03 Differential Equations

1. Differential Equation:

An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**.

A differential equation is said to be **ordinary**, if the differential coefficients have reference to a single independent variable only e.g. $\frac{d^2y}{dx^2}-\frac{2dy}{dx}+\cos x=0$ and it is said to be **partial** if there are two or more independent variables. e.g. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$ is a partial differential equation. We are concerned with ordinary differential equations only.

2. Order of Differential Equation:

The order of a differential equation is the order of the highest differential coefficient occurring in it.

3. Degree of Differential Equation:

The highest exponent of the highest order differential coefficient, when the differential equation is expressed as a polynomial in all the differential coefficient is called degree of the differential equation.

Thus the differential equation :

$$a_{0,0}+\sum_{i=1}^{n_1}a_{i,1}\left(\frac{d^my}{dx^m}\right)^i+\sum_{i=1}^{n_2}a_{i,2}\left(\frac{d^{m-1}y}{dx^{m-1}}\right)^i+\ldots\ =0 \text{ is of order } m \text{ \& degree } n_1.$$

where $a_{i,j}$ is some expression in x, y and $a_{n_1,1}\neq 0$

Note:

(i) The exponents of all the differential coefficient should be free from radicals and fraction.

(ii) The degree is always positive natural number.

(iii) The degree of differential equation may or may not exist.

Illustration 1:

Find the order and degree of the following differential equation :

(i) $\sqrt{\frac{d^2y}{dx^2}}=\sqrt[3]{\frac{dy}{dx}+3}$ (ii) $\frac{d^2y}{dx^2}=\sin\left(\frac{dy}{dx}\right)$ (iii) $\frac{dy}{dx}=\sqrt{3x+5}$

Solution:

(i) The given differential equation has the order 2. Since the given differential equation is not a polynomial in the differential coefficients, the degree of the equation is **not defined**.

(ii) The given differential equation has the order 2. Since the given differential equation is not a polynomial in the differential coefficients, the degree of the equation is **not defined**.

(iii) Its order is 1 and degree 1.

Illustration 2:

The order and degree of the differential equation $\left(\frac{d^2s}{dt^2}\right)^2+3\left(\frac{ds}{dt}\right)^3+4=0$ are -

(A) 2 , 2 (B) 2 , 3 (C) 3 , 2 (D) none of these

Ans. (A)

Solution:

Clearly order is 2 and degree is 2 (from the definition of order and degree of differential equations).

4. Formation of a Differential Equation:

In order to obtain a differential equation whose solution is

$f(x_1, y_1, c_1, c_2, c_3 \ldots\ldots\ldots, c_n) = 0$

where $c_1, c_2, \ldots\ldots. c_n$ are 'n' arbitrary constants, we have to eliminate the 'n' constants for which we require $(n+1)$ equations.

A differential equation is obtained as follows :

(a) Differentiate the given equation w.r.t the independent variable (say x) as many times as the number of independent arbitrary constants in it.

(b) Eliminate the arbitrary constants.

(c) The eliminant is the required differential equation.

Note:

(i) A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

(ii) For there being n differentiation, the resulting equation must contain a derivative of n^{th} order i.e. equal to number of independent arbitrary constant.

Illustration 3:

Find the differential equation of all parabolas whose axes is parallel to the x-axis and having latus rectum a.

Solution:

Equation of parabola whose axes is parallel to x-axis and having latus rectum 'a' is

$(y-\beta)^2 = a(x-\alpha)$

Differentiating both sides, we get $2(y-\beta)\,\frac{dy}{dx} = a$

Again differentiating, we get

$$\Rightarrow 2(y-\beta)\,\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2 = 0 \Rightarrow a\,\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^3 = 0.$$

Illustration 4:

Find the differential equation whose solution represents the family : $c(y+c)^2 = x^3$

Solution:

$c(y+c)^2 = x^3$ ...(i)

Differentiating, we get, c. $[2(y+c)]\frac{dy}{dx} = 3x^2$

Writing the value of c from (i), we have

$$\frac{2x^3}{(y+c)^2}\,(y+c)\,\frac{dy}{dx} = 3x^2 \Rightarrow \frac{2x^3}{y+c}\frac{2x^3}{y+c} = 3x^2$$

i.e. $\frac{2x}{y+c}\frac{dy}{dx}=3 \Rightarrow \frac{2x}{3}\left[\frac{dy}{dx}\right]=y+c$

Hence c = $\frac{2x}{3}\left[\frac{dy}{dx}\right]$ − y

Substituting value of c in equation (i), we get $\left[\frac{2x}{3}\left(\frac{dy}{dx}\right)-y\right]\left[\frac{2x}{3}\frac{dy}{dx}\right]^2=x^3$

which is the required differential equation.

Illustration 5:

Find the differential equation whose solution represents the family : y = a cos(θx) + b sin(θx), where θ = fixed constant

Solution:

$y = a\cos(\theta x) + b\sin(\theta x), \theta$ = fixed constant ...(i)

Differentiating, we get $\frac{dy}{dx} = -\theta a\sin(\theta x) + \theta b\cos(\theta x)$

Again differentiating, we get $\frac{d^2y}{dx^2} = -\theta^2 a\cos(\theta x) - \theta^2 b\sin(\theta x)$

using equation (i), we get $\frac{d^2y}{dx^2} = -\theta^2 y$

5. Solution of Differential Equation:

The solution of the differential equation is a relation between the variables of the equation not containing the derivatives, but satisfying the given differential equation (i.e., from which the given differential equation can be derived).

