

# 02

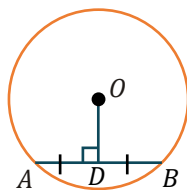
## Circle

### Basic Geometries:

#### Circle

##### Theorem 1:

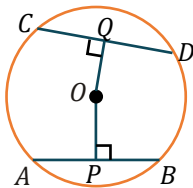
If a straight line drawn from the center of a circle bisects a chord, not passing through the centre, then it cuts the chord at right angles. Conversely, if it cuts the chord at right angles, then it bisects the chord.



##### Theorem 2:

Equal chords of a circle are equidistant from the center.

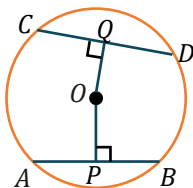
$$AB = CD, OP \perp AB \text{ and } OQ \perp CD \Rightarrow OP = OQ$$



Conversely,

Chords which are equidistant from the centre are equal.

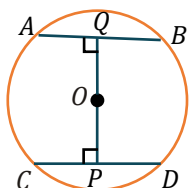
$$OP = OQ, OP \perp AB \text{ and } OQ \perp CD \Rightarrow AB = CD$$



##### Theorem 3:

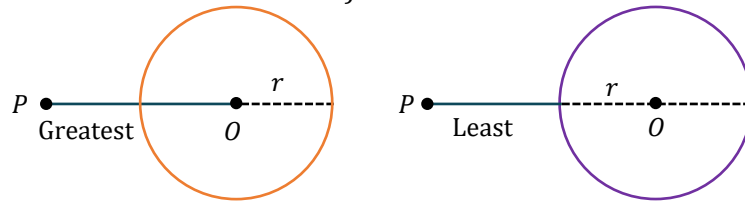
Of any two chords of a circle, that which is nearer to the center is greater than one lying remote. Conversely, the greater of two chords is nearer to the center.

$$OP > OQ \Rightarrow CD < AB$$



**Theorem 4:**

If  $P$  be any point (interior, exterior or on the circle) then greatest distance between point  $P$  and circumference of the circle is  $PO + r$  and the smallest distance between point  $P$  and circumference of the circle is  $|PO - r|$  (where  $r$  is radius and  $O$  is centre)

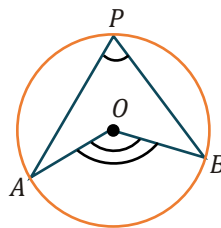


**Angles in a Circle:**

**Theorem 1:**

The angle at the centre of a circle is double of an angle at the circumference subtended by the same arc.

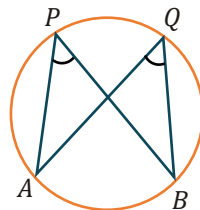
$$\angle AOB = 2\angle APB$$



**Theorem 2:**

Angles in the same segment of a circle are equal.

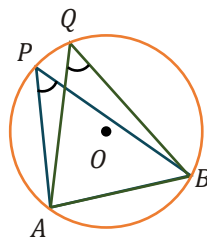
$$\angle APB = \angle AQB$$



**Converse of Theorem 2:**

If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, then four points lie on a circle (i.e. they are concyclic).

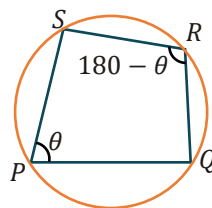
If  $\angle APB = \angle AQB \Rightarrow AB$  is a chord of circle and  $A, P, Q, B$  are concyclic points



**Theorem 3:**

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.

$$\angle P + \angle R = 180^\circ \text{ and } \angle Q + \angle S = 180^\circ$$

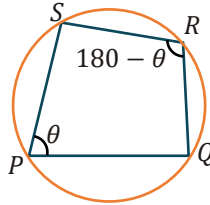


**Converse of Theorem 3:**

If a pair of opposite angles of a quadrilateral are supplementary, then its vertices are concyclic.

If  $\angle P + \angle R = 180^\circ$  or  $\angle Q + \angle S = 180^\circ$

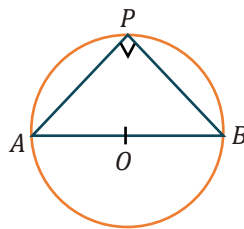
$\Rightarrow$  Then points  $P, Q, R$  and  $S$  are concyclic.



**Theorem 4:**

The angle in a semi-circle is a right angle.

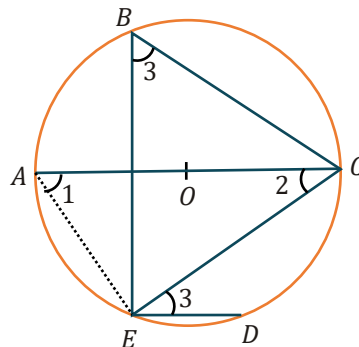
$\angle APB = 90^\circ$



**Illustration 1:**

In the given figure, the chord  $ED$  is parallel to the diameter  $AC$  and  $\angle CBE = 50^\circ$ . Find  $\angle CED$ .

**Solution:**



$\angle CBE = \angle CAE$  ( $\because$  angles in the same segment)

$\angle CAE = \angle 1 = 50^\circ$  ...(1)  $(\because \angle CBE = 50^\circ)$

$\angle AEC = 90^\circ$  ...(2)  $(\because$  Angle in a semi-circle is a right angle)

Now in  $\triangle AEC$

$\angle 1 + \angle AEC + \angle 2 = 180^\circ$  [ $\because$  sum of angles of a  $\Delta = 180^\circ$ ]

$\therefore 50^\circ + 90^\circ + \angle 2 = 180^\circ$

$\Rightarrow \angle 2 = 40^\circ$  ...(3)

Also,  $ED \parallel AC$  (Given)

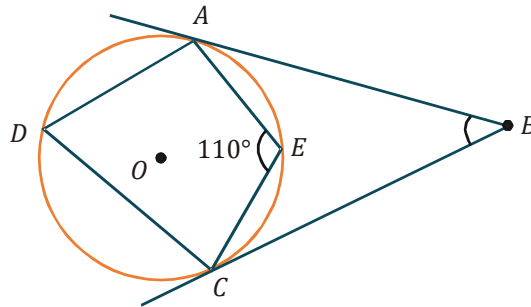
$\therefore \angle 2 = \angle 3$  (Alternate angles)

$\therefore 40^\circ = \angle 3$  i.e.,  $\angle 3 = 40^\circ$

Hence  $\angle CED = 40^\circ$

**Illustration 2:**

In the figure below,  $BA$  and  $BC$  are tangents to the circle at points  $A$  and  $C$ , respectively. If  $\angle AEC = 110^\circ$ . Find  $\angle ABC$ .



**Solution:**

Consider the above diagram.

- Opposite angles of a cyclic quadrilateral sum up to  $180^\circ$ .

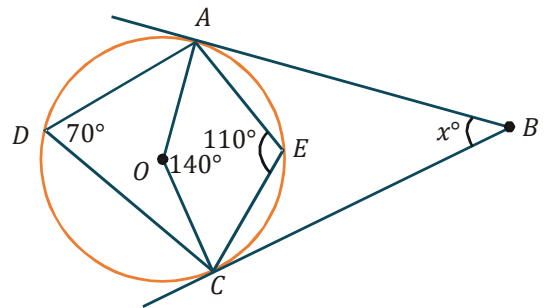
So  $\angle ADC = 180^\circ - 110^\circ = 70^\circ$ .

- From the inscribed angle theorem, we can see that

$$\angle AOC = 2\angle ADC = 2(70^\circ) = 140^\circ$$

Since the sum of interior angle of a quadrilateral is  $360^\circ$ , we have

$$\begin{aligned} \angle ABC &= 360^\circ - \angle BCO - \angle BAO - \angle AOC \\ &= 360^\circ - 90^\circ - 90^\circ - 140^\circ = 40^\circ \end{aligned}$$



**Illustration 3:**

An acute isosceles triangle,  $ABC$  is inscribed in a circle. Through  $B$  and  $C$ , tangents to the circle are drawn, meeting at point  $D$ . If  $\angle ABC = \angle ACB = 2\angle D$ , find the measure of  $\angle A$ .

**Solution:**

Let  $\angle D = \theta$

$$\Rightarrow \angle ABC = \angle ACB = 2\theta$$

In  $\triangle BCD$ .

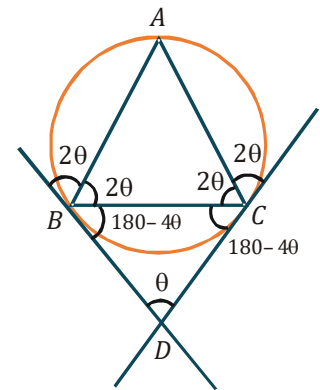
$$180^\circ - 4\theta + 180^\circ - 4\theta + \theta = 180^\circ$$

$$\Rightarrow 7\theta = 180^\circ \Rightarrow \theta = \frac{180^\circ}{7}$$

$$\therefore \angle A = 180^\circ - 4\theta$$

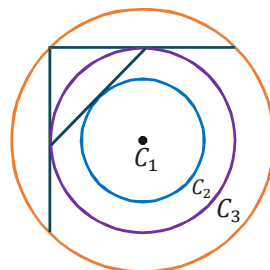
$$= 180^\circ - \frac{4 \times 180^\circ}{7}$$

$$\angle A = \frac{3 \times 180^\circ}{7}$$



**Illustration 4:**

In the given figure if  $C_1, C_2, C_3$  are three concentric circles such that radius of  $C_1$  and  $C_2$  is 1 and 3 unit respectively, then radius of  $C_3$  is



**Solution:**

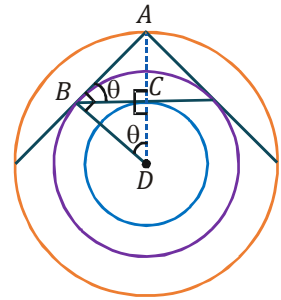
Let  $\angle BDC = \theta$ , then

$$\angle DBC = \frac{\pi}{2} - \theta$$

$\therefore \angle CBA = \theta$ , So

$\triangle ABD \sim \triangle BCD$ , so

$$\frac{AD}{BD} = \frac{BD}{CD} \Rightarrow R_3 = \frac{(3)^2}{1} = 9$$



**Illustration 5:**

In the given figure,  $O$  is the centre of the circle. Prove that  $\angle x + \angle y = \angle z$ .

**Solution:**

Given : In figure,  $O$  is the centre of the circle.

To Prove :  $\angle x + \angle y = \angle z$

Proof:  $\angle z = 2\angle 2$  [angle at the centre is double to the angle in the remaining part.]

or  $\angle z = \angle 2 + \angle 2$

But,  $\angle 2 = \angle 3$  [angles in the same segment]

$$\Rightarrow \angle z = \angle 2 + \angle 3 \quad \dots(i)$$

Now, we will determine the values of  $\angle 2$  and  $\angle 3$  in terms of  $\angle x$  and  $\angle y$ .

$\therefore \angle 3$  is an exterior angle of  $\triangle AEB$ ,

$$\therefore \angle 3 = \angle 1 + \angle x \quad \dots(ii) \text{ [ext. } \angle = \text{sum of two int. opp. } \angle s]$$

Again,  $\angle y$  is an exterior angle of  $\triangle DEP$

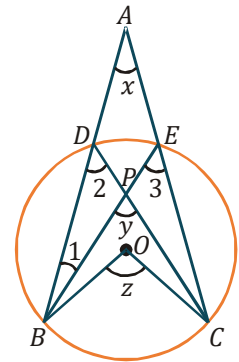
$$\therefore \angle y = \angle 1 + \angle 2$$

$$\Rightarrow \angle 2 = \angle y - \angle 1 \quad \dots(iii)$$

(i), (ii) and (iii)

$$\Rightarrow \angle z = (\angle y - \angle 1) + (\angle 1 + \angle x) = \angle x + \angle y$$

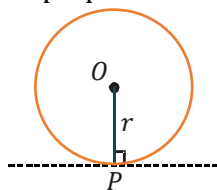
Hence,  $\angle x + \angle y = \angle z$ .



**Tangent:**

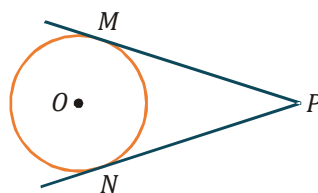
**Theorem 1:**

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.



