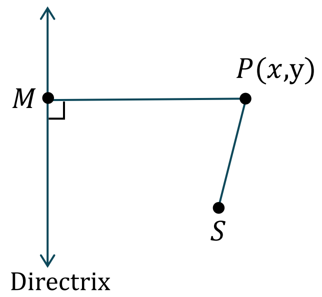
05 Hyperbola

General Equation of Hyperbola:

In the previous chapter, we have learned that hyperbola is one of the conic sections.

Let (α, β) be the focus S, and $\ell x + my + n = 0$ is the equation of directrix.

Let $P(x, y)$ be any point on the hyperbola. Then by definition.



$\frac{PS}{PM} = e \Rightarrow SP = e.PM$ (e is the eccentricity) $\Rightarrow (x-\alpha)^2 + (y-\beta)^2 = e^2\frac{(\ell x+my+n)^2}{(\ell^2+m^2)}$

$\Rightarrow (\ell^2 + m^2)\{(x-\alpha)^2 + (y-\beta)^2\} = e^2\{\ell x + my + n\}^2$

This simplifies to $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

in which $\Delta \neq 0, h^2 > ab$. $(\Delta = abc + 2fgh - af^2 - bg^2 - ch^2)$

Illustration 1:

Find the equation of a hyperbola whose focus is (1, 2) eccentricity is $\sqrt{3}$ and the directrix $2x + 3y = 0$.

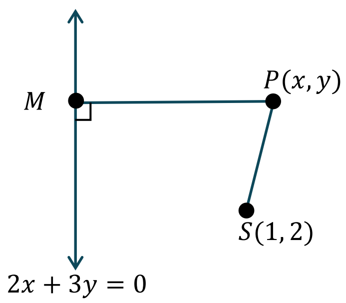
Solution:

Let $P(x, y)$ be any point on the hyperbola

Whose focus is $S(1, 2)$ and the directrix is $2x + 3y = 0$

PM is perpendicular from $P(x, y)$ on the directrix $2x + 3y = 0$

Then by definition $SP = e.PM$



$\Rightarrow (SP)^2 = e^2(PM)^2 \Rightarrow (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x+3y}{\sqrt{13}}\right\}^2$

$\Rightarrow 13(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + 9y^2 + 12xy)$

$\Rightarrow x^2 - 14y^2 - 36xy - 26x - 52y + 65 = 0$

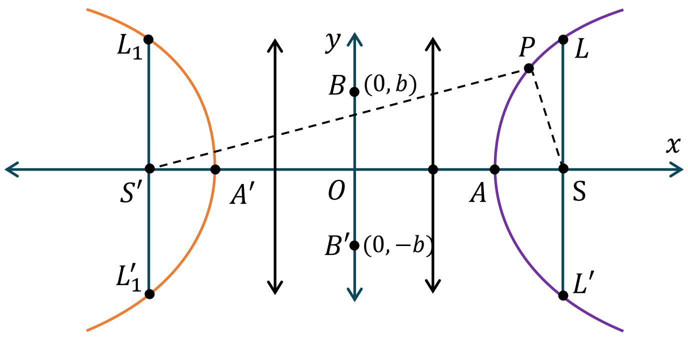
which is the required equation of the hyperbola.

Two Standard Hyperbola:

Standard Equation and Definitions:

Standard equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $b^2 = a^2(e^2-1)$

or $a^2e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \frac{b^2}{a^2} = 1 + \left(\frac{\text{Length of Conjugate Axis}}{\text{Length of Transverse Axis}}\right)^2$



(a) Foci : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(b) Equations of directrices: $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

(c) Vertices : $A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

(d) Latus rectum:

(i) Equation : $x = \pm ae$

(ii) Length $= \frac{2b^2}{a} = \frac{(\text{Length of conjugate Axis})}{(\text{Length of Transverse Axis})} = 2a(e^2-1) = 2e$

(distance from focus to directrix)

(iii) Ends : $\left(ae, \frac{b^2}{a}\right), \left(ae, \frac{-b^2}{a}\right); \left(-ae, \frac{b^2}{a}\right). \left(-ae, \frac{-b^2}{a}\right)$

(e) (i) Transverse Axis :

The line segment $A'A$ of length $2a$ in which the foci S' & S both lie is called the Transverse Axis of the Hyperbola.

(ii) Conjugate Axis :

The line segment $B'B$ between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the Conjugate Axis of the Hyperbola.

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the Principal axes of the hyperbola.

(f) Focal Property:

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left| |PS| - |PS'| \right| = 2a$. The distance SS' = focal length.

(g) Focal distance :

Distance of any point $P(x, y)$ on Hyperbola from foci $PS = ex - a$ & $PS' = ex + a$.

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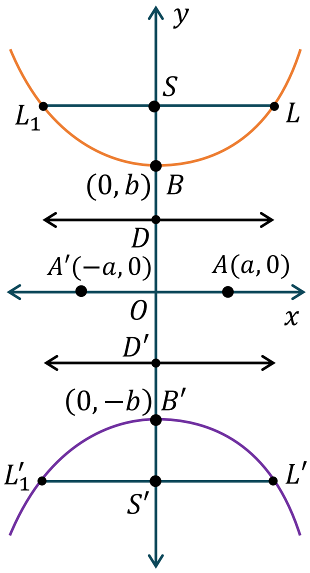
Another Standard Hyperbola:



$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

Conjugate axis $= AA' = 2a$

Transverse axis $= BB' = 2b$

$$a^2 = b^2 (e^2 - 1) \Rightarrow e = \sqrt{1 + \frac{a^2}{b^2}}$$

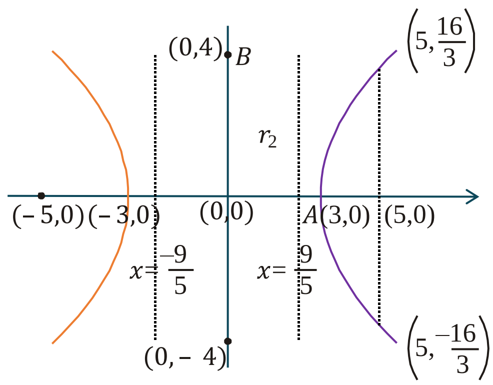
Foci : $(0, \pm be)$ $SS' = 2be$ Equation of directrix $y = \pm \frac{b}{e}$

Latus rectum $LL' = L_1L_1' = \frac{2a^2}{b}$ equation of latus rectum $y = \pm be$

Illustration 2:

Find all parameters of hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

Solution:



$$a = 3, b = 4, \ e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$$

Foci : $(\pm ae, 0), (\pm 5, 0)$

Vertices : $(\pm a, 0), (\pm 3, 0)$

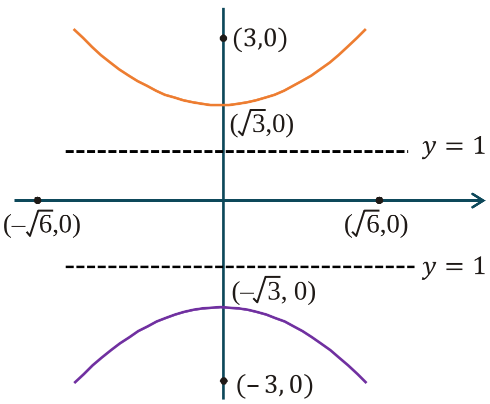
Latus rectum $= \frac{2b^2}{a} = \frac{32}{3}$

Directrix: $x = \pm \frac{a}{e} = \pm \frac{9}{5}$

Illustration 3:

Find all parameters of hyperbola $3x^2 - 6y^2 = -18$.

Solution:



$\frac{x^2}{6} - \frac{y^2}{3} = -1$, Conjugate hyperbola

$a^2 = 6, b^2 = 3,$

Transverse Axis $= 2b = 2\sqrt{3}$,

Conjugate Axis = $2\sqrt{6}$

$$e=\sqrt{1+\frac{a^2}{b^2}}=\sqrt{3}$$

Foci: $(0,\pm be) \equiv (0,\pm 3)$

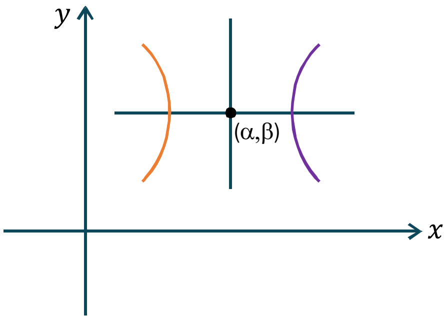
Vertices: $(0,\pm b) \equiv (0,\pm\sqrt{3})$

Latus Rectum = $\frac{2a^2}{b}=4\sqrt{3}$

Directrix: $y=\pm\frac{b}{e} \Rightarrow y=\pm 1$.

