} - 2\tan^{-1}\sqrt{e^x-1} + C$

$A^3 + B^2 = 2^3 + (-2)^2 = 12$

10. Ans. (11)

$I = \int \frac{2\sin 2\phi - \cos\phi}{6 - \cos^2\phi - 4\sin\phi}d\phi$

$= \int \frac{(4\sin\phi - 1)\cos\phi}{6 - (1-\sin^2\phi) - 4\sin\phi}d\phi$

put $\sin\phi = t \Rightarrow \cos\phi\, d\phi = dt$

$\Rightarrow I = \int \frac{(4t-1)\,dt}{5+t^2-4t} = 2\int \frac{(2t-4)+7/2}{t^2-4t+5}dt = 2[\ell n|t^2 - 4t + 5|] + 7\frac{1}{(t-2)^2+(1)^2}dt$

$= 2[\ell n|t^2 - 4t + 5|] + 7\tan^{-1}\left(\frac{t-2}{1}\right) + C = 2\,\ell n\left|\sin^2\phi - 4\sin\phi + 5\right| + 7\tan^{-1}(\sin\phi - 2) + C$

JEE (Advanced) Practice Paper

1. Ans. (A)

$\int \frac{dx}{\sin(x-a)\cos(x-b)}$

$= \frac{1}{\cos(b-a)}\int \frac{\cos((x-a)-(x-b))}{\sin(x-a)\cos(x-b)}dx$

$= \frac{1}{\cos(b-a)}\int \left[\frac{\cos(x-a)\cos(x-b)}{\sin(x-a)\cos(x-b)} + \frac{\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}\right]dx$

$= \frac{1}{\cos(b-a)}\left[\int \cot(x-a)\ dx + \int \tan(x-b)\ dx\right]$

$= \frac{1}{\cos(a-b)}$ $[\ell n\,|\sin(x-a)| - \ell n\,|\cos(x-b)|] + C$ $= \frac{1}{\cos(a-b)}\ \ell n\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C$

2. Ans. (C)

$I = \int x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\ dx$

Put $x^2 = a^2 \cos 2\theta \Rightarrow 2x\,dx = -2a^2 \sin 2\theta\ d\theta$

$\Rightarrow I = \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\ (-a^2 \sin 2\theta)\,d\theta = -a^2 \int \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\,d\theta$

$= -a^2 \int (1-\cos 2\theta)\ d\theta = -a^2\theta + \frac{a^2 \sin 2\theta}{2} + C_1 = \frac{a^2}{2}\sqrt{1-\frac{x^4}{a^4}} - \frac{a^2}{2}\cos^{-1}\left(\frac{x^2}{a^2}\right) + C_1$

$= \frac{1}{2}\sqrt{a^4-x^4} + \frac{a^2}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) + C$

3. Ans. (A)

$I = \int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}}\,dx$

$= \int \frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}\,dx$

put $\frac{x-1}{x+2} = t \Rightarrow \frac{3}{(x+2)^2}\,dx = dt$

$\Rightarrow I = \frac{1}{3}\int t^{-3/4}dt = \frac{1}{3}\frac{t^{1/4}}{(1/4)} + C = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + C$

4. Ans. (A)

$I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Put $x = a\tan^2\theta \Rightarrow dx = 2a\tan\theta\sec^2\theta\,d\theta$

$\therefore I = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a\sec^2\theta}} \times 2a\tan\theta\sec^2\theta\,d\theta$

$= 2a\int \theta\ \tan\theta\sec^2\theta\ d\theta = 2a[\theta\int \tan\theta\sec^2\theta d\theta - \int(1\int \tan\theta\ \sec^2\theta\ d\theta)d\theta]$

$= 2a\left[\theta\ \frac{\sec^2\theta}{2} - \frac{1}{2}\int \sec^2\theta\ d\theta\right]$

$= a\,[\theta\tan^2\theta - \tan\theta + \theta] + C = a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C$

$= (a+x)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$

5. **Ans. (A)**

$I = \int \frac{2\ell nx}{x}(x^{\ell nx})(\ell n(x^{\ell nx}))dx$

put $x^{\ell nx} = t \Rightarrow \frac{2\ell nx}{x}(x^{\ell nx}) = dt$

$I = \int \ell nt dt = t\ell nt - t + C = x^{\ell nx}(\ell nx)^2 - x^{\ell nx} + C$

