# Definite Integration

## SOLUTIONS

### EXERCISE - O

**1.** **Ans. (B)**

$I=\int_{\pi}^{2\pi}[2\cos x]dx$ king $x \to 3\pi - x$

$I=\int_{\pi}^{2\pi}[-2\cos x]dx$

$\therefore\ 2I=\int_{\pi}^{2\pi}[2\cos x]+[-2\cos x]dx$

$\Rightarrow 2I=\int_{\pi}^{2\pi}(-1)dx \Rightarrow I=-\frac{\pi}{2}$

$\left(\because [x]+[-x]=\begin{cases}0 & x\in \text{Integer}\\ -1 & x\notin \text{Integer}\end{cases}\right)$

**2.** **Ans. (B)**

$\because$ $\{x\}$ and $\sin 2\pi x$ both have period equals to 1.

$\therefore I = (37-19)\int_0^1(\{x\}^2+3\sin 2\pi x)dx$

$= 18\left(\int_0^1 x^2dx+3\int_0^1\sin 2\pi x\,dx\right)$

$= 18\left(\frac{1}{3}+3.0\right)=6$

**3.** **Ans. (B)**

$\int_{-1}^{1}\frac{x^3+|x|+1}{x^2+2|x|+1}dx$

$= \underbrace{\int_{-1}^{1}\frac{x^3dx}{x^2+2|x|+1}}_{\text{Odd function}}+\underbrace{\int_{-1}^{1}\frac{|x|+1}{x^2+2|x|+1}dx}_{\text{Even function}}$

$= 0 + 2\int_0^1\frac{x+1}{(x+1)^2}dx$

$= 2[\ell n(x+1)]_0^1=2\ell n2$

$\Rightarrow a = 2, b = 0$

**4.** **Ans. (B)**

$I=\int_2^4[x(3-x)(4+x)(6-x)(10-x)]dx+\int_2^4\sin x\,dx$

$\Rightarrow I=-\int_2^4\underbrace{(6-x)(3-x)(10-x)x(4+x)}_{\text{King prop.}}+[-\cos x]_2^4$

$\Rightarrow 2I = 2[-\cos 4 + \cos 2] \Rightarrow I = \cos 2 - \cos 4$

**5. Ans. (D)**

$$I = \int_0^{\frac{\pi}{2}} \sin x \sin 2x \sin 3x \, dx$$

$$= -\int_0^{\frac{\pi}{2}} \cos x \sin 2x \cos 3x \, dx \qquad \text{(king prop.)}$$

$$\Rightarrow 2I = \int_0^{\pi/2} \sin 2x(\sin 3x \sin x - \cos 3x \cos x)dx$$

$$= -\int_0^{\pi/2} \sin 2x \cos 4x \, dx$$

$$= -\frac{1}{2}\int_0^{\pi/2} [\sin 6x - \sin 2x]dx$$

$$= \frac{1}{2}\left[\frac{\cos 6x}{6} - \frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(-\frac{1}{6}+\frac{1}{2}\right)-\left(\frac{1}{6}-\frac{1}{2}\right)\right]$$

$$= \frac{1}{3} \Rightarrow I = \frac{1}{6}$$

**6. Ans. (A)**

$$I_1 = \int_{-\pi/4}^{\pi/4} \ell n(\sin x + \cos x)dx$$

$$= \int_{-\pi/4}^{\pi/4} \ell n(-\sin x + \cos x)dx \qquad \text{(using prop.)}$$

$$2I_1 = \int_{-\pi/4}^{\pi/4} \ell n(\cos^2 x - \sin^2 x)dx = \int_{-\pi/4}^{\pi/4} \ell n(\cos 2x)dx$$

$$= 2\int_0^{\pi/4} \ell n(\cos 2x)dx \qquad \text{(even function)}$$

put $2x = t$

$$\Rightarrow I_1 = \frac{1}{2}\int_0^{\pi/2} \ell n(\cos t)\, dt = \frac{1}{2}\int_0^{\pi/2} \ell n(\sin t)\, dt \qquad \text{(king)}$$

$$\Rightarrow I_1 = \frac{I}{2}$$

**7. Ans. (D)**

$$I_1 = \int_{1-k}^{k} x.f(x(1-x))dx$$

$$= \int_{1-k}^{k} (1-x)f((1-x)x)dx \qquad \text{(king prop.)}$$

$$\Rightarrow 2I_1 = \int_{1-k}^{k} f(x(1-x))dx = I_2$$

$$\Rightarrow \frac{I_2}{I_1} = 2$$

**8. Ans. (D)**

$\ell nx = t \Rightarrow x = e^t$

$\therefore f'(t) = \begin{cases} 1 \; ; & 0<e^t \le 1 \\ e^t \; ; & e^t > 1 \end{cases} \Rightarrow f'(t) = \begin{cases} 1 \; ; & t\in(-\infty,0] \\ e^t \; ; & t\in(0,\infty) \end{cases}$

$\therefore$ We observe, $f(x)$ is derivable at $x = 0$

$\Rightarrow$ continuous also at $x = 0$

$\therefore f(t) = \begin{cases} t+c_1 \; ; & t\in(-\infty,0] \\ e^t + c_2 \; ; & t\in(0,\infty) \end{cases}$

$\because \; f(0) = 0 \Rightarrow o + c_1 = 0 \Rightarrow c_1 = 0$

Also, f(x) is continuous at x = 0

$\therefore 0 + c_1 = e^\circ + c^2 \Rightarrow c_2 = -1$

$\therefore f(t) = \begin{cases} t \; ; & t\in(-\infty,0] \\ e^t - 1 \; ; & t\in(0,\infty) \end{cases}$

**9. Ans. (A)**

Put $x = \frac{1}{t} \Rightarrow \; dx = -\frac{1}{t^2}dt$

$$\Rightarrow I = \int_\infty^0 f\left(t+\frac{1}{t}\right)\cdot\frac{\ell n\left(\frac{1}{t}\right)}{\frac{1}{t}}\cdot\left(-\frac{1}{t^2}\right)dt$$

$$= -\int_0^\infty f\left(t+\frac{1}{t}\right)\cdot\frac{\ell nt}{t}dt = -I$$

$\Rightarrow I = -I$

$\Rightarrow I = 0$

**10. Ans. (C)**

differentiate both the sides w.r.t. '$x$'

$f(x) = 1 + 0 - x^2 f(x)$

$$\Rightarrow \; f(x) = \frac{1}{1+x^2} \Rightarrow \int_{-1}^{1}\frac{1}{1+x^2}dx = 2[\tan^{-1}x]_0^1 = \frac{\pi}{2}$$

**11. Ans. (B)**

$$\frac{dx}{dy} = \frac{1}{\sqrt{1+y^2}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{1+y^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right)\cdot\frac{dy}{dx}$$

$$= \frac{2y}{2\sqrt{1+y^2}}\cdot\sqrt{1+y^2} = y$$

**12. Ans. (D)**

$$\sqrt{5x-6-x^2}+\frac{\pi}{2}\cdot x > x\cdot 2\cdot\int_0^{\pi/2}\sin^2 x\,dx \qquad \text{(Queen property)}$$

$$\Rightarrow \sqrt{5x-6-x^2}+\frac{\pi}{2}x > x\cdot 2\cdot\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) \qquad \text{(Walli's formula)}$$

$\Rightarrow \sqrt{5x-6-x^2} > 0$ which is true in domain except $x = 2, 3$

$\Rightarrow 5x - 6 - x^2 \geq 0$ (domain)

$\Rightarrow (x-2)(x-3) \leq 0$ but $x \neq 2, 3$

$\Rightarrow x \in (2, 3)$

**13. Ans. (A)**

$$\lim_{n\to\infty}\sum_{r=1}^{3n}\frac{1}{\sqrt{n}\sqrt{n+r}}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{3n}\frac{1}{n\cdot\sqrt{1+\frac{r}{n}}}$$

$$= \int_0^3\frac{1}{\sqrt{1+x}}dx$$

$$= \left(2\sqrt{1+x}\right)_0^3$$

$= 4 - 2 = 2$

**14. Ans. (B,C,D)**

$$f(x)=\begin{cases}x+1 & ; \quad 0\leq x\leq 1\\ 2x^2-6x+6 & ; \quad 1<x\leq 2\end{cases}; \; g(t)=\int_{t-1}^{t} f(x)dx \quad for\, t\in[1,2]$$

$\Rightarrow f(1^-) = 2 = f(1)$

$f(1^+) = 2 - 6 + 6 = 2$

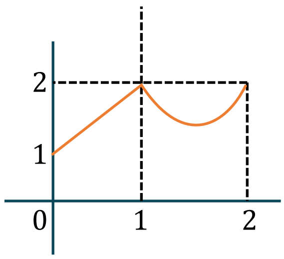
R.H.S = L.H.S = $f(1)$

So continuous

$$\Rightarrow f'(x)=\begin{cases}1 & ; \quad 0\leq x<1\\ 4x-6 & ; \quad 1<x\leq 2\end{cases}$$

$$\left.\begin{aligned}f'(1^+)&=-2\\ f'(1^-)&=1\end{aligned}\right\} \text{So non differentiable}$$

$\Rightarrow g(t) = \int_{t-1}^{t} f(x)dx$




$g(t) = \int_{(t-1)}^{1} f(x)dx + \int_{1}^{t} f(x)dx$

$g'(t) = f(t) - f(t-1) = 0$

$f(t) = f(t-1)$

$2t^2 - 6t + 6 = (t-1) + 1$

$2t^2 - 6t + 6 = t$

$2t^2 - 7t + 6 = 0$

$t = 2, \frac{3}{2}$

$g'\left(\frac{3}{2}\right) = f\left(\frac{3}{2}\right) - f\left(\frac{1}{2}\right)$

$= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 6 - \left(\frac{1}{2} + 1\right)$

$= \frac{9}{2} - 9 + 6 - \frac{3}{2} = 0$

$g'(2) = f(2) - f(1)$

$= 2 - 2 = 0$

$g'(t)$ vanishes for $t = \frac{3}{2}, 2$

$g''(t) = f'(t) - f'(t-1)$

$g''(3/2) = \left(4\left(\frac{3}{2}\right) - 6\right) - (1)$

$= 6 - 6 - 1 = -1$

$g(t)$ is maximum at $t = \frac{3}{2}$

$\Rightarrow g(t) = \int_{t-1}^{t} f(x)dx$

$g(1) = \int_0^1 f(x)dx = \int_0^1 (x+1)dx$

$= \left(\frac{x^2}{2} + x\right)_0^1 = \frac{1}{2} + 1 = \frac{3}{2}$

$g(2) = \int_1^2 (2x^2 - 6x + 6)dx$

$= \left[\frac{2x^3}{3} - \frac{6x^2}{2} + 6x\right]_1^2$

$= \frac{10}{6}$

So minimum at $t = 1$

**15.** **Ans. (A,B,C)**

**(A)** Cont. on $(0, \pi)$

**(B)** $g(x) = \int_0^x \underbrace{\underset{cont.}{t} \; \underbrace{\sin(1/t)}_{cont.}}_{cont.} dt$

$\therefore$ $g(x)$ will be cont. function by fundamental theorem of calculus.

**(C)** $h\left(\left(\frac{3\pi}{4}\right)^+\right) = 2\sin\left(\frac{2}{9} \times \frac{3\pi}{4}\right) = 2\sin\left(\frac{\pi}{6}\right) = 1$

$h\left(\left(\frac{3\pi}{4}\right)^-\right) = 1$ $\therefore$ continuous

**(D)** $\ell\left(\left(\frac{\pi}{2}\right)^{-}\right)=\frac{\pi}{2}\sin\frac{\pi}{2}=\frac{\pi}{2}$

$\ell\left(\left(\frac{\pi}{2}\right)^{+}\right)=\frac{\pi}{2}\sin\left(\frac{\pi}{2}+\pi\right)=-\frac{\pi}{2}$

$\therefore\ \ell(x)$ is $DC$ at $x=\frac{\pi}{2}$

**16. Ans. (B,C,D)**

$F(x) = \int_{\sin x}^{\cos x} e^{(1+\sin^{-1}t)^2}dt\ \left[0,\frac{\pi}{2}\right]$

$F'(x)=e^{(1+\sin^{-1}\cos x)^2}(-\sin x)-e^{(1+\sin^{-1}\sin x)^2}\cos x$

$= -\left[e^{\left(1+\sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right)\right)^2}(\sin x)+e^{(1+x)^2}\cos x\right]$

$=-\left[\underbrace{e^{\left(1+\frac{\pi}{2}-x\right)^2}\sin x}_{+ve}+\underbrace{e^{(1+x)^2}\cos x}_{+ve}\right]$

$F'(x) = -ve$

$F'(0) = -e$

$F'\left(\frac{\pi}{2}\right) = -\left[e^{\left(1+\frac{\pi}{2}-\frac{\pi}{2}\right)^2}\right] = -e$

For option (D) $\rightarrow$ at $x=\frac{\pi}{4}$ both $\sin x = \cos x = \frac{1}{\sqrt{2}}$ so $F(c) = 0$

For option (C) $\rightarrow$ $F'(x)$ is always $-\ ve$

For option (B) $\rightarrow F'(x)$ is continuous in $\left(0,\frac{\pi}{2}\right)$

$F'(0) = F'(\pi/2)$

$F''(x) = 0$ for some $c\in\left(0,\frac{\pi}{2}\right)$ from Rolle's theorem.

**17. Ans. (B)**

$f(x)\cdot f'(-x) = f(-x)\cdot f'(x)$

$\Rightarrow \frac{f'(-x)}{f(-x)}=\frac{f'(x)}{f(x)}$

$\Rightarrow \int\frac{f'(-x)}{f(-x)}dx=\int\frac{f'(x)}{f(x)}dx$

$\Rightarrow -\ell n\, f(-x) + C = \ell n(f(x))$

$\Rightarrow \ell n\, f(x)\cdot f(-x) = C$

$\Rightarrow f(x)\cdot f(-x) = a$

At $x = 0 : C_1 = f(0)\cdot f(0) \Rightarrow a = 9$

$\therefore f(x)\cdot f(-x) = 9$

**18. Ans. (A)**

$$I=\int_{-51}^{51}\frac{dx}{3+f(x)} \quad ...(1)$$

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$$I=\int_{-51}^{51}\frac{dx}{3+f(-x)}$$

$$I=\int_{-51}^{51}\frac{dx}{3+\frac{9}{f(x)}}$$

$$I=\int_{-51}^{51}\frac{f(x)}{3f(x)+9}dx$$

$$\Rightarrow I=\frac{1}{3}\int_{-51}^{51}\frac{f(x)}{f(x)+3}dx$$

$$\Rightarrow I=\frac{1}{3}\int_{-51}^{51}\frac{f(x)+3-3}{f(x)+3}dx$$

$$\Rightarrow I=\frac{1}{3}\left(\int_{-51}^{51}dx-3\int_{-51}^{51}\frac{dx}{f(x)+3}\right)$$

$$\Rightarrow I=\frac{1}{3}(51-(-51))-I$$

$$\Rightarrow 2I=\frac{102}{3}\Rightarrow I=17$$

**19. Ans. (A)**

$\because f(x)\cdot f(-x)=9\ \forall\ x\in\mathbb{R}$

$\therefore f(x)$ can never be zero for any $x$ as this product is non zero.

