02 Definite Integration

A definite integral is denoted by $\int_a^b f(x)dx$ which represents the algebraic area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis.

Geometrical Interpretation of Definite Integral:

$\int_a^b f(x)dx$ represents algebraic sum of the areas of the figure bounded by curve $y = f(x)$, the x-axis and lines $x = a$ and $x = b$. The areas above the x –axis enter into this sum with plus sign, while those below the x-axis enter it with a minus sign. i.e. If $f(x) > 0\ \forall\ x \in [a, b]$, then $\int_a^b f(x)dx$ is always > 0 (when $a < b$)

& if $f(x) < 0\ \forall\ x \in [a, b]$, then $\int_a^b f(x)\ dx$ is always < 0 (when $a < b$).

Let $\int_a^{c_1} f(x)dx = A_1 > 0$

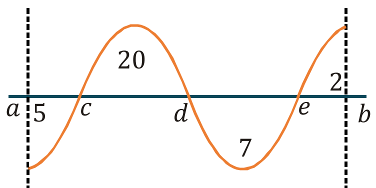
$\int_{c_1}^{c_2} f(x)dx = A_2 < 0$

$\int_{c_2}^{c_3} f(x)dx = A_3 < 0$ & $\int_{c_3}^{b} f(x)dx = A_4 > 0$ then $\int_a^b f(x)dx = A_1 + A_2 + A_3 + A_4$

Note :

If $\int_a^b f(x)dx = 0$ then the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b) while the converse is not true.

Illustration 1:



then find :-

(i) $\int_a^b f(x)dx$ (ii) $\int_b^a f(x)dx$ (iii) $\int_e^c f(x)dx$ (iv) $\int_a^d f(x)dx + \int_b^e f(x)dx$

Solution:

A (Sign convention) :

(a) Left to right : above x-axis $\to +ve$

Left to right : below x-axis $\to -ve$

(b) Right to left : above x-axis $\to -ve$

Right to left : below x-axis $\to +ve$

(i) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^e f(x)dx + \int_e^b f(x)dx$

$= -5 + 20 - 7 + 2$

$= -12 + 2 + 20 = -10 + 20 = 10$

(ii) $\int_b^a f(x)dx = (-2) + (7) + (-20) + 5 = -10$

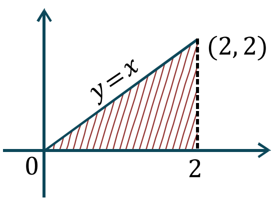
(iii) $\int_e^c f(x)dx = 7 + (-20) = -13$

(iv) $\int_a^d f(x)dx + \int_b^e f(x)dx = (-5+20) + (-2) = 13$

Illustration 2:

$\int_0^2 xdx =$

Solution:



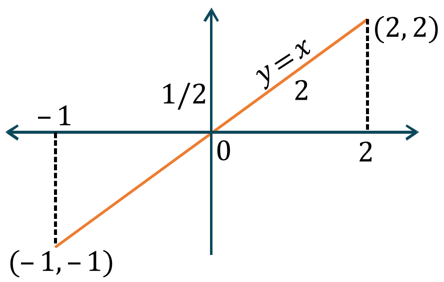
Area = $\frac{1}{2} \times 2 \times 2 = 2$

Illustration 3:

$\int_{-1}^2 xdx =$

Solution:

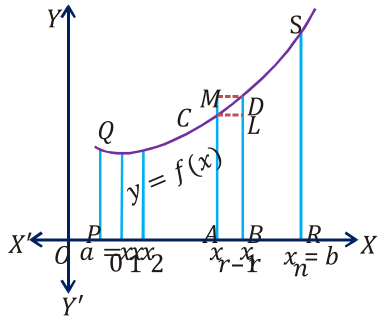
Sum of algebraic area $= \frac{-1}{2} + 2 = \frac{3}{2}$



on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Definite Integral as The Limit of Sum :



If $f(x) > 0$ for all $x \in [a, b]$ then the definite integral $\int_a^b f(x)dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the ordinates $x = a, x = b$ and the x −axis.
To evaluate area of the region $PRSQP$ between this curve, x −axis and the ordinates $x = a$ and $x = b$, divide the interval $[a, b]$ into n equal subintervals denoted by
$[x_0, x_1], [x_1, x_2], \ldots\ldots, [x_{r-1}, x_r] \ldots\ldots\ldots [x_{n-1}, x_n]$, where $x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots\ldots, x_r = a + rh$ and $x_n = b = a + nh$ or $h = \dfrac{b-a}{n}$. We note that as $n \to \infty, h \to 0$.

The region $PRSQP$ under consideration is the sum of n subregions, where each subregion is defined on subintervals $[x_{r-1}, x_r], r = 1, 2, 3, \ldots, n$.
From Figure, we have area of the rectangle $(ABLC)$ <area of the region $(ABDCA)$ < area of the rectangle $(ABDM)$ ...(1)
Evidently as $x_r - x_{r-1} \to 0$, i.e., $h \to 0$ all the three areas shown in (1) become nearly equal to each other. Now we form the following sums.

$$s_n = h[f(x_0) + \ldots. + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \quad \ldots(2)$$

and $$S_n = h[f(x_1) + f(x_2) + \ldots. + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \quad \ldots(3)$$

Hence, s_n and S_n denote the sum of areas of all lower rectangles and upper rectangle raised over subintervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, \ldots\ldots, n$, respectively.
In view of the inequality (1) for an arbitrary subinterval $[x_{r-1}, x_r]$, we have

$$s_n < \text{area of the region } PRSQP < S_n \quad \ldots(4)$$

As $n \to \infty$ strips become narrower and narrower, it is assumed that the limiting values of (2) and (3) are the same in both cases and the common limiting values is the required area under the curve.
Symbolically, we write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{area of the region } PRSQP = \int_a^b f(x)dx \quad \ldots(5)$$

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve, For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (5) as

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots. + f(a+(n-1)h)]$$

or $$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a) + f(a+h) + \ldots. + f(a+(n-1)h)] \quad \ldots(6)$$

where $h = \dfrac{b-a}{n} \to 0$ as $n \to \infty$

The above expression (6) is known as the definition of definite integral as the limit of sum.

Evaluating a definite integral by evaluating the limit of a sum is called evaluating definite integral by first principle or by ab-initio method.

Remark : The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable. If the independent variable is denoted by t or u instead of x, we simply write the integral as $\int_a^b f(t)dt$ or $\int_a^b f(u)du$ instead of $\int_a^b f(x)dx$. Hence, the variable of integration is called a dummy variable.

Illustration 4:

Find $\int_0^2 (x^2+1)dx$ as the limit of a sum.

Solution:

By definition $\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+......+f(a+(n-1)h)]$, where $h=\frac{b-a}{n}$

Here $a=0, b=2, f(x)=x^2+1, h=\frac{2-0}{n}=\frac{2}{n}$

Therefore, $\int_0^2 (x^2+1)dx = 2\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{2}{n}\right)+f\left(\frac{4}{n}\right)+....+f\left(\frac{2(n-1)}{n}\right)\right]$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{2^2}{n^2}+1\right)+\left(\frac{4^2}{n^2}+1\right)+....+\left(\frac{(2n-2)^2}{n^2}+1\right)\right]$$

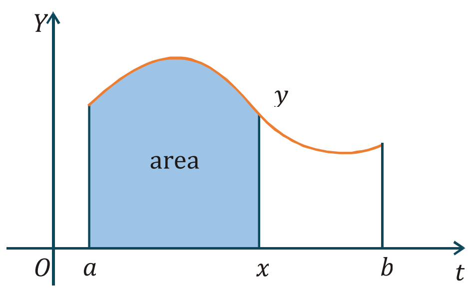
$$= 2\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{(1+1+.....+1)}_{\text{n-terms}}+\frac{1}{n^2}(2^2+4^2+.....+(2n-2)^2)\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2^2}{n^2}(1^2+2^2+.....+(n-1)^2)\right]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2^2}{n^2}\frac{(n-1)n(2n-1)}{6}\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{2}{3}\frac{(n-1)(2n-1)}{n}\right]$$

$$= 2\lim_{n\to\infty}\left[1+\frac{2}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right] = 2\left[1+\frac{4}{3}\right] = \frac{14}{3}.$$

The Fundamental Theorem of Calculus:



The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus : differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's teacher at Cambridge, Isaac Barrow (1630-1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus given the precise inverse relationship between the derivative and the integral. It was Newton and Leibnitz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums.

The Fundamental Theorem of Calculus

Part 1 : If f is continuous on $[a, b]$, then the function g defined by $g(x)=\int_a^x f(t)dt \quad a \le x \le b$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Proof (Desirable) :

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt-\int_a^x f(t)dt}{h}=\lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h} \quad \text{...(i)}$$

According to the mean value theorem for integrals the value of the expression in equation (i) is one of the value taken on by f in the interval joining x and $x + h$. That is, for some number c in this interval $[a, b]$,

$$\frac{1}{h}\int_x^{x+h} f(t)dt = f(c) \quad \text{...(ii)}$$

We can therefore find out what happens to $(1/h)$ times the integral as $h \to 0$ by watching what happens to $f(c)$ as $h \to 0$. As $h \to 0$, c approaches x, and, since f is continuous at x, $f(c)$ approaches $f(x)$:

$$\Rightarrow \lim_{h\to 0} f(c)=f(x) \quad \text{...(iii)}$$

from (i), (ii) & (iii)

$$g'(x)=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt=\lim_{h\to 0} f(c)=f(x)$$

Part 2 : If f is continuous on $[a, b]$, then $\int_a^b f(x)dx=F(b)-F(a)$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Evaluating Definite Integrals by Finding Antiderivatives:

Illustration 5:

$$\int_1^2 2x dx=[x^2]_1^2 = 4 - 1 = 3$$

Illustration 6:

$$\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx$$

Solution:

$\sin^{-1}x = t$

$$\frac{dx}{\sqrt{1-x^2}} = dt$$

at $x = 0$

$t = 0$

at $x = 1$

$t = \pi/2$

$$\int_0^{\pi/2} t\,dt = \left(\frac{t^2}{2}\right)_0^{\pi/2} = \frac{\pi^2}{8}$$

Illustration 7:

$$f(x) = \begin{cases} x, & x < 1 \\ \sin x, & x \ge 1 \end{cases}$$

find $\int_0^{\pi} f(x)dx$.