**Theorem 2:**

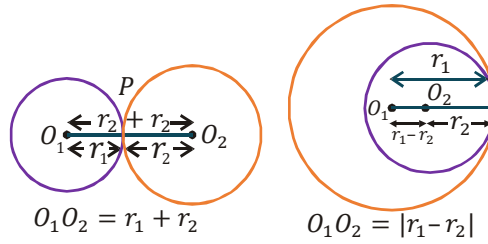
Two tangents can be drawn to a circle from an external point and length of these tangents are equal.



So,  $PM = PN$

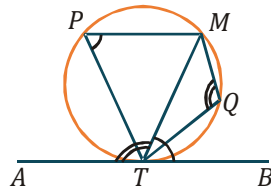
**Theorem 3:**

If two circles touch one another, then centers and the point of contact are collinear.



**Theorem 4 (Alternate Segments Theorem):**

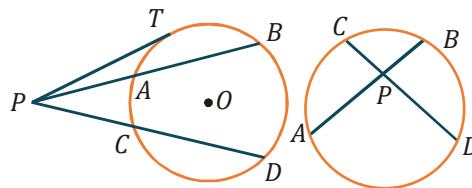
The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.



$\angle MTB = \angle TPM, \angle MTA = \angle MQT$

**Theorem 5:**

Suppose a line through a point P intersects a circle in two points, A and B, for all such lines, the product (PA)(PB) is constant.



**P lies outside the circle P lies inside the circle**

$PT^2 = (PA)(PB) = (PC)(PD) \Rightarrow PA \cdot PB = PC \cdot PD$   
 = The power of point P

**Proof:** Given : A secant PAB to a circle C(O, r)

intersecting it in points A and B and PT is a tangent segment of the circle.

To Prove :  $PA \times PB = PT^2$ .

Construction : Draw  $OD \perp AB$ . Join OP, OT and OA.

Proof :  $OD \perp AB \Rightarrow AD = DB$

Now,  $PA \times PB = (PD - AD)(PD + DB) = (PD - AD)(PD + AD) \quad [\because DB = AD]$

$= PD^2 - AD^2$

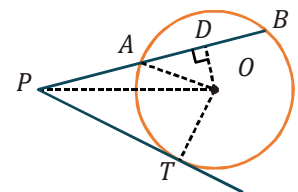
$= (OP^2 - OD^2) - AD^2 \quad [OP^2 = OD^2 + PD^2] \quad [\text{Pythagoras Theorem}]$

$= OP^2 - (OD^2 + AD^2)$

$= OP^2 - OA^2 \quad [\because OA^2 = OD^2 + AD^2]$

$= OP^2 - OT^2 \quad [\because OA = OT] \quad [\text{radii of same circle}]$

$= PT^2 \quad [\because \angle OTP = 90^\circ, OP^2 = OT^2 + PT^2]$



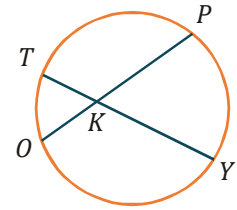
**Illustration 6:**

Chords  $\overline{TY}$  and  $\overline{OP}$  meet at point  $K$  such that  $TK = 2, KY = 16$  and  $KP = 2(KO)$ . Find  $OP$ .

**Solution:**

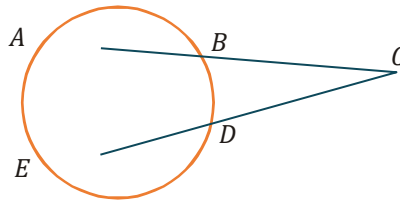
We have  $(KP)(KO) = (KT)(KY)$ .

Substituting the given information in this equation yields  $2(KO)(KO) = 2(16)$ , from which we find  $KO = 4$ . Therefore,  $KP = 2KO = 8$ , so  $OP = KO + KP = 12$ . from which we find  $KO = 4$ . Therefore,  $KP = 2KO = 8$ , so  $OP = KO + KP = 12$ .



**Illustration 7:**

In the diagram,  $CB = 9, BA = 11$  and  $CE = 18$ . Find  $DE$ .



**Solution:**

We have two intersecting secants,

So  $(CD)(CE) = (CB)(CA)$ .

Therefore,  $(CD)(18) = 9(20)$ , so  $CD = 10$  and  $DE = CE - CD = 8$ .

**Illustration 8:**

Chords  $AB$  and  $CD$  of a circle intersect each other at  $P$  as shown in fig., if  $AP = 6$  cm,  $PB = 2$  cm,  $PD = 2.5$  cm, find  $CD$ .

**Solution:**

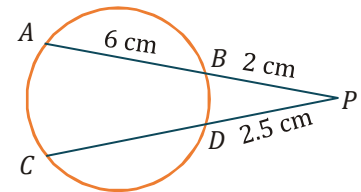
Since chords  $AB$  and  $CD$  of a circle meet in  $P$  (when produced),

$$\therefore PA \cdot PB = PC \cdot PD$$

$$\Rightarrow (6 + 2) \cdot 2 = PC(2.5)$$

$$\Rightarrow PC = \frac{16}{2.5} = \frac{16 \times 2}{5} = \frac{32}{5} = 6.4 \text{ cm}$$

$$\therefore CD = PC - PD = 6.4 - 2.5 = 3.9 \text{ cm.}$$



**Illustration 9:**

If two circles of radius 1 cm and 4 cm touch each other, then find the length of common tangent.

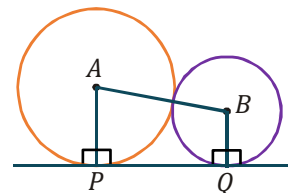
**Solution:**

Distance between centres  $AB = r_1 + r_2 = 5$

Length of tangent

$$PQ = \sqrt{d^2 - (r_1 - r_2)^2} = \sqrt{5^2 - (4 - 1)^2}$$

$$PQ = 4 \text{ cm.}$$



**Definition:**

A circle is a locus of a point in a plane whose distance from a fixed point (called centre) is always constant (called radius).

**Equation of a Circle in Various Forms:**

- (a) The circle with centre as origin & radius 'r' has the equation;  $x^2 + y^2 = r^2$ .
- (b) The circle with centre (h, k) & radius 'r' has the equation;  $(x - h)^2 + (y - k)^2 = r^2$ .
- (c) The general equation of a circle is  $x^2 + y^2 + 2gx + 2fy + c = 0$

with centre as (-g, -f) & radius =  $\sqrt{g^2 + f^2 - c}$ .

This can be obtained from the equation  $(x - h)^2 + (y - k)^2 = r^2$

$$\Rightarrow x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

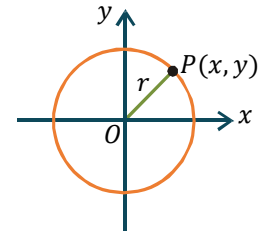
Take  $-h = g, -k = f$  and  $h^2 + k^2 - r^2 = c$

Condition to define circle :

$$g^2 + f^2 - c > 0 \Rightarrow \text{real circle.}$$

$$g^2 + f^2 - c = 0 \Rightarrow \text{point circle.}$$

$$g^2 + f^2 - c < 0 \Rightarrow \text{imaginary circle, with real centre, that is } (-g, -f)$$



**Note:** That every second degree equation in x & y, in which coefficient of  $x^2$  is equal to coefficient of  $y^2$  & the coefficient of  $xy$  is zero, always represents a circle.

- (d) The equation of circle with  $(x_1, y_1)$  &  $(x_2, y_2)$  as extremities of its diameter is:

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

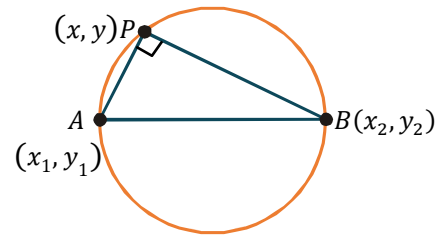
This is obtained by the fact that angle in a semicircle is a right angle.

$$\therefore (\text{Slope of } PA)(\text{Slope of } PB) = -1$$

$$\Rightarrow \frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1$$

$$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Note that this will be the circle of least radius passing through  $(x_1, y_1)$  &  $(x_2, y_2)$ .



**Illustration 10:**

Find the equation of the circle whose centre is (0, 3) and radius is 3.

**Solution:**

The equation of the circle is  $(x - 0)^2 + (y - 3)^2 = 3^2$

$$\Rightarrow x^2 + y^2 - 6y = 0$$

**Illustration 11:**

Find the equation of the circle which passes through (1, -1) and two of its diameter are  $x + 2y - 5 = 0$  and  $x - y + 1 = 0$

**Solution:**

Let P be the point of intersection of the lines

$$x + 2y - 5 = 0 \quad \dots(i)$$

and  $x - y + 1 = 0 \quad \dots(ii)$

Solving (i) and (ii), we get  $x = 1, y = 2$ . So, coordinates of centre are (1, 2). since circle passes through (1, -1)

$$\text{so radius} = \sqrt{(1-1)^2 + (2+1)^2} \Rightarrow \text{radius}=3$$

Hence the equation of the required circle is  $(x - 1)^2 + (y + 2)^2 = 9$

**Illustration 12:**

If the equation  $ax^2 + (b - 3)xy + 3y^2 + 6ax + 2by - 3 = 0$  represents the equation of a circle then find  $a, b$ .

**Solution:**

$$ax^2 + (b - 3)xy + 3y^2 + 6ax + 2by - 3 = 0$$

above equation will represent a circle if

coefficient of  $x^2 =$  coefficient of  $y^2$

$$a = 3$$

coefficient of  $xy = 0$

$$b = 3$$

**Illustration 13:**

Find the equation of a circle whose diametric end points are  $(x_1, y_1)$  and  $(x_2, y_2)$  where  $x_1, x_2$  are the roots of  $x^2 - ax + b = 0$  and  $y_1, y_2$  are the roots of  $y^2 - by + a = 0$ .

**Solution:**

We know that the equation of the circle described on the line segment joining  $(x_1, y_1)$  and  $(x_2, y_2)$  as a diameter is  $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ .

$$x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + x_1x_2 + y_1y_2 = 0$$

Here,  $x_1 + x_2 = a, x_1x_2 = b$

$y_1 + y_2 = b, y_1y_2 = a$

So, the equation of the required circle is

$$x^2 + y^2 - ax - by + a + b = 0.$$

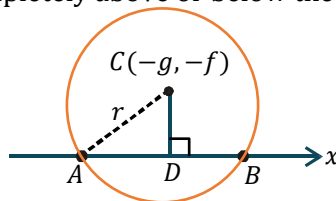
**Intercepts (Made by a Circle on the Axes):**

The intercepts made by the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  on the co-ordinate axes are  $2\sqrt{g^2 - c}$  (on  $x$ -axis) &  $2\sqrt{f^2 - c}$  (on  $y$ -axis) respectively.

If  $g^2 > c \Rightarrow$  circle cuts the  $x$ -axis at two distinct points.

$g^2 = c \Rightarrow$  circle touches the  $x$ -axis.

$g^2 < c \Rightarrow$  circle lies completely above or below the  $x$ -axis.



$$AB = 2AD = 2\sqrt{r^2 - CD^2} = 2\sqrt{r^2 - f^2} = 2\sqrt{g^2 + f^2 - c - f^2} = 2\sqrt{g^2 - c}$$

**Illustration 14:**

Find the locus of the centre of the circle whose  $x$  and  $y$  intercepts are  $a$  and  $b$  respectively.

**Solution:**

Equation of circle is  $x^2 + y^2 + 2gx + 2fy + c = 0$

$x$  intercept =  $a$

$$2\sqrt{g^2 - c} = ag^2 - c = \frac{a^2}{4} \quad \dots(i)$$

$y$  intercept =  $b$

$$2\sqrt{f^2 - c} = bf^2 - c = \frac{b^2}{4} \quad \dots(ii)$$

subtracting equation (i) and (ii)

$$g^2 - f^2 = \frac{a^2 - b^2}{4}$$

Hence locus of centre is  $x^2 - y^2 = \frac{a^2 - b^2}{4}$

**Illustration 15:**

If a circle of constant radius  $3k$  passes through the origin and meets the axes at  $A$  and  $B$ , the locus of the centroid of the triangle  $OAB$  is the circle

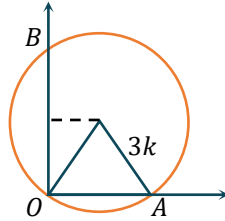
- (A)  $x^2 + y^2 = k^2$       (B)  $x^2 + y^2 + k^2 = 0$       (C)  $x^2 + y^2 + 4k^2 = 0$       (D)  $x^2 + y^2 = 4k^2$

**Ans. (D)**

**Solution:**

Equation of circle is  $x^2 + y^2 + 2gx + 2fy = 0$

Radius =  $3k = \sqrt{g^2 + f^2}$



∴ Intercepts are of length  $2\sqrt{g^2}$  and  $2\sqrt{f^2}$ .

