Hyperbola

SOLUTIONS

EXERCISE - O

1. **Ans. (B)**

$\frac{x^2}{5}-\frac{y^2}{5\cos^2\alpha}=1$

$e_1^2=1+\frac{b^2}{a^2}=1+\frac{5\cos^2\alpha}{5}=1+\cos^2\alpha;$

eccentricity of the ellipse

$\frac{x^2}{25\cos^2\alpha}+\frac{y^2}{25}=1$ is $e_2^2=1-\frac{25\cos^2\alpha}{25}=\sin^2\alpha$; put $e_1=\sqrt{3}\,e_2 \Rightarrow e_1^2=3e_2^2$

$\Rightarrow 1+\cos^2\alpha=3\sin^2\alpha \Rightarrow 2=4\sin^2\alpha$

$\Rightarrow \sin\alpha=\frac{1}{\sqrt{2}}$

2. **Ans. (B)**

$e_H=\frac{5}{4};\ e_E=\frac{3}{4} \Rightarrow \frac{9}{16}=1-\frac{b^2}{16} \Rightarrow b^2=7$

3. **Ans. (C)**

$x^2-4x-3(y^2+2y)-11=0$

$=(x-2)^2-3(y+1)^2=12$

$=\frac{(x-2)^2}{12}-\frac{(y+1)^2}{4}=1$

$a^2=12,\ b^2=4,\ e=\sqrt{1+\frac{4}{12}}=\frac{2}{\sqrt{3}}$

focal length $=2ae=2\left(\sqrt{12}\right)\left(\frac{2}{\sqrt{3}}\right)=8$

4. **Ans. (B)**

For ellipse $29-p>0$ and $4-p>0 \Rightarrow p<4$

for hyperbola $29-p>0$ and $4-p<0 \Rightarrow p\in(4,29)$

5. **Ans. (B)**

$T:\ \frac{xx_1}{a^2}-\frac{yy_1}{b^2};\ \frac{x\cdot ae}{a^2}-\frac{y\cdot b^2}{a\cdot b^2}=1$ or $\frac{ex}{a}-\frac{y}{a}=1$ or $ex-y=a \Rightarrow m=e$

6. **Ans. (A)**

$y=-(5/2)x+5 \Rightarrow m=2/5 \Rightarrow a^2m^2-b^2=9\,.\,4/25-4=(36-100)/25<0$

Note that the slope of the tangent (2/5) is less than the slope of the asymptote which is 2/3 which is not possible

7. **Ans. (D)**

$\frac{y^2}{1/16}-\frac{x^2}{1/9}=1$

Locus will be the auxiliary circle $x^2+y^2=1/16$

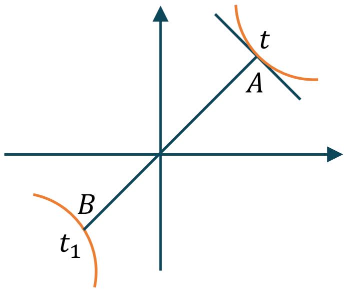
Feet of perpendiculars to a variable tangent lies on auxiliary circle.

8. Ans. (A)

$A = ab = a^2 \tan\lambda \Rightarrow b/a = \tan\lambda$, hence $e^2 = 1 + (b^2/a^2)$

$\Rightarrow e^2 = 1 + \tan^2\lambda \Rightarrow e = \sec\lambda$

9. Ans. (B)



$x = ct \Rightarrow \frac{dx}{dt} = c$

$y = \frac{c}{t} \Rightarrow \frac{dy}{dt} = -\frac{c}{t^2}$

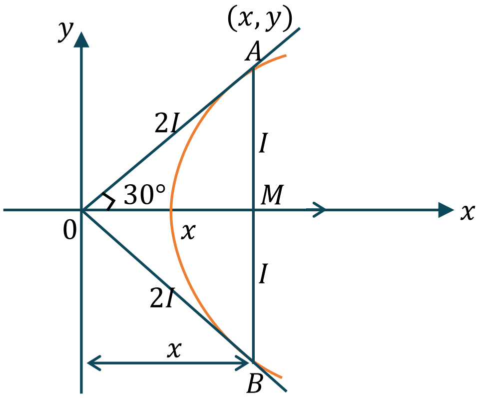
$\frac{dy}{dx} = -\frac{1}{t^2}$

$\therefore m_N = t^2$

$\therefore t^2 = m_{AB} = -\frac{1}{t_1 t}$

$\therefore t^3 t_1 = -1$

10. Ans. (D)



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $y = l$

$\frac{x^2}{a^2} = 1 + \frac{l^2}{b^2} \Rightarrow x^2 = (b^2 + l^2)\frac{a^2}{b^2}$ ...(1)

now $x^2 + l^2 = 4l^2 \Rightarrow x^2 = 3l^2$ ...(2)

from (1) and (2) $\frac{a^2(b^2 + l^2)}{b^2} = 3l^2$

$\Rightarrow a^2b^2 + a^2l^2 = 3b^2l^2$

$l^2(3b^2 - a^2) = a^2b^2$

$l^2 = \frac{a^2b^2}{3b^2 - a^2} > 0 \Rightarrow 3b^2 - a^2 > 0 \Rightarrow \frac{b^2}{a^2} > \frac{1}{3}$; $1 + \frac{b^2}{a^2} > \frac{4}{3}$

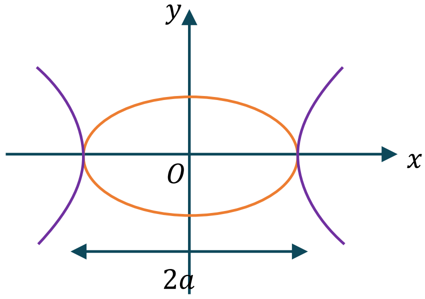
$\Rightarrow e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$

Note : $\frac{b}{a} > \frac{1}{\sqrt{3}} \Rightarrow 1 + \frac{b^2}{a^2} > \frac{4}{3} \Rightarrow e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$

11. Ans. (B)

$OT = a\cos\theta$; $ON = a\sec\theta \Rightarrow OT.ON = a^2$

12. Ans. (C)



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(1);

$\frac{x^2}{a^2} - \frac{y^2}{b_1^2} = 1$ ....(2)

$R = \sqrt{a^2 - b_1^2}$

$2R = \sqrt{a^2 + b^2}$

$\therefore 2\sqrt{a^2 - b_1^2} = \sqrt{a^2 + b^2}$ $\left[e_1^2 = 1 - \frac{b^2}{a^2}; e_2^2 = 1 + \frac{b_1^2}{a^2}\right]$

$4\left(a^2 - b_1^2\right) = a^2 + b^2$

$4\left(1 - \dfrac{b_1^2}{a^2}\right) = 1 + \dfrac{b^2}{a^2}$

$4[(1 - (e_2^2 - 1)] = 1 + 1 - e_1^2$

$8 - 4e_2^2 = 2 - e_1^2$

$4e_2^2 - e_1^2 = 6$

13. Ans. (C)

Tangent : $\dfrac{x}{ct} + \dfrac{yt}{c} = 2$

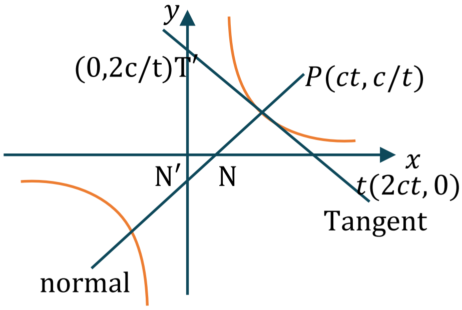
put $y = 0$; $x = 2ct$ (T)

$x = 0$; $y = \dfrac{2c}{t}$ (T')

normal is $y - \dfrac{c}{t} = t^2(x - ct)$

put $y = 0$; $x = ct - \dfrac{c}{t^3}$ (N)

$x = 0$; $\dfrac{c}{t} - ct^3$ (N')



Area of $\Delta PNT = \dfrac{c}{2t}\left(ct + \dfrac{c}{t^3}\right) \Rightarrow \Delta = \dfrac{c^2(1+t^4)}{2t^4}$

area of $\Delta PN'T' = \dfrac{ct}{2}\left(\dfrac{c}{t} + ct^3\right) \Rightarrow \Delta' = \dfrac{c^2(1+t^4)}{2}$

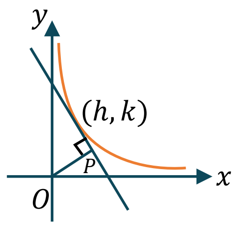
$\therefore \dfrac{1}{\Delta} + \dfrac{1}{\Delta'} = \dfrac{2t^4}{c^2(1+t^4)} + \dfrac{2}{c^2(1+t^4)} = \dfrac{2}{c^2(1+t^4)}(t^4 + 1) = \dfrac{2}{c^2}$

which is independent of t.

14. Ans. (D)

$hx + ky = h^2 + k^2$. Solve it with $xy = c^2$ & $D = 0$

Put $y = \dfrac{c^2}{x}$ in $hx + ky = h^2 + k^2$



$\Rightarrow hx + \dfrac{kc^2}{x} = h^2 + k^2 \Rightarrow hx^2 - (h^2 + k^2)x + kc^2 = 0$

$\Rightarrow D = 0 \Rightarrow (h^2 + k^2)^2 - 4hkc^2 = 0$

Now replace (h, k) by (x, y)

$\Rightarrow (x^2 + y^2)^2 - 4xyc^2 = 0$

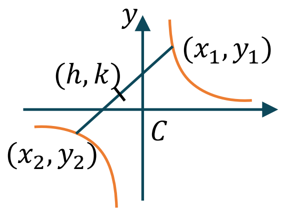
15. Ans. (A)

note that chord of $xy = c^2$ whose middle point is (h, k) in $\dfrac{x}{h} + \dfrac{y}{k} = 2$

further, now $2h = x_1 + x_2$ and $2k = y_1 + y_2$

$\dfrac{x}{2h} + \dfrac{y}{2k} = 1$

$\Rightarrow \dfrac{x}{x_1 + x_2} + \dfrac{y}{y_1 + y_2} = 1$



16. Ans. (B)

$\sqrt{3}x - y = 4\sqrt{3}t$ ...(1)

$\sqrt{3}x + y = \frac{4\sqrt{3}}{t}$ ...(2)

Now multiplying (1) & (2)

$\Rightarrow 3x^2 - y^2 = 48$

$\frac{x^2}{16} - \frac{y^2}{48} = 1$

$e = \sqrt{1 + \frac{48}{16}} = 2$

17. Ans. (C)

$e_H = 5/4;\ e_C = 5/3$

area = $\frac{d_1 d_2}{2}$ = $\frac{100}{2}$ = 50

$A_C : x^2 + y^2 = 16;\ A_H = x^2 + y^2 = 9$

18. Ans. (A)

Let a point on given hyperbola

(3secθ, 2tanθ)

Equation of chord of contact

$(3\sec\theta)x + (2\tan\theta)y = 9$ ...(1)

Let (h, k) is mid point of chord of contact

then

$hx + ky - 9 = h^2 + k^2 - 9$

$hx + ky = h^2 + k^2$ ...(2)

