COMPLEX NUMBER

INTRODUCTION

Indian mathematician Mahavira (850 A.D.) was first to mention in his work 'Ganitasara Sangraha'; 'As in nature of things a negative (quantity) is not a square (quantity), it has, therefore, no square root'. Hence there is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$.

A symbol $\sqrt{-1}$, denoted by letter i was intrdouced by Swiss Mathematician, Leonhard Euler (1707-1783) in 1748 to provide solutions of equation $x^2 + 1 = 0$. i was regarded as a fictitious or imaginary number which could be manipulated algebrically like an ordinary real number, except that its square was – 1. The letter i was used to denote $\sqrt{-1}$, possibly because i is the first letter of the Latin word 'imaginarius'.

DEFINITION

Complex numbers are definited as expressions of the form a + ib where $a, b \in R$ & $i = \sqrt{-1}$. It is denoted by z i.e. $z = a + ib$. 'a' is called as real part of z (Re z) and 'b' is called as imaginary part of z (Im z).

EVERY COMPLEX NUMBER CAN BE REGARDED AS

Purely real	Purely imaginary	Imaginary
if $b = 0$	if $a = 0$	if $b \neq 0$

(A) The set R of real numbers is a proper subset of the Complex Numbers. Hence the complete number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(B) Zero is purely real as well as purely imaginary but not imaginary.

(C) $i = \sqrt{-1}$ is called the imaginary unit.

Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

(D) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of a or b is non - negative.

(E) If $z = a + ib$, then $a - ib$ is called complex conjugate of z and written as $\bar{z} = a - ib$

(F) Real numbers satisfy order relations where as imaginary numbers do not satisfy order relations i.e. $i > 0$, $3 + i < 2$ are meaningless.

Ex. Write the following as complex number

(i) $\sqrt{-16}$ **(ii)** $\sqrt{x}$, $(x > 0)$ **(iii)** $-b + \sqrt{-4ac}$, $(a, c > 0)$

Sol. **(i)** $0 + 4i$ **(ii)** $\sqrt{x} + 0i$ **(iii)** $-b + i\sqrt{4ac}$

Ex. The value of $i^{57} + 1/i^{125}$ is :-

Sol. $i^{57} + 1/i^{125} = i^{56}.\ i + \frac{1}{i^{124}.i}$

$$= \left(i^4\right)^{14} i + \frac{1}{\left(i^4\right)^{31} i}$$

$$= i + \frac{1}{i} = i + \frac{i}{i^2} = i - i = 0$$

ALGEBRAIC OPERATIONS

Fundamental operations with complex numbers

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by – 1 when it occurs.

1. Addition $(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d) i$

2. Subtraction $(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d) i$

3. Multiplication $(a + bi) (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$

4. Division $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} . \frac{c-di}{c-di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2i^2}$

$$= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$$

Inequalities in imaginary numbers are not defined. There is no validity if we say that imaginary number is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

In real numbers if $a^2 + b^2 = 0$ then $a = 0 = b$ however in complex numbers,
$z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

(i) The algebraic operations on complex numbers are similar to those on real numbers treating i as a polynomial.

(ii) Inequalities in complex numbers (non-real) are not defined. There is no validity if we say that complex number (non-real) is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

(iii) In real numbers, if $a^2 + b^2 = 0$, then $a = 0 = b$ but in complex numbers, $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

Ex. Find multiplicative inverse of 3 + 2i.

Sol. Let z be the multiplicative inverse of 3 + 2i. then

$\Rightarrow$ $z\,.\,(3 + 2i) = 1$

$\Rightarrow$ $z = \dfrac{1}{3+2i} = \dfrac{3-2i}{(3+2i)(3-2i)}$

$\Rightarrow$ $z = \dfrac{3}{13} - \dfrac{2}{13}i$

$\left(\dfrac{3}{13} - \dfrac{2}{13}i\right)$

Ex. $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if $\theta =$

Sol. $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if the real part vanishes, i.e.,

$$\frac{(3+2i\sin\theta)}{(1-2i\sin\theta)} \times \frac{(1+2i\sin\theta)}{(1+2i\sin\theta)} = \frac{(3-4\sin^2\theta) + i(8\sin\theta)}{(1+4\sin^2\theta)}$$

$\dfrac{3-4\sin^2\theta}{1+4\sin^2\theta} = 0 \quad \Rightarrow \quad 3 - 4\sin^2\theta = 0$ (only if θ be real)

$\Rightarrow$ $\sin^2\theta = \left(\dfrac{\sqrt{3}}{2}\right)^2 = \left(\sin\dfrac{\pi}{3}\right)^2$

$\Rightarrow$ $\theta = n\pi \pm \dfrac{\pi}{3}, n \in I$

EQUALITY IN COMPLEX NUMBER

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real and imaginary parts are equal respectively

i.e. $z_1 = z_2 \quad \Leftrightarrow \quad Re(z_1) = Re(z_2)$ and $I_m(z_1) = I_m(z_2)$.

Ex. Find the value of x and y for which $(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$ where $x, y \in R$.

Sol. $(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i$

$\Rightarrow$ $2x^2 - 3y = 2x - 3y$

$\Rightarrow$ $x^2 - x = 0$

$\Rightarrow$ $x = 0, 1$ and $3x^2 + 2y = 5$

$\Rightarrow$ if $x = 0,\ y = \dfrac{5}{2}$ and if $x = 1, y = 1$

$\therefore$ $x = 0, y = \dfrac{5}{2}$ and $x = 1, y = 1$

are two solutions of the given equation which can also be represented as $\left(0, \dfrac{5}{2}\right)$ & (1, 1)

$\left(0, \dfrac{5}{2}\right)$, (1, 1)

Ex. If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Sol. We have , $x = -5 + 2\sqrt{-4}$

$\Rightarrow \quad x + 5 = 4i \qquad \Rightarrow \quad (x+5)^2 = 16i^2$

$\Rightarrow \quad x^2 + 10x + 25 = -16 \qquad \Rightarrow \quad x^2 + 10x + 41 = 0$

Now,

$x^4 + 9x^3 + 35x^2 - x + 4$

$\Rightarrow \quad x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$

$\Rightarrow \quad x^2(0) - x(0) + 4(0) - 160$

$\Rightarrow \quad -160$

Ex. Find square root of 9 + 40i

Sol. Let $x + iy = \sqrt{9 + 40i}$

$(x + iy)^2 = 9 + 40i$

$\therefore \quad x^2 - y^2 = 9$ .....**(i)**

and $\quad xy = 20$ ......**(ii)**

squaring **(i)** and adding with 4 times the square of **(ii)**

we get $x^4 + y^4 - 2x^2y^2 + 4x^2y^2 = 81 + 1600$

$\Rightarrow \quad (x^2 + y^2)^2 = 1681$

$\Rightarrow \quad x^2 + y^2 = 41$ ......**(iii)**

from **(i)** + **(iii)** we get $x^2 = 25 \Rightarrow x = \pm 5$

and $y^2 = 16 \Rightarrow y = \pm 4$

from equation **(ii)** we can see that

x & y are of same sign

$\therefore \quad x + iy = (5 + 4i)$ or $-(5 + 4i)$

$\therefore \quad$ sq. roots of $9 + 40i = \pm(5 + 4i)$ **and** $\pm(5 + 4i)$

CONJUGATE OF A COMPLEX NUMBER

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

IMPORTANT PROPERTIES OF CONJUGATE

(A) $z + \bar{z} = 2\,\text{Re}(z)$ **(B)** $z - \bar{z} = 2i\,\text{Im}(z)$ **(C)** $\overline{(\bar{z})} = z$

(D) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ **(E)** $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(F) $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$. In general $\overline{z_1 z_2 \ldots\ldots\ldots z_n} = \bar{z}_1 . \bar{z}_2 \ldots\ldots\ldots \bar{z}_n$

(G) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$; $z_2 \neq 0$ **(H)** If $f(\alpha + i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

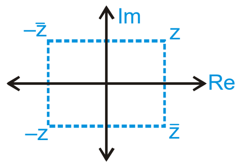
Note that

(iii) $z\bar{z} = a^2 + b^2$, which is purely real

(iv) If z is purely real, then $z - \bar{z} = 0$

(v) If z is purely imaginary, then $z + \bar{z} = 0$

(vi) If z lies in the 1^{st} quadrant, then $\bar{z}$ lies in the 4^{th} quadrant and $-\bar{z}$ lies in the 2^{nd} quadrant.



