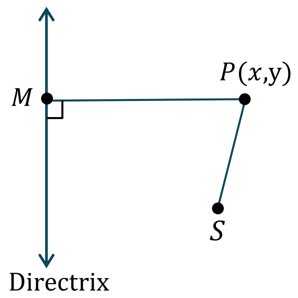
04 Ellipse

General Equation of an Ellipse:

In the previous chapter, we have learned that ellipse is one of the conic sections.

Let (α, β) be the focus S, and $\ell x + my + n = 0$ is the equation of directrix.

Let $P(x, y)$ be any point on the ellipse. Then by definition.



$\frac{PS}{PM} = e \Rightarrow SP = e\,PM$ (e is the eccentricity) $\Rightarrow (x-\alpha)^2 + (y-\beta)^2 = e^2 \frac{(\ell x + my + n)^2}{(\ell^2 + m^2)}$

$\Rightarrow (\ell^2 + m^2)\{(x-\alpha)^2 + (y-\beta)^2\} = e^2\{\ell x + my + n\}^2$

This simplifies to $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

in which $\Delta \neq 0, h^2 < ab$ & if ($h = 0$ then $a \neq b$)

Illustration 1:

Find the equation of an ellipse whose focus is (2, 3) eccentricity is 1/3 and the directrix $x + 2y - 1 = 0$

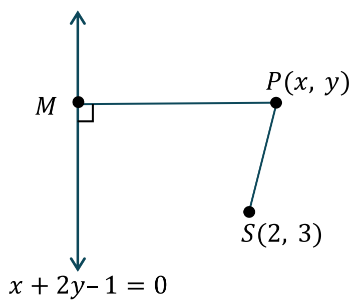
Solution:

Let $P(x, y)$ be any point on the ellipse

whose focus is $S(2, 3)$ and the directrix is $x + 2y - 1 = 0$

PM perpendicular from $P(x, y)$ on the directrix $x + 2y - 1 = 0$

Then by definition $SP = ePM$



$\Rightarrow (SP)^2 = e^2(PM)^2 \Rightarrow (x-2)^2 + (y-3)^2 = \frac{1}{9}\left\{\frac{x+2y-1}{\sqrt{5}}\right\}^2$

$\Rightarrow 45(x^2 + y^2 - 4x - 6y + 13) = x^2 + 4y^2 + 1 + 4xy - 2x - 4y$

$\Rightarrow 44x^2 + 41y^2 - 4xy - 178x - 266y + 584 = 0$

which is the required equation of the ellipse.

Standard Equation of Ellipse:

Standard Equation & Definition:

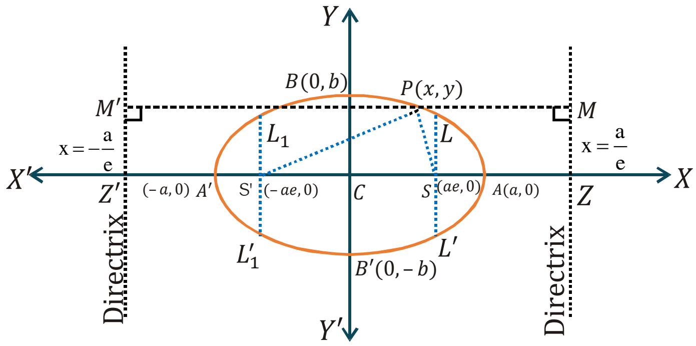
Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Where

$a > b$ & $b^2 = a^2(1-e^2) \Rightarrow a^2 - b^2 = a^2 e^2$.

where e = eccentricity $(0 < e < 1)$.

Foci : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.



(a) Equation of directrices:

$$x = \frac{a}{e} \quad \& \quad x = -\frac{a}{e}.$$

(b) Vertices:

$A' \equiv (-a, 0)$ & $A' \equiv (a, 0)$.

(c) Major axis: The line segment $A'A$ in which the foci S' and S lie is of length $2a$ and is called the major axis $(a > b)$ of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix $(x) \equiv \left(\pm\frac{a}{e}, 0\right)$.

(d) Minor Axis: The y −axis intersects the ellipse in the points $B' \equiv (0, -b)$ & $B \equiv (0, b)$. The line segment $B'B$ of length $2b(b < a)$ is called the Minor Axis of the ellipse.

(e) Principal Axes: The major & minor axis together are called Principal Axes of the ellipse.

(f) Centre: The point which bisects every chord of the conic drawn through it is called the centre of the conic. $C \equiv (0,0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(g) Diameter: A chord of the conic which passes through the centre is called a diameter of the conic.

(h) Focal Chord: A chord which passes through a focus is called a focal chord.

(i) Double Ordinate: A chord perpendicular to the major axis is called a double ordinate.

(j) Latus Rectum: The focal chord perpendicular to the major axis is called the latus rectum.

(i) Length of latus rectum $(LL') = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$

(ii) Equation of latus rectum : $x = \pm ae$.

(iii) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right), L'\left(ae, -\frac{b^2}{a}\right), L_1\left(-ae, \frac{b^2}{a}\right)$ and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

(k) Focal radii: $SP = a - ex$ & $S'P = a + ex \Rightarrow SP + S'P = 2a$ = Major axis.

(l) Eccentricity: $e = \sqrt{1-\frac{b^2}{a^2}}$

Note :

(i) The sum of the focal distances of any point on the ellipse is equal to the major Axis. Hence distance of focus from the extremity of a minor axis is equal to semi major axis. **i.e.** $\boldsymbol{BS = CA}$**.**

(ii) If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & nothing is mentioned, then the rule is to assume that $a > b$.

Another form of Ellipse:

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a < b)$

(a) AA' = Minor axis $= 2a$

(b) BB' = Major axis $= 2b$

(c) $a^2 = b^2(1 - e^2)$

(d) Latus rectum $LL' = L_1L_1' = \frac{2a^2}{b}$, equation $y = \pm$ be

(e) Ends of the latus rectum are:

$L\left(\frac{a^2}{b}, be\right), L'\left(-\frac{a^2}{b}, be\right), L_1\left(\frac{a^2}{b}, -be\right), L_1'\left(-\frac{a^2}{b}, -be\right)$

(f) Equation of directrix $y = \pm$ b/e

(g) Eccentricity: $e=\sqrt{1-\frac{a^2}{b^2}}$

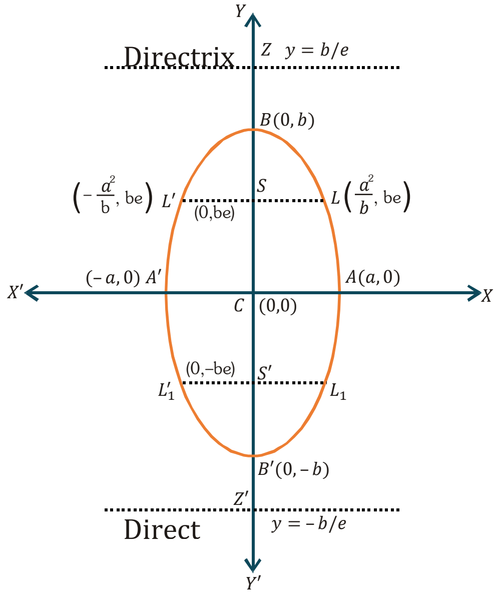


Illustration 2:

Find all parameters of the ellipse $16x^2 + 25y^2 = 400$

Solution:

$\frac{x^2}{25}+\frac{y^2}{16}=1$, $a = 5, b = 4$, $e=\sqrt{1-\frac{b^2}{a^2}}=\frac{3}{5}$

$\Rightarrow ae = 3$, $\frac{a}{e}=\frac{25}{3}$

Centre $(0, 0)$

Vertex : $(\pm a, 0) \equiv (\pm 5, 0)$

Foci : $(\pm ae, 0) \equiv (\pm 3, 0)$

Ends of latus rectum $\equiv \left(\pm ae, \pm\frac{b^2}{a}\right) \equiv \left(\pm 3, \pm\frac{16}{5}\right)$

Directrix: $x = \pm a/e \equiv x = \pm\frac{25}{3}$

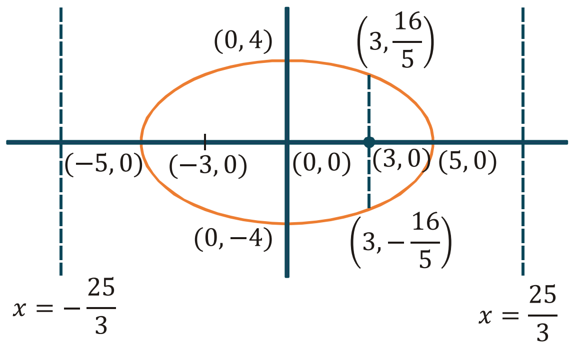


Illustration 3:

If LR of an ellipse is half of its minor axis, then its eccentricity is -

(A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{2}}{3}$

Ans. (C)

Solution:

As given $\frac{2b^2}{a}=b \Rightarrow 2b = a \Rightarrow 4b^2 = a^2$

$\Rightarrow 4a^2(1 - e^2) = a^2 \Rightarrow 1 - e^2 = 1/4$

$\therefore e = \frac{\sqrt{3}}{2}$.