(1) & (2) represent same line, so

$\frac{3\sec\theta}{h} = \frac{2\tan\theta}{k} = \frac{9}{h^2 + k^2}$

$\sec\theta = \frac{3h}{h^2 + k^2}$, $\tan\theta = \frac{9k}{2(h^2 + k^2)}$

$\sec^2\theta - \tan^2\theta = 1$

$\frac{9h^2}{(h^2 + k^2)^2} - \frac{81k^2}{4(h^2 + k^2)^2} = 1$

$4h^2 - 9k^2 = \frac{4}{9}(h^2 + k^2)^2$

$\frac{h^2}{9} - \frac{k^2}{4} = \frac{(h^2 + k^2)^2}{81}$

Locus of (h, k) is $\frac{x^2}{9} - \frac{y^2}{4} = \left(\frac{x^2 + y^2}{9}\right)^2$

19. Ans. (C,D)

$a\sec\theta = y + x\tan\theta$

$b\sec\theta = x - y\tan\theta$

$(a^2 + b^2)\sec^2\theta = x^2(1 + \tan^2\theta) + y^2(1 + \tan^2\theta)$

$\Rightarrow x^2 + y^2 = a^2 + b^2 \Rightarrow$ (C) and (D)

20. Ans. (A,B,C,D)

solving $xy = c^2$ and $x^2 + y^2 = a^2$

$$x^2 + \frac{c^4}{x^2} = a^2$$

$x^4 - a^2x^2 + c^4 = 0$

$\Rightarrow \sum x_i = 0 \quad ; \quad \sum y_i = 0$

$x_1x_2x_3x_4 = c^4 \Rightarrow y_1y_2y_3y_4 = c^4$

21. Ans. (A,C,D)

(A) $\frac{2x}{a} = t + \frac{1}{t}$ ...(1)

$\frac{2y}{b} = t - \frac{1}{t}$ ...(2)

$(1)^2 - (2)^2 \Rightarrow \frac{4x^2}{a^2} - \frac{4y^2}{b^2} = 4$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(B) $t\left(\frac{x}{a}+1\right) = \frac{y}{b}$ and $\left(\frac{x}{a}-1\right) = -\frac{ty}{b}$

$\Rightarrow \left(\frac{x}{a}+1\right)\left(\frac{x}{a}-1\right) = \frac{-y^2}{b^2} \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(C) $x^2 - y^2 = 4$

(D) $x^2 - 6 = 2\left(2\cos^2\frac{t}{2} - 1\right)$

$x^2 - 6 = 4\cos^2\frac{t}{2} - 2$

$x^2 - 6 = y^2$

$x^2 - y^2 = 6$

22. Ans. (B,C)

$\frac{x^2}{8} + \frac{y^2}{4} = 1 \Rightarrow a_1 = 2\sqrt{2}, b_1 = 2,\ e_1 = \frac{1}{\sqrt{2}}$

$\frac{x^2}{a^2} - \frac{y^2}{3} = 1 \Rightarrow a_2 = a, b_2 = \sqrt{3}, e_2 = \sqrt{1 + \frac{3}{a^2}}$

both are confocal so $a_1e_1 = a_2e_2$

$\Rightarrow a = 1$

$\Rightarrow e_2 = 2$

It is obvious that A, B, C, D are the vertices of a rectangle so concyclic.

$y = mx + \sqrt{a_1^2 m^2 + b_1^2}$ tangent to ellipse

& $y = mx + \sqrt{a_2^2 m^2 - b_2^2}$ tangent to hyperbola

Making constant terms equal

$\Rightarrow 8m^2 + 4 = m^2 - 3$

$\Rightarrow 7m^2 = -7 \Rightarrow m^2 = -1$ not possible

so no common tangent can be drawn.

23. Ans. (B,D)

Given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

ellipse is $\frac{x^2}{2^2} + \frac{y^2}{1} = 1$

eccentricity of ellipse = $\sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

eccentricity of hyperbola = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{4}{3}}$

$\Rightarrow \frac{b^2}{a^2} = \frac{1}{3} \Rightarrow 3b^2 = a^2$ ...(1)

also hyperbola passes through foci of ellipse $(\pm\sqrt{3}, 0)$

$\frac{3}{a^2} = 1 \Rightarrow a^2 = 3$ ...(2)

from (1) & (2)

$b^2 = 1$

equation of hyperbola is $\frac{x^2}{3} - \frac{y^2}{1} = 1$

$\Rightarrow x^2 - 3y^2 = 3$

eccentricity of hyperbola $= \sqrt{1 + \frac{1}{3}} = \sqrt{\frac{4}{3}}$

focus of hyperbola = $\left(\pm\sqrt{3}.\frac{2}{\sqrt{3}}, 0\right) \equiv (\pm 2, 0)$

24. Ans. (A,B,C)

foci are (1,1) & (3,3)

$2ae = \sqrt{8}$ & $2a = 2 \Rightarrow a = 1$

$e = \sqrt{2}$

Given hyperbola is a rectangular hyperbola distance between directrices are $\frac{2a}{e} = \sqrt{2}$ centre is (2,2) & lies on $y = x$

Asymptotes are always perpendicular.

25. Ans. (A,B,C)

Put $y=\frac{c}{x}$ in equation of ellipse

$\Rightarrow x^4 - 4x^2 + 2c^2 = 0$

Put $x^2 = t \Rightarrow t^2 - 4t + 2c^2 = 0$

ellipse touches hyperbola

$\Rightarrow D = 0 \Rightarrow 16 - 8c^2 = 0$

$\Rightarrow c^2 = 2 \Rightarrow c=\pm\sqrt{2}$

Let $c=\sqrt{2}$ $\therefore$ $P\left(\sqrt{2},1\right)$ and $Q\left(-\sqrt{2},-1\right)$

$\Rightarrow PQ=2\sqrt{3} \Rightarrow OP = \sqrt{3} = OQ$

$\Rightarrow PQ = OP + OQ$

$\Rightarrow O, P, Q$ are colinear points.

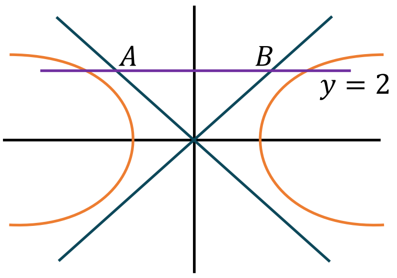
26. Ans. (A,B)

For two distinct tangents on different branches the point should lie on the line $y = 2$ and between A and B (where A and B are the points on the asymptotes).

Equation of asymptotes are $4x = \pm 3y$

Solving with $y = 2$

$x = \pm \frac{3}{2}$



$\therefore -\frac{3}{2} < \alpha < \frac{3}{2}$.

27. Ans. (A,B,C,D)

Extremities of L.R. = $\left(\pm ae, \pm\frac{b^2}{a}\right)$

$e=\sqrt{1-\frac{12}{16}}=\frac{1}{2}$

= $(\pm 2, \pm 3) = (x_1, y_1)$

Equation of circle $\Rightarrow (x-x_1)^2+(y-y_1)^2+\lambda\left(\frac{xx_1}{16}+\frac{yy_1}{12}-1\right)=0$

It passes through (0, 0) $\Rightarrow$ $\lambda=x_1^2+y_1^2$

Put value of $(x_1, y_1) = (\pm 2, \pm 3)$

$\Rightarrow \lambda = 4 + 9 = 13$

equation of required circles are

$\Rightarrow \left(x\pm 2\right)^2+\left(y\pm 3\right)^2\pm 13\left(\frac{2x}{16}+\frac{3y}{12}-1\right)=0$

So option A, B, C, D are

28. **Ans. (A,D)**

Hyperbola passes through $(0, \pm b) = (0, \pm\sqrt{12})$

So length of transverse axis $= 2\sqrt{12}$

eccentricity of hyperbola = 2

$a^2 = b^2(e^2 - 1)$

$a^2 = 12(3) = 36$

Equation of hyperbola $= \frac{x^2}{36} - \frac{y^2}{12} = -1$

Focus $(0, +be)$ and $(0, -be)$

$(0, 4\sqrt{3})$ and $(0, -4\sqrt{3})$

29. **Ans. (A)**

30. **Ans. (A)**

Solution for Q.29 & Q.30

The equation of required hyperbola is

$$\frac{\left(\frac{2x-y+4}{\sqrt{5}}\right)^2}{\left(\frac{\sqrt{2}}{2}\right)^2} - \frac{\left(\frac{x+2y-3}{\sqrt{5}}\right)^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1 \Rightarrow x^2 - 4xy - 2y^2 + 10x + 4y = 0$$

$\therefore h + b + g + f + c = -2 - 2 + 5 + 2 + 0 = 3$

Now let directrix of hyperbola $2x - y + \lambda = 0$ (as it is parallel to conjugate axis)

using distance between conjugate axis and directrix $= \left(\frac{a}{e}\right) = \frac{\text{semi } T.A.}{\text{Eccentricity}}$

$$\Rightarrow \left|\frac{\lambda-4}{\sqrt{5}}\right| = \frac{\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{1+\left(\frac{1/3}{1/2}\right)}} \Rightarrow \left|\frac{\lambda-4}{\sqrt{5}}\right| = \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{\sqrt{5}} \Rightarrow \lambda - 4 = \pm\sqrt{\frac{3}{2}} \quad \therefore \lambda = 4 \pm \sqrt{\frac{3}{2}}$$

Since centre of hyperbola is intersection of axis $C(-1, 2)$ and foci lies on T.A. at a distance ae from centre

$\therefore$ focus $((-1 \pm ae\cos\theta), (2 \pm ae\sin\theta))$ by transverse axis slope $-\frac{1}{2}$

we get $\cos\theta = \frac{-2}{\sqrt{5}}$, $\sin\theta = \frac{1}{\sqrt{5}} \Rightarrow \left(-1 \mp \frac{2}{\sqrt{6}}, 2 \pm \frac{1}{\sqrt{6}}\right)$

31. **Ans. (C)**

(A) Since $3x^2 - 5xy - 2y^2 + 5x + 11y + c = 0$ are **asympotes**

$\therefore$ It reprsents a pair of a stright lines

$\therefore 3(-2)c + 2 \cdot \frac{11}{2}\left(\frac{5}{2}\right)\left(\frac{-5}{2}\right) - 3\left(\frac{11}{2}\right)^2 - (-2)\left(\frac{5}{2}\right)^2 - c\left(-\frac{5}{2}\right)^2 = 0$

i.e. $-6c - \frac{275}{4} - \frac{363}{4} + \frac{25}{2} - \frac{25}{4}c = 0$

i.e. $-24c - 275 - 363 + 50 - 25c = 0$

i.e. $49c = -588$

i.e. $c = -12$

(B) Let the point be (h, k). Then equation of the chord of contact is $hx + ky = 4$

Since $hx + ky = 4$ is tangent to $xy = 1$

$\therefore\ x\left(\frac{4-hx}{k}\right) = 1$ has two equal roots

i.e. $hx^2 - 4x + k = 0$

i.e. $hk = 4$

$\therefore$ locus of (h, k) is $xy = 4$

i.e. $c^2 = 4$

(C) Equation of the hyperbola is $\frac{x^2}{c/a} - \frac{y^2}{c/b} = 1$ eccentricity $e = \sqrt{\frac{a+b}{b}}$

$\therefore\ \sqrt{\frac{c}{b}} = \frac{5}{2}$ and $\frac{13}{2} = \sqrt{\frac{c}{a}} \cdot \sqrt{\frac{a+b}{b}} \Rightarrow \frac{13}{2} = \frac{5}{2}\sqrt{1+\frac{b}{a}} \Rightarrow \frac{b}{a} = \frac{144}{25}$

$\therefore\ \frac{c}{a} = 36$

$\therefore$ the hyperbola is $25x^2 - 144y^2 = 900$

$\therefore\ a = 25, b = 144, c = 900$

$\therefore\ \frac{ab}{c} = 4$

(D) Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

then $2a = ae$ i.e. $e = 2$

$\therefore\ \frac{b^2}{a^2} = e^2 - 1 = 3$

$\therefore\ \frac{(2b)^2}{(2a)^2} = 3$

EXERCISE - S

1. **Ans. (4)**

Point of intersection of lines

$7x + 13y - 87 = 0$ and $5x - 8y + 7 = 0$ is $(5, 4)$.