Modulus

If P denotes complex number z = x + iy, then the length OP is called modulus of complex number z. It is denoted by |z|.

$$OP = |z| = \sqrt{x^2 + y^2}$$

Geometrically $|z|$ represents the distance of point P from origin. ($|z| \geq 0$)

IMPORTANT PROPERTIES OF MODULUS

(A) $|z| \geq 0$

(B) $|z| \geq \text{Re}(z)$ **(C)** $|z| \geq \text{Im}(z)$

(D) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$ **(E)** $z\bar{z} = |z|^2$

(F) $|z_1 z_2| = |z_1|.|z_2|$. In general $|z_1 z_2 ....z_n| = |z_1| .|z_2| ....|z_n|$

(G) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(H) $|z^n| = |z|^n$, $n \in I$

(I) $|z_1 + z_2|^2 = |z|^2 + |z_2|^2 + 2\,\text{Re}\left(z_1\bar{z}_2\right)$

(J) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\alpha - \beta)$, where α, β are $\arg(z_1), \arg(z_2)$ respectively.

(K) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

(L) $\left||z_1| - |z_2|\right| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

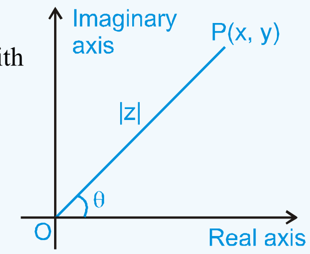
(M) $\left||z_1| - |z_2|\right| \leq |z_1 - z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

Unlike real numbers, $|z| = \begin{cases} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{cases}$ is not correct.

(C) Argument or Amplitude :

If P denotes complex number $z = x + iy$ and if OP makes an angle θ with real axis, then θ is called one of the arguments of z.

$\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive real axis)



IMPO7RTANT PROPERTIES OF AMPLITUDE

(A) $\text{amp}\,(z_1 . z_2) = \text{amp}\ z_1 + \text{amp}\ z_2 + 2\,k\pi\ ;\ k \in I$

(B) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}\ z_1 - \text{amp}\ z_2 + 2\,k\pi\ ;\ k \in I$

(C) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi\ ; n,k \in I$

where proper value of k must be chosen so that RHS lies in $(-\pi\,,\pi\,]$.

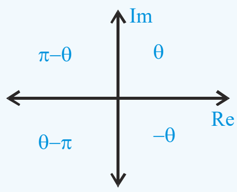
(i) Argument of a complex number is a many valued function. If θ is the argument of a complex number, then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \le \pi$ is called Amplitude (principal value of the argument).

(iii) Principal argument of a complex number $z = x + iy$ can be found out using method given below :

(A) Find $\theta = \tan^{-1}\left|\dfrac{y}{x}\right|$ such that $\theta \in \left(0, \dfrac{\pi}{2}\right)$.

(B) Use given figure to find out the principal argument according as the point lies in respective quadrant.



(iv) Unless otherwise stated, amp z implies principal value of the argument.

(v) The unique value of $\theta = \tan^{-1}\dfrac{y}{x}$ such that $0 < \theta \le 2\pi$ is called least positive argument.

(vi) If $z = 0$, arg(z) is not defined

(vii) If z is real & negative, $\arg(z) = \pi$.

(viii) If z is real & positive, $\arg(z) = 0$

(ix) If $\theta = \dfrac{\pi}{2}$, z lies on the positive side of imaginary axis.

(x) If $\theta = -\dfrac{\pi}{2}$, z lies on the negative side of imaginary axis.

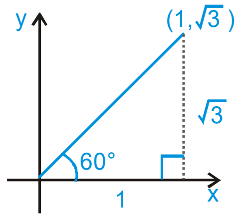
By specifying the modulus & argument a complex number is defined completely. Argument impart direction & modulus impart distance from origin.

For the complex number $0 + 0i$ the argument is not defined and this is the only complex number which is given by its modulus only.

Ex. Find the modulus, argument, principal value of argument, least positive argument of complex numbers

(A) $1+i\sqrt{3}$ **(B)** $-1+i\sqrt{3}$ **(C)** $1-i\sqrt{3}$ **(D)** $-1-i\sqrt{3}$

Sol. **(A)** For $z = 1+i\sqrt{3}$



$|z| = \sqrt{1^2+(\sqrt{3})^2} = 2$

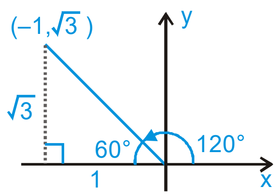
$\arg(z) = 2n\pi + \frac{\pi}{3}$, $n \in I$

Least positive argument is $\frac{\pi}{3}$

If the point is lying in first or second quadrant then amp(z) is taken in anticlockwise direction.

In this case amp(z) $= \frac{\pi}{3}$

(B) For $z = -1 + i\sqrt{3}$



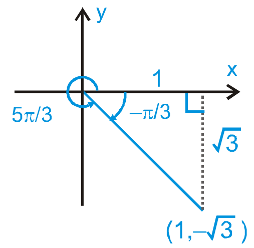
$|z| = 2$

$\arg(z) = 2n\pi + \frac{2\pi}{3}$, $n \in I$

Least positive argument $= \frac{2\pi}{3}$

$\text{amp}(z) = \frac{2\pi}{3}$

(C) For $z = 1 - i\sqrt{3}$



$|z| = 2$

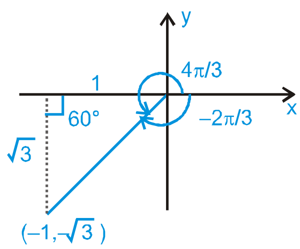
$\arg(z) = 2n\pi - \frac{\pi}{3}$, $n \in I$

Least positive argument $= \frac{5\pi}{3}$

If the point lies in third or fourth quadrant then consider amp(z) in clockwise direction.

In this case amp(z) $= -\frac{\pi}{3}$

(D) For $z = -1 - i\sqrt{3}$



$|z| = 2$

$\arg(z) = 2n\pi - \frac{2\pi}{3}$, $n \in I$

Least positive argument $= \frac{4\pi}{3}$

$\text{amp}(z) = -\frac{2\pi}{3}$

Ex. Find amp z and |z| if $z=\left[\dfrac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right]^2$.