Illustration 4:

The equation of the standard ellipse with respect to coordinate axes whose minor axis is equal to the distance between its foci and whose $LR = 10$, will be-

(A) $2x^2 + y^2 = 100$ (B) $x^2 + 2y^2 = 100$

(C) $2x^2 + 3y^2 = 80$ (D) none of these

Ans. (A, B)

Solution:

When $a > b$

As given $2b = 2ae \Rightarrow b = ae$ ...(i)

Also $\frac{2b^2}{a} = 10 \Rightarrow b^2 = 5a$ ...(ii)

Now since $b^2 = a^2 - a^2e^2 \Rightarrow b^2 = a^2 - b^2$ [From (i)]

$\Rightarrow 2b^2 = a^2$ ...(iii)

(ii), (iii) $\Rightarrow a^2 = 100, b^2 = 50$

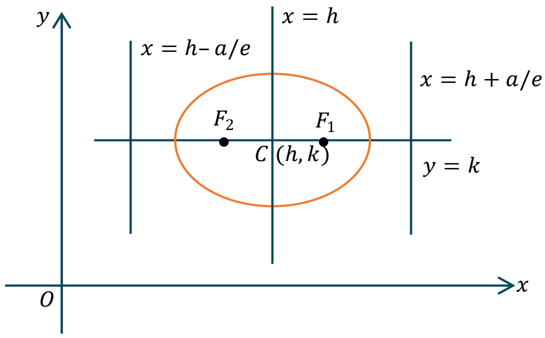
Hence equation of the ellipse will be $\frac{x^2}{100} + \frac{y^2}{50} = 1 \Rightarrow x^2 + 2y^2 = 100$

Similarly, when $a < b$ then required ellipse is $2x^2 + y^2 = 100$.

Shifted Ellipse:

Equation of Ellipse having Axes Parallel to Coordinate axes:

If the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$, is shifted (translated) such that its centre is at (h, k), then equation of ellipse becomes $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.



Since $a > b$,

major axis is along $y = k$ and minor axis along $x = h$.

Eccentricity is given by relation $b^2 = a^2(1 - e^2)$.

Foci are $F_1(h + ae, k)$ and $F_2(h - ae, k)$.

Equation of directrices are $x = h \pm \frac{a}{e}$.

If equation of the ellipse having $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$,

$a < b$, then $a^2 = b^2(1 - e^2)$. Foci are $F_1(h, k + be)$ and $F_2(h, k - be)$. Equation of two directrices are $y = k \pm \frac{b}{e}$.

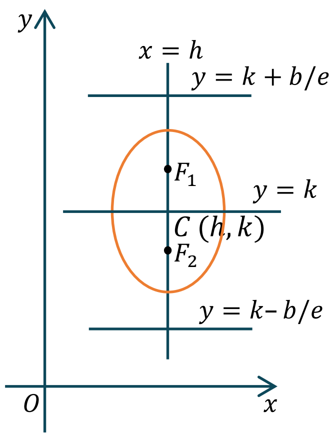


Illustration 5:

Find the equation of the ellipse whose foci are $(2, 3), (-2, 3)$ and whose semi minor axis is of length $\sqrt{5}$.

Solution:

Here S is $(2, 3)$ & S' is $(-2, 3)$ and $b = \sqrt{5} \Rightarrow SS' = 4 = 2ae \Rightarrow ae = 2$

but $b^2 = a^2(1 - e^2) \Rightarrow 5 = a^2 - 4 \Rightarrow a = 3$.

Hence the equation to major axis is $y = 3$

Centre of ellipse is midpoint of SS' i.e. $(0, 3)$

$\therefore$ Equation to ellipse is $\frac{x^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$ or $\frac{x^2}{9} + \frac{(y-3)^2}{5} = 1$

Illustration 6:

Find the equation of the ellipse having centre at (1,2), one focus at (6,2) and passing through the point (4,6).

Solution:

With centre at $(1, 2)$, the equation of the ellipse is $\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$. It passes through the point $(4, 6)$

$\Rightarrow \frac{9}{a^2} + \frac{16}{b^2} = 1$ ...(i)

Distance between the focus and the centre = $(6 - 1) = 5 = ae$

$\Rightarrow b^2 = a^2 - a^2e^2 = a^2 - 25$ ...(ii)

Solving for a^2 and b^2 from the equations (i) and (ii), we get $a^2 = 45$ and $b^2 = 20$.

Hence the equation of the ellipse is $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$.

Illustration 7:

Find the equation of axes, directrix, co-ordinates of foci, center, vertices, length of latus-rectum and eccentricity of an ellipse $\frac{(x-2)^2}{25} + \frac{(y-4)^2}{16} = 1$.

Solution:

Let $x - 2 = X, y - 4 = Y$, so equation of ellipse becomes as $\frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$.

equation of major axis is $Y = 0 \Rightarrow y = 4$.

equation of minor axis is $X = 0 \Rightarrow x = 2$.

center $(X = 0, Y = 0) \Rightarrow x = 2, y = 4$

$C \equiv (2, 4)$

Length of semi-major axis $a = 5$

Length of major axis $2a = 10$

Length of semi-minor axis $b = 4$

Length of minor axis = $2b = 8$.

Let 'e' be eccentricity

$\therefore b^2 = a^2(1 - e^2)$

$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{25-16}{25}} = \frac{3}{5}$.

Length of latus rectum = $LL' = \frac{2b^2}{a} = \frac{2\times16}{5} = \frac{32}{5}$

Co-ordinates foci are $X = \pm ae, Y = 0$

$\Rightarrow S \equiv (X = 3, Y = 0)$ & $S' \equiv (X = -3, Y = 0)$

$\Rightarrow S \equiv (5,4)$ &$S' \equiv (-1,4)$

Position of a Point w.r.t. an Ellipse:

Position of a Point w.r.t. Standard Ellipse:

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Consider the point $P(x_1, y_1)$

If $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < 1$,then P lies inside the ellipse

If $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$,then P lies on the ellipse

If $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} > 1$,then P lies outside the ellipse.

Position of a Point w.r.t. Shifted Ellipse:

Consider the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Consider the point $P(x_1, y_1)$

If $\frac{(x_1-h)^2}{a^2} + \frac{(y_1-k)^2}{b^2} < 1$, then P lies inside the ellipse

If $\frac{(x_1-h)^2}{a^2} + \frac{(y_1-k)^2}{b^2} = 1$,then P lies on the ellipse

If $\frac{(x_1-h)^2}{a^2} + \frac{(y_1-k)^2}{b^2} > 1$,then P lies outside the ellipse.

Illustration 8:

The position of the point (2, 6) with respect to the ellipse $x^2 + 5y^2 = 20$ is

(A) Outside the ellipse
(B) On the ellipse
(C) On the major axis
(D) None of these

Ans. (A)

Solution:

Since $S_1 = (2)^2 + 5(6)^2 - 20 = 164 > 0$.

Hence the point is outside the ellipse.