Then $\frac{25}{a^2} - \frac{16}{b^2} = 1$ ...(i)

Also, latus rectum L.R = $= \frac{2b^2}{a} = \frac{32\sqrt{2}}{5}$

$\Rightarrow b^2 = \frac{16\sqrt{2}a}{5}$ ...(ii)

From (i) & (ii) $a^2 = \frac{25}{2}, b^2 = 16$

$\Rightarrow b = 4$

2. **Ans. (2)**

As directrix cut the x–axis at $(\pm a/e, 0)$

Hence, $\frac{2a}{e} + 0 = 1$ (for nearer directrix)

$\Rightarrow 2a = e$ ...(i)

Now, $b^2 = a^2(e^2 - 1) = a^2(4a^2 - 1)$

$\Rightarrow \frac{b^2}{a^2} = 4a^2 - 1$ ...(ii)

Given line $y = -2x + 1$ is a tangent to the hyperbola condition of tangency is $c^2 = a^2m^2 - b^2$

$\Rightarrow 1 = 4a^2 - b^2$

$\Rightarrow 4a^2 - 1 = b^2$ ...(iii)

from (ii) & (iii), $a^2 = 1$

$\Rightarrow$ from (ii), $b^2 = 3$

$\Rightarrow e = \sqrt{\frac{1+3}{1}} = 2$

3. **Ans. (8)**

Hyperbola is $x^2 - 9y^2 = 9$

or $\frac{x^2}{9} - \frac{y^2}{1} = 1$

equation of tangent is $y = mx \pm \sqrt{a^2m^2 - b^2}$ ...(1)

it passes through (3, 2)

$2 = 3m \pm \sqrt{9m^2 - 1}$

or $4 + 9m^2 - 12m = 9m^2 - 1$

or $\boxed{m_1 = \frac{5}{12}}$ & $\boxed{m_2 = \infty}$

equation of tangent (1) for $m_1 = \frac{5}{12} \Rightarrow y = \frac{5}{12}x \pm \sqrt{9\left(\frac{5}{12}\right)^2 - 1}$

or $y = \frac{5}{12}x \pm \frac{9}{12}$

or $y = \frac{5}{12}x \pm \frac{3}{4}$

on taking (–)ve sign point $P(3, 2)$ does not satisfy the equation of tangent therefore rejecting (–)ve sign.

Now the first tangent is

$\boxed{5x - 12y = -9} \Rightarrow \boxed{y = \frac{5x}{12} + \frac{3}{4}}$ ...(2)

equation of tangent from $P(3,2)$ for $m_2 \to \infty$ is

$\boxed{x - 3 = 0}$ ...(3)

Now equation of chord of contact w.r.t. point $P(3, 2)$ is

$T = 0$

$xx_1 - 9yy_1 = 9$

or $3x - 18y = 9$

or $x - 6y = 3$ ....(4)

solving (2) & (4) $x = -5, y = -\frac{4}{3}$

solving (3) & (4)

$x = 3, y = 0$

Now vertices of triangle:-

(3, 2), (3, 0), (–5, –4/3)

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ -5 & -\frac{4}{3} & 1 \end{vmatrix}$$

$$= \frac{1}{2}\times\left|3\left(\frac{4}{3}\right)-2(3+5)+1(-4)\right|$$

$$= \frac{1}{2}\,|\,4-16-4\,|$$

= 8 sq. units

4. **Ans. (5)**

$a = 10, b = 5 \Rightarrow e = \sqrt{1+\frac{b^2}{a^2}} = \frac{\sqrt{5}}{2}$

S and S' are $(5\sqrt{5}, 0)$ and $(-5\sqrt{5}, 0)$; vertex $A(10, 0)$

$\Rightarrow SA \cdot S'A = \sqrt{(5\sqrt{5}-10)^2(-5\sqrt{5}-10)^2} = (5\sqrt{5}-10)(5\sqrt{5}+10) = 25$

$\Rightarrow \sqrt{SA \cdot S'A} = 5$

5. **Ans. (3)**

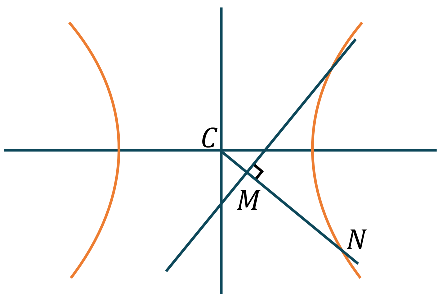
Since tangents drawn from the point $A(a, 2)$ are perpendicular, A must lie on the director circle $x^2 + y^2 = 7$. Putting $y = 2$, we get $a^2 = 3$.

6. **Ans. (9)**

$x^2 - y^2 = 9$

Equation of tangent

$\Rightarrow y = mx \pm 3\sqrt{m^2-1}$ ...(1)



Equation of $CM/CN \Rightarrow y = -\frac{1}{m}x$

$\Rightarrow x + my = 0$ ...(2)

For $CM \Rightarrow (y - mx)^2 = 9(m^2 - 1)$

$(x + my)^2 = 0$

$\Rightarrow (y^2 + x^2) = \frac{9(m^2-1)}{(m^2+1)} = cm^2$ ...(3)

For $CN \Rightarrow$ put $x = -my$ in $x^2 - y^2 = 9$

$y^2 = \frac{9}{m^2-1}$

& $x^2 = \frac{9m^2}{m^2 - 1}$

$x^2 + y^2 = \frac{9(m^2+1)}{m^2-1} = CN^2$ ...(4)

$CM^2.CN^2 = 81$

$CM.CN = 9$

7. Ans. (81.00)

Asymptotes $\Rightarrow (x + y + 1)(2x - y + 2) = 0$

hyperbola $\Rightarrow (x + y + 1)(2x - y + 2) = \lambda$ ...(1)

put $x = 2 \Rightarrow (3 + y)(6 - y) - \lambda_1 = 0$

$y^2 - 3y - 18 + \lambda_1 = 0$

$\Rightarrow D = 0$

$9 - 4(-18 + \lambda_1) = 0$

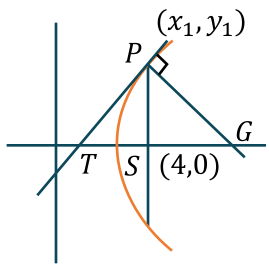
$\lambda_1 = \frac{81}{4}$

from (1) $(x + y + 1)(2x - y + 2) = \frac{81}{4}$

So $\lambda = 81$.

8. Ans. (45.00)

$\frac{x^2}{4} - \frac{y^2}{12} = 1$, $a = 2$, $b = \sqrt{12}$, $e = 2$



$P(4, 6)$

tangent at $(4, 6) \Rightarrow x - \frac{y}{2} = 1 \Rightarrow T(1, 0)$

normal at $(4, 6) \Rightarrow x + 2y = 16 \Rightarrow G(16, 0)$

Area of $\Delta PTG = \frac{1}{2}(PS)(TG) = 45$

9. Ans. (77)

Let $P(x, y)$ be any point on the hyperbola

Then by focus directrix property $\frac{\text{distance of } P \text{ from the focus}}{\text{distance of } P \text{ from the directrix}} = e = 3$

$\therefore \left| \frac{\sqrt{(x+1)^2 + (y-1)^2}}{\frac{x-y+3}{\sqrt{1^2+(-1)^2}}} \right| = 3$ or $(x + 1)^2 + (y - 1)^2 = 9 \cdot \left(\frac{x-y+3}{\sqrt{2}}\right)^2$

or $7x^2 - 18xy + 7y^2 + 50x - 50y + 77 = 0$

10. **Ans.** $45\sqrt{5}$

Equation PQ : chord of contact $T = 0$

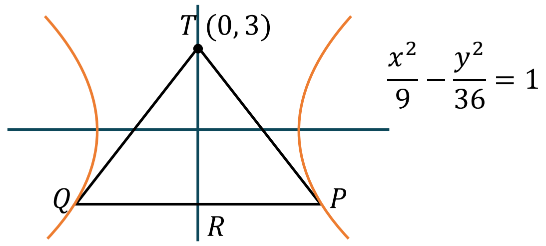
$y = -12$

Area : $\frac{1}{2} PQ \cdot TR$

$TR = 3 + 12 = 15$, Point $P(3\sqrt{5}, -12)$

$\Rightarrow PQ = 6\sqrt{5}$

Area of $\Delta PTQ = \frac{1}{2} \cdot 15 \cdot 6\sqrt{5} = 45\sqrt{5}$ sq. units



EXERCISE - JEE (Main) PYQ

1. **Ans. (2)**

$e = \sqrt{1 + \tan^2\theta} = \sec\theta$

As, $\sec\theta > 2 \Rightarrow \cos\theta < \frac{1}{2}$

$\Rightarrow \theta \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

Now, $\ell(\text{L·R}) = \frac{2b^2}{a} = 2\frac{(1-\cos^2\theta)}{\cos\theta}$

$= 2(\sec\theta - \cos\theta)$

Which is strictly increasing, so

$\ell(\text{L·R}) \in (3, \infty)$.

2. **Ans. (3)**

Hyperbola $\frac{x^2}{5} - \frac{y^2}{4} = 1$

slope of tangent = 1

equation of tangent $y = mx \pm \sqrt{a^2m^2 - b^2}$

$\Rightarrow y = x \pm \sqrt{5-4} \Rightarrow y = x \pm 1$

$\Rightarrow y = x + 1$ or $y = x - 1$

3. **Ans. (4)**

$\frac{y^2}{1+r} - \frac{x^2}{1-r} = 1$

for $r > 1$, $\frac{y^2}{1+r} + \frac{x^2}{r-1} = 1$ represents an ellipse.