Thus, the solution of $\frac{dy}{dx}=e^x$ could be obtained by simply integrating both sides, i.e., $y = e^x + c$ and that of, $\frac{dy}{dx} = px + q$ is $y = \frac{px^2}{2} + qx + c$, where c is arbitrary constant.

(i) A general solution or an integral of a differential equation is a relation between the variables (not involving the derivatives) which contains the same number of the arbitrary constants as the order of the differential equation.

For example, a general solution of the differential equation $\frac{d^2x}{dt^2} = -4x$ is $x = A\cos 2t + B\sin 2t$ where A and B are the arbitrary constants.

(ii) Particular solution or particular integral is that solution of the differential equation which is obtained from the general solution by assigning particular values to the arbitrary constant in the general solution.

For example, $x = 10\cos 2t + 5\sin 2t$ is a particular solution of differential equation $\frac{d^2x}{dt^2}=-4x$.

Note:

(i) The general solution of a differential equation can be expressed in different (but equivalent) forms. For example

$\log x - \log(y + 2) = k$ ...(i)

where k is an arbitrary constant is the general solution of the differential equation $xy' = y + 2$.

The solution given by equation (i) can also be re-written as

$\log\left(\frac{x}{y+2}\right) = k$ or $\frac{x}{y+2} = e^k = c_1$ ...(ii)

Or $x = c_1 \quad (y + 2)$ ...(iii)

where $c_1 = e^k$ is another arbitrary constant. The solution (iii) can also be written as

$y + 2 = c_2 x$

where $c_2 = 1/c_1$ is another arbitrary constant.

(ii) All differential equations that we come across have unique solutions or a family of solutions.

For example, the differential equation $\left|\frac{dy}{dx}\right| + |y| = 0$ has only the trivial solution, i.e. $y = 0$.

The differential equation $\left|\frac{dy}{dx}\right| + |y| + c = 0,\ c > 0$ has no solution.

6. Elementary Types of First Order & First Degree Differential Equations:

(a) Separation of Variables:

Some differential equations can be solved by the method of separation of variables (or "variable separable").

This method is only possible, if we can express the differential equation in the form

$A(x)dx + B(y)\,dy = 0$

where $A(x)$ is a function of 'x' only and $B(y)$ is a function of 'y' only.

A general solution of this is given by,

$\int A(x)dx + \int B(y)dy = c$

where 'c' is the arbitrary constant.

Illustration 6:

Solve the differential equation $xy\frac{dy}{dx} = \frac{1+y^2}{1+x^2}(1 + x + x^2)$.

Solution:

Differential equation can be rewritten as

$$xy\frac{dy}{dx} = (1 + y^2)\left(1 + \frac{x}{1+x^2}\right) \Rightarrow \frac{y}{1+y^2}dy = \left(\frac{1}{x} + \frac{1}{1+x^2}\right)dx$$

Integrating, we get

$$\frac{1}{2}\ell n(1 + y^2) = \ell n\, x + \tan^{-1}x + \ell n\, c \Rightarrow \sqrt{1+y^2} = cxe^{\tan^{-1}x}$$

Illustration 7:

Solve the differential equation $(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$.

Solution:

The given equation $(x^3 - y^2x^3)\ \dfrac{dy}{dx} + y^3 + x^2y^3 = 0$

Case-I : $y = 0$ is one solution of differential equation

Case-II : If $y \neq 0$, then

$$\Rightarrow \frac{1-y^2}{y^3}dy + \frac{1+x^2}{x^3}dx = 0 \Rightarrow \int\left(\frac{1}{y^3} - \frac{1}{y}\right)dy + \int\left(\frac{1}{x^3} + \frac{1}{x}\right)dx = 0$$

$$\Rightarrow \log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2} + \frac{1}{x^2}\right) + c$$

$y = 0$ or $\log\left(\dfrac{x}{y}\right) = \dfrac{1}{2}\left(\dfrac{1}{y^2} + \dfrac{1}{x^2}\right) + c$ **Ans.**

Illustration 8:

Solve: $\dfrac{dy}{dx} = (x-3)(y+1)^{\frac{2}{3}}$

Solution:

Case-I : $y = -1$ is one solution of differential equation

Case-II : If $y \neq -1$, then

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3}$$

$$\int \frac{dy}{(y+1)^{2/3}} = \int (x-3)dx$$

Integrate and solve for y: $3(y+1)^{\frac{1}{3}} = \dfrac{1}{2}(x-3)^2 + C$

$$(y+1)^{\frac{1}{3}} = \frac{1}{6}(x-3)^2 + C_0 \Rightarrow y+1 = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3$$

$y = -1$ or $y = \left(\dfrac{1}{6}(x-3)^2 + C_0\right)^3 - 1$

Illustration 9:

Solve $\dfrac{dy}{dx} = \cos(x+y) - \sin(x+y)$.