Solution:

$$\int_0^1 f(x)dx + \int_1^{\pi} f(x)dx$$

$$\int_0^1 x\,dx + \int_1^{\pi} \sin x\,dx$$

$= (x)_0^1 + (-\cos x)_1^{\pi} = 2 + \cos 1$

Illustration 8:

$$\int_0^{\ell n 2} \frac{e^x}{1+e^x}dx$$

Solution:

$1 + e^x = t \qquad x = 0 \Rightarrow t = 2$

$e^x dx = dt \qquad x = \ell n 2 \Rightarrow t = 1 + 2 = 3$

$$\int_2^3 \frac{dt}{t} \Rightarrow \left(\ell n|t|\right)_2^3 = \ell n\left(\frac{3}{2}\right)$$

Illustration 9:

$$\int_0^a \frac{x^2 dx}{\sqrt{a^6 - x^6}}, a > 0$$

Solution:

M-1 : Put $x^3 = t$

$3x^2 dx = dt$

M-2 : $x^3 = a^3 \sin\theta \Rightarrow \sin^{-1}\left(\frac{x^3}{a^3}\right) = \theta$

$$3x^2 = a^3 \cos\theta \cdot \frac{d\theta}{dx}$$

$$\frac{1}{3}\int_0^{\pi/2} \frac{a^3 \cos\theta d\theta}{\sqrt{a^6 - a^6 \sin^2\theta}} = \frac{1}{3}\int_0^{\pi/2} \frac{a^3 \cos\theta d\theta}{a^3 \cos\theta}$$

$$\frac{1}{3}(\theta)_0^{\pi/2} = \frac{\pi}{6}$$

Illustration 10:

$$\int_{\alpha}^{\beta}\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}\quad \beta>\alpha$$

Solution:

Put $\sqrt{x-\alpha}=t$

$x-\alpha=t^2$

$dx=2tdt$

$$\int\frac{2t\,dt}{t\sqrt{(\beta-(t^2+\alpha)}}\Rightarrow 2\int\frac{dt}{\sqrt{(\beta-\alpha)-t^2}}$$

$$\Rightarrow\ 2\sin^{-1}\left(\frac{t}{\sqrt{\beta-\alpha}}\right)\ \Rightarrow\ \left[2\sin^{-1}\left(\frac{\sqrt{x-\alpha}}{\sqrt{\beta-\alpha}}\right)\right]_{\alpha}^{\beta}$$

$$\Rightarrow\ 2\left(\frac{\pi}{2}-0\right)\Rightarrow\ \pi$$

Walli's Theorem:

(a) $$\int_0^{\pi/2}\sin^n x\,dx=\int_0^{\pi/2}\cos^n x\,dx=\frac{(n-1)(n-3).....(1\ or\ 2)}{n(n-2).....(1\ or\ 2)}K$$

where $K=\begin{cases}\pi/2 & \text{if } n \text{ is even}\\ 1 & \text{if } n \text{ is odd}\end{cases}$

(b) $$\int_0^{\pi/2}\sin^n x.\cos^m x\,dx=\frac{[(n-1)(n-3)(n-5)....1\ or\ 2][(m-1)(m-3)....1\ or\ 2]}{(m+n)(m+n-2)(m+n-4)....1\ or\ 2}K$$

Where $K=\begin{cases}\frac{\pi}{2} & \text{if both } m \text{ and } n \text{ are even } (m,n\in N)\\ 1 & \text{otherwise}\end{cases}$

Illustration 11:

$$\int_0^{\pi/2}\cos^6 x\,dx$$

Solution:

$n=6=\dfrac{5\cdot3\cdot1}{6\cdot4\cdot2}\cdot\dfrac{\pi}{2}$

Illustration 12:

$$\int_0^{\pi/4}\sin^5 2x\,dx$$

Solution:

put $2x=t$

$$=\int_0^{\pi/2}\sin^5 t\frac{dt}{2}\ =\frac{1}{2}\int_0^{\pi/2}\sin^5 t\,dt\ =\frac{1}{2}\left[\frac{4\cdot2\cdot1}{5\cdot3\cdot1}\right]=\frac{4}{15}$$

Illustration 13:

$$\int_0^{\pi/2} \sin^4 2x\,dx$$

Solution:

$$\int_0^{\pi/2} (2\sin\cos x)^4\,dx \;=16\int_0^{\pi/2} \sin^4 x\cdot\cos^4 x\,dx \;=16\left[\frac{(3\cdot1\cdot)(3\cdot1)}{8\cdot6\cdot4\cdot2}\frac{\pi}{2}\right]$$

Illustration 14:

$$\int_{-\pi/2}^{\pi/2} \sin^4 x\;\cos^6 x\;dx =$$

(A) $\frac{3\pi}{64}$ (B) $\frac{3\pi}{572}$ (C) $\frac{3\pi}{256}$ (D) $\frac{3\pi}{128}$

Ans. (C)

Solution:

$$\text{I} = \int_{-\pi/2}^{\pi/2} \sin^4 x\;\cos^6 x\;dx = 2\int_0^{\pi/2} \sin^4 x\;\cos^6 x.dx \;= 2\,\frac{(3.1)(5.3.1)}{10.8.6.4.2}.\frac{\pi}{2} = \frac{3\pi}{256}$$

Properties of Definite Integral:

P–1 : $\int_a^b f(x)dx = \int_a^b f(t)dt$ (change of variable does not change value of integral)

P–2 : $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Illustration 15:

If $\frac{d}{dx}f(x) = \frac{e^{\sin x}}{x}$, $x > 0$ & $\int_1^4 \frac{2e^{\sin x^2}}{x}dx = f(k) - f(1)$, then find possible values of k.

Ans. (16)

Solution:

$x^2 = t$

$2xdx = dt$

$$\int_1^4 \frac{2x\cdot e^{\sin(x^2)}}{x^2}dx = \int_1^{16} 2\cdot\frac{e^{\sin t}}{t}dt\cdot\frac{dt}{2} = \int_1^{16}\frac{d}{dt}f(t)dt = f(16) - f(1)$$

$k = 16$

Illustration 16:

$\int_0^1 \frac{\tan^{-1}x}{x}dx = \lambda\int_0^{\pi/2}\frac{\theta}{\sin\theta}d\theta$, then find λ.

Ans. 1/2

Solution:

L.H.S. $\int\frac{\tan^{-1}x}{x}dx$

Put $\tan^{-1}x = t$

$x = \tan t$

$dx = \sec^2 t\,dt$

$$=\int_0^{\pi/4}\frac{t}{\tan t}\cdot\sec^2 t\,dt=\int_0^{\pi/4}\frac{t}{\sin t\cos t}dt=\int_0^{\pi/4}\frac{2t\,dt}{\sin 2t}dt$$

$2t = \theta$

$2dt = d\theta$

$$=\frac{1}{2}\int_0^{\pi/2}\frac{\theta}{\sin\theta}d\theta$$

$$\therefore\ \lambda=\frac{1}{2}$$

P–3 : $\int_a^b f(x)=\int_a^c f(x)\ dx+\int_c^b f(x)dx$

This property is useful when $f(x)$ is not continuous in $[a,b]$ because we can break up the integral into several integrals at the points of discontinuity so that the function is continuous in the subinterval.

Illustration 17:

If $f(x)=\begin{cases}x+3 & : & x<3\\ 3x^2+1 & : & x\geq 3\end{cases}$, then find $\int_2^5 f(x)\ dx$.

Solution:

$$\int_2^5 f(x)\,dx=\int_2^3 f(x)dx+\int_3^5 f(x)dx=\int_2^3(x+3)\,dx+\int_3^5(3x^2+1)\,dx=\left[\frac{x^2}{2}+3x\right]_2^3+\left[x^3+x\right]_3^5$$

$$=\frac{9-4}{2}+3(3-2)+5^3-3^3+5-3=\frac{211}{2}$$

Illustration 18:

Evaluate $\int_2^8 |x-5|\,dx$.

Solution:

$$\int_2^8|x-5|\,dx=\int_2^5(-x+5)\,dx+\int_5^8(x-5)\,dx=9$$

Illustration 19:

Show that $\int_0^2(2x+1)\,dx=\int_0^5(2x+1)\,dx+\int_5^2(2x+1)\,dx$

Solution:

L.H.S. = $\left.x^2+x\right]_0^2$ = 4 + 2 = 6 ; R.H.S. = 25 + 5 – 0 + (4 + 2) – (25 + 5) = 6

∴ L.H.S. = R.H.S

Illustration 20:

If $f(x)=\begin{cases}x^2, & 0<x<2\\ 3x-4, & 2\leq x<3\end{cases}$ then evaluate $\int_0^3 f(x)dx$

Solution:

$$\int_0^3 f(x)dx=\int_0^2 f(x)dx+\int_2^3 f(x)dx=\int_0^2 x^2dx+\int_2^3(3x-4)dx$$

$$=\left(\frac{x^3}{3}\right)_0^2+\left(\frac{3x^2}{2}-4x\right)_2^3=\frac{8}{3}+\frac{27}{2}-12-6+8=37/6$$

Illustration 21:

If $f(x) = \begin{cases} 3[x]-5\frac{|x|}{x}, & x \neq 0 \\ 2, & x=0 \end{cases}$ then $\int_{-3/2}^{2} f(x)dx$ is equal to ([.] denotes the greatest integer function)

(A) $-\frac{11}{2}$ (B) $-\frac{7}{2}$ (C) -6 (D) $-\frac{17}{2}$

Ans. (A)

Solution:

$3[x]-5, \frac{|x|}{x} = 3[x]-5$ if $x > 0$

$= 3[x]+5,$ if $x < 0$

$\Rightarrow \int_{-3/2}^{2} f(x)dx = \int_{-3/2}^{-1}(-1)dx + \int_{-1}^{0}(2)dx + \int_{0}^{1}(-5)dx + \int_{1}^{2}(-2)dx$

$= -1\left(-1+\frac{3}{2}\right)+2(1)+1(-5)+(-2) = -\frac{1}{2}+2-5-2 = -\frac{11}{2}$

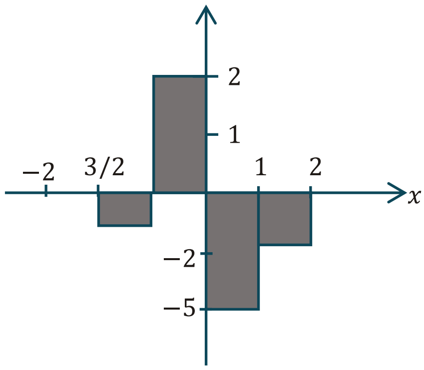


Illustration 22:

The value of $\int_{1}^{2}(x^{[x^2]}+[x^2]^x)dx$, where [.] denotes the greatest integer function, is equal to -

(A) $\frac{5}{4}+\sqrt{3}+(2^{\sqrt{3}}-2^{\sqrt{2}})+\frac{1}{\log 3}(9-3^{\sqrt{3}})$

(B) $\frac{5}{4}+\sqrt{3}+\frac{\sqrt{2}}{3}+\frac{1}{\log 2}(2^{\sqrt{3}}-2^{\sqrt{2}})+\frac{1}{\log 3}(9-3^{\sqrt{3}})$

(C) $\frac{5}{4}+\frac{\sqrt{2}}{3}+\frac{1}{\log 2}(2^{\sqrt{3}}-2^{\sqrt{2}})+\frac{1}{\log 3}(9-3^{\sqrt{3}})$

(D) none of these

Ans. (B)

Solution:

We have, $I = \int_{1}^{2}(x^{[x^2]}+[x^2]^x)dx = \int_{1}^{\sqrt{2}}(x+1)dx + \int_{\sqrt{2}}^{\sqrt{3}}(x^2+2^x)dx + \int_{\sqrt{3}}^{2}(x^3+3^x)dx$

$= \left(\frac{x^2}{2}+x\right)_1^{\sqrt{2}} + \left(\frac{x^3}{3}+\frac{2^x}{\log 2}\right)_{\sqrt{2}}^{\sqrt{3}} + \left(\frac{x^4}{4}+\frac{3^x}{\log 3}\right)_{\sqrt{3}}^{2}$

$= \frac{5}{4}+\sqrt{3}+\frac{\sqrt{2}}{3}+\frac{1}{\log 2}(2^{\sqrt{3}}-2^{\sqrt{2}})+\frac{1}{\log 3}(3^2-3^{\sqrt{3}})$

Illustration 23:

Evaluate: $\int_{-10}^{20}[\cot^{-1}x]dx$. Here [.] is the greatest integer function.