$e = \sqrt{1 - \left(\frac{r-1}{r+1}\right)} = \sqrt{\frac{(r+1)-(r-1)}{(r+1)}} = \sqrt{\frac{2}{r+1}} = \sqrt{\frac{2}{r+1}}$

For $r < 1$, $\frac{y^2}{1+r} - \frac{x^2}{1-r} = 1$ represents a hyperbola

$e = \sqrt{1 + \frac{1-r}{1+r}} = \sqrt{\frac{2}{1+r}}$

4. Ans. (4)

$2b = 5$ and $2ae = 13$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{25}{4} = \frac{169}{4} - a^2$

$\Rightarrow a = 6 \quad \Rightarrow e = \frac{13}{12}$

5. Ans. (3)

$\frac{x^2}{24} - \frac{y^2}{18} = 1 \Rightarrow a = \sqrt{24}; b = \sqrt{18}$

Parametric equation of normal: $\sqrt{24}\cos\theta.x + \sqrt{18}.y\cot\theta = 42$

At $x = 0$: $y = \frac{42}{\sqrt{18}}\tan\theta = 7\sqrt{3}$ (from given equation)

$\Rightarrow \tan\theta = \sqrt{\frac{3}{2}} \Rightarrow \sin\theta = \pm\sqrt{\frac{3}{5}}$

slope of parametric normal $= \frac{-\sqrt{24}\cos\theta}{\sqrt{18}\cot\theta} = m$

$\Rightarrow m = -\sqrt{\frac{4}{3}}\sin\theta = -\frac{2}{\sqrt{5}}$ or $\frac{2}{\sqrt{5}}$

6. Ans. (3)

$V(\pm 6, 0) \Rightarrow 2a = 12 \Rightarrow a = 6$

hyperbola $\frac{x^2}{36} - \frac{y^2}{b^2} = 1$ ...(i)

$P(10,16)$ lies on (i)

We get $b^2 = 144$

$\frac{x^2}{36} - \frac{y^2}{144} = 1$

Equation of normal at (x_1, y_1) is

$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2e^2 \Rightarrow 2x + 5y = 100$

7. Ans. (2)

Slope of tangent is 2, Tangent of hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ at the point (x_1, y_1) is

$\frac{xx_1}{4} - \frac{yy_1}{2} = 1$ $(T = 0)$

Slope : $\frac{1}{2}\frac{x_1}{y_1} = 2 \Rightarrow x_1 = 4y_1$ ...(i)

(x_1, y_1) lies on hyperbola

$\Rightarrow \frac{x_1^2}{4} - \frac{y_1^2}{2} = 1$ ...(ii)

From (i) & (ii)

$\frac{(4y_1)^2}{4} - \frac{y_1^2}{2} = 1 \Rightarrow 4y_1^2 - \frac{y_1^2}{2} = 1$

$\Rightarrow \quad 7y_1^2 = 2 \Rightarrow y_1^2 = \frac{2}{7}$

Now $x_1^2 + 5y_1^2 = (4y_1)^2 + 5y_1^2$

$= (21)y_1^2 = 21 \times \frac{2}{7} = 6$

8. **Ans. (2)**

Ellipse : $\frac{x^2}{4} + \frac{y^2}{3} = 1$

eccentricity = $\sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

$\therefore$ foci = $(\pm 1, 0)$

for hyperbola, given $2a = \sqrt{2} \Rightarrow a = \frac{1}{\sqrt{2}}$

$\therefore$ hyperbola will be $\frac{x^2}{1/2} - \frac{y^2}{b^2} = 1$

eccentricity = $\sqrt{1 + 2b^2}$

$\therefore$ foci = $\left(\pm\sqrt{\frac{1+2b^2}{2}}, 0\right)$

$\because$ Ellipse and hyperbola have same foci

$\Rightarrow \sqrt{\frac{1+2b^2}{2}} = 1 \Rightarrow b^2 = \frac{1}{2}$

$\therefore$ Equation of hyperbola : $\frac{x^2}{1/2} - \frac{y^2}{1/2} = 1$

$\Rightarrow x^2 - y^2 = \frac{1}{2}$

Clearly $\left(\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}\right)$ does not lie on it.

9. **Ans. (2)**

$y = mx + c$ is tangent to

$\frac{x^2}{100} - \frac{y^2}{64} = 1$ and $x^2 + y^2 = 36$

For circle $c = \sqrt{36(1 + m^2)}$

For hyperbola $c = \sqrt{100m^2 - 64}$

By applying conditions of tangency

$\Rightarrow 100\,m^2 - 64 = 36 + 36m^2$

$m^2 = \frac{100}{64} \Rightarrow m = \pm\frac{10}{8}$

$c^2 = 36\left(1 + \frac{100}{64}\right) = \frac{36 \times 164}{64}$

$4c^2 = 369$

10. Ans. (1)

For ellipse $\frac{x^2}{25}+\frac{y^2}{b^2}=1\ (b<5)$

Let e_1 is eccentricity of ellipse

$\therefore b^2 = 25(1-e_1^2)$ ...(1)

Again for hyperbola

$\frac{x^2}{16}-\frac{y^2}{b^2}=1$

Let e_2 is eccentricity of hyperbola.

$\therefore b^2 = 16(e_2^2-1)$ ...(2)

by (1) & (2)

$25(1-e_1^2) = 16(e_2^2-1)$

Now $e_1.e_2 = 1$ (given)

$\therefore 25(1-e_1^2) = 16\left(\frac{1-e_1^2}{e_1^2}\right)$

or $e_1=\frac{4}{5}\ \therefore e_2=\frac{5}{4}$

Now distance between foci is $2ae$

$\therefore$ distance for ellipse $= 2\times5\times\frac{4}{5} = 8 = \alpha$

distance for hyperbola $= 2\times4\times\frac{5}{4} = 10 = \beta$

$\therefore (\alpha, \beta) \equiv (8, 10)$

11. Ans. (4)

Tangent to hyperbola of slope $m = -2$ is given by

$y=-2x\pm\sqrt{3(4)-3}$

$\left(y=mx\pm\sqrt{a^2m^2-b^2}\right)$

$\Rightarrow y+2x=\pm3 \Rightarrow 2x+y=3\ (k>0)$

For parabola $y^2 = \alpha x$ equation of tangent with slope $m=-2$ is

$y = mx + \frac{\alpha}{4m}$

$\Rightarrow y = -2x + \frac{\alpha}{-8} \Rightarrow \frac{\alpha}{-8}=3$

$\Rightarrow \alpha = -24.$

12. Ans. (1)

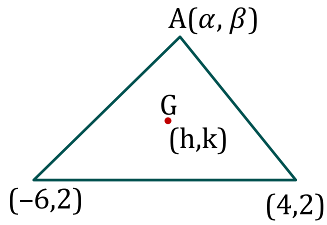
Given hyperbola is

$16(x+1)^2 - 9(y-2)^2 = 164 + 16 - 36 = 144$

$\Rightarrow \frac{(x+1)^2}{9}-\frac{(y-2)^2}{16}=1$

Eccentricity, $e=\sqrt{1+\frac{16}{9}}=\frac{5}{3}$

$\Rightarrow$ foci are $(4, 2)$ and $(-6, 2)$



Let the centroid be (h, k)

and $A(\alpha, \beta)$ be point on hyperbola

So $h = \frac{\alpha - 6 + 4}{3}, k = \frac{\beta + 2 + 2}{3}$

$\Rightarrow \alpha = 3h + 2, \beta = 3k - 4$

(α, β) lies on hyperbola so

$16(3h + 2 + 1)^2 - 9(3k - 4 - 2)^2 = 144$

$\Rightarrow 144(h + 1)^2 - 81(k - 2)^2 = 144$

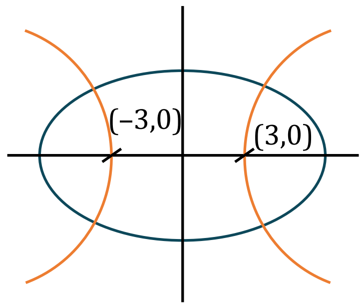
$\Rightarrow 16(h^2 + 2h + 1) - 9(k^2 - 4k + 4) = 16$

$\Rightarrow 16x^2 - 9y^2 + 32x + 36y - 36 = 0$

13. Ans. (2)

For ellipse $e_1 = \sqrt{1 - \frac{b^2}{a^2}} = \frac{3}{5}$

for hyperbola $e_2 = \frac{5}{3}$



Let hyperbola be

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\because$ it passes through (3,0) $\Rightarrow \frac{9}{a_2} = 1$

$\Rightarrow a^2 = 9 \Rightarrow b^2 = a^2(e^2 - 1) = 9\left(\frac{25}{9} - 1\right) = 16$

$\therefore$ Hyperbola is

$\frac{x^2}{9} - \frac{y^2}{16} = 1$

14. Ans. (2)

$K = \frac{4\sqrt{3}}{\sqrt{3}x + y} = \frac{\sqrt{3}x - y}{4\sqrt{3}}$

$\Rightarrow 3x^2 - y^2 = 48 \Rightarrow \frac{x^2}{16} - \frac{y^2}{48} = 1$

Now, $48 = 16(e^2 - 1)$

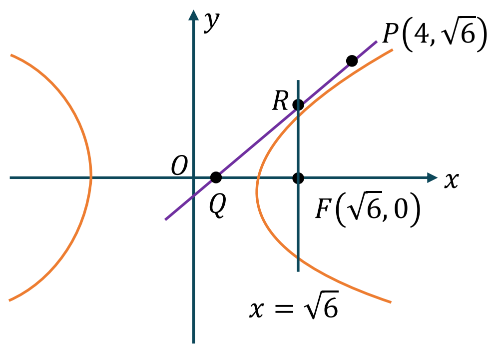
$\Rightarrow e = \sqrt{4} = 2$

15. Ans. (3)

$\frac{x^2}{4} - \frac{y^2}{2} = 1$

$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{3}{2}}$

$\therefore$ Focus $F(ae, 0) \Rightarrow F(\sqrt{6}, 0)$



equation of tangent at P to the hyperbola is

$2x - y\sqrt{6} = 2$

tangent meet x – axis at $Q(1, 0)$

& latus rectum $x = \sqrt{6}$ at $R\left(\sqrt{6}, \frac{2}{\sqrt{6}}\left(\sqrt{6}-1\right)\right)$

$\therefore$ Area of $\Delta_{QFR} = \frac{1}{2}\left(\sqrt{6}-1\right).\frac{2}{\sqrt{6}}\left(\sqrt{6}-1\right)$

$= \frac{7}{\sqrt{6}} - 2$

16. Ans. (3)

$\frac{x^2}{a^2} - \frac{y^2}{9} = 1$; $\left(8, 3\sqrt{3}\right)$ lie on hyperbola then $\frac{64}{a^2} - \frac{27}{9} = 1$

$\Rightarrow a^2 = \frac{64}{4} = 16$

equation of normal at $\left(8, 3\sqrt{3}\right)$:

$\frac{16x}{8} + \frac{9y}{3\sqrt{3}} = 16 + 9$

$2x + \sqrt{3}y = 25$

17. Ans. (4)

Tangent at (α, β) has slope 1

$\beta^2 = 24\alpha$

Equation of tangent $y\beta = 12(x + \alpha)$, $\frac{12}{\beta} = 1$

$\Rightarrow \alpha = 6, \beta = 12$

$\therefore$ $(\alpha + 4, \beta + 4) = (10, 16)$

Normal at $(10, 16)$ to $\frac{x^2}{36} - \frac{y^2}{144} = 1$ is $2x + 5y = 100$

18. Ans. (42)

$\frac{x^2}{a^2} - \frac{y^2}{1} = 1$ $\qquad$ $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$e_H = \sqrt{1 + \frac{1}{a^2}}$ $\qquad$ $e_E = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

$\ell.R. = \frac{2}{a}$ $\qquad$ $\ell R = \frac{2 \times 3}{2} = 3$

$\frac{2}{a} = 3$

$\boxed{a = \frac{2}{3}}$

$e_H = \sqrt{1 + \frac{9}{4}} = \frac{\sqrt{13}}{2}$

$12\left(e_H^2 + e_E^2\right) = 12\left(\frac{13}{4} + \frac{1}{4}\right) = \frac{12 \times 14}{4} = 42$