Shifted Hyperbola:

In case the centre of the hyperbola is (α, β) and axes parallel to coordinate axes then its equation can be taken as $\frac{(x-\alpha)^2}{a^2}-\frac{(y-\beta)^2}{b^2}=1$



Note: In case the transverse and conjugate axes of the hyperbola are parallel to y and x axis respectively then the equation of hyperbola becomes $\frac{(x-\alpha)^2}{a^2}-\frac{(y-\beta)^2}{b^2}=-1$.

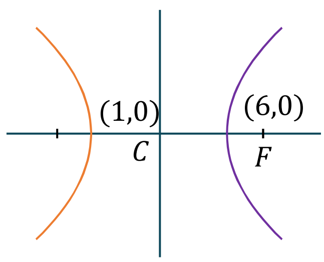
Illustration 4:

Find the equation of hyperbola whose centre is (1, 0); focus is (6, 0) and transverse axis 6.

Solution:

Equation of hyperbola with centre (1, 0)

$$\frac{(x-1)^2}{a^2}-\frac{y^2}{a^2(e^2-1)} = 1 \quad ...(1)$$



Given $a = 3$ and $ae = 5$ hence $e=\frac{5}{3}$

∴ equation (1) becomes

$$\frac{(x-1)^2}{9}-\frac{y^2}{9\left(\frac{25}{9}-1\right)}=1 \Rightarrow \frac{(x-1)^2}{9}-\frac{y^2}{16}=1.$$

Illustration 5:

Find the equation of hyperbola whose centre is (– 3, 2), one vertex is (– 3, 4) and eccentricity is 5/2.

Solution:

Equation of hyperbola is $\frac{(x+3)^2}{a^2}-\frac{(y-2)^2}{b^2}=-1$

(as the line joining centre to the vertex is parallel to y-axis)

now $b = 2$ (distance between centre and vertex)

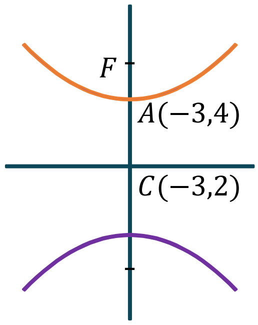
focus = $be = 2 \cdot \frac{5}{2} = 5$

Also $a^2 = b^2(e^2-1) = 4\left(\frac{25}{4}-1\right) = 21$

$\therefore \frac{(x+3)^2}{21}-\frac{(y-2)^2}{4}=-1$

$4(x+3)^2 - 21(y-2)^2 = -84$

$4x^2 - 21y^2 + 24x + 84y + 36 = 0.$



Conjugate Hyperbola:

Corresponding to every hyperbola there exist a hyperbola such that, the conjugate axis and transverse axis of one is equal to the transverse axis and conjugate axis of other, such hyperbolas are known as conjugate to each other.

Hence for the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1)

the conjugate hyperbola is, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ ...(2)

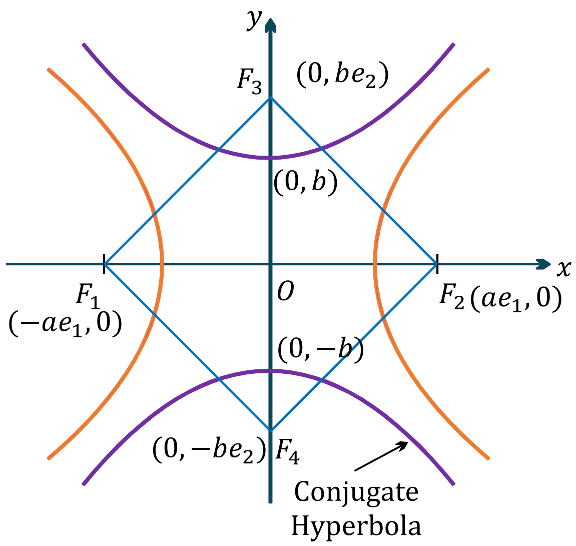
if e_1 and e_2 are the eccentricities of a hyperbola and its conjugate respectively, then $\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$

Proof:

For hyperbola $e_1^2=1+\frac{b^2}{a^2}=\frac{a^2+b^2}{a^2}$

For conjugate hyperbola = $e_2^2=1+\frac{a^2}{b^2}=\frac{a^2+b^2}{b^2}$

$\therefore \frac{1}{e_1^2}+\frac{1}{e_2^2}=1$



Note: The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

Proof:

All the four sides of the quadrilateral $F_1F_3F_2F_4$ obviously are equal with their diagonals at right angles $\Rightarrow$ a rhombus. Now to prove that $F_1F_3F_2F_4$ is a square it is sufficient to prove that,

$ae_1 = be_2$ or $a^2e_1^2 = b^2e_2^2 = a^2(e_1^2-1)e_2^2$ or $e_1^2 = e_1^2e_2^2 - e_2^2$

or $e_1^2 + e_2^2 = e_1^2e_2^2$ or $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$ which is True

Hence $ae_1 = be_2 \Rightarrow F_1F_3F_2F_4$ is a square.

Illustration 6:

The eccentricity of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is-

(A) 2 (B) $2/\sqrt{3}$ (C) 4 (D) 4/3

Solution:

Equation of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is

$-x^2 + 3y^2 = 1 \Rightarrow -\frac{x^2}{1} + \frac{y^2}{1/3} = 1$

Here $a^2 = 1, b^2 = 1/3$

$\therefore$ Eccentricity $e = \sqrt{1 + a^2/b^2} = \sqrt{1+3} = 2$.

Position of a Point 'P' w.r.t. a Hyperbola:

Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

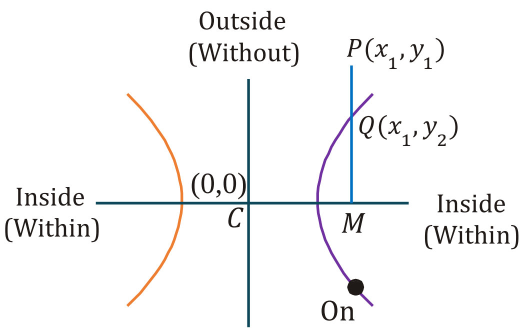
The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is positive, zero or negative according as the point (x_1, y_1) lies within, upon or without the curve.

Explanation: (x_1, y_2) lies on the hyperbola

$\therefore \frac{x_1^2}{a^2} - \frac{y_2^2}{b^2} = 1 \Rightarrow \left(\frac{x_1^2}{a^2} - 1\right)b^2 = y_2^2$

Now if $P(x_1, y_1)$ lies outside then $y_1^2 > y_2^2$

$y_1^2 > \left(\frac{x_1^2}{a^2} - 1\right)b^2 \Rightarrow \frac{y_1^2}{b^2} > \frac{x_1^2}{a^2} - 1$ or $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$



Note : $S_1 > 0$ Inside

$S_1 = 0$ On

$S_1 < 0$ Outside

Illustration 7:

Find the position of the point (2, 4) relative to the hyperbola $3x^2 - 5y^2 = 15$.

Solution:

Here, $x_1 = 2$ and $y_1 = 4$

$S_1 = 3(2)^2 - 5(4)^2 - 15 = -83 < 0$

Hence the point lies outside the hyperbola.

Illustration 8:

Find the position of the point $(-1, 2)$ relative to the hyperbola $(x-1)^2 - 2(y+1)^2 = 1$.

Solution:

Here, $x_1 = -1$ and $y_1 = 2$

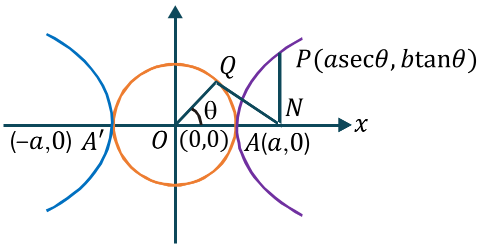
$S_1 = (-1-1)^2 - 2(2+1)^2 - 1 = -15 < 0$

Hence the point lies outside the hyperbola.