$\therefore$ No root for $f(x)=0$

**20. Ans. (A)**

**(P)** $$\ell=\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right).....\left(1+\frac{n^2}{n^2}\right)\right]^{\frac{1}{n}}$$

$$\ell n(\ell)=\lim_{n\to\infty}\frac{1}{n}\left[\ell n\left(1+\frac{1}{n^2}\right)+\ell n\left(1+\frac{2^2}{n^2}\right).....+\ell n\left(1+\frac{n^2}{n^2}\right)\right]$$

$$\ell n(\ell)=\lim_{n\to\infty}\sum_{r=1}^{n}\ell n\left(1+\frac{r^2}{n^2}\right)\frac{1}{n}$$

$$\ell n(\ell)=\int_0^1\underset{II}{1}.\underset{I}{\ell n(1+x^2)}dx$$

$$\ell n(\ell)=\left(x\ell n(1+x^2)\right)_0^1-\int_0^1 x\frac{2x}{1+x^2}dx$$

$$=\ell n2-0-2\int_0^1\frac{x^2+1-1}{x^2+1}dx$$

$$=\ell n2-2\left[x-\tan^{-1}x\right]_0^1$$

$$\ell n\ell = \ell n2 - 2 + \frac{\pi}{2}$$

$$\ell = e^{\ell n2 + \frac{\pi}{2} - 2}$$

$$\boxed{\ell = 2e^{\frac{1}{2}(\pi-4)}}$$

**(Q)** $y = \lim\limits_{n\to\infty}\left[\frac{n!}{n^n}\right]^{\frac{1}{n}}$

$$\ell ny = \lim_{n\to\infty}\frac{1}{n}\ell n\left[\frac{n!}{n^n}\right]$$

$$\ell ny = \lim_{n\to\infty}\frac{1}{n}\ell n\left[\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\ldots\cdot\frac{n}{n}\right]$$

$$\ell ny = \lim_{n\to\infty}\frac{1}{n}\left[\ell n\frac{1}{n} + \ell n\frac{2}{n} + \ell n\frac{3}{n} + \ldots + \ell n\frac{n}{n}\right]$$

$$\ell ny = \lim_{n\to\infty}\sum_{n=1}^{n}\frac{1}{n}\cdot\ell n\frac{r}{n}$$

$$\ell ny = \int_0^1 \ell nx\,dx$$

$$\ell ny = [x\ell nx - x]_0^1$$

$$\ell ny = [0 - 1 - [0 - 0]]$$

$$y = \frac{1}{e}$$

**(R)** $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{3n}\frac{1}{n}\left(\frac{r}{n+r}\right)$

$$= \lim_{n\to\infty}\sum_{r=1}^{3n}\frac{\frac{r}{n}}{\left(1+\frac{r}{n}\right)}\left(\frac{1}{n}\right)$$

$$= \int_0^3 \frac{x}{1+x}dx$$

$$= \int_0^3 \frac{x+1-1}{x+1}dx$$

$$= \int_0^3 1dx - \int_0^3 \frac{1}{1+x}dx$$

$$= [x]_0^3 - [\ell n(x+1)]_0^3$$

$$= 3 - \ell n\, 4$$

**(S)** $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{2n}\frac{n}{n^2+r^2} = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{2n}\frac{1/n}{1+\left(\frac{r}{n}\right)^2}$

$$= \int_0^2 \frac{dx}{1+x^2}$$

## EXERCISE - S

**1.** **Ans. (2012)**

$$= \lim_{n\to\infty}\frac{\int_{-1/n}^{1/n}(2010\sin x+2012\cos x)|x|\,dx}{\frac{1}{n^2}}\quad\left[\frac{0}{0}\right]$$

$$= \lim_{n\to\infty}\frac{\left(2010\sin\left(\frac{1}{n}\right)+2012\cos\left(\frac{1}{n}\right)\right)\left(\frac{1}{n}\right)\left(-\frac{1}{n^2}\right)-\left(2010\sin\left(-\frac{1}{n}\right)+2012\cos\left(-\frac{1}{n}\right)\left(\frac{1}{n}\right)\frac{1}{n^2}\right)}{-\frac{2}{n^3}}$$

$$= \lim_{n\to\infty}\frac{-\left(2010\sin\left(\frac{1}{n}\right)+2012\cos\left(\frac{1}{n}\right)\right)-\left(-2010\sin\left(\frac{1}{n}\right)+2012\cos\left(\frac{1}{n}\right)\right)}{-2}$$

$$= \lim_{n\to\infty}\frac{-2(2012)\cos\left(\frac{1}{n}\right)}{-2}$$

$$= \lim_{n\to\infty}2012\cos\left(\frac{1}{n}\right)=2012$$

**2.** **Ans. (90)**

$$U_{10}=\int_0^{\frac{\pi}{2}}x\sin^8 x\sin^2 x\,dx$$

$$U_{10}=\int_0^{\frac{\pi}{2}}x\sin^8 x\left(1-\cos^2 x\right)dx$$

$$U_{10}=\int_0^{\frac{\pi}{2}}x\sin^8 x\,dx-\int_0^{\frac{\pi}{2}}x\sin^8 x\cos^2 x\,dx$$

$$U_{10}=U_8-\int_0^{\frac{\pi}{2}}\underset{I}{(x\cos x)}\ \underset{II}{\left(\sin^8 x\cdot\cos x\right)}dx$$

$$U_{10}=U_8-\left(x\cos x\frac{\sin^9 x}{9}\right)_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}(\cos x-x\sin x)\frac{\sin^9 x}{9}dx$$

$$U_{10}=U_8-0+\int_0^{\pi/2}\cos x\frac{\sin^9 x}{9}dx-\int_0^{\pi/2}\frac{x\sin^{10}x}{9}dx$$

$$U_{10}=U_8+\frac{\left[\sin^{10}x\right]_0^{\frac{\pi}{2}}}{90}-\frac{1}{9}U_{10}$$

$$10\frac{U_{10}}{9}=U_8+\frac{1}{90}$$

$$\boxed{\frac{100U_{10}-1}{U_8}=90}$$

**3. Ans. (8)**

$$I=2f(x)(-)\cos\left(\frac{x}{2}-1\right)\Big|_{2-\pi}^{2+\pi}+\int_{2-\pi}^{2+\pi}f'(x)(+)2\cos\left(\frac{x}{2}-1\right)dx$$

$$=0+2\left[2f'(x)\sin\left(\frac{x}{2}-1\right)\Big|_{2-\pi}^{2+\pi}-\int_{2-\pi}^{2+\pi}f''(x)2\sin\left(\frac{x}{2}-1\right)dx\right]$$

$$=2\left[2f'(2+\pi)+2f'(2-\pi)-2f''(x)\int_{2-\pi}^{2+\pi}\sin\left(\frac{x}{2}-1\right)dx\right]$$

$$\left(\because f(x)=ax^2+bx+c\to f''(x)=\text{constant}\right)$$

$$=4\left(f'(2+\pi)+f'(2-\pi)\right)-4f''(x)(-2)\cos\left(\frac{x}{2}-1\right)\Big|_{2-\pi}^{2+\pi}$$

$$=4\left(f'(2+\pi)+f'(2-\pi)\right)$$

If $f(x)=ax^2+bx+c \Rightarrow f'(x)=2ax+b$

$\to f'(2+\pi)=2a(2+\pi)+b$

$\to f'(2-\pi)=2a(2-\pi)+b$

$f'(2+\pi)+f'(2-\pi)=8a+2b$

$=2(4a+b)$

$f'(2)=1\Rightarrow 4a+b=1$

$\Rightarrow f'(2+\pi)+f'(2-\pi)=2(1)=2$

$\Rightarrow I=4\left[f'(2+\pi)+f'(2-\pi)\right]$

$=4\ (2)$

$=8$

**4. Ans. (5250)**

$$\int_0^{\pi/2}\sin^3 x+a^3\cos^3 x+3a\sin^2 x\cos x+3a^2\sin x\cos^2 x\,dx-\frac{4a}{\pi-2}\int_0^{\pi/2}\frac{x\cdot\cos x\cdot dx}{I\qquad II}=2$$

[ $\sin 3x = 3\sin x - 4\sin^3 x$ & $\cos 3x = 4\cos^3 x - 3\cos x$]

$$\Rightarrow\int_0^{\pi/2}\left(\frac{3\sin x-\sin 3x}{4}+\frac{a^3(\cos 3x+3\cos x)}{4}+3a\sin^2 x\cos x+3a^2\sin x\cos^2 x\right)dx$$

$$-\frac{4a}{\pi-2}\left[(x\sin x)_0^{\pi/2}-\int_0^{\pi/2}\sin x\,dx\right]=2$$

$$\Rightarrow\left[-\frac{3\cos x}{4}+\frac{\cos 3x}{12}+\frac{a^3\sin 3x}{12}+\frac{3a^3\sin x}{4}+a\sin^3 x-a^2\cos^3 x\right]_0^{\pi/2}-\frac{4a}{\pi-2}\left[\frac{\pi}{2}+(\cos x)_0^{\pi/2}\right]=2$$

$$\Rightarrow\left[0+0-\frac{a^3}{12}+\frac{3a^3}{4}+a\right]-\left[-\frac{3}{4}+\frac{1}{12}-a^2\right]-\frac{4a}{\pi-2}\left(\frac{\pi}{2}-1\right)=2$$

$$\Rightarrow\frac{8a^3}{12}+a+\frac{8}{12}+a^2-2a=2$$

$$\Rightarrow\frac{2a^3}{3}+a^2-a+\frac{2}{3}-2=0$$

$\Rightarrow 2a^3 + 3a^2 - 3a - 4 = 0$

$a_1 + a_2 + a_2 = -3/2$ & $a_1a_2 + a_2a_3 + a_3a_1 = -3/2$

$a_1^2 + a_2^2 + a_3^2 = (-3/2)^2 - 2(-3/2) = 21/4$

$1000 \times \frac{21}{4} = 250 \times 21 = \boxed{5250}$

**5. Ans. (125)**

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$

$$= \int_1^{1/2} \frac{\left(\frac{1}{t^2}-1\right)\left(-\frac{1}{t^2}\right)dt}{\left(\frac{1}{t^3}\right)\sqrt{\frac{2}{t^4}-\frac{2}{t^2}+1}} = \int_1^{1/2} \frac{\frac{(1-t^2)(-1)}{t^4}}{\frac{1}{t^3}\frac{\sqrt{t^4-2t^2+2}}{t^2}}dt$$

$$= \int_1^{1/2} \frac{(t^2-1)t}{\sqrt{t^4-2t^2+2}}dt$$

$$\begin{bmatrix} Put\, t^4 - 2t^2 + 2 = x^2 \\ (4t^3 - 4t)dt = 2x\,dx \\ (t^2-1)t\,dt = \frac{1}{2}x\,dx \end{bmatrix}$$

$$= = \int_1^{5/4} \frac{xdx}{2x}$$

$$= \frac{1}{2}\left[\frac{5}{4}-1\right]$$

$$= \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

$$\therefore \frac{U}{V} = \frac{1}{8}$$

$$\therefore \frac{1000U}{v} = \frac{1000}{8} = \boxed{125}$$

**6. Ans. (153)**

$$\int_0^{\pi} \sqrt{1+1+1+2(\cos x + \cos x + \cos 2x)}\ dx = \int_0^{\pi} \sqrt{3+2(2\cos x + \cos 2x)}\ dx$$

$$= \int_0^{\pi} \sqrt{3+4\cos x + 2\cos 2x}dx$$

$$= \int_0^{\pi} \sqrt{3+4\cos x + 4\cos^2 x - 2}\ dx$$

$$= \int_0^{\pi} \sqrt{4\cos^2 x + 4\cos x + 1}dx$$

$$= \int_0^{\pi} |2\cos x + 1|\,dx$$

$$= \int_0^{2\pi/3} 2\cos x + 1\, dx + \int_{\frac{2\pi}{3}}^{\pi} -(2\cos x + 1)\, dx$$

$$= (2\sin x + x)_0^{\frac{2\pi}{3}} - (2\sin x + x)_{\frac{2\pi}{3}}^{\pi}$$

$$= \left(2\times\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right) - \left(\pi - 2\times\frac{\sqrt{3}}{2} - \frac{2\pi}{3}\right)$$

$$= \sqrt{3} + \frac{2\pi}{3} - \pi + \left(\sqrt{3} + \frac{2\pi}{3}\right)$$

$$= 2\sqrt{3} + \frac{4\pi}{3} - \pi$$

$$= 2\sqrt{3} + \frac{\pi}{3}$$

$$= \sqrt{12} + \pi/3$$

$\Rightarrow k = 3$ & $w = 12$

$\therefore \boxed{k^2 + w^2 = 153}$

**7. Ans. (5)**

Let $\int_1^2 f(x)dx = 1$

$$I + 1 = \int_0^1 f(x)dx + \int_1^2 f(x)dx$$

$$= \int_0^2 f(x)dx$$

Put $x = 2t \Rightarrow dx = 2dt$

$$= \int_0^1 f(2t)dt \cdot 2$$

$$= 2\int_0^1 3f(t)dt \qquad \because \{f(2t) = 3f(t)\}$$

$$I + 1 = 6\int_0^1 f(t)dt = 6$$

$$I = 5 \Rightarrow \boxed{\int_1^2 f(x)dx = 5}$$

**8. Ans. (9)**

$f(x) = x + \sin x$

$f'(x) = 1 + \cos x \geq 0$

$$I = \int_{\pi}^{2\pi} \left(f^{-1}(x) + \sin x\right)dx$$

$$I = \int_{\pi}^{2\pi} f^{-1}(x)dx + \int_{\pi}^{2\pi} \sin x\, dx$$

$$I=\left[(2\pi\times 2\pi)-(\pi\times\pi)\right]-\int_{\pi}^{2\pi} f(x)dx+\left[-\cos x\right]_{\pi}^{2\pi}$$

$$I = 3\pi^2 - \int_{\pi}^{2\pi}(x+\sin x)dx+\left[-\cos x\right]_{\pi}^{2\pi}$$

$$I=3\pi^2-\frac{1}{2}\left[(2\pi)^2-(\pi)^2\right]-\left[-\cos x\right]_{\pi}^{2\pi}+\left[-\cos x\right]_{\pi}^{2\pi}$$

$$I=3\pi^2-\frac{3\pi^2}{2}$$

$$I=\frac{3\pi^2}{2}$$

So, $\left[\frac{2I}{3}\right]=\left[\pi^2\right]=9$

**9.** **Ans. (0.5)**

Put $nx = t$ we get

$$= \lim_{n\to\infty}\frac{\frac{1}{n}\int_{n/n+1}^{1}\tan^{-1}(t)dt}{\frac{1}{n}\int_{n/n+1}^{1}\sin^{-1}(t)dt} = \lim_{n\to\infty}\frac{\tan^{-1}\left(\frac{n}{n+1}\right)}{\sin^{-1}\left(\frac{n}{n+1}\right)} = \frac{\pi/4}{\pi/2} = \frac{1}{2}$$

**10.** **Ans. (0.66 or 0.67)**

$$I_1=\int_{-1}^{1}\{x\}\{x^2\}+\{x^2\}\{x^3\}dx$$

$$I_1=\int_{-1}^{1}\{x^2\}\left[\{x\}+\{x^3\}\right]dx \qquad ...(1)$$

$$I_1=\int_{-1}^{1}\left\{(-x)^2\right\}\left[\{-x\}+\left\{(-x)^3\right\}\right]dx \quad \text{[Apply king]}$$

$$I_1=\int_{-1}^{1}\{x^2\}\left[1-\{x\}+1-\{x^3\}\right]dx \qquad ...(2)$$

(1) + (2)

$$\Rightarrow 2I_1=\int_{-1}^{1}\{x^2\}(2)dx$$

$$\Rightarrow I_1=\int_{-1}^{1}\{x^2\}dx$$

$$\Rightarrow I_1=2\int_{0}^{1}x^2\,dx$$

$$\Rightarrow I_1=2\left(\frac{x^3}{3}\right)_0^1$$

$$\boxed{I_1=\frac{2}{3}}$$

## EXERCISE - JEE (Main) PYQ

**1.** **Ans. (2)**

$$\int_0^x f(t)dt = x^2 + \int_x^1 t^2 f(t)dt \qquad f'\left(\frac{1}{2}\right) = ?$$

Differentiate w.r.t. '$x$'

$f(x) = 2x + 0 - x^2 f(x)$

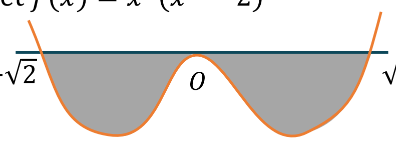
$$f(x) = \frac{2x}{1+x^2} \Rightarrow f'(x) = \frac{(1+x^2)2 - 2x(2x)}{(1+x^2)^2}$$

$$f'(x) = \frac{2x^2 - 4x^2 + 2}{(1+x^2)^2}$$

$$f'\left(\frac{1}{2}\right) = \frac{2 - 2\left(\frac{1}{4}\right)}{\left(1+\frac{1}{4}\right)^2} = \frac{\left(\frac{3}{2}\right)}{\frac{25}{16}} = \frac{48}{50} = \frac{24}{25}$$

**2.** **Ans. (2)**

Let $f(x) = x^2(x^2 - 2)$




As long as $f(x)$ lie below the $x$-axis, definite integral will remain negative, so correct value of $(a, b)$ is $(-\sqrt{2}, \sqrt{2})$ for minimum of $I$.