Solution:

$I = \int_{-10}^{20} [\cot^{-1} x]dx$, we know $\cot^{-1}x \in (0,\pi)\ \forall\ x \in R$

Thus $[\cot^{-1} x] = \begin{cases} 3, & x \in (-\infty, \cot 3) \\ 2, & x \in (\cot 3, \cot 2) \\ 1, & x \in (\cot 2, \cot 1) \\ 0 & x \in (\cot 1, \infty) \end{cases}$

Hence I = $\int_{-10}^{\cot 3} 3dx + \int_{\cot 3}^{\cot 2} 2dx + \int_{\cot 2}^{\cot 1} 1dx + \int_{\cot 1}^{20} 0dx$ = 30 + cot1 + cot2 + cot3 **Ans.**

P–4 : $\int_{-a}^{a} f(x)dx = \int_{0}^{a} (f(x) + f(-x))\ dx = \begin{bmatrix} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_{0}^{a} f(x)dx & \text{if } f(x) \text{ is even} \end{bmatrix}$

Proof : $I = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$

Put $x = -t$ in $\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-t)dt = \int_{0}^{a} f(-x)dx \Rightarrow I = \int_{0}^{a} (f(x) + f(-x))dx$

Note:

Odd function : $f(-x) = -f(x)$

Even function : $f(-x) = f(x)$

Even function × Even function = Even function

Odd function × Odd function = Even function

Odd function × Even function = Odd function

Illustration 24:

Evaluate $\int_{-1}^{1} \frac{3^x + 3^{-x}}{1+3^x} dx$

Solution:

$\int_{-1}^{1} \frac{3^x + 3^{-x}}{1+3^x} dx = \int_{0}^{1} \left(\frac{3^x + 3^{-x}}{1+3^x} + \frac{3^{-x} + 3^x}{1+3^{-x}} \right) dx = \int_{0}^{1} \left(\frac{3^x + 3^{-x}}{1+3^x} + \frac{3^x(3^{-x} + 3^x)}{1+3^x} \right)$

$= \int_{0}^{1} (3^x + 3^{-x})\ dx = \left(\frac{3^x}{\ell n3} - \frac{3^{-x}}{\ell n3} \right)_0^1 = \left(\frac{3}{\ell n3} - \frac{3^{-1}}{\ell n3} \right) - \left(\frac{1}{\ell n3} - \frac{1}{\ell n3} \right) = \frac{1}{\ell n3}\left[3 - \frac{1}{3}\right] = \frac{8}{3\ell n3}$

Illustration 25:

Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx$.

Solution:

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx = 2\int_{0}^{\frac{\pi}{2}} \cos x\ dx = 2$ ($\because$ cos x is even function)

Illustration 26:

Evaluate $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx$.

Solution:

Let $f(x) = \log_e\left(\frac{2-x}{2+x}\right) \Rightarrow f(-x) = \log_e\left(\frac{2+x}{2-x}\right) = -\log_e\left(\frac{2-x}{2+x}\right) = -f(x)$

i.e. $f(x)$ is odd function $\quad \therefore \int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx = 0$

Illustration 27:

Evaluate $\int_{-1/2}^{1/2} \cos x \, \ell n\left(\frac{1+x}{1-x}\right) dx$

Solution:

$f(-x) = \cos(-x)\ \ell n\left(\frac{1-x}{1+x}\right) = -\cos\ \ell n\left(\frac{1+x}{1-x}\right) = -f(x) \Rightarrow f(x)$ is odd

Hence, the value of the given integral = 0.

Illustration 28:

If $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x + \tan x \end{vmatrix}$, then the value of $\int_{-\pi/2}^{\pi/2} (x^2+1)(f(x)+f''(x))dx$

(A) 1 (B) –1 (C) 2 (D) none of these

Ans. (D)

Solution:

As, $f(x) = \begin{vmatrix} \cos x & e^{x^2} & 2x\cos^2 x/2 \\ x^2 & \sec x & \sin x + x^3 \\ 1 & 2 & x + \tan x \end{vmatrix}$

$\Rightarrow f(-x) = -f(x) \Rightarrow f(x)$ is odd

$\Rightarrow f'(x)$ is even $\Rightarrow f''(x)$ is odd

Thus, $f(x) + f''(x)$ is odd function let,

$\phi(x) = (x^2 + 1).\{f(x) + f''(x)\} \Rightarrow \phi(-x) = -\phi(x)$

i.e. $\phi(x)$ is odd

P–5 : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ or $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ (King)

Proof : $\int_a^b f(x)dx$

Put $x = a + b - t$

$dx = -dt$

$\int_b^a f(a+b-t)(-dt) = \int_a^b f(a+b-t)\,dt = \int_a^b f(a+b-x)\,dx$

Illustration 29:

Prove that $\int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\sin x)+g(\cos x)}\,dx = \frac{\pi}{4}$.

Solution:

Let I = $\int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\,dx \Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)}{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)+g\left(\cos\left(\frac{\pi}{2}-x\right)\right)}\,dx$

$= \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\cos x)+g(\sin x)}\,dx$

on adding, we obtain

$2I = \int_0^{\frac{\pi}{2}} \left(\frac{g(\sin x)}{g(\sin x)+g(\cos x)}+\frac{g(\cos x)}{g(\cos x)+g(\sin x)}\right)dx = \int_0^{\frac{\pi}{2}} dx \Rightarrow I = \frac{\pi}{4}$

Illustration 30:

$\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{5}}\,dx$

Solution:

$I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$

$I = \int_2^3 \frac{\sqrt{3+2-x}}{\sqrt{5-(3+2-x)}+\sqrt{3+2-x}}\,dx$

$I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$

$2I = \int_2^3 \left(\frac{\sqrt{5-x}+\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}\right)dx \Rightarrow 2I = \int_2^3 dx$

$2I = (x)_2^3 \Rightarrow 2I = 1$

$I = \frac{1}{2}$

Illustration 31:

Evaluate $\int_{-\pi}^{\pi} \frac{x\sin x}{e^x+1}\,dx$

Solution:

$I = \int_{-\pi}^{0} \frac{x\sin x}{e^x+1}\,dx + \int_0^{\pi} \frac{x\sin x}{e^x+1}\,dx = I_1 + I_2$

where $I_1 = \int_{-\pi}^{0} \frac{x\sin x}{e^x+1}\,dx$

Put $x = -t \Rightarrow dx = -dt$

$$\Rightarrow I_1 = \int_{\pi}^{0} \frac{(-t)\sin(-t)(-dt)}{e^{-t}+1} = \int_0^{\pi} \frac{t\sin t\,dt}{e^{-t}+1} = \int_0^{\pi} \frac{e^t t\sin t\,dt}{e^t+1} = \int_0^{\pi} \frac{e^x x\sin x\,dx}{e^x+1}$$

Hence $I = I_1 + I_2 = \int_0^{\pi} \frac{e^x x\sin x}{e^x+1}dx + \int_0^{\pi} \frac{x\sin x}{e^x+1}dx$

$$I = \int_0^{\pi} x\sin x dx = \int_0^{\pi} (\pi - x)\sin(\pi - x)\,dx = \pi\int_0^{\pi} \sin x dx - I$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \sin x dx = \pi|-\cos x|_0^{\pi} = 2\pi \Rightarrow I = \pi$$

Illustration 32:

Evaluate $\int_0^2 \frac{dx}{(17+8x-4x^2)[e^{6(1-x)}+1]}$

Solution:

Let $I = \int_0^2 \frac{dx}{(17+8x-4x^2)[e^{6(1-x)}+1]}$

Also $I = \int_0^2 \frac{dx}{(17+8x-4x^2)[e^{-6(1-x)}+1]} \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

Adding, we get

$$2I = \int_0^2 \frac{1}{17+8x-4x^2}\left(\frac{1}{e^{6(1-x)}+1} + \frac{1}{e^{-6(1-x)}+1}\right)dx$$

$$= \int_0^2 \frac{1}{17+8x-4x^2}dx = -\frac{1}{4}\int_0^2 \frac{dx}{x^2-2x-17/4}$$

$$= -\frac{1}{4}\int_0^2 \frac{dx}{(x-1)^2-21/4} = -\frac{1}{4}\times\frac{1}{2\times\frac{\sqrt{21}}{2}}\left[\log\left|\frac{x-1-\frac{\sqrt{21}}{2}}{x-1+\frac{\sqrt{21}}{2}}\right|\right]_0^2$$

$$= -\frac{1}{4\sqrt{21}}\left[\log\left|\frac{2x-2-\sqrt{21}}{2x-2+\sqrt{21}}\right|\right]_0^2 \Rightarrow I = -\frac{1}{8\sqrt{21}}\left[\log\left|\frac{2-\sqrt{21}}{2+\sqrt{21}}\right| - \log\left|\frac{2+\sqrt{21}}{\sqrt{21}-2}\right|\right]$$

$$= -\frac{1}{4\sqrt{21}}\left[\log\left|\frac{\sqrt{21}-2}{2+\sqrt{21}}\right|\right]$$

Illustration 33:

$\int_0^1 \cot^{-1}(1-x+x^2)dx$ equals -

(A) $\frac{\pi}{2}+\log 2$ (B) $\frac{\pi}{2}-\log 2$ (C) $\pi - \log 2$ (D) none of these

Ans. (B)

Solution:

$$I=\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx=\int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$$

$$=\int_0^1[\tan^{-1}x+\tan^{-1}(1-x)]\ dx=\int_0^1 \tan^{-1}x\ dx+\int_0^1 \tan^{-1}(1-x)\ dx$$

$$=2\int_0^1 \tan^{-1}x\ dx=2\left[x\ \tan^{-1}x-\frac{1}{2}\log(1+x^2)\right]_0^1=2\frac{\pi}{4}-\log 2=\frac{\pi}{2}-\log 2$$

Illustration 34:

$$\int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin x+\cos x}dx$$

Solution:

$$I=\int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin x+\cos x}dx \qquad ...(i)$$

$$I=\int_0^{\pi/2}\frac{a\sin(\pi/2-x)+b\cos(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}dx=\int_0^{\pi/2}\frac{a\cos x+b\sin x}{\sin x+\cos x}dx \qquad ...(ii)$$

$$\therefore 2I=\int_0^{\pi/2}\frac{(a+b)(\sin x+\cos x)}{\sin x+\cos x}dx=\int_0^{\pi/2}(a+b)dx=(a+b)\pi/2 \Rightarrow I=(a+b)\pi/4$$

Illustration 35:

$$\int_0^{\pi/2}\frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}dx \text{ equals -}$$

(A) 2 (B) π (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

Ans. (C)

Solution:

$$I=\int_0^{\pi/2}\frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}dx=\int_0^{\pi/2}\frac{2^{\sin(\pi/2-x)}}{2^{\sin x(\pi/2-x)}+2^{\cos(\pi/2-x)}}dx=\int_0^{\pi/2}\frac{2^{\cos x}}{2^{\cos x}+2^{\sin x}}dx$$

$$2I=\int_0^{\pi/2}dx=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

P–6 : $\int_0^{2a} f(x)dx=\int_0^a f(x)dx+\int_0^a f(2a-x)dx$ **=** $\begin{bmatrix} 0 & \text{if } f(2a-x)=-f(x) \\ 2\int_0^a f(x)\ dx & \text{if } f(2a-x)=f(x)\end{bmatrix}$ (Queen)

Learn : $\int_0^{\pi/2} ln(\sin x)dx=-\frac{\pi}{2}\ell n2=\int_0^{\pi/2}\ell n(\cos x)dx$

Proof: $\int_0^{2a} f(x)dx=\int_0^a f(x)dx+\int_a^{2a} f(x)dx$

Put $x = a + t$

$$=\int_0^a f(x)dx+\int_0^a f(a+t)dt=\int_0^a f(x)dx+\int_0^a f(2a-t)dt=\int_0^a f(x)dx+\int_0^a f(2a-x)dx$$

Illustration 36:

Evaluate $\int_0^{\pi} \cot x . \cos 2x \, dx$

Solution:

Let $f(x) = \cot x \cos 2x$

$\Rightarrow f(\pi - x) = \cot(\pi - x) \cos 2(\pi - x) = -\cot x \cos 2x = -f(x)$

$\therefore \int_0^{\pi} \cot x \cos 2x \, dx = 0$

Illustration 37:

Evaluate $\int_0^{\pi} \frac{dx}{1+3\cos^2 x} dx.$

Solution:

Let $f(x) = \frac{1}{1+3\cos^2 x} \Rightarrow f(\pi - x) = f(x) \Rightarrow \int_0^{\pi} \frac{dx}{1+3\cos^2 x}$

$= 2\int_0^{\frac{\pi}{2}} \frac{dx}{1+3\cos^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x \; dx}{1+\tan^2 x+3} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x \; dx}{4+\tan^2 x} = \left[\tan^{-1}\left(\frac{\tan x}{2}\right)\right]_0^{\frac{\pi}{2}}$

$\because \tan\frac{\pi}{2}$ is undefined, we take limit $= \lim_{x\to\pi/2-} \tan^{-1}\left(\frac{\tan x}{2}\right) - \tan^{-1}\left(\frac{\tan 0}{2}\right) = \pi/2 - 0 = \pi/2$

Illustration 38:

Evaluate : $\int_0^{\infty} (\cot^{-1} x)^2 dx$

Solution:

Let I = $\int_0^{\infty} (\cot^{-1} x)^2 dx \Rightarrow$ Let $x = \cot\theta \Rightarrow dx = -\operatorname{cosec}^2\theta \, d\theta$

$\therefore I = \int_{\frac{\pi}{2}}^{0} \theta^2(-\operatorname{cosec}^2\theta) d\theta \;\; \Rightarrow I = \int_0^{\frac{\pi}{2}} \theta^2(\operatorname{cosec}^2\theta) d\theta$

$= \left(\theta^2(-\cot\theta)\right)_0^{\pi/2} + 2\int_0^{\frac{\pi}{2}} \theta\cot\theta \, d\theta \Rightarrow I = 0 + 2\int_0^{\frac{\pi}{2}} \theta\cot\theta \, d\theta$

$= \left(2\theta\ell n\sin\theta\right)_0^{\pi/2} - 2\int_0^{\frac{\pi}{2}} \ell n\sin\theta \, d\theta \left\{ \begin{array}{l} \text{Standard result} \\ \int_0^{\frac{\pi}{2}} \ell n \sin\theta \, d\theta = \frac{-\pi}{2}\ell n2 \end{array} \right. = 0 - 2 \times \left(-\frac{\pi}{2}\right)\ell n2 = \pi\ell n2.$

Illustration 39:

Evaluate $\int_0^{\pi} \frac{xdx}{1+\cos^2 x}$

Solution:

Let $I = \int_0^{\pi} \frac{x\,dx}{1+\cos^2 x} = \int_0^{\pi} \frac{(\pi - x)dx}{1+\cos^2(\pi - x)} = \int_0^{\pi} \frac{\pi dx}{1+\cos^2 x} - I$

$\Rightarrow 2I = \int_0^{\pi} \frac{\pi dx}{1+\cos^2 x} = 2\pi \int_0^{\pi/2} \frac{dx}{1+\cos^2 x} = 2\pi \int_0^{\pi/2} \frac{\sec^2 x dx}{2+\tan^2 x}$

Let $\tan x = t$ so that for $x \to 0, t \to 0$ and for $x \to \pi/2, t \to \infty$. Hence we can write,

$I = \pi \int_0^{\infty} \frac{dt}{2+t^2} = \pi \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^{\infty} = \frac{\pi^2}{2\sqrt{2}}$ **Ans.**

Illustration 40:

Prove that $\int_0^{\pi/2} \log(\sin x)dx = \int_0^{\pi/2} \log(\cos x)dx = -\frac{\pi}{2}\log 2$

Solution:

Let $I = \int_0^{\pi/2} \log(\sin x)dx$ ...(i)

then $I = \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - x\right)dx = \int_0^{\pi/2} \log(\cos x)dx$ ...(ii)

adding (i) and (ii), we get

$2I = \int_0^{\pi/2} \log\sin x\,dx + \int_0^{\pi/2} \log\cos x\,dx = \int_0^{\pi/2} (\log\sin x + \log\cos x)dx$

$\Rightarrow 2I = \int_0^{\pi/2} \log(\sin x\cos x)dx = \int_0^{\pi/2} \log\left(\frac{2\sin x\cos x}{2}\right)dx$

$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx = \int_0^{\pi/2} \log(\sin 2x)dx - \int_0^{\pi/2} (\log 2)dx = \int_0^{\pi/2} \log\sin 2x.dx - (\log 2)(x)_0^{\pi/2}$

$\Rightarrow 2I = \int_0^{\pi/2} \log(\sin 2x)dx - \frac{\pi}{2}\log 2$ ...(iii)

Let $I_1 = \int_0^{\pi/2} \log(\sin 2x)dx,$ putting $2x = t$, we get

$I_1 = \int_0^{\pi} \log(\sin t)\frac{dt}{2} = \frac{1}{2}\int_0^{\pi} \log(\sin t)dt = \frac{1}{2}.2\int_0^{\pi/2} \log(\sin t)dt$

$I_1 = \int_0^{\pi/2} \log(\sin x)dx$

$\therefore$ (iii) becomes; $2I = I - \frac{\pi}{2}\log 2$

Hence $\int_0^{\pi/2} \log\sin x\,dx = -\frac{\pi}{2}\log 2$

Illustration 41:

$\int_0^{\pi/2}(2\log\sin x - \log\sin 2x)dx$ equals -

(A) $\pi \log 2$ (B) $-\pi \log 2$ (C) $(\pi/2) \log 2$ (D) $-(\pi/2) \log 2$

Ans. (D)

Solution:

$$I = \int_0^{\pi/2}(2\log\sin x - \log 2\sin x\cos x)dx = \int_0^{\pi/2}(2\log\sin x - \log 2 - \log\sin x - \log\cos x)dx$$

$$= \int_0^{\pi/2}\log\sin x dx - \int_0^{\pi/2}\log 2 dx - \int_0^{\pi/2}\log\cos x dx = -(\pi/2)\log 2$$

P – 7 Let $f(x)$ be periodic function with period 'T' i.e. $f(T + x) = f(x)$ (Jack)

Eg. $\sin x, \cos x$: Period 2π

$\tan x, \cot x, |\sin x|, |\cos x|, \sin^2 x$: Period π

$\{x\}, x - [x], e^{\{x\}}$: Period 1

(i) $\int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx, n \in I$

(ii) $\int_a^{T+a} f(t)dt$ will be independent of a and equal to $\int_0^{T} f(t)dt$

(iii) $\int_a^{a+nT} f(x)dx = \int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx$

(iv) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^{b} f(x)dx$, where $n \in I$.

(v) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^{T} f(x)dx$, where $n, m \in I$.

Proof: $\int_{mT}^{nT} f(x)dx = \int_0^{nT} f(x)dx - \int_0^{mT} f(x)dx$

$= n\int_0^{T} f(x)dx - m\int_0^{T} f(x)dx$

$= (n-m)\int_0^{T} f(x)dx$

Illustration 42:

Evaluate $\int_{-3}^{5} e^{\{x\}}\, dx$, where {.} denotes the fractional part function.