∴  $A$  is  $(2g, 0)$ ,  $B(0, 2f)$  and centroid of  $\Delta$  is  $\left(\frac{2g}{3}, \frac{2f}{3}\right)$  and its locus is

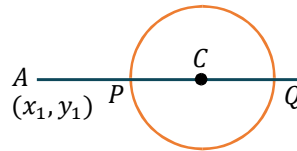
$$\left(\frac{9}{4}\right)(x^2 + y^2) = 9k^2 \Rightarrow x^2 + y^2 = 4k^2.$$

**Position of a Point with Respect to a Circle:**

The point  $(x_1, y_1)$  is inside, on or outside the circle  $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ .

according as  $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < =$  or  $> 0$ .

**Note :** The greatest & the least distance of a point  $A$  (lies outside the circle) from a circle with centre  $C$  & radius  $r$  is  $AC + r$  &  $AC - r$  respectively.



**Illustration 16:**

The point  $P(10, 7)$  lies outside the circle  $x^2 + y^2 - 4x - 2y - 20 = 0$ . The greatest distance of  $P$  from the circle is

- (A) 5      (B)  $\sqrt{3}$       (C)  $\sqrt{5}$       (D) 15

**Ans. (D)**

**Solution:**

$$r = \sqrt{4 + 1 + 20} = 5$$

$$C \equiv (2, 1)$$

$$\therefore CP = \sqrt{(10 - 2)^2 + (7 - 1)^2} = 10$$

$$\therefore \text{Maximum distance} = 10 + 5 = 15$$

**Illustration 17:**

Check whether the point  $(1, 2)$  lies in smaller or larger region made by circle  $x^2 + y^2 - 4x + 2y - 11 = 0$  & the line  $x + y = 0$

**Circle**

**Solution:**

We have  $x^2 + y^2 - 4x + 2y - 11 = 0$  or  $S = 0$ ,

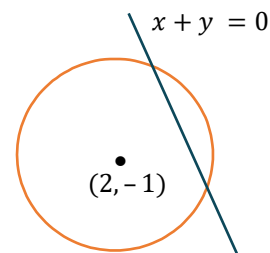
where  $S = x^2 + y^2 - 4x + 2y - 11$ .

For the point  $(1, 2)$ , we have  $S_1 = 1^2 + 2^2 - 4 \times 1 + 2 \times 2 - 11 < 0$

Hence, the point  $(1, 2)$  lies inside the circle

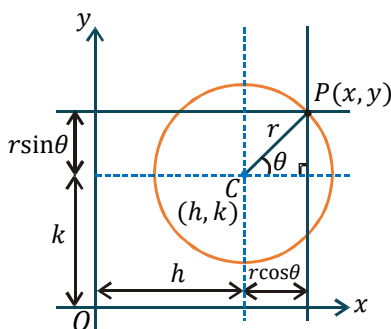
Points  $(1, 2)$  and  $(2, -1)$  lie on same side of the line  $x + y = 0$

Hence the point  $(1, 2)$  lies in the larger region.



**Parametric Equations of a Circle:**

The parametric equations of  $(x - h)^2 + (y - k)^2 = r^2$  are:  $x = h + r \cos \theta$ ;  $y = k + r \sin \theta$ ;  $-\pi < \theta \leq \pi$  where  $(h, k)$  is the centre,  $r$  is the radius &  $\theta$  is a parameter.



**Illustration 18:**

Find the parametric equations of the circle  $x^2 + y^2 + 4x + 6y + 9 = 0$ .

**Solution:**

We have :  $x^2 + y^2 + 4x + 6y + 9 = 0$

$$\Rightarrow (x + 2)^2 + (y + 3)^2 = 2^2$$

So, the parametric equations of this circle are

$$x = -2 + 2 \cos \theta, y = -3 + 2 \sin \theta.$$

**Illustration 19:**

Find the equation of the following curve in cartesian form  $x + y = \cos \theta, x - y = \sin \theta$  where  $\theta$  is the parameter.

**Solution:**

$$\text{We have : } x + y = \cos \theta \quad \dots(i)$$

$$x - y = \sin \theta \quad \dots(ii)$$

$$(i)^2 + (ii)^2$$

$$\Rightarrow (x + y)^2 + (x - y)^2 = 1$$

$$x^2 + y^2 = \frac{1}{2}$$

Clearly, it is a circle with centre at  $(0, 0)$  and radius  $\frac{1}{\sqrt{2}}$ .

**Line and a Circle:**

Let  $L = 0$  be a line &  $S = 0$  be a circle. If  $r$  is the radius of the circle &  $p$  is the length of the perpendicular from the centre on the line, then :

(i)  $p > r \Leftrightarrow$  the line does not meet the circle i.e. passes outside the circle.

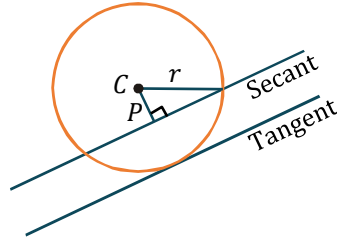
(ii)  $p = r \Leftrightarrow$  the line touches the circle. (It is tangent to the circle)

(iii)  $p < r \Leftrightarrow$  the line is a secant of the circle.

(iv)  $p = 0 \Rightarrow$  the line is a diameter of the circle.

Also, if  $y = mx + c$  is line and  $x^2 + y^2 = a^2$  is circle then

- (i)  $c^2 < a^2(1 + m^2) \Leftrightarrow$  the line is a secant of the circle.
- (ii)  $c^2 = a^2(1 + m^2) \Leftrightarrow$  the line touches the circle. (It is tangent to the circle)
- (iii)  $c^2 > a^2(1 + m^2) \Leftrightarrow$  the line does not meet the circle i.e. passes outside the circle.



These conditions can also be obtained by solving  $y = mx + c$  with  $x^2 + y^2 = a^2$  and making the discriminant of the quadratic greater than zero for secant, equal to zero for tangent and less the zero for the last case.

**Illustration 20:**

For what value of  $\lambda$ , does the line  $x + y = \lambda$  touch the circle  $x^2 + y^2 - 2x - 2y = 0$

**Solution:**

We have :  $x + y = \lambda$  ... (i)

and  $x^2 + y^2 - 2x - 2y = 0$  ... (ii)

If the line (i) touches the circle (ii), then

length of the  $\perp$  from the centre  $(1, 1) =$  radius of circle (ii)

$$\Rightarrow \left| \frac{1+1-\lambda}{\sqrt{1^2+1^2}} \right| = \sqrt{2} \Rightarrow |2-\lambda|=2 \Rightarrow \lambda = 0 \text{ or } 4$$

Hence, the line (i) touches the circle (ii) for  $\lambda = 0$  or  $4$

**Illustration 21:**

The straight line  $(x - 2) + (y + 3) = 0$  cuts the circle  $(x - 2)^2 + (y - 3)^2 = 11$  at

- (A) One point
- (B) Two points
- (C) No points
- (D) None of these

**Ans. (A)**

**Solution:**

Equation of line is  $x + y + 1 = 0$ . Since the perpendicular distance from centre to line is greater than radius, hence it does not cut the circle.

**Illustration 22:**

The equation of the circle whose centre is  $(3, -1)$  and which cuts off a chord of length 6 on the line  $2x - 5y + 18 = 0$  is

- (A)  $(x - 3)^2 + (y + 1)^2 = 38$
- (B)  $(x + 3)^2 + (y - 1)^2 = 38$
- (C)  $(x - 3)^2 + (y + 1)^2 = \sqrt{38}$
- (D) None of these

**Ans. (A)**

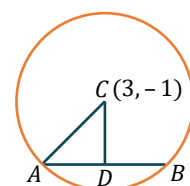
**Solution:**

Let  $AB(= 6)$  be the chord intercepted by the line  $2x - 5y + 18 = 0$  from the circle and let  $CD$  be the perpendicular drawn from centre  $(3, -1)$  to the chord  $AB$ .

i.e.  $AD=3, CD = \frac{2 \cdot 3 - 5(-1) + 18}{\sqrt{2^2 + 5^2}} = \sqrt{29}$

Therefore,  $CA^2 = 3^2 + (\sqrt{29})^2 = 38$

Hence required equation is  $(x - 3)^2 + (y + 1)^2 = 38$ .



**Tangent:****Slope form of Tangent:**

$y = mx + c$  is always a tangent to the circle  $x^2 + y^2 = a^2$  if  $c^2 = a^2(1 + m^2)$ .

Hence, of tangent is  $y = mx \pm a\sqrt{1 + m^2}$  and the point of contact is  $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$ .

**Point form of Tangent:**

- (i) The equation of the tangent to the circle  $x^2 + y^2 = a^2$  at its point  $(x_1, y_1)$  is,  $xx_1 + yy_1 = a^2$ .  
 (ii) The equation of the tangent to the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  at its point  $(x_1, y_1)$  is :  
 $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ .

**Note:**

In general the equation of tangent to any second degree curve at point  $(x_1, y_1)$  on it can be obtained by replacing  $x^2$  by  $xx_1$ ,  $y^2$  by  $yy_1$ ,  $x$  by  $\frac{x + x_1}{2}$ ,  $y$  by  $\frac{y + y_1}{2}$ ,  $xy$  by  $\frac{x_1y + xy_1}{2}$  and  $c$  remains as  $c$ .

**Parametric form of Tangent:**

The equation of a tangent to circle  $x^2 + y^2 = a^2$  at  $(a \cos \alpha, a \sin \alpha)$  is  $x \cos \alpha + y \sin \alpha = a$ .

**Note:** The point of intersection of the tangents at the points  $P(\alpha)$  &  $Q(\beta)$  is  $\left(\frac{a \cos \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2}}, \frac{a \sin \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2}}\right)$

**Illustration 23:**

Find the equation of the tangent to the circle  $x^2 + y^2 - 2x - 2y - 11 = 0$  at  $(3, 4)$ .

**Solution:**

$$\text{Equation of tangent is } 3x + 4y - 2\left(\frac{x+3}{2}\right) - 2\left(\frac{y+4}{2}\right) - 11 = 0$$

$$\text{or } 2x + 3y - 18 = 0$$

Hence, the required equation of the tangent is  $2x + 3y - 18 = 0$ .

**Illustration 24:**

Find the equation of tangents to the circle  $x^2 + y^2 - 4x + 2y = 0$  which are perpendicular to the line  $x + 2y + 4 = 0$

**Solution:**

$$\text{Given circle is } x^2 + y^2 - 4x + 2y = 0 \quad \dots(i)$$

$$\text{and given line is } x + 2y + 4 = 0 \quad \dots(ii)$$

Centre of circle (i) is  $(2, -1)$  and its radius  $\sqrt{5}$  is Equation of any line

$$2x - y + k = 0 \text{ perpendicular to the line (ii)} \quad \dots(iii)$$

If line (iii) is tangent to circle (i) then

$$\frac{|4 + 1 + k|}{\sqrt{5}} = \sqrt{5} \text{ or } |k + 5| = 5 \text{ or } k = 0, -10$$

Hence equation of required tangents are  $2x - y = 0$  and  $2x - y - 10 = 0$ .

**Normal:**

If a line is normal / orthogonal to a circle, then it must pass through the centre of the circle. Using this fact

normal to the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  at  $(x_1, y_1)$  is;  $y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$ .

**Illustration 25:**

Two normals of a circle are  $2x + 3y = 5$  and  $3x - 4y + 1 = 0$ . Find its equation having radius 2.