Solution:

$$\frac{dy}{dx} = \cos(x+y) - \sin(x+y)$$

Substituting, $x + y = t$, we get $\dfrac{dy}{dx} = \dfrac{dt}{dx} - 1$

Therefore $\dfrac{dt}{dx} - 1 = \cos t - \sin t$

$$\Rightarrow \int \frac{dt}{1+\cos t - \sin t} = \int dx \Rightarrow \int \frac{\sec^2\frac{t}{2}dt}{2\left(1-\tan\frac{t}{2}\right)} = \int dx \Rightarrow -\ell n\left|1-\tan\frac{x+y}{2}\right| = x + c.$$

Illustration 10:

Solve : $y' = (x + y + 1)^2$

Solution:

$y' = (x + y + 1)^2$ ...(i)

Let $t = x + y + 1$

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}$$

Substituting in equation (i) we get

$$\frac{dt}{dx} = t^2 + 1 \Rightarrow \int \frac{dt}{1+t^2} = \int dx \Rightarrow \tan^{-1} t = x + C \Rightarrow t = \tan(x + C)$$

$x + y + 1 = \tan(x + C) \Rightarrow y = \tan(x + C) - x - 1$

(ii) Equation of the form :

$$\Rightarrow \frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

Case I : If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ **then**

Let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$ then $a_1 = \lambda a_2$ ...(i)

$b_1 = \lambda b_2$ ...(ii)

from (i) and (ii), differential equation becomes

$$\frac{dy}{dx} = \frac{\lambda a_2x + \lambda b_2y + c_1}{a_2x + b_2y + c_2} \Rightarrow \frac{dy}{dx} = \frac{\lambda(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2}$$

or we can say, $\frac{dy}{dx} = f(a_2x + b_2y)$

which can be solved by substituting $t = a_2x + b_2y$

Illustration 11:

Solve : $(x + y)dx + (3x + 3y - 4)dy = 0$

Solution:

Let $t = x + y$

$\Rightarrow dy = dt - dx$

So we get, $tdx + (3t - 4)(dt - dx) = 0$

$$2dx + \left(\frac{3t-4}{2-t}\right)dt = 0 \Rightarrow 2dx - 3dt\frac{2}{2-t} + dt = 0$$

Integrating and replacing t by $x + y$, we get

$2x - 3t - 2[\ell n|(2 - t)|] = c_1$

$\Rightarrow 2x - 3(x + y) - 2[\ell n|(2 - x - y)|] = c_1$

$\Rightarrow x + 3y + 2\ell n|(2 - x - y)| = c$ **Ans.**

Case II :

If $a_2 + b_1 = 0$, then a simple cross multiplication and substituting $d(xy)$ for $xdy + ydx$ and integrating term by term, yield the results easily.

Illustration 12:

Solve $\frac{dy}{dx}=\frac{x-2y+1}{2x+2y+3}$

Solution:

$\frac{dy}{dx}=\frac{x-2y+1}{2x+2y+3}$

$\Rightarrow 2xdy + 2y\,dy + 3dy = xdx - 2y\,dx + dx$

$\Rightarrow (2y + 3)dy = (x + 1)dx - 2(xdy + ydx)$

On integrating, we get

$\Rightarrow \int(2y+3)dy=\int(x+1)dx-\int 2d(xy)$

Solving : $y^2+3y=\frac{x^2}{2}+x-2xy+c$

(iii) Equation of the form :

$\Rightarrow \boldsymbol{yf(xy)dx + xg(xy)dy = 0}$ ...(i)

The substitution $xy = z$, reduces differential equation of this form to the form in which the variables are separable.

Let $xy = z$ ...(ii)

$dy = \left[\frac{xdz-zdx}{x^2}\right]$ ...(iii)

using equation (ii) & (iii), equation (i) becomes

$\Rightarrow \frac{z}{x}f(z)dx+xg(z)\left[\frac{xdz-zdx}{x^2}\right]=0 \Rightarrow \frac{z}{x}\{f(z)-g(z)\}dx+g(z)dz=0$

$\Rightarrow \frac{1}{x}dx+\frac{g(z)dz}{z\{f(z)-g(z)\}}=0$

Illustration 13:

Solve $y(xy + 1)dx + x(1 + xy + x^2y^2)dy = 0$

Solution:

Let $xy = v$

$y=\frac{v}{x} \Rightarrow dy=\frac{xdv-vdx}{x^2}$

Now, differential equation becomes $\Rightarrow \frac{v}{x}(v+1)dx+x(1+v+v^2)\left(\frac{xdv-vdx}{x^2}\right)=0$

On solving, we get

$v^3dx - x(1 + v + v^2)dv = 0$

separating the variables & integrating we get

$\Rightarrow \int\frac{dx}{x}-\int\left(\frac{1}{v^3}+\frac{1}{v^2}+\frac{1}{v}\right)dv=0 \Rightarrow \ell nx+\frac{1}{2v^2}+\frac{1}{v}-\ell nv=c$

$\Rightarrow 2v^2\ell n\left(\frac{v}{x}\right)-2v-1=-2cv^2 \Rightarrow 2x^2y^2\ell ny - 2xy - 1 = Kx^2y^2$ where $K = -2c$

(iv) Transformation to polar-co-ordinates :

Sometimes conversion of cartesian co-ordinates into polar coordinates helps us in separating the variables.