Solution:

$\int_{-3}^{5} e^{\{x\}}\, dx = (5-(-3))\int_0^{1} e^{\{x\}}dx$

$= 8\int_0^{1} e^{x}dx = 8\left(e^{x}\right)_0^1 = 8\,(e-1)$

Illustration 43:

Evaluate $\int_0^{4\pi} |\cos x| dx$

Solution:

Note that |cos x| is a periodic function with period π. Hence the given integral.

$$I = 4\int_0^{\pi} |\cos x| dx = 4\left[\int_0^{\frac{\pi}{2}} \cos x dx - \int_{\frac{\pi}{2}}^{\pi} \cos x dx\right] = 4\left[[\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}\right] = 4[1+1] = 8$$

Illustration 44:

Evaluate $\int_0^{16\pi/3} |\sin x| dx$

Solution:

$$\int_0^{16\pi/3} |\sin x| dx = \int_0^{5\pi} |\sin x| dx + \int_{5\pi}^{5\pi+\pi/3} |\sin x| dx = 5\int_0^{\pi} |\sin x| dx + \int_0^{\pi/3} |\sin x| dx$$

$$= 5[-\cos x]_0^{\pi} + [-\cos x]_0^{\pi/3} = 10 + \left(-\frac{1}{2}+1\right) = \frac{21}{2}$$

Illustration 45:

Evaluate: $\int_0^{2n\pi} [\sin x + \cos x] dx$. Here [.] is the greatest integer function.

Solution:

Let $I = \int_0^{2n\pi} [\sin x + \cos x] dx = n\int_0^{2\pi} [\sin x + \cos x] dx$

($\because$ [sin$x$ + cos$x$] is periodic function with period 2π]

$$[\sin x + \cos x] = \begin{cases} 1, & 0 \le x \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x \le \frac{3\pi}{4} \\ -1, & \frac{3\pi}{4} < x \le \pi \\ -2, & \pi < x \le \frac{3\pi}{2} \\ -1, & \frac{3\pi}{2} < x \le \frac{7\pi}{4} \\ 0, & \frac{7\pi}{4} < x \le 2\pi \end{cases}$$

Hence $I = n\left[\int_0^{\pi/2} 1dx + \int_{\pi/2}^{3\pi/4} 0dx + \int_{3\pi/4}^{\pi} -1dx + \int_{\pi}^{3\pi/2} -2dx + \int_{3\pi/2}^{7\pi/4} -1dx + \int_{7\pi/4}^{2\pi} 0dx\right]$

$$I = n\left[\frac{\pi}{2} + 0 - \pi + \frac{3\pi}{4} - 3\pi + 2\pi - \frac{7\pi}{4} + \frac{3\pi}{2} + 0\right] = -n\pi$$

Illustration 46:

Evaluate $\sum_{n=1}^{1000}\int_{n-1}^{n}|\cos 2\pi x|dx$

Solution:

$$\int_0^1|\cos 2\pi x|\ dx + \int_1^2|\cos 2\pi x|\ dx + ...+ \int_{999}^{1000}|\cos 2\pi x|\ dx = \int_0^{1000}|\cos 2\pi x|\ dx$$

Now $|\cos 2\pi x|$ is a periodic function of period 1/2

$$I = 2000\int_0^{\frac{1}{2}}|\cos 2\pi x|dx \Rightarrow I = 2000 \times 2 = 4000$$

Illustration 47:

$$\int_{-1}^{2} x[x]dx$$

Solution:

$$= \int_{-1}^{2} x[x]dx \Rightarrow \int_{-1}^{0} x[x]dx + \int_0^1 x[x]dx + \int_1^2 x[x]dx$$

$$= \int_{-1}^{0} x(-1)dx + \int 0 + \int_1^2 x(1)dx = -\int_{-1}^{0} x\,dx + \int_1^2 x\,dx \Rightarrow -\left(\frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2}\right)_1^2$$

$$= -\left(0-\left(\frac{1}{2}\right)\right) + \left(\frac{4}{2}-\frac{1}{2}\right) = \frac{1}{2}+\frac{3}{2} = \frac{4}{2} = 2$$

Leibnitz Theorem:

If $F(x) = \int_{g(x)}^{h(x)} f(t)dt$, then $\frac{dF(x)}{dx} = h'(x)f\big(h(x)\big) - g'(x)\, f(g(x))$

Proof : Let $P(t) = \int f(t)dt \Rightarrow F(x) = \int_{g(x)}^{h(x)} f(t)dt = P\big(h(x)\big) - P(g(x))$

$$\Rightarrow \frac{dF(x)}{dx} = P'\big(h(x)\big)h'(x) - P'\big(g(x)\big)g'(x) = f\big(h(x)\big)h'(x) - f(g(x))g'(x)$$

Illustration 48:

If $F(x) = \int_x^{x^2}\sqrt{\tan t}\,dt$, then find $F'(x)$.

Solution:

$F'(x) = 2x.\sqrt{\tan x^2} - 1.\sqrt{\tan x}$

Illustration 49:

If $F(x) = \int_{x^2}^{x^3}\frac{1}{\ell nt}dt$ then find $F'(e)$

Solution:

$$F'(x) = \frac{3x^2}{\ell nx^3} - \frac{2x}{\ell nx^2} = \frac{x^2}{\ell nx} - \frac{x}{\ell nx} = \frac{x(x-1)}{\ell nx},\ \text{now } F'(e) = \frac{e(e-1)}{\ell ne} = e(e-1)$$

Illustration 50:

Evaluate : $\lim_{x\to 0^+} \int_0^{x^2} \frac{\sin\sqrt{t}\ \tan\sqrt{t}dt}{x^4}$

Solution:

Applying L'hospital rule

$$\lim_{x\to 0^+} \frac{2x\sin x\tan x}{4x^3}$$

$$\Rightarrow \lim_{x\to 0} \frac{1}{2}\left(\frac{\sin x}{x}\right)\left(\frac{\tan x}{x}\right) = \frac{1}{2}$$

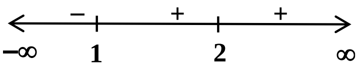
Illustration 51:

Let $f(x) = \int_0^x (t-1)(t-2)^2\, dt$, then find a point of minimum

Solution:

$$f(x) = \int_0^x (t-1)(t-2)^2\, dt$$

$$f'(x) = (x - 1)(x - 2)^2$$



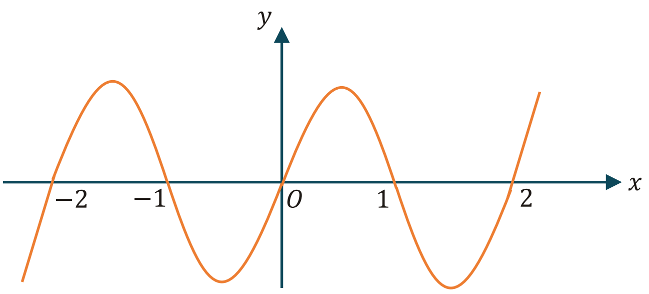
$\Rightarrow x = 1$ is the point of minimum

$f(1) = \int_0^1 (t^3 - 5t^2 + 8t - 4)dt = \frac{1}{4} - \frac{5}{3} + 4 - 4 = -\frac{17}{12}$. Hence $\left(1, -\frac{17}{12}\right)$ is a point of minimum

Illustration 52:

Find the points of maxima/minima of $\int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$

Solution:



Graph of $f'(x)$

Let $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$

$$f'(x) = \frac{x^4 - 5x^2 + 4}{2 + e^{x^2}} 2x - 0 = \frac{(x-1)(x+1)(x-2)(x+2)2x}{2 + e^{x^2}}$$

From the wavy curve, it is clear that $f'(x)$ changes its sign at $x = \pm 2, \pm 1, 0$ and hence the points of maxima are –1, 1 and of the minima are –2, 0, 2.

Illustration 53:

Evaluate $\frac{d}{dt}\int_{t^2}^{t^3}\frac{1}{\log\ x}dx$

Solution:

$$\frac{d}{dt}\int_{t^2}^{t^3}\frac{1}{\log\ x}dx=\frac{1}{\log\ t^3}.\frac{d}{dt}(t^3)-\frac{1}{\log\ t^2}.\frac{d}{dt}(t^2)=\frac{3t^2}{3\ \log\ t}-\frac{2t}{2\log\ t}=\frac{t(t-1)}{\log\ t}$$

Illustration 54:

Evaluate , $\int_0^1\frac{x^b-1}{\ell nx}dx$ 'b' being parameter.

Solution:

Let $I(b)=\int_0^1\frac{x^b-1}{\ell nx}dx \Rightarrow \frac{dI(b)}{db}=\int_0^1\frac{x^b\ell nx}{\ell nx}dx+0-0$ (using modified Leibnitz Theorem)

$=\int_0^1 x^b dx=\left.\frac{x^{b+1}}{b+1}\right]_0^1 \Rightarrow I(b)=\ell n(b+1)+c$

$b=0\Rightarrow I(0)=0$

$\therefore\ c=0 \qquad \therefore\ I(b)=\ell n\,(b+1)$

Illustration 55:

If $f(x)=5^{g(x)}$ and $g(x)=\int_2^{x^2}\frac{t}{\ell n(1+t^2)}dt$, then find the value of $f'(\sqrt{2})$.

Solution:

$f'(x)=5^{g(x)}\ell n5\, g'(x)$

Now $g'(x)=\frac{x^2\cdot 2x}{\ell n(1+x^4)}$

$\therefore f'(x)=5^{g(x)}\ell n5\frac{2x^3}{\ell n(1+x^4)}$

$\Rightarrow f'(\sqrt{2})=5^{g(\sqrt{2})}\ell n5\frac{2(\sqrt{2})^3}{\ell n(1+(\sqrt{2})^4)}=4\sqrt{2}$ {since $g(\sqrt{2})=0$}

Illustration 56:

Find the slope of the tangent to the curve $y=\int_x^{x^2}\cos^{-1}t^2\,dt$ at $x=\frac{1}{\sqrt[4]{2}}$

Solution:

Given curve is $y=\int_x^{x^2}\cos^{-1}t^2dt$; $\frac{dy}{dx}=\frac{d}{dx}\int_x^{x^2}\cos^{-1}t^2dt$

using Leibnitz theorem,

$$\frac{dy}{dx}=2x\cos^{-1}x^4-\cos^{-1}x^2$$

$$\left(\frac{dy}{dx}\right)_{x=\frac{1}{\sqrt[4]{2}}}=\frac{2}{2^{1/4}}\cos^{-1}\frac{1}{2}-\cos^{-1}\frac{1}{\sqrt{2}}=2^{3/4}\frac{\pi}{3}-\frac{\pi}{4}=\left(\frac{\sqrt[4]{8}}{3}-\frac{1}{4}\right)\pi$$

Illustration 57:

If $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}dt$, then prove that $f'(x) = 0 \ \forall \ x \in R$.