19. Ans. (4)

$e=\sqrt{1+\frac{b^2}{a^2}},\ \ell=\frac{2b^2}{a}$

Given $e^2=\frac{11}{14}\ell$

$1+\frac{b^2}{a^2}=\frac{11}{14}.\frac{2b^2}{a}$

$\frac{a^2+b^2}{a^2}=\frac{11}{7}.\frac{b^2}{a}$ ...(1)

Also $e'=\sqrt{1+\frac{a^2}{b^2}},\ \ell'=\frac{2a^2}{b}$

Given $(e')^2=\frac{11}{8}\ell'$

$1+\frac{a^2}{b^2}=\frac{11}{8}.\frac{2a^2}{b}$

$\frac{a^2+b^2}{b^2}=\frac{11}{4}.\frac{a^2}{b}$ ...(2)

New (1) ÷ (2)

$\frac{b^2}{a^2}=\frac{4}{7}.\frac{b^3}{a^3}$

$\therefore 7a = 4b$ ...(3)

From (2)

$\frac{\frac{16b^2}{49}+b^2}{b^2}=\frac{11}{4}.\frac{16b^2}{49b}$

$\frac{65}{49}=\frac{11}{4}.\frac{16}{49}.b$

$\therefore b=\frac{4\times 65}{11\times 16}$ ...(4)

We have to find value of

$77a + 44b$

$11(7a + 4b) = 11(4b + 4b) = 11 \times 8b$

$\therefore$ Value of $11 \times 8b = 11\times 8\times\frac{4\times 65}{16\times 11}=130$

20. Ans. (85)

$e^2=1+\frac{b^2}{a^2}=\frac{25}{16}\Rightarrow\frac{b^2}{a^2}=\frac{9}{16}$ ...(1)

$A\left(\frac{8}{\sqrt{5}},\frac{12}{5}\right)$ satisfies $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\Rightarrow \frac{64}{5a^2}-\frac{144}{25b^2}=1$ ...(2)

Solving (1) & (2) $b=\frac{6}{5}\ \ a=\frac{8}{5}$

Normal at A is $\frac{\sqrt{5}a^2x}{8}+\frac{5b^2y}{12}=a^2+b^2$

Comparing it $8\sqrt{5}x+\beta y=\lambda$

Gives $\lambda=100, \beta=15$

$\lambda-\beta=85$

21. Ans. (306)

$H_n \Rightarrow \frac{x^2}{1+n}-\frac{y^2}{3+n}=1$

$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{3+n}{1+n}}=\sqrt{\frac{2n+4}{n+1}}$

$e=\sqrt{\frac{2n+4}{n+1}}$

$n=48$ (smallest even value for which $e\in Q$)

$\boxed{e=\frac{10}{7}}$

$a^2=n+1$, $b^2=n+3$

$=49$, $=51$

1 = length of LR $=\frac{2b^2}{a}$

$L=2\cdot\frac{51}{7}$

$1=\frac{102}{7}$

$\boxed{21\ell=306}$

22. Ans. (12)

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$ae = 2$ & $e=\frac{3}{2}\Rightarrow a=\frac{4}{3}$

also $b^2 = a^2e^2 - a^2 \Rightarrow 4-\frac{16}{9}$

$\Rightarrow b^2=\frac{20}{9}$

Slope of tangent = $\frac{3}{2}$

So tangent equation will be

$y = mx \pm \sqrt{a^2m^2-b^2}$

$\Rightarrow y=\frac{3x}{2}\pm\sqrt{\frac{16}{9}\cdot\frac{9}{4}-\frac{20}{9}} \Rightarrow y=\frac{3x}{2}\pm\frac{4}{3}\Rightarrow |x_{\text{intercept}}|=\frac{8}{9}$

$|y_{\text{intercept}}|=\frac{4}{3}\Rightarrow |6a|+|5b|=\frac{48}{9}+\frac{60}{9}=\frac{108}{9}=12$

23. **Ans. (3)**

$3x^2 - 4y^2 = 36 \qquad 3x + 2y = 1$

$$m = -\frac{3}{2}$$

$$m = +\frac{\sec\theta\,3}{\sqrt{12}.\tan\theta} \Rightarrow \frac{3}{\sqrt{12}} \times \frac{1}{\sin\theta} = \frac{-3}{2}$$

$$\sin\theta = -\frac{1}{\sqrt{3}}$$

$$\left(\sqrt{12}.\sec\theta, 3\tan\theta\right)$$

$$\left(\sqrt{12}.\frac{\sqrt{3}}{\sqrt{2}}, -3\times\frac{1}{\sqrt{2}}\right) \Rightarrow \left(\frac{6}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)$$

24. **Ans. (2)**

$$e_H = \sqrt{2}$$

$$e_E = \frac{1}{\sqrt{2}}$$

Since the curves intersect each other orthogonally
The ellipse and the hyperbola are confocal

$$H : \frac{x^2}{1/2} - \frac{y^2}{1/2} = 1$$

$\Rightarrow$ foci $= (1,0)$

For ellipse $a.e_E = 1$

$$\Rightarrow a = \sqrt{2}$$

$$(e_E)^2 = \frac{1}{2} \Rightarrow 1 - \frac{b^2}{a^2} = \frac{1}{2} \Rightarrow \frac{b^2}{a^2} = \frac{1}{2}$$

$$\Rightarrow b^2 = 1$$

Length of $L.R. = \frac{2b^2}{a} = \frac{2}{\sqrt{2}} = \sqrt{2}$

25. **Ans. (216)**

$$H : \frac{x^2}{36} - \frac{y^2}{9} = 1$$

equation of normal is $6x\cos\theta + 3y\cot\theta = 45$

slope $= -2\sin\theta = -\sqrt{2}$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Equation of normal is $\sqrt{2}x + y = 15$

$P : (a\sec\theta, b\tan\theta)$

$\Rightarrow P(6\sqrt{2}, 3)$ and $K(0,15)$

$d^2 = 216$

EXERCISE - JEE (Advanced) PYQ

1. Ans. (B)

Equation of normal at $P(6,3)$ on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{a^2x}{6}+\frac{b^2y}{3}=a^2e^2$

It intersects x-axis at $(9,0)$

$\Rightarrow\ a^2\frac{9}{6}=a^2e^2 \Rightarrow e=\sqrt{\frac{3}{2}}$

2. Ans. (A, B)

Let parametric coordinates be $P(3\sec\theta, 2\tan\theta)$

Equation of tangent at point P will be

$\frac{x\sec\theta}{3}-\frac{y\tan\theta}{2}=1$

$\because$ tangent is parallel to $2x-y=1$

$\Rightarrow \frac{2\sec\theta}{3\tan\theta}=2 \Rightarrow \sin\theta=\frac{1}{3}$

$\therefore$ coordinates are

$\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$.

3. Ans. (A, B, D)

Given $H:\ x^2-y^2=1$

Now, equation of family of circle touching hyperbola at (x_1,y_1) is

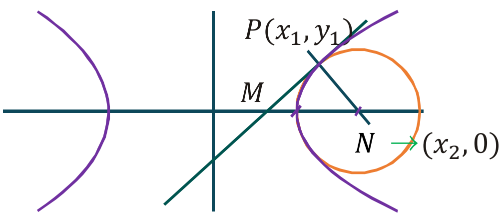
$(x-x_1)^2+(y-y_1)^2+\lambda(xx_1-yy_1-1)=0$

Now, its centre is $(x_2,0)$

$\therefore 2y_1+\lambda y_1=0 \Rightarrow \lambda=-2$

$\therefore\ x_2=\frac{2x_1-\lambda x_1}{2}=\frac{4x_1}{2}=2x_1$

$\therefore\ P\equiv(x_1,\sqrt{x_1^2-1})$



$N\equiv(2x_1,0)$ and $M\equiv\left(\frac{1}{x_1},0\right)$

$\therefore\ \ell=\frac{3x_1+\frac{1}{x_1}}{3}=x_1+\frac{1}{3x_1}$

$\Rightarrow \frac{d\ell}{dx_1}=1-\frac{1}{3x_1^2} \qquad x_1>1$

$m=\frac{\sqrt{x_1^2-1}}{3} \Rightarrow \frac{dm}{dx_1}=\frac{x_1}{3\sqrt{x_1^2-1}}$

Also $m=\frac{y_1}{3} \Rightarrow \frac{dm}{dy_1}=\frac{1}{3}\quad y_1>0\quad \therefore$ (A), (B) and (D)

4. Ans. (D)

$P\left(\sqrt{3},\frac{1}{2}\right)$; tangent $\sqrt{3}x+2y=4$

$\Rightarrow\left(\sqrt{3}\right)x+4\left(\frac{1}{2}\right)y=4$ comparing with (II)

$\Rightarrow a=2\ \therefore\ y=mx+\sqrt{a^2m^2+1}$ is tangent for $m=-\frac{\sqrt{3}}{2}$ i.e. (ii)

$\therefore$ point of contact for $a = 2,\ m=-\frac{\sqrt{3}}{2}$ is R.

5. Ans. (A)

$y = x + 8$ is tangent $\Rightarrow m = 1$; $P(8, 16)$

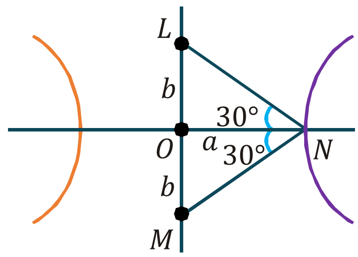
Comparing tangent with (i) of column 2, $m = 1$ satisfied and $a = 8$ obtained which matches for point of contact (P) of column 3 and (III) of column I.

6. Ans. (D)

For $a=\sqrt{2}$ and point $(-1,1)$ only I of column-1 satisfies. Hence equation of tangent is

$-x + y = 2$ or $y = x + 2 \Rightarrow m = 1$ which matches with (ii) of column 2 and also with Q of column 3.