**3.** **Ans. (3)**

$$4\alpha\left[\int_{-1}^{0} e^{\alpha x}dx + \int_0^2 e^{-\alpha x}dx\right] = 5$$

$$\Rightarrow 4\alpha\left(\left[\frac{e^{\alpha x}}{\alpha}\right]_{-1}^{0} + \left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^2\right) = 5$$

$$\Rightarrow 4e^{-2\alpha} + 4e^{-\alpha} - 3 = 0$$

Let $e^{-\alpha} = t, 4t^2 + 4t - 3 = 0, t = \frac{1}{2}, \frac{-3}{2}$ (Rejected)

$$e^{-\alpha} = \frac{1}{2} \Rightarrow \alpha = \ell n2$$

**4.** **Ans. (1)**

$$f(x) = \frac{1}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$$

$$f'(x) = \frac{-6(x-1)(x-2)}{2(2x^3 - 9x^2 + 12x + 4)^{3/2}}$$

$\therefore f(x)$ is decreasing in $(1, 2)$

$$f(1) = \frac{1}{3};\ f(2) = \frac{1}{\sqrt{8}}$$

$$\frac{1}{3} < I < \frac{1}{\sqrt{8}} \Rightarrow I^2 \in \left(\frac{1}{9}, \frac{1}{8}\right)$$

**5. Ans. (2)**

$a > 0$

Let $n \le a < n+1, n \in W$

$\therefore a = [a] + \{a\}$

$\Downarrow \quad \Downarrow$

G.I.F Fractional part

Here $[a] = n$

Now, $\int_0^a e^{x-[x]}\,dx = 10e - 9$

$\Rightarrow \int_0^n e^{\{x\}}\,dx + \int_n^a e^{x-[x]}\,dx = 10e - 9$

$\therefore n\int_0^1 e^x\,dx + \int_n^a e^{x-n}\,dx = 10e - 9$

$\Rightarrow n(e-1) + (e^{a-n} - 1) = 10e - 9$

$\therefore \boxed{n = 10}$ and $\{a\} = \log_e 2$

So, $a = [a] + \{a\} = (10 + \log_e 2)$

$\Rightarrow$ Option (2) is correct.

**6. Ans. (3)**

$f(x) = \int_0^1 (5 + (1-t))\,dt + \int_1^x (5 + (t-1))\,dt$

$= 6 - \frac{1}{2} + \left(4t + \frac{t^2}{2}\right)\Bigg|_1^x$

$= \frac{11}{2} + 4x + \frac{x^2}{2} - 4 - \frac{1}{2}$

$= \frac{x^2}{2} + 4x + 1$

$f(2^+) = 2 + 8 + 1 = 11$

$f(2) = f(2^-) = 5 \times 2 + 1 = 11$

$\Rightarrow$ continuous at $x = 2$

Clearly differentiable at $x = 1$

$Lf'(2) = 5$

$Rf'(2) = 6$

$\Rightarrow$ not differentiable at $x = 2$

**7. Ans. (3)**

$f(x) = \int_1^x \frac{\log_e t}{(1+t)}\,dt$

$f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ell nt}{1+t}\,dt$, let $t = \frac{1}{y}$

$= +\int_1^x \frac{\ell ny}{1+y} \cdot \frac{y}{y^2}\,dy$

$$= \int_1^x \frac{\ell ny}{y(1+y)} dy$$

hence

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{(1+t)\ell nt}{t(1+t)} dt = \int_1^x \frac{\ell nt}{t} dt$$

$$= \frac{1}{2} \ell n^2(x)$$

so $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$

**8.** **Ans. (3)**

$$\int_0^{\pi} \frac{e^{\cos x} \sin x}{(1+\cos^2 x)(e^{\cos x} + e^{-\cos x})} dx \quad ...(1)$$

Use King's property

$$I = \int_0^{\pi} \frac{e^{-\cos x} \sin x}{(1+\cos^2 x)(e^{-\cos x} + e^{\cos x})} dx \quad ...(2)$$

On adding equation (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx = 2\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$$

On putting $\cos x = t$, we get

$$I = \int_0^1 \frac{dt}{1+t^2} = \left(\tan^{-1} t\right)_0^1 = \frac{\pi}{4}$$

**9.** **Ans. (4)**

$$I = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx \quad ...(1)$$

$x \rightarrow -x$

$$I = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{-x + \frac{\pi}{4}}{2 - \cos 2x} dx \quad ...(2)$$

(1) + (2)

$$2I = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\frac{\pi}{2}}{2 - \cos 2x} dx$$

$$I = \frac{\pi}{4}.2\int_0^{\frac{\pi}{4}} \frac{(1+\tan^2 x)dx}{2(1+\tan^2 x) - (1-\tan^2 x)}$$

$$I = \frac{\pi}{4}\int_0^1 \frac{dt}{3t^2+1}$$

$$\Rightarrow \quad I = \frac{\pi}{2\sqrt{3}} \tan^{-1}\sqrt{3}$$

$$I = \frac{\pi^2}{6\sqrt{3}}$$

**10. Ans. (1)**

$$I=\int_0^1\left[2x-|3x^2-3x-2x+2|+1\right]dx$$

$$I=\int_0^1\left[2x-|(3x-2)(x-1)|\right]dx+\int_0^1 1\,dx \quad I=\int_0^{2/3}\left[\left(2x-\left(3x^2-5x+2\right)\right)\right]dx+\int_{2/3}^{1}\left(2x+\left(3x^2-5x+2\right)\right)dx+1$$

$$I=\int_0^{2/3}\left[-3x^2+7x-2\right]dx+\int_{2/3}^{1}\left(3x^2-3x+2\right)dx+1$$

$$\int_0^{\alpha}(-2)dx+\int_{\alpha}^{1/3}(-1)dx+\int_{1/3}^{\beta}0\,dx+\int_{\beta}^{2/3}1.dx$$

$$=-2\alpha-\left(\frac{1}{3}-\alpha\right)+\frac{2}{3}-\beta=-\alpha-\beta+\frac{1}{3}$$

When $x\in\left(\frac{2}{3},1\right)$

$$3x^2-3x+2\in\left(\frac{4}{3},2\right)$$

$[3x^2 - 3x + 2] = 1$

$$\therefore \int_{2/3}^{1}\left[3x^2-3x+2\right]dx=1\left(1-\frac{2}{3}\right)=\frac{1}{3}$$

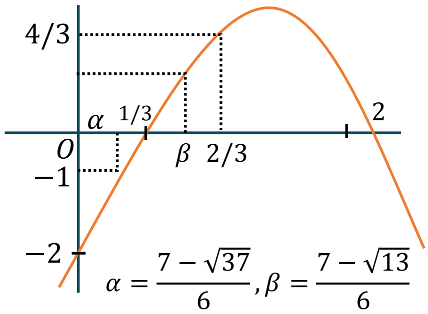
Hence $I=\left(\frac{1}{3}-(\alpha+\beta)\right)+\left(\frac{1}{3}\right)+1$

$$=\frac{5}{3}-\left(\frac{7-\sqrt{37}}{6}+\frac{7-\sqrt{13}}{6}\right)$$

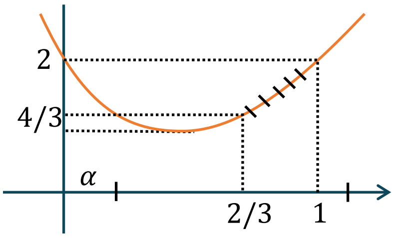
$$=\frac{-2}{3}+\frac{\sqrt{37}+\sqrt{13}}{6}$$

$$=\frac{\sqrt{37}+\sqrt{13}-4}{6}$$

$y=-3x^2+7x-2$




$y=3x^2-3x+2$




**11. Ans. (4)**

$$I=\int_{-\ln 2}^{\ln 2}e^x\left(\ln\left(e^x+\sqrt{1+e^{2x}}\right)\right)dx$$

Put $e^x=t\Rightarrow e^x dx=dt$

$$I=\int_{1/2}^{2}\ln\left(t+\sqrt{1+t^2}\right)dt$$

Applying integration by parts.

$$=\left[t\ln\left(t+\sqrt{1+t^2}\right)\right]_{\frac{1}{2}}^{2}-\int_{1/2}^{2}\frac{t}{t+\sqrt{1+t^2}}\left(1+\frac{2t}{2\sqrt{1+t^2}}\right)dt \;=2\ln\left(2+\sqrt{5}\right)-\frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right)-\int_{1/2}^{2}\frac{t}{\sqrt{1+t^2}}dt$$

$$=2\ln\left(2+\sqrt{5}\right)-\frac{1}{2}\ln\left(\frac{1+\sqrt{5}}{2}\right)-\frac{\sqrt{5}}{2}$$

$$= \ln\left(\frac{\left(2+\sqrt{5}\right)^2}{\left(\frac{\sqrt{5}+1}{2}\right)^{\frac{1}{2}}}\right) - \frac{\sqrt{5}}{2}$$

**12. Ans. (3)**

Minimum $\{x^2, \{x\}\} = x^2$ ; $x \in [0,1)$

$[x - \log_e x] = 1$ ; $x \in [1,2)$

$$\therefore f(x) = \begin{cases} e^{x^2} ; x \in [0,1) \\ e ; x \in [1,2) \end{cases}$$

$$\int_0^2 xf(x)\,dx = \int_0^1 xe^{x^2}\,dx + \int_1^2 ex\,dx$$

$$= \frac{1}{2}(e-1) + \frac{1}{2}(4-1)e$$

$$= 2e - \frac{1}{2}$$

**13. Ans. (13)**

$$I = \int_0^{\pi} \frac{5^{\cos x}(1+\cos x\cos 3x+\cos^2 x+\cos^3 x\cos 3x)}{1+5^{\cos x}}dx$$

$$I = \int_0^{\pi} \frac{5^{-\cos x}(1+\cos x\cos 3x+\cos^2 x+\cos^3 x\cos 3x)}{1+5^{-\cos x}}dx$$

$$2I = \int_0^{\pi} (1+\cos x\cos 3x+\cos^2 x+\cos^3 x\cos 3x)dx$$

$$\not{2}I = \not{2}\int_0^{\frac{\pi}{2}} (1+\cos x\cos 3x+\cos^2 x+\cos^3 x\cos 3x)dx$$

$$I = \int_0^{\frac{\pi}{2}} (1+\sin x(-\sin 3x)+\sin^2 x-\sin^3 x\sin 3x)dx$$

$$2I = \int_0^{\frac{\pi}{2}} (3+\cos 4x+\cos^3 x\cos 3x-\sin^3 x\sin 3x)dx$$

$$2I = \int_0^{\frac{\pi}{2}} 3+\cos 4x+\left(\frac{\cos 3x+3\cos x}{4}\right)\cos 3x-\sin 3x\left(\frac{3\sin x-\sin 3x}{4}\right)dx$$

$$2I = \int_0^{\frac{\pi}{2}} \left(3+\cos 4x+\frac{1}{4}+\frac{3}{4}\cos 4x\right)dx$$

$$2I = \frac{13}{4}\times\frac{\pi}{2}+\frac{7}{4}\left(\frac{\sin 4x}{4}\right)_0^{\frac{\pi}{2}} \Rightarrow I = \frac{13\pi}{16}$$

**14. Ans. (1)**

$S_1 : \lim\limits_{n\to\infty} \frac{n(n+1)}{n^2} = 1 \Rightarrow$ True

$S_2 : \lim\limits_{n\to\infty} \frac{1}{n^{16}}\left(\sum r^{15}\right) = \lim\limits_{n\to\infty} \frac{1}{n} \sum \left(\frac{r}{n}\right)^{15}$

$= \int_0^1 x^{15} dx = \frac{1}{16} \Rightarrow$ True

**15. Ans. (1)**

For $x \le 0$

$$f(x) = \int_0^2 e^{t-x} dt = e^{-x}\left(e^2 - 1\right)$$

For $0 < x < 2$

$$f(x) = \int_0^x e^{x-t} dt + \int_x^2 e^{t-x} dt = e^x + e^{2-x} - 2$$

For $x \ge 2$

$$f(x) = \int_0^2 e^{x-t} dt = e^{x-2}(e^2 - 1)$$

For $x \le 0$, $f(x)$ is $\downarrow$ and $x \ge 2$, $f(x)$ is $\uparrow$

$\therefore$ Minimum value of $f(x)$ lies in $x \in (0,2)$

Applying $A.M \ge G.M$,

minimum value of $f(x)$ is $2(e-1)$

## EXERCISE - JEE (Advanced) PYQ

**1. Ans. (B)**

$$L = \lim_{n\to\infty} \frac{1^a + 2^a + \ldots\ldots\ldots\ldots + n^a}{(n+1)^{a-1}\left[\underbrace{na + na + na + \ldots\ldots\ldots + na}_{n\,times} + 1 + 2 + 3 + \ldots\ldots + n\right]} = \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^a}{(n+1)^{a-1}\left[n^2 a + \frac{n(n+1)}{2}\right]}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\sum_{r=1}^{n} \frac{r^a}{n^a}\right) n^{a+1}}{(n+1)^{a-1}\left[n^2 a + \frac{n(n+1)}{2}\right]} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\sum_{r=1}^{n} \frac{r^a}{n^a}\right)}{\left(\frac{n+1}{n}\right)^{a-1}\left[\frac{n^2 a + \frac{n(n+1)}{2}}{n^2}\right]}$$

$$= \frac{\int_0^1 x^a dx}{\left(a + \frac{1}{2}\right)} = \frac{1}{60} \Rightarrow \frac{2}{(a+1)(2a+1)} = \frac{1}{60} \Rightarrow 2a^2 + 3a - 119 = 0 \Rightarrow a = 7 \,\&\, -\frac{17}{2}$$

$a = -\frac{17}{2}$ will be rejected as $\int_0^1 x^{-\frac{17}{2}} dx$ is not defined.