Solution:

$f'(x) = \sin^{-1}\sqrt{\sin^2 x} \,.\, 2\sin x \cos x - \cos^{-1}\sqrt{\cos^2 x} \,.\, 2\cos x . \sin x$

$= x . \sin 2x - x\sin 2x = 0.$

Sum of Series Using Definite Integration:

Using definite integration as limit of sum

$$\lim_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + ... + f\left(a+\overline{n-1}h\right)] = \int_a^b f(x)dx$$

$$\text{or } \lim_{h\to\infty} h\sum_{r=0}^{n-1} f(a+rh) = \int_a^b f(x)dx \text{, where } b - a = nh$$

If $a = 0$ & $b = 1$ then, $\lim_{n\to\infty} h\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)dx$; where $nh = 1$

Hence we have $\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$

Steps to express the infinite series as definite integral :

Step I : Express the given series in the form $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$

Step II : Then the limit is its sum when $n \to \infty$, i.e. $\lim_{n\to\infty} \frac{1}{n} f\left(\frac{r}{n}\right)$

Step III : Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx and $\lim_{n\to\infty}\sum$ by the sign of $\int$

Step IV : The lower and the upper limit of integration are the limiting values of $\frac{r}{n}$ for the first and the last term of r respectively.

Illustration 58:

$$\lim_{n\to\infty} \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{\boxed{4n}_{\to n+3n}}$$

Solution:

$$\lim_{n\to\infty} \sum_{r=0}^{3n} \frac{1}{n+r}$$

$$= \lim_{n\to\infty} \sum_{r=0}^{3n} \frac{1}{n\left(1+\frac{r}{n}\right)} = \int_0^3 \frac{dx}{1+x} = \left[\ell n(1+x)\right]_0^3 = \ell n 4$$

Illustration 59:

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r}{n^2}\sec^2\left(\frac{r^2}{n^2}\right)$$

Solution:

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\frac{r}{n}\left[\sec\left(\frac{r}{n}\right)^2\right]^2$$

$$\int_0^1 x(\sec(x^2))^2\,dx$$

Put $x^2 = t$ $\quad 2x=\frac{dt}{dx}$

$$= \frac{1}{2}\int_0^1 \sec^2 t\,dt \Rightarrow \frac{1}{2}(\tan t)_0^1 = \frac{1}{2}(\tan 1 - \tan 0) = \frac{\tan 1}{2}$$

Illustration 60:

$$\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+.....+2\sqrt{n}}{n\sqrt{n}}$$

Solution:

$$\lim_{n\to\infty}\sum_{r=1}^{4n}\frac{\sqrt{r}}{n\sqrt{n}} = \lim_{n\to\infty}\sum_{r=1}^{4n}\frac{1}{n}\sqrt{\frac{r}{n}} = \int_0^4\sqrt{x}\,dx \Rightarrow \frac{\left((x)^{3/2}\right)_0^4}{(3/2)}$$

$$=\frac{2}{3}(4)^{3/2} \Rightarrow \frac{2}{3}(2)^{2\times\frac{3}{2}}$$

$$\Rightarrow \frac{2}{3}\times 8 = \frac{16}{3}$$

Illustration 61:

$$\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+.....+\frac{3}{5n}\right]$$

Solution:

$$\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{n+r}{n^2+r^2} = \lim_{n\to\infty}\sum_{r=1}^{2n}\frac{n\left(1+\frac{r}{n}\right)}{n^2\left(1+\left(\frac{r}{n}\right)^2\right)} = \int_0^2\frac{1+x}{1+x^2}dx$$

$$=\int_0^2\frac{dx}{1+x^2}+\int_0^2\frac{xdx}{1+x^2} \Rightarrow \left(\tan^{-1}(x)\right)_0^2+\left(\frac{1}{2}\ell n\left(1+x^2\right)\right)_0^2$$

$$\Rightarrow \tan^{-1}(2)+\frac{1}{2}\ell n5$$

Illustration 62:

Evaluate $\lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}+.........+\frac{1}{6n}\right)$

Solution:

Let $S_n = \frac{1}{2n+1}+\frac{1}{2n+2}+.........+\frac{1}{6n} = \sum_{r=1}^{4n}\frac{1}{2n+r} = \sum_{r=1}^{4n}\frac{1}{n}\cdot\frac{1}{2+\left(\frac{r}{n}\right)}$

$$\Rightarrow S = \lim_{n\to\infty}S_n = \int_0^4\frac{dx}{2+x} = [\ell n|2+x|]_0^4 = \ell n6-\ell n2 = \ell n3$$

Illustration 63:

Evaluate $\lim_{n\to\infty}\left[\frac{\sqrt{n}}{(3+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{3}(3\sqrt{3}+4\sqrt{n})^2}+.......+\frac{1}{49n}\right]$

Solution:

Let $p=\lim_{n\to\infty}\left[\frac{\sqrt{n}}{(3+4\sqrt{n})^2}+\frac{\sqrt{n}}{\sqrt{2}(3\sqrt{2}+4\sqrt{n})^2}+.........+\frac{\sqrt{n}}{\sqrt{n}(3\sqrt{n}+4\sqrt{n})^2}\right]$

Analyzing the expression with the view of increasing integral value we get the expression in terms of r as

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}(3\sqrt{r}+4\sqrt{n})^2}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\sqrt{\frac{r}{n}}\left(3\sqrt{\frac{r}{n}}+4\right)^2}=\int_0^1\frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$$

Put $3\sqrt{x}+4=t,\ \therefore\ \frac{3}{2\sqrt{x}}dx=dt$

Hence $p=\frac{2}{3}\int_4^7\frac{dt}{t^2}=\frac{2}{3}\left[-\frac{1}{t}\right]_4^7=\frac{2}{3}\left(-\frac{1}{7}+\frac{1}{4}\right)=\frac{1}{14}$

Determination of Function:

Illustration 64:

$\int f'(x)dx=x\cos\pi x$ and $f(0)=0$

then value of $f\left(\frac{1}{3}\right)$

Solution:

$f(x)+C=x\cos\pi x$

$f(0)+C=0\cos 0$

$0+C=0$

$\therefore\ C=0$

$\therefore\ f(x)=x\cos\pi x$

$f\left(\frac{1}{3}\right)=\frac{1}{3}\cos\frac{\pi}{3}\Rightarrow\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$

Illustration 65:

$\int_0^x f(t)dt=x\cos\pi x$, $x>0$ find $f(4)$

Solution:

Differentiating or applying Newton Labneitz

$f(x)=\cos\pi x+x(-\sin\pi x)\cdot\pi$

$f(4)=\cos4\pi+4\pi(-\sin4\pi)=1$

Estimation of D.I. & General Inequality

(a) If $f(x)$ is continuous in $[a,b]$ and it's range in this interval is $[m,M]$, then

$$m(b-a)\le\int_a^b f(x)dx\le M(b-a)$$

Illustration 66:

Prove that $4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$

Solution:

Since the function $f(x) = \sqrt{3+x^3}$ increases monotonically on the interval $[1,3], m = 2, M = \sqrt{30}$, $b - a = 2$.

Hence, $2.2 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30} \Rightarrow 4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$

(b) If $f(x) \le \varphi(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b \phi(x)\,dx$

Illustration 67:

Prove that $\frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$

Solution:

Since $4 - x^2 \ge 4 - x^2 - x^3 \ge 4 - 2x^2 > 0\ \forall\ x \in [0, 1]$

$\sqrt{4-x^2} \ge \sqrt{4-x^2-x^3} \ge \sqrt{4-2x^2} > 0\ \forall\ x \in [0, 1]$

$\Rightarrow 0 < \frac{1}{\sqrt{4-x^2}} \le \frac{1}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{4-2x^2}}\ \forall\ x \in [0\ 1]$

$\Rightarrow \int_0^1 \frac{dx}{\sqrt{4-x^2}} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \int_0^1 \frac{dx}{\sqrt{4-2x^2}}\ \forall x \in [0, 1]$

$\Rightarrow \left[\sin^{-1}\frac{x}{2}\right]_0^1 \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1 \Rightarrow \frac{\pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\pi}{4\sqrt{2}}$

(c) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$.

Illustration 68:

Prove that $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}\,dx\right| < 10^{-7}$

Solution:

To find $I = \left|\int_{10}^{19} \frac{\sin x}{1+x^8}\,dx\right| \le \int_{10}^{19} \left|\frac{\sin x}{1+x^8}\right| dx$ ...(i)

Since $|\sin x| \le 1$ for $x \ge 10$

The inequality $\left|\frac{\sin x}{1+x^8}\right| \le \frac{1}{|1+x^8|}$ ...(ii)

also, $10 \le x \le 19$

$\Rightarrow 1 + x^8 > 10^8$

$\Rightarrow \frac{1}{1+x^8} < \frac{1}{10^8}$ or $\frac{1}{|1+x^8|} < 10^{-8}$ ...(iii)

from (ii) and (iii) ;

$\left|\frac{\sin x}{1+x^8}\right| < 10^{-8}$

$$\left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right|<\int_{10}^{19}10^{-8}dx$$

$$\therefore\left|\int_{10}^{19}\frac{\sin x}{1+x^8}dx\right|<(19-10).10^{-8}<10^{-7}$$

Illustration 69:

If $f(x)$ is integrable function such that $|f(x)-f(y)| \le |x^2 - y^2|$, $\forall\ x,y \in [a,b]$ then prove that $\left|\int_a^b\frac{f(x)-f(a)}{x+a}dx\right|\le\frac{(a-b)^2}{2}$.

Solution:

Given, $\left|\int_a^b\frac{f(x)-f(a)}{x+a}dx\right|\le\int_a^b\left|\frac{f(x)-f(a)}{x+a}\right|dx$

$$\le\int_a^b\left|\frac{x^2-a^2}{x+a}\right|dx=\int_a^b|x-a|\,dx=\int_a^b(x-a)dx=\frac{(a-b)^2}{2}$$

(d) If $f(x)\ge 0$ on the interval $[a,b]$, then $\int_a^b f(x)dx\ge 0$.

Illustration 70:

If $f(x)$ is a continuous function such that $f(x)\ge 0\ \forall\ x\in[2,10]$ and $\int_4^8 f(x)dx=0$, then find $f(6)$.

Solution:

$f(x)$ is above the x-axis or on the x-axis for all $x\in[2,10]$. If $f(x)$ is greater than zero for any sub interval of $[4,8]$, then $\int_4^8 f(x)dx$ must be greater than zero.