**Solution:**

Since point of intersection of normals is the centre of the circle,  
point of intersection of lines  $2x + 3y = 5$  and  $3x - 4y + 1 = 0$  is  $(1, 1)$   
equation of circle having centre  $(1, 1)$  and radius 2 is  $(x - 1)^2 + (y - 1)^2 = 4$

**Illustration 26:**

If the straight line  $ax + by = 2$ ;  $a, b \neq 0$  touches the circle  $x^2 + y^2 - 2x = 3$  and is normal to the circle  $x^2 + y^2 - 4y = 6$ , then the values of  $a$  and  $b$  are respectively

- (A) 1, -1                      (B) 1, 2                      (C)  $-\frac{4}{3}, 1$                       (D) 2, 1

**Ans. (C)**

**Solution:**

Given,  $x^2 + y^2 - 2x = 3$   
 $\therefore$  Centre is  $(1, 0)$  and radius is 2 and  $x^2 + y^2 - 4y = 6$   
 $\therefore$  Centre is  $(0, 2)$  and radius is  $\sqrt{10}$ .  
 Since line  $ax + by = 2$  touches the first circle  
 $\therefore \frac{a(1)+b(0)-2}{\sqrt{a^2+b^2}} = 2$  or  $(a - 2) = [2\sqrt{a^2 + b^2}] \dots(i)$

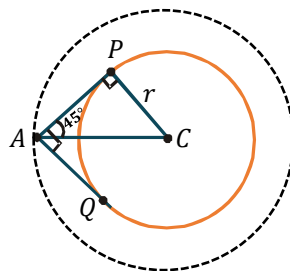
Also, the given line is normal to the second circle. Hence it will pass through the centre of the second circle.  
 $\therefore a(0) + b(2) = 2$  or  $2b = 2$  or  $b = 1$

Putting this value in equation (i) we get  
 $a - 2 = 2\sqrt{a^2 + 1^2}$  or  $(a - 2)^2 = 4(a^2 + 1)$   
 or  $a^2 + 4 - 4a = 4a^2 + 4$  or  $3a^2 + 4a = 0$   
 or  $a(3a + 4) = 0$  or  $a = 0, -\frac{4}{3}$   
 $\therefore$  Values of  $a$  and  $b$  are  $\left(-\frac{4}{3}, 1\right)$  respectively.

**Director Circle:**

The locus of the point of intersection of two perpendicular tangents is called the director circle of the given circle. The director circle of a circle is the concentric circle having radius equal to  $\sqrt{2}$  times the original circle.

**Proof:**



$AC = r \operatorname{cosec} 45^\circ = r\sqrt{2}$

**Illustration 27:**

Find the equation of director circle of the circle  $x^2 + y^2 + 6x + 8y - 2 = 0$ .

**Solution:**

Centre & radius of given circle are  $(-3, -4)$  &  $\sqrt{27}$  respectively.

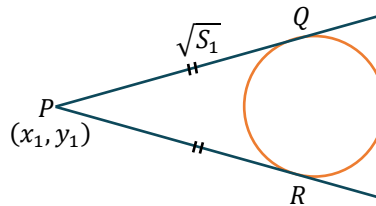
Centre and radius of the director circle will be  $(-3, -4)$  &  $\sqrt{27} \cdot \sqrt{2} = \sqrt{54}$  respectively.

$\therefore$  equation of director circle is  $(x + 3)^2 + (y + 4)^2 = 54$

$\Rightarrow x^2 + y^2 + 6x + 8y - 29 = 0$ .

**Length of Tangent:**

Let  $PQ$  and  $PR$  be two tangents drawn from  $P(x_1, y_1)$  to the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$ .



Then  $PQ = PR$  is called the length of tangent drawn from point  $P$  and is given by

$$PQ = PR = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}.$$

**Illustration 28:**

If the length of the tangents drawn from the point  $(1, 2)$  to the circles  $x^2 + y^2 + x + y - 4 = 0$  and  $3x^2 + 3y^2 - x - y + k = 0$  be in the ratio  $4 : 3$ , then  $k =$

- (A)  $7/2$                       (B)  $21/2$                       (C)  $-21/4$                       (D)  $7/4$

**Ans. (C)**

**Solution:**

Given  $\frac{T_1}{T_2} = \frac{4}{3}$ , where  $T_1$  and  $T_2$  are the length of tangents drawn to the given circle.

$$\Rightarrow \frac{\sqrt{1+4+1+2-4}}{\sqrt{(1)^2+(2)^2-\frac{1}{3}-\frac{2}{3}+\frac{k}{3}}} = \frac{4}{3} \Rightarrow k = -\frac{21}{4}.$$

**Illustration 29:**

Two tangents  $PQ$  and  $PR$  drawn to the circle  $x^2 + y^2 - 2x - 4y - 20 = 0$  from point  $P(16,7)$ . If the centre of the circle is  $C$ , then the area of quadrilateral  $PQCR$  will be

- (A) 75 sq. units                      (B) 150 sq. units                      (C) 15 sq. units                      (D) None of these

**Ans. (A)**

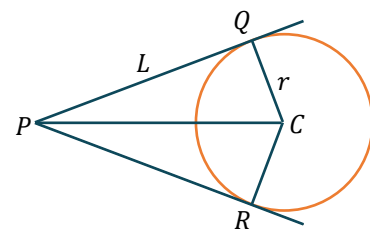
**Solution:**

$$\text{Area } PQCR = 2 \cdot \Delta PQC = 2 \times \frac{1}{2} L \times r$$

Where  $L =$  length of tangent and  $r =$  radius of circle.

$$L = \sqrt{S_1} \text{ and } r = \sqrt{1 + 4 + 20} = 5$$

Hence the required area = 75 sq. units.



**Power of a Point:**

For circle  $\equiv x^2 + y^2 + 2gx + 2fy + c = 0$

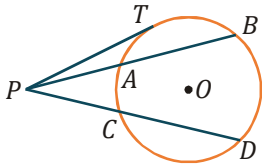
Power of a point  $P(x_1, y_1)$  is equal to  $S_1$

$$S_1 = (x_1)^2 + (y_1)^2 + 2gx_1 + 2fy_1 + c = 0$$

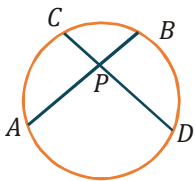
Power of Point  $> 0 \quad \leftrightarrow \quad$  Point lies outside the circle

Power of Point  $< 0 \quad \leftrightarrow \quad$  Point lies inside the circle

Power of Point  $= 0 \quad \leftrightarrow \quad$  Point lies on circle



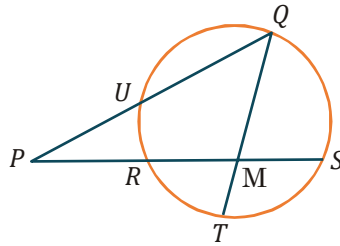
**P lies outside the circle,  $PT^2 = (PA)(PB) = (PC)(PD)$  = The power of point P**



**P lies inside the circle,  $PA \cdot PB = PC \cdot PD$**

**Illustration 30:**

Points  $R$  and  $M$  trisect, so  $PR = RM = MS$ . Point  $U$  is the midpoint of  $TQ$ ,  $TM = 2$  and  $MQ = 8$ . Find  $PU$ .



**Solution:**

The power of point  $M$  gives us  $(MR)(MS) = (MT)(MQ)$ .

We know that  $RM = MS$ , so substitution gives  $MR^2 = (2)(8)$ , i.e.,  $MR = 4$ . Therefore,  $PR = MR = 4$  and  $PS = 3(MR) = 12$ .

Since  $U$  is the midpoint of  $\overline{TQ}$ , we have  $PQ = 2PU$ . Now we can apply the power of point  $P$  to find :  $(PU)(PQ) = (PR)(PS)$ .

Substitution gives  $(PU)(2PU) = 4(12)$ , so  $PU = 2\sqrt{6}$ .

**Illustration 31:**

In the diagram, we have  $BP = 8, AB = 10, CD = 7$  and  $\angle APC = 60^\circ$ . Find the area of the circle.

**Solution:**

It's not immediately obvious how we will find the radius,

So we start by finding what we can. The power of a point  $P$  gives us

$$(PC)(PD) = (PB)(PA),$$

So  $(PC)(PC + 7) = 144$ . Therefore,  $PC^2 + 7PC - 144 = 0$ , so  $(PC + 16)(PC - 9) = 0$ .  $PC$  must be positive, so  $PC = 9$ .

Seeing that  $\angle APC = 60^\circ$  makes us wonder if there are any equilateral or  $30 - 60 - 90$  triangle lurking about.

**Circle**

Since  $CP = AP/2$  and the angle between these sides is  $60^\circ$ , the sides adjacent to the  $60^\circ$  angle in  $\triangle ACP$  are in the same ratio as the sides adjacent to the  $60^\circ$  angle in a  $30 - 60 - 90$  triangle with right angle at  $\angle ACP$ !

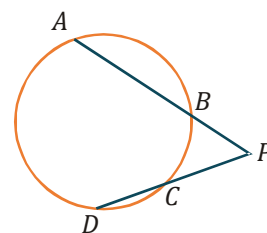
$$\Rightarrow \angle ACP = 90^\circ \Rightarrow \angle ACD = 90^\circ$$

Since  $\angle ACD$  is right and inscribed in  $AD$ , we know  $AD$  is a semicircle. Therefore,  $\overline{AD}$  is a diameter of the circle.

Since  $AC = CP\sqrt{3} = 9\sqrt{3}$  from our  $30 - 60 - 90$  triangle, we have.

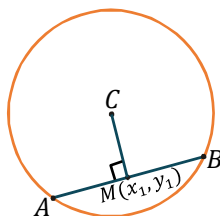
$$AD = \sqrt{AC^2 + CD^2} = \sqrt{243 + 49} = 2\sqrt{73}$$

Finally, the radius of the circle is  $AD/2 = \sqrt{73}$ , so the area is  $(\sqrt{73})^2 \pi = 73\pi$ .



**Equation of Chord with given Middle Point:**

The equation of the chord of the circle  $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$  in terms of its mid-point  $M(x_1, y_1)$  is  $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$  which is designated by  $T = S_1$ .



**Note:**

- (i) The shortest chord of a circle passing through a point 'M' inside the circle is one chord whose middle point is M.
- (ii) The chord passing through a point 'M' inside the circle and which is at a maximum distance from the centre is a chord with middle point M.

**Illustration 32:**

Find the equation of the chord of the circle  $x^2 + y^2 + 2x - 2y - 4 = 0$ , whose middle point is  $(0, 0)$ .

**Solution:**

Equation of given circle is  $S \equiv x^2 + y^2 + 2x - 2y - 4 = 0$

Let  $L \equiv (0, 0)$

For point  $L(0, 0), S_1 = -4$  and

$$T \equiv x \cdot 0 + y \cdot 0 + (x + 0) - (y + 0) - 4 \text{ i.e. } T \equiv x - y - 4$$

Now equation of the chord of circle (i) whose middle point is  $L(0, 0)$  is

$$T = S_1 \text{ or } x - y = 0$$

**Second Method:**

Let  $C$  be the centre of the given circle, then  $C \equiv (-1, 1), L \equiv (0, 0)$  slope of  $CL = -1$ .

$\therefore$  Equation of chord of circle whose middle point is  $L$ , is  $y - 0 = 1(x - 0)$

( $\because$  chord is perpendicular to  $CL$ ) or  $x - y = 0$

**Illustration 33:**

A variable chord is drawn through the origin to the circle  $x^2 + y^2 - 2ax = 0$ . The locus of the centre of the circle drawn on this chord as diameter, is

- (A)  $x^2 + y^2 + ax = 0$
- (B)  $x^2 + y^2 + ay = 0$
- (C)  $x^2 + y^2 - ax = 0$
- (D)  $x^2 + y^2 - ay = 0$

**Ans. (C)**

**Solution:**

Suppose mid-point of chord be  $(h, k)$ , then its equation is  $T = S_1$   
 $\Rightarrow hx + ky - a(x + h) = h^2 + k^2 - 2ah$   
 It passes through origin, then  $hx + ky - ax = 0$   
 Hence locus of mid-point is  $x^2 + y^2 - ax = 0$ .

**Chord of Contact:**

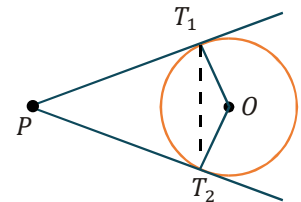
If two tangents  $PT_1$  &  $PT_2$  are drawn from the point  $P(x_1, y_1)$  to the circle  $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ , then the equation of the chord of contact  $T_1T_2$  is:  
 $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ .

**Note:** Here  $R$  = radius;  $L$  = length of tangent.

(a) Chord of contact exists only if the point 'P' is not inside.

(b) Length of chord of contact  $T_1T_2 = \frac{2LR}{\sqrt{R^2 + L^2}}$ .

(c) Area of the triangle formed by the pair of the tangents & its chord of contact =  $\frac{RL^3}{R^2 + L^2}$



(d) Tangent of the angle between the pair of tangents from  $(x_1, y_1) = \left( \frac{2RL}{L^2 - R^2} \right)$

(e) Equation of the circle circumscribing the triangle  $PT_1T_2$  is:  
 $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$ .