**2. Ans. (A,C)**

Given that $1:[a,b]\to[1,\infty)$

$$g(x)=\begin{cases} 0 & ; \quad x<a \\ \int_a^x f(t)dt & ; \quad a\le x\le b \\ \int_a^b f(t)dt & ; \quad x>b \end{cases}$$

Now $g(a^-)=0=g(a^+)=g(a)$

$$g(b^-)=g(b^+)=g(b)=\int_a^b f(t)dt$$

$\Rightarrow$ $g$ is continuous $\forall x\in R$

Now $g'(x)=\begin{cases} 0 & : x<a \\ f(x) & : a<x<b \\ 0 & : x>b \end{cases}$

$g'(a^-)=0$ but $g'(a^+)=f(a)\ge 1$

$\Rightarrow g$ is non differentiable at $x=a$

and $g'(b^+)=0$ but $g'(b^-)=f(b)\ge 1$

$\Rightarrow g$ is non differentiable at $x=b$

**3. Ans. (2)**

using integration by part

$$\int_0^1 4x^3\left(\left(1-x^2\right)^5\right)'' dx$$

$$=4x^3\left(\left(1-x^2\right)^5\right)'\Big]_0^1-\int_0^1 12x^2\left(\left(1-x^2\right)^5\right)' dx$$

using integration by part

$$=-12\left[x^2\left(\left(1-x^2\right)^5\right)_0^1-\int_0^1 2x\left(1-x^2\right)^5 dx\right]$$

$$=12.2\int_0^1 x(1-x^2)^5 dx$$

Let $1-x^2=t\Rightarrow xdx=-\frac{dt}{2}$

$$=24\int_1^0 t^5\left(-\frac{dt}{2}\right)$$

$$=12\int_0^1 t^5 dt=2$$

**4.** **Ans. (A)**

Let $\cos ecx + \cot x = e^{u}$

$\underline{\cos ecx - \cot x = e^{-u}}$

cosec$x$ = $\frac{1}{2}\left(e^{u}+e^{-u}\right) \& \cot x = \frac{1}{2}\left(e^{u}-e^{-u}\right)$

$\text{cosec}^{2} x dx = -\frac{1}{2}\left(e^{u}+e^{-u}\right) du$

$\Rightarrow \int_{\ell n(\sqrt{2}+1)}^{0} 2^{17}\left(\frac{1}{2}\left(e^{u}+e^{-u}\right)\right)^{15}\left\{-\frac{1}{2}\left(e^{u}+e^{-u}\right)\right\} du$

$\Rightarrow \int_{0}^{\ell n(\sqrt{2}+1)} 2\left(e^{u}+e^{-u}\right)^{16} du.$

**5.** **Ans. (B)**

$F(x) = \int_{0}^{x^{2}} f\left(\sqrt{t}\right) dt$

$F'(x) = 2xf(x) \because x \in [0,2]$

$\Rightarrow f'(x) = 2xf(x) \qquad \because F'(x) = f'(x)$

$\Rightarrow \frac{f'(x)}{f(x)} = 2x \Rightarrow \ell n(f(x)) = x^{2} + c$

$c = 0 \quad (\because f(0) = 1)$

$\Rightarrow f(x) = e^{x^{2}}$

$f(x) = \int_{0}^{x^{2}} f\left(\sqrt{t}\right) dt = \int_{0}^{x^{2}} e^{t}.dt$

$\therefore f(2) = \int_{0}^{x^{2}} e^{t} dt = e^{4} - 1.$

**6.** **Ans. (A)**

$g\left(\frac{1}{2}\right) = \lim_{h \to 0^{+}} \int_{h}^{1-h} \frac{dt}{\sqrt{t(1-t)}}$

$= \lim_{h \to 0^{+}} \int_{h}^{1-h} \frac{dt}{\sqrt{\frac{1}{4} - \left(t - \frac{1}{2}\right)^{2}}}$

$= \sin^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\Bigg|_{0}^{1} = \sin^{-1}(1) - \sin^{-1}(-1) = \pi$

**7. Ans. (D)**

Given

$$g(a)=\lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$$

$$g'(a)=\lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}(-\ln t+\ln(1-t))\,dt$$

$$g'\left(\frac{1}{2}\right)=\lim_{h\to 0^+}\int_h^{1-h}\frac{\ln\left(\frac{1-t}{t}\right)dt}{\sqrt{t(1-t)}} \qquad ...(1)$$

$$g'\left(\frac{1}{2}\right)=\lim_{h\to 0^+}\int_h^{1-h}\frac{\ln\left(\frac{1-(1-t)}{1-t}\right)}{\sqrt{(1-t)t}}dt \qquad ...(2) \qquad \left(\text{Apply } \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$$

$$\Rightarrow 2g'\left(\frac{1}{2}\right)=\lim_{h\to 0^+}\int_h^{1-h} 0\,dt \Rightarrow g'\left(\frac{1}{2}\right)=0$$

**8. Ans. (A,C,D)**

$$f(x)=\int_{\frac{1}{x}}^{x}\frac{e^{-\left(-t+\frac{1}{t}\right)}}{t}dt$$

$$f'(x)=1.\frac{e^{-\left(x+\frac{1}{x}\right)}}{x}-\left(\frac{-1}{x^2}\right)\frac{e^{-\left(\frac{1}{x}+x\right)}}{1/x}$$

$$=\frac{e^{-\left(x+\frac{1}{x}\right)}}{x}+\frac{e^{-\left(x+\frac{1}{x}\right)}}{x}=\frac{2e^{-\left(x+\frac{1}{x}\right)}}{x}$$

$\therefore$ $f(x)$ is monotonically increasing on $(0, \infty) \Rightarrow$ A is correct & B is wrong.

Now $f(x)+f\left(\frac{1}{x}\right)=\int_{1/x}^{x}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt+\int_{x}^{1/x}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt=0\ \forall x\in(0,\infty)$

Now let $g(x)=f(2^x)=\int_{2^{-x}}^{2^x}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt$

$g(-x)=f(2^{-x})=\int_{2^x}^{2^{-x}}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt=-g(x)$

$\therefore$ $f(2^x)$ is an odd function.

**9. Ans. (D)**

**(P)** Let $f(x)=ax^2+bx+c$

{but $f(0)=0$}

$\therefore$ $\boxed{f(x)=ax^2+bx}$

$$So,\ \int_0^1(ax^2+bx).dx=1$$

$$\Rightarrow\left[\frac{ax^3}{3}+\frac{bx^2}{2}\right]_0^1=1$$

$\Rightarrow \frac{a}{3}+\frac{b}{2}=1$

$a = 3$ or $a = 0$

$b = 0$ $\quad$ $b = 2$

so two polynomials

& $\left.\begin{matrix} f(x)=3x^2 \\ f(x)=2x \end{matrix}\right\}$ are possible

**(Q)** Max. of $\sin(x^2)+\cos(x^2)$ is $\sqrt{2}$

$\Rightarrow \sin(x^2)+\cos(x^2)=\sqrt{2};\ x\in\left[-\sqrt{13},\sqrt{13}\right]$

$\Rightarrow \sin\left(x^2+\frac{\pi}{4}\right)=1;\ x^2\in[0,13]$

$\Rightarrow x^2+\frac{\pi}{4}=2n\pi+\frac{\pi}{2};n\in I$

$\Rightarrow x^2=2n\pi+\frac{\pi}{4};\ n\in I;\ x^2\in[0,13]$

if $n = 0$ if $n = 1$

$x^2=\frac{\pi}{4}$ $\qquad$ $x^2 = 2\pi+\frac{\pi}{4}$

$x=\pm\sqrt{\frac{\pi}{4}}$ $\qquad$ $x=\pm\sqrt{\frac{9\pi}{4}}$

$\therefore$ No. of solution = (4) **Ans.**

**(R)** $I=\int_{-2}^{2}\frac{3x^2}{1+e^x}dx$ ...(1)

King

$I=\int_{-2}^{2}\frac{3x^2\cdot e^2}{1+e^x}dx$ ...(2)

(1) + (2)

$2I=\int_{-2}^{2}3x^2\cdot dx$ (even)

$2I=2\int_{0}^{2}3x^2\cdot dx$

$\Rightarrow I=\left[x^3\right]_0^2=8$

**(S)** $$\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2x)\log\left(\frac{1+x}{1-x}\right)dx}{\int_0^{1/2} \cos(2x)\cdot\log\left(\frac{1+x}{1-x}\right).dx}$$

$$=\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} (\text{odd function})dx}{\int_0^{1/2} \cos(2x)\cdot\log\left(\frac{1+x}{1-x}\right)dx} = 0$$

**10. Ans. (0)**

Given $f(x)=\begin{bmatrix} [x] & , & x\le 2 \\ 0 & , & x>2 \end{bmatrix}$

where $[x]$ denotes greatest integer function.

Now $I=\int_{-1}^{2}\frac{xf(x^2)}{2+f(x+1)}dx$

$$I=\underbrace{\int_{-1}^{0}\frac{xf(x^2)}{2+f(x+1)}dx}_{I_1}+\underbrace{\int_{0}^{1}\frac{xf(x^2)}{2+f(x+1)}dx}_{I_2}+\underbrace{\int_{1}^{\sqrt{2}}\frac{xf(x^2)}{2+f(x+1)}dx}_{I_3}+\underbrace{\int_{\sqrt{2}}^{\sqrt{3}}\frac{xf(x^2)}{2+f(x+1)}dx}_{I_4}+\underbrace{\int_{\sqrt{3}}^{3}\frac{xf(x^2)}{2+f(x+1)}dx}_{I_5}$$

$\therefore I = I_1 + I_2 + I_3 + I_4 + I_5$

Clearly $I_1, I_2, I_4$ & $I_5$ are zero using by definition of $f(x)$.

$$\therefore\ I=I_3=\int_1^{\sqrt{2}}\frac{xf(x^2)}{2+f(x+1)}dx=\int_1^{\sqrt{2}}\frac{x.1}{2+0}dx=\left.\frac{x^2}{4}\right|_1^{\sqrt{2}}=\frac{2}{4}-\frac{1}{4}=\frac{1}{4}$$

$\therefore 4I - 1 = 0$

**11. Ans. (9)**

$$\alpha=\int_0^1 e^{\left(9x+\tan^{-1}x\right)}\left(9+\frac{3}{1+x^2}\right)dx$$

$$\alpha=\left(e^{9x+\tan^{-1}x}\right)_0^1$$

$$=e^{9+\frac{3x}{4}}-1$$

$$\log|\alpha+1|=9+\frac{3\pi}{4}\Rightarrow \text{Ans.} = 9$$

**12. Ans. (7)**

$$\lim_{x\to1}\frac{F(x)}{G(x)}=\lim_{x\to1}\frac{F'(x)}{G'(x)}=\lim_{x\to1}\frac{f(x)}{x\left|f\left(f(x)\right)\right|}=\frac{1}{14}$$

$$\Rightarrow\frac{\frac{1}{2}}{\left|f\left(\frac{1}{2}\right)\right|}=\frac{1}{14}\Rightarrow\left|f\left(\frac{1}{2}\right)\right|=7$$

$f\left(\frac{1}{2}\right)=7$

$f\left(\frac{1}{2}\right)\neq -7$ as $f(x)$ vanishes exactly at one point.

**13. Ans. (A,C)**

Let

$$I_1=\int_0^{4\pi} e^t\left(\sin^6 at+\cos^4 at\right)dt$$

$$\therefore I_1=\int_0^{\pi} e^t\left(\sin^6 at+\cos^4 at\right)dt+\underbrace{\int_{\pi}^{2\pi} e^t\left(\sin^6 at+\cos^4 at\right)dt}_{\substack{t=\pi+x\\ \text{transforms time}\\ \text{to}\\ \downarrow}}+\underbrace{\int_{2\pi}^{3\pi} e^t\left(\sin^6 at+\cos^6 at\right)dt}_{\substack{t=2\pi+y\\ \text{transforms this}\\ \text{to}\\ \downarrow}}+\underbrace{\int_{3\pi}^{4\pi} e^t\left(\sin^6 at+\cos^6 at\right)dt}_{\substack{t=3\pi+z\\ \text{transforms this}\\ \text{to}\\ \downarrow}}$$

$$\therefore\ I_1=\int_0^{\pi} e^t\left(\sin^6 at+\cos^4 at\right)dt+\int_0^{\pi} e^{\pi+t}\left(\sin^6 at+\cos^4 at\right)dt+\int_0^{\pi} e^{2\pi+t}\left(\sin^6 at+\cos^4 at\right)+$$

$$\int_0^{\pi} e^{t+3\pi}\left(\sin^6 at+\cos^4 at\right)dt$$

$$\therefore I_1=\left(1+e^{\pi}+e^{2\pi}+e^{3\pi}\right)\int_0^{\pi} e^t\left(\sin^6 at+\cos^4 at\right)dt$$

$$\therefore \frac{\int_0^{4\pi} e^t\left(\sin^6 at+\cos^4 at\right)dt}{\int_0^{\pi} e^t\left(\sin^6 at+\cos^4 at\right)dt}=1+e^{\pi}+e^{2\pi}+e^{3\pi}=\frac{e^{4\pi}-1}{e^{\pi}-1}$$

**14. Ans. (A,B)**

Given $f(x)=(7\tan^6 x-3\tan^2 x)\sec^2 x$

$$\therefore \int_0^{\pi/4} \underset{\text{I}}{x}\underbrace{\left(7\tan^6 x-3\tan^2 x\right)\sec^2 x}_{\text{II}}\,dx$$

Using I.B.P.

$$=x.\left(\tan^7 x-\tan^3 x\right)_0^{\pi/4}-\int_0^{\pi/4}\left(\tan^7 x-\tan^3 x\right)dx$$

$$=-\int_0^{\pi/4}\tan^3 x\left(\tan^2 x-1\right)\sec^2 x dx$$

Put $\tan x=t$

$$=\int_0^1\left(t^3-t^5\right)dt=\frac{1}{4}-\frac{1}{6}=\frac{3-2}{12}=\frac{1}{12}$$

Also, $=\int_0^{\pi/4}\left(7\tan^6 x-3\tan^2 x\right)\sec^2 x dx=\tan^7 x-\tan^3 x\Big|_0^{\pi/4}=0$

**15.** **Ans. (D)**

$$f'(x)=\frac{192x^3}{2+\sin^4 \pi x}$$

$$\frac{192x^3}{3}\le f'(x)\le\frac{192x^3}{2}$$

$$\frac{192}{12}\left(x^4-\frac{1}{16}\right)\le\int_{1/2}^{\pi} f'(x)dx\le\frac{192}{8}\left(x^4-\frac{1}{16}\right)$$

$$16x^4-1\le f(x)\le 24x^4-\frac{3}{2}$$

$$2.6\le\int_{1/2}^{1} f(x)dx\le 3.9$$

out of given options only option D is correct.