Sol. amp z = $2\left[\text{amp}(3+4i)+\text{amp}(1+i)+\text{amp}(1+\sqrt{3}i)-\text{amp}(1-i)-\text{amp}(4-3i)-\text{amp}(2i)\right]+2k\pi$ where $k\in I$ and k chosen so that amp z lies in $(-\pi,\pi]$.

$$\Rightarrow \quad \text{amp } z=2\left[\tan^{-1}\frac{4}{3}+\frac{\pi}{4}+\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)-\tan^{-1}\left(-\frac{3}{4}\right)-\frac{\pi}{2}\right]+2k\pi$$

$$\Rightarrow \quad \text{amp } z=2\left[\tan^{-1}\frac{4}{3}+\cot^{-1}\frac{4}{3}+\frac{\pi}{3}\right]+2k\pi \Rightarrow \text{amp } z=2\left[\frac{\pi}{2}+\frac{\pi}{3}\right]+2k\pi$$

$$\Rightarrow \quad \text{amp } z=-\frac{\pi}{3} \qquad [\text{at } k=-1]$$

Also,

$$|z|=\left|\frac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right|^2$$

$$\Rightarrow \quad |z|=\left(\frac{|3+4i||1+i||1+\sqrt{3}i|}{|1-i|\,|4-3i|\,|2i|}\right)^2$$

$$\Rightarrow \quad |z|=\left(\frac{5\times\sqrt{2}\times 2}{\sqrt{2}\times 5\times 2}\right)^2=1$$

Ex. If $\dfrac{z-1}{z+1}$ is purely imaginary, then prove that $|z|=1$

Sol. $\text{Re}\left(\dfrac{z-1}{z+1}\right)=0$

$$\Rightarrow \quad \frac{z-1}{z+1}+\overline{\left(\frac{z-1}{z+1}\right)}=0$$

$$\Rightarrow \quad \frac{z-1}{z+1}+\frac{\bar{z}-1}{\bar{z}+1}=0$$

$\Rightarrow \quad z\bar{z}-\bar{z}+z-1+z\bar{z}-z+\bar{z}-1=0$

$\Rightarrow \quad z\bar{z}=1$

$\Rightarrow \quad |z|^2=1$

$\Rightarrow \quad |z|=1$ **Hence proved**

Ex. z_1 and z_2 are two complex numbers such that $\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular (whose modulus is one), while z_2 is not unimodular. Find $|z_1|$.

Sol. Here $\left|\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1 \Rightarrow \dfrac{|z_1 - 2z_2|}{|2 - z_1\bar{z}_2|} = 1$

$\Rightarrow \quad |z_1 - 2z_2| = |2 - z_1\bar{z}_2| \Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow \quad (z_1 - 2z_2)\left(\overline{z_1 - 2z_2}\right) = (2 - z_1\bar{z}_2)\left(\overline{2 - z_1\bar{z}_2}\right)$

$\Rightarrow \quad (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$

$\Rightarrow \quad z_1\bar{z}_1 - 2z_1\bar{z}_2 - 2z_2\bar{z}_1 + 4z_2\bar{z}_2 = 4 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1 z_2\bar{z}_2$

$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2 \quad \Rightarrow \quad |z_1|^2 - |z_1|^2|z_2|^2 + 4|z_2|^2 - 4 = 0$

$\Rightarrow \quad \left(|z_1|^2 - 4\right)\left(1 - |z_2|^2\right) = 0$

But $|z_2| \neq 1$ (given)

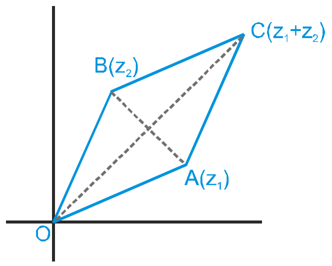
$\therefore \quad |z_1|^2 = 4$

Hence, $|z_1| = 2$.

DISTANCE, TRIANGULAR INEQUALITY

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then distance between points z_1, z_2 in argand plane is

$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$



In triangle OAC

$OC \leq OA + AC$

$OA \leq AC + OC$

$AC \leq OA + OC$

using these in equalities we have $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

Similarly from triangle OAB

we have $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

(A) $||z_1| - |z_2|| = |z_1 + z_2|$, $|z_1 - z_2| = |z_1| + |z_2|$ if origin, z_1 and z_2 are collinear and origin lies between z_1 and z_2.

(B) $|z_1 + z_2| = |z_1| + |z_2|$, $||z_1| - |z_2|| = |z_1 - z_2|$ if origin, z_1 and z_2 are collinear and z_1 and z_2 lies on the same side of origin.

Ex. $\left|z-\frac{2}{z}\right| = 1$ then find the maximum and minimum value of $|z|$

Sol. $\left|z-\frac{2}{z}\right| = 1$ $\quad \left||z|-\left|\frac{2}{z}\right|\right| \le \left|z-\frac{2}{2}\right| \le |z| + \left|-\frac{2}{z}\right|$

Let $|z| = r$

$\Rightarrow \quad \left|r-\frac{2}{r}\right| \le 1 \le r+\frac{2}{r}$

$r+\frac{2}{r} \ge 1 \quad \Rightarrow \quad r \in R^{+}$ ......**(i)**

and $\left|r-\frac{2}{r}\right| \le 1 \quad \Rightarrow \quad -1 \le r-\frac{2}{r} \le 1$

$\Rightarrow \quad r \in [1, 2]$ ......**(ii)**

$\therefore$ from **(i)** and **(ii)** $r \in [1, 2]$

$r \in [1, 2]$

$|z|_{max} = 2, |z|_{min} = 1$

Ex. If $\left|z-\frac{4}{z}\right| = 2$, then the greatest value of $|z|$ is -

Sol. We have $|z| = \left|z-\frac{4}{z}+\frac{4}{z}\right| \le \left|z-\frac{4}{z}\right| + \frac{4}{|z|} = 2+\frac{4}{|z|}$

$\Rightarrow \quad |z|^2 \le 2|z|+4 \quad \Rightarrow \quad (|z|-1)^2 \le 5$

$\Rightarrow \quad |z|-1 \le \sqrt{5} \quad \Rightarrow \quad |z| \le \sqrt{5}+1$

Therefore, the greatest value of $|z|$ is $\sqrt{5}+1$.

REPRESENTATION OF A COMPLEX NUMBER

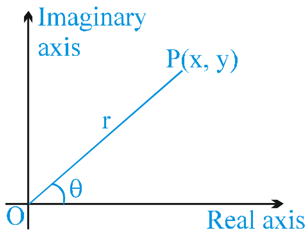
Cartesian Form (Geometric Representation)

Every complex number $z = x + i y$ can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).

length OP is called modulus of the complex number denoted by $|z|$ & θ is called the argument or amplitude .

eg. $|z| = \sqrt{x^2+y^2}$ &

$\theta = \tan^{-1}\frac{y}{x}$ (angle made by OP with positive x–axis)



(i) $|z|$ is always non negative . Unlike real numbers $|z| = \begin{bmatrix} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{bmatrix}$ is not correct

(ii) Argument of a complex number is a many valued function . If θ is the argument of a complex number then $2\,n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(iii) The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument.

(iv) Unless stated, amp z implies principal value of the argument.

(v) By specifying the modulus & argument a complex number is defined completely. For the complex number $0 + 0\,i$ the argument is not defined and this is the only complex number which is given by its modulus.

(vi) There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.

Trigonometric / Polar Representation

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as CiS θ.

Euler's formula :

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

It was introduced by Euler in 1748, and is used as a method of expressing complex numbers.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Euler's identities.

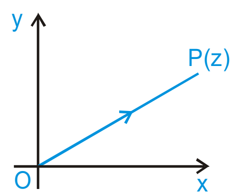
Exponential Representation

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

VECTORIAL REPRESENTATION OF A COMPLEX NUMBER

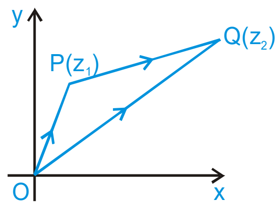
(A) In complex number every point can be represented in terms of position vector.

If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

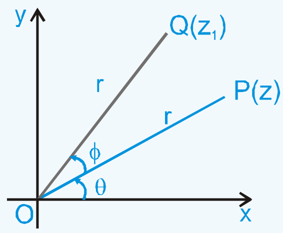


(B) If $P(z_1)$ & $Q(z_2)$ be two complex numbers on argand plane then

$\vec{PQ}$ represents complex number $z_2 - z_1$.

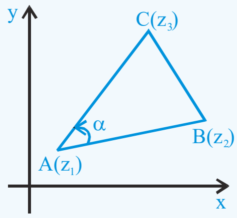


(i) If $\vec{OP} = z = re^{i\theta}$ then $\vec{OQ} = z_1 = re^{i(\theta+\phi)} = z.\,e^{i\phi}$. If $\vec{OP}$ and $\vec{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}\,e^{i\phi}$ i.e. $\frac{z_1}{|z_1|} = \frac{z}{|z|}e^{i\phi}$



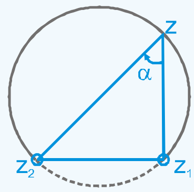
(ii) In general, if z_1, z_2, z_3 be the three vertices of ΔABC then

$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\alpha}$. Here $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \alpha$.