6. **Ans. (D)**

$I = \int e^{\tan\theta}(\sec\theta - \sin\theta)\ d\theta = \int e^{\tan\theta}\left(\frac{\sec\theta - \sin\theta}{\sec^2\theta}\right)\ \sec^2\theta\ \ d\theta$

$I = \int e^{\tan\theta}\left(\frac{1}{\sec\theta} - \frac{\sin\theta}{\sec^2\theta}\right)\ \sec^2\theta\, d\theta = \int e^{\tan\theta}\left(\frac{1}{\sqrt{1+\tan^2\theta}} - \frac{\tan\theta}{(1+\tan^2\theta)^{3/2}}\right)\sec^2\theta\, d\theta$

$= e^{\tan\theta}.\ \frac{1}{\sqrt{1+\tan^2\theta}} = e^{\tan\theta}.\cos\theta + C$

7. **Ans. (A)**

$\int\left\{\ln(1+\sin x) + x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\}dx = \int \ell n(1+\sin x)\ \ dx + \int x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\ dx$

$= x.\ell n(1+\sin x) - \int \frac{1}{1+\sin x}\cos x.\ \ x\ \ dx + \int x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\ dx$

$= x.\ell n\ (1+\sin x) - \int \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}.xdx + \int x\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\ dx = x\ \ell n\ (1+\sin x) + C$

8. **Ans. (A)**

$\int x.\frac{\ell n(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\ dx$

$I = \int \ell n(x+\sqrt{1+x^2})\ .\ \frac{x}{\sqrt{1+x^2}}\ dx$

$= \sqrt{1+x^2}\ \ \ell n(x+\sqrt{1+x^2}) - \int \frac{1}{\sqrt{1+x^2}}.\sqrt{1+x^2}dx$ (using integration by parts)

$= \sqrt{1+x^2}\ \ \ell n(x+\sqrt{1+x^2}) - x + C$

Alternate:

Let $x + \sqrt{1+x^2} = e^t$ then $\sqrt{1+x^2} - x = e^{-t}$

$\frac{dx}{\sqrt{1+x^2}} = dt$

$I = \int x\ \ t\ \ dt = \int \frac{(e^t - e^{-t})t\ \ dt}{2} = \frac{1}{2}\ t(e^t + e^{-t}) - \frac{1}{2}(e^t - e^{-t}) + C$

$= \sqrt{1+x^2}\ \ \ell n\ (x + \sqrt{1+x^2}\ ) - x + C.$

9. **Ans. (A,C)**

$f\left(\frac{1-x}{1+x}\right)=x \Rightarrow f(x) = \frac{1-x}{1+x}$

$g(x) = \int f(x)dx = \int \frac{1-x}{1+x}dx$

$g(x) = -x + 2\ell n|1 + x| + c$; $\int g(x)dx = -\frac{x^2}{2}+2x\ell n(x+1)-2x+2\ell n(x+1)+c$

$g'(x) = \frac{1-x}{1+x}$

$\lim_{x\to\infty} g'(x) = -1$

10. **Ans. (B,D)**

$I = \int \frac{\ell n\left(\frac{x-1}{x+1}\right)}{x^2-1}\,dx$

put $\ell n\left(\frac{x-1}{x+1}\right) = t \Rightarrow \frac{2}{x^2-1}dx = dt$

$\Rightarrow I = \int t\frac{dt}{2} = \frac{t^2}{4} + C$

$= \log^2\left(\frac{x-1}{x+1}\right) + C = \frac{1}{4}\log^2\left(\frac{x+1}{x-1}\right) + C$

11. **Ans (B,C)**

$I = \int \frac{\cos^3 x}{\sin^2 x+\sin x}dx = \int \frac{\cos x\ .\ (1-\sin^2 x)}{\sin x(1+\sin x)}dx$

Put $\sin x = t$, then $\cos x\ dx = dt$

$\Rightarrow I = \int \frac{(1-t)(1+t)\ \ dt}{t\ \ (1+t)} = \ell n\,|t| - t + C = \ell n\,|\sin x| - \sin x + C$

12. **Ans. (A,C)**

$I = \int \frac{(x-1)\ \ dx}{x^2\sqrt{2x^2-2x+1}} = \int \frac{(x-1)\ \ dx}{x^3\sqrt{2-\frac{2}{x}+\frac{1}{x^2}}}$