7. Ans. (B)



$\tan 30°=\frac{b}{a}$

$\Rightarrow a=b\sqrt{3}$

Now area of $\Delta LMN=\frac{1}{2}.2b.b\sqrt{3}$

$4\sqrt{3}=\sqrt{3}b^2$

$\Rightarrow b = 2$ and $a=2\sqrt{3}$

$\Rightarrow e=\sqrt{1+\frac{b^2}{a^2}}=\frac{2}{\sqrt{3}}$

P. Length of conjugate axis = $2b = 4$

So P $\rightarrow$ 4

Q. Eccentricity $e=\frac{2}{\sqrt{3}}$

So Q $\rightarrow$ 3

R. Distance between foci = $2ae$

$= 2\left(2\sqrt{3}\right)\left(\frac{2}{\sqrt{3}}\right)=8$

So R $\rightarrow$ 1

S. Length of latus rectum

$=\frac{2b^2}{a}=\frac{2(2)^2}{2\sqrt{3}}=\frac{4}{\sqrt{3}}$

So S $\rightarrow$ 2

8. **Ans. (A, D)**

Since Normal at point P makes equal intercept on co-ordinate axes, therefore slope of Normal = –1

Hence slope of tangent = 1

Equation of tangent

$y - 0 = 1(x - 1)$

$y = x - 1$

Equation of tangent at (x_1, y_1)

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

$x - y = 1$ (equation of Tangent)

on comparing $x_1 = a^2, y_1 = b^2$

Also $a^2 - b^2 = 1$ ...(1)

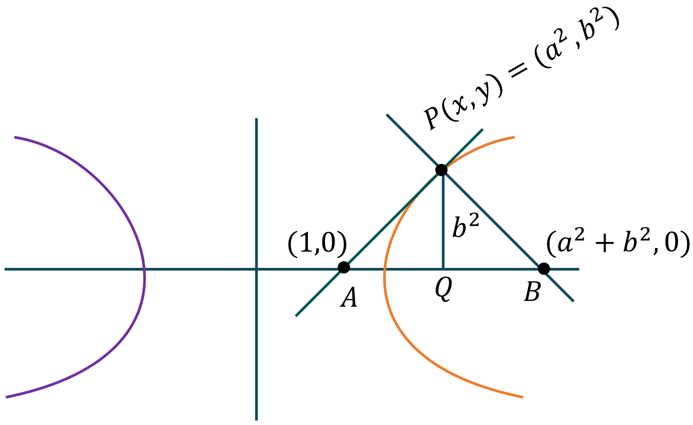
Now equation of normal at $(x_1, y_1) \equiv (a^2, b^2)$

$y - b^2 = -1(x - a^2)$

$x + y = a^2 + b^2$ ...(Normal)

point of intersection with x-axis is

$(a^2 + b^2, 0)$



Now $e = \sqrt{1 + \frac{b^2}{a^2}}$

$e = \sqrt{1 + \frac{b^2}{b^2 + 1}}$ $\left[\text{from}(1)\ \frac{b^2}{b^2+1} < 1\right]$

$1 < e < \sqrt{2}$

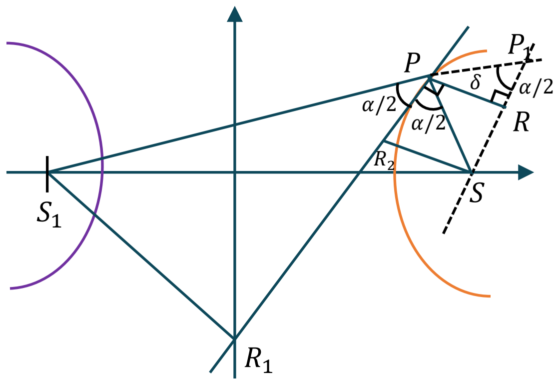
$\Delta = \frac{1}{2}.AB.PQ$

and $\Delta = \frac{1}{2}(a^2 + b^2 - 1).b^2$

$\Delta = \frac{1}{2}(2b^2)b^2$ (from (1) $a^2 - 1 = b^2$)

$\Delta = b^4$

9. **Ans. (7)**



$PR = SR_2 = \delta$

$S_1R_1 = S_1$ ($P \sin \alpha/2 = B \sin \alpha/2$)

$\frac{\beta\delta}{9} \sin\left(\frac{\alpha}{2}\right) = \frac{SR_2 \cdot S_1R_1}{9} = \frac{b^2}{9} = 7$

JEE (Main) Practice Paper

SECTION–A

1. **Ans. (3)**

If e_1 & e_2 are eccentircities of two conjugate hyperbolas

then $\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$

$\therefore$ e_1 = secα & e_2 = cosecα

2. **Ans. (3)**

$\frac{2b^2}{a} = 8$ ...(1)

and $2b = \frac{2ae}{2}$ ...(2)

and $e^2 = 1 + \frac{b^2}{a^2}$ ...(3)

by (1), (2), (3)

$e = \frac{2}{\sqrt{3}}$

3. **Ans. (1)**

$2b = 5$

$\Rightarrow b = \frac{5}{2}$

$2ae = 13$

$\therefore$ $ae = \frac{13}{2}$

$\therefore$ $b^2 = a^2e^2 - a^2$

$\Rightarrow \frac{25}{4} = \frac{169}{4} - a^2 \Rightarrow a^2 = \frac{144}{4} = 36$

$a = 6$

$\therefore$ Equation of Hyperbola is

$\frac{x^2}{36} - \frac{y^2}{\frac{25}{4}} = 1$

4. **Ans. (2)**

Centre of hyperbola ≡ (5, 0)

$\therefore$ $2a = 10$

$\therefore$ $a = 5$

$\therefore$ $ae = 13$

$b^2 = a^2e^2 - a^2$

$b^2 = 169 - 25$

$\therefore$ $b^2 = 144$

$\therefore$ $\frac{(x-5)^2}{25} - \frac{y^2}{144} = 1$

5. Ans. (1)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$e^2 = 1 + \frac{b^2}{a^2} = \frac{a^2+b^2}{a^2} \Rightarrow \frac{y^2}{b^2}-\frac{x^2}{a^2} = 1\ , \ (e')^2 = 1 + \frac{a^2}{b^2} = \frac{b^2+a^2}{b^2}$$

$$\frac{1}{e^2}+\frac{1}{(e')^2} = \frac{a^2}{a^2+b^2}+\frac{b^2}{b^2+a^2} = \frac{a^2+b^2}{a^2+b^2} = 1$$

So the point lie on $x^2+y^2=1$

6. Ans. (1)

$$\sqrt{2}^2\sec^2\theta + \sqrt{2}^2\tan^2\theta = 6$$

$$\Rightarrow 1 + 2\tan^2\theta = 3$$

$\therefore\ \theta = \pi/4$ for first quadrant

7. Ans. (1)

Curve $xy = c^2$

Point $P\left(ct, \frac{c}{t}\right)$ Point $Q\left(ct', \frac{c}{t'}\right)$

Equation of normal $xt^3 - yt = c\left(t^4-1\right)$

Point Q satisfy the equation $ct't^3 - \frac{c}{t'}t = c\left(t^4-1\right)$

$$t't^3 - \frac{t}{t'} = t^4 - 1$$

$$\left(t'\right)^2 t^3 - t = t'\left(t^4-1\right)$$

$$t'^2 t^4 + t' - t - t't^4 = 0$$

$$\Rightarrow\ t'\left(t't^3+1\right) - t\left(1+t't^3\right) = 0$$

$$t' = t \quad \text{or} \quad t' = -\frac{1}{t^3}$$

so only possibility $t' = -\frac{1}{t^3}$

8. Ans. (1)

Since $x + y = a$ touches the hyperbola

$$x^2 - 2y^2 = 18$$

$\therefore\ x^2 - 2(a-x)^2 = 18$ has equal roots

i.e. $x^2 - 4ax + 18 + 2a^2 = 0$ has equal roots

$$\therefore\ 16a^2 - 4\left(18+2a^2\right) = 0$$

$8a^2 - 72 = 0$ $\qquad a = \pm 3$

$\therefore\ |b| = 3$

9. Ans. (1)

Let tangent given by

$y = mx + \sqrt{m^2 - 5}$

$\because$ it passes through (2, 8)

$(8 - 2m)^2 = m^2 - 5$

$3m^2 - 32m + 69 = 0$

$\Rightarrow m = 3$ or $\frac{23}{3}$

$\therefore$ tangent can be

$3x - y + 2 = 0$

or $23x - 3y - 22 = 0$

10. Ans. (1)

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

tangent at point P $(a \sec\theta, b \tan\theta)$

$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ or $\frac{x}{a\cos\theta} + \frac{y}{(-b\cot\theta)} = 1$

Point $A(a\cos\theta, 0)$, $B(0, -b\cot\theta)$

Cordinate of point P is

$(h, k) \equiv (a\cos\theta, -b\cot\theta)$

$\cos\theta = \frac{h}{a}$, $\cot\theta = -\frac{k}{b}$

$\cot\theta = \frac{h}{\sqrt{a^2 - h^2}} = -\frac{k}{b}$

$\frac{h^2}{a^2 - h^2} = \frac{k^2}{b^2}$

$\frac{a^2}{h^2} - 1 = \frac{b^2}{k^2}$

So locus is

$\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

11. Ans. (1)

$4x^2 - 9y^2 = 36 \Rightarrow \frac{x^2}{9} - \frac{y^2}{4} = 1$

$5x + 2y - 10 = 0$

$m = \frac{-5}{2}$ $\qquad$ $m' = \frac{2}{5}$

Equation of tangent $y = m'x \pm \sqrt{a^2 m^2 - b^2}$

$y = \frac{2}{5}x \pm \sqrt{9 \times \frac{4}{25} - 16}$

$y = \frac{2}{5}x \pm \sqrt{-ve}$ so not possible

12. **Ans. (2)**

Equation of the hyperbola can be written as $\frac{X^2}{5^2} - \frac{Y^2}{4^2} = 1$

where $X = x - 3$ and $Y = y - 2$.

$\therefore$ tangent $Y = X \pm \sqrt{25-16}$

$y-2=(x-3)\pm 3$

$\Rightarrow \; y-2=x-3+3$ or $y-2=x-3-3$

$\Rightarrow y = x + 2$ or $y = x - 4$

13. **Ans. (3)**

Equation of chord joining given points

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

If $(ae, 0)$ satisfies it

$$\frac{1}{e} = \frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}$$

Now by componendo dividendo

$$\frac{1-e}{1+e} = \tan\frac{\theta}{2}\tan\frac{\phi}{2} \qquad \text{...(B)}$$

again if $(-ae, 0)$ satisfies it

$$-e\cos\left(\frac{\theta-\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow \frac{1+e}{1-e} = \tan\frac{\theta}{2}\;\tan\frac{\phi}{2} \qquad \text{...(D)}$$

14. **Ans. (1)**

by $T = S_1$

$3xh-2yk+2(x+h)-3(y+k)$

$=3h^2-2k^2+4h-6k$

$\Rightarrow x(3h+2)+y(-2k-3) = 3h^2-2k^2+2h-3k$

If is parallel to $y = 2x$

$\therefore \frac{(3h+2)}{(2k+3)} = 2$

$\Rightarrow 3x - 4y = 4$

15. **Ans. (2)**

Locus of R will be

$T = 0$

$\frac{x.2}{16} - \frac{y.1}{9} - 1 = 0$

$9x - 8y - 72 = 0$

16. Ans. (4)

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$$

locus of perpendicular tangents

(Director circle) $x^2 + y^2 = a^2 - b^2$

$x^2 + y^2 = \cos^2\alpha - \sin^2\alpha = \cos 2\alpha$

But $0 < \alpha < \frac{\pi}{4} \Rightarrow 0 < \cos 2\alpha < 1$

$0 < x^2 + y^2 < 1$

So there are infinite points.