(iii) Note that the locus of z satisfying $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ is:

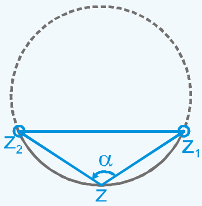
Case (A) $0 < \alpha < \pi/2$
Locus is major arc of circle as shown excluding z_1 & z_2



Case (B) $\frac{\pi}{2} < \alpha < \pi$

Locus is minor arc of circle as shown

excluding z_1 & z_2



(iv) If A, B, C & D are four points representing the complex numbers

z_1, z_2, z_3 & z_4 then AB || CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely real ;

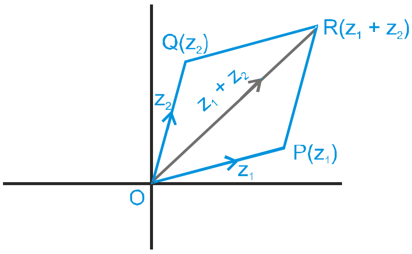
$AB \perp CD$ if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary.

(v) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

(i) $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$ **(ii)** $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

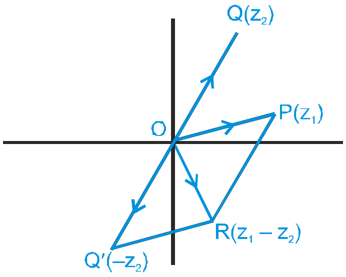
GEOMETRICAL REPRESENTATION OF FUNDAMENTAL OPERATIONS

(i) Geometrical representation of addition.



If two points P and Q represent complex numbers z_1 and z_2 respectively in the Argand plane, then the sum $z_1 + z_2$ is represented by the extremity R of the diagonal OR of parallelogram OPRQ having OP and OQ as two adjacent sides.

(ii) Geometric representation of substraction.



(iii) Modulus and argument of multiplication of two complex numbers.

Theorem For any two complex numbers z_1, z_2 we have $|z_1\, z_2| = |z_1|\, |z_2|$ and $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$.

Proof $z_1 = r_1\ e^{i\theta_1}$, $z_2 = r_2\ e^{i\theta_2}$

$z_1z_2 = r_2r_2\ e^{i(\theta_1+\theta_2)}$

$\Rightarrow$ $|z_1z_2| = |z_1|\,|z_2|$

$\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

i.e. to multiply two complex numbers, we multiply their absolute values and add their arguments.

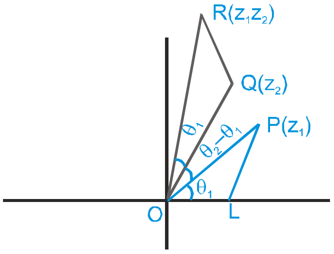
(i) P.V. $\arg(z_1z_2) \neq$ P.V. $\arg(z_1)$ + P.V. $\arg(z_2)$

(ii) $|z_1\, z_2 \,....\, z_n| = |z_1|\, |z_2| \,.....\, |z_n|$

(iii) $\arg(z_1z_2 \,....\, z_n) = \arg z_1 + \arg z_2 + \,.....\, + \arg z_n$

(iv) Geometrical representation of multiplication of complex numbers.

Let P, Q be represented by $z_1 = r_1\ e^{i\theta_1}$, $z_2 = r_2\ e^{i\theta_2}$ repectively. To find point R representing complex number z_1z_2 we take a point L on real axis such that OL = 1 and draw triangle OQR similar to triangle OLP. Therefore



$\dfrac{OR}{OQ} = \dfrac{OP}{OL}$ $\Rightarrow$ $OR = OP.OQ$ i.e. $OR = r_1r_2$ and $Q\hat{O}R = \theta_1$

$L\hat{O}R = L\hat{O}P + P\hat{O}Q + Q\hat{O}R = \theta_1 + \theta_2 - \theta_1 + \theta_1 = \theta_1 + \theta_2$

Hence, R is represented by $z_1z_2 = r_1r_2\ e^{i(\theta_1+\theta_2)}$

(v) Modulus and argument of division of two complex numbers.

Theorem : If z_1 and z_2 ($\neq 0$) are two complex numbers, then $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ and $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Note : P.V. $\arg\left(\dfrac{z_1}{z_2}\right) \neq$ P.V. $\arg(z_1)$ – P.V. $\arg(z_2)$

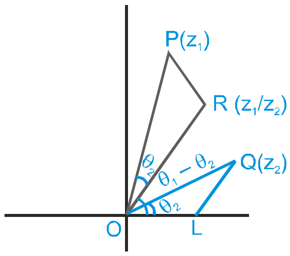
(vi) Geometrical representation of the division of complex numbers.

Let P, Q be represented by $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2}$ respectively. To find point R representing complex number $\dfrac{z_1}{z_2}$, we take a point L on real axis such that OL = 1 and draw a triangle OPR similar to OQL.

Therefore $\dfrac{OP}{OQ} = \dfrac{OR}{OL}$ $\Rightarrow$ $OR = \dfrac{r_1}{r_2}$

and $L\hat{O}R = L\hat{O}P - R\hat{O}P = \theta_1 - \theta_2$

Hence, R is represented by $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\ e^{i(\theta_1-\theta_2)}$.

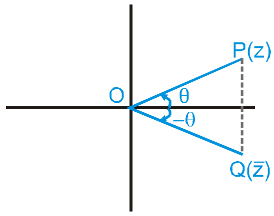


CONJUGATE OF A COMPLEX NUMBER

Conjugate of a complex number z = a + ib is denoted and defined by $\overline{z} = a - ib$.

In a complex number if we replace i by – i, we get conjugate of the complex number. $\overline{z}$ is the mirror image of z about real axis on Argand's Plane.

Geometrical representation of conjugate of complex number.



$|z| = |\bar{z}|$

$\arg(\bar{z}) = -\arg(z)$

General value of $\arg(\bar{z}) = 2n\pi - \text{P.V.} \arg(z)$

Properties

(i) If $z = x + iy$, then $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

(ii) $z = \bar{z} \Leftrightarrow$ z is purely real

(iii) $z + \bar{z} = 0 \Leftrightarrow$ z is purely imaginary

(iv) Relation between modulus and conjugate. $|z|^2 = z\bar{z}$

(v) $\bar{\bar{z}} = z$

(vi) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

(vii) $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\bar{z}_1)}{(\bar{z}_2)}$ $(z_2 \neq 0)$

Theorem Imaginary roots of polynomial equations with real coefficients occur in conjugate pairs

Proof If z_0 is a root of $a_0z^n + a_1z^{n-1} + \ldots\ldots + a_{n-1} z + a_n = 0$,

$a_0, a_1, \ldots\ldots. a_n \in R$, then $a_0z_0^n + a_1z_0^{n-1} + \ldots\ldots. + a_{n-1} z_0 + a_n = 0$

By using property **(vi)** and **(vii)** we have $a_0\bar{z}_0^n + a_1\bar{z}_0^{n-1} + \ldots\ldots. + a_{n-1}\bar{z}_0 + a_n = 0$

$\Rightarrow$ $\bar{z}_0$ is also a root.

Note If $w = f(z)$, then $\bar{w} = f(\bar{z})$

Theorem $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm (z_1 \bar{z}_2 + \bar{z}_1 z_2)$

$= |z_1|^2 + |z_2|^2 \pm 2\,\text{Re}(z_1 \bar{z}_2)$

$= |z_1|^2 + |z_2|^2 \pm 2\,|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

Ex. Express the complex number $z = -1 + \sqrt{2}\,i$ in polar form.