Illustration 9:

The point $(-4, 1)$ with respect to the ellipse $9(x-2)^2 + 4(y+1)^2 = 36$

(A) Lies on the ellipse
(B) Is inside the ellipse
(C) Is outside the ellipse
(D) Is focus of the ellipse

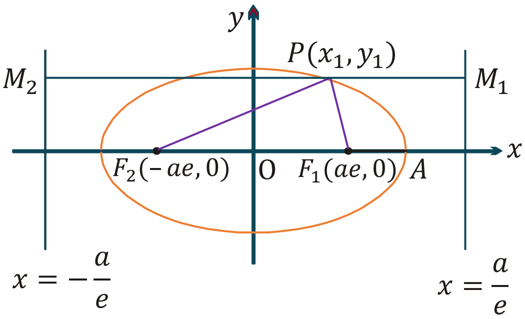
Ans. (C)

Solution:

Since $S_1 = 9(-4-2)^2 + 4(1+1)^2 - 36 = 9(-6)^2 + 4(2)^2 - 36 = 304 > 0$.

Hence the point is outside the ellipse.

Focal Distance / Focal Radii:



For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b)$

$F_1(ae, 0), F_2(-ae, 0)$

$PF_1 = e\left(\frac{a}{e} - x_1\right) = a - ex_1$

$PF_2 = e\left(\frac{a}{e} + x_1\right) = a + ex_1$

$\Rightarrow \quad PF_1 + PF_2 = 2a$

Similarly, for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a < b)$

$F_1(0, be), F_2(0, -be)$

$PF_1 = e\left(\frac{b}{e} - y_1\right) = b - ey_1$

$PF_2 = e\left(\frac{b}{e} + y_1\right) = b + ey_1$

$\Rightarrow PF_1 + PF_2 = 2b$

Note :

Greatest focal length = Length of semi-major axis.

Least focal length = Length of semi-minor axis.

Illustration 10:

Find the focal distance, from focus on the negative x –axis, of a point on the ellipse $\frac{x^2}{49} + \frac{y^2}{36} = 1$ whose abscissa is 4.

Solution:

P(x_1, y_1), hence $x_1 = 4, e = \frac{\sqrt{13}}{7}$

Focal distance $= a + ex_1 = 7 + \frac{\sqrt{13}}{7} \times 4 = \frac{49+4\sqrt{13}}{7}$

Illustration 11:

Find the focal distance, from focus on the positive y –axis, of a point on the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ whose ordinate is 1.

Solution:

P(x_1, y_1), hence $y_1 = 1, e = \frac{3}{5}$

Focal distance $= b - ey_1 = 5 - 1 \times \frac{3}{5} = \frac{22}{5}$.

Illustration 12:

Find the product of focal distances of a point on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ whose abscissa is 2.

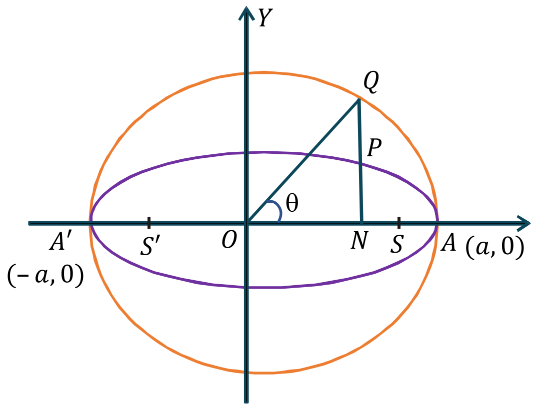
Solution:

$P(x_1, y_1)$, hence $x_1=2, e^2=1-\frac{4}{9}=\frac{5}{9}$

product of focal distances $=(a-ex_1)(a+ex_1)=a^2-e^2x_1^2=9-\frac{25}{81}\cdot 4=\frac{629}{81}$.

Auxiliary Circle/Eccentric Angle/Parametric Coordinates:

Auxiliary Circle/Eccentric Angle:



A circle described on major axis as diameter is called the **auxiliary circle.** Let Q be a point on the auxiliary circle $x^2+y^2=a^2$ such that QP produced is perpendicular to the x −axis then P & Q are called as the CORRESPONDING POINTS on the ellipse & the auxiliary circle respectively. 'θ' is called the **ECCENTRIC ANGLE** of the point P on the ellipse $(0 \le \theta < 2\pi)$.

Note: $\frac{l(PN)}{l(QN)}=\frac{b}{a}=\frac{\text{Semi minor axis}}{\text{Semi major axis}}$

Parametric Representation:

The equations $x=a\cos\theta$ & $y=b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

where θ is a parameter (eccentric angle).

Note that if $P(\theta)\equiv(a\cos\theta, b\sin\theta)$ is on the ellipse then;

$\boldsymbol{Q}(\theta)\equiv(\boldsymbol{a}\cos\boldsymbol{\theta}, \boldsymbol{a}\sin\boldsymbol{\theta})$ is on the auxiliary circle.

Parametric Coordinates of Shifted Ellipse:

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

Parametric equation $\Rightarrow x=h+a\cos\theta, y=k+b\sin\theta$

Auxiliary circle $\Rightarrow (x-h)^2+(y-k)^2=b^2$

Illustration 13:

Find the equation of auxiliary circle of the ellipse having foci $S(2,3)$ and $S'(5,7)$ and eccentricity $\frac{3}{4}$.

Solution:

Semi-major axis $= a$, semi − minor axis $= b$, $e = \frac{3}{4}$

$SS' = 5 = 2ae \Rightarrow a = \frac{5\times4}{2\times3} = \frac{10}{3}$

Centre of the auxiliary circle is mid-point of SS' which is $\left(\frac{7}{2}, 5\right)$ and radius is a $= \frac{10}{3}$

Hence equation is $\left(x-\frac{7}{2}\right)^2 + (y-5)^2 = \frac{100}{9}$.

Illustration 14:

Find the equation of ellipse in cartesian form whose parametric equation is
$x = 2 + 4\cos\theta$ and $y = -3 + 5\sin\theta$.

Solution:

Here $a = 4,\ b = 5,\ h = 2,\ k = -3$

Center is $(2, -3)$

$\Rightarrow$ Equation is $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{25} = 1$

Chord Joining Two Points Ehose Eccentric Angles are $P(\alpha)$ & $Q(\beta)$:

Equation of Chord of an Ellipse:

Equation of a chord of an ellipse joining two points $P(\alpha)$ and $Q(\beta)$ on it is equal to

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

Proof

$P(\alpha) \equiv (a\cos\alpha, b\sin\alpha) Q(\beta) \equiv (a\cos\beta, b\sin\beta)$

equation of chord PQ is

$$(y - b\sin\alpha) = \frac{b\sin\beta - b\sin\alpha}{a\cos\beta - a\cos\alpha}(x - a\cos\alpha)$$

$$\Rightarrow \frac{y}{b} - \sin\alpha = \frac{-2\cos\frac{(\alpha+\beta)}{2}\sin\frac{(\alpha-\beta)}{2}}{2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}\left(\frac{x}{a} - \cos\alpha\right)$$

$$\Rightarrow \frac{y}{b}\sin\frac{(\alpha+\beta)}{2} + \frac{x}{a}\cos\frac{(\alpha+\beta)}{2} = \cos\alpha\cos\frac{(\alpha+\beta)}{2} + \sin\alpha\sin\frac{(\alpha+\beta)}{2} = \cos\frac{(\alpha-\beta)}{2}$$

If this particular chord passes through $(d, 0)$ then we have

$$\frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right);\ \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} = \frac{a}{d}$$

By C & D rule

$$\frac{\cos\left(\frac{\alpha+\beta}{2}\right) - \cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha-\beta}{2}\right)} = \frac{a-d}{a+d}$$

Or $-\dfrac{2\sin\alpha/2\,\sin\beta/2}{2\cos\alpha/2\,\cos\beta/2}=\dfrac{a-d}{a+d}$ i.e. $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{d-a}{d+a}$

if $d = ae$ i.e. PQ is a focal chord then $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{e-1}{e+1}$

Illustration 15:

Find the equation of chord of an ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{2}=1$ joining two points $P\left(\frac{\pi}{6}\right)$ and $Q\left(\frac{2\pi}{3}\right)$.