Put $2 - \frac{2}{x} + \frac{1}{x^2} = t^2$, then $\left(\frac{x-1}{x^3}\right) dx = t\,dt$

$\Rightarrow I = \int \frac{t\ \ dt}{t} = t + C = \sqrt{2-\frac{2}{x}+\frac{1}{x^2}} + C = \frac{\sqrt{2x^2-2x+1}}{x} + C$

So $f(x) = \sqrt{2x^2-2x+1}$ and $g(x) = x$

13. **Ans. (A,B)**

$\because \sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$

$\therefore I = \int \frac{2}{\pi}\left(\frac{\pi}{2}-2\cos^{-1}\sqrt{x}\right)dx$

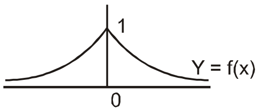
$= x - \frac{4}{\pi}\int \cos^{-1}\sqrt{x}\,dx = x - \frac{4}{\pi}\left[\cos^{-1}\sqrt{x}.x - \int x.\frac{-1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}dx\right.$

$= x - \frac{4x}{\pi}\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int\frac{\sqrt{x}}{\sqrt{1-x}}dx$

$= x - \frac{4x}{\pi}\cos^{-1}\sqrt{x} - \frac{2}{\pi}(\sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x}) + C$

14. Ans. (A,B,C)

$I = \int e^x\left(\frac{1}{\sqrt{1+x^2}} - \frac{x}{(1+x^2)^{3/2}}\right)dx$



$I = \int e^x(f(x) + f'(x))dx = f(x)e^x + C$

$I = e^x.\frac{1}{\sqrt{1+x^2}} + C$

$f(x) = \frac{1}{\sqrt{1+x^2}}$

15. Ans. (A)

Let $P = \sin^3x\cos^3x$

$\frac{dp}{dx} = 3\sin^2x\cos^4x - 3\sin^4x\cos^2x = 3\sin^2x(1-\sin^2x)\cos^2x - 3\sin^4x\cos^2x$

$= 3\sin^2x\cos^2x - 6\sin^4x\cos^2x$

$\therefore P = 3I_{2,2} - 6I_{4,2}$

$\therefore I_{4,2} = \frac{1}{6}(-P + 3I_{2,2})$

16. Ans. (A)

Let $P = \sin^5x\cos^3x$

$\therefore \frac{dp}{dx} = 5\sin^4x\cos^4x - 3\sin^6x\cos^2x$

$= 5\sin^4x(1-\sin^2x)\cos^2x - 3\sin^6x\cos^2x = 5\sin^4x\cos^2x - 8\sin^6x\cos^2x$

$\therefore P = 5I_{4,2} - 8I_{6,2}$

$\therefore I_{4,2} = \frac{1}{5}(P + 8I_{6,2})$

17. Ans. (B)

Let $P = \sin^5x\cos^3x$

$\therefore \frac{dp}{dx} = 5\sin^4x\cos^4x - 3\sin^6x\cos^2x$

$= 5\sin^4x\cos^4x - 3\sin^4x(1-\cos^2x)\cos^2x$

$= 8\sin^4x\cos^4x - 3\sin^4x\cos^2x$

$\therefore P = 8I_{4,4} - 3I_{4,2}$

$\therefore I_{4,2} = \frac{1}{3}(-P + 8I_{4,4})$

18. Ans. (13)

$$I=\int\frac{(x-1)^2}{x^4+x^2+1}\,dx = \int\frac{x^2+1}{x^4+x^2+1}\,dx-\int\frac{2x}{x^4+x^2+1}dx = I_1 - I_2$$

Now $I_1 = \int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3}\,dx$

put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx=dt$

$$\Rightarrow I_1 = \int\frac{dt}{t^2+3}=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)+C_1$$

Also $I_2 = \int\frac{2x}{x^4+x^2+1}\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\Rightarrow I_2 = \int\frac{dt}{t^2+t+1}=\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C_2$$

$$\int\frac{(x-1)^2}{x^4+x^2+1}dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right)-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$$