17. Ans. (4)

The product of the lengths of the perpendiculars from the two focii on any tangent to the hyperbola

$\frac{x^2}{25}-\frac{y^2}{3}=1$ is 3

$\therefore 3 = \sqrt{k}$, hence $k = 9$

18. Ans. (2)

$2x - 3y - 2(x + 2) + 2(y + 3) = 0$

$-y + 2 = 0 \Rightarrow y = 2$

19. Ans. (4)

$\frac{16x^2}{225}=1 \Rightarrow x=\pm\frac{15}{4}$

Hence intersection points are $P\left(\frac{15}{4},\frac{15}{4}\right)$ and $Q\left(-\frac{15}{4},-\frac{15}{4}\right)$

$2a = PQ = 2\sqrt{2}\times\frac{15}{4} = \frac{15}{\sqrt{2}} \Rightarrow a = \frac{15}{2\sqrt{2}}$

given $b = \frac{5}{2\sqrt{2}}$

$e = \sqrt{1-\frac{1}{9}} = \frac{2\sqrt{2}}{3}$

20. Ans. (1)

$c = \frac{a'}{m} = \sqrt{a^2m^2-b^2}$

$\frac{2}{m} = \sqrt{m^2-3}$

$\Rightarrow 4 = m^2(m^2-3) \Rightarrow m^4 - 3m^2 - 4 = 0$

$(m^2-4)(m^2+1) = 0$

$\Rightarrow m = \pm 2$

$y = \pm 2x \pm 1$

$\Rightarrow \pm y = 2x + 1$

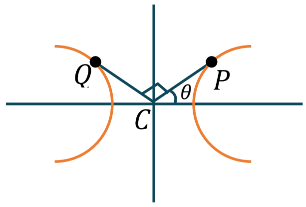
SECTION–B

1. Ans. (1)

$CP = \frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta} = r_1$ where $CP = r_1$

$\therefore P(r_1\cos\theta, r_1\sin\theta)$

Similarly $Q\left(r_2\cos\left(\frac{\pi}{2}+\theta\right), r_2\sin\left(\frac{\pi}{2}+\theta\right)\right)$



$Q\ (-r_2\sin\theta, r_2\cos\theta)$

P & Q lies on Hyperbola

$\therefore\ r_1^2\left(\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}\right)=1$

$\therefore\ r_1^2=\frac{a^2b^2}{(b^2\cos^2\theta-a^2\sin^2\theta)}$ & $r_2^2=\frac{a^2b^2}{(b^2\sin^2\theta-a^2\cos^2\theta)}$

$\therefore\ \frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{b^2-a^2}{a^2b^2}=\frac{1}{a^2}-\frac{1}{b^2}$ H.P.

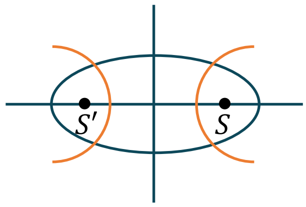
2. Ans. (2)

ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Hyperbola, $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$

$\therefore\ e_1^2=1-\frac{b^2}{a^2},\ e_2^2=1+\frac{B^2}{A^2}$



and $2ae_1=2Ae_2$

Also, $b = B$

So, $\frac{b}{ae_1}=\frac{B}{Ae_2}$

$\therefore\ e_1^2=1-\frac{B^2}{A^2}\frac{e_1^2}{e_2^2}$

$= 1 - \frac{(e_2^2-1)e_1^2}{e_2^2}$

$e_1^2e_2^2=e_2^2-e_1^2e_2^2+e_1^2$

$\Rightarrow\ e_1^{-2}+e_2^{-2}=2$

3. Ans. (4)

By property, orthocentre always lie on rect. hyperbola

$\therefore\ \lambda \times 4 = 16$

$\therefore\ \lambda = 4$

4. **Ans. (22)**

Let (x_1, y_1) be the point, of contact of tangent

$3x - 4y = 5$ to $x^2 - 4y^2 = 5$ Solving we have $\Rightarrow (x_1, y_1) = (3, 1)$

Now any tangent to $\frac{x^2}{25} - \frac{y^2}{16} = 1$ is

$y = mx \pm \sqrt{25m^2 - 16}$

$\Rightarrow y^2 + m^2x^2 - 2mxy = 25m^2 - 16$ ...(i)

$\because$ (1) passes through (3, 1)

$\therefore 16m^2 + 6m - 17 = 0$ ...(ii)

Let m_1 & m_2 be the roots of (ii) and $m_1 + m_2 = -\frac{3}{8}$ and $m_1m_2 = \frac{-17}{16}$

$\therefore 32(m_1 + m_2 - m_1m_2) = 22$

5. **Ans. (4)**

$3x^2 - 2y^2 = 6$

$\frac{x^2}{2} - \frac{y^2}{3} = 1$

Let the equation of tangent

$y = mx + \sqrt{a^2m^2 - b^2}$

passes through (α, β)

$(\beta - m\alpha)^2 = a^2m^2 - b^2$

$m^2\alpha^2 + \beta^2 - 2m\alpha\beta = a^2m^2 - b^2$

$m^2(\alpha^2 - a^2) - 2m\alpha\beta + \beta^2 + b^2 = 0$

$m_1m_2 = \frac{\beta^2 + b^2}{\alpha^2 - a^2} = 2$

$2\alpha^2 - 2a^2 = \beta^2 + b^2$

or $2\alpha^2 - 4 = \beta^2 + 3$

$\beta^2 = 2\alpha^2 - 7$

6. **Ans. (25)**

P is $(3\sec\theta, 4\tan\theta)$

Tangent at P is $\frac{x}{3}\sec\theta - \frac{y}{4}\tan\theta = 1$

It meets $4x - 3y = 0$ i.e. $\frac{x}{3} = \frac{y}{4}$ in Q

$\therefore Q$ is $\left(\frac{3}{\sec\theta - \tan\theta}, \frac{4}{\sec\theta - \tan\theta}\right)$

It meets $4x + 3y = 0$

i.e. $\frac{x}{3} = -\frac{y}{4}$ in R

$\therefore R\left(\frac{3}{\sec\theta + \tan\theta}, \frac{-4}{\sec\theta + \tan\theta}\right)$ is

$\therefore CQ.CR = \left(\frac{\sqrt{3^2 + 4^2}}{\sec\theta - \tan\theta}\right)\left(\frac{\sqrt{3^2 + 4^2}}{\sec\theta + \tan\theta}\right) = 25$

7. Ans. (6)

Equation of director circles of ellipse and hyperbola are respectively.

$x^2 + y^2 = a^2 + b^2$

and $x^2 + y^2 = a^2 - b^2$

$a^2 + b^2 = 4r^2$ ...(1)

$a^2 - b^2 = r^2$ ...(2)

So $2a^2 = 5r^2$

$a^2 = \frac{5r^2}{2}$

$b^2 = 4r^2 - \frac{5r^2}{2}$

$b^2 = \frac{3r^2}{2}$

$\because\ e_e^2 = 1 - \frac{b^2}{a^2} \Rightarrow e_e^2 = 1 - \frac{3r^2}{2} \times \frac{2}{5r^2} = 1 - \frac{3}{5} = \frac{2}{5}$

$\because\ e_h^2 = 1 + \frac{b^2}{a^2} \Rightarrow e_h^2 = 1 + \frac{3}{5} = \frac{8}{5}$

So $4e_h^2 - e_e^2 = 4 \times \frac{8}{5} - \frac{2}{5} = \frac{30}{5} = 6$

8. Ans. (16)

The line $y = mx + c\sqrt{m^2+1}$ is a tangent to the given circle. Since it passes through the foucs $\left(\sqrt{2}c, \sqrt{2}c\right)$ of the hyperbola, $\sqrt{2}c = m\sqrt{2}\,c + c\sqrt{m^2+1} \Rightarrow m^2 + 1 = 4m$ ...(1)

The line $y = mx + c\sqrt{m^2+1}$ intersects the hyperbola $xy = c^2$ at the points given by

$\frac{c^2}{x} = mx + c\sqrt{m^2+1}$

or $mx^2 + c\sqrt{m^2+1}\,x - c^2 = 0$.

If x_1, x_2 are the roots of this equation, then

$x_1 + x_2 = \frac{-c\sqrt{m^2+1}}{m},\ x_1x_2 = \frac{-c^2}{m}$ so that

$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = \frac{c^2(m^2+1)}{m^2} + \frac{4c^2}{m} = \frac{8c^2}{m}$ (using (1)).

Also $(y_1 - y_2)^2 = m^2(x_1 - x_2)^2 = \frac{m^2 \cdot 8c^2}{m} = 8c^2m$.

Hence the length of the focal chord = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= c\sqrt{\frac{8}{m} + 8m} = c\sqrt{\frac{8(1+m^2)}{m}} = \sqrt{32}\,c = 4\sqrt{2}\,c$.

9. Ans. (6)

Let S and S′ be the focii and b be the length of semi-minor axis of the hyperbola.

Let A and B be the feet of perpendiculars from the focii upon any tangent.

We have to prove that $AS + BS' \geq 2b$

We know that, $AS \times BS' = b^2$

Applying $A.M \geq G.M$

$$\frac{AS+BS'}{2} \geq \sqrt{b^2}$$

$AS + BS' \geq 2b$.

10. Ans. (3)

Let A is $\left(t, \frac{1}{t}\right)$, then B is $\left(\frac{1}{t}, t\right)$

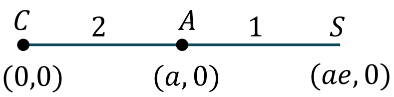
Now $\left(t-\frac{1}{t}\right)^2+\left(\frac{1}{t}-t\right)^2=14$

$$2\left(t^2+\frac{1}{t^2}\right)-4=14 \Rightarrow t^2+\frac{1}{t^2}=9=r^2$$

$\Rightarrow r = 3$

JEE (Advanced) Practice Paper

1. Ans. (A)



Clearly $\frac{2ae}{3}$ = a $\Rightarrow$ e = $\frac{3}{2}$

$\therefore S = \left(\frac{3a}{2}, 0\right)$

$$b^2 = a^2e^2 - a^2 \Rightarrow \frac{9}{4}a^2 - a^2 = \frac{5a^2}{4}$$

Equation of hyperbola

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{\left(\frac{5a^2}{4}\right)} = 1$$

Which reduces to $5x^2 - 4y^2 = 5a^2$

2. Ans. (C)

Homogenizing the equation of hyperbola with the help of line

We have $\frac{x^2}{a^2} - \frac{y^2}{2a^2} = \left(\frac{x \cos\alpha + y \sin\alpha}{p}\right)^2$

Now this subtends an angle of 90° at origin

so coefficient of x^2 + coefficient of y^2 = 0 i.e. $\frac{1}{a^2}-\frac{\cos^2\alpha}{p^2}-\frac{1}{2a^2}-\frac{\sin^2\alpha}{p^2}=0$

So $\frac{1}{2a^2}=\frac{1}{p^2}$ $\qquad p=\sqrt{2}\,a$

3. **Ans. (C)**

Let equation of directrix be $y = 2x + C$ ...(i)

$\because$ (1) passes through centre (2, 3) of the circle

$\Rightarrow C = -1$

$\therefore$ (i) reduces to $y = 2x - 1$

$\therefore$ equation of required hyperbola will be

$\sqrt{(x)^2+(y-3)^2} = 2\left(\frac{2x-y-1}{\sqrt{5}}\right)$

which reduces to $\Rightarrow 11x^2-y^2-16xy-16x+38y-41=0$

4. **Ans. (D)**

Circle $x^2+y^2=a^2e^2$ ...(1)

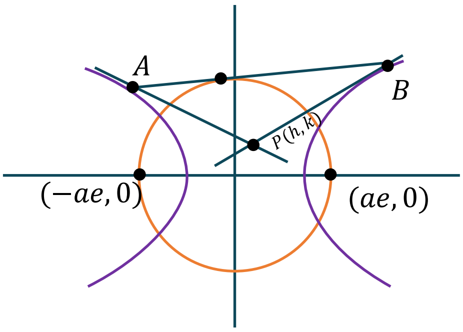
For P, AB is

given by $T = 0 \Rightarrow \frac{xh}{a^2}-\frac{yk}{b^2}=1$

It is tangent to (1)

So, by $p = r$ we get

$\frac{1}{\sqrt{\frac{h^2}{a^4}+\frac{k^2}{b^4}}}$ = ae ; $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$ H.P.