Solution:

Equation of a chord of an ellipse joining two points $P(\alpha)$ and $Q(\beta)$ on it is equal to

$\dfrac{x}{a}\cos\left(\dfrac{\alpha+\beta}{2}\right)+\dfrac{y}{b}\sin\left(\dfrac{\alpha+\beta}{2}\right)=\cos\left(\dfrac{\alpha-\beta}{2}\right)$, $\alpha=\left(\frac{\pi}{6}\right)$and $\beta=\left(\frac{2\pi}{3}\right)$, $a=2, b=\sqrt{2}$

$$\Rightarrow \frac{x}{2}\cos\left(\frac{\frac{\pi}{6}+\frac{2\pi}{3}}{2}\right)+\frac{y}{\sqrt{2}}\sin\left(\frac{\frac{\pi}{6}+\frac{2\pi}{3}}{2}\right)=\cos\left(\frac{\frac{\pi}{6}-\frac{2\pi}{3}}{2}\right)$$

$$\Rightarrow \frac{x}{2}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+\frac{y}{\sqrt{2}}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)=\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow x(\sqrt{3}-1)+y(\sqrt{2})(\sqrt{3}+1)=4$$

Illustration 16:

Find the eccentricity of the ellipse if chord joining two points P, Q whose eccentric angles are $-\dfrac{\pi}{3}$ and $\dfrac{\pi}{2}$ passes through the focus.

Solution:

PQ is a focal chord then $\tan\dfrac{-\frac{\pi}{3}}{2}\tan\dfrac{\frac{\pi}{2}}{2}=\dfrac{e-1}{e+1}$

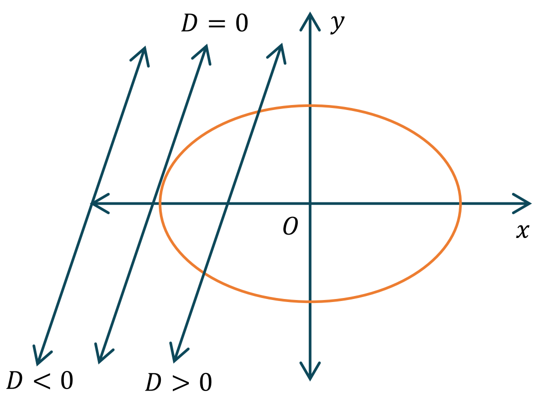
$$\Rightarrow \tan\left(-\frac{\pi}{6}\right)\tan\frac{\pi}{4}=\frac{e-1}{e+1}=\left(-\frac{1}{\sqrt{3}}\right) \Rightarrow e=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Tangents to the Ellipse:

Equation of Tangent in Slope form

Line $y = mx + c$ ...(1)

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ...(2)



solving (1) and (2)

$b^2x^2 + a^2(mx + c)^2 = a^2b^2$

i.e. $(a^2m^2 + b^2)x^2 + 2a^2cmx + a^2(c^2 - b^2) = 0$

Here $D \equiv 4a^4m^2c^2 - 4a^2(c^2 - b^2)\,(b^2 + a^2m^2)$

For tangent

$D = 0 \Rightarrow a^2m^2c^2 = c^2b^2 + c^2a^2m^2 - b^4 - a^2b^2m^2$

$\Rightarrow c^2 = b^2 + a^2m^2$

Hence $\boxed{y = mx \pm \sqrt{a^2m^2 + b^2}}$ is always a tangent to the ellipse for all $m \in R$.

Equation of Tangent in Cartesian form

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of tangent at $P(x_1,y_1)$ is $T=0$

$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Slope of tangent $=-\frac{b^2x_1}{a^2y_1}$

Equation of Tangent in Parametric form

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$x_1=a\cos\theta \qquad y_1=b\sin\theta$

Equation of tangent at $P(\theta)$ is

$\Rightarrow \frac{x(a\cos\theta)}{a^2}+\frac{y(b\sin\theta)}{b^2}=1$

$\Rightarrow \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

Slope of tangent$=-\frac{b\cos\theta}{a\sin\theta}=-\frac{b}{a}\cot\theta$

Illustration 17:

Find equation of tangent to an ellipse $3x^2+4y^2=12$, parallel to the line $y+2x=4$.

Solution:

$a^2=4, b^2=3, m=-2$

$\Rightarrow\ y=-2x\pm\sqrt{4(-2)^2+3}\Rightarrow y=-2x\pm\sqrt{19}$

Illustration 18:

Find the equation of the tangent to an ellipse $9x^2+16y^2=144$ passing from $(2,3)$

[Ans. $y=3, x+y=5$]

Solution:

$y-3=m(x-2)\Rightarrow y=mx+3-2m$

Now

$c^2=a^2m^2+b^2\Rightarrow(3-2m)^2=16m^2+9\Rightarrow m=-1$ and 0

OR

Solve $y=mx\pm\sqrt{a^2m^2+b^2}$ put $x=2$ and $y=3$

$\Rightarrow 3=2m\pm\sqrt{16m^2+9}\Rightarrow m=-1$ and 0 but now don't put in above equation because it will give four equation. Rather use $y-3=m(x-2)$ and put values of m.

Illustration 19:

Find equation of tangent at the point $(3,2)$ on the ellipse $x^2+4y^2=25$.

Solution:

Here $x_1=3$ and $y_1=2$

Equation of tangent is $T=0$

$\Rightarrow x(3)+4(y)(2)=25\Rightarrow 3x+8y=25$

Illustration 20:

Find the equation of the tangents to the ellipse $x^2 + 16y^2 = 16$ each one of which makes an angle of 60° with the x-axis

Solution:

We have, $x^2 + 16y^2 = 16 \Rightarrow \frac{x^2}{4^2}+\frac{y^2}{1^2} = 1$

comparing this equation $\frac{x^2}{4^2}+\frac{y^2}{1^2} = 1$ where $a^2 = 16$ and $b^2 = 1$

Slope of tangent $m = \sqrt{3}$

So, the equations of the tangents are

$y = mx \pm \sqrt{a^2m^2 + b^2}$ i.e. $y = \sqrt{3}x \pm \sqrt{48+1} \Rightarrow y = \sqrt{3}x \pm 7$

Illustration 21:

Find the equations of tangents to the ellipse $x^2 + 4y^2 = 25$ at the points whose ordinate is 2.

Solution:

$x^2 + 4(2)^2 = 25 \Rightarrow x_1 = \pm 3, y_1 = 2$

Equation of tangent is $T = 0$

$\Rightarrow \pm 3x + 4y(2) = 25 \Rightarrow 3x + 8y = 25, -3x + 8y = 25$

Illustration 22:

For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Solution:

$\because$ Equation of ellipse is $9x^2 + 16y^2 = 144$ or $\frac{x^2}{16}+\frac{y^2}{9}=1$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then we get $a^2 = 16$ and $b^2 = 9$

and comparing the line $y = x + \lambda$ with $y = mx + c$ $\therefore m = 1$ and $c = \lambda$

If the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$, then $c^2 = a^2m^2 + b^2$

$l \Rightarrow \lambda^2 = 16 \times 12 + 9 \Rightarrow \lambda^2 = 25$ $\therefore \lambda = \pm 5$

Illustration 23:

Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Solution:

Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$.

$m = \frac{1}{2}$

Since $3x^2 + 4y^2 = 12$ or $\frac{x^2}{4}+\frac{y^2}{3}=1$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore a^2 = 4$ and $b^2 = 3$

So the equation of the tangent are $y=\frac{1}{2}x\pm\sqrt{4\times\frac{1}{4}+3}$

$\Rightarrow y = \frac{1}{2}x \pm 2 \Rightarrow x - 2y \pm 4 = 0.$

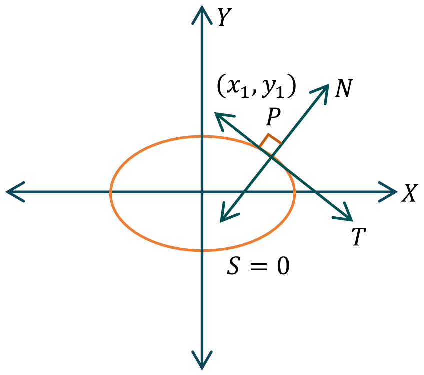
Normals:

Point form:

The equation of normal at $P \equiv (x_1, y_1)$ on ellipse $S = 0$ is given by $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.

Proof:

The equation of tangent at $P(x_1, y_1)$ is $T = 0$



$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$

So, slope of tangent = $\frac{-b^2x_1}{a^2y_1}$

$\Rightarrow$ Slope of normal = $\frac{a^2y_1}{b^2x_1}$

Hence, equation of normal at pint P is

$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$ or $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

If $(x_1, y_1) \equiv (a\cos\theta, b\sin\theta)$ then equation of normal is

$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

Note: Major axis and minor axis are two normal to ellipse.