**Illustration 34:**

Find the equation of the chord of contact of the tangents drawn from  $(0, 1)$  to the circle  $x^2 + y^2 - 2x + 4y = 0$

**Solution:**

Given circle is  $x^2 + y^2 - 2x + 4y = 0$  ... (i)

Let  $P = (0, 1)$

For point  $P(0, 1), T = x \cdot 0 + y \cdot 1 - (x + 0) + 2(y + 1)$

i.e.  $T = x - 3y - 2$

Now equation of the chord of contact of point  $P(0, 1)$  w.r.t. circle (i) will be

$$x - 3y - 2 = 0$$

**Illustration 35:**

If the chord of contact of the tangents drawn from  $(\alpha, \beta)$  to the circle  $x^2 + y^2 = a^2$  subtends right angle at the centre then prove that  $\alpha^2 + \beta^2 = 2a^2$ .

**Solution:**

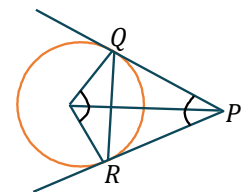
$$\angle QOR = \angle QPR = \frac{\pi}{2}$$

So  $OQPR$  is a square

$$OQ^2 = PQ^2$$

$$a^2 = \alpha^2 + \beta^2 - a^2$$

$$\alpha^2 + \beta^2 = 2a^2$$

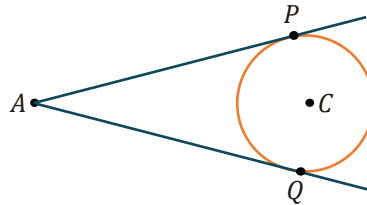


**Pair of Tangents from a Point:**

The equation of a pair of tangents drawn from the point  $A(x_1, y_1)$  to the circle  $x^2 + y^2 + 2gx + 2fy + c = 0$  is :  $SS_1 = T^2$ .

Where  $S \equiv x^2 + y^2 + 2gx + 2fy + c$ ;  $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$ .



**Illustration 36:**

Find the equation of the pair of tangents drawn to the circle  $x^2 + y^2 + 4x - 6y + 9 = 0$  from the point  $(2, 1)$ .

**Solution:**

Given circle is  $S = x^2 + y^2 + 4x - 6y + 9 = 0$

Let  $P \equiv (2, 1)$

For point  $P, S_1 = 16$

Clearly  $P$  lies outside the circle

and  $T \equiv 2x + y + 2(x + 2) - 3(y + 1) + 9 = 0$

i.e.  $T \equiv 2(2x - y + 5)$

Now equation of pair of tangents from  $P(2, 1)$  to circle (1) is  $SS_1 = T^2$

or  $16(x^2 + y^2 + 4x - 6y + 9) = 4(2x - y + 5)^2$

or  $12y^2 - 16x - 56y + 16xy + 44 = 0$

or  $3y^2 - 4x - 14y + 4xy + 11 = 0$

**Family of Circles (Part-I):**

This article is aimed at obtaining the equation of a group of circles having a specific characteristic.

For example, the equation  $x^2 + y^2 + 4x + 2y + \lambda = 0$  where  $\lambda$  is arbitrary, represents a family of circles with fixed centre  $(-2, -1)$  but variable radius. We have the following results for some other families of circles.

(a) The equation of the family of circles passing through the points of intersection of two circles  $S_1 = 0$  &  $S_2 = 0$  is:  $S_1 + K S_2 = 0$

(b) The equation of the family of circles passing through the point of intersection of a circle  $S = 0$  & a line  $L = 0$  is given by  $S + KL = 0$ .

**Illustration 37:**

Find the equation of the circle passing through the point  $(1, 1)$  and points of intersection of the circles  $x^2 + y^2 + 13x - 3y = 0$  and  $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ .

**Solution:**

Any circle through the intersection of given circles is  $S_1 + \lambda S_2 = 0$

Or  $x^2 + y^2 + 13x - 3y + \lambda(x^2 + y^2 + 2x - 7y/2 - 25/2) = 0$

This circle passes through  $(1, 1)$

$1 + 1 + 13 - 3 + \lambda(1 + 1 + 2 - 7/2 - 25/2) = 0$

$\lambda = 1$

Putting the value of  $\lambda$  in (i) the required circle is  $4x^2 + 4y^2 + 30x - 13y - 25 = 0$

**Illustration 38:**

Find the equations of smallest circle which passes through the points of intersection of the line  $x + y = 1$  and the circle  $x^2 + y^2 = 9$ .

**Solution:**

The required circle by  $S + \lambda L = 0$  is

$$x^2 + y^2 - 9 + \lambda(x + y - 1) = 0 \quad \dots(i)$$

$$\text{centre } (-g, -f) = \left( -\frac{\lambda}{2}, -\frac{\lambda}{2} \right)$$

centre lies on the line  $x + y = 1$

$$-\frac{\lambda}{2} - \frac{\lambda}{2} = 1$$

$$\lambda = -1$$

Putting the value of  $\lambda$  in (i) the required circle is

$$x^2 + y^2 - x - y - 8 = 0$$

**Illustration 39:**

If  $y = 2x$  is a chord of the circle  $x^2 + y^2 - 10x = 0$ , then the equation of the circle of which this chord is a diameter, is

(A)  $x^2 + y^2 - 2x + 4y = 0$

(B)  $x^2 + y^2 + 2x + 4y = 0$

(C)  $x^2 + y^2 + 2x - 4y = 0$

(D)  $x^2 + y^2 - 2x - 4y = 0$

**Ans. (D)**

**Solution:**

Family of circles passing from points of intersection of circle and chord are  $x^2 + y^2 - 10x + \lambda(y - 2x) = 0$ .

Its centre is  $\left\{ (5 + \lambda), -\frac{\lambda}{2} \right\}$ . It lies on  $y - 2x = 0$ .

$$\text{i.e. } -\frac{\lambda}{2} - 10 - 2\lambda = 0 \Rightarrow -5\lambda = 20 \Rightarrow \lambda = -4$$

Hence equation of circle is  $x^2 + y^2 - 2x - 4y = 0$ .

**Family of Circles (Part-II):**

(a) The equation of a family of circles passing through two given points  $(x_1, y_1)$  &  $(x_2, y_2)$  can be written in the form :

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \text{ where } K \text{ is a parameter.}$$

(b) The equation of a family of circles touching a fixed line  $y - y_1 = m(x - x_1)$  at the fixed point  $(x_1, y_1)$  is  $(x - x_1)^2 + (y - y_1)^2 + K(y - y_1 - m(x - x_1)) = 0$ , where  $K$  is a parameter.

**Illustration 40:**

Find the equation of circle passing through the points  $A(1, 1)$  &  $B(0, 3)$  and whose radius is  $\sqrt{\frac{5}{2}}$ .

**Solution:**

Equation of  $AB$  is  $2x + y - 3 = 0$

$\therefore$  equation of circle is  $(x - 1)(x) + (y - 1)(y - 3) + \lambda(2x + y - 3) = 0$

or  $x^2 + y^2 + (2\lambda - 1)x + (\lambda - 4)y + 3 - 3\lambda = 0$

$$\sqrt{\left(\frac{2\lambda - 1}{2}\right)^2 + \left(\frac{\lambda - 4}{2}\right)^2} + 3\lambda - 3 = \sqrt{\frac{5}{2}}$$

$\lambda = 1$

$\therefore$  equation of circle is  $x^2 + y^2 + x - 3y = 0$

**Common Tangents to Two Circles:**

Different cases of intersection of two circles:

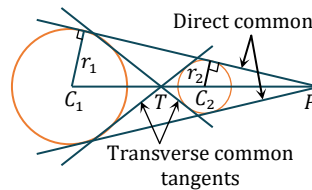
Let the two circles be  $(x - x_1)^2 + (y - y_1)^2 = r_1^2$  ... (i)

and  $(x - x_2)^2 + (y - y_2)^2 = r_2^2$  ... (ii)

with centres  $C_1(x_1, y_1)$  and  $C_2(x_2, y_2)$  and radii  $r_1$  and  $r_2$  respectively. Then following cases may arise :

**Case-I:**

When  $|C_1C_2| > r_1 + r_2$  i.e., the distance between the centres is greater than the sum of radii.



In this case four common tangents can be drawn to the two circles, in which two are direct common tangents and the other two are transverse common tangents.

The points  $P, T$  of intersection of direct common tangents and transverse common tangents respectively, always lie on the line joining the centres of the two circles and divide it externally and internally respectively in the ratio of their radii.

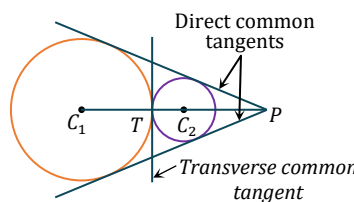
$$\frac{C_1P}{C_2P} = \frac{r_1}{r_2} \text{ (externally) and } \frac{C_1T}{C_2T} = \frac{r_1}{r_2} \text{ (internally)}$$

Hence, the ordinates of  $P$  and  $T$  are

$$P \equiv \left( \frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2} \right) \text{ and } T \equiv \left( \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2} \right).$$

**Case-II:**

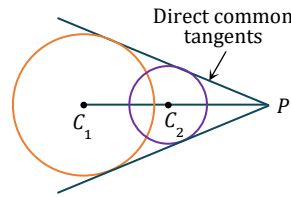
When  $|C_1C_2| = r_1 + r_2$  i.e., the distance between the centres is equal to the sum of radii.



In this case two direct common tangents are real and distinct while the transverse tangents are coincident.

**Case-III:**

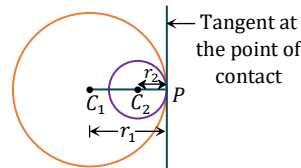
When  $|C_1C_2| < r_1 + r_2$  i.e., the distance between the centres is less than sum of radii.



In this case two direct common tangents are real and distinct while the transverse tangents are imaginary.

**Case-IV:**

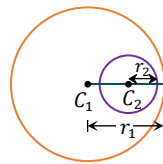
When  $|C_1C_2| = |r_1 - r_2|$ , i.e., the distance between the centres is equal to the difference of the radii.



In this case two tangents are real and coincident while the other two tangents are imaginary.

**Case-V:**

When  $|C_1C_2| < |r_1 - r_2|$ , i.e., the distance between the centres is less than the difference of the radii.



In this case, all the four common tangents are imaginary

**Illustration 41:**

Prove that the circles  $x^2 + y^2 + 2ax + c^2 = 0$  and  $x^2 + y^2 + 2by + c^2 = 0$  touch each other,

if  $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ .

**Solution:**

Given circles are  $x^2 + y^2 + 2ax + c^2 = 0$  ... (i)

and  $x^2 + y^2 + 2by + c^2 = 0$  ... (ii)

Let  $C_1$  and  $C_2$  be the centres of circles (i) and (ii), respectively and  $r_1$  and  $r_2$  be their radii, then

$C_1 = (-a, 0), C_2 = (0, -b), r_1 = \sqrt{a^2 - c^2}, r_2 = \sqrt{b^2 - c^2}$

Here we find the two circles touch each other internally or externally.

For touch,  $|C_1C_2| = |r_1 \pm r_2|$

Or  $\sqrt{(a^2 + b^2)} = |\sqrt{(a^2 - c^2)} \pm \sqrt{(b^2 - c^2)}|$

On squaring  $a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2 - c^2)}\sqrt{(b^2 - c^2)}$

Or  $c^2 = \pm\sqrt{a^2b^2 - c^2(a^2 + b^2)} + c^4$

Again squaring,  $c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$

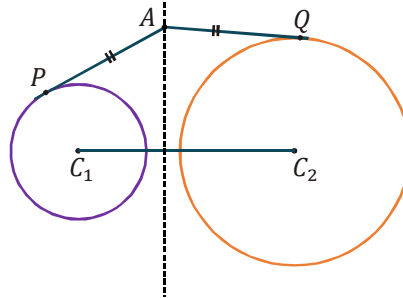
Or  $c^2(a^2 + b^2) = a^2b^2$  or  $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$

**Radical Axis & Radical Centre:**

The radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal.

The equation of radical axis of the two circles  $S_1 = 0$  &  $S_2 = 0$  is given by

$$S_1 - S_2 = 0 \text{ i.e. } 2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0.$$



The common point of intersection of the radical axes of three circles taken two at a time is called the radical centre of three circles. Note that the length of tangents from radical centre to the three circles are equal.

**Note:**

- If two circles intersect, then the radical axis is the common chord of the two circles.
- If two circles touch each other, then the radical axis is the common tangent of the two circles at the common point of contact.
- Radical axis is always perpendicular to the line joining the centres of the two circles.
- Radical axis will pass through the mid-point of the line joining the centres of the two circles only if the two circles have equal radii.
- Radical axis bisects a common tangent between the two circles.
- A system of circles, every two which have the same radical axis, is called a coaxial system.
- Pairs of circles which do not have radical axis are concentric.