Auxiliary Circle/Eccentric Angle/Parametric Coordinates:

Definition:

Circle described on the transverse axis as diameter is called the auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



has the equation $x^2 + y^2 = a^2$ ...(i)

Corresponding Points:

Perpendicular drawn from the point P on the hyperbola, to the transverse axis meet it at the point N then tangent drawn from the point N to auxiliary circle, touches it at Q then Q is known as corresponding point. From the above figure P & Q are corresponding points of each other.

Eccentric Angle of Point P:

Angle made by radius vector through Q (corresponding point of P) with transverse axis is eccentric angle of point 'P'. $\angle QON = \theta \neq \angle PON$ $0 \le \theta < 2\pi, \theta \neq \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$.

Parametric Coordinates of Any Point P:

If $Q \equiv (a\cos\theta, a\sin\theta)$

$\frac{ON}{OQ} = \sec\theta \Rightarrow ON = a\sec\theta$

Put in hyperbola: $\frac{a^2\sec^2\theta}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow y = b\tan\theta$

Hence $P \equiv (a\sec\theta, b\tan\theta)$

P and Q are corresponding points.

Note:

The equation $x = a\ sec\theta$ & $y = b\ tan\theta$ together represents the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where θ is a parameter. If $\theta\in\left(0,\frac{\pi}{2}\right)$, $(x\to+, y\to+)$ then P lies on right upper branch ; if $\theta\in\left(\frac{\pi}{2},\pi\right)$ $(x\to-, y\to-)$ then P lies on left lower branch; if $\theta\in\left(\pi,\frac{3\pi}{2}\right)$ $(x\to-, y\to+)$ then P lies on left upper branch and if $\theta\in\left(\frac{3\pi}{2},2\pi\right)$, $(x\to+, y\to-)$ then P lies on right lower branch.

Illustration 9:

Find the parametric equation of the hyperbola $\frac{x^2}{36}-\frac{y^2}{25}=1$.

Solution:

Here, $a = 6, b = 5$

$x = 6\ sec\theta$ and $y = 5\ tan\theta$

Illustration 10:

Find the parametric equation of the hyperbola $\frac{(x-5)^2}{36}-\frac{(y+3)^2}{25}=1$.

Solution:

Here, $a = 6, b = 5$ and centre $(5, -3)$

$x = 5 + 6sec\theta$ and $y = -3 + 5tan\theta$

Illustration 11:

Find eccentric angle of the points lie on the hyperbola $\frac{x^2}{16}-\frac{y^2}{25}=1$ which is/are at a distance of $\sqrt{139}$ from the centre.

Solution:

$16sec^2\theta + 25tan^2\theta = 139$

$\Rightarrow 41tan^2\theta = 123$

$\Rightarrow tan\theta = \pm\sqrt{3}$

$\Rightarrow \theta = 60°, 120°, 240°, 300°$.

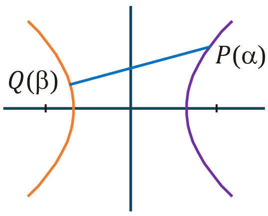
Chord Joining Two Points of Hyperbola:

Chord of Hyperbola:

Joining two points with eccentric angles α and β is given by

$$y - b\ tan\beta = \frac{b\tan\alpha-b\tan\beta}{a\sec\alpha-a\sec\beta}(x-a\sec\beta)$$

$$\Rightarrow y - b\ tan\beta = \frac{b\sin(\alpha-\beta)}{a(\cos\beta-\cos\alpha)}(x-a\sec\beta)$$



$$\Rightarrow \left(\frac{y}{b}-\tan\beta\right)=\frac{2\sin\frac{(\alpha-\beta)}{2}\cos\frac{(\alpha-\beta)}{2}}{2\sin\frac{(\alpha+\beta)}{2}\sin\frac{(\alpha-\beta)}{2}}\left(\frac{x}{a}-\sec\beta\right)$$

$$\Rightarrow \frac{y}{b}\sin\frac{(\alpha+\beta)}{2}-\sin\frac{(\alpha+\beta)}{2}\tan\beta$$

$\Rightarrow \frac{x}{a}\cos\frac{(\alpha-\beta)}{2} - \sec\beta \cdot \cos\frac{(\alpha-\beta)}{2}$

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha-\beta}{2} = \cos\frac{\alpha+\beta}{2} \qquad ...(1)$$

If (1) passes through $(d, 0)$ then

$$\frac{d}{a}\cos\frac{\alpha-\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

$$\frac{d}{a} = \frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

$$\frac{d+a}{d-a} = -\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} \Rightarrow \boxed{\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{a-d}{a+d}}$$

Illustration 12:

Find the equation of chord of a hyperbola $\frac{x^2}{36} - \frac{y^2}{25} = 1$ joining two points $P\left(\frac{\pi}{3}\right)$ and $Q\left(\frac{2\pi}{3}\right)$.

Solution:

Here, $\alpha = \frac{\pi}{3}$, $\beta = \frac{2\pi}{3}$ $a = 6, b = 5$

Equation of the chord PQ

$$\Rightarrow \frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha-\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

$$\Rightarrow \frac{x}{6}\cos\frac{\left(\frac{\pi}{3}-\frac{2\pi}{3}\right)}{2} - \frac{y}{5}\sin\frac{\left(\frac{\pi}{3}+\frac{2\pi}{3}\right)}{2} = \cos\frac{\left(\frac{\pi}{3}+\frac{2\pi}{3}\right)}{2}$$

$$\Rightarrow \frac{x}{6}\left(\frac{\sqrt{3}}{2}\right) - \frac{y}{5}(1) = 0$$

$$\Rightarrow 5\sqrt{3}x - 12y = 60$$

Illustration 13:

If $(a\ sec\theta, b\ tan\theta)$ and $(asec\phi, b\ tan\phi)$ are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ equal to

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{1+e}{1-e}$ (D) $\frac{e+1}{e-1}$

Ans. (B, C)

Solution:

Equation of chord connecting the points $(asec\theta, b\ tan\theta)$ and $(asec\phi, b\ tan\phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right) \qquad ...(i)$$

If it passes through $(ae, 0)$; we have, $e\cos\left(\frac{\theta-\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$

$$\Rightarrow e=\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}=\frac{1-\tan\frac{\theta}{2}.\tan\frac{\phi}{2}}{1+\tan\frac{\theta}{2}.\tan\frac{\phi}{2}} \Rightarrow \tan\frac{\theta}{2}.\tan\frac{\phi}{2}=\frac{1-e}{1+e}$$

Similarly if (i) passes through $(-ae, 0)$, $\tan\frac{\theta}{2}.\tan\frac{\phi}{2}=\frac{1+e}{1-e}$

Tangents to The Hyperbola:

Line and a Hyperbola:

Consider the line $y = mx + c$ and hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$b^2x^2 - a^2(mx + c)^2 = a^2b^2$

$(b^2 - a^2m^2)x^2 - 2a^2mc\, x - a^2(b^2 + c^2) = 0$ ...(1)

$D = (2a^2mc)^2 - 4(b^2 - a^2m^2)(-a^2b^2 - a^2c^2)$

$= 4a^2b^2[b^2 + c^2 - a^2m^2]$

If $D > 0$ then line is secant

If $D = 0$ the line is tangent

If $D < 0$ the line does not intersect the hyperbola

For the line to be tangent $D = 0 \Rightarrow c^2 = a^2m^2 - b^2$.

Tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

(a) Point form:

Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

Note: In general two tangents can be drawn from an external point $(x_1\ y_1)$ to the hyperbola and they are $y - y_1 = m_1(x - x_1)$ & $y - y_1 = m_2(x - x_1)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$. If $D < 0$, then no tangent can be drawn from $(x_1\ y_1)$ to the hyperbola.

(b) Slope form:

The equation of tangents of slope m to the given hyperbola is $y = mx \pm\sqrt{a^2m^2-b^2}$. Point of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2-b^2}}\right)$

Note: There are two parallel tangents having the same slope m.

(c) Parametric form:

Equation of the tangent to the given hyperbola at the point $(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$

Note: Point of intersection of the tangents at θ_1 & θ_2 is $x = a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}$, $y = b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$.