19. Ans. (1)

$$\int\frac{1+x\cos x}{x\left(1-x^2e^{2\sin x}\right)}dx$$

Put $xe^{\sin x} = t \Rightarrow (xe^{\sin x}.\cos x + e^{\sin x})\,dx = dt \Rightarrow e^{\sin x}(x\cos x+1)dx = dt$

$$\Rightarrow I = \int\frac{dt}{t(1-t^2)} = \int\frac{1}{t(1-t)(1+t)}dt$$

Let $\frac{1}{t(1-t)(1+t)} = \frac{A}{t}+\frac{B}{(1-t)}+\frac{C}{1+t}$

$1 = A(1-t)(1+t) + B(t)(1+t) + C(t)(1-t)$

put $t = 0 \Rightarrow A = 1$

Put $t = 1 \Rightarrow B = 1/2$

Put $t = -1 \Rightarrow C = -1/2$

$$\Rightarrow I = \int\left\{\frac{1}{t}+\frac{1}{2(1-t)}-\frac{1}{2(1+t)}\right\}dt = \ell n\,|t|-\frac{1}{2}\ell n\,|1-t|-\frac{1}{2}\ell n\,|1+t|+C$$

$$= \ell n\,|x\,e^{\sin x}|-\frac{1}{2}\log|1-x^2e^{2\sin x}|+C = \ell n\sqrt{\frac{x^2e^{2\sin x}}{1-x^2e^{2\sin x}}}+C$$

20. **Ans. (2)**

$$\int \frac{x^4+1}{x(x^2+1)^2}\,dx = A\,\ell n\,|x| + \frac{B}{1+x^2} + C$$

$$\int \frac{x^4+1}{x(x^2+1)^2}dx = \int \frac{(x^2+1)^2-2x^2}{x(x^2+1)^2}dx = \int \frac{1}{x}\,dx - \int \frac{2x}{(x^2+1)^2}dx = \ell n\,|x| + \frac{1}{1+x^2} + C$$

21. **Ans. (1)**

$$I = \int \frac{1}{1-\sin^4 x}dx = \int \frac{\sec^4 x}{\sec^4 x-\tan^4 x}dx = \int \frac{\sec^2 x.\ \sec^2 x\ \ dx}{\sec^2 x+\tan^2 x}$$

$$= \int \frac{(1+\tan^2 x)\ \ \sec^2 x\ \ dx}{2\tan^2 x+1}$$

put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\Rightarrow I = \frac{1}{2}\int \frac{2+2t^2}{2t^2+1}\ dt = \frac{1}{2}\left[\int dt+\int \frac{1}{2t^2+1}dt\right] = \frac{1}{2}t+\frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\ \ t)+C$$

$$= \frac{1}{2}\tan x+\frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+C$$

22. **Ans. (11)**

$$I = \int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}dx = \int \frac{(2-3\sin^2 x+\sin^4 x)\cos x}{\sin^2 x(1+\sin^2 x)}dx$$

Let $sinx = t$

$$I = \int \frac{2-3t^2+t^4}{t^2+t^4} = \int\left(1+\frac{2}{t^2}-\frac{6}{t^2+1}\right)dt = t-\frac{2}{t} - 6\tan^{-1}(t) + c = \sin x - \frac{2}{\sin x} - 6\tan^{-1}(\sin x) + C$$

23. **Ans. (10)**

$$I = \int \frac{dx}{\sqrt{\sin^3 x\cos^5 x}} = \int \frac{dx}{\sqrt{\tan^3 x\cos^8 x}} = \int \frac{\sec^4 x\ \ dx}{\sqrt{\tan^3 x}}$$

$$= \int \frac{(1+\tan^2 x)\ \ \sec^2 x}{\sqrt{\tan^3 x}}\,dx$$

Let $\tan x = t^{2/3} \Rightarrow \sec^2 x\,dx = \frac{2}{3}t^{-1/3}dt$

$$\Rightarrow I = \int \frac{(1+t^{4/3})}{t}\frac{2}{3}t^{-1/3}dt = \frac{2}{3}\int(1+t^{4/3})\ .t^{-4/3}dt = \frac{2}{3}\int(t^{-4/3}+1)\ \ dt = \frac{2}{3}\left[\frac{t^{-1/3}}{(-1/3)}+t\right]+C$$

$$= -2\ \ t^{-1/3}+\frac{2}{3}t+C = -2\ \sqrt{\cot x}+\frac{2}{3}\tan^{3/2}x+C \Rightarrow a = -2,\ b = 2/3$$