**Illustration 42:**

If the circle  $x^2 + y^2 + 2x + 3y + 1 = 0$ , cuts  $x^2 + y^2 + 4x + 3y + 2 = 0$  at  $A$  and  $B$ , then find the equation of the circle on  $AB$  as diameter

**Solution:**

$AB$  is the radical axis of the two circles

The equation of  $AB$  is  $2x + 1 = 0$  (using  $S_1 - S_2 = 0$ ).

The equation of the required circle is

$$(x^2 + y^2 + 2x + 3y + 1) + \lambda(2x + 1) = 0 \text{ [Using } S_1 + \lambda(S_2 - S_1) = 0 \text{]}$$

$$\text{Or } x^2 + y^2 + 2x(\lambda + 1) + 3y + \lambda + 1 = 0$$

Since  $AB$  is diameter of this circle, the center lies on it. So,

$$-2\lambda - 2 + 1 = 0 \text{ or } \lambda = -\frac{1}{2}$$

Thus, the required circle is

$$x^2 + y^2 + x + 3y + \frac{1}{2} = 0 \text{ or } 2x^2 + 2y^2 + 2x + 6y + 1 = 0$$

**Illustration 43:**

Find the co-ordinates of the point from which the lengths of the tangents to the following three circles be equal.

$$x^2 + y^2 = 1$$

$$x^2 + y^2 - 8x + 15 = 0$$

$$x^2 + y^2 + 10y + 24 = 0$$

**Solution:**

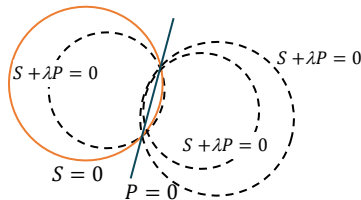
Here we have to find the radical centre of the three circles. First reduce them to standard form in which coefficients of  $x^2$  and  $y^2$  be each unity. Subtracting in pairs the three radical axes are

$$\begin{aligned} x &= 2; \\ 8x + 10y + 9 &= 0 \\ 10y + 25 &= 0 \end{aligned}$$

Solving any two, we get the point  $\left(2, -\frac{5}{2}\right)$  which satisfies the third also. This point is called the radical centre and by definition the length of the tangents from it to the three circles are equal.

**Coaxial System of Circles:**

A system (or a family) of circles, every pair of which have the same radical axis, are called co-axial circles.



- (1) The equation of a system of co-axial circles, when the equation of the radical axis and of one circle of the system are  $P \equiv lx + my + n = 0, S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$  respectively, is  $S + \lambda P = 0$  ( $\lambda$  is an arbitrary constant).

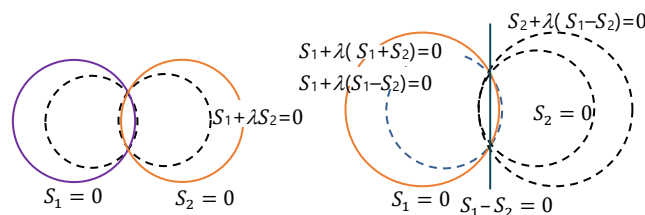
- (2) The equation of a co-axial system of circles, where the equation of any two circles of the system are

$$\begin{aligned} S_1 &\equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \text{ and} \\ S_2 &\equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \end{aligned}$$

Respectively, is  $S_1 + \lambda(S_1 - S_2) = 0$

$$\text{or } S_2 + \lambda_1(S_1 - S_2) = 0$$

Other form  $S_1 + \lambda S_2 = 0, (\lambda \neq -1)$



- (3) The equation of a system of co-axial circles in the simplest form is  $x^2 + y^2 + 2gx + c = 0$ , where  $g$  is a variable and  $c$  is a constant.

**Illustration 44:**

The gradient of the radical axis of the circles  $x^2 + y^2 - 3x - 4y + 5 = 0$  and  $3x^2 + 3y^2 - 7x + 8y + 11 = 0$  is

- (A)  $\frac{1}{3}$                       (B)  $-\frac{1}{10}$                       (C)  $-\frac{1}{2}$                       (D)  $-\frac{2}{3}$

Ans. (B)

**Circle**

**Solution:**

Equation of radical axis is  $S_1 - S_2 = 0$

$$S_1 \equiv x^2 + y^2 - 3x - 4y + 5 = 0, S_2 \equiv x^2 + y^2 - \frac{7}{3}x + \frac{8y}{3} + \frac{11}{3} = 0$$

∴ Radical axis is  $-2x - 20y - 4 = 0$

Hence gradient of radical axis  $= -\frac{1}{10}$

**Illustration 45:**

The coordinates of the radical centre of the three circles  $x^2 + y^2 - 4x - 2y + 6 = 0$ ,

$x^2 + y^2 - 4x - 2y + 6y = 0$ ,  $x^2 + y^2 - 12x + 2y + 30 = 0$  are

- (A) (6, 30)                      (B) (0, 6)                      (C) (3, 0)                      (D) None of these

**Ans. (D)**

**Solution:**

Radical axis of first two circles is

$$6y - 6 = 0 \text{ i.e., } y = 1 \quad \dots(i)$$

Radical axis of last two circles is

$$8x + 2y = 30 \quad \dots(ii)$$

Solving (i) and (ii), we get radical centre  $\left(\frac{7}{2}, 1\right)$ .

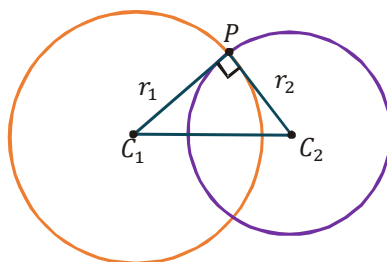
**Orthogonality of Two Circles:**

Two circles  $S_1 = 0$  &  $S_2 = 0$  are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is :

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2.$$

**Proof:**

$$\begin{aligned} (C_1C_2)^2 &= (C_1P)^2 + (C_2P)^2 \\ \Rightarrow (g_1 - g_2)^2 + (f_1 - f_2)^2 &= g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2 \\ \Rightarrow 2g_1g_2 + 2f_1f_2 &= c_1 + c_2 \end{aligned}$$



**Note :**

- (a) The centre of a variable circle orthogonal to two fixed circles lies on the radical axis of two circles.
- (b) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles  $S_1 = 0$ ,  $S_2 = 0$  &  $S_3 = 0$  are concurrent in a circle which is orthogonal to all the three circles.
- (c) The centre of a circle which is orthogonal to three given circles is the radical centre provided the radical centre lies outside all the three circles.

**Illustration 46:**

If the circles  $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$  and  $2x^2 + 2y^2 + 2g_2x + 2f_2y + c_2 = 0$  are orthogonal to each other then prove that  $g_1g_2 + f_1f_2 = c_1 + \frac{c_2}{2}$

**Solution:**

Given circles are  $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$  ... (i)

And  $2x^2 + 2y^2 + 2g_2x + 2f_2y + c_2 = 0$

Or  $x^2 + y^2 + g_2x + f_2y + \frac{c_2}{2} = 0$  ... (ii)

Since circles (i) and (ii) cut orthogonally

$$\therefore 2g_1\left(\frac{g_2}{2}\right) + 2f_1\left(\frac{f_2}{2}\right) = c_1 + \frac{c_2}{2}$$

$$g_1g_2 + f_1f_2 = c_1 + \frac{c_2}{2}$$

**Illustration 47:**

If the circles  $x^2 + y^2 + 2x + 2ky + 6 = 0$  and  $x^2 + y^2 + 2ky + k = 0$  intersect orthogonally, then  $k$  is

- (A) 2 or  $-\frac{3}{2}$       (B)  $-2$  or  $\frac{3}{2}$       (C) 2 or  $\frac{3}{2}$       (D)  $-2$  or  $-\frac{3}{2}$

**Ans. (A)**

**Solution:**

$$2gg' + 2ff' = c + c' \text{ i.e., } 2 \cdot 1 \cdot 0 + 2 \cdot k \cdot k = 6 + k$$

$$\text{or } 2k^2 - k - 6 = 0$$

$$\text{or } (2k + 3)(k - 2) = 0$$

$$\therefore k = 2, -\frac{3}{2}$$

**Illustration 48:**

The circle  $x^2 + y^2 = 4x + 8y + 5$  intersects the line  $3x - 4y = m$  at two distinct points if :

- (A)  $-85 < m < -35$     (B)  $-35 < m < 15$     (C)  $15 < m < 65$     (D)  $35 < m < 85$

**Ans. (B)**

**Solution:**

$$s: x^2 + y^2 - 4x - 8y - 5$$

$$(x - 2)^2 + (y - 4)^2 = 5^2$$

for  $m \perp$  distance from  $(2, 4)$  to  $3x - 4y - m = 0$  is  $P$

$$0 < |P| < 5$$

$$\Rightarrow \frac{|3(2) - 4(4) - m|}{5} < 5$$

$$\Rightarrow |10 + m| < 25$$

$$-35 < m < 15$$

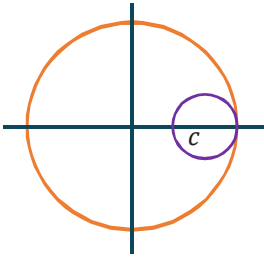
**Illustration 49:**

The two circles  $x^2 + y^2 = ax$  and  $x^2 + y^2 = c^2 (c > 0)$  touch each other if :

- (A)  $a = 2c$       (B)  $|a| = 2c$       (C)  $2|a| = c$       (D)  $|a| = c$

**Ans. (D)**

**Solution:**



$$\left| \frac{a}{2} \right| = c - \left| \frac{a}{2} \right|$$

$$|a| = c$$

**Illustration 50:**

The equation of the circle passing through the points (1, 0) and (0, 1) and having the smallest radius is:

- (A)  $x^2 + y^2 + x + y - 2 = 0$                       (B)  $x^2 + y^2 - 2x - 2y + 1 = 0$   
 (C)  $x^2 + y^2 - x - y = 0$                               (D)  $x^2 + y^2 + 2x + 2y - 7 = 0$

**Ans. (C)**

**Solution:**

(1, 0) and (0, 1) will be ends of diameter

So equation of circle

$$(x - 1)(x - 0) + (y - 0)(y - 1)$$

$$x^2 + y^2 - x - y = 0$$

**Illustration 51:**

The length of the diameter of the circle which touches the  $x$ -axis at the point (1, 0) and passes through the point (2, 3) is :

- (A) 5/3                      (B) 10/3                      (C) 3/5                      (D) 6/5

**Ans. (B)**

**Solution:**

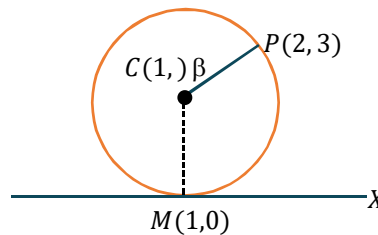
Let center of the circle be  $C(1, \beta)$

$$\beta^2 = (2 - 1)^2 + (3 - \beta)^2$$

$$\Rightarrow \beta^2 = -6\beta + 10 + \beta^2$$

$$\Rightarrow \beta = \frac{5}{3} \quad \therefore r = \frac{5}{3}$$

$$\text{diameter} = \frac{10}{3}$$



**Illustration 52:**

The circle passing through (1, -2) and touching the axis of  $x$  at (3, 0) also passes through the point :

- (A) (-5, 2)                      (B) (2, -5)                      (C) (5, -2)                      (D) (-2, 5)

**Ans. (C)**

**Solution:**

Let equation of circle be  $(x - 3)^2 + (y + r)^2 = r^2$

$\therefore$  it passes through (1, -2)

$$\Rightarrow r = 2$$

$$\Rightarrow \text{circle is } (x - 3)^2 + (y + 2)^2 = 4$$

$$\Rightarrow (5, -2)$$

**Aliter :**

$$(x - 3)^2 + y^2 + \lambda y = 0 \quad \dots(1)$$

Putting (1, -2) in (1)

$$\Rightarrow \lambda = 4$$

Required circle is

$$x^2 + y^2 - 6x + 4y + 9 = 0$$

point (5, -2) satisfies the equation the equation.