Illustration 14:

Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Solution:

Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y + 4 = 0$

$\therefore m \times 1 = -1 \Rightarrow m = -1$

Since $x^2 - 4y^2 = 36$ or $\frac{x^2}{36} - \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore a^2 = 36$ and $b^2 = 9$

So the equation of tangents are $y = (-1)x \pm \sqrt{36 \times (-1)^2 - 9}$

$y = -x \pm \sqrt{27} \Rightarrow x + y \pm 3\sqrt{3} = 0.$

Illustration 15:

Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

Solution:

The given line is $x\cos\alpha + y\sin\alpha = p \Rightarrow y\sin\alpha = -x\cos\alpha + p$

$\Rightarrow y = -x\cot\alpha + p\,\mathrm{cosec}\,\alpha$

Comparing this line with $y = mx + c$

$m = -\cot\alpha, c = p\,\mathrm{cosec}\,\alpha$

Since the given line touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then

$c^2 = a^2m^2 - b^2 \Rightarrow p^2\,\mathrm{cosec}^2\alpha = a^2\cot^2\alpha - b^2$ or $p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$

Illustration 16:

A common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$ is -

(A) $y = 3\sqrt{\frac{2}{7}}x - \frac{15}{\sqrt{7}}$ (B) $y = 3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$ (C) $y = -3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$ (D) $y = -3\sqrt{\frac{2}{7}}x - \frac{15}{\sqrt{7}}$

Ans. (A,B,C,D)

Solution:

$\frac{x^2}{16} - \frac{y^2}{9} = 1, x^2 + y^2 = 9$

Equation of tangent $y = mx + \sqrt{16m^2 - 9}$ (for hyperbola)

Equation of tangent $y = m'x + 3\sqrt{1 + m'^2}$ (circle)

For common tangent $m = m'$ and $3\sqrt{1 + m'^2} = \sqrt{16m^2 - 9}$

or $9 + 9m^2 = 16m^2 - 9$ or $7m^2 = 18 \Rightarrow m = \pm 3\sqrt{\frac{2}{7}}$

required equation is $y = \pm 3\sqrt{\frac{2}{7}}\,x \pm 3\sqrt{1 + \frac{18}{7}}$ or $y = \pm 3\sqrt{\frac{2}{7}}\,x \pm \frac{15}{\sqrt{7}}$

Asymptotes:

Definition:

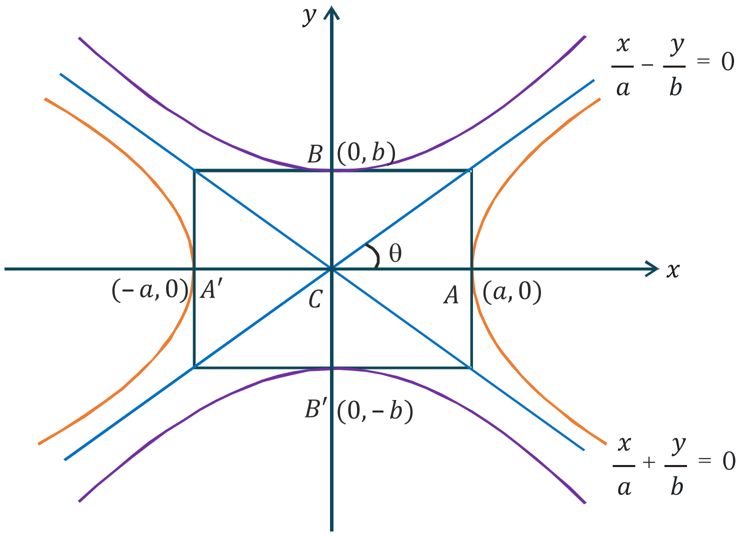
If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.

To find the asymptote of the hyperbola:

Let $y = mx + c$ is the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving these two we get the quadratic as $(b^2 - a^2m^2)x^2 - 2a^2 mcx - a^2 (b^2 + c^2) = 0$ ...(1)

In order that $y = mx + c$ be an asymptote, both roots of equation (1) must approach infinity, the conditions for which are : coefficient of $x^2 = 0$ & coefficient of $x = 0$.



$\Rightarrow b^2 - a^2m^2 = 0$ or $m = \pm\frac{b}{a}$ & $a^2 mc = 0 \Rightarrow c = 0$.

equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$.

combined equation to the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Note :

(i) A hyperbola and its conjugate have the same asymptote.

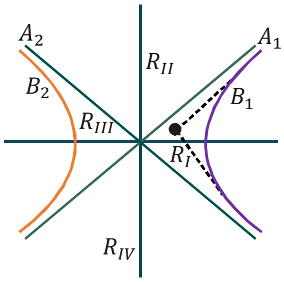
(ii) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

(iii) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

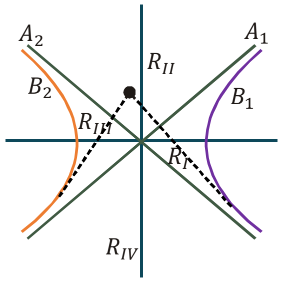
(iv) A simple method to find the co-ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as : Let $f(x, y) = 0$ represents a hyperbola. Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ gives the centre of the hyperbola.

An Important Concept:

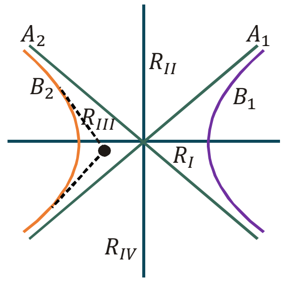
Region from where tangents drawn to same/different branches of hyperbola



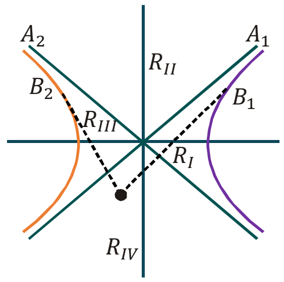
If point lies in region-I (R_I) then both tangents drawn will be on the same branch (B_1) of the hyperbola.



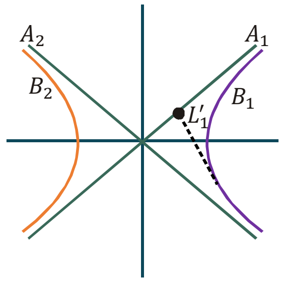
If point lies in region-II (R_{II}) then both tangents drawn will be on the different branches (B_1, B_2) of the hyperbola.



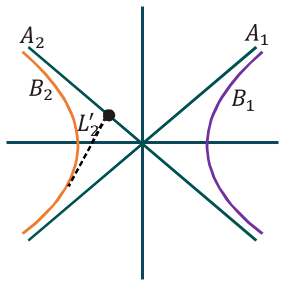
If point lies in region-III (R_{III}) then both tangents drawn will be on the same branch (B_2) of the hyperbola.



If point lies in region-IV (R_{IV}) then both tangents drawn will be on the different branches (B_1, B_2) of the hyperbola.



If point lies on the line A_1 then one of the tangent will be A_1 itself and another will be $L_1{'}$ on the branch (B_1) of the hyperbola.



If point lies on the line A_2 then one of the tangent will be A_2 itself and another will be $L_2{'}$ on the branch (B_2) of the hyperbola.

Illustration 17:

Find the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Find also the general equation of all the hyperbolas having the same set of asymptotes.

Solution:

Let $2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ be asymptotes. This will represent two straight line so

$4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0 \Rightarrow \lambda = 2$

$\Rightarrow 2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$ are asymptotes

$\Rightarrow (2x + y + 2) = 0$ and $(x + 2y + 1) = 0$ are asymptotes

And $2x^2 + 5xy + 2y^2 + 4x + 5y + c = 0$ is general equation of hyperbola.

Illustration 18:

Find the hyperbola whose asymptotes are $2x - y = 3$ and $3x + y - 7 = 0$ and which passes through the point (1, 1).

Solution:

The equation of the hyperbola differs from the equation of the asymptotes by a constant

$\Rightarrow$ The equation of the hyperbola with asymptotes $3x + y - 7 = 0$ and $2x - y = 3$ is

$(3x + y - 7)(2x - y - 3) + k = 0$

It passes through $(1, 1) \Rightarrow k = -6$.

Hence the equation of the hyperbola is $(2x - y - 3)(3x + y - 7) = 6$.

Normal to the Hyperbola:

Normal to the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(a) Point form : The equation of the normal to the given hyperbola at the point $P(x_1, y_1)$ on it is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$.