(1) $x = r\cos\theta,\ \ y = r\sin\theta$

then $x^2 + y^2 = r^2$

$xdx + ydy = rdr$

$xdy - ydx = r^2 d\theta$

(2) $x = r\sec\theta,\ y = r\tan\theta$

then $x^2 - y^2 = r^2$

$xdx - ydy = rdr$

$xdy - ydx = r^2 \sec\theta\, d\theta$

Illustration 14:

Solve : $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}$

Solution:

The given equation can be reduced to

$$\frac{xdx + ydy}{ydx - xdy} = \frac{(x^2+y^2)^2}{x^2}$$

Substituting $x = r\cos\theta$

$y = r\sin\theta$

we get, $\dfrac{rdr}{r^2 d\theta} = \dfrac{-(r^2)^2}{r^2\cos^2\theta} \Rightarrow \displaystyle\int \frac{dr}{r^3} = -\int \sec^2\theta d\theta \Rightarrow -\frac{1}{2r^2} = -\tan\theta + c$

Substituting, $\dfrac{1}{2(x^2+y^2)} = \dfrac{y}{x} + K$

(b) Homogeneous equations :

A function $f(x, y)$ is said to be a homogeneous function of degree n, if the substitution

$x = \lambda x, y = \lambda y, \lambda > 0$ produces the equality

$f(\lambda x, \lambda y) = \lambda^n f(x, y)$

The degree of homogeneity 'n' can be any real number.

Illustration 15:

Find the degree of homogeneity of function

(i) $f(x, y) = x^2 + y^2$ (ii) $f(x, y) = (x^{\frac{3}{2}} + y^{\frac{3}{2}})/(x + y)$ (iii) $f(x, y) = \sin\left(\dfrac{x}{y}\right)$

Solution:

(i) $f(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2(x^2 + y^2) = \lambda^2 f(x, y)$

degree of homogeneity $\to 2$

(ii) $f(\lambda x, \lambda y) = \dfrac{\lambda^{3/2}x^{3/2} + \lambda^{3/2}y^{3/2}}{\lambda x + \lambda y}$

$f(\lambda x, \lambda y) = \lambda^{\frac{1}{2}} f(x, y)$

degree of homogeneity $\to 1/2$

(iii) $f(\lambda x, \lambda y) = \sin\left(\frac{\lambda x}{\lambda y}\right) = \lambda^{\circ}\sin\left(\frac{x}{y}\right) = \lambda^{\circ} f(x,y)$

degree of homogeneity $\rightarrow 0$

Illustration 16:

Determine whether or not each of the following functions is homogeneous.

(i) $f(x,y) = x^2 - xy$ (ii) $f(x,y) = \frac{xy}{x+y^2}$ (iii) $f(x,y) = \sin xy$

Solution:

(i) $f(\lambda x, \lambda y) = \lambda^2 x^2 - \lambda^2 xy = \lambda^2(x^2 - xy) = \lambda^2 f(x,y)$ homogeneous.

(ii) $f(\lambda x, \lambda y) = \frac{\lambda^2 xy}{\lambda x + \lambda^2 y^2} \neq \lambda^n f(x,y)$ not homogeneous.

(iii) $f(\lambda x, \lambda y) = \sin(\lambda^2 xy) \neq \lambda^n f(x,y)$ not homogeneous.

(i) Homogeneous First Order Differential Equation

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

where $f(x,y)$ and $g(x,y)$ are homogeneous functions of x, y and of the same degree, is said to be homogeneous. Such equations can be solved by substituting

$y = vx,$

so that the dependent variable y is changed to another variable v.

Since $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree say, n, they can be written as

$$f(x,y) = x^n f_1\left(\frac{y}{x}\right) \text{ and } g(x,y) = x^n g_1\left(\frac{y}{x}\right).$$

As $y = vx$, we have $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

The given differential equation, therefore, becomes

$$v + x\frac{dv}{dx} = \frac{f_1(v)}{g_1(v)} \Rightarrow \frac{g_1(v)dv}{f_1(v) - vg_1(v)} = \frac{dx}{x},$$

so that the variables v and x are now separable.

Note : Sometimes homogeneous equation can be solved by substituting $x = vy$ or by using polar coordinate substitution.

Illustration 17:

The solution of the differential equation $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$ is -

(A) $\sin^2 y = x\sin y + \frac{x^2}{2} + c$

(B) $\sin^2 y = x\sin y - \frac{x^2}{2} + c$

(C) $\sin^2 y = x + \sin y + \frac{x^2}{2} + c$

(D) $\sin^2 y = x - \sin y + \frac{x^2}{2} + c$

Ans. (A)

Solution:

Here, $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$

$\Rightarrow \cos y \frac{dy}{dx} = \frac{\sin y + x}{2\sin y - x}$, (put $\sin y = t$)

$\Rightarrow \frac{dt}{dx} = \frac{t+x}{2t-x}$ (put $t = vx$)

$\frac{xdv}{dx} + v = \frac{vx+x}{2vx-x} = \frac{v+1}{2v-1}$

$\therefore x\frac{dv}{dx} = \frac{v+1}{2v-1} - v = \frac{v+1-2v^2+v}{2v-1}$

or $\frac{2v-1}{-2v^2+2v+1}dv = \frac{dx}{x}$ on solving, we get

$\sin^2 y = x\sin y + \frac{x^2}{2} + c.$

Illustration 18:

Solve the differential equation $\left(1 + 2e^{\frac{x}{y}}\right)dx + 2e^{\frac{x}{y}}(1 - x/y)dy = 0.$

Solution:

The equation is homogeneous of degree 0.