Normal other than major axis, never passes through focus.

Equation of Normal to Ellipse with given Slope

The equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point $p(a\cos\theta,\ b\sin\theta)$ is

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$

or $y = \left(\frac{a}{b}\tan\theta\right)x - \frac{(a^2 - b^2)}{b}\sin\theta$

Let $\frac{a}{b}\tan\theta = m \Rightarrow \sin\theta = \pm\frac{bm}{\sqrt{a^2 + b^2m^2}}$

$\Rightarrow y = mx \pm \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}},\ \ m \in R$

Illustration 24:

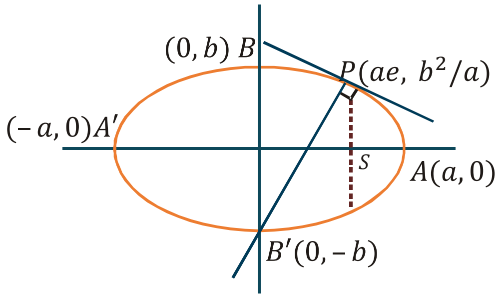
If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by $e=\sqrt{\frac{\sqrt{5}-1}{2}}$

Solution:

The co-ordinates of an end of the latus-rectum are $(ae, b^2/a)$.

The equation of normal at $P(ae, b^2/a)$ is

$$\frac{a^2x}{ae}-\frac{b^2(y)}{b^2/a}=a^2-b^2 \text{ or } \frac{ax}{e}-ay=a^2-b^2$$



It passes through one extremity of the minor axis whose co-ordinates are $(0,-b)$

$\therefore\ 0+ab = a^2-b^2 \Rightarrow (a^2b^2)=(a^2-b^2)^2$

$\Rightarrow\ a^2.a^2(1-e^2)=(a^2e^2)^2 \Rightarrow 1-e^2=e^4$

$\Rightarrow\ e^4+e^2-1=0 \Rightarrow (e^2)^2+e^2-1=0$

$\therefore\ e^2=\frac{-1\pm\sqrt{1+4}}{2} \Rightarrow e=\sqrt{\frac{\sqrt{5}-1}{2}}$ (taking positive sign)

Illustration 25:

Find the equation of normal to the ellipse $9x^2+16y^2=288$ at the point (4,3).

Solution:

Ellipse is $\frac{x^2}{32}+\frac{y^2}{18}=1, x_1=4, y_1=3$

Equation of normal is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

$\frac{32x}{4}-\frac{18y}{3}=32-18 \Rightarrow 8x-6y=14$

$\Rightarrow 4x-3y=7$

Illustration 26:

Find the equation of normal of with slope 1 to the ellipse $x^2+3y^2=6$

Solution:

$m=1,\ a^2=6,\ b^2=2$

Equation of normal with slope m is $y=mx\pm\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}} \Rightarrow y = x\pm\frac{(6-2)}{\sqrt{6+2}}$

$\Rightarrow y=x\pm\sqrt{2}$

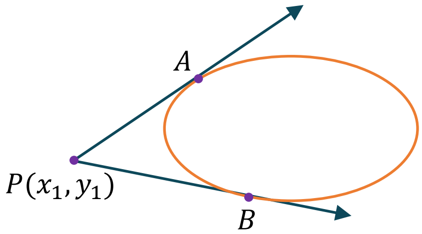
Common Articles:

Equation of Pair of Tangent from an External Point:

If $P(x_1, y_1)$ be any point lies outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and a pair of tangents PA, PB can be drawn to it from P. Then the equation of pair of tangents of PA and PB is $SS_1=T^2$

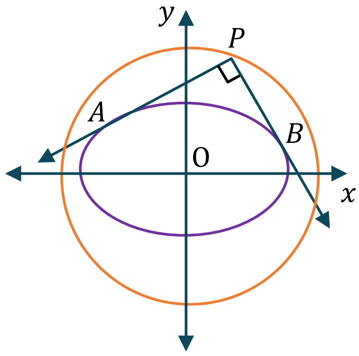
where $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$, $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$ c

i.e. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$



Director Circle:

Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.**



Consider the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

Let perpendicular tangents to ellipse intersect at point $P(h, k)$.

So, the equation of pair to tangents through point P is.

$$\left(\frac{hx}{a^2} + \frac{ky}{b^2} = 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{h^2}{a^2} + \frac{k^2}{b^1} - 1\right)$$

Since tangents are perpendicular, in above equation sum of coefficients of x^2 and y^2 is zero.

$$\therefore \frac{1}{a^2}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) + \frac{1}{b^2}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) - \frac{h^2}{a^4} - \frac{k^2}{b^4} = 0$$

$$\Rightarrow \frac{k^2}{a^2b^2} - \frac{1}{a^2} + \frac{h^2}{a^2b^2} - \frac{1}{b^2} = 0$$

$\Rightarrow h^2 + k^2 = a^2 + b^2$

Thus, locus of point is P is $x^2 + y^2 = a^2 + b^2$, which is circle.

This is called director circle of ellipse.

Note: Director circle is always concentric with the ellipse

For the shifted ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ equation of director circle is

$(x-h)^2 + (y-k)^2 = a^2 + b^2$

Illustration 27:

The angle between the pair of tangents drawn to the ellipse $2x^2 + 3y^2 = 6$ from the point $(-1, 2)$, is

Solution:

$SS_1 = T^2 \Rightarrow (2x^2+3y^2-6)(2+12-6) = (-2x+6y-6)^2 \Rightarrow x^2+2xy-y^2-2x+6y-7=0$

$a=1, b=-1, \Rightarrow a+b=0$

Hence angle between tangents is $\frac{\pi}{2}$.

Illustration 28:

The equations of the tangents of the ellipse $3x^2+4y^2=12$ which pass through the point $(2,1)$ is

Solution:

Tangent will be of the form $y=mx\pm\sqrt{a^2m^2+b^2} \Rightarrow y=mx\pm\sqrt{4m^2+3}$.

Put (2,1) in above equation $\Rightarrow 1=2m\pm\sqrt{4m^2+3}$

$\Rightarrow 4m^2-4m+1=4m^2+3 \Rightarrow m=-\frac{1}{2}$.

Also $x=2$ is tangent to ellipse at its vertex.

Hence tangents are $y=-\frac{x}{2}+2$ and $x=2$.

Illustration 29:

If the distance between directrices of an ellipse is 12 and its eccentricity is $\frac{1}{3}$, then find the radius of director circle of this ellipse.

Solution:

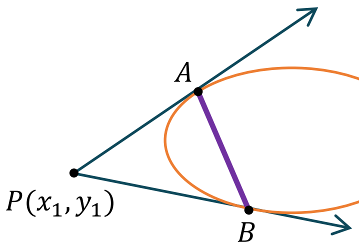
Distance between directrix $=\frac{2a}{e}=12, e=\frac{1}{3} \Rightarrow a=2, b=a\sqrt{1-e^2}=2\sqrt{1-\frac{1}{9}}=\frac{4\sqrt{2}}{3}$

Radius of director circle is $\sqrt{a^2+b^2}=\sqrt{4+\frac{32}{9}}=\frac{2\sqrt{17}}{3}$.

Chord of Contact:

If PA and PB be the tangents from point $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the chord of contact AB is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $T=0$ $(at\ x_1,y_1)$.

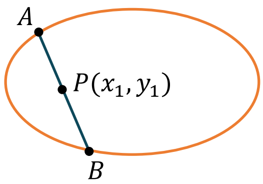


Equation of Chord With Point (x_1,y_1):

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid-point is (x_1,y_1) *is* $\boldsymbol{T=S_1}$

where $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$, $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, i.e. $\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$

P is mid-point of chord AB



Property of Segments of Focal Chord:

$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$

The harmonic mean of segments of any focal chord is semi latus rectum.

Let AB be a focal chord passing through focus F_1

$\Rightarrow \frac{2(AF_1)(BF_1)}{AF_1 + BF_1} = \frac{b^2}{a}$

Illustration 30:

Find the locus of the mid-point of focal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

Let $P \equiv (h, k)$ be the mid-point

$\therefore$ Equation of chord whose mid-point is (h, k) is $\frac{xh}{a^2} + \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ It passes through focus, either $(ae, 0)$ or $(-ae, 0)$

If it passes through $(ae, 0)$

$\Rightarrow \frac{eh}{a} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \Rightarrow \frac{ex}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

If it passes through $(-ae, 0)$

$\therefore$ locus is $-\frac{ex}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Illustration 31:

Tangents are drawn from the point $(-2, 2)$ to the ellipse $x^2 + 2y^2 = 6$. Find the equation of chord of contact.