Sol. $z = -1 + i\sqrt{2}$

$|z| = \sqrt{(-1)^2 + \left(\sqrt{2}\right)^2} = \sqrt{1+2} = \sqrt{3}$

$\text{Arg } z = \pi - \tan^{-1}\left(\frac{\sqrt{2}}{1}\right) = \pi - \tan^{-1}(\sqrt{2}) = \theta$ (say)

$\therefore$ $z = \sqrt{3}\,(\cos\theta + i \sin\theta)$ where $\theta = \pi - \tan^{-1}\sqrt{2}$

Ex. Express the following complex numbers in polar and exponential form :

(i) $\dfrac{1+3i}{1-2i}$ **(ii)** $\dfrac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

Sol. **(i)** Let $z=\dfrac{1+3i}{1-2i}=\dfrac{1+3i}{1-2i}\times\dfrac{1+2i}{1+2i}=-1+i$

$|z|=\sqrt{(-1)^2+1^2}=\sqrt{2}$

$\tan\alpha=\left|\dfrac{1}{-1}\right|=1=\tan\dfrac{\pi}{4}\Rightarrow\alpha=\dfrac{\pi}{4}$

$\because$ Re(z) < 0 and Im(z) > 0 $\Rightarrow$ z lies in second quadrant.

$\therefore$ $\theta=\arg(z)=\pi-\alpha=\pi-\dfrac{\pi}{4}=\dfrac{3\pi}{4}$

Hence Polar form is $z=\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$ and exponential form is $z=\sqrt{2}\,e^{3\pi/4}$

(ii) Let $z=\dfrac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}=\dfrac{i-1}{\frac{1}{2}+\frac{i\sqrt{3}}{2}}=\dfrac{2(i-1)}{(1+i\sqrt{3})}$

$\Rightarrow$ $z=\dfrac{2(i-1)}{(1+i\sqrt{3})}\times\dfrac{(1-i\sqrt{3})}{(1-i\sqrt{3})}\Rightarrow z=\left(\dfrac{\sqrt{3}-1}{2}\right)+i\left(\dfrac{\sqrt{3}+1}{2}\right)$

$\because$ Re(z) > 0 and Im(z) > 0 $\Rightarrow$ z lies in first quadrant.

$\therefore$ $|z|=\sqrt{\left(\dfrac{\sqrt{3}-1}{2}\right)^2+\left(\dfrac{\sqrt{3}+1}{2}\right)^2}=\sqrt{\dfrac{2(3+1)}{4}}=\sqrt{2}$.

$\tan\theta=\left|\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\right|=\tan\dfrac{5\pi}{12}\Rightarrow\alpha=\dfrac{5\pi}{12}$

Hence Polar form is $z=\sqrt{2}\left(\cos\dfrac{5\pi}{12}+i\sin\dfrac{5\pi}{12}\right)$ and exponential form is $z=\sqrt{2}e^{5\pi/12}$

Ex. Find the locus of :

(A) $|z-1|^2+|z+1|^2=4$ **(B)** $\text{Re}(z^2)=0$

Sol. **(A)** Let $z=x+iy$

$\Rightarrow$ $(|x+iy-1|)^2+(|x+iy+1|)^2=4$

$\Rightarrow$ $(x-1)^2+y^2+(x+1)^2+y^2=4$

$\Rightarrow$ $x^2-2x+1+y^2+x^2+2x+1+y^2=4\Rightarrow x^2+y^2=1$

Above represents a circle on complex plane with center at origin and radius unity.

(B) Let $z = x + iy$

$\Rightarrow$ $z^2 = x^2 - y^2 + 2xyi$

$\therefore$ $\text{Re}(z^2) = 0$

$\Rightarrow$ $x^2 - y^2 = 0 \Rightarrow y = \pm x$

Thus $\text{Re}(z^2) = 0$ represents a pair of straight lines passing through origin.

Ex. Find the minimum value of $|1 + z| + |1 - z|$.

Sol. $|1 + z| + |1 - z| \ge |1 + z + 1 - z|$ (triangle inequality)

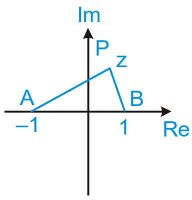
$\Rightarrow$ $|1 + z| + |1 - z| \ge 2$

$\therefore$ minimum value of $(|1 + z| + |1 - z|) = 2$

Geometrically $|z + 1| + |1 - z| = |z + 1| + |z - 1|$

which represents sum of distances of z from 1 and -1

it can be seen easily that minimum $(PA + PB) = AB = 2$



Ex. Among the complex number z which satisfies $|z - 25i| \le 15$, find the complex numbers z having

(A) least positive argument **(B)** maximum positive argument

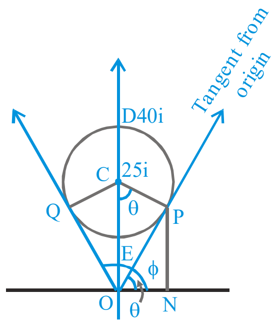
(C) least modulus **(D)** maximum modulus

Sol. The complex numbers z satisfying the condition

$|z - 25i| \le 15$

are represented by the points inside and on the circle of radius 15 and centre at the point C(0, 25).

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle



Here θ = least positive argument

and ϕ = maximum positive argument

$\therefore$ In $\Delta OCP, OP = \sqrt{(OC)^2 - (CP)^2} = \sqrt{(25)^2 - (15)^2} = 20$

and $\sin\theta = \dfrac{OP}{OC} = \dfrac{20}{25} = \dfrac{4}{5}$

$\therefore$ $\tan\theta = \dfrac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

Thus, complex number at P has modulus 20 and argument $\theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

$\therefore$ $z_p = 20(\cos\theta + i\sin\theta) = 20\left(\dfrac{3}{5} + i\dfrac{4}{5}\right)$

$\therefore$ $z_p = 12 + 16i$

Similarly $z_Q = -12 + 16i$

From the figure, E is the point with least modulus and D is the point with maximum modulus.

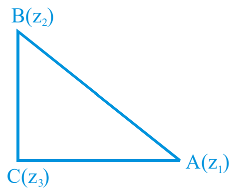
Hence, $z_E = \overrightarrow{OE} = \overrightarrow{OC} - \overrightarrow{EC} = 25i - 15i = 10i$

and $z_D = \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = 25i + 15i = 40i$

Ex. Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Sol. In the isosceles triangle ABC, AC = BC and BC⊥AC. It means that AC is rotated through angle $\pi/2$ to occupy the position BC.

Hence we have, $\dfrac{z_2 - z_3}{z_1 - z_3} = e^{+i\pi/2} = +i \Rightarrow z_2 - z_3 = +i(z_1 - z_3)$



$$\Rightarrow \quad z_2^2 + z_3^2 - 2z_2z_3 = -\left(z_1^2 + z_3^2 - 2z_1z_3\right)$$

$$\Rightarrow \quad z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_1z_2 - 2z_3^2$$

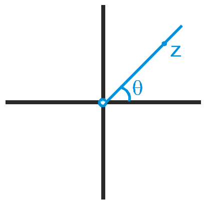
$$= 2\left(z_1 - z_3\right)\left(z_3 - z_2\right)$$

$$\Rightarrow \quad \left(z_1 - z_2\right)^2 = 2\left(z_1 - z_3\right)\left(z_3 - z_2\right)$$

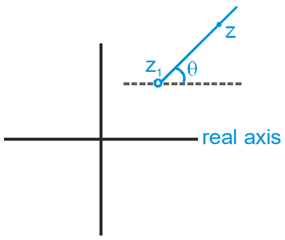
ROTATION

Important results

(i) arg z = θ represnets points (non-zero) on ray eminating from origin making an angle θ with positive direction of real axis

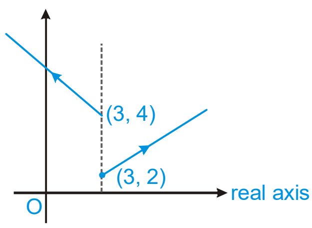


(ii) arg $(z - z_1) = \theta$ represents points $(\neq z_1)$ on ray eminating from z_1 making an angle θ with positive direction of real axis

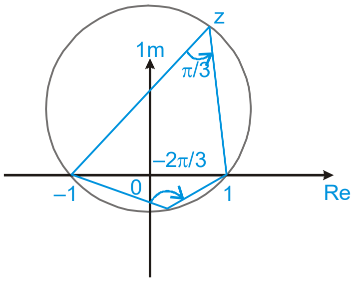


Ex. Solve for z, which satisfy Arg $(z - 3 - 2i) = \dfrac{\pi}{6}$ and Arg $(z - 3 - 4i) = \dfrac{2\pi}{3}$.