Put $x = vy, dx = v\,dy + y\,dv,$

Then, differential equation becomes

$(1 + 2e^v)(vdy + ydv) + 2e^v(1 - v)dy = 0 \Rightarrow (v + 2e^v)dy + y(1 + 2e^v)dv = 0$

$\frac{dy}{y} + \frac{1+2e^v}{v+2e^v}dv = 0$

Integrating and replacing v by x/y, we get $\ell n\, y + \ell n\,(v + 2e^v) = \ell n c$ and $x + 2\,ye^{\frac{x}{y}} = c$ **Ans.**

(ii) Equations Reducible to Homogeneous Form

The equation of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

can be reduced to homogeneous form by changing the variable x, y to u, v as

$$x = u + h, y = v + k$$

where h, k are the constants to be chosen so as to make the given equation homgeneous.

We have

$$\frac{dy}{dx} = \frac{dv}{du}$$

$\therefore$ The equation becomes, $\frac{dv}{du} = \frac{a_1u+b_1v+(a_1h+b_1k+c_1)}{a_2u+b_2v+(a_2h+b_2k+c_2)}$

Let h and k be chosen so as to satisfy the equation

$a_1h + b_1k + c_1 = 0$ ...(i)

$a_2h + b_2k + c_2 = 0$ ...(ii)

Solve for h and k from (i) and (ii)

Now $\frac{du}{dv} = \frac{a_1u + b_1v}{a_2u + b_2v}$

is a homogeneous equation and can be solved by substituting $v = ut$.

Illustration 19:

Solve $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

Solution:

Put $x = X + h, y = Y + k$

We have $\frac{dY}{dX} = \frac{X+2Y+(h+2k+3)}{2X+3Y+(2h+3k+4)}$

To determine h and k, we write

$h + 2k + 3 = 0, 2h + 3k + 4 = 0 \Rightarrow h = 1, k = -2$

So that $\frac{dY}{dX} = \frac{X+2Y}{2X+3Y}$

Putting $Y = VX$, we get

$V + X\frac{dV}{dX} = \frac{1+2V}{2+3V} \Rightarrow \frac{2+3V}{3V^2-1}\, dV = -\frac{dX}{X}$

$\Rightarrow \left[\frac{2+\sqrt{3}}{2(\sqrt{3}V-1)} - \frac{2-\sqrt{3}}{2(\sqrt{3}V+1)}\right] dV = -\frac{dX}{X}$

$\Rightarrow \frac{2+\sqrt{3}}{2\sqrt{3}} \log(\sqrt{3}\,V - 1) - \frac{2-\sqrt{3}}{2\sqrt{3}} \log(\sqrt{3}\,V + 1) = (-\log X + c)$

$\frac{2+\sqrt{3}}{2\sqrt{3}} \log(\sqrt{3}\,Y - X) - \frac{2-\sqrt{3}}{2\sqrt{3}} \log(\sqrt{3}\,Y + X) = A$ where A is another constant and $X = x - 1, Y = y + 2$.

(c) Linear Differential Equations:

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together.

The n^{th} order linear differential equation is of the form ;

$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_n(x)\, y = \phi(x)$, where $a_0(x), a_1(x) \ldots . a_n(x)$ are called the coefficients of the differential equation.

Note that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear. e.g. the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is not linear, though its degree is 1.

Illustration 20:

Which of the following equation is linear?

(A) $\frac{dy}{dx}+xy^2=1$ (B) $x^2\frac{dy}{dx}+y=e^x$ (C) $\frac{dy}{dx}+3y=xy^2$ (D) $x\frac{dy}{dx}+y^2=\sin x$

Ans. (B)

Solution:

Clearly answer is (B)

Illustration 21:

Which of the following equation is non-linear ?

(A) $\frac{dy}{dx}=\cos x$ (B) $\frac{d^2y}{dx^2}+y=0$ (C) $dx+dy=0$ (D) $x\left(\frac{dy}{dx}\right)+\frac{3}{\left(\frac{dy}{dx}\right)}=y^2$

Ans. (D)

Solution:

Clearly answer is (D)

(i) Linear differential equations of first order:

The most general form of a linear differential equation of first order is $\frac{dy}{dx}+Py=Q$, where P & Q are functions of x.

To solve such an equation multiply both sides by $e^{\int Pdx}$.

So that we get $e^{\int Pdx}\left[\frac{dy}{dx}+Py\right]=Qe^{\int Pdx}$ ...(i)

$$\frac{d}{dx}\left(e^{\int Pdx}.y\right)=Qe^{\int Pdx} \quad \text{...(ii)}$$

On integrating equation (ii), we get

$$ye^{\int Pdx}=\int Qe^{\int Pdx}dx+c$$

This is the required general solution.

Note :

(i) The factor $e^{\int Pdx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y, is called integrating factor of the differential equation popularly abbreviated as I.F.

(ii) Sometimes a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation; $(x+y+1)\frac{dy}{dx}=y^2+3$ can be written as $(y^2+3)\frac{dx}{dy}=x+y+1$ which is a linear differential equation.

Illustration 22:

Solve $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$.