(b) Slope form : The equation of normal of slope m to the given hyperbola is $\boldsymbol{y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2 - m^2b^2}}}$, where $m \in \left(-\frac{a}{b}, \frac{a}{b}\right)$ foot of normal are $\left(\pm\frac{\boldsymbol{a^2}}{\sqrt{\boldsymbol{a^2 - m^2b^2}}}, \mp\frac{\boldsymbol{mb^2}}{\sqrt{\boldsymbol{a^2 - m^2b^2}}}\right)$

(c) Parametric form : The equation of the normal at the point $P(a\sec\boldsymbol{\theta}, b\tan\boldsymbol{\theta})$ to the given hyperbola is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$ or $ax\cos\theta + by\cot\theta = a^2 + b^2 = a^2e^2$.

Illustration 19:

Find the equation of tangent and normal to the hyperbola $16x^2 - 9y^2 = 144$ at the point $\left(5, \frac{16}{3}\right)$.

Solution:

$\frac{x^2}{9} - \frac{y^2}{16} = 1$

Tangent : $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \Rightarrow \frac{x\cdot(5)}{9} - \frac{y\left(\frac{16}{3}\right)}{16} = 1 \Rightarrow 5x - 3y = 9$

Normal : $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = 25 \Rightarrow \frac{9x}{5} + \frac{16y}{16/3} = 25 \Rightarrow 9x + 15y = 125$

Illustration 20:

Find the equation of normal to the hyperbola $3x^2 - 4y^2 = 12$ having slope 1.

Solution:

Here, $a^2 = 4, b^2 = 3$ and $m = 1$

Equation of the normal in slope form is $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2-m^2b^2}}$, where $m \in \left(-\frac{a}{b}, \frac{a}{b}\right)$

$\Rightarrow y = x \pm \frac{(3+4)}{\sqrt{4-3}} \quad \Rightarrow y = x \pm 7$.

Hence equation of normal is $y = x \pm 7$.

Illustration 21:

If a chord joining the points $P(a \sec\theta, a\tan\theta)$ & $Q(a\sec\phi, a\tan\phi)$ on the hyperbola $x^2 - y^2 = a^2$ is a normal to it at P, then show that $\tan\phi = \tan\theta\,(4\sec^2\theta - 1)$.

Solution:

$x^2 - y^2 = a^2$ ...(1)

Equation of normal at P is

$\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a$ ...(2)

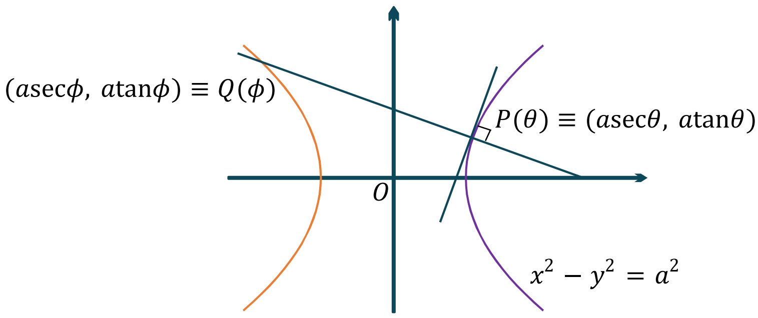
$\frac{x}{\sec\theta} = \left(2a - \frac{y}{\tan\theta}\right)$

$\sec^2\theta\left(2a - \frac{y}{\tan\theta}\right)^2 - y^2 = a^2$

$\sec^2\theta\left[4a^2 + \frac{y^2}{\tan^2\theta} - \frac{4ay}{\tan\theta}\right] - y^2 = a^2$

$\left(\frac{\sec^2\theta}{\tan^2\theta} - 1\right)y^2 - \left(\frac{4a\sec\theta}{\tan\theta}\right)y$

$+ a^2(4\sec^2\theta - 1) = 0$



$y_1y_2 = a^2(4\sec^2\theta - 1)\cdot\tan^2\theta = \tan\theta\tan\phi\; a^2$ (y_1 & y_2 are two roots of above quadratic equation)

$\tan\phi = \tan\theta(4\sec^2\theta - 1)$ Hence proved.

Common Articles:

Equation of Pair of Tangents from a Point:

Equation of pair of tangents to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ from a point $P(x_1, y_1)$ not lying on it is given by

$\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right)$

Or $T^2 = SS_1$

Director Circle:

Like ellipse in hyperbola also, locus of point of intersection of perpendicular tangents is circle which is called director circle.

Equation of director circle for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ is $x^2 + y^2 = a^2 - b^2$

Director circle is only possible when $a > b$.

Chord of Contact:

Equation of chord of contact for point $P(x_1, y_1)$ with respect to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$ is $T = 0$, which is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1=0$.

Chord with Given Mid-Point:

Equation of chord of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$ with mid-point $P(x_1, y_1)$ is $T = S_1$.

Illustration 22:

On which curve does the perpendicular tangents drawn to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ intersect?

Solution:

The locus of the point of intersection of perpendicular tangents to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is the director circle given by $x^2 + y^2 = a^2 - b^2$

Hence, the perpendicular tangents drawn intersect on the curve $x^2 + y^2 = 25 - 16 = 9$.

Illustration 23:

Find the equation of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from $(3, 2)$.

Solution:

Here $P(x_1, y_1) \equiv (3, 2)$

Hence equation of pair of tangents is $T^2 = SS_1$

$(3x - 18y - 9)^2 = (x^2 - 9y^2 - 9)(9 - 36 - 9)$

Factorizing above equation we get equation of tangents as $5x - 12y + 26 = 0$ and $x = 3$.

Illustration 24:

Find the equation to the locus of the middle points of the chords of the hyperbola $2x^2 - 3y^2 = 1$, each of which makes an angle of 45° with the x-axis.

Solution:

Let $P(h, k)$ be the mid-point

$T = S_1$

$\Rightarrow 2xh - 3ky - 1 = 2h^2 - 3k^2 - 1$

Now slope = $\frac{2h}{3k} = 1$

$\Rightarrow 2x = 3y$

Illustration 25:

Show that the mid points of focal chords of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ lie on another similar hyperbola.

Solution:

Equation of chord AB with $T = S_1$

$$\frac{hx}{a^2}-\frac{ky}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2}$$

It passes through $(ae, 0)$ $\frac{he}{a}=\frac{h^2}{a^2}-\frac{k^2}{b^2}$

Locus is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{ex}{a}$

$$\Rightarrow \frac{1}{a^2}[x^2-aex]-\frac{y^2}{b^2}=0 \;\Rightarrow\; \frac{\left(x-\frac{ae}{e}\right)^2-\frac{a^2e^e}{4}}{a^2}-\frac{y^2}{b^2}=0 \;\Rightarrow\; \frac{\left(x-\frac{ae}{2}\right)^2}{a^2}-\frac{y^2}{b^2}=\frac{e^2}{4}$$

which is also a hyperbola with eccentricity e.

Rectangular Hyperbola (Part-1):

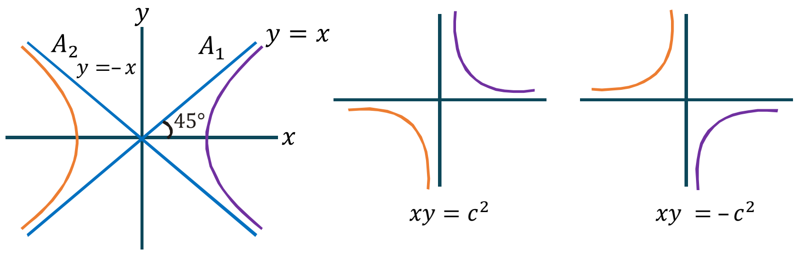
Rectangular Hyperbola:

A hyperbola is said to be rectangular hyperbola if its length of transverse axis is equal to its length of conjugate axis

i.e. $b = a \Rightarrow e = \sqrt{2}$

In rectangular hyperbola asymptotes are perpendicular to each other. If $x^2-y^2=a^2$ its asymptotes are $y=\pm x$ then rotating the axes by an angle $-\frac{\pi}{4}$ or $\frac{\pi}{4}$ about the same origin, equation of the rectangular hyperbola $x^2-y^2=a^2$ is reduced to $xy=\frac{a^2}{2}$ (i.e. $xy=c^2$) or $xy=-\frac{a^2}{2}$ (i.e. $xy=-c^2$) respectively.

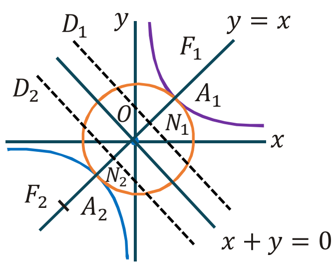
In $xy = c^2$ or $xy = -c^2$ asymptotes are coordinates axes.