Solution:

$x_1 = -2, y_1 = 2$.

Equation of chord of contact is $T = 0$

$\Rightarrow x(-2) + 2y(2) - 6 = 0 \Rightarrow x - 2y + 3 = 0$.

Illustration 32:

If $x + 2y = 4$ intersect the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at P and Q, then the point of intersection of tangents at P and Q is

Solution:

Let point of intersection is $R(h, k)$

PQ is chord of contact

$\frac{hx}{25} + \frac{ky}{16} = 1 \qquad$ ...(i)

Equation of PQ given is $\frac{x}{4} + \frac{y}{2} = 1 \qquad$ ...(ii)

Comparing (i) and (ii) $\Rightarrow \frac{h}{25} = \frac{1}{4}, \frac{k}{16} = \frac{1}{2} \Rightarrow h = \frac{25}{4}, k = 8$

Illustration 33:

Find the equation of chord of ellipse $4x^2 + 5y^2 = 20$ whose mid-point is $(2, -1)$

Solution:

Equation of chord with given mid-point is $T = S_1$.

$x_1 = 2, y_1 = -1$

$T = S_1 \Rightarrow 4x(2) + 5y(-1) - 20 = 16 + 5 - 20 \ \Rightarrow 8x - 5y = 21$

Important Highlights

Property - I:

The ratio of area of any triangle inscribed in an ellipse to the triangle formed by corresponding points on the auxiliary circle is equal to the ratio of semi minor axis to semi major axis.

Let X, Y, Z be three points on an ellipse and P, Q, R be three corresponding points on its auxiliary circle.

$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$

$\frac{\text{Area of } \Delta XYZ}{\text{Area of } \Delta PQR} = \frac{b}{a}$

Let the three point on the ellipse be $X(a\cos\alpha, b\sin\alpha), Y(a\cos\beta\ , b\sin\beta)$ and $Z(a\cos\gamma, b\sin\gamma)$.

Then corresponding points on the auxiliary circle are

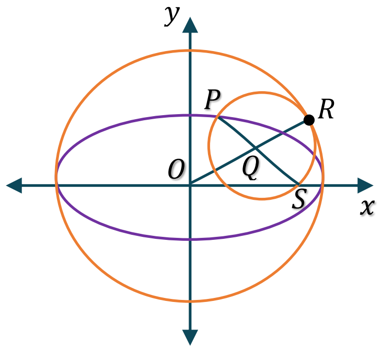
$P(a\cos\alpha, a\sin\alpha), Q(a\cos\beta, a\sin\beta)$ and $R(a\cos\gamma, a\sin\gamma)$

$$\text{Now, } \frac{\text{area of } \Delta XYZ}{\text{area of } \Delta PQR} = \frac{\frac{1}{2}\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ a\cos\gamma & b\sin\gamma & 1 \end{vmatrix}}{\frac{1}{2}\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ a\cos\gamma & b\sin\gamma & 1 \end{vmatrix}}$$

$$= \frac{ab\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}}{a^2\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}} = \frac{b}{a}$$

Property - II:

The circle on any focal distance as diameter touches the auxiliary circle.



$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ (a > b)$

PS is diameter $\Rightarrow\ PQ = QS$

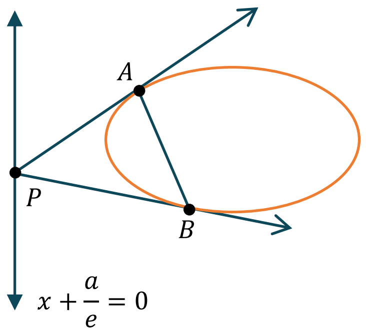
O, Q, R are collinear.

Property - III:

Chord of contact w.r.t. any point on the directrix passes thought the corresponding focus.

$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b)$

AB is focal chord



Property - IV:

Feet of perpendiculars from foci upon any tangent lie on the auxiliary circle.

Product of the lengths of perpendiculars from foci upon any tangent of ellipse is equal to the square of the length of semi minor axis.

$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b)$

$(ST)(S'T') = b^2$

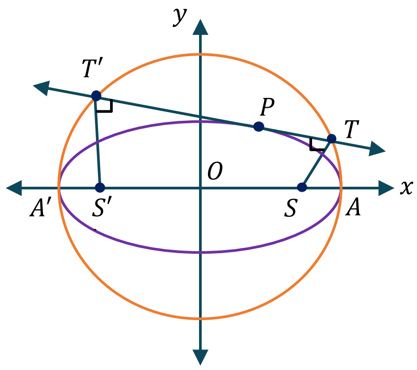


Illustration 34:

The ratio of the area of triangle inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1(a > b)$ to that of triangle formed by the corresponding points on its auxiliary circle is $1/3$ then, find the eccentricity of the ellipse.

Solution:

Ratio of area of triangle $= \frac{b}{a} = \frac{1}{3}$

$\Rightarrow e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

Illustration 35:

Find the locus of point of intersection of tangents at the end points of any focal chord of ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Solution:

Locus of point of intersection of tangents at the end points of any focal chord is its directrix.

$$e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

$\Rightarrow$ Equation of directrices are $x = \pm\frac{4}{\sqrt{7}/4} = \pm\frac{16}{\sqrt{7}}$

Hence locus is $x = \pm\frac{16}{\sqrt{7}}$

Illustration 36:

If F_1 and F_2 are the feet of the perpendiculars from the foci S_1 and S_2, respectively, of an ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ on the tangent at any point P on the ellipse, then prove that $S_1F_1 + S_2F_2 \geq 8$.

Solution:

We know that the product of perpendiculars from two foci of an ellipse upon any tangent is equal to the square of the semi-minor axis.

Then $(S_1F_1) . (S_2F_2) = 16$

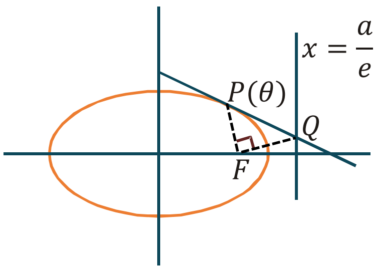
Now, A.M. $\geq$ G.M.

$$\Rightarrow \frac{S_1F_1 + S_2F_2}{2} \geq \sqrt{S_1F_1 \times S_2F_2}$$

$\Rightarrow S_1F_1 + S_2F_2 \geq 8.$

Property-V:

Portion of tangent between the point of contact and the point where it meets the directrix subtends right angle at the corresponding focus.



$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b)$

By solving equation of tangent and directrix we get

$$Q\left(\frac{a}{e}, \frac{b}{\sin\theta}\left(1 - \frac{\cos\theta}{e}\right)\right)$$

Slope of $PF = m_1 = \frac{b\sin\theta}{a\cos\theta - ae}$

Slope of $FQ = m_2 = \dfrac{\dfrac{b}{\sin\theta}\left(1-\dfrac{\cos\theta}{e}\right)}{\dfrac{a}{e}-ae} = \dfrac{b(e-\cos\theta)}{a\sin\theta(1-e^2)}$

Now, $m_1 m_2 = \dfrac{b\sin\theta}{a\cos\theta - ae} \times \dfrac{b(e-\cos\theta)}{a\sin\theta(1-e^2)}$

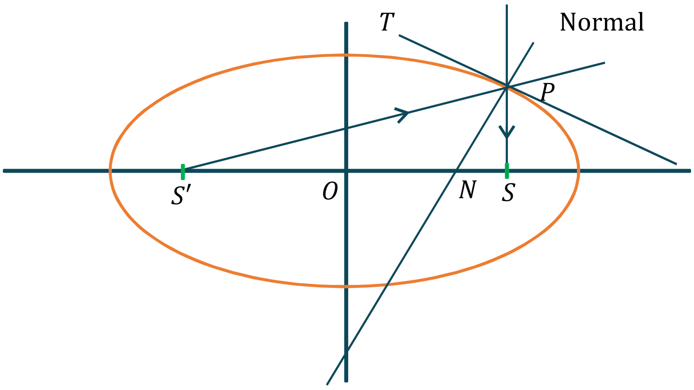
$= -\dfrac{b^2}{a^2(1-e^2)} = -1$

Property - VI:

The tangent and normal at a point P on the ellipse bisect the external and internal angles between the focal distances of P. If incident ray emanating from one of the foci gets reflected by the ellipse, then reflected ray passes through the other focus.