**16.** **Ans. (A, B, C)**

According to given data

$F(x)<0\;\forall\, x\in(1,3)$

$f(x)=xF(x)$

$f'(x)=F(x)+xF'(x)$ ...(i)

$f'(1)=F(1)+F'(1)<0$ (use $F(1)=0$ & $F'(x)<0$)

$f(2)=2F(2)<0$ (use $F(x)<0\;\forall\, x\in(1,3)$ )

$f'(x)=F(x)+x\,F'(x)<0$ (use $F(x)<0\;\forall\, x\in(1,3)$ )

$F'(x)<0$

**17.** **Ans. (C,D)**

Given

$$\int_1^3 x^2F'(x)dx=-12$$

$$\Rightarrow[x^2F(x)]_1^3-2\int_1^3 xF(x)dx=-12$$

$$\Rightarrow\int_1^3 f(x)dx=-12 \qquad (\text{Use } x\,F\,(x)=f(x))$$

Given

$$\int_1^3 x^3F''(x)dx=40$$

$$\Rightarrow[x^3F'(x)]_1^3-3\int_1^3 x^2F'(x)dx=40$$

$\Rightarrow[x^2(f'(x)-F(x)]_1^3=4$ (use equation (i) from Q.No.16)

$9\big(f'(3)-F(3)\big)-(f'(1)-F(1))=4$

$9f'(3)+36-f'(1)=4$

$9f'(3)-f'(1)+32=0$

**18. Ans. (A)**

Let

$$I=\int_{-\pi/2}^{\pi/2}\frac{x^2\cos x}{1+e^x}dx=\int_0^{\pi/2}\left(\frac{1}{1+e^x}+\frac{1}{1+e^{-x}}\right)x^2\cos x\,dx$$

$$=\int_0^{\pi/2}x^2\cos x\,dx=\left(x^2\sin x\right)_0^{\pi/2}-2\int_0^{\pi/2}x.\sin x\,dx$$

(I) (II) (I) (II)

$$=\frac{\pi^2}{4}-2\left[-\left(x.\cos x\right)_0^{\pi/2}+\int_0^{\pi/2}1.\cos xdx\right]=\frac{\pi^2}{4}-2[0+1]=\left(\frac{\pi^2}{4}-2\right)$$

**19. Ans. (1)**

Let $f(x)=\int_0^x\frac{t^2}{1+t^4}dt-2x+1$

$$f'(x)=\frac{x^2}{1+x^4}-2$$

$$0\le\frac{x^2}{1+x^4}<1\quad\forall x\in[0,1]$$

So, $f'(x)<0\quad\forall x\in[0,1]$

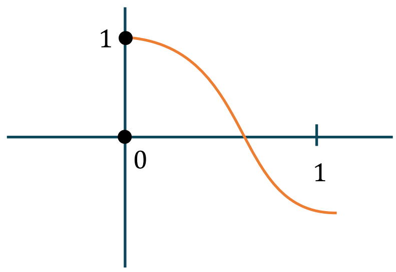
$f(0)=1$

$$f(1)=\int_0^1\frac{t^2}{1+t^4}dt-1$$

$$\because\quad 0\le\frac{t^2}{1+t^4}<1\Rightarrow 0<\int_0^1\frac{t^2}{1+t^4}dt<1$$

So, $f(1)\in(-1,0)$

Number of solution = 1




**20. Ans. (B,C)**

$$\ell nf(x)=\lim_{n\to\infty}\frac{x}{n}\ell n\left[\frac{\prod_{r=1}^{n}\left(x+\frac{1}{r/n}\right)}{\prod_{r=1}^{n}\left(x^2+\frac{1}{(r/n)^2}\right)}\frac{1}{\prod_{r=1}^{n}(r/n)}\right]$$

$$=x\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ell n\left(\frac{x\frac{r}{n}+1}{\left(x\frac{r}{n}\right)^2+1}\right)$$

$$=x\int_0^1\ell n\left(\frac{1+tx}{1+t^2x^2}\right)dt\qquad\text{put } tx=z$$

$$\ell nf(x)=\int_0^x\ell n\left(\frac{1+z}{1+z^2}\right)dz$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ell n\left(\frac{1+x}{1+x^2}\right)$$

Here $f'(1) = 0$. Sign scheme of $f'(x)$ :

+ −
———|———
1

$$\Rightarrow f\left(\frac{1}{2}\right) < f(1), f\left(\frac{1}{3}\right) < f\left(\frac{2}{3}\right), f'(2) < 0$$

Also $\frac{f'(3)}{f(3)} - \frac{f'(2)}{f(2)} = \ell n\left(\frac{4}{10}\right) - \ell n\left(\frac{3}{5}\right)$

$$= \ell n\left(\frac{4}{6}\right) < 0 \Rightarrow \frac{f'(3)}{f(3)} < \frac{f'(2)}{f(2)}$$

**21. Ans. (2)**

$$g(x) = \int_x^{\pi/2} \left( f'(t)\text{cosec}t - f(t)\text{cosec}t \cot t \right) dt$$

$$= \int_x^{\pi/2} \left( f(t)\text{cosec}t \right)' dt = \left[ f(t)\text{cosec}t \right]_x^{\pi/2}$$

$$= f\left(\frac{\pi}{2}\right)\text{cosec}\left(\frac{\pi}{2}\right) - \frac{f(x)}{\sin x} = 3 - \frac{f(x)}{\sin x}$$

$$\therefore \lim_{x\to 0} g(x) = 3 - \lim_{x\to 0}\frac{f(x)}{\sin x}; as\ f'(0) = 1$$

$$= 3 - \lim_{x\to 0}\frac{f'(x)}{\cos(x)}$$

$$\Rightarrow \lim_{x\to 0} g(x) = 3 - 1 = 2$$

**22. Ans. (B,C)**

$$I = \sum_{k=1}^{98}\left(\int_k^{k+1}\frac{(k+1)}{x(x+1)}dx\right)$$

$$= \sum_{k=1}^{98}(k+1)\left(\int_k^{k+1}\left(\frac{1}{x} - \frac{1}{x+1}\right)dx\right)$$

$$= \sum_{k=1}^{98}(k+1)\left((\ell nx - \ell n(x+1))_k^{k+1}\right)$$

$$= \sum_{k=1}^{98}((k+1)\ell n(k+1) - k.\ell nk) - \sum_{k=1}^{98}((k+1).\ell n(k+2) - k.\ell n(k+1)) + \sum_{k=1}^{98}(\ell n(k+1) - \ell nk)$$

(Difference series)

$$\therefore I = (99\ \ell n\ 99) + (-99\ \ell n\ 100 + \ell n\ 2) + (\ell n\ 99) = \ell n\left(\frac{2\times(99)^{100}}{(100)^{99}}\right) \quad \text{..... (1)}$$

For option (B) :

Now, consider $(100)^{99} = (1+99)^{99}$

$$={}^{99}C_0+{}^{99}C_1(99)+{}^{99}C_2(99)^2+.......{}^{99}C_{97}(99)^{97}+\underbrace{{}^{99}C_{98}(99)^{98}}_{(\text{value}(99)^{99})}+\underbrace{{}^{99}C_{99}(99)^{99}}_{(\text{value}(99)^{99})}$$

$$\Rightarrow (100)^{99}>2.(99)^{99}\Rightarrow \frac{2\times(99)^{99}}{(100)^{99}}<1$$

$$\therefore \quad \frac{2\times(99)^{100}}{(100)^{99}}<99 \text{ (on multiplying by 99)}$$

$$\Rightarrow \quad I<\ell n99$$

For option (C) :

Since, $\sum_{k=1}^{98}\int_{k}^{k+1}\frac{k+1}{(x+1)^2}dx<\sum_{k=1}^{98}\int_{k}^{k+1}\frac{(k+1)dx}{x(x+1)}$

$$\Rightarrow \sum_{k=1}^{98}\left(\frac{1}{k+2}\right)<I$$

(on integration)

$$\Rightarrow \underbrace{\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+......+\frac{1}{100}\right)}_{98\text{ terms}}<I$$

$$\Rightarrow \frac{98}{100}<\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+......+\frac{1}{100}<I$$

$$\therefore \quad I>\frac{49}{50}$$

Hence option (C) is correct.

**23.** **Ans. (Bonus)**

$g(x)=\int_{\sin x}^{\sin 2x}\sin^{-1}t\,dt\Rightarrow g'(x)=2\sin^{-1}(\sin 2x)\times\cos 2x-\sin^{-1}(\sin x)\cos x$

$$\Rightarrow g'\left(\frac{\pi}{2}\right)=0\ \&\ g'\left(-\frac{\pi}{2}\right)=0$$

No option matches the result

$\Rightarrow$ BONUS

**24.** **Ans. (B,D)**

For option (A) :

Let $g(x)=e^x-\int_0^x f(t)\sin t\,dt$

$$\therefore\ g'(x)=e^x-\left(f(x).\sin x\right)>0\ \forall x\in(0,1)$$

$\Rightarrow g(x)$ is strictly increasing function.

Also, $g(0)=1$

$$\Rightarrow\ g(x)>1\ \forall x\in(0,1)$$

$\therefore$ option (A) is not possible.

For option (B) :

Let $k(x)=x^9-f(x)$

Now, $k(0) = -f(0) < 0$ (As $f \in (0, 1)$)

Also, $k(1) = 1 - f(1) > 0$ (As $f \in (0, 1)$)

$\Rightarrow k(0).k(1) < 0$

So, option (B) is correct.

For option (C) :

Let $T(x) = f(x) + \int_0^{\pi/2} f(t) \cdot \sin t \, dt$

$\Rightarrow T(x) > 0 \,\forall x \in (0,1)$ (As $f \in (0, 1)$)

so, option (C) is not possible.

For option (D) :

Let $M(x) = x - \int_0^{\frac{\pi}{2}-x} f(t) \cos t \, dt$

$\therefore\ M(0) = 0 - \int_0^{\pi/2} f(t) \cdot \cos t \, dt < 0$

Also, $M(1) = 1 - \int_0^{\frac{\pi}{2}-1} f(t).\cos t \, dt > 0$

$\Rightarrow M(0) . M(1) < 0$

$\therefore$ option (D) is correct.

**25. Ans. (1)**

$$y_n = \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)......\left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}}$$

$$y_n = \prod_{r=1}^{n}\left(1+\frac{r}{n}\right)^{1/n}$$

$$\log\ y_n = \frac{1}{n}\prod_{r=1}^{n}\ell n\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \lim_{n\to\infty}\log\ y_n = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\ell n\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \log L = \int_0^1 \ell n(1+x)dx$$

$$\Rightarrow \log L = \log\frac{4}{e}$$

$$\Rightarrow L = \frac{4}{e}$$

$$\Rightarrow [L] = 1$$

**26.** **Ans. (2)**

$$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{\left[(1+x)^2(1-x)^6\right]^{1/4}}$$

$$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{(1+x)^2\left[\frac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$$

Put $\frac{1-x}{1+x}=t \Rightarrow \frac{-2dx}{(1+x)^2}=dt$

$$I=\int_1^{1/3}\frac{\left(1+\sqrt{3}\right)dt}{-2t^{6/4}}=\frac{-\left(1+\sqrt{3}\right)}{2}\times\left|\frac{-2}{\sqrt{t}}\right|^{1/3}=\left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right)=2$$

**27.** **Ans. (A,B,C)**

$f(x) = (x-1)(x-2)(x-5)$

$F(x)=\int_0^x f(t)dt, x>0$

$F'(x)=f(x)=(x-1)(x-2)(x-5), x>0$

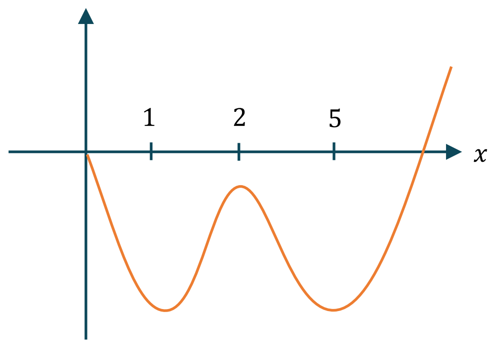
clearly $F(x)$ has local minimum at $x = 1, 5$

$F(x)$ has local maximum at $x = 2$

$f(x) = x^3 - 8x^2 + 17x - 10$

$\Rightarrow F(x)=\int_0^x\left(t^3-8t^2+17t-10\right)dt$

$F(x)=\frac{x^4}{4}-\frac{8x^3}{3}+\frac{17x^2}{2}-10x$




from the graph of $y = F(x)$, clearly $F(x) \neq 0 \ \forall \ x \in (0, 5)$

**28.** **Ans. (4.00)**

$$2I=\frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\left[\frac{1}{\left(1+e^{\sin x}\right)(2-\cos 2x)}+\frac{1}{\left(1+e^{-\sin x}\right)(2-\cos 2x)}\right]dx \text{ (using King's Rule)}$$

$$\Rightarrow I=\frac{1}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{2-\cos 2x}$$

$$\Rightarrow I=\frac{2}{\pi}\int_0^{\pi/4}\frac{dx}{2-\cos 2x}=\frac{2}{\pi}\int_0^{\pi/4}\frac{\sec^2 dx}{1+3\tan^2 x}$$

$$=\frac{2}{\sqrt{3}\pi}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi/4}=\frac{2}{3\sqrt{3}}$$

$$\Rightarrow 271^2=24\times\frac{4}{24}=4$$

**29. Ans. (A,B)**

$$\lim_{n\to\infty}\frac{n^{1/3}\left(\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}\right)}{n^{7/3}\left(\sum_{r=1}^{n}\frac{1}{(an+r)^2}\right)}=54\Rightarrow\lim_{n\to\infty}\frac{\left(\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}\right)}{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(a+r/n)^2}\right)}=54\Rightarrow\frac{\int_0^1 x^{1/3}dx}{\int_0^1\frac{1}{(a+x)^2}dx}=54\Rightarrow\frac{\frac{3}{4}}{\frac{1}{a(a+1)}}=54$$

$$\Rightarrow a(a+1)=72 \quad\Rightarrow\quad a^2+a-72=0 \quad\Rightarrow\quad a=-9,8$$

**30. Ans. (0.50)**

$$I=\int_0^{\pi/2}\frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta$$

$$=\int_0^{\pi/2}\frac{3\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta$$

$$2I=\int_0^{\pi/2}\frac{3d\theta}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^4}$$

$$=3\int_0^{\pi/2}\frac{\sec^2\theta d\theta}{\left(1+\sqrt{\tan\theta}\right)^4}$$

Let $1+\sqrt{\tan\theta}=t$

$$\frac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta=dt$$

$\sec^2\theta d\theta=2(t-1)dt$

$$=3\int_t^{\infty}\frac{2(t-1)dt}{t^4}$$

$$=6\int_t^{\infty}\left(t^{-3}-t^{-4}\right)dt$$

$$2I=6\left(\frac{t^{-2}}{-2}-\frac{t^{-3}}{-3}\right)_1^{\infty}=6\left[0-0-\left\{-\frac{1}{2}+\frac{1}{3}\right\}\right]$$

$I=0.50$

**31. Ans. (A,B,D)**

(A) $\int_0^1 x\cos x dx\ge\int_0^1 x\left(1-\frac{x^2}{2!}\right)dx\ge\frac{3}{8}$

(B) $\int_0^1 x\sin x\, dx\ge\int_0^1 x\left(x-\frac{x^3}{3!}\right)dx\ge\frac{3}{10}$

(C) $\int_0^1 x^2\cos x\, dx\le\int_0^1 x^2dx\le\frac{1}{3}$

(D) $\int_0^1 x^2\sin x\, dx\ge\int_0^1 x^2\left(x-\frac{x^3}{3!}\right)dx\ge\frac{2}{9}$

**32. Ans. (4)**

$$F(x)=\int_0^x f(t).dt$$

$$\Rightarrow F'(x)=f(x)$$

$$I=\int_0^\pi f'(x).\cos x dx+\int_0^\pi F(x)\cos(x)dx=2 \qquad ...(1)$$

$$I_1=\int_0^\pi f'(x).\cos x\, dx \qquad \text{(Let)}$$

Using by parts

$$I_1=\left(\cos x.f(x)\right)_0^\pi+\int_0^\pi \sin x.f(x)dx$$

$$I_1=6-f(0)+\int_0^\pi \sin x.F'(x)dx$$

$$I_1=6-f(0)+I_2 \qquad ...(2)$$

$$I_2=\int_0^\pi \sin x.F'(x).dx$$

Using by part we get

$$I_2=\left(\sin x.F(x)\right)_0^\pi-\int_0^\pi \cos x.F(x)dx$$

$$I_2=\int_0^\pi \cos x.F(x)dx$$

$$(1) \Rightarrow I=6-f(0)=2 \Rightarrow f(0)=4$$

$$(2) \Rightarrow I_1=6-f(0)-\int_0^\pi \cos x.F(x)\,dx$$

**33. Ans. (A,B,C)**

**(A)** Let $g(x)=f(x)-3\cos 3x$

Now $\int_0^{\pi/3} g(x)dx=\int_0^{\pi/3} f(x)dx-3\int_0^{\pi/3}\cos 3x\,dx=0$

Hence $g(x)=0$ has a root in $\left(0,\frac{\pi}{3}\right)$

**(B)** Let $h(x)=f(x)-3\sin 3x+\frac{6}{\pi}$

Now $\int_0^{\pi/3} g(x)dx=\int_0^{\pi/3} f(x)dx-3\int_0^{\pi/3}\sin 3x dx+\int_0^{\pi/3}\frac{6}{\pi}dx$