Sol. From the figure, it is clear that there is no z, which satisfy both ray



Ex. If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$ then interpret the locus.



Sol. $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$

$\Rightarrow \arg\left(\frac{1-z}{-1-z}\right) = \frac{\pi}{3}$

Here $\arg\left(\frac{1-z}{-1-z}\right)$ represents the angle between lines joining –1 and z, and 1 and z. As this angle is constant, the locus of z will be a larger segment of circle. (angle in a segment is constant).

LOGARITHM OF A COMPLEX QUANTITY

(i) $\text{Log}_e(\alpha + i\beta) = \frac{1}{2}\text{Log}_e(\alpha^2+\beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$ where $n \in I$.

(ii) i^i represents a set of positive real numbers given by $e^{-\left(2n\pi+\frac{\pi}{2}\right)}$, $n \in I$.

DEMOIVRE'S THEOREM

Case I

Statement

If n is any integer then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\,.....(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 + \,.........\, \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \,.......\, + \theta_n)$

Case II

Statement

If $p, q \in Z$ and $q \neq 0$ then

$$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where $k = 0, 1, 2, 3, \ldots\ldots, q-1$

Continued product of the roots of a complex quantity should be determined using theory of equations.

Ex. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and also $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that

(A) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(B) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(C) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

Sol. Let $z_1 = \cos\alpha + i\sin\alpha,\ z_2 = \cos\beta + i\sin\beta\ \ \&\ \ z_3 = \cos\gamma + i\sin\gamma$.

$\therefore$ $z_1 + z_2 + z_3 = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i\,.\,0 = 0$ ...... **(i)**

(A) Also $\dfrac{1}{z_1} = (\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha$

$\dfrac{1}{z_2} = \cos\beta - i\sin\beta,\ \dfrac{1}{z_3} = \cos\gamma - i\sin\gamma$

$\therefore$ $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} = (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$ ...... **(ii)**

$= 0 - i\,.\,0 = 0$

Now $z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$

$= 0 - 2z_1z_2z_3\left(\dfrac{1}{z_3} + \dfrac{1}{z_1} + \dfrac{1}{z_2}\right) = 0 - 2z_1z_2z_3\,.\,0 = 0$ {using **(i)** and **(ii)**}

or $(\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

or $\cos 2\alpha + i\sin 2\alpha + \cos 2\beta + i\sin 2\beta + \cos 2\gamma + i\sin 2\gamma = 0 + i.0$

Equating real and imaginary parts on both sides,

$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ and $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(B) If $z_1 + z_2 + z_3 = 0$ then $z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$

$\therefore$ $(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

or $\cos 3\alpha + i\sin 3\alpha + \cos 3\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma$

$= 3\{\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)\}$

Equating imaginary parts on both sides, $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(C) Equating real parts on both sides, $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

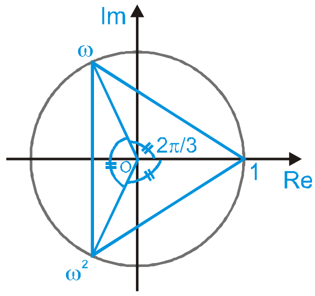
CUBE ROOT OF UNITY

(A) The cube roots of unity are $1\ ,\ \dfrac{-1 + i\sqrt{3}}{2}(\omega)\ ,\ \dfrac{-1 - i\sqrt{3}}{2}(\omega^2)$.

(B) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3 & $1 + \omega^r + \omega^{2r} = 3$ if $r = 3\lambda\ ;\ \lambda \in I$

(C) In polar form the cube roots of unity are :

$$1 = \cos 0 + i \sin 0\ ;\quad \omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\quad \omega^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$



(D) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

(E) The following factorisation should be remembered :

($a, b, c \in R$ & ω is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

Ex. Find the value of $\omega^{192} + \omega^{194}$

Sol. $\omega^{192} + \omega^{194}$

$= 1 + \omega^2 = -\omega$

Ex. If α & β are imaginary cube roots of unity then $\alpha^n + \beta^n$ is equal to -

Sol. $\alpha = \frac{\cos 2\pi}{3} + \frac{i\sin 2\pi}{3}$ $\qquad \beta = \frac{\cos 2\pi}{3} - \frac{i\sin 2\pi}{3}$

$$\alpha^n + \beta^n = \left(\frac{\cos 2\pi}{3} + \frac{i\sin 2\pi}{3}\right)^n + \left(\frac{\cos 2\pi}{3} - \frac{i\sin 2\pi}{3}\right)^n$$

$$= \left(\frac{\cos 2n\pi}{3} + \frac{i\sin 2n\pi}{3}\right) + \left(\frac{\cos 2n\pi}{3} - i\sin\left(\frac{2n\pi}{3}\right)\right) = 2\cos\left(\frac{2n\pi}{3}\right)$$

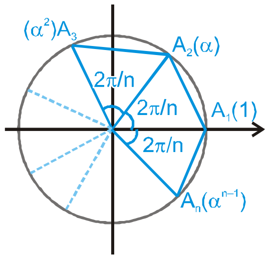
n^{th} ROOTS OF UNITY

If $1, \alpha_1, \alpha_2, \alpha_3 .... \alpha_{n-1}$ are the n, n^{th} root of unity then :

(A) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(B) Their arguments are in A.P. with common difference $\frac{2\pi}{n}$

(C) The points represented by n, n^{th} roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having center at origin, one vertex being on positive real axis.



(D) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

$= n$ if p is an integral multiple of n

(E) $(1 - \alpha_1)(1 - \alpha_2)......(1 - \alpha_{n-1}) = n$

(F) $(1 + \alpha_1)(1 + \alpha_2).......(1 + \alpha_{n-1}) = 0$ if n is even and

$= 1$ if n is odd.

(G) $1.\ \alpha_1.\ \alpha_2.\ \alpha_3.........\ \alpha_{n-1} = 1$ or -1 according as n is odd or even.

Ex. Find the roots of the equation $z^6 + 64 = 0$ where real part is positive.