Solution:

Differential equation can be rewritten as $(1+y^2)\dfrac{dx}{dy}+x=e^{\tan^{-1}y}$

or $\dfrac{dx}{dy}+\dfrac{1}{1+y^2}.x=\dfrac{e^{\tan^{-1}y}}{1+y^2}$ ...(i)

I. F = $e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$

so solution is $xe^{\tan^{-1}y}=\int\dfrac{e^{\tan^{-1}y}e^{\tan^{-1}y}}{1+y^2}dy$

Let $e^{\tan^{-1}y}=t\Rightarrow\dfrac{e^{\tan^{-1}y}}{1+y^2}\,dy=dt$

$xe^{\tan^{-1}y}=\int t\,dt$ [Putting = t]

or $x\,e^{\tan^{-1}y}=\dfrac{t^2}{2}+\dfrac{c}{2}\Rightarrow 2x\,e^{\tan^{-1}y}=e^{2\tan^{-1}y}+c.$

Illustration 23:

The solution of differential equation $(x^2-1)\dfrac{dy}{dx}+2xy=\dfrac{1}{x^2-1}$ is -

(A) $y(x^2-1)=\dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right|+C$

(B) $y(x^2+1)=\dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right|-C$

(C) $y(x^2-1)=\dfrac{5}{2}\log\left|\dfrac{x-1}{x+1}\right|+C$

(D) none of these

Ans.(A)

Solution:

The given differential equation is

$(x^2-1)\dfrac{dy}{dx}+2\ xy=\dfrac{1}{x^2-1}\ \Rightarrow\dfrac{dy}{dx}+\dfrac{2x}{x^2-1}y=\dfrac{1}{(x^2-1)^2}$ ...(i)

This is linear differential equation of the form

$\dfrac{dy}{dx}+Py=Q,$ where $P=\dfrac{2x}{x^2-1}$ and $Q=\dfrac{1}{(x^2-1)^2}$

$\therefore$ I.F. = $e^{\int P\,dx}=e^{\int 2x/(x^2-1)dx}=e^{\log(x^2-1)}=(x^2-1)$

multiplying both sides of (i) by I.F. = (x^2-1), we get

$(x^2-1)\dfrac{dy}{dx}+2\ xy=\dfrac{1}{x^2-1}$

integrating both sides we get

$y(x^2-1)=\int\dfrac{1}{x^2-1}dx+C$ [Using : y (I.F.) = $\int Q.(I.F.)\ dx+C$]

$\Rightarrow y(x^2-1)=\dfrac{1}{2}\log\left|\dfrac{x-1}{x+1}\right|+C$.

This is the required solution.

(ii) Equation reducible to linear form :

The equation of the form $\frac{dy}{dx}+Py=Qy^n$, where P and Q are functions of x,

is called **Bernoulli's equation.**

On dividing by y^n, we get $y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q$

Let $y^{-n+1}=t$, so that $(-n+1)y^{-n}\frac{dy}{dx}=\frac{dt}{dx}$

then equation becomes $\frac{dt}{dx}+P(1-n)t=Q(1-n)$

which is linear with t as a dependent variable.

Illustration 24:

Solve the differential equation $x\frac{dy}{dx}+y=x^3y^6$.

Solution:

The given differential equation can be written as

$$\frac{1}{y^6}\frac{dy}{dx}+\frac{1}{xy^5}=x^2$$

Putting $y^{-5}=v$ so that

$-5\,y^{-6}\frac{dy}{dx}=\frac{dv}{dx}$ *or* $y^{-6}\frac{dy}{dx}=-\frac{1}{5}\frac{dv}{dx}$ we get

$$-\frac{1}{5}\frac{dv}{dx}+\frac{1}{x}v=x^2\Rightarrow\frac{dv}{dx}-\frac{5}{x}v=-5x^2 \qquad ......(i)$$

This is the standard form of the linear deferential equation having integrating factor

$$\text{I.F} = e^{\int-\frac{5}{x}dx}=e^{-5\,\log x}=\frac{1}{x^5}$$

Multiplying both sides of (i) by I.F. and integrating w.r.t. x

We get $v.\frac{1}{x^5}=\int-5x^2.\frac{1}{x^5}dx$

$$\Rightarrow\frac{v}{x^5}=\frac{5}{2}x^{-2}+c$$

$\Rightarrow y^{-5}x^{-5}=\frac{5}{2}x^{-2}+c$ which is the required solution.

Illustration 25:

Find the solution of differential equation $\frac{dy}{dx}-y\tan x=-y^2\sec x$.

Solution:

$$\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}\tan x=-\sec x$$

$\frac{1}{y}=v\,;\;\frac{-1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$ $\qquad\qquad$ $\therefore\;\frac{-dv}{dx}-v\tan x=-\sec x$

$\frac{dv}{dx} + v\tan x = \sec x$, Here $P = \tan x, Q = \sec x$

I.F. = $e^{\int \tan x dx} = |\sec x|$ $v|\sec x| = \int \sec^2 x\, dx + c$

Hence the solution is $y^{-1}|\sec x| = \tan x + c$

7. Trajectories:

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

The trajectory will be called **Orthogonal** if each trajectory cuts every member of given family at right angle.

Working rule for finding orthogonal trajectory

1. Form the differential equation of family of curves
2. Write $-\frac{1}{dy/dx}$ for $\frac{dy}{dx}$ or $-\frac{r^2 d\theta}{dr}$ for $\frac{dr}{d\theta}$ if differential equation is in the polar form.
3. Solve the new differential equation to get the equation of orthogonal trajectories.

Note : A family of curves is self-orthogonal if it is its own orthogonal family.