$S \equiv \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ (a > b)$

$\dfrac{PS'}{PS} = \dfrac{S'N}{NS}$



Property-VII:

Image of one of the foci in the tangent lies on the line joining other focus and point of contact.

$S \equiv \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad (a > b)$

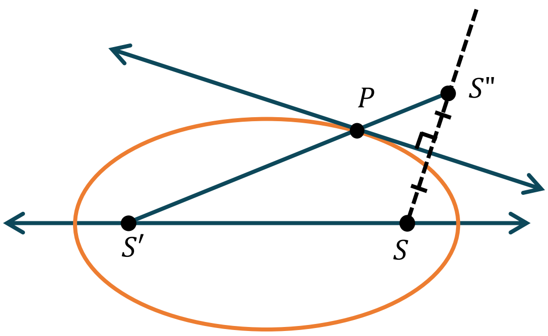
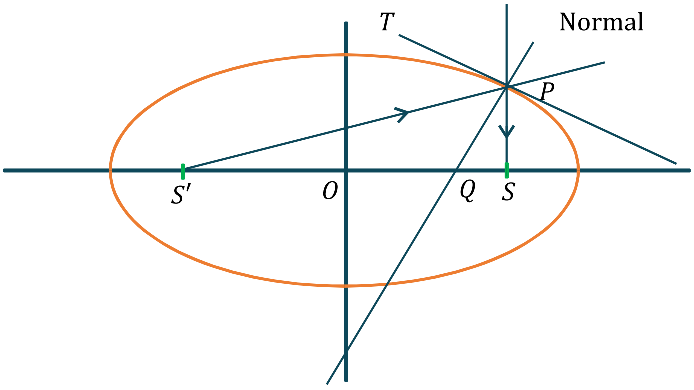


Illustration 37:

The normal at point $P\left(\dfrac{\pi}{3}\right)$ on the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ meets the major axis of the ellipse at Q. if S and S' are foci of given ellipse, then find the ratio $SQ : S'Q$.

Solution:



$P\left(\frac{\pi}{3}\right) \equiv P\left(\frac{5}{2}, 2\sqrt{3}\right)$ Equation of the ellipse is

$\frac{x^2}{25} + \frac{y^2}{16} = 1$

$\therefore\ e^2 = 1 - \frac{16}{25} = \frac{9}{25} \quad \Rightarrow e = \frac{3}{5}$

Hence, foci are $S(3,0)$ and $S'(-3,0)$.

Normal at P bisects the angle $\angle SPS'$.

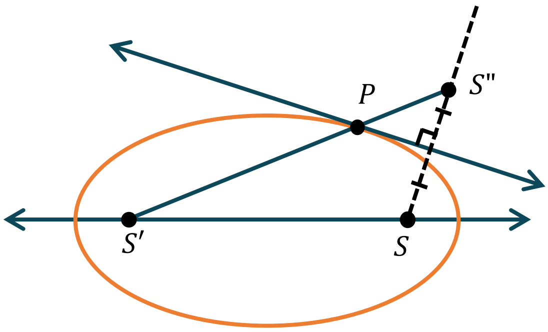
Therefore, in triangle SPS' using angle bisector theorem, we get

$$\frac{SQ}{S'Q} = \frac{SP}{S'P} = \frac{a-ex}{a+ex} = \frac{5-\frac{3}{5}\times\frac{5}{2}}{5+\frac{3}{5}\times\frac{5}{2}} = \frac{7}{13}$$

Illustration 38:

An ellipse has the points (2,2) and (1, -1) as its foci and $x + y - 3 = 0$ as one of its tangents. Find the point where this line touches the ellipse.

Solution:



$S(2,2), S'(1,-1)$, tangent is $x + y - 3 = 0$

Image of focus $S(2,2)$ in the tangent at P lies on the line $S'P$.

Let image $S'' \equiv (h,k)$

$\therefore\ \frac{h-2}{1} = \frac{k-2}{1} = \frac{-2(2+2-3)}{2} = -1$

$\Rightarrow S'' = (1,1)$

Equation of line $S'S''$ is $x = 1$

Solving tangent and $S'S''$, we get $P \equiv (1,2)$.

Illustration 39:

Find the equation of tangents to the ellipse $\frac{x^2}{50} + \frac{y^2}{32} = 1$ which pass through the point $(15, -4)$.

Solution:

Tangent passes through $(15, -4)$

$(y + 4) = m(x - 15)$

It is tangent to $\frac{x^2}{50} + \frac{y^2}{32} = 1$, so

$(4 + 15m)^2 = 50\,m^2 + 32$

$\Rightarrow 225\,m^2 + 120m + 16 = 50m^2 + 32$

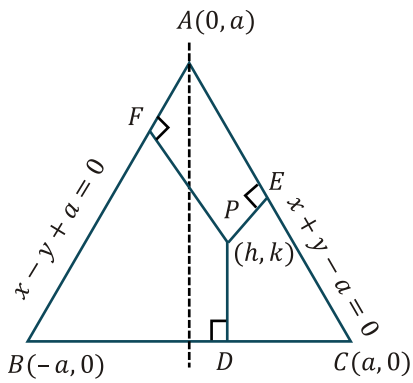
$\Rightarrow (5m + 4)\,(35m - 4) = 0$

$m = \frac{-4}{5}\ \&\ \frac{4}{35}$

$\therefore$ tangents are $4x + 5y = 40\ \&\ 4x - 35y = 200$.

Illustration 40:
ABC is an isosceles triangle with its base BC twice its altitude, A point P moves within the triangle such that the square of its distance from BC is half the area of rectangle contained by its distances from the two sides. Show that the locus of P is an ellipse with eccentricity $\sqrt{\frac{2}{3}}$ passing through B & C.

Solution:



$(PD)^2 = \frac{1}{2}PE.PF$

$k^2 = \frac{1}{2}\left|\frac{h+k-a}{\sqrt{2}}\right|\left|\frac{h-k+a}{\sqrt{2}}\right|$

$4k^2 = -(h+k-a)\,(h-k+a)$

$4k^2 = -\{h^2-(k-a)^2\}$

$4k^2 = -\{h^2-k^2+2ak-a^2\}$

$h^2+3k^2+2ak-a^2=0$

$\therefore$ Locus of (h, k) is

$x^2+3y^2+2ay-a^2=0$

$x^2+3(y^2+\frac{2}{3}ay+\frac{1}{9}a^2) = a^2+\frac{a^2}{3}$

$x^2+3\left(y+\frac{1}{3}a\right)^2 = \frac{4a^2}{3}$

$$\frac{x^2}{\frac{4a^2}{3}}+\frac{(y+\frac{1}{3}a)^2}{\frac{4a^2}{9}}=1$$

$\therefore e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{1}{3}} = \sqrt{\frac{2}{3}}$

Illustration 41:
If focus and corresponding directrix of an ellipse are $(3, 4)$ and $x+y-1=0$ and eccentricity is $\frac{1}{2}$ then find the co-ordinates of extremities of major axis.

Solution:
Let point $P(5\cos\theta, 2\sin\theta)$

Equation of tangent at P to ellipse is

$$\frac{x\cos\theta}{5}+\frac{y\sin\theta}{2}=1$$

If it is also tangent to circle $x^2+y^2=16$, then

$$\frac{0+0-1}{\sqrt{\frac{\cos^2\theta}{125}+\frac{\sin^2\theta}{4}}} = 4 = (p=r)$$

$4\cos 2\theta + 25\sin 2\theta = \frac{25}{4} \Rightarrow \sin^2\theta = \frac{9}{4\times 21}$

$\Rightarrow \sin\theta = \frac{3}{2\sqrt{21}} = \frac{\sqrt{3}}{2\sqrt{7}}$ & $\cos\theta = \frac{5}{2\sqrt{7}}$

$\therefore$ Equation of tangent is $\frac{x}{2\sqrt{7}}+\frac{y\sqrt{3}}{4\sqrt{7}}=1$

$\Rightarrow\ y=\frac{-2}{\sqrt{3}}x+4\sqrt{\frac{7}{3}}$

Intercept between coordinate axis i.e., distance between.