Sol. $z^6 = -64$

$z^6 = 2^6 . e^{i(2n+1)\pi}$ n = 0, 1, 2, 3, 4, 5

$\Rightarrow \quad z = 2\, e^{i(2n+1)\frac{\pi}{6}}$

$\therefore \quad z = 2\, e^{i\frac{\pi}{6}}, 2\, e^{i\frac{\pi}{2}}, 2\, e^{i\frac{5\pi}{6}}, 2\, e^{i\frac{7\pi}{6}}, 2\, e^{i\frac{3\pi}{2}}, 2\, e^{i\frac{11\pi}{6}}$

$\therefore$ roots with +ve real part are $= 2\, e^{\frac{i\pi}{6}}, 2\, e^{i\frac{11\pi}{6}}$

Ex. Find the value $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - \cos\frac{2\pi k}{7}\right)$

Sol. $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}\right) - \sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7}\right) = \sum_{k=1}^{6}\sin\frac{2\pi k}{7} - \sum_{k=0}^{6}\cos\frac{2\pi k}{7} + 1$

$= \sum_{k=0}^{6}$ (Sum of imaginary part of seven seventh roots of unity)

$-\sum_{k=0}^{6}$ (Sum of real part of seven seventh roots of unity) +1 = 0 – 0 + 1 = 1

THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED

(A) $\cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\frac{n+1}{2}\right)\theta.$

(B) $\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\frac{n+1}{2}\right)\theta.$

❖ If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

GEOMETRICAL PROPERTIES

Section formula

If z_1 and z_2 are affixes of the two points P and Q respectively and point C divides the line segment joining P and Q internally in the ratio m : n then affix z of C is given by

$$z = \frac{mz_2 + nz_1}{m+n} \qquad \text{where } m, n > 0$$

If C devides PQ in the ratio m : n externally then $z = \frac{mz_2 - nz_1}{m-n}$

If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(1) If the vertices A, B, C of a Δ are represented by complex numbers z_1, z_2, z_3 respectively and a, b, c are the length of sides then,

(i) Centroid of the Δ ABC $= \dfrac{z_1 + z_2 + z_3}{3}$:

(ii) Orthocentre of the Δ ABC =

$$\frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C} \text{ or } \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\tan A + \tan B + \tan C}$$

(iii) Incentre of the Δ ABC $= (az_1 + bz_2 + cz_3) \div (a + b + c)$.

(iv) Circumcentre of the Δ ABC = :

$(Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C) \div (\sin 2A + \sin 2B + \sin 2C)$.

(2) amp(z) = θ is a ray emanating from the origin inclined at an angle θ to the positive x– axis.

(3) $|z - a| = |z - b|$ is the perpendicular bisector of the line joining a to b.

(4) The equation of a line joining z_1 & z_2 is given by, $z = z_1 + t(z_1 - z_2)$ where t is a real parameter.

(5) $z = z_1(1 + it)$ where t is a real parameter is a line through the point z_1 & perpendicular to the line joining z_1 to the origin.

(6) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as

$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$. This is also the condition for three complex numbers z, z_1, z_2 to be collinear. The above

equation on manipulating, takes the form $\bar{\alpha}z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.

If we replace z by $ze^{i\theta}$ and $\bar{z}$ by $\bar{z}e^{-i\theta}$ then we get equation of a straight line which makes an angle θ with the given straight line.

(7) The equation of circle having centre z_0 & radius ρ is :

$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form

$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + k = 0$, k is real. Centre is $-\alpha$ & radius $= \sqrt{|\alpha|^2 - k}$.

Circle will be real if $|\alpha|^2 - k \geq 0$.

(8) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is $\arg\dfrac{z - z_2}{z - z_1} = \pm\dfrac{\pi}{2}$ or

$(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.

(9) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is the number $\frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}$ should be real.

Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as $\frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real

$$\Rightarrow \quad \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)} = \frac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}.$$

(10) $\operatorname{Arg}\left(\frac{z-z_1}{z-z_2}\right) = \theta$ represent **(i)** a line segment if $\theta = \pi$

(ii) Pair of ray if $\theta = 0$ **(iii)** a part of circle, if $0 < \theta < \pi$.

(11) Area of triangle formed by the points z_1, z_2 & z_3 is $\left|\frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|$

(12) Perpendicular distance of a point z_0 from the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\frac{|\bar{\alpha}z_0 + \alpha\bar{z}_0 + r|}{2|\alpha|}$

(13) **(i)** Complex slope of a line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ is $\omega = -\frac{\alpha}{\bar{\alpha}}$.

(ii) Complex slope of a line joining by the points z_1 & z_2 is $\omega = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

(iii) Complex slope of a line making θ angle with real axis $\omega = e^{2i\theta}$

(14) **Dot and cross product**

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers [vectors]. The dot product [also called the scalar product] of z_1 and z_2 is defined by

$$z_1 \bullet z_2 = |z_1|\,|z_2| \cos\theta = x_1 x_2 + y_1 y_2 = \operatorname{Re}\{\bar{z}_1 z_2\} = \frac{1}{2}\{\bar{z}_1 z_2 + z_1 \bar{z}_2\}$$

where θ is the angle between z_1 and z_2 which lies between 0 and π

If vectors z_1, z_2 are perpendicular then $z_1 \bullet z_2 = 0 \quad \Rightarrow \quad \frac{z_1}{\bar{z}_1} + \frac{z_2}{\bar{z}_2} = 0.$

i.e. Sum of complex slopes = 0

The cross product of z_1 and z_2 is defined by

$$z_1 \times z_2 = |z_1|\,|z_2| \sin\theta = x_1 y_2 - y_1 x_2 = \operatorname{Im}\{\bar{z}_1 z_2\} = \frac{1}{2i}\{\bar{z}_1 z_2 - z_1 \bar{z}_2\}$$

If vectors z_1, z_2 are parallel then $z_1 \times z_2 = 0 \quad \Rightarrow \quad \frac{z_1}{\bar{z}_1} = \frac{z_2}{\bar{z}_2}.$

i.e. Complex slopes are equal.

ω_1 & ω_2 are the compelx slopes of two lines.

(i) If lines are parallel then $\omega_1 = \omega_2$

(ii) If lines are perpendicular then $\omega_1 + \omega_2 = 0$

(15) If $|z - z_1| + |z - z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose focii are z_1 & z_2

(16) If $|z - z_0| = \left|\dfrac{\bar{\alpha}z + \alpha\bar{z} + r}{2|\alpha|}\right|$ then locus of z is parabola whose focus is z_0 and directrix is the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ (Provided $\bar{\alpha}z_0 + \alpha\bar{z}_0 + r \neq 0$)

(17) If $\left|\dfrac{z - z_1}{z - z_2}\right| = k \neq 1, 0$, then locus of z is circle.

(18) If $\big|\,|z - z_1| - |z - z_2|\,\big| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose focii are z_1 & z_2.

Reflection points for a straight line

Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if; $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non zero complex constant.

Inverse points w.r.t. a circle

Two points P & Q are said to be inverse w.r.t. a circle with centre 'O' and radius ρ, if :

(i) The point O, P, Q are collinear and on the same side of O.

(ii) $OP \cdot OQ = \rho^2$.

Note that the two points z_1 & z_2 will be the inverse points w.r.t. the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ if and only if $z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$.

PTOLEMY'S THEOREM

It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the lengths of the two pairs of its opposite sides.

i.e. $|z_1 - z_3|\,|z_2 - z_4| = |z_1 - z_2|\,|z_3 - z_4| + |z_1 - z_4|\,|z_2 - z_3|$.

TIPS & FORMULAS

1. Definition

Complex numbers are defined as expressions of the form a+ ib where a, b $\in$ R & $i = \sqrt{-1}$. It is denoted by z i.e. z = a + ib. 'a' is called real part of z (Re z) and 'b' is called imaginary part of z (lm z).

Every Complex Number Can be Regarded As

- Purely real if $b = 0$
- Purely imaginary if $a = 0$
- imaginary if $b \neq 0$

Note

(i) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(ii) Zero is both purely real as well as purely imaginary but not imaginary.

(iii) $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$

(iv) $\sqrt{a}\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.

2. Conjugate Complex

If z = a + ib then its conjugate complex is obtained by chamging the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

Note that :

(i) $z + \bar{z} = 2\,Re(z)$ **(ii)** $z - \bar{z} = 2i\,Im(z)$ **(iii)** $z\bar{z} = a^2 + b^2$ which is real

(iv) If z is purely real then $z - \bar{z} = 0$ **(v)** if z is purely imaginary then $z + \bar{z} = 0$

3. Representations of a Complex Number in Various Forms

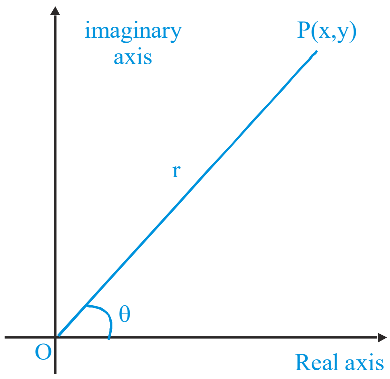
(A) Cartesian Form (Geometrical Representation)

Every complex nimber z = x + iy can be represented by a point on the cartesian plane know as complex plane (Argand diagram) by the ordered pair (x, y).