Note : (i) Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

Basics of the Rectangular Hyperbola $xy = c^2$.



1.	Eccentricity	$\sqrt{2}$
	(angle between the two asymptotes = 90°)	
2.	Asymptotes	$x = 0;\ y = 0$
3.	Transverse axis	$y = x$
4.	Conjugate axis	$y = -x$
5.	Centre	$(0, 0)$
6.	Vertex	(c, c) and $(-c, -c)$
7.	Foci	$(c\sqrt{2}, c\sqrt{2})$ and $(-c\sqrt{2}, -c\sqrt{2})$
8.	Length of Latus rectum	$=\frac{2b^2}{a}=2a=2\sqrt{2}c$

9. Equation of auxiliary circle $x^2 + y^2 = 2c^2$
10. Equation of director circle $x^2 + y^2 = 0$
11. Equation of the directrices $x + y = \pm\sqrt{2}c$

Illustration 26:

Find everything for the conic $xy = -36$.

Solution:

Eccentricity $= \sqrt{2}$,

Asymptotes: $x = 0$ & $y = 0$,

Transverse axis $y = -x$, Conjugate axis: $y = x$,

Centre (0,0),

Vertices (6, – 6) & (– 6,6),

Foci $\left(6\sqrt{2},-6\sqrt{2}\right)$ & $\left(-6\sqrt{2},6\sqrt{2}\right)$,

Length of $LR = 12\sqrt{2}$,

Equation of auxiliary circle $x^2 + y^2 = 72$,

Equation of director circle $x^2 + y^2 = 0$,

Equation of directrices $x - y = \pm 6\sqrt{2}$.

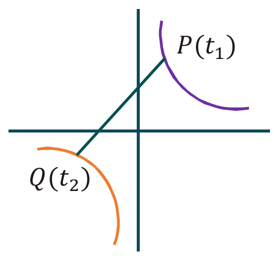
Rectangular Hyperbola (Part-2):

Parametric Coordinates:

If equation is $xy = c^2$ then its parametric representation is $x = ct, y = \frac{c}{t}, t \in R - \{0\}$

Chord Joining Two Points $P(t_1)$ & $Q(t_2)$:

Equation of a chord joining the points $P(t_1)$ and $Q(t_2)$ is



$$y-\frac{c}{t_1}=\frac{\left(\frac{c}{t^2}-\frac{c}{t_1}\right)}{ct_2-ct_1}(x-ct_1)$$

$\Rightarrow x + t_1 t_2 y = c(t_1 + t_2)$ with slope $m = -\frac{1}{t_1t_2}$

Tangent to The Hyperbola:

(i) Cartesian form:

Equation of the tangent at the point $P(x_1, y_1)$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$ to the hyperbola $xy = c^2$.

Proof: $\frac{xy_1+x_1y}{2}=c^2=x_1y_1$

(ii) Parametric form:

Equation of the tangent at the point $P(t)$ is $\frac{x}{t}+ty=2c$ to the hyperbola $xy = c^2$, $\left(\text{slope}=\frac{-1}{t^2}\right)$

Normal to the Hyperbola: (Slope $= t^2$)

Equation of the normal at the point $P(t)$ is $y-\frac{c}{t}=t^2(x-ct) \Rightarrow xt^3 - yt = c(t^4 - 1)$ to the hyperbola $xy = c^2$.

Note: If $P(t_1), Q(t_2), R(t_3)$ & $S(t_4)$ are four co – normal points then $t_1.t_2.t_3.t_4 = -1$.

Chord With Given Mid-Point: $(T = S_1)$

Chord with a given middle point i.e. (h, k) to the hyperbola $xy = c^2$ can be written as

$$\frac{xk + yh}{2} - c^2 = hk - c^2$$

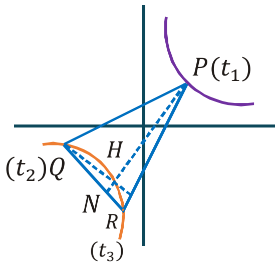
$$\Rightarrow kx + hy = 2hk$$

Illustration 27:

A rectangular hyperbola $xy = c^2$ circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \frac{c}{t_i}\right) i=1,2,3$ be the angular points P, Q, R then, prove that orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

Solution:

Slope of $QR = -\frac{1}{t_2t_3}$



$\therefore$ slope of $PN = t_2t_3$

$\therefore$ equation of altitude through P

$$y-\frac{c}{t_1}=t_2t_3(x-ct_1)$$

$$y + c\,t_1\,t_2\,t_3 = \frac{c}{t_1}+xt_2t_3$$

$$y+ct_1t_2t_3 = t_2t_3\left(x+\frac{c}{t_1t_2t_3}\right) \quad \text{...(i)}$$

Similarly, equation of altitude from Q is

$$y+ct_1t_2t_3 = t_1t_3\left(x+\frac{c}{t_1t_2t_3}\right) \quad \text{...(ii)}$$

Solving (i) and (ii) we get orthocentre as $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$.

Illustration 28:

Consider hyperbola $xy = 16$ to find the following:

(i) Equation of tangent at point $(2, 8)$

(ii) Equation of chord of contact w.r.t. point $(2, 3)$

(iii) Equation of chord which gets bisected at point $(5, 6)$

(iv) Equation of normal having slope 2.

Solution:

(i) Equation of tangent at point $(2,8)$ is $T = 0$

$\Rightarrow \dfrac{2y+8x}{2}-16=0$

$\Rightarrow 4x + y - 16 = 0$

(ii) Equation of chord of contact w.r.t. point $(2,3)$ is $T = 0$

$\Rightarrow \dfrac{2y+3x}{2}-16=0$

$\Rightarrow 3x + 2y - 32 = 0$

(iii) Equation of chord which gets bisected at point $(5,6)$ is $T = S_1$

$\Rightarrow \dfrac{5y+6x}{2}-16=(5)(6)-16$

$\Rightarrow 6x + 5y = 60$

(iv) We have $\dfrac{dy}{dx}=-\dfrac{y}{x}$

Thus, slope of normal at any point on the curve is

$-\dfrac{dx}{dy}=\dfrac{x}{y}=2$

$\therefore x = 2y$

Solving this with hyperbola, we get points $A(2\sqrt{8},\sqrt{8})$ and $B(-2\sqrt{8},-\sqrt{8})$ on the hyperbola where slope of normal is 2.

Equation of normal at point A is

$y-\sqrt{8}=2(x-2\sqrt{8})$

$\Rightarrow 2x-y=3\sqrt{8}$

Equation of normal at point B is

$y+\sqrt{8}=2(x+2\sqrt{8})$

$\Rightarrow 2x-y=-3\sqrt{8}$

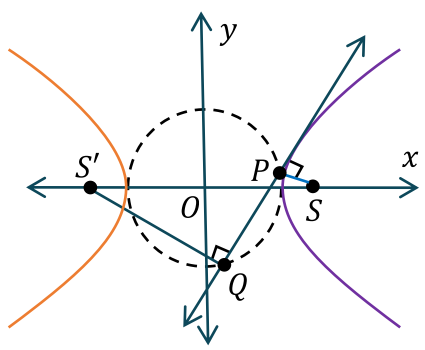
Important Highlights (JEE Advanced) (Part-1):

(a) Feet of perpendiculars from foci upon any tangent lie on auxiliary circle.

Product of perpendicular distances from foci upon any tangent of hyperbola is equal to square of the semi-conjugate axis.

$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

$(SP)(S'Q)=b^2$



(b) Length of tangent between the point of contact and the point where it meets the directrix subtends right angle at the corresponding focus.

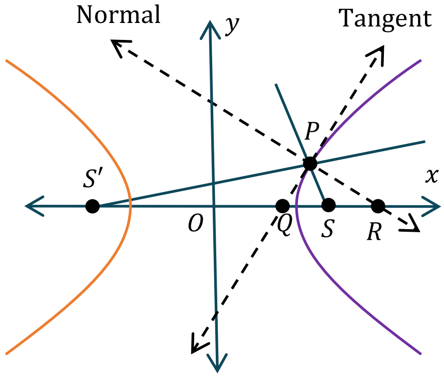
(c) Chord of contact with respect to any point on directrix passes through the corresponding focus.