$\left(0,4\sqrt{\frac{7}{3}}\right)$ & $\left(2\sqrt{7},0\right)$ is $\frac{14}{\sqrt{3}}$.

Illustration 42:

The area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is $\pi/4$ is -

(A) $\frac{(a^2-b^2)ab}{a^2+b^2}$ (B) $\frac{(a^2+b^2)ab}{a^2-b^2}$ (C) $\frac{(a^2-b^2)}{ab(a^2+b^2)}$ (D) $\frac{(a^2+b^2)}{(a^2-b^2)ab}$

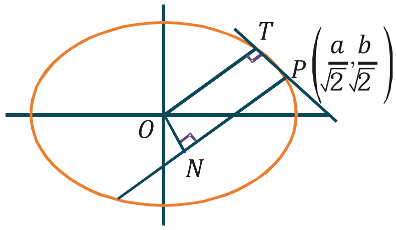
Ans. (A)

Solution:

Let equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of tangent at $P\left(a\cos\frac{\pi}{4},\ b\sin\frac{\pi}{4}\right)$ is

$\frac{x}{a}+\frac{y}{b}=\sqrt{2}$



Equation of normal at P is $\sqrt{2}ax-\sqrt{2}by=a^2-b^2$

Now $OT=\left|\frac{-\sqrt{2}ab}{\sqrt{a^2+b^2}}\right|$

and $ON=\left|\frac{-\left(a^2-b^2\right)}{\sqrt{2}\sqrt{a^2+b^2}}\right|$

Area of rectangle = OT. ON = $\frac{(a^2-b^2)ab}{a^2+b^2}$

Illustration 43:

Q is a point on the auxiliary circle of an ellipse. P is the corresponding point on ellipse. N is the foot of perpendicular from focus S, to the tangent of auxiliary circle at Q. Then -

(A) $SP=SN$ (B) $SP=PQ$ (C) $PN=SP$ (D) $NQ=SP$

Ans. (A)

Solution:

Let ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then

$P(a\cos\theta, b\sin\theta)$

$Q(a\cos\theta, a\sin\theta)$

S is $(ae,0)$ &

tangent at Q will be

$x\cos\theta+y\sin\theta=a$

Now check for $SP=SN$

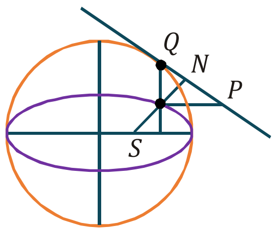


Illustration 44:

Find the locus of the point the chord of contact of the tangent drawn from which to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2 + y^2 = c^2$, where $c < b < a$.

Solution:

Let point be (h, k)

chord of contact to ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{hx}{a^2}+\frac{ky}{b^2}=1$

It is tangent to circle $x^2 + y^2 = c^2$, then its perpendicular distance from centre = radius

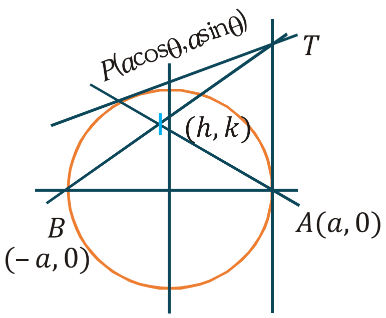
$$\left|\frac{0+0-1}{\sqrt{\left(\frac{h}{a^2}\right)^2+\left(\frac{k}{b^2}\right)^2}}\right|=|c|$$

Locus is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{C^2}$

Illustration 45:

The tangent at any point P of a circle $x^2 + y^2 = a^2$ meets the tangent at a fixed point $A(a, 0)$ in T and T is joined to B, the other end of the diameter through A. Prove that the locus of the intersection of AP and BT is an ellipse whose eccentricity is $\frac{1}{\sqrt{2}}$.

Solution:



Equation of tangent at P

$x\cos\theta + y\sin\theta = a$

which meets $x = a$ at T

$\therefore T\left(a, a\tan\frac{\theta}{2}\right)$

Equation of $AP \rightarrow y=-\cot\left(\frac{\theta}{2}\right)(x-a)$ ...(1)

Equation of $BT \rightarrow y = \frac{\tan(\theta/2)}{2}(x+a)$ ...(2)

From (1) & (2)

$y^2 = -\frac{1}{2}(x^2 - a^2)$

$x^2 + 2y^2 = a^2$ whose eccentricity is $\frac{1}{\sqrt{2}}$.

Illustration 46:

The tangent at $P\left(4\cos\theta\,,\,\frac{16}{\sqrt{11}}\sin\theta\right)$ to the ellipse $16x^2+11y^2=256$ is also a tangent to the circle $x^2+y^2-2x-15=0$. Find θ. Find also the equation to the common tangent.

Solution:

Tangent at $P\left(4\cos\theta\,,\,\frac{16}{\sqrt{11}}\sin\theta\right)$ to ellipse

$16x^2+11y^2=(16)^2$

$\Rightarrow 16.4\cos\theta.x+11\times\frac{16}{\sqrt{11}}\sin\theta\,y=16^2$

$\Rightarrow 4.x\cos\theta.x+\sqrt{11}\,y\sin\theta=16$ ...(1)

If it is tangent to circle $x^2+y^2-2x-15=0$, then

$$p=r\Rightarrow\frac{4\cos\theta-16}{\sqrt{(4\cos\theta)^2+\left(\sqrt{11}\sin\theta\right)^2}}=4$$

On solving $\cos\theta=\frac{1}{2}$ i.e. $\theta=\frac{\pi}{3}$ or $\frac{5\pi}{3}$,

So line will be $4x\pm\sqrt{33}\,y=32$

Illustration 47:

Find the equation of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2+4y^2=16$.

Solution:

Let radius of circle be r then equation of circle is

$(x-1)^2+y^2=r^2$ ...(1)

Let it touch the ellipse $x^2+4y^2=16$ at

$P(4\cos\theta,2\sin\theta)$, then equation of tangent to ellipse at P is

$4x\cos\theta+8y\sin\theta=16$

Now CP is perpendicular to tangent, so

$$\left(\frac{2\sin\theta-0}{4\cos\theta-1}\right)\times\frac{\cos\theta}{2\sin\theta}=-1\;\Rightarrow\;\cos\theta=\frac{-1}{3}$$

$\because\; r=CP=\sqrt{(4\cos\theta-1)^2+(2\sin\theta-0)^2}$

$$=\sqrt{\left(\frac{4}{3}-1\right)^2+\left(2\times\frac{2\sqrt{2}}{3}\right)^2}\;\Rightarrow r=\sqrt{\frac{11}{3}}$$

$\therefore$ equation of circle is $(x-1)^2+y^2=\frac{11}{3}$

Illustration 48:

The tangents from (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect at right angles. Show that the normals at the points of contact meet on the line $\frac{y}{y_1}=\frac{x}{x_1}$.

Solution:

$$(x_1, y_1)=\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

Tangent at $P(\alpha)$ $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$

Tangent at $Q(\beta)$ $\frac{x}{a}\cos\beta+\frac{y}{b}\sin\beta=1$

Both are perpendicular so $\frac{b^2\cos\alpha\cos\beta}{a^2\sin\alpha\sin\beta}=-1$ ...(1)

Normal at $p(x) \Rightarrow \frac{ax}{\cos x}-\frac{by}{\sin\alpha}=a^2-b^2$

if both normals interest at (h, k), then

$$\frac{ah}{\cos\alpha}-\frac{bk}{\sin\alpha}=\frac{ah}{\cos\beta}-\frac{bk}{\sin\beta}$$

$$\frac{ah.\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\alpha\cos\beta}=\frac{-bk\ \cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\alpha\ \sin\beta}$$

$$\frac{ah}{bk}\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}=\frac{-\cos\alpha\ \cos\beta}{\sin\alpha\ \sin\beta} \qquad ...(2)$$

Use (1) & (2) $\Rightarrow \frac{ah}{bk}, \frac{ay}{bx_1}=\frac{-\cos\alpha\cos\beta}{\sin\alpha\sin\beta} \Rightarrow \frac{hy_1}{kx_1}=1$