$=0-2+2=0$

Hence $h(x)=0$ has a root in $\left(0,\frac{\pi}{3}\right)$

**(C)** $\lim_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=\lim_{x\to 0}\underbrace{\left(\frac{x^2}{1-e^{x^2}}\right)}_{-1}\underbrace{\frac{\int_0^x f(t)dt}{x}}_{\text{Apply L'Hopital's Rule}}$

$=-\lim_{x\to 0}\frac{f(x)}{1}=-1$

**(D)** $\lim_{x\to 0}\frac{(\sin x)\int_0^x f(t)dt}{x^2}$

$=\lim_{x\to 0}\underbrace{\left(\frac{\sin x}{x}\right)}_{1}\underbrace{\frac{\int_0^x f(t)dt}{x}}_{\text{Apply L'Hopitals Rule}}$

$=\lim_{x\to 0}\frac{f(x)}{1}=1$

**34.** **Ans. (2.00)**

$S_1=\int_{\pi/8}^{3\pi} f(x)dx=\int_{\pi/8}^{3\pi/8}\sin^2 x dx=\int_{\pi/8}^{3\pi/8}\sin^2\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)dx=\int_{\pi/8}^{3\pi/8}\cos^2 x\, dx$

$2S_1=\int_{\pi/8}^{3\pi/8}\left(\sin^2 x+\cos^2 x\right)dx=\frac{3\pi}{8}-\frac{\pi}{8}=\frac{\pi}{4}$

$\Rightarrow\frac{16S_1}{\pi}=2$

**35.** **Ans. (1.50)**

$S_2=\int_{\pi/8}^{3\pi/8} f(x)g_2(x)dx=\int_{\pi/8}^{3\pi/8} sin^2x|4x-\pi|dx$

$=\int_{\pi/8}^{3\pi/8}\sin^2\left(\frac{\pi}{2}-x\right)\left|4\left(\frac{\pi}{2}-x\right)-\pi\right|dx$

$=\int_{\pi/8}^{3\pi/8}\left(\cos^2 x\right)|\pi-4x|dx$

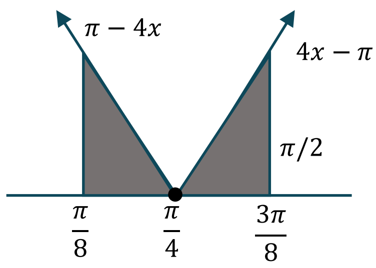
$\Rightarrow 2S_2=\int_{\pi/8}^{3\pi/8}|4x-\pi|\left(\sin^2 x+\cos^2 x\right)dx$

$=\int_{\pi/8}^{3\pi 8}|4x-\pi|dx$

$=2\times\frac{1}{2}\times\frac{\pi}{8}\times\frac{\pi}{2}=\frac{\pi^2}{16}$

$\Rightarrow S_2=\frac{\pi^2}{32}$

$\Rightarrow\frac{48S_2}{\pi^2}=\frac{3}{2}=1.5$



**36. Ans. (C)**

$f'(x) = (|x|-x^2)e^{-x^2} + (|x|-x^2)e^{-x^2}, x \ge 0$

$f'(x) = 2(x-x^2)e^{-x^2}$

$$\underset{0}{|}\ +++\ \underset{1}{|}\ ---$$

hence option (D) is wrong

$g'(x) = xe^{-x^2}2x$

$f'(x) + g'(x) = 2xe^{-x^2}$

$f(x) + g(x) = -e^{-x^2} + c$

$f(x) + g(x) = -e^{-x^2} + 1$

$f(\ell n3) + g(\sqrt{\ell n3}) = 1 - \frac{1}{3} = \frac{2}{3}$ (option (A) is wrong)

$H(x) = \Psi_1(x) - 1 - \alpha x = e^{-x} + x - 1 - \alpha x,\ x \ge 1\ \&\ \alpha \in (1, x)$

$H(1) = e^{-1} + 1 - 1 - \alpha < 0$

$H'(x) = -e^{-x} + 1 - \alpha > 0 \Rightarrow H(x)$ is $\downarrow \Rightarrow$ option (B) is wrong

(C) Appying L.M.V.T. to $\Psi_2(x)$ in $[0, x]$

$\Psi_2'(\beta) = \frac{\Psi_2(x) - \Psi_2(0)}{x}$

$2\beta - 2 + 2e^{-\beta} = \frac{\Psi_2(x) - 0}{x}$

$\Rightarrow \Psi_2(x) = 2x(\Psi_1(\beta - 1))$ has one solution

option (C) is correct.

**37. Ans. (D)**

**(A)** $\psi_1(x) = e^{-x} + x, x \ge 0$

$\psi_1'(x) = 1 - e^{-x} > 0 \Rightarrow \psi_1(x)$ is $\uparrow$

$\psi_1(x) \ge \psi_1(0)\ \forall x \ge 0 \Rightarrow \psi_1(x) \ge 1$

**(B)** $\psi_2(x) = x^2 - 2x + 2 - 2e^{-x}\ \ x \ge 0$

$\psi_2'(x) = 2x - 2 + 2e^{-x} = 2\psi_1(x) - 2 \ge 0\ \forall x \ge 0$

$\Rightarrow \psi_2(x) is \uparrow \Rightarrow \psi_2(x) \ge \psi_2(0) \Rightarrow \psi_2(x) \ge 0$

**(C)** $f(x) = 2\int_0^x (t - t^2)e^{-t^3} dt \quad \& \quad x \in \left(0, \frac{1}{2}\right)$

$\Rightarrow f(0) = 0, f'(x) = 2(x - x^2)e^{-x^2}$

Let $H(x) = f(x) - 1 + e^{-x^2} + \frac{2}{3}x^3 - \frac{2}{5}x^5, x \in \left(0, \frac{1}{2}\right)$

$H(0) = 0$

$H'(x) = 2(x - x^2)\ e^{-x^2} - 2xe^{-x^2} + 2x^2 - 2x^4$

$$= -2x^2 e^{-x^2} + 2x^2 - 2x^4$$

$$= 2x^2\left(1 - x^2 - e^{-x^2}\right)$$

$$\because e^{-x} \geq 1 - x \ \forall \ x \geq 0$$

$$\Rightarrow H'(x) \leq 0 \Rightarrow H(x) \leq H(0) \ \ \forall x \in \left(0, \frac{1}{2}\right)$$

**(D)** Let $P(x) = g(x) - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{7}x^7 \ ; \ \ x \in \left(0, \frac{1}{2}\right)$

$$P'(x) = 2x^2 e^{-x^2} - 2x^2 + 2x^4 - x^6$$

$$= 2x^2\left(1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{3} + .....\right) - 2x^2 + 2x^4 - x^6$$

$$= -\frac{x^8}{3} + \frac{x^{10}}{12} ..........$$

$$\Rightarrow P'(x) \leq 0$$

$$\Rightarrow P(x) is \downarrow$$

$$\Rightarrow P(x) \leq 0$$

Option (D) is correct.

**38.** **Ans. (182)**

Let $f(x) = \left(\frac{10x}{x+1}\right)$

So, $f'(x) = 10\left(\frac{(x+1) - x}{(x+1)^2}\right) = \frac{10}{(x+1)^2} > 0 \ \forall \ x \in [0,10]$

So, $f(x)$ is an increasing function

So, range of $f(x)$ is $\left[0, \sqrt{\frac{100}{11}}\right]$

$$\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{1/9}^{2/3}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{9}^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$$

$$I = \int_0^{1/9} = 0 + \int_{1/9}^{2/3} dx + 2\int_{2/3}^{9} dx + 3\int_9^{10} dx$$

$$= \frac{2}{3} - \frac{1}{9} + 2\left(9 - \frac{2}{3}\right) + 3(10 - 9)$$

$$= \frac{6-1}{9} + 2 \times \frac{25}{3} + 3 = \frac{5}{9} + \frac{50}{3} + 3$$

$$= \frac{5 + 150 + 27}{9} = \frac{182}{9}$$

So, $9I = 182$

**39. Ans. (C,D)**

$$\int_1^e \frac{(\log_e x)^{1/2}}{x\left(a-(\log_e x)^{3/2}\right)^2}=1$$

Let $a-(\log_e x)^{3/2}=t$

$$\frac{(\log_e x)^{1/2}}{x}dx=-\frac{2}{3}dt$$

$$=\frac{2}{3}\int_a^{a-1}\frac{-dt}{t^2}=\frac{2}{3}\left(\frac{1}{t}\right)_a^{a-1}=1$$

$$\frac{2}{3a(a-1)}=1$$

$3a^2-3a-2=0$

$$a=\frac{3\pm\sqrt{33}}{6}$$

**40. Ans. (5)**

$$f(x)=\log_2(x^3+1)=y$$

$$x^3+1=2^y \Rightarrow x=(2^y-1)^{1/3}=f^{-1}(y)$$

$$f^{-1}(x)=(2^x-1)^{1/3}$$

$$=\int_1^2\log_2(x^3+1)dx+\int_1^{\log_2 9}(2^x-1)^{1/3}dx$$

$$=\int_1^2 f(x)dx+\int_1^{\log_2 9}f^{-1}(x)dx=2\log_2 9-1$$

$$8<9<2^{7/2}\Rightarrow 3<\log_2 9<\frac{7}{2}$$

$$\Rightarrow 5<2\log_2 9-1<6$$

$$\therefore [2\log_2 9-1]=5$$

**41. Ans. (B)**

$$f(n)=n+\sum_{r=1}^{n}\frac{16r+(9-4r)n-3n^2}{4rn+3n^2}$$

$$f(n)=n+\sum_{r=1}^{n}\frac{(16r+9n)-(4rn+3n^2)}{4rn+3n^2}$$

$$f(n)=n+\left(\sum_{r=1}^{n}\frac{16r+9n}{4rn+3n^2}\right)-n$$

$$\lim_{n\to\infty}f(n)=\lim_{n\to\infty}\sum\frac{16r+9n}{4rn+3n^2}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\left(16\left(\frac{r}{n}\right)+9\right)\frac{1}{n}}{4\left(\frac{r}{n}\right)+3}$$

$$= \int_0^1 \frac{16x+9}{4x+3} dx = \int_0^1 4\,dx - \int_0^1 \frac{3\,dx}{4x+3}$$

$$= 4 - \frac{3}{4}\left(\ell n|4x+3|\right)_0^1$$

$$= 4 - \frac{3}{4}\ell n\frac{7}{3}$$

**42.** **Ans. (C)**

Diff. wr.t '$x$'

$3f(x) = f(x) + xf'(x) - x^2$

$$\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x$$

$$IF = e^{-2\ell nx} = \frac{1}{x^2}$$

$$y\left(\frac{1}{x^2}\right) = \int x.\frac{1}{x^2}dx$$

$y = x^2\ell nx + cx^2$

$$\therefore\; y(1) = \frac{1}{3} \Rightarrow c = \frac{1}{3}$$

$$y(e) = \frac{4e^2}{3}$$

**43.** **Ans. (0)**

$$f(x) = \int_0^{x\tan^{-1}x} \frac{e^{t-\cos t}}{1+t^{2023}}dt$$

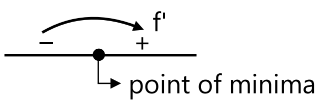
$$f'(x) = \frac{e^{x\tan^{-1}x - \cos(x\tan^{-1}x)}}{1+(x\tan^{-1}x)^{2023}}\cdot\left(\frac{x}{1+x^2} + \tan^{-1}x\right)$$

For $x < 0,\ \tan^{-1}x \in \left(-\frac{\pi}{2}, 0\right)$

For $x \geq 0,\ \tan^{-1}x \in \left[0, \frac{\pi}{2}\right)$

$\Rightarrow x\tan^{-1}x \geq 0\ \forall\ x \in \mathrm{R}$

And $\frac{x}{1+x^2} + \tan^{-1}x = \begin{cases} >0 & \text{For } x>0 \\ <0 & \text{For } x<0 \\ 0 & \text{For } x=0 \end{cases}$




Hence minimum value is $f(0) = \int_0^0 = 0$

## JEE (Main) Practice Paper

### SECTION–A

**1.** **Ans. (3)**

$\because I = \int \frac{dx}{\sqrt{e^x - 1}} = \int \frac{e^{x/2}\ dx}{e^{x/2}\sqrt{(e^{x/2})^2 - 1}}$

Let $e^{\frac{x}{2}} = z \Rightarrow I = 2\int \frac{dz}{|z|\sqrt{z^2 - 1}} = 2\sec^{-1}z + c = 2\sec^{-1}(e^{x/2}) + c$

Given $\frac{\pi}{6} = \int_{\ell n2}^{x} \frac{dx}{\sqrt{e^x - 1}} = 2\,[\sec^{-1}(e^{x/2}) - \sec^{-1}\sqrt{2}]$

$\therefore \sec^{-1}(e^{x/2}) = \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}$

$\therefore e^{\frac{x}{2}} = 2 \Rightarrow \frac{x}{2} = \ell n\, 2 \Rightarrow x = \ell n\, 4$

**2.** **Ans. (3)**

Dividing $N^r$ & $D^r$ by $x^2$ we get

$\int_0^{\infty} \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2} + 7} dx = \int_0^{\infty} \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 9} dx$ , substitute $x - \frac{1}{x} = t$

$= \int_{-\infty}^{\infty} \frac{dt}{t^2 + 9} = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)\Big|_{-\infty}^{\infty} = \frac{\pi}{3}$

**3.** **Ans. (1)**

$\int_0^{na} f(x)\, dx = n\int_0^{a} f(x)\, dx$

where a is period of $f(x)$.

$\therefore I = \int_0^{[x]} \{x\}\, dx = [x]\int_0^{1} x\, dx$ $\quad \{\because \{x\} = x, \text{ when } x \in [0, 1)\} \Rightarrow I = \frac{[x]}{2}$

**4.** **Ans. (3)**

$I = \int_0^{11} 11^{\{x\}} dx = \int_0^{11\times 1} 11^{\{x\}} dx = 11\int_0^{1} 11^{\{x\}} dx$ $\quad \{\because \{x\} \text{ is periodic with period } 1\}$

$= 11\int_0^1 11^x dx = 11\left[\frac{11^x}{\ell n 11}\right]_0^1 = 11\left[\frac{11}{\ell n 11} - \frac{1}{\ell n 11}\right] = \frac{110}{\ell n 11} = \frac{k}{\ell n 11}$

$k = 110$

**5.** **Ans. (1)**

$\int_0^{\pi/2} \ell n\, |\tan x + \cot x|\, dx = \int_0^{\pi/2} \ell n \left|\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right| dx = \int_0^{\pi/2} -\ell n \left|\left(\frac{\sin 2x}{2}\right)\right| dx$

$= -\int_0^{\pi/2} \ell n\left|\left(\frac{\sin 2x}{2}\right)\right| dx = -\int_0^{\pi/2} \ell n \sin 2x\ dx + \int_0^{\pi/2} \ell n 2\, dx$

$= \left(\frac{\pi}{2}\ell n\ 2\right) + \ell n\, 2 \,.\left(\frac{\pi}{2}\right) = \pi\, \ell n\, 2$

**6.** **Ans. (2)**

$$f'(x) = \cos x \int_1^x \frac{\cos y}{y^2+y+1} dy + \sin x . \frac{\cos x}{x^2+x+1}$$

Note that $f'(x) = 0 \ \forall \ x \in (2n+1)\ \frac{\pi}{2}, n \in Z$

**7.** **Ans. (1)**

Put $x = \sin^2\theta$ & apply Wallis formula or solve directly.