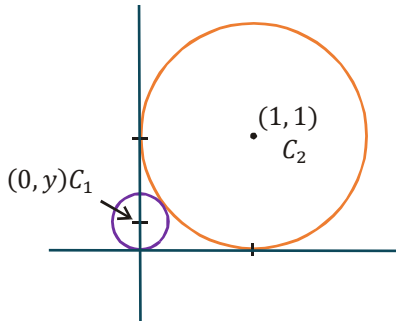
**Illustration 53:**

Let  $C$  be the circle with centre at (1, 1) and radius = 1. If  $T$  is the circle centred at (0,  $y$ ), passing through origin and touching the circle  $C$  externally, then the radius of  $T$  is equal to :

- (A)  $\frac{\sqrt{3}}{\sqrt{2}}$       (B)  $\frac{\sqrt{3}}{2}$       (C)  $\frac{1}{2}$       (D)  $\frac{1}{4}$

**Ans. (D)**

**Solution:**



$$C_1 C_2 = r_1 + r_2$$

$$\sqrt{1 + (1 - y)^2} = 1 + y$$

$$\therefore y = \frac{1}{4}$$

$$\therefore \text{radius} = \frac{1}{4}$$

**Illustration 54:**

Let  $PQ$  and  $RS$  be tangents at the extremities of the diameter  $PR$  of a circle of radius  $r$ . If  $PS$  and  $RQ$  intersect at a point  $X$  on the circumference of the circle then  $2r$  equals

- (A)  $\sqrt{PQ \cdot RS}$       (B)  $\frac{PQ + RS}{2}$       (C)  $\frac{2PQ \cdot RS}{PQ + RS}$       (D)  $\sqrt{\frac{(PQ)^2 + (RS)^2}{2}}$

**Ans. (A)**

**Solution:**

Let  $\angle RPS = \theta$

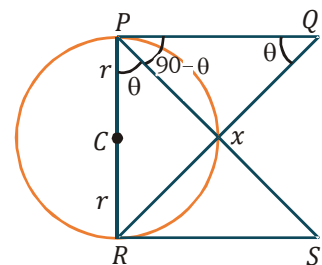
$\angle XPQ = 90 - \theta$

$\therefore \angle PQX = \theta$  ( $\because \angle PXQ = 90^\circ$ )

$\therefore \Delta PRS \sim \Delta QPR$  (AAA similarity)

$$\therefore \frac{PR}{QP} = \frac{RS}{PR} \Rightarrow PR^2 = PQ \cdot RS$$

$$\Rightarrow PR = \sqrt{PQ \cdot RS}$$



**Illustration 55:**

Let  $2x^2 + y^2 - 3xy = 0$  be the equation of a pair of tangents drawn from the origin 'O' to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, if the length of OA is  $3(a + \sqrt{b})$ , then find  $a + b$

**Solution:**

The equation  $2x^2 - 3xy + y^2 = 0$  represents pair of tangents OA and OA'.

Let angle between these two tangents be  $2\theta$ .

$$\text{Then } \tan 2\theta = \frac{2\sqrt{\left(\frac{-3}{2}\right)^2 - 2 \times 1}}{2 + 1}$$

$$[\text{Using } \tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}]$$

$$\frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{1}{3} \Rightarrow \tan^2 \theta + 6 \tan \theta - 1 = 0$$

$$\tan \theta = \frac{-6 \pm \sqrt{36 + 4}}{2} = -3 \pm \sqrt{10}$$

As  $\theta$  is acute  $\therefore \tan \theta = \frac{3}{OA}$

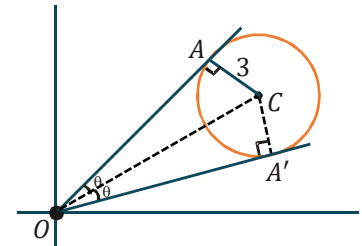
Now we know that line joining the point through which tangents are drawn to the centre bisects the angle between the tangents,

$$\therefore \angle AOC = \angle A'OC = \theta$$

In  $\Delta OAC$   $\tan \theta = \sqrt{10} - 3$

$$\Rightarrow OA = \frac{3}{\sqrt{10} - 3} \times \frac{\sqrt{10} + 3}{\sqrt{10} + 3}$$

$$\therefore OA = 3(3 + \sqrt{10})$$



**Illustration 56:**

Find the equation of the circle which passes through the points of intersection of circles  $x^2 + y^2 - 2x - 6y + 6 = 0$  and  $x^2 + y^2 + 2x - 6y + 6 = 0$  and intersects the circle  $x^2 + y^2 + 4x + 6y + 4 = 0$  orthogonally.

**Solution:**

$$S_1: x^2 + y^2 - 2x - 6y + 6 = 0$$

$$S_2: x^2 + y^2 + 2x - 6y + 6 = 0$$

Family of circle through point of intersection

$$\text{of } S_1 = 0 \text{ \& } S_2 = 0$$

$$S_1 + \lambda(S_2 - S_1) = 0$$

$$S: x^2 + y^2 - 2x - 6y + 6 + \lambda(4x) = 0$$

It is orthogonal to  $S_3: x^2 + y^2 + 4x + 6y + 4 = 0$

$$\text{for } \lambda \text{ apply } 2g_1g_2 + 2f_1f_2 = C_1 + C_2$$

**Illustration 57:**

Tangents TP and TQ are drawn from a point T to the circle  $x^2 + y^2 = a^2$ . If the point T lies on the line  $px + qy = r$ , find the locus of centre of the circumcircle of triangle TPQ.

**Solution:**

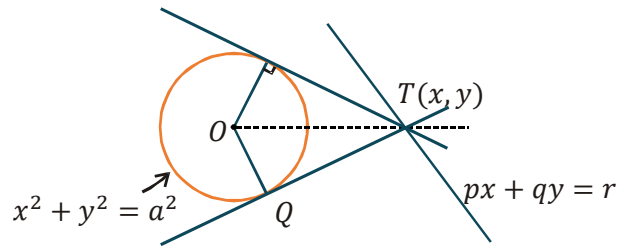
Circum centre of  $TPQ$  is also circumcentre of  $POQT$

∴ Circumcentres is mid-point of  $OT$

$$(h, k) \equiv \left( \frac{0+x_1}{2}, \frac{0+y_1}{2} \right)$$

$$(x_1, y_1) \equiv (2h, 2y)$$

$$\text{or } p(2h) + q(2k) = r$$



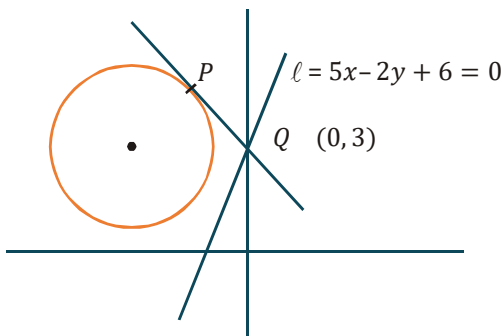
**Illustration 58:**

If the tangent at the point  $P$  on the circle  $x^2 + y^2 + 6x + 6y = 2$  meets the straight line  $5x - 2y + 6 = 0$  at a point  $Q$  on the  $y$ -axis, then the length of  $PQ$  is

- (A) 4                                      (B)  $2\sqrt{5}$                                       (C) 5                                      (D)  $3\sqrt{5}$

**Ans. (C)**

**Solution:**



$$S: (x + 3)^2 + (y + 3)^2 = 42$$

$$Q \equiv (0, 3)$$

$$L_{PQ} = \sqrt{S_1} = \sqrt{0+9+18-2} = \sqrt{25} = 5$$

**Illustration 59:**

If  $a > 2b > 0$  then the positive value of  $m$  for which  $y = mx - b\sqrt{1 + m^2}$  is a common tangent to  $x^2 + y^2 = b^2$  and  $(x - a)^2 + y^2 = b^2$  is

- (A)  $\frac{2b}{\sqrt{a^2 - 4b^2}}$                                       (B)  $\frac{\sqrt{a^2 - 4b^2}}{2b}$                                       (C)  $\frac{2b}{a - 2b}$                                       (D)  $\frac{b}{a - 2b}$

**Ans. (A)**

**Solution:**

$$S_1: x^2 + y^2 = b^2$$

$$S_2: (x - a)^2 + y^2 = b^2$$

$$T: y = mx - b\sqrt{1 + m^2} \Rightarrow mx - y - b\sqrt{1 + m^2} = 0$$

∴ ⊥ distance from  $C_1$  &  $C_2 = b$

$$\frac{|am - b\sqrt{1 + m^2}|}{\sqrt{1 + m^2}} = b$$

$$\Rightarrow am - b\sqrt{1 + m^2} = b\sqrt{1 + m^2}$$

$$\Rightarrow am = 2b\sqrt{1 + m^2}$$

$$\Rightarrow a^2 m^2 = 4b^2 (1 + m^2)$$

$$\Rightarrow m^2(a^2 - 4b^2) = 4b^2$$

$$\Rightarrow m^2 = \frac{4b^2}{a^2 - 4b^2} \Rightarrow m = \frac{2b}{\sqrt{a^2 - 4b^2}}$$

**Illustration 60:**

The radius of the circle, having centre at (2, 1), whose one of the chord is a diameter of the circle  $x^2 + y^2 - 2x - 6y + 6 = 0$

- (A) 1 (B) 2 (C) 3 (D)  $\sqrt{3}$

**Ans. (C)**

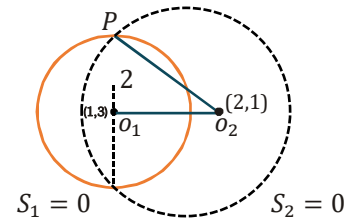
**Solution:**

$$S_1 : (x - 1)^2 + (y - 3)^2 = 2^2$$

$$O_1O_2 = \sqrt{(2-1)^2 + (1-3)^2} = \sqrt{1+4} = \sqrt{5}$$

$$PO_1 = 2$$

$$\therefore r = PO_2 = \sqrt{2^2 + (\sqrt{5})^2} = \sqrt{4+5} = 3$$



**Illustration 61:**

Line  $2x + 3y + 1 = 0$  is a tangent to a circle at (1, -1). This circle is orthogonal to a circle which is drawn having diameter as a line segment with end points (0, -1) and (-2, 3). Find equation of circle.

**Solution:**

Family of circle whose tangent at (1, -1) is

$$2x + 3y - 1 = 0$$

$$(x - 1)^2 + (y + 1)^2 + \lambda(2x + 3y - 1) = 0$$

$$S : x^2 - 2x(1 - \lambda) + y^2 + 2y(1 + 3/2) + 2 - \lambda = 0$$

$$S_1 : (x - 0)(x + 2) + (y + 1)(y - 3) = 0$$

$$x^2 + y^2 + 2x - 2y - 3 = 0$$

Apply condition

$$2g_1g_2 + 2f_1f_2 = C_1C_2 \text{ to get } \lambda$$

**Illustration 62:**

Let ABCD be a quadrilateral with area 18, with side AB parallel to the side CD and  $AB = 2CD$ . Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral ABCD touching all the sides, then its radius is

- (A) 3 (B) 2 (C) 3/2 (D) 1

**Ans. (B)**

**Solution:**

$$a + b = \frac{1}{2}(a + c)$$

$$a + 2b = c \quad \dots(i)$$

Area of quad.

$$\frac{1}{2} \times 2a[a + b + a + c] = 18$$

$$\Rightarrow a(a + b + a + a + 2b) = 18$$

$$\Rightarrow 3a(a + b) = 18$$

$$\Rightarrow a(a + b) = 6 \quad \dots(ii)$$

$$BM^2 + CM^2 = BC^2$$

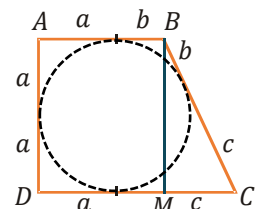
$$\Rightarrow (2a)^2 + (c - b)^2 = (b + c)^2$$

$$\Rightarrow 4a^2 + c^2 + b^2 - 2bc = b^2 + c^2 + 2bc$$

$$\Rightarrow a^2 = bc \quad \dots(iii)$$

by solving (i), (ii) & (iii)  $a = 2, b = 1, c = 4$

$\therefore$  radius of circle is  $r = a = 2$



**Illustration 63:**

Consider the two curves  $C_1: y^2 = 4x$ ;  $C_2: x^2 + y^2 - 6x + 1 = 0$ . Then,

- (A)  $C_1$  and  $C_2$  touch each other only at one point
- (B)  $C_1$  and  $C_2$  touch each other exactly at two points
- (C)  $C_1$  and  $C_2$  intersect (but do not touch) at exactly two points
- (D)  $C_1$  and  $C_2$  neither intersect nor touch each other

**Ans. (B)**

**Solution:**

$$C_1: y^2 = 4x$$

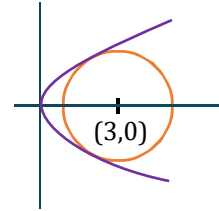
$$C_2: x^2 + y^2 - 6x + 1 = 0$$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)^2 = 0 \Rightarrow x = 1$$

$$y = \pm 2$$

So, the curves touches each other at two points  $(1, 2)$  &  $(1, -2)$



**Comprehension (Illustration 64 & 65) :**

A circle  $C$  of radius 1 is inscribed in an equilateral triangle  $PQR$ . The points of contact of  $C$  with the sides  $PQ, QR, RP$  are  $D, E, F$  respectively. The line  $PQ$  is given by the equation  $\sqrt{3}x + y - 6 = 0$  and the point  $D$

is  $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ . Further, it is given that the origin and the centre of  $C$  are on the same side of the line  $PQ$ .