5. **Ans. (B)**

Let $P(x_1, y_1)$ be a point on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$\Rightarrow \frac{x_1{}^2}{a^2} - \frac{y_1{}^2}{b^2} = 1$

Chord of contact of $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 2 from (x_1, y_1) is

$\frac{xx_1}{a^2}-\frac{yy_1}{b^2} = 2$ ...(1)

equations of asymptotes are $\frac{x}{a}-\frac{y}{b}=0$ and $\frac{x}{a}+\frac{y}{b}=0$

pts. of Intersection of (1) with two asymptotes are

$$x_1 = \frac{2a}{\frac{x_1}{a} - \frac{y_1}{b}}, \; y_1 = \frac{2b}{\frac{x_1}{a} - \frac{y_1}{b}}$$

$$x_2 = \frac{2a}{\frac{x_1}{a} + \frac{y_1}{b}}, \; y_2 = \frac{-2b}{\frac{x_1}{a} + \frac{y_1}{b}}$$

$$\therefore \text{ or } \Delta = \frac{1}{2}(x_1 y_2 - x_2 y_1) = \frac{1}{2}\left(\frac{4ab \times 2}{\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}}\right) = 4ab$$

6. **Ans. (B,C,D)**

$$\frac{x^2}{3} - \frac{y^2}{2} = 1$$

$\Rightarrow 3\sin^2\alpha - 2\cos^2\alpha - 1 < 0 \Rightarrow 3\sin^2\alpha - 2 + 2\sin^2\alpha - 1 < 0$

$\Rightarrow 5\sin^2\alpha < 3 \Rightarrow |\sin\alpha| < \sqrt{\frac{3}{5}} \Rightarrow -\sqrt{\frac{3}{5}} < \sin\alpha < \sqrt{\frac{3}{5}}$

7. **Ans. (B,D)**

On solving

$xy = c^2$ with

circle

$x^2 + y^2 + 2gx + 2fy + \lambda = 0$

$x^2 + \frac{c^4}{x^2} + 2gx + \frac{2fc^2}{x} + \lambda = 0$

$x^4 + 2gx^3 + \lambda x^2 + 2fc^2x + c^4 = 0$

$\therefore \sum x_1 = -2g$

$\sum x_1x_2 = \lambda$

and again by eliminating x from equation of circle and hyperbola we have

$\Rightarrow y^4 + 2fy^3 + \lambda y^2 + 2gc^2y + c^4 = 0$

$\therefore \sum y_1 = -2f$

$\sum y_1y_2 = \lambda$

Now $CP^2 + CQ^2 + CR^2 + CS^2$

$\sum x_1^2 + \sum y_1^2 \Rightarrow \left(\sum x_1\right)^2 + \left(\sum y_1\right)^2 - 2\left(\sum x_1x_2 + \sum y_1y_2\right) \Rightarrow 4g^2 + 4f^2 - 4\lambda \Rightarrow 4r^2$

8. **Ans. (A,C,D)**

(i) Points α and β on the hyperbola are (a secα, b tanα) and (a secβ, b tanβ).

Equation of chord joining α and β is

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

By hypothesis this chord is focal chord and is passing through $(\pm ae, 0)$

$$\therefore \pm e\cos\left(\frac{\alpha-\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

(ii) Let $k = \pm 1$

$\therefore\ ke\cos\left(\frac{\alpha-\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$

$$\frac{ke}{1} = \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

By componendo and dividendo method

$$\left(\frac{ke-1}{ke+1}\right)=\frac{\cos\left(\frac{\alpha+\beta}{2}\right)-\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)+\cos\left(\frac{\alpha-\beta}{2}\right)} = \frac{-2\sin(\alpha/2)\sin(\beta/2)}{2\cos(\alpha/2)\cos(\beta/2)}$$

$\Rightarrow \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) + \left(\frac{ke-1}{ke+1}\right) = 0$

9. **Ans. (A,D)**

Equation of chord joining $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ is

$x + t_1t_2\, y = (t_1 + t_2)c$ ...(1)

$\because$ (i) is parallel to $y = x$

$\Rightarrow\ t_1t_2 = -1$

Now equation of circle will be $(x - ct_1)\,(x - ct_2) + \left(y-\frac{c}{t_1}\right)\left(y-\frac{c}{t_2}\right) = 0$

$\Rightarrow\ x^2+y^2-cx(t_1+t_2)+yc(t_1+t_2)-2c^2=0$

$\Rightarrow\ x^2+y^2-2c^2-c(t_1+t_2)(x-y)=0$ ...(2)

equation (2) is of the form $S + \lambda L = 0$ where S is $x^2+y^2-2c^2=0$ and L is $x - y = 0$

Solving S and L we get (c, c) and $(-c, -c)$

$\therefore$ (2) will always pass through (c, c) and $(-c, -c)$

10. **Ans. (A,B,D)**

Let the equation of the circle is

(A) $x^2+y^2+2gx+2fy+k=0$

and the equation of the rectangular hyperbola is $xy = c^2$

put $x = ct$ and $y = \frac{c}{t}$

$c^2t^2 + \frac{c^2}{t^2} + 2gct + \frac{2fc}{t} + k = 0$

$c^2t^4+2gct^3+kt^2+2fct+c^2=0$

$t_1t_2t_3t_4=\frac{c^2}{c^2}=1$

(B) A.M. of 4 points is $\left(\dfrac{ct_1+ct_2+ct_3+ct_4}{4},\left(\dfrac{\dfrac{c}{t_1}+\dfrac{c}{t_2}+\dfrac{c}{t_3}+\dfrac{c}{t_4}}{4}\right)\right)$

$\left(\dfrac{c}{4}(t_1+t_2+t_3+t_4),\dfrac{c}{4}\left(\dfrac{1}{t_1}+\dfrac{1}{t_2}+\dfrac{1}{t_3}+\dfrac{1}{t_4}\right)\right)$

$\left(\dfrac{-g}{2},\dfrac{-f}{2}\right)$

mid-point of centre of circle $(-g,-f)$ and hyperbola (0, 0) is $\left(\dfrac{-g}{2},\dfrac{-f}{2}\right)$

(D) Centre of circle is $(-g,-f)$

$\left(\dfrac{c}{2}\left(\dfrac{-2g}{c}\right),\dfrac{c}{2}\left(\dfrac{-2f}{2}\right)\right)\equiv\left(\dfrac{c}{2}(t_1+t_2+t_3+t_4),\dfrac{c}{2}\left(\dfrac{1}{t_1}+\dfrac{1}{t_2}+\dfrac{1}{t_3}+\dfrac{1}{t_4}\right)\right)$

$t_1t_2t_3t_4=1$

$\left(\dfrac{c}{2}\left(t_1+t_2+t_3+\dfrac{1}{t_1t_2t_3}\right),\dfrac{c}{2}\left(\dfrac{1}{t_1}+\dfrac{1}{t_2}+\dfrac{1}{t_3}+t_1t_2t_3\right)\right)$

11. Ans. (B)

Equation of tangent is $y = mx \pm \sqrt{a^2m^2-b^2}$ where $a^2m^2 - b^2 > 0 \Rightarrow y = mx \pm \sqrt{3m^2-2}$

As it passes through (2, 1) $\Rightarrow 1 = 2m \pm \sqrt{3m^2-2}$

$\Rightarrow 1 - 2m = \pm\sqrt{3m^2-2}$ $\quad\Rightarrow m = 1, 3$

so tangents are $y = x - 1$ and $y = 3x - 5$

The corresponding points of contact are (3, 2) and $\left(\dfrac{9}{5},\dfrac{2}{5}\right)$

12. Ans. (A)

Equation of tangent is $y - 2 = m(x - 1) \Rightarrow y = mx + (2 - m)$

it will be tangent if $c^2 = a^2m^2 - b^2$ provided $c \neq 0$

$\Rightarrow (2 - m)^2 = m^2 - 4 \Rightarrow m = 2$ or $\infty \Rightarrow$ m $= \infty \Rightarrow$ tangent is $x = 1$

13. Ans. (C)

Since equation of a hyperbola and its asymptotes differ in constant terms only,

$\therefore$ Pair of asymptotes is given by $xy - 3y - 2x + \lambda = 0$

where λ is any constant such that it represents two straight lines.

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow 0 + 2 \times \left(-\dfrac{3}{2}\right) \times (-1) \times \dfrac{1}{2} - 0 - 0 - \lambda\left(\dfrac{1}{2}\right)^2 = 0$

$\therefore \lambda = 6$

From (1), the asymptotes of given hyperbola are given by

$xy - 3y - 2x + 6 = 0$ $\quad$ or $(y - 2)(x - 3) = 0$

$\therefore$ Asymptotes are $x - 3 = 0$ and $y - 2 = 0$

14. Ans. (12)

Tangent to $\frac{x^2}{5}-\frac{y^2}{9}=1$ is

$y = mx\pm\sqrt{5m^2-9}$ comparing it with $y = 3x + \lambda$ we get $\lambda = \pm 6$.

15. Ans. (0)

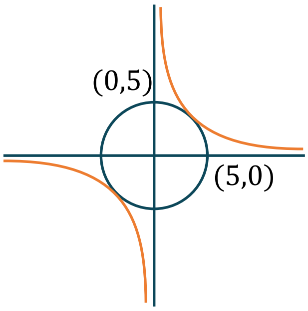
$\because\ y = -\sqrt{2}x-\frac{n}{\sqrt{2}}$ touches

$\frac{x^2}{9}-\frac{y^2}{16}=1$ hence,

$\frac{-n}{\sqrt{2}} = \pm\sqrt{9(-\sqrt{2})^2-16}\ \therefore\ n = \pm 2$.

16. Ans. (10)

It is clear from the diagram distance



between point of contacts is 10

17. Ans. (30)

Tangent on $(3\sec\phi, 4\tan\phi)$ is

$\frac{\sec\phi}{3}x - \frac{\tan\phi}{4}y = 1$ ...(i)

given that (i) is $\perp$ to $3x + 8y - 12 = 0$

$\Rightarrow \frac{4}{3}\left(\frac{\sec\phi}{\tan\phi}\right)\left(\frac{-3}{8}\right) = -1 \Rightarrow \phi = 30°$

18. Ans. (264)

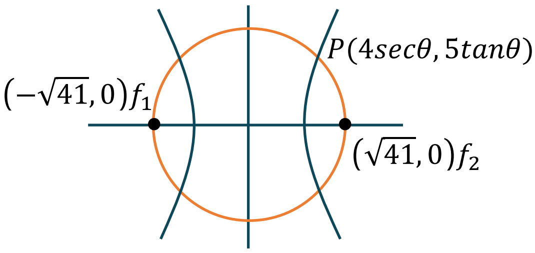
Focii of hyperbola are $(\pm\sqrt{41}, 0)$

$\because P$ lies on the circle $x^2+y^2=41$

Any point on hyperbola is $(4\sec\theta, 5\tan\theta)$

$\Rightarrow 16\sec^2\theta + 25\tan^2\theta = 41$

$\Rightarrow \tan\theta = \frac{5}{\sqrt{41}}$ and $\sec\theta = \sqrt{\frac{66}{41}}$



$\therefore\ pf_1 + pf_2 = e\left(4\sec\theta-\frac{a}{e}\right) + e\left(4\sec\theta+\frac{a}{e}\right)$

$=8e\sec\theta$

$=8\times\frac{\sqrt{41}}{4}\times\frac{\sqrt{66}}{\sqrt{41}}=2\sqrt{66} = \sqrt{264}$ (where $e = \frac{\sqrt{41}}{4}$)