**8.** **Ans. (3)**

Note that $\sqrt{2} < \sqrt{x^4+x^2} < 3\sqrt{10} < \forall \ x \in (1, 3)$

Hence $2\sqrt{2} < \int_1^3 \sqrt{x^4+x^3} dx < 6\sqrt{10}$

**9.** **Ans. (3)**

$$I = \lim_{n \to \infty} \sum_{r=1}^{n-1} \frac{\pi}{n} \ \sin \frac{r\pi}{n} = \pi \int_0^1 \sin \pi x \ \ dx = \pi = [-\cos\pi + 1] = 2$$

**10.** **Ans. (4)**

$$I_n = \int_0^1 (1-x^3)^n .1 \ dx = (1-x^3)^n . \ x\Big|_0^1 - 3n\int_0^1 (1-x^3)^{n-1}(1-x^3-1)dx$$

$$\Rightarrow I_n = 0 - 3n(I_n - I_{n-1}) \Rightarrow (3n+1) \ I_n = 3nI_{n-1}$$

**11.** **Ans. (2)**

$$A = \int_0^1 \frac{e^t dt}{t+1}$$

$$\therefore I = \int_{a-1}^{a} \frac{e^{-t} dt}{t-a-1}$$

Put $t = a - y \Rightarrow dt = -dy$

then $I = -e^{-a} \int_0^1 \frac{e^y dy}{y+1} = -A \ e^{-a}$

**12.** **Ans. (3)**

Substitute $x^{x^2} = t$

$$\Rightarrow x^{x^2}(x + 2x\ell nx).dx = dt \Rightarrow x^{x^2+1} \ (1 + 2\ell nx).dx = dt$$

Hence $I = \int_1^{16} t dt = \frac{t^2}{2}\Big|_1^{16} = \frac{255}{2}$

**13.** **Ans. (4)**

$$f\left(\frac{1}{x}\right) + x^2 f(x) = 0 \Rightarrow f(x) = -\frac{1}{x^2} \ f\left(\frac{1}{x}\right)$$

$$\Rightarrow \text{I} = \int_{\sin\theta}^{\text{cosec}\theta} f(x) \ dx = \int_{\sin\theta}^{\text{cosec}\theta} -\frac{1}{x^2} f\left(\frac{1}{x}\right) dx$$

$$\frac{1}{x} = t \Rightarrow -\frac{1}{x^2} dx = dt$$

$$\Rightarrow I = \int_{\text{cosec}\theta}^{\sin\theta} f(t) \ dt \Rightarrow I = -\int_{\sin\theta}^{\text{cosec}\theta} f(t) \ dt = -\text{I} \Rightarrow 2I = 0 \Rightarrow I = 0$$

**14. Ans. (1)**

$\int_0^1 (C_2x^2 + C_1x + C_0)\, dx$

$= \left[\frac{C_2x^3}{3} + \frac{C_1x^2}{2} + C_0x\right]_0^1 = \frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} = 0$ (given)

$\Rightarrow$ graph $y = C_2x^2 + C_1x + C_0$ crosses x-axis atleast once.

$\Rightarrow$ at least one root of the equation $C_2x^2 + C_1x + C_0 = 0$ is present in $(0, 1)$

**15. Ans. (2)**

$f(x + \pi) = \int_0^{x+\pi} (2\cos^2 3t + 3\sin^2 3t)\, dt = \int_0^{x} (2\cos^2 3t + 3\sin^2 3t)\, dt + \int_x^{x+\pi} (2\cos^2 3t + 3\sin^2 3t)\, dt$

$t = x + y$

$= f(x) + \int_0^{\pi} (2\cos^2 3y + 3\sin^2 3y)\, dy = f(x) + 2\int_0^{\pi/2} (2\cos^2 3y + 3\sin^2 3y)\, dy = f(x) + 2f\left(\frac{\pi}{2}\right)$

**16. Ans. (2)**

$f'(x) = \frac{1}{\sqrt{1+x^3}}$ & $f''(x) = \frac{-3x^2}{2(1+x^3)^{3/2}}$, Also $g(f(x)) = x$

$\Rightarrow g''(f(x)) = -\frac{f''(x)}{(f'(x))^3} = \frac{3x^2}{2(1+x^3)^{3/2}}.(1+x^3)^{3/2} \Rightarrow g''(f(x)) = \frac{3x^2}{2}$

Let $f(x) = t \Rightarrow g(t) = x$

so $2g'' = 3g^2$.

**17. Ans. (3)**

$\frac{2}{x}\int_x^{2x} f(y)dy = x + 2 \Rightarrow 2\int_x^{2x} f(y)dy = x^2 + 2x$

differentiate

$\Rightarrow 2[2f(2x) - f(x)] = 2x + 2 \Rightarrow 2f(2x) - f(x) = x + 1$ ...(i)

$\Rightarrow 4f(4x) - 2f(2x) = 2\,(2x + 1) = 4x + 2$ ...(ii)

$\Rightarrow 8f(8x) - 4f(4x) = 2(8x + 2) = 16x + 4$ ...(iii)

Add (i) + (ii) and (iii) we get

$\Rightarrow 8f(8x) - f(x) = 21x + 7$

$\Rightarrow \int_0^1 (8f(8x) - f(x) - 21x)dx = \int_0^1 7dx = 7$

**18. Ans. (2)**

$I_n = \left.\frac{(\tan^{-1}x).x^{n+1}}{n+1}\right|_0^1 - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}dx \Rightarrow (n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{x^2+1}dx$

Similary $(n-1)I_{n-2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n-1}}{x^2+1}dx \Rightarrow (n+1)I_n + (n-1)I_{n-2} = \frac{\pi}{2} - \int_0^1 \frac{x^{n-1}(x^2+1)}{x^2+1}dx = \frac{\pi}{2} - \frac{1}{n}$

**19. Ans. (3)**

$$I = \int_{1/e}^{\tan x} \frac{t}{1+t^2}\, dt + \int_{1/e}^{\cot x} \frac{1}{t(1+t^2)}\, dt$$

put $t = \frac{1}{x}$

$$I = \int_{1/e}^{\tan x} \frac{t}{1+t^2}\, dt + \int_{e}^{\tan x} \frac{1}{\frac{1}{x}\left(1+\frac{1}{x^2}\right)} \cdot \frac{-1}{x^2} \,.\, dx$$

$$I = \int_{1/e}^{\tan x} \frac{t}{1+t^2} dt + \int_{\tan x}^{e} \frac{x}{1+x^2}\, dx = \int_{1/e}^{e} \frac{t}{1+t^2}\, dt = \left(\frac{1}{2}\ell n\ (1+t^2))\right)_{1/e}^{e} = 1$$

**20. Ans. (2)**

L.H.S. = $\int_0^x \left\{ \int_0^u f(t)\ dt \right\} du$

Integrating by parts taking 1 as 2nd function

L.H.S. = $\left[ u \int_0^u f(t)\ dt \right]_0^x - \int_0^x f(u)\ .udu = x\int_0^x f(t)\ dt - \int_0^x f(u)\ .\ udu$

$= x\int_0^x f(u)\ du - \int_0^x f(u)\ .\ udu = \int_0^x f(u).(x-u)du = R.H.S.$

**SECTION-B**

**1. Ans. (61)**

Note that $(x + 1)^3 - (x - 1)^3 = 2(3x^2 + 1)$

Hence $I = \frac{1}{2}\left[\int_2^4 \frac{(x+1)^3 - (x-1)^3}{(x+1)^3(x-1)^3} dx\right] = \frac{1}{2}\left[\int_2^4 \frac{dx}{(x-1)^3} - \int_2^4 \frac{dx}{(x+1)^3}\right]$

$= \frac{1}{2}\left[\frac{1}{2(x+1)^2} - \frac{1}{2(x-1)^2}\right]_2^4 = \frac{1}{4}\left[\left(\frac{1}{25}-\frac{1}{9}\right)-\left(\frac{1}{9}-1\right)\right] = \frac{46}{225}$

**2. Ans. (64)**

$$U = \int_{\pi/6}^{\pi/2} \cos x dx = \sin x\Big|_{\pi/6}^{\pi/2} = \frac{1}{2}$$

$$V = \int_{-3}^{1} (-x^2)dx + \int_1^5 x^2 dx = \frac{1}{3}[-(1+27)+(125-1)] = \frac{1}{3}[-28+124] = 32$$

$\therefore V = \lambda U \Rightarrow \lambda = 64$

**3. Ans. (10)**

$$\int_1^{100} \frac{f(x)}{x} dx = \int_1^{10} \frac{f(x)}{x} dx + \int_{10}^{100} \frac{f(x)}{x} dx = 5 + \int_{10}^{1} \frac{f\left(\frac{100}{t}\right)}{100} .t\left(\frac{-100dt}{t^2}\right) \text{ (substituting } x = \frac{100}{t})$$

$= 5 + \int_t^{10} f\left(\frac{100}{t}\right).\frac{dt}{t} = 5 + \int_1^{10} \frac{f(t)}{t} dt = 10$

**4. Ans. (29)**

$$\frac{2005\int_0^{1002}\frac{\sqrt{1003^2-x^2}-\sqrt{1002^2-x^2}}{2005}+\int_{1002}^{1003}\sqrt{1003^2-x^2}\,dx}{\int_0^1\sqrt{1-x^2}\,dx}$$

$$=\frac{\int_0^{1003}\sqrt{1003^2-x^2}\,dx-\int_0^{1002}\sqrt{1002^2-x^2}\,dx}{\int_0^1\sqrt{1-x^2}\,dx}$$

Put $x = 1003t$

Hence $2^2 + 5^2 = 29$

**5. Ans. (4)**

$I = \int_0^{\pi/2}\sqrt{\sin 2\theta}\cos\theta d\theta$ ; $I = \int_0^{\pi/2}\sqrt{\sin 2\theta}\sin\theta d\theta$ (By property)

Let $\sin\theta - \cos\theta = t$ ; $2I = \int_{-1}^{1}\sqrt{1-t^2} = 2\int_0^1\sqrt{1-t^2}\,dt$ ;

$I = \int_0^1\sqrt{1-t^2}\,dt = \frac{\pi}{4}$

**6. Ans. (4)**

Substitute $x = \tan\theta$ in $I_2$.

$I_2 = \int_0^{\pi/4}\frac{d\theta}{(1+\tan\theta)^2} = \int_0^{\pi/4}\frac{d\theta}{\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)^2}$ $\left[\int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

$= \int_0^{\pi/4}\frac{(1+\tan\theta)^2}{4}d\theta \Rightarrow I_2 = \frac{I_1}{4}$

**7. Ans. (2)**

$I = \int_0^1 \underbrace{e^{t^2+t}}_{f(t)} + \underbrace{t.e^{t^2+t}(2t+1)}_{f'(t)}dt \;=\; t.e^{t^2+t}\Big|_0^1 = e^2$

**8. Ans. (2)**

Let $I = \int_{-1}^{0}\frac{xe^{-x}}{(x+1)e^{-x}+1}dx$

substituting $1 + (x + 1)e^{-x} = t \Rightarrow -xe^{-x}\,dx = dt$

$I = -\int_1^2\frac{dt}{t} = -\ell n2$

**9. Ans. (0)**

$$I = \int_0^a f(x)\ g(x)\ h(x)\ dx$$

$$I = \int_0^a f(a-x)\ g(a-x)\ h(a-x)\ dx = \int_0^a f(x).(-g(x))\ \frac{3h(x)-5}{4}\ dx$$

$$I = -\frac{3}{4}\int_0^a f(x)g(x)h(x)+\frac{5}{4}\int_0^a f(x)g(x)\ dx$$

$$\frac{7}{4}\ I = 0 \Rightarrow I = 0$$

$$\because f(x)\ g(x)\ dx = 0$$

$$I_1 = \int_0^a f(x)\ g(x)\ dx \qquad \text{...(i)}$$

$$I_1 = \int_0^a f(a-x)\ g(a-x)\ dx$$

$$I_1 = \int_0^a f(x)\ (-g(x))\ dx \qquad \text{...(ii)}$$

(i) + (ii) $2I_1 = 0 \Rightarrow I_1 = 0$

**10. Ans. (2)**

$$I = \int_0^\pi f(x)\ dx = \int_0^\pi \frac{\sin x}{x}\ dx \qquad \text{...(i)}$$

$$I = \int_0^\pi f(\pi-x)\ dx = \int_0^\pi \frac{\sin(\pi-x)}{\pi-x}\ dx = \int_0^\pi \frac{\sin x}{\pi-x}dx \qquad \text{...(ii)}$$

(i) + (ii)

$$\Rightarrow 2I = \int_0^\pi \left\{\frac{\sin x}{x}+\frac{\sin x}{\pi-x}\right\}\ dx \ \Rightarrow I = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{x(\pi-x)}dx \qquad \text{...(iii)}$$

$$\text{Now } \frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(x) f\left(\frac{\pi}{2}-x\right)dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\times\frac{\sin\left(\frac{\pi}{2}-x\right)}{\frac{\pi}{2}-x}\ dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\cdot\frac{\cos x}{\frac{\pi}{2}-x}\ dx = \cdot\ \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin\ 2x}{x\left(\frac{\pi}{2}-x\right)}\ dx$$

$$= \frac{\pi}{8}\int_0^\pi \frac{\sin t}{\frac{t}{2}\left(\frac{\pi}{2}-\frac{t}{2}\right)}dt,\ \text{where } t = 2x \qquad \text{... (iv)}$$

(iii) + (iv)

$$\Rightarrow \frac{\pi}{2}\int_0^\pi f(x)\ \ f\left(\frac{\pi}{2}-x\right)dx = \int_0^\pi f(x)\ dx$$

## JEE (Advanced) Practice Paper

**1.** **Ans. (C)**

$$\int_0^1 (\{2x\}-1)(\{3x\}-1)\,dx$$

$$= \int_0^{1/3}(2x-1)(3x-1)dx + \int_{1/3}^{1/2}(2x-1)(3x-2)dx + \int_{1/2}^{2/3}(2x-2)(3x-2)dx + \int_{2/3}^{1}(2x-2)(3x-3)dx$$

$$= \int_0^{1/3}(6x^2-5x+1)\,dx + \int_{1/3}^{1/2}(6x^2-7x+2)\,dx + \int_{1/2}^{2/3}(6x^2-10x+4)\,dx + \int_{2/3}^{1}(6x^2-12x+6)\,dx = \frac{19}{72}.$$

**2.** **Ans. (B)**

$$\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)\,dx\right) = \int_0^1 f(x)\,dx+\int_0^1 f(1+x)\,dx+\int_0^1 f(2+x)\,dx+.......+\int_0^1 f(99+x)\,dx$$

$$= \int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx+.....+\int_{99}^{100} f(x)\,dx$$

{using shifting property}

$$= \int_0^{100} f(x)dx = a$$

**3.** **Ans. (A)**

$$\lim_{t\to\left(\frac{\pi}{2}\right)^-}\int_0^t \frac{\sin\theta}{\sqrt{\cos\theta}}.\ln(\cos\theta)d\theta$$ , let $\cos\theta = y^2$

$$2\lim_{a\to 0^+}\int_0^t \frac{\ell n(y^2)}{y}ydy = 4.\lim_{a\to 0^+}\int_0^t \ell nydy = 4\{y\ell ny-y\}_a^1 = 4.\lim_{a\to 0^+}\{-1+a-a\ell na\} = -4$$

**4.** **Ans. (C)**

$$\int_0^2 f(x)\,dx = \int_0^{1/2}1.dx+\int_{1/2}^{2/3}1.dx+\int_{2/3}^{3/4}1.dx+.....+\int_{\frac{n-1}{n}}^{\frac{n}{n+1}}1.dx + .........+\int_1^2 1.\,dx$$

$$= \left(\frac{1}{2}\right)+\left(\frac{2}{3}-\frac{1}{2}\right)+\left(\frac{3}{4}-\frac{2}{3}\right)+....+\left(\frac{n}{n+1}-\frac{n-1}{n}\right)+....+1=\frac{n}{n+1}+....+1 \text{ as } n\to\infty$$

taking limit $n\to\infty$

we get $\int_0^2 f(x)\,dx = 1+1=2$

**5.** **Ans. (D)**

Let $I = \int_0^\infty e^{-ax^2}dx$

Put $\sqrt{a}x = t \Rightarrow dx = \frac{dt}{\sqrt{a}}$

then $I = \frac{1}{\sqrt{a}}\int_0^\infty e^{-t^2}dt = \frac{1}{\sqrt{a}}\int_0^\infty e^{-x^2}dx = \frac{1}{\sqrt{a}}\frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