Illustration 26:

Find the value of k such that the family of parabolas $y = cx^2 + k$ is the orthogonal trajectory of the family of ellipses $x^2 + 2y^2 - y = c$.

Solution:

Differentiate both sides of $x^2 + 2y^2 - y = c$ w.r.t. x, We get

$2x + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0$

or $2x + (4y - 1)\frac{dy}{dx} = 0$, is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to obtain the differential equation of the orthogonal trajectories, we get

$2x + \frac{(1-4y)}{\frac{dy}{dx}} = 0 \Rightarrow \frac{dy}{dx} = \frac{4y-1}{2x}$

$\Rightarrow \int \frac{dy}{4y-1} = \int \frac{dx}{2x} \Rightarrow \frac{1}{4}\ell n(4y-1) = \frac{1}{2}\ell n\, x + \frac{1}{2}\ell n\, a$, where a is any constant.

$\Rightarrow \ell n(4y - 1) = 2\,\ell n\, x + 2\,\ell n\, a$ or , $4y - 1 = a^2x^2$

or, $y = \frac{1}{4}a^2x^2 + \frac{1}{4}$, is the required orthogonal trajectory, which is of the form $y = cx^2 + k$ where $c = \frac{a^2}{4}$, $k = \frac{1}{4}$

Illustration 27:

Prove that $\frac{x^2}{1+\lambda}+\frac{y^2}{4+\lambda}=1$ are self orthogonal family of curves.

Solution:

$$\frac{x^2}{1+\lambda}+\frac{y^2}{4+\lambda}=1 \qquad \text{...(i)}$$

Differentiating (i) with respect to x, we have

$$\frac{x}{1+\lambda}+\frac{y}{4+\lambda}\frac{dy}{dx}=0 \qquad \text{...(ii)}$$

From (i) and (ii), we have to eliminate λ.

Now, (ii) gives

$$\lambda=\frac{-\left[4x+y\frac{dy}{dx}\right]}{x+y\frac{dy}{dx}}$$

$$\Rightarrow 1+\lambda=\frac{(1-4)x}{x+y(dy/dx)},\quad 4+\lambda=\frac{-(1-4)y(dy/dx)}{x+y(dy/dx)}$$

Substituting these values in (i), we get

$$\left(x+y\frac{dy}{dx}\right)\left(x-y\frac{dx}{dy}\right)=-3 \qquad \text{...(iii)}$$

as the differential equation of the given family.

Changing dy/dx to $-dx/dy$ in (iii), we obtain

$$\left(x-y\frac{dx}{dy}\right)\left(x+y\frac{dy}{dx}\right)=-3 \qquad \text{...(iv)}$$

which is the same as (iii). Thus we see that the family (i) as self-orthogonal, i.e., every member of the family (i) cuts every other member of the same family orthogonally.

Note :

Following exact differentials must be remembered :

(i) $xdy + y\,dx = d(xy)$ (ii) $\frac{xdy-ydx}{x^2}=d\left(\frac{y}{x}\right)$ (iii) $\frac{ydx-xdy}{y^2}=d\left(\frac{x}{y}\right)$

(iv) $\frac{xdy+ydx}{xy}=d\left(\ell nxy\right)$ (v) $\frac{dx+dy}{x+y}=d\left(\ell n(x+y)\right)$ (vi) $\frac{xdy-ydx}{xy}=d\left(\ell n\frac{y}{x}\right)$

(vii) $\frac{ydx-xdy}{xy}=d\left(\ell n\frac{x}{y}\right)$ (viii) $\frac{xdy-ydx}{x^2+y^2}=d\left(\tan^{-1}\frac{y}{x}\right)$ (ix) $\frac{ydx-xdy}{x^2+y^2}=d\left(\tan^{-1}\frac{x}{y}\right)$

(x) $\frac{xdx+ydy}{x^2+y^2}=d\left[\ell n\sqrt{x^2+y^2}\right]$ (xi) $d\left(-\frac{1}{xy}\right)=\frac{xdy+ydx}{x^2y^2}$ (xii) $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$

(xiii) $d\left(\frac{e^y}{x}\right)=\frac{xe^ydy-e^ydx}{x^2}$

Illustration 28:

Solve $\dfrac{y+\sin x\cos^2(xy)}{\cos^2(xy)}dx + \left(\dfrac{x}{\cos^2(xy)}+\sin y\right)dy = 0.$

Solution:

The given differential equation can be written as;

$\dfrac{y\,dx+x\,dy}{\cos^2(xy)} + \sin x\,dx + \sin y\,dy = 0.$

$\Rightarrow \sec^2(xy)d(xy) + \sin x\,dx + \sin y\,dy = 0$

$\Rightarrow d(\tan(xy)) + d(-\cos x) + d(-\cos y) = 0$

$\Rightarrow \tan(xy) - \cos x - \cos y = c.$

(c) Geometrical applications:

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then slope of the tangent at point P is $\left(\dfrac{dy}{dx}\right)_{(x_1,\,y_1)}$

(i) The equation of the tangent at P is $y-y_1=\dfrac{dy}{dx}(x-x_1)$

x-intercept of the tangent = $x_1 - y_1\left(\dfrac{dx}{dy}\right)$

y-intercept of the tangent = $y_1 - x_1\left(\dfrac{dy}{dx}\right)$

(ii) The equation of normal at P is $y-y_1=-\dfrac{1}{(dy/dx)}(x-x_1)$

x and y-intercepts of normal are ; $x_1+y_1\dfrac{dy}{dx}$ and $y_1+x_1\dfrac{dx}{dy}$