(d) The tangent and normal at a point P on the hyperbola bisect the internal and external angles between the focal distances of P.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\angle S'PQ = \angle SPQ$

$\frac{PS'}{PS} = \frac{S'Q}{SQ} = \frac{S'R}{SR}$



(e) Reflection Property

An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.

Illustration 29:

Find the equation of hyperbola having foci $S(2,1)$ and $S(10,1)$ and a straight line $x + y - 9 = 0$ as its tangent. Also, find the equation of its director circle.

Solution:

Foci are $S(2,1), S(10,1)$

So transverse axis is horizontal and has the equation $y = 1$.

Centre is midpoint of SS', which is $(6,1)$

So, equation of hyperbola is

$$\frac{(x-6)^2}{a^2} - \frac{(y-1)^2}{b^2} = 1$$

Now, tangent is $x + y - 9 = 0$

Product of length of perpendicular from foci on the tangent is b^2.

$$\therefore\ b^2 = \frac{|2+1-9|}{\sqrt{2}} \times \frac{|10+1-9|}{\sqrt{2}} = \left(\frac{6}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right)$$

$\therefore b^2 = 6$

Distance between foci, $2ae = 8$

$\therefore ae = 4$

Now, $a^2 + b^2 = a^2e^2$

$\Rightarrow a^2 = 10$

Therefore, equation of hyperbola is

$$\frac{(x-6)^2}{10} - \frac{(y-1)^2}{6} = 1$$

So, equation of director circle is $(x-6)^2 + (y-1)^2 = a^2 - b^2$

$\therefore (x-6)^2 + (y-1)^2 = 4$

Illustration 30:

A ray emanating from the point $(5, 0)$ is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point $P(8, 3\sqrt{3})$. Find the equation of the reflected ray after first reflection.

Solution:

Given hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Here, $a = 4$ and $b = 3$.

So, foci are $(\pm\sqrt{a^2+b^2}, 0) \equiv (\pm 5, 0)$

Incident ray through $F_1(5, 0)$ strikes the ellipse at point $P(8, 3\sqrt{3})$. Therefore, reflected ray will go through another focus $F_2(-5, 0)$. So, reflected ray is line through points F_2 and P, which is $3\sqrt{3}x - 13y + 15\sqrt{3} = 0$.

Illustration 31:

Normal to a rectangular hyperbola at P meets the transverse axis at N. If foci of hyperbola are S and S', then find the value of $\frac{SN}{SP}$.

Solution:

We know that PN is external angle bisector of focal radii SP and $S'P$.

$\therefore \frac{SP}{S'P} = \frac{SN}{S'N}$

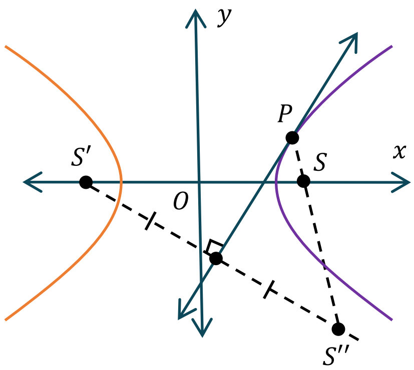
$\therefore \frac{SN}{SP} = \frac{S'N}{S'P} = \left|\frac{SN - S'N}{SP - S'P}\right|$ (Using theorem on equation ratios)

$= \frac{SS'}{|SP - S'P|} = e = \sqrt{2}$

Important Highlights (JEE Advanced) (Part-2):

Image of focus in a tangent property:

Image of one of the foci in any tangent to the hyperbola lies on the line joining the other focus and point of contact.



Orthogonal Intersection of Ellipse and Hyperbola:

If an ellipse and a hyperbola are confocal, then they intersect orthogonally.

If they intersect orthogonally, then they are confocal.

Highlights on Asymptotes:

(a) If the angle between the asymptotes of a hyperbola is 2θ, then its eccentricity is $\sec\theta$.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$e = \sec\theta$

(b) The length of any tangent intercepted between the asymptotes is bisected at the point of contact.

(c) The area of triangle formed by any tangent to a hyperbola and its asymptotes is constant and is equal to product of semi-transverse and semi-conjugate axis.

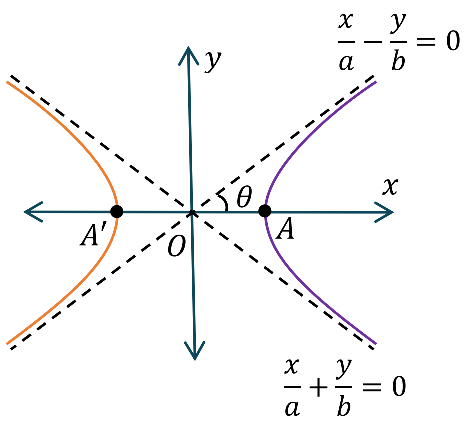


Illustration 32:

If ellipse $\frac{x^2}{25}+\frac{y^2}{b^2}=1$ and hyperbola $\frac{x^2}{7}-\frac{y^2}{9}=1$ intersect orthogonally. Find the value of b^2.

Solution:

As they intersect orthogonally, they are confocal

$\Rightarrow 25 - b^2 = 7 + 9$

$\Rightarrow b^2 = 9$

Illustration 33:

If eccentricity of a hyperbola is $\frac{2}{\sqrt{3}}$, then find the acute angle between the asymptotes.

Solution:

Let the angle between asymptotes be 2θ

$\Rightarrow e = \sec\theta \Rightarrow \frac{2}{\sqrt{3}}=\sec\theta \quad \Rightarrow \theta=\frac{\pi}{6}$

$\therefore\ 2\theta = \frac{\pi}{3}$

Illustration 34:

A tangent at $P\left(\frac{\pi}{6}\right)$ to $\frac{x^2}{3}-\frac{y^2}{27}=1$, intersects its asymptotes at A and B. If O is centre of hyperbola, find the equation of median in ΔOAB through vertex O.

Solution:

Intersect of any tangent on asymptotes is bisected at the point of contact.

Hence, $P\left(\frac{\pi}{6}\right)$ is the mid-point of AB.

$$P\left(\frac{\pi}{6}\right)\equiv\left(\sqrt{3}.\frac{2}{\sqrt{3}},3\sqrt{3}.\frac{1}{\sqrt{3}}\right)\equiv(2,3)$$

$\therefore$ Equation of median through O i.e., OP is $3x - 2y = 0$.

JEE Advanced (Part-3):

Illustration 35:

From any point on the hyperbola $H_1: (x^2/a^2) - (y^2/b^2) = 1$ tangents are drawn to the hyperbola $H_2 : (x^2/a^2) - (y^2/b^2) = 2$. The area cut-off by the chord of contact on the asymptotes of H_2 is equal to

(A) $ab/2$ (B) ab (C) $2\,ab$ (D) $4\,ab$

Ans. (C)

Solution:

Let any point on hyperbola H_1 is $(a\sec\theta, b\tan\theta)$.

Equation of chord of contact is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 2 \qquad \text{...(i)}$$

Equation of asymptotes is $y = \pm\frac{b}{a}x$ ...(ii)

From (i) & (ii) we get two intersection point

$P(2a\,(sec\,\theta\ +\ tan\,\theta), 2b(sec\,\theta\ +\ tan\,\theta))$

$Q(2a\,(sec\,\theta\ -\ tan\,\theta), -\ 2b(sec\,\theta\ -\ tan\,\theta))$

Then area of triangle OPQ is $\Delta\ =\ 2ab$.

Illustration 36:

Find the asymptotes of the hyperbola $2x^2 - 3xy - 2y^2 + 3x - y + 8 = 0$. Also find the equation to the conjugate hyperbola & the equation of the principal axes of the curve.

Solution:

Let equation of asymptotes are

$2x^2 - 3xy - 2y^2\ +\ 3x - y\ +\ 8\ +\lambda\ =\ 0$

As it represents two straight lines

$\therefore -4(8\ +\lambda)\ +\ \frac{9}{4}-\frac{1}{2}+\frac{9}{2}\ -\ (8\ +\lambda)\ \frac{9}{4}\ =\ 0$

$\Rightarrow \lambda\ = -7$

So asymptotes are $2x^2 - 3xy - 2y^2\ +\ 3x - y\ +\ 1 = 0$

$\Rightarrow 2y - x - 1\ =\ 0$ & $2x\ +\ y\ +\ 1\ =\ 0$

and the equation of conjugate hyperbola will be

$2x^2 - 3xy - 2y^2\ +\ 3x - y\ +\ 8 - 14\ =\ 0.$

Illustration 37:

Find the equation of the standard hyperbola passing through the point $(-\sqrt{3},3)$ and having the asymptotes as straight lines $\sqrt{5}\ x\ \pm\ y\ =\ 0.$

Solution:

Let hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

By given conditions $\frac{b}{a}=\sqrt{5}$ and $\frac{3}{a^2}-\frac{9}{b^2}=1$

$a^2=\frac{6}{5}$ and $b^2\ =\ 6$

Illustration 38:

The tangents and normal at a point on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cut the y – axis at A & B. Prove that the circle on AB as diameter passes through the foci of the hyperbola.