**6.** **Ans. (A)**

$$\sum_{i=1}^{4}\left(\sin^{-1}x_i+\cos^{-1}y_i\right) = 6\pi \Rightarrow x_1 = x_2 = x_3 = x_4 = 1 \ \& \ y_1 = y_2 = y_3 = y_4 = -1$$

hence $I = \int_{-4}^{4} x\ln(1+x^2)\left(\frac{e^x}{1+e^{2x}}\right)dx = 0$ (as $f(x)$ is an odd function)

**7.** **Ans. (A,B)**

$$I_n = \int_0^1 \frac{dx}{(1+x^2)^n} = \int_0^1 (1+x^2)^{-n}dx = \left[\frac{x}{(1+x^2)^n}\right]_0^1 - \int_0^1 (-n)(1+x^2)^{-n-1}2x^2\,dx$$

$$= \frac{1}{2^n}+2\ n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}} = \frac{1}{2^n}+2\ n\int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}}\ dx = \frac{1}{2^n}+2\ n\ \ I_n-2n\ \ I_{n+1}$$

$$\therefore\ 2nI_{n+1}=2^{-n}+(2n-1)I_n$$

$$\therefore\ 2I_2 = \frac{1}{2}+I_1 = \frac{1}{2}+\left[\tan^{-1}x\right]_0^1 \Rightarrow I_2 = \frac{1}{4}+\frac{\pi}{8}$$

**8.** **Ans. (A,B,C)**

$f(-x) = -f(x)$ ...(1)

$f(x+2) = f(x)$ ...(2)

$g(2n) = \int_0^{2n} f(t)\ dt = n\int_0^2 f(t)\ dt \Rightarrow g(2n) = n\,g(2)$ ...(3)

Now $g(-x) = \int_0^{-x} f(t)dt$

Put $t = -z \Rightarrow dt = -dz = \int_0^x f(-z)\ (-dz) = -\int_0^x f(-z)\ dz$ (from (1))

$= \int_0^x f(t)\ dt = g(x)$

$\therefore\ g(-x) = g(x)$

Again $g(x+2) = \int_0^x f(t)dt + \int_x^{x+2} f(t)dt$

$\therefore\ g(x+2) = \int_0^x f(t)\ dt + \int_0^2 f(t)\ dt$ ($\because\ f \to$ period)

$\Rightarrow g(x+2) = g(x)+g(2)$ ...(4)

Putting $x = 0, 2, ......$

$g(2) = g(0) + g(2) \Rightarrow g(0) = 0$

$g(4) = g(2) + g(2) \Rightarrow g(4) = 2g(2)$

putting $x \to -x$ we get

$g(2-x) = g(-x) + g(2) = g(x) + g(2)$

at $x = 2$

$g(0) = 2g(2) \Rightarrow g(2) = 0$

$\therefore\ g(0) = g(\pm 2) = g(\pm 4) = ..... = 0$

from (3) $g(2n) = 0$

& from (4) $g(x+2) = g(x) \Rightarrow$ prd. of $g(x)$ is 2

**9. Ans. (A,B)**

$$f(x) = \int_{-1}^{e^x}\frac{dt}{1+t^2} + \int_{1}^{e^{-x}}\frac{dt}{1+t^2}$$

$$f'(x) = \frac{e^x}{1+e^{2x}} - \frac{e^x}{1+e^{2x}} = \frac{e^x}{1+e^{2x}} - \frac{e^x}{1+e^{2x}} = 0$$

Hence $f(x)$ = costant = $f(0) = \int_{-1}^{1}\frac{dt}{1+t^2} = \frac{\pi}{2}$

**10. Ans. (B,D)**

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\left(a+\frac{k}{n}\right)\left(b+\frac{k}{n}\right)}\int_0^1\frac{dx}{(a+x)(b+x)} = \text{I (say)}$$

If $a = b$

$$I = \int_0^1\frac{dx}{(a+x)^2} = -\frac{1}{a+x}\Big|_0^1 = -\left(\frac{1}{a+1}-\frac{1}{a}\right) = \frac{1}{a(a+1)}$$

If $a \neq b$

$$I = \frac{1}{a-b}\int_0^1\left(\frac{1}{b+x}-\frac{1}{a+x}\right)dx = \frac{1}{a-b}\ell n\left(\frac{b+x}{a+x}\right)\Bigg|_0^1$$

$$= \frac{1}{a-b}\ell n\left(\frac{a(b+1)}{b(a+1)}\right)$$

**11. Ans. (B,C,D)**

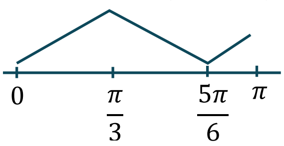
Since |sinθ| is continuous $f(x)$ will be differentiable

$$f'(x) = \left|\sin\left(x+\frac{\pi}{3}\right)\right| - |\sin x|$$

$$f'(x) = 0 \Rightarrow \sin^2\left(x+\frac{\pi}{3}\right) = \sin^2 x \Rightarrow \sin\left(2x+\frac{\pi}{3}\right).\sin\frac{\pi}{3} = 0$$

$$\Rightarrow 2x + \frac{\pi}{3} = \pi, 2\pi \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{6}$$

Indicator diagram of $f(x)$ is




$$f(0) = \frac{1}{2}$$

$$f\left(\frac{\pi}{3}\right) = 1$$

$$f\left(\frac{5\pi}{6}\right) = 2\int_{5\pi/6}^{\pi}\sin x\,dx = 2-\sqrt{3}$$

$$f(\pi) = \frac{1}{2}$$

**12. Ans. (A,C)**

Consider $I_{n+2} - 2I_{n+1} + I_n = \int_0^{\pi} \frac{\sin^2(n+2)x - 2\sin^2(n+1)x + \sin^2(nx)}{\sin^2 x} dx$

$= \int_0^{\pi} \frac{\{\sin^2(n+2)x - \sin^2(n+1)x\} - \{\sin^2(n+1)x - \sin^2 nx\}}{\sin^2 x} dx = \int_0^{\pi} \frac{\sin x\{\sin(2n+3)x - \sin(2n+1)x\}}{\sin^2 x} dx$

$= \int_0^{\pi} \frac{2\cos(2n+2)x \cdot \sin x}{\sin x} dx = \frac{1}{n+1}\sin(2n+2)x\Big|_0^{\pi} = 0$

Hence $I_1, I_1, I_2$, ......... are in A.P.

$I_1 = \int_0^{\pi} \frac{\sin^2 x}{\sin^2 x} dx = \pi$, $I_2 = \int_0^{\pi} \frac{4\sin^2 x\cos^2 x}{\sin^2 x} dx = 2\pi$

Hence $I_n = \pi + (n-1)\pi = n\pi$

**13. Ans. (A,B,C)**

$\lim_{n\to\infty} \frac{(1^k + 2^k + 3^k + ..... + n^k)}{(1^2 + 2^2 + ..... + n^2)(1^3 + 2^3 + ........ + n^3)}$

Now $\lim_{n\to\infty} \frac{1^k + 2^k + ......n^k}{n^{k+1}} = \int_0^1 x^k dx = \frac{1}{k+1} \quad \forall p \in N$

Hence the given limit is finite for $k \le 6$ & equals 12/7 for $k = 6$.

**14. Ans. (C)**

At $x = 1, y = 0$

$\frac{dy}{dx} = 2x \cdot x^8 - x^4 = 2 - 1 = 1$

$\therefore$ equation of tangent $y - 0 = 1\ (x - 1)$

**15. Ans. (A)**

$F(x) = \int_1^x e^{t^2/2}\ (1 - t^2)\ dt$

$F'(x) = \left(e^{\frac{x^2}{2}}(1 - x^2)\right)$

$\therefore F'(1) = 0$

**16. Ans. (A)**

$\frac{dy}{dx} = 4x^3\ (\ell n\ x^4)^2 - 3x^2\ (\ell n\ x^3)^2$

$= 4x^3\ (4\ \ell n\ x)^2 - 3x^2\ (3\ \ell n\ x)^2$

$= 64x^3\ (\ell n\ x)^2 - 27\ x^2\ (\ell n\ x)^2$

$\therefore \lim_{x\to 0^+} \frac{dy}{dx} = 64 \lim_{x\to 0^+} x^3\ (\ell n\ x)^2 - 27 \lim_{x\to 0^+} x^2\ (\ell n\ x)^2 = 0$

**17. Ans. (A→q; B→s; C→p; D→t)**

Let $I = \dfrac{\lim\limits_{T\to\infty}\int\limits_0^T (\sin x + \sin ax)^2\,dx}{T}$

(A) If $a = 0, I = \lim\limits_{T\to\infty}\dfrac{\int\limits_0^T \sin^2 x\,dx}{T} = \lim\limits_{T\to\infty}\dfrac{\int\limits_0^T (1-\cos 2x)\,dx}{2T} = \lim\limits_{T\to\infty}\dfrac{T-\dfrac{\sin 2T}{2}}{2T} = \dfrac{1}{2}$

(B) If $a = 1, I = \lim\limits_{T\to\infty}\dfrac{4\int\limits_0^T \sin^2 x\,dx}{T} = 2$

(C) If $a = -1$ $I = 0$

(D) If $a \in R - \{-1, 0\}$

$$I = \lim_{T\to\infty}\frac{\int\limits_0^T (\sin^2 x + \sin^2 ax + 2\sin x \sin ax)dx}{T}$$

$$= \lim_{T\to\infty}\frac{1}{2T}\left\{\int\limits_0^T (1-\cos 2x)dx + \int\limits_0^T (1-\cos 2ax)dx + \int\limits_0^T \{\cos(a-1)x - \cos(a+1)x\}\,dx\right\}$$

$$= \lim_{T\to\infty}\frac{1}{2T}\left\{2T - \frac{\sin 2T}{2} - \frac{\sin 2aT}{2a} + \frac{\sin(a-1)T}{a-1} - \frac{\sin(a+1)T}{a+1}\right\} = 1$$

**18. Ans. (25)**

Let $I = \int\limits_{-\pi+a}^{3\pi+a} |x-a-\pi|\sin\left(\dfrac{x}{2}\right)dx$

Let $x - a - \pi = t$

$\Rightarrow I = \int\limits_{-2\pi}^{2\pi} |t|\sin\left(\dfrac{t+a}{2}+\dfrac{\pi}{2}\right)dt = \int\limits_{-2\pi}^{2\pi} |t|\cos\left(\dfrac{a+t}{2}\right)dt = \int\limits_0^{2\pi} t\left\{\cos\left(\dfrac{a+t}{2}\right)+\cos\left(\dfrac{a-t}{2}\right)\right\}dt$

$= \int\limits_0^{2\pi} t\left\{2\cos\dfrac{a}{2}\cos\dfrac{t}{2}\right\}dt = 2\cos\left(\dfrac{a}{2}\right)\int\limits_0^{2\pi} t\cdot\cos\left(\dfrac{t}{2}\right)dt$ Let $\dfrac{t}{2} = y$

$8\cos\left(\dfrac{a}{2}\right)\int\limits_0^{\pi} y\cos y\,dy = 8\cos\left(\dfrac{a}{2}\right)\{y\sin y + \cos y\}_0^{\pi} = -16\cos\left(\dfrac{a}{2}\right)$

Now $I = -16 \Rightarrow \cos\left(\dfrac{a}{2}\right) = 1 \Rightarrow a = 4k\pi, k \in \mathrm{I}$

Hence number of value of '$a$' is 25

**19. Ans. (4)**

$$\int\limits_0^1 \frac{dx}{1+x^2} = \int\limits_0^1 (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + .........)dx$$

$$\Rightarrow \tan^{-1} x\Big|_0^1 = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}.......\Big|_0^1 \Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}.......$$

**20. Ans. (65)**

$f(x) = x + \int_0^1 t(x+t)\ f(t)dt\ ;\ f(x) = x + x\int_0^1 t\ f(t)dt + \int_0^1 t^2\ f(t)dt$

Let $\int_0^1 t\ f(t)dt = a$ & $\int_0^1 t^2\ f(t)dt = b$

Hence $f(x) = (a+1)x + b$

so $a = \int_0^1 t\{(a+1)t+b\}dt = \frac{a+1}{3} + \frac{b}{2} \Rightarrow 4a - 3b = 2$ ...(1)

& $b = \int_0^1 t^2\{(a+1)t+b\}dt = \frac{a+1}{4} + \frac{b}{3} \Rightarrow 8b - 3a = 3$ ...(2)

from (1) & (2) $a = \frac{25}{23}$ & $b = \frac{18}{23}$

so $\int_0^1 f(t)dt = \frac{6}{23}\int_0^1 (8t+3)dt = \frac{6}{23}.7 = \frac{42}{23} \Rightarrow p + q = 65$

**21. Ans. (5)**

$f(x).e^{-x} = \int_0^x e^{-y}.e^{y}(ay+b+a)dy - (x^2+x+1)$

$f'(x)\ .e^{-x} - f(x).e^{-x} = ax + (a+b) - (2x+1)$

$f'(x) - f(x) = e^{x}\{(a-2)x + (a+b-1)\}$

$\Rightarrow \{(ax+b+a) - (ax+b)\}e^{x} = e^{x}\{(a-2)x + (a+b-1)\}$

$\Rightarrow a - 2 = 0$ & $a = a + b - 1 \Rightarrow a = 2$ & $b = 1$

$\Rightarrow a^2 + b^2 = 5$

**22. Ans. (11)**

$f(x) = \int_0^x \frac{d\theta}{\cos\theta} + \int_x^{\pi/2} \frac{d\theta}{\sin\theta} \Rightarrow f'(x) = \frac{1}{\cos x} - \frac{1}{\sin x} = \frac{\sin x - \cos x}{\sin x \cdot \cos x}$

$f'(x) = 0 \Rightarrow x = \frac{\pi}{4} \Rightarrow f'(x)$ changes sign from (–) to (+) hence it is a point of minima.

Now $f\left(\frac{\pi}{4}\right) = \int_0^{\pi/4} \sec\theta d\theta + \int_{\pi/4}^{\pi/2} cosec\ d\theta = 2\int_0^{\pi/4} \sec\theta d\theta = 2\ln(\sec\theta + \tan\theta)\Big|_0^{\pi/4} = 2\ell n(\sqrt{2} + 1)$

$= \ell n(3 + \sqrt{8})$

Hence $a + b = 11$

**23. Ans. (3)**

$5 = \int_0^{\pi} f(x).\sin x\ dx + \int_0^{\pi} \sin x.f''(x)\ dx$

$5 = -(f(x).\cos x)_0^{\pi} + \int_0^{\pi} f'(x)\ \cos x dx + (\sin x.f'(x))_0^{\pi} - \int_0^{\pi} \cos x f'(x)\ dx$

$5 - 2 = f(0) \Rightarrow f(0) = 3$

**24.** **Ans.(2)**

$f'(x) = 6(x^2 - 5x + 4)$

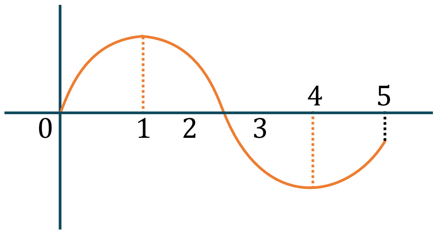
$g'(x) = f(x) - f(5 - x)$

for function $g(x)$ to be increasing

$g'(x) > 0$

Now graph of $f(x)$ will be as shown

If $0 < x < \frac{5}{2}$




$f(x) > f(5 - x)$

So $g(x)$ is increasing in $\left(0, \frac{5}{2}\right]$. Hence 2 integers 1 and 2.