But $\int_4^8 f(x)dx=0\Rightarrow f(x)=0\ \forall\ x\in[4,8]$

$\Rightarrow f(6)=0.$

Illustration 71:

For $x\in(0,1)$ arrange $f_1(x)=\frac{1}{9-x^2}$, $f_2(x)=\frac{1}{9-2x^2}$ and $f_3(x)=\frac{1}{9-x^2-x^3}$ in ascending order and hence prove that $\frac{1}{6}\ell n2<\int_0^1\frac{1}{9-x^2-x^3}dx<\frac{1}{6\sqrt{2}}\ell n5$

Solution:

$\because 0<x^3<x^2$, for all $x\in(0,1)\Rightarrow x^2<x^2+x^3<2x^2$

$\Rightarrow -2x^2<-x^2-x^3<-x^2\Rightarrow 9-2x^2<9-x^2-x^3<9-x^2$

$$\Rightarrow\frac{1}{9-x^2}<\frac{1}{9-x^2-x^3}<\frac{1}{9-2x^2}$$

$f_1(x)<f_3(x)<f_2(x)$ for $x\in(0,1)$

$$\Rightarrow\int_0^1 f_1(x)dx<\int_0^1 f_3(x)dx<\int_0^1 f_2(x)dx\Rightarrow\int_0^1\frac{dx}{9-x^2}<\int_0^1\frac{dx}{9-x^2-x^3}<\int_0^1\frac{dx}{9-2x^2}$$

$$\Rightarrow\frac{1}{6}\left(\ell n\left|\frac{3+x}{3-x}\right|\right)_0^1<\int_0^1\frac{dx}{9-x^2-x^3}<\frac{1}{6\sqrt{2}}\left(\ell n\left|\frac{3+2x}{3-2x}\right|\right)_0^1$$

$$\Rightarrow\frac{1}{6}\ell n2<\int_0^1\frac{1}{9-x^2-x^3}dx<\frac{1}{6\sqrt{2}}\ell n5$$

Illustration 72:

Prove that $1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right)dx < \frac{6}{5}$

Solution:

Let $f(x) = \frac{5-x}{9-x^2}$

$\therefore f'(x) = -\frac{(x-9)(x-1)}{(9-x^2)^2} \Rightarrow f'(x) = 0$ or not defined $\Rightarrow x = 1$

Then $f(0) = \frac{5}{9}, f(1) = \frac{1}{2}, f(2) = \frac{3}{5}$. The greatest and least values of the integrand in the interval [0,2] are respectively, equal to $f(2) = \frac{3}{5}$ and $f(1) = \frac{1}{2}$

$(2-0)\frac{1}{2} < \int_0^2 \left(\frac{5-x}{9-x^2}\right)dx < (2-0)\frac{3}{5}$.

Hence $1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right)dx < \frac{6}{5}$

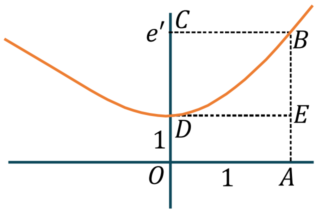
Illustration 73:

Estimate the value of $\int_0^1 e^{x^2}dx$ using (i) rectangle, (ii) triangle.

Solution:

(i) By using rectangle

Area $OAED < \int_0^1 e^{x^2}dx <$ Area $OABC$



$1 < \int_0^1 e^{x^2}dx < 1.e$

$1 < \int_0^1 e^{x^2}dx < e$

(ii) By using triangle

Area $OAED < \int_0^1 e^{x^2}dx <$ Area $OAED$ + Area of triangle DEB

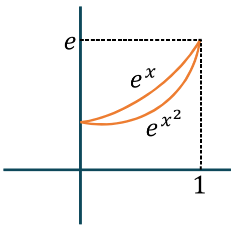
$1 < \int_0^1 e^{x^2}dx < 1 + \frac{1}{2}.1.(e-1) 1 < \int_0^1 e^{x^2}dx < \frac{e+1}{2}$

Illustration 74:

Estimate the value of $\int_0^1 e^{x^2}dx$ by using $\int_0^1 e^{x}dx$.

Solution:

For $x \in (0,1)$, $e^{x^2} < e^x$



$\Rightarrow 1 \times 1 < \int_0^1 e^{x^2}dx < \int_0^1 e^{x}dx$

$1 < \int_0^1 e^{x^2}dx < e-1$

Reduction Formulae:

Illustration 75:

If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$, then show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

Solution:

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\ dx$$

$$I_n = \left[-\sin^{n-1} x\ \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\ \sin^{n-2} x\ .\ \cos^2 x\ dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\ .\ (1-\sin^2 x)\ dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\ dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\ dx \Rightarrow I_n + (n-1)I_n = (n-1)I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

Note :

1. $\int_0^{\frac{\pi}{2}} \sin^n x\ dx = \int_0^{\frac{\pi}{2}} \cos^n x\ dx$

2. $I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)$..... I_0 or I_1 according as n is even or odd. $I_0 = \frac{\pi}{2}, I_1 = 1$

$$\text{Hence } I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)......\left(\frac{1}{2}\right).\frac{\pi}{2}, & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)......\left(\frac{2}{3}\right).1, & \text{if } n \text{ is odd} \end{cases}$$

Illustration 76:

If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\ dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$

Solution:

$$I_n = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}\ .\ \tan^2 x\ dx = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}\ (\sec^2 x - 1)dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \sec^2 x\ dx - \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}\ dx = \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$I_n = \frac{1}{n-1} - I_{n-2} \qquad \therefore \qquad I_n + I_{n-2} = \frac{1}{n-1}$$

Illustration 77:

If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \,.\, \cos^n x dx$, then show that $I_{m,n} = \frac{m-1}{m+n} \; I_{m-2,n}$

Solution:

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m-1} x \; (\sin x \; \cos^n x)\, dx$$

$$= \left[-\frac{\sin^{m-1} x \,.\, \cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos^{n+1} x}{n+1} (m - 1)\sin^{m-2}x \cos x \, dx$$

$$= \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \sin^{m-2} x.\cos^n x.\cos^2 x \, dx$$

$$= \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \left(\sin^{m-2} x.\cos^n x - \sin^m x.\cos^n x\right) dx$$

$$= \left(\frac{m-1}{n+1}\right) I_{m-2,n} - \left(\frac{m-1}{n+1}\right) I_{m,n} \Rightarrow \left(1+\frac{m-1}{n+1}\right) I_{m,n} = \left(\frac{m-1}{n+1}\right) I_{m-2,n}$$

$$I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$$

Illustration 78:

Let $I_n = \int_0^1 \underset{II}{1} \cdot \underbrace{(1-x^4)}_{I}{}^n dx$, $n \in N$ then prove that $\frac{I_n}{I_{n-1}} = \frac{4n}{4n+1}$

Solution:

$$I_n = \left[(1-x^4)^n \cdot x\right] - \int_0^1 n(1-x^4)^{n-1}(-4x^3)\cdot x \, dx$$

$$= \left[x(1-x^4)^n\right]_0^1 + 4n\int x^4(1-x^4)^{n-1} dx = 0 + 4n\int_0^1 (1-x^4)^{n-1}(1-(1-x^4))dx$$

$$= 4n\left[\int_0^1 (1-x^4)^{n-1} dx - \int_0^1 (1-x^4)^n dx\right]$$

$I_n = 4nI_{n-1} - 4nI_n$

$(4n + 1)I_n = 4nI_{n-1}$

$$\frac{I_n}{I_{n-1}} = \frac{4n}{4n+1}$$

Illustration 79:

The value of $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ is

(A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^2}{8}$ (C) $\frac{\pi^2}{16}$ (D) $\frac{3\pi^2}{16}$

Ans. (C)

Solution:

Let $I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\ dx$

$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\cos^4 x+\sin^4 x}dx$

$\therefore\ 2I = \int_0^{\pi/2} \frac{\left(x+\frac{\pi}{2}-x\right)\sin x\cos x}{\cos^4 x+\sin^4 x}dx$

$= \frac{\pi}{4}\int_0^{\pi/2} \frac{2\sin x\cos x\ \ dx}{\sin^4 x+\cos^4 x}dx = \frac{\pi}{4}\int_0^{\pi/2} \frac{2\tan x\sec^2 xdx}{1+\tan^4 x}dx$

$= \frac{\pi}{4}\int_0^{\infty} \frac{dt}{1+t^2}$. Put $t = \tan^2 x$ $= \frac{\pi}{4}\left[\tan^{-1} t\right]_0^{\infty} = \frac{\pi}{4}\left(\frac{\pi}{2}-0\right) = \frac{\pi^2}{8}$

$\therefore\ I = \frac{\pi^2}{16}$

Illustration 80:

The value of $\int_0^1 \frac{x\tan^{-1} x}{(1+x^2)^{3/2}}\ dx$ is

(A) $\frac{4+\pi}{4\sqrt{2}}$ (B) $\frac{4-\pi}{4\sqrt{2}}$ (C) $\frac{\pi}{2}$ (D) $-\frac{\pi}{2}$

Ans. (B)

Solution:

Let $\tan^{-1} x\ =\ t \Rightarrow \frac{dx}{1+x^2} = dt$

$\therefore\ I = \int_0^{\pi/4} \frac{t\tan t\cdot dt}{\sqrt{1+\tan^2 t}} = \int_0^{\pi/4} t\cdot\sin t\, dt$

$= -\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4-\pi}{4\sqrt{2}}$

Illustration 81:

If f is a continuous function and $\phi(x) = \int_0^x \left((3t+4)\int_t^3 f(u)du\right)dt$ and $\int_0^3 f(x)dx = 3$, then :

(A) $\phi'(0) = 0$ (B) $\phi'(0) = 13$ (C) $\phi''(3) = 13f(3)$ (D) $\phi''(3) = -13f(3)$

Ans. (D)

Solution:

$\phi'(x) = (3x + 4) \int_x^3 f(u)du$

$\phi''(x) = 3\int_x^3 f(u)du + (3x + 4)(0 - f(x))$

so $\phi'(0) = (0 + 4) \int_x^3 f(u)\, du = 12$

$\phi''(3) = -13f(3)$

Illustration 82:

If $f(x) = \int_0^x (2\cos^2 3t + 3\sin^2 3t)\, dt$, $f(x+\pi)$ is equal to :

(A) $f(x) + 2f(\pi)$ (B) $f(x) + 2f\left(\frac{\pi}{2}\right)$ (C) $f(x) + 4f\left(\frac{\pi}{4}\right)$ (D) $2f(x)$

Ans. (B)

Solution:

$f(x+\pi) = \int_0^{x+\pi} (2\cos^2 3t + 3\sin^2 3t)dt = \int_0^x (2\cos^2 3t + 3\sin^2 3t)dt + \int_x^{x+\pi} (2\cos^2 3t + 3\sin^2 3t)dt$

$t = x + y$

$= f(x) + \int_0^{\pi} (2\cos^2 3y + 3\sin^2 3y)\, dy = f(x) + 2\int_0^{\pi/2} (2\cos^2 3y + 3\sin^2 3y)dy = f(x) + 2f\left(\frac{\pi}{2}\right)$

Illustration 83:

Let $f(x) = \int_0^x \frac{dt}{\sqrt{1+t^3}}$ and $g(x)$ be the inverse of $f(x)$, then which one of the following holds good?