(iii) Length of tangent = $PT=|y_1|\sqrt{1+(dx/dy)^2_{(x_1,\,y_1)}}$

(iv) Length of normal = $PN=|y_1|\sqrt{1+(dy/dx)^2_{(x_1,\,y_1)}}$

(v) Length of sub-tangent = $ST=\left|y_1\left(\dfrac{dx}{dy}\right)_{(x_1,\,y_1)}\right|$

(vi) Length of sub-normal = $SN=\left|y_1\left(\dfrac{dy}{dx}\right)_{(x_1,\,y_1)}\right|$

(vii) Length of radius vector = $\sqrt{x_1^2+y_1^2}$

Illustration 29:

Solve $(y\log x-1)\,ydx = xdy$.

Solution:

The given differential equation can be written as

$$x\frac{dy}{dx}+y=y^2\log x \qquad \text{...(i)}$$

Divide by xy^2. Hence $\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=\frac{1}{x}\log x$

Let $\frac{1}{y}=v \Rightarrow -\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$ so that $\frac{dv}{dx}-\frac{1}{x}v=-\frac{1}{x}\log x$ ...(ii)

(ii) is the standard linear differential equation with $P=-\frac{1}{x}$, $Q=-\frac{1}{x}\log x$

I.F. = $e^{\int pdx}=e^{\int -1/x\,dx}=1/x$

The solution is given by

$$v.\frac{1}{x}=\int\frac{1}{x}\left(-\frac{1}{x}\log x\right)dx=-\int\frac{\log x}{x^2}dx=\frac{\log x}{x}-\int\frac{1}{x}.\frac{1}{x}dx=\frac{\log x}{x}+\frac{1}{x}+C$$

$\Rightarrow v=1+\log x+cx=\log ex+cx$

Or $\frac{1}{y}=\log ex+cx$ or $y(\log ex+cx)=1$.

Illustration 30:

For a certain curve $y=f(x)$ satisfying $\frac{d^2y}{dx^2}=6x-4$, $f(x)$ has a local minimum value 5 when $x=1$. Find the equation of the curve and also the global maximum and global minimum values of $f(x)$ given that $0\le x\le 2$.

Solution:

Integrating $\frac{d^2y}{dx^2}=6x-4$, we get $\frac{dy}{dx}=3x^2-4x+A$

When $x=1$, $\frac{dy}{dx}=0$, so that $A=1$. Hence

$$\frac{dy}{dx}=3x^2-4x+1 \qquad \text{...(i)}$$

Integrating, we get $y=x^3-2x^2+x+B$

When $x=1, y=5$, so that $B=5$.

Thus we have $y=x^3-2x^2+x+5$.

From (i), we get the critical points $x=1/3, x=1$

At the critical point $x=\frac{1}{3}$, $\frac{d^2y}{dx^2}$ is negative.

Therefore at $x=1/3$, y has a local maximum.

At $x=1$, $\frac{d^2y}{dx^2}$ is positive.

Therefore at $x = 1$, y has a local minimum.

Also $f(1) = 5, f\left(\frac{1}{3}\right)=\frac{139}{27}$. $f(0) = 5, f(2) = 7$

Hence the global maximum value = 7, and the global minimum value = 5.

Illustration 31:

Solve $\frac{dy}{dx}$ = tan$y$ cot$x$ – sec$y$ cos$x$.

Solution:

$\frac{dy}{dx}$ = tan$y$ cot$x$ – sec$y$ cos$x$.

Rearrange it :

$(\sin x - \sin y)\cos x\, dx + \sin x \cos y\, dy = 0$.

Put $u = \sin y$, So, $du = \cos y\, dy$:

Substituting, we get

$(\sin x - u)\cos x\, dx + \sin x\, du = 0$, $\frac{du}{dx}-u\frac{\cos x}{\sin x}=-\cos x$

The equation is first-order linear in u.

The integrating factor is

$$I=\exp\int -\frac{\cos x}{\sin x}dx=\exp\{-\ln(\sin x)\}=\frac{1}{\sin x}.$$

Hence, $u\frac{1}{\sin x}=-\int\frac{\cos x}{\sin x}dx=-\ln|\sin x|+C$.

Solve for u : $u = -\sin x\, \ell n\, |\sin x| + C \sin x$.

Put y back : $\sin y = -\sin x\, \ell n\, |\sin x| + C \sin x$.

Illustration 32:

Solve the equation $x\int_0^x y(t)dt=(x+1)\int_0^x t\,y(t)dt, x>0$

Solution:

Differentiating the equation with respect to x, we get

$$xy(x)+1.\int_0^x y(t)dt=(x+1)xy(x)+1.\int_0^x ty(t)dt$$

i.e., $\int_0^x y(t)dt=x^2y(x)+\int_0^x ty(t)dt$

Differentiating again with respect to x, we get $y(x) = x^2y'(x) + 2xy(x) + xy(x)$

i.e., $(1-3x)y(x) = \frac{x^2dy(x)}{dx}$

i.e., $\frac{(1-3x)dx}{x^2}=\frac{dy(x)}{y(x)}$, integrating we get

i.e., $y=\frac{c}{x^3}e^{-1/x}$