**Illustration 64:**

The equation of circle  $C$  is

(A)  $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$

(B)  $(x - 2\sqrt{3})^2 + \left(y + \frac{1}{2}\right)^2 = 1$

(C)  $(x - \sqrt{3})^2 + (y + 1)^2 = 1$

(D)  $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

**Ans. (D)**

**Solution:**

$$m_{PQ} = -\sqrt{3}$$

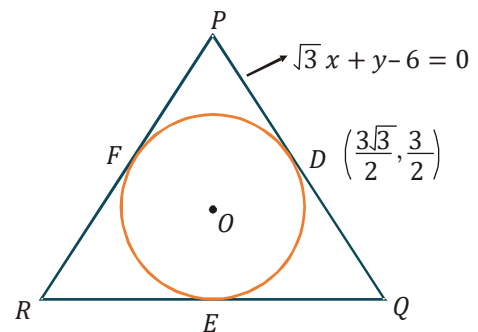
so, slope of  $OD = \frac{1}{\sqrt{3}}$

$$\tan \theta = \frac{1}{\sqrt{3}}$$

$$\therefore \frac{x - \frac{3\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = \frac{y - \frac{3}{2}}{\frac{1}{2}} = \pm 1$$

$(2\sqrt{3}, 2)$  (not possible) &  $(\sqrt{3}, 1)$

hence circle is  $(x - \sqrt{3})^2 + (y - 1)^2 = 1$



**Illustration 65:**

Points  $E$  and  $F$  are given by

(A)  $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$

(B)  $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3}, 0)$

(C)  $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

(D)  $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

**Ans. (A)**

**Solution:**

For point  $E$   $\frac{x-\sqrt{3}}{\frac{\sqrt{3}}{2}} = \frac{y-1}{\frac{1}{2}} = 1$   $\left[ \therefore E\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right) \right]$

For point  $F$   $\frac{x-\sqrt{3}}{0} = \frac{y-1}{-1} = 1$   $\left[ \therefore F(\sqrt{3}, 0) \right]$

**Illustration 66:**

Equations of the sides  $RP, RQ$  are

(A)  $y = \frac{2}{\sqrt{3}}x + 1, y = -\frac{2}{\sqrt{3}}x - 1$

(B)  $y = \frac{1}{\sqrt{3}}x, y = 0$

(C)  $y = \frac{\sqrt{3}}{2}x + 1, y = -\frac{\sqrt{3}}{2}x - 1$

(D)  $y = \sqrt{3}x, y = 0$

**Ans. (D)**

**Solution:**

Equation of line  $RP$   $y = 0$

Equation of line  $QR$   $y - \frac{3}{2} = \sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right) \Rightarrow y = \sqrt{3}x$

**Illustration 67:**

Tangents drawn from the point  $P(1, 8)$  to the circle  $x^2 + y^2 - 6x - 4y - 11 = 0$  touch the circle at the points  $A$  and  $B$ . The equation of the circumcircle of the triangle  $PAB$  is

(A)  $x^2 + y^2 + 4x - 6y + 19 = 0$

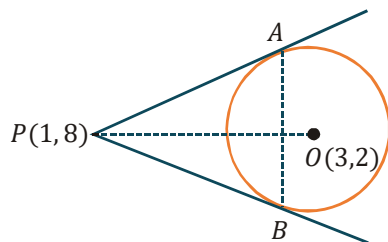
(B)  $x^2 + y^2 - 4x - 10y + 19 = 0$

(C)  $x^2 + y^2 - 2x + 6y - 29 = 0$

(D)  $x^2 + y^2 - 6x - 4y + 19 = 0$

**Ans. (B)**

**Solution:**



The required circle is a circle described on  $OP$  as diameter.

**Illustration 68:**

The centres of two circles  $C_1$  and  $C_2$  each of unit radius are at a distance of 6 units from each other. Let  $P$  be the mid point of the line segment joining the centres of  $C_1$  and  $C_2$  and  $C$  be a circle touching circles  $C_1$  and  $C_2$  externally. If a common tangent to  $C_1$  and  $C$  passing through  $P$  is also a common tangent to  $C_2$  and  $C$ , then the radius of the circle  $C$  is

**Solution:**

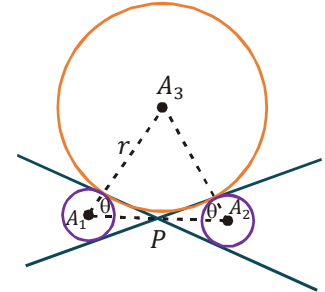
In triangle  $A_1A_2A_3$

$$A_1A_3 = A_3A_2$$

$$\text{Let angle } A_3A_1A_2 = \theta, \cos \theta = \frac{1}{3}, \sin \theta = \frac{2\sqrt{2}}{3}$$

Apply sine rule in triangle  $A_1A_2A_3$

$$\frac{6}{\sin(\pi-2\theta)} = \frac{r+1}{\sin \theta} \Rightarrow r = 8$$



**Illustration 69:**

Two parallel chords of a circle of radius 2 are at a distance  $\sqrt{3} + 1$  apart. If the chords subtend at the center, angles of  $\frac{\pi}{k}$  and  $\frac{2\pi}{k}$ , where  $k > 0$ , then the value of  $[k]$  is

[Note :  $[k]$  denotes the largest integer less than or equal to  $k$ ]

**Solution:**

$$OA = 2 \cos \frac{\pi}{k}$$

$$OB = 2 \cos \frac{\pi}{2k}$$

$$2 \cos \frac{\pi}{k} + 2 \cos \frac{\pi}{2k} = \sqrt{3} + 1$$

$$\Rightarrow 2 \cos^2 \frac{\pi}{2k} - 1 + \cos \frac{\pi}{2k} = \frac{\sqrt{3} + 1}{2}$$

$$\text{Let } \cos \frac{\pi}{2k} = t$$

$$2t^2 + t - 1 - \frac{\sqrt{3} + 1}{2} = 0$$

$$\Rightarrow 4t^2 + 2t - (3 + \sqrt{3}) = 0 \Rightarrow t = \frac{\sqrt{3}}{2}, -\frac{1 + \sqrt{3}}{2}$$

$$t = -\frac{1 + \sqrt{3}}{2} \text{ (not possible)}$$

$$t = \frac{\sqrt{3}}{2} = \cos 30^\circ = \cos \frac{\pi}{6} \Rightarrow \cos \frac{\pi}{2k} = \cos \frac{\pi}{6}$$

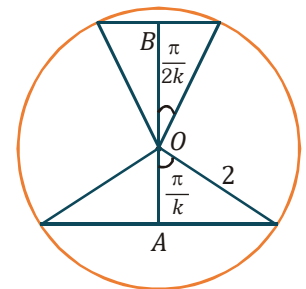
$$k = 3$$

**Illustration 70:**

The circle passing through the point  $(-1, 0)$  and touching the  $y$ -axis at  $(0, 2)$  also passes through the point -

- (A)  $\left(-\frac{3}{2}, 0\right)$       (B)  $\left(-\frac{5}{2}, 2\right)$       (C)  $\left(-\frac{3}{2}, \frac{5}{2}\right)$       (D)  $(-4, 0)$

Ans. (D)



**Solution:**

Family of circle which touches  $y$  – axis at  $(0,2)$  is

$$x^2 + (y-2)^2 + \lambda x = 0$$

Passing through  $(-1,0)$

$$\Rightarrow 1 + 4 - \lambda = 0 \Rightarrow \lambda = 5$$

$$\therefore x^2 + y^2 + 5x - 4y + 4 = 0$$

which satisfy the point  $(-4, 0)$ .

**Illustration 71:**

The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line  $4x-5y = 20$  to the circle  $x^2 + y^2 = 9$  is-

- (A)  $20(x^2 + y^2) - 36x + 45y = 0$                       (B)  $20(x^2 + y^2) + 36x - 45y = 0$   
 (C)  $36(x^2 + y^2) - 20x + 45y = 0$                       (D)  $36(x^2 + y^2) + 20x - 45y = 0$

**Ans. (A)**

**Solution:**

Let mid-point be  $(h, k)$ , then chord of contact:

$$hx + ky = h^2 + k^2 \quad \dots(i)$$

Let any point on the line  $4x-5y = 20$  be  $\left(x_1, \frac{4x_1-20}{5}\right)$

$\therefore$  Chord of contact :

$$5x_1x + (4x_1 - 20)y = 45 \quad \dots(ii)$$

(i) and (ii) are same

$$\therefore \frac{5x_1}{h} = \frac{4x_1 - 20}{k} = \frac{45}{h^2 + k^2}$$

$$\Rightarrow x_1 = \frac{9h}{h^2 + k^2} \text{ and } x_1 = \frac{45k + 20(h^2 + k^2)}{4(h^2 + k^2)}$$

$$\Rightarrow \frac{9h}{h^2 + k^2} = \frac{45k + 20(h^2 + k^2)}{4(h^2 + k^2)}$$

$$\Rightarrow 20(h^2 + k^2) - 36h + 45k = 0$$

$$\therefore \text{Locus is } 20(x^2 + y^2) - 36x + 45y = 0$$

**Paragraph (for Illustration 72 & 73)**

A tangent  $PT$  is drawn to the circle  $x^2 + y^2 = 4$  at the point  $P(\sqrt{3}, 1)$ . A straight line  $L$ , perpendicular to  $PT$  is a tangent to the circle  $(x - 3)^2 + y^2 = 1$ .

**Illustration 72:**

A common tangent of the two circles is

- (A)  $x = 4$                       (B)  $y = 2$                       (C)  $x + \sqrt{3}y = 4$                       (D)  $x + 2\sqrt{2}y = 6$

**Ans. (D)**

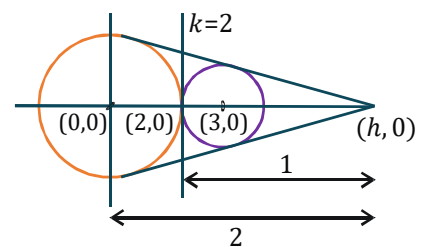
**Solution:**

$$h = \frac{2 \times 3 - 1 \times 0}{2 - 1} = 6$$

equation of tangents from  $(6, 0)$ :

$$y - 0 = m(x - 6) \Rightarrow y - mx + 6m = 0$$

use  $p = r$



$$\left| \frac{6m}{\sqrt{1+m^2}} \right| = 2 \Rightarrow 36m^2 = 4 + 4m^2$$

$$\Rightarrow 32m^2 = 4$$

$$\Rightarrow m^2 = 1/8 \Rightarrow m = \pm \frac{1}{2\sqrt{2}}$$

$$\text{at } m = -\frac{1}{2\sqrt{2}}$$

equation of tangent will be  $x + 2\sqrt{2}y = 6$

**Illustration 73:**

A possible equation of  $L$  is

(A)  $x - \sqrt{3}y = 1$       (B)  $x + \sqrt{3}y = 1$       (C)  $x - \sqrt{3}y = -1$       (D)  $x + \sqrt{3}y = 5$

**Ans. (A)**

**Solution:**

Equation of tangent at  $P$  will be  $\sqrt{3}x + y = 4$

Slope of line  $L$  will be  $\frac{1}{\sqrt{3}}$

Let equation of  $L$  be :  $y = \frac{x}{\sqrt{3}} + c$

$$\Rightarrow x - \sqrt{3}y + \sqrt{3}c = 0$$

Now this  $L$  is tangent to 2<sup>nd</sup> circle

$$\text{So } \frac{3 + \sqrt{3}c}{2} = \pm 1 \Rightarrow c = -\frac{1}{\sqrt{3}}$$

$$\text{or } c = -\frac{5}{\sqrt{3}}$$

$$\text{using } c = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow y = \frac{x}{\sqrt{3}} - \frac{1}{\sqrt{3}} \Rightarrow x - \sqrt{3}y = 1.$$