(A) $2g'' = g^2$ (B) $2g'' = 3g^2$ (C) $3g'' = 2g^2$ (D) $3g'' = g^2$

Ans. (B)

Solution:

$f'(x) = \frac{1}{\sqrt{1+x^3}}$ & $f''(x) = \frac{-3x^2}{2(1+x^3)^{3/2}}$. Also $g(f(x)) = x$

$\Rightarrow g''(f(x)) = -\frac{f''(x)}{(f'(x))^3} = \frac{3x^2}{2(1+x^3)^{3/2}}.(1+x^3)^{3/2} \Rightarrow g''(f(x)) = \frac{3x^2}{2}$

Let $f(x) = t \Rightarrow g(t) = x$

so, $2g'' = 3g^2$.

Illustration 84:

DIRECTIONS :

Each question has 4 choices (1), (2), (3) and (4) out of which ONLY ONE is correct.

(A) Both the statements are true.

(B) Statement-I is true, but Statement-II is false.

(C) Statement-I is false, but Statement-II is true.

(D) Both the statements are false.

1. **STATEMENT-1 : If {.} represents fractional part function, then** $\int_0^{5.5} \{x\}dx = \frac{21}{8}$

   STATEMENT-2 : If [.] and {.} represent greatest integer and frational part functions respectively, then

   $\int_0^t \{x\}\, dx = \frac{[t]}{2} + \frac{\{t\}^2}{2}$

2. **STATEMENT-1 :** $\int_0^{10\pi} |\cos x|\, dx = 20$

   STATEMENT-2 : $\int_a^b f(x)dx \ge 0$, then $f(x) \ge 0$, $\forall\, x \in (a, b)$

3. **STATEMENT-1 :** $\int_0^{2\pi} \tan^2 x\, dx = 4\int_0^{\pi/2} \tan^2 x\, dx$

STATEMENT-2 : $\int_0^{nT} f(x)dx = n\int_0^{T} f(x)\, dx$ **, where n is an integer and T is a period of $f(x)$**

Solution:

1. **Ans. (A)**

$\int_0^t \{x\}\, dx = \int_0^{[t]} \{x\}\, dx + \int_{[t]}^{t} \{x\}\, dx = [t]\int_0^1 x\, dx + \int_0^{\{t\}} x\, dx = \frac{[t]}{2} + \frac{\{t\}^2}{2}$

$\therefore$ statement-2 is true.

$\int_0^{5.5} \{x\}dx = \frac{5}{2} + \frac{(.5)^2}{2} = \frac{21}{8}$

$\therefore$ statement-1 is true and is explained by statement-2.

2. **Ans. (B)**

Statement -1 : $10\int_0^{\pi} |\cos x|\, dx = 10\left[\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx\right] = 10\,.\,2 = 20$

Statement - 2 : $\int_0^{3\pi/4} \cos x\, dx = \sin x\Big|_0^{3\pi/4} = \frac{1}{\sqrt{2}}$

but $\cos x < 0, \forall\, x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

$\therefore$ statement-2 is false.

3. **Ans. (A)**

$\int_0^{2\pi} \tan^2 x\, dx = 2\int_0^{\pi} \tan^2 x\, dx$

$= 2\left[\int_0^{\pi/2} \tan^2 x dx + \int_0^{\pi/2} \tan^2(\pi - x)dx\right] = 4\int_0^{\pi/2} \tan^2 x dx$

$\therefore$ Statement 1 is true

statement-2 $\int_0^{nT} f(x)dx = \int_0^{T} f(x)dx + \int_T^{2T} f(x)dx + ... + \int_{(n-1)T}^{nT} f(x)dx$

$= \int_0^{T} f(x)dx + \int_0^{T} f(x+T)dx + ... + \int_0^{T} f(x+(n-1)T)dx$

$= \int_0^{T} f(x)\, dx + \int_0^{T} f(x)dx + ... + \int_0^{T} f(x)dx$ $\quad (\because\ f$ has a period $T)$

$= n\int_0^{T} f(x)dx$

Illustration 85:

If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non – zero x, then $\int_{\sin\theta}^{\text{cosec}\theta} f(x)dx$ equals to :

(A) $\sin\theta + \text{cosec}\theta$ (B) $\sin^2\theta$ (C) $\text{cosec}^2\theta$ (D) none of these

Ans. (D)

Solution:

$f\left(\frac{1}{x}\right) + x^2 f(x) = 0 \Rightarrow f(x) = -\frac{1}{x^2}\ f\left(\frac{1}{x}\right)$

$\Rightarrow I = \int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)dx\ = \int_{\sin\theta}^{\operatorname{cosec}\theta} -\frac{1}{x^2} f\left(\frac{1}{x}\right)dx$

$\frac{1}{x} = t \Rightarrow -\frac{1}{x}dx\ = \ dt \Rightarrow I = \int_{\operatorname{cosec}\theta}^{\sin\theta} f(t)dt\ \Rightarrow I\ = -\int_{\sin\theta}^{\operatorname{cosec}\theta} f(t)dt\ = -I \Rightarrow 2I\ = \ 0 \Rightarrow I\ = \ 0$

Illustration 86:

If $\frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} = 0$, where C_0, C_1, C_2 are all real, the equation $C_2x^2\ +\ C_1x\ +\ C_0 = 0$ has:

(A) atleast one root in $(0, 1)$

(B) one root in $(1, 2)$ & other in $(3, 4)$

(C) one root in $(-1, 1)$ & the other in $(-5, -2)$

(D) both roots imaginary

Ans. (A)

Solution:

$\int_0^1 (C_2x^2 + C_1x + C_0)\,dx$

$= \left[\frac{C_2x^3}{3} + \frac{C_1x^2}{2} + C_0x\right]_0^1 = \frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} = 0$ (given)

$\Rightarrow$ graph $y = C_2x^2 + C_1x + C_0$ crosses x-axis atleast once.

$\Rightarrow$ at least one root of the equation $C_2x^2\ +\ C_1x\ +\ C_0 = 0$ is present in $(0, 1)$.

Illustration 87:

If $I_1 = \int_0^{\pi} x\, f(\sin^3 x + \cos^2 x)dx$ and $I_2 = \pi\int_0^{\pi/2} f(\sin^3 x + \cos^2 x)dx$ then

(A) $I_1 = I_2$ (B) $I_1 + I_2 = 0$ (C) $I_1 = 2I_2$ (D) $2I_1 = I_2$

Ans. (A)

Solution:

$I_1 = \int_0^{\pi} x\ \ f(\sin^3 x + \cos^2 x)dx$ ...(1)

$= \int_0^{\pi} (\pi - x) f(\sin^3(\pi - x) + \cos^2(\pi - x)dx$ ...(2)

(1) + (2)

$2I_1 = \pi \int_0^{\pi} f(\sin^3 x + \cos^2 x)dx$

$2I_1 = 2\pi \int_0^{\pi/2} f(\sin^3 x + \cos^2 x)dx$

$\therefore\ I_1 = \pi \int_0^{\pi/2} f(\sin^3 x + \cos^2 x)dx$

Illustration 88:

$\int_{-1}^{1} \frac{e^x+1}{e^x-1} dx$ equals -

(A) $\ell n(e^x + 1)$ (B) $\ell n(e^x - 1)$ (C) 1 (D) 0

Ans. (D)

Solution:

Using properties ...... $\int_{-1}^{1} \frac{e^x+1}{e^x-1} dx = 0$

Illustration 89:

If $[x]$ stands for the greatest integer function, the value of $\int_{4}^{10} \frac{[x^2]}{[x^2-28x+196]+[x^2]} dx$ is :

(A) 0 (B) 1 (C) 3 (D) $\frac{3}{2}$

Ans. (C)

Solution:

$$I = \int_{4}^{10} \frac{[(x-14)^2]}{[x^2]+[(x-14)^2]} dx$$

$$\Rightarrow 2I = \int_{4}^{10} dx = 6 \Rightarrow I = 3$$

Illustration 90:

The tangent to the graph of the function $y = f(x)$ at the point with abscissa $x = 1$ form an angle of $\pi/6$ and at the point $x = 2$, an angle of $\pi/3$ and at the point $x = 3$, an angle of $\pi/4$ with positive x-axis.

The value of $\int_{1}^{3} f'(x) f''(x) dx + \int_{2}^{3} f''(x) dx$ ($f''(x)$ is supposed to be continuous) is :

(A) $\frac{4\sqrt{3}-1}{3\sqrt{3}}$ (B) $\frac{3\sqrt{3}-1}{2}$ (C) $\frac{4-\sqrt{3}}{3}$ (D) $\frac{4}{3}-\sqrt{3}$

Ans. (D)

Solution:

Given,

$$f'(1) = \left(\frac{dy}{dx}\right)_{x=1} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$f'(2) = \left(\frac{dy}{dx}\right)_{x=2} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$f'(3) = \left(\frac{dy}{dx}\right)_{x=3} = \tan\frac{\pi}{4} = 1$$

Let, $I = \int_{1}^{3} f'(x).f''(x)dx + \int_{2}^{3} f''(x)dx = I_1 + I_2$

$$\therefore \quad I_1 = \int_{1}^{3} f'(x).f''(x)dx$$

$I_1 = f'(x).f'(x)\Big|_1^3 - \int_1^3 f''(x)\ f'(x)\,dx$

$2I_1 = \{f'(3)\}^2 - \{f'(1)\}^2$

$2I_1 = 1 - \frac{1}{3}$

$I_1 = \frac{1}{3}$

and, $I_2 = \int_2^3 f''(x)dx = f'(x)\Big|_2^3 = f'(3) - f'(2) = 1 - \sqrt{3}$

$\therefore\ I = I_1 + I_2 = \frac{1}{3} + 1 - \sqrt{3} = \frac{4}{3} - \sqrt{3}$

Illustration 91:

Let $A = \int_0^1 \frac{e^t}{1+t}dt$, then $\int_{a-1}^{a} \frac{e^{-t}}{t-a-1}dt$ has the value :

(A) Ae^{-a} (B) $-Ae^{-a}$ (C) $-ae^{-a}$ (D) Ae^{a}

Ans. (B)

Solution:

$A = \int_0^1 \frac{e^t dt}{t+1}$ $\therefore$ $I = \int_{a-1}^{a} \frac{e^{-t}dt}{t-a-1}$

Put $t = a - y \Rightarrow dt = -dy$

then $I = -e^{-a}\int_0^1 \frac{e^y dy}{y+1} = -Ae^{-a}$