Length OP is called modulus of the complex number denoted by |z| & q is called the argument or amplitude.

e.g. $|z| = \sqrt{x^2 + y^2}$ & $\theta = \tan^{-1}\frac{y}{x}$ (angle made by OP with positive x - axis).

Geometrically |z| represents the distance of point P from origin. ($|z| \geq 0$)



(B) Trignometric / Polar Representation

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as CiS θ

Euler's formula

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} + e^{-ix}}{2i}$ are known as Euler's identities.

(C) Exponential Representation

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

4. IMPORTENT PROPERTIES OF CONJUGATE

(A) $\overline{(\bar{z})} = z$ **(B)** $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ **(C)** $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ **(D)** $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

(E) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}; z \neq 0$ **(F)** If $f(\alpha + i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

5. IMPORTANT PROPERTIES OF MODULUS

(A) $|z| \geq 0$ **(B)** $|z| \geq \text{Re}(z)$ **(C)** $|z| \geq \text{Im}(z)$ **(D)** $z = |\bar{z}| = |-z| = |-\bar{z}|$

(E) $z\bar{z} = |z^2|$ **(F)** $|z_1 z_2| = |z_1|.|z_2|$ **(G)** $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$ **(H)** $|z^n| = |z|^n$

(I) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1 \bar{z}_2)$ **or** $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(J) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

(K) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

(L) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$ **[Triangle Inequality]**

(M) If $\left|z + \frac{1}{z}\right| = 0 (a > 0)$, then $\max |z| = \frac{a + \sqrt{a^2 + 4}}{2}$ and $\min |z| = \frac{1}{2}\left(\sqrt{a^2 + 4} - a\right)$

6. IMPORTANT PROPERTIES OF AMPLITUDE

(A) $\text{amp}(z_1 . z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi ; k \in I$

(B) $\left(\frac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi; \ k \in I$

(C) $\text{amp}(z^n) = n \text{ amp}(z) + 2k\pi$, where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(D) $\log(z) = \log(re^{i\theta}) = \log r + i\theta = \log|z| + i \text{ amp}(z)$

7. DE'MOIVER'S THEOREM

The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ if 'n' is integer & it is one of the values of $(\cos\theta + i\sin\theta)^n$ if n is a rational number of the form p/q, where p & q are co-prime.

Note : Continued product of roots of a complex quantity should be determined using theory of equation.

8. CUBE ROOT OF UNITY

(A) The cube roots ofb unity are 1, $\omega = \frac{-1 + i\sqrt{3}}{2} = e^{i2\pi/3}$ & $\omega^2 = \frac{-1 - i\sqrt{3}}{2} = e^{i4\pi/3}$

(B) $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$, in general

$$1 + \omega^r + \omega^{2r} = \begin{cases} 0 & r \text{ is not integral multiple of } 3 \\ 3 & r \text{ is multiple of } 3 \end{cases}$$

(C) $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$a^3 + b^3 = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

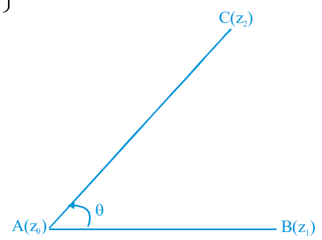
$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$

$x^2 + x + 1 = (x - \omega)(x - \omega^2)$

9. SQUARE ROOT OF COMPLEX NUMBER

$$\sqrt{a+ib} = \pm\left\{\frac{\sqrt{|z|+a}}{2} + i\frac{\sqrt{|z|-a}}{2}\right\} \text{ for } b>0 \;\; \& \;\; \pm\left\{\frac{\sqrt{|z|+a}}{2} - i\frac{\sqrt{|z|-a}}{2}\right\} \text{ for } b>0 \text{ where } |z| = \sqrt{a^2+b^2}.$$

10. ROTATION

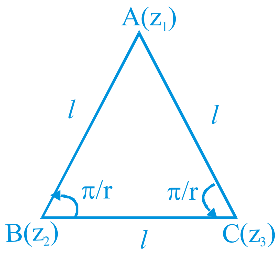


$$\frac{z_2 - z_0}{|z_2 - z_0|} = \frac{z_1 - z_0}{|z_1 - z_0|} e^{i\theta}$$

Take θ in anticlockwisev direction.

11. RESULT RELATED WITH TRIANGLE

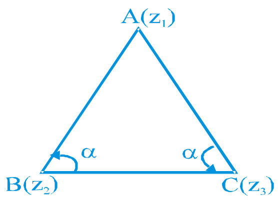
(A) Equilateral Triangle



$$\frac{z_1 - z_2}{\ell} = \frac{z_3 - z_2}{\ell} e^{i\pi/3} \qquad \text{.........(i)}$$

Also $$\frac{z_3 - z_2}{\ell} = \frac{z_1 - z_3}{\ell} . e^{i\pi/3} \qquad \text{......... (ii)}$$

from (i) & (ii)



⇒ $$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

or $$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

(B) Isosceles triangle

$4\cos^2 a\,(z_1 - z_2)(z_3 - z_1) = (z_3 - z_2)^2$

(C) Area of triangle ΔABC given by modulus of $\frac{1}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$

12. EQUATION OF LINE THROUGH POINTS Z_1 & Z_2

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_2 - z_1) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

⇒ $$z(\bar{z}_1 - \bar{z}_2)i + \bar{z}(z_2 - z_1)i + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

Let $(z_2 - z_1)\, i = a$, then equation of line is $[\bar{a}z + a\bar{z} + b = 0]$ where $a \in C$ & $b \in R$.

Note :

(i) Complex slope of line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{a}{\bar{a}}$

(ii) Two lines with slope μ_1 & μ_2 are parallel or perpendicular if $\mu_1 = \mu_2$ or $\mu_1 + \mu_2 = 0$

(iii) Length of perpendicular from point A(α) to line $\bar{a}z + a\bar{z} + b = 0$ is $\frac{|\bar{a}\alpha + a\bar{\alpha} + b|}{2|a|}$.

13. EQUATION OF CIRCLE

(A) Circle whose centre is z_0 & radii = r

$|z - z_0| r$

(B) General equation of circle

$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$

centre '–a' & radii = $\sqrt{|a|^2 - b}$

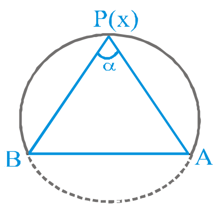
(C) Diameter form $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

or $\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\frac{\pi}{2}$

(D) Equation $\left|\frac{z - z_1}{z - z_2}\right| = k$ represent a circle if $k \neq 1$ and a straight line if $k = 1$.

(E) Equation $|z - z_1|^2 + |z - z_2|^2 = k$

represent circle if $k \geq \frac{1}{2}|z_1 - z_2|^2$



(F) $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha \quad 0 < \alpha < \pi, \alpha \neq \frac{\pi}{2}$

represent a segment of circle passing through $A(z_1)$ & $B(z_2)$

14. STANDARD LOCI

(A) $|z - z_1| + |z - z_2| = 2k$ (a constant) represent

(i) if $2k > |z_1 - z_2|$ ⇒ An ellipse

(ii) if $2k = |z_1 - z_2|$ ⇒ An line segment

(iii) if $2k < |z_1 - z_2|$ ⇒ No solution

(B) Equation $\left|\,|z - z_1| - |z - z_2|\,\right| = 2k$ (a constant) representy

(i) if $2k > |z_1 - z_2|$ ⇒ A hyperbola

(ii) if $2k = |z_1 - z_2|$ ⇒ A line ray

(iii) if $2k < |z_1 - z_2|$ ⇒ No solution