Solution:

Tangent at $(asec\theta, btan\theta)$ is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$. So, $A\ =\ (0, -bcot\theta)$

Normal at $(asec\theta, btan\theta)$ is $axcos\theta\ +\ bycot\theta\ =\ a^2e^2$. So, $B=\left(0,\frac{a^2e^2}{b}tan\theta\right)$

Circle AB as diameter is

$x^2+(y+b\cot\theta)\left(y-\frac{a^2e^2}{b}\tan\theta\right)=0$

It passes through $(\pm ae, 0)$. Put $x = \pm ae$ and $y = 0$ in above equation.

$$(\pm ae)^2 + (0 + b\cot\theta)\left(0 - \frac{a^2e^2}{b}\tan\theta\right) = 0 \Rightarrow 0 = 0$$

So, foci of the hyperbola lies on the circle.

Illustration 39:

Show that the locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.

Solution:

If (h, k) be mid-point of any chord of hyperbola

$x^2 - y^2 = a^2$, then its equation is

$hx - ky = h^2 - k^2$ ...(i)

But (i) is normal to hyperbola, then its equation is

$x\cos\theta + y\cot\theta = 2a$ ...(ii)

Comparing (i) & (ii)

$$\frac{h}{\cos\theta} = \frac{-k}{\cot\theta} = \frac{h^2 - k^2}{2a}$$

$$\Rightarrow \cot\theta = -\frac{2ak}{h^2 - k^2}, \quad \cos\theta = \frac{2ah}{h^2 - k^2}$$

$$\Rightarrow \tan\theta = \frac{k^2 - h^2}{2ak}, \quad \sec\theta = \frac{h^2 - k^2}{2ah}$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \left(\frac{h^2 - k^2}{2ah}\right)^2 - \left(\frac{k^2 - h^2}{2ak}\right)^2 = 1$$

$$\Rightarrow \frac{(h^2 - k^2)^2}{4a^2}\left[\frac{k^2 - h^2}{k^2h^2}\right] = 1$$

So we get $(y^2 - x^2)^3 = 4a^2x^2y^2$.

JEE Advanced (Part-4):

Illustration 40:

Chords of the circle $x^2 + y^2 = a^2$ touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Prove that locus of their middle point is the curve $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$.

Solution:

Let (h, k) be the mid – point of the chord of the circle $x^2 + y^2 = a^2$,

so that its equation by $T = S_1$ is $hx + ky = h^2 + k^2$

or $y = -\frac{h}{k}x + \frac{h^2 + k^2}{k}$ i.e. of the form $y = mx + c$

It will touch the hyperbola if $c^2 = a^2m^2 - b^2$

$$\therefore \left(\frac{h^2 + k^2}{k}\right)^2 = a^2\left(-\frac{h}{k}\right)^2 - b^2 \text{ or } (h^2 + k^2)^2 = a^2h^2 - b^2k^2$$

Generalising, the locus of mid-point (h, k) is $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$

Illustration 41:

C is the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

. The tangent at any point P on this hyperbola meets the straight lines $bx - ay = 0$ and $bx + ay = 0$ in the points Q and R respectively. Show that $CQ.CR = a^2 + b^2$.

Solution:

Consider $P \equiv (a\sec\theta, b\tan\theta)$

Tangent at P is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$

It meets $bx - ay = 0$ i.e. $\frac{x}{a}=\frac{y}{b}$ at Q

$\therefore Q$ is $\left(\frac{a}{\sec\theta-\tan\theta}, \frac{b}{\sec\theta-\tan\theta}\right)$

It meets $bx + ay = 0$ i.e. $\frac{x}{a}=-\frac{y}{b}$ at R.

$\therefore R$ is $\left(\frac{a}{\sec\theta+\tan\theta}, \frac{-b}{\sec\theta+\tan\theta}\right)$

$\therefore CQ.CR = \frac{\sqrt{(a^2+b^2)}}{\sec\theta-\tan\theta}.\frac{\sqrt{(a^2+b^2)}}{\sec\theta+\tan\theta} = a^2 + b^2$ $\quad (\because \sec^2\theta - \tan^2\theta = 1)$

Illustration 42:

A circle with centre (h, k) of variable radius cuts the rectangular hyperbola $x^2 - y^2 = 9a^2$ in points P, Q, R and S. Determine the equation of the locus of the centroid of triangle PQR.

Solution:

Let the circle be $(x - h)^2 + (y - k)^2 = r^2$ where r is variable. Its intersection with $x^2 - y^2 = 9a^2$ is obtained by putting $y^2 = x^2 - 9a^2$.

$x^2 + x^2 - 9a^2 - 2hx + h^2 + k^2 - r^2 = 2k\sqrt{(x^2-9a^2)}$

Or $[2x^2 - 2hx + (h^2 + k^2 - r^2 - 9a^2)]^2 = 4k^2(x^2 - 9a^2)$

Or $4x^4 - 8hx^3 + \ldots.. = 0$

$\therefore$ Above gives the abscissas of the four points of intersection.

$\therefore \Sigma x_1 = \frac{8h}{4} = 2h$

$x_1 + x_2 + x_3 + x_4 = 2h$

Similarly, $y_1 + y_2 + y_3 + y_4 = 2k$.

Now if (α, β) be the centroid of ΔPQR, then $3\alpha = x_1 + x_2 + x_3, 3\beta = y_1 + y_2 + y_3$

$\therefore x_4 = 2h - 3\alpha, y_4 = 2k - 3\beta$

But (x_4, y_4) lies on $x^2 - y^2 = 9a^2$

$\therefore (2h - 3\alpha)^2 + (2k - 3\beta)^2 = 9a^2$

Hence the locus of centroid (α, β) is $(2h - 3x)^2 + (2k - 3y)^2 = 9a^2$

Or $\left(x-\frac{2h}{3}\right)^2+\left(y-\frac{2k}{3}\right)^2=a^2$

Illustration 43:

If a circle cuts a rectangular hyperbola $xy = c^2$ in A, B, C, D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively, then prove that :

(a) $t_1 t_2 t_3 t_4 = 1$

(b) The centre of mean position of the four points bisects the distance between the centres of the two curves.

Solution:

(a) Let the equation of the hyperbola referred to rectangular asymptotes as axes be $xy = c^2$ or its parametric equation be

$x = ct, y = c/t$ ...(i)

and that of the circle be

$x^2 + y^2 + 2gx + 2fy + k = 0$ ...(ii)

Solving (i) and (ii), we get

$$c^2t^2 + \frac{c^2}{t^2} + 2gct + 2f\frac{c}{t} + k = 0$$

Or $c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$ ...(iii)

Above equation being of fourth degree in t gives us the four parameters t_1, t_2, t_3, t_4 of the points of intersection.

$\therefore t_1 + t_2 + t_3 + t_4 = -\frac{2gc}{c^2} = -\frac{2g}{c}$ ...(iv)

$t_1t_2t_3 + t_1t_2t_4 + t_3t_4t_1 + t_3t_4t_2$

$= -\frac{2fc}{c^2} = -\frac{2f}{c}$ ...(v)

$t_1t_2t_3t_4 = \frac{c^2}{c^2} = 1$. It proves (a) ...(vi)

Dividing (v) by (vi), we get

$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = -\frac{2f}{c}$ ...(vii)

(b) The centre of mean position of the four points of intersection is

$$\left[\frac{c}{4}(t_1 + t_2 + t_3 + t_4), \frac{c}{4}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right)\right] = \left[\frac{c}{4}\left(-\frac{2g}{c}\right), \frac{c}{4}\left(-\frac{2f}{c}\right)\right], \text{ by (iv) and (vii)}$$

$= (-g/2, -f/2)$

Above is clearly the mid-point of $(0, 0)$ and $(-g, -f)$ i.e. the join of the centres of the two curves.