02 Binomial Theorem

Binomial Expression:

An algebraic expression consisting of two different terms is called a **Binomial Expression.**

e.g. (i) $x + y$ (ii) $x - y$ (iii) $3x - 2y$ (iv) $x^2 + \frac{1}{x^2}$

Note:

(i) $x + y + z$ is trinomial.

(ii) The expression containing more than two different terms is multinomial.

Binomial Theorem:

The formula by which any positive integral index (power) of a binomial expression can be expanded in the form of a series is known as **Binomial Theorem**. (This theorem was given by Newton)

Binomial Theorem

- $n \in N$ (JEE)
- Any index (not in JEE) (negative or fractional index) (C.B.S.E.)

Observations:

$(x+y)^0$	1	1	${}^0C_0$
$(x+y)^1$	$x + y$	1 1	${}^1C_0, {}^1C_1$
$(x+y)^2$	$x^2 + 2xy + y^2$	1 2 1	${}^2C_0, {}^2C_1, {}^2C_2$
$(x+y)^3$	$x^3 + 3x^2y + 3xy^2 + y^3$	1 3 3 1	${}^3C_0, {}^3C_1, {}^3C_2, {}^3C_3$
$(x+y)^4$	$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^3$	1 4 6 4 1	${}^4C_0, {}^4C_1, {}^4C_2, {}^4C_3, {}^4C_4$

Pascal's Triangle

(i) Pascal's triangle - A triangular arrangement of numbers as shown. The numbers give the binomial coefficients for the expansion of $(x + y)^n$. The first row is for $n = 0$, the second for $n = 1$, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

(ii) Pascal triangle is formed by binomial coefficient.

(iii) The number of terms in the expansion of $(x + y)^n$ is $(n + 1)$ i.e. one more than the index.

(iv) The sum of the indices of x & y in each term is n.

(v) Power of first variable (x) decreases while of second variable (y) increases.

(vi) Binomial coefficients are also called **combinatorial coefficients** ($\boldsymbol{{}^nC_r}$)

(vii) Binomial coefficients of the terms equidistant from the beginning and end are equal.

(viii) Binomial coefficients of middle term is greatest.

General Expansion of Binomial Theorem:

Statement: $n \in N$

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots\ldots + {}^nC_rx^{n-r}y^r + \ldots\ldots + {}^nC_{n-1}x^1y^{n-1} + {}^nC_ny^n$$

$$= \sum_{r=0}^{n} {}^nC_r\, x^{n-r}y^r.$$

OR

$$(\text{I} + \text{II})^n = \sum_{r=0}^{n} {}^nC_r\ (\text{I})^{n-r}(\text{II})^r$$

Proof: (Desirable)

$\because$ $(x+y_1)(x+y_2)\ldots\ldots(x+y_n) = x^n + (\sum y_1)x^{n-1} + (\sum y_1y_2)x^{n-2} + \ldots\ldots + (y_1y_2\ldots\ldots y_n)$

Put $y_1 = y_2 = \ldots\ldots = y_n = y$

and number of terms in $\sum y_1$ is ${}^nC_1$, number of terms in $\sum y_1y_2$ is ${}^nC_2$ and so on.

$\Rightarrow$ $(x+y)^n = x^n + {}^nC_1yx^{n-1} + {}^nC_2y^2x^{n-2} + \ldots\ldots + {}^nC_ny^n.$

General Term:

General term in the expansion of $(I+II)^n$ is $T_{r+1} = {}^nC_r(I)^{n-r}(II)^r, 0 \le r \le n$

Illustration 1:

Expand $\left(x+\frac{1}{x}\right)^6$ and write general term:

Solution:

$$\left(x+\frac{1}{x}\right)^6 = \sum_{r=0}^{6} {}^6C_r\, x^{6-r}\left(\frac{1}{x}\right)^r$$

$$= {}^6C_0(x)^6\left(\frac{1}{x}\right)^0 + {}^6C_1(x)^5\left(\frac{1}{x}\right)^1 + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2 + {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)^1\left(\frac{1}{x}\right)^5 + {}^6C_6(x)^0\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Illustration 2:

Expand the following binomials:

(i) $(x-3)^5$ (ii) $\left(1-\frac{3x^2}{2}\right)^4$

Solution:

(i) $(x-3)^5 = {}^5C_0x^5 + {}^5C_1x^4(-3)^1 + {}^5C_2x^3(-3)^2 + {}^5C_3x^2(-3)^3 + {}^5C_4x(-3)^4 + {}^5C_5(-3)^5$

$= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

(ii) $\left(1-\frac{3x^2}{2}\right)^4 = {}^4C_0 + {}^4C_1\left(-\frac{3x^2}{2}\right) + {}^4C_2\left(-\frac{3x^2}{2}\right)^2 + {}^4C_3\left(-\frac{3x^2}{2}\right)^3 + {}^4C_4\left(-\frac{3x^2}{2}\right)^4$

$= 1 - 6x^2 + \frac{27}{2}x^4 - \frac{27}{2}x^6 + \frac{81}{16}x^8$

Illustration 3:

Expand the binomial $\left(\frac{2x}{3}+\frac{3y}{2}\right)^{20}$ up to four terms

Solution:

Expanding upto four terms in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^{20}$

$$= {}^{20}C_0\left(\frac{2x}{3}\right)^{20} + {}^{20}C_1\left(\frac{2x}{3}\right)^{19}\left(\frac{3y}{2}\right) + {}^{20}C_2\left(\frac{2x}{3}\right)^{18}\left(\frac{3y}{2}\right)^{2} + {}^{20}C_3\left(\frac{2x}{3}\right)^{17}\left(\frac{3y}{2}\right)^{3}$$

$$=\left(\frac{2x}{3}\right)^{20} + 20.\left(\frac{2}{3}\right)^{18} x^{19}y + 190.\left(\frac{2}{3}\right)^{16} x^{18}y^2 + 1140.\left(\frac{2}{3}\right)^{14} x^{17}y^3$$

Illustration 4:

The number of dissimilar terms in the expansion of $(1-3x+3x^2-x^3)^{20}$ is

(A) 21 (B) 31 (C) 41 (D) 61

Ans. (D)

Solution:

$(1-3x+3x^2-x^3)^{20} = [(1-x)^3]^{20} = (1-x)^{60}$

Therefore, number of dissimilar terms in the expansion of $(1-3x+3x^2-x^3)^{20}$ is 61.

Illustration 5:

$$\sum_{r=0}^{10} 10C_r(2)^{10-r}(3)^r = ?$$

(A) 5^{10} (B) $(-1)^{10}$ (C) 6^{10} (D) $\left(\frac{2}{3}\right)^{10}$

Ans. (A)

Solution:

$[2+3]^{10} = (5)^{10}$

Illustration 6:

$$\sum_{r=0}^{20} 20C_r(2)^{21-r}(3)^r = ?$$

(A) $3(5)^{20}$ (B) $2(5)^{20}$ (C) $2(-1)^{20}$ (D) None of these

Ans. (B)

Solution:

$$\sum_{r=0}^{20} 20C_r(2)^{21-r}(3)^r$$

$$= 2^1\sum_{r=0}^{20}[20C_r(2)^{20-r}(3)^r]$$

$= 2[2+3]^{20} = 2(5)^{20}$

Illustration 7:

If $6^{th}, 7^{th}, 8^{th}$ terms in the expansion of $(x+y)^n$ are respectively 112, 7 and $\frac{1}{4}$. Find x, y, n.

Solution:

$^{n}C_5\, x^{n-5} y^5 = 112$ ...(1)

$^{n}C_6 x^{n-6}\, y^6 = 7$ ...(2)

$^{n}C_7 x^{n-7} y^7 = \dfrac{1}{4}$ ...(3)

Divide (1) by (2) and (2) by (3)

$\Rightarrow \dfrac{^{n}C_5}{^{n}C_6}\left(\dfrac{x}{y}\right) = 16$ ...(4)

$\Rightarrow \dfrac{^{n}C_6}{^{n}C_7}\left(\dfrac{x}{y}\right) = 28$ ...(5)

Divide (4) by (5)

$$\frac{^{n}C_5 \cdot {}^{n}C_7}{^{n}C_6 \cdot {}^{n}C_6} = \frac{16}{28} = \frac{4}{7}$$

$$\Rightarrow \frac{\dfrac{n!}{(n-5)!5!} \cdot \dfrac{n!}{(n-7)!7!}}{\left(\dfrac{n!}{(n-6)!6!}\right)^2} = \frac{4}{7}$$

$$\Rightarrow \frac{(n-6)\cdot 6}{(n-5)\cdot 7} = \frac{4}{7}$$

$\Rightarrow n = 8$

Put in (5)

$$\frac{^{8}C_6}{^{8}C_7}\left(\frac{x}{y}\right) = 28$$

$$\Rightarrow \frac{\dfrac{8\times 7}{2!}}{8}\left(\frac{x}{y}\right) = 28$$

$\Rightarrow \dfrac{x}{y} = 8 \Rightarrow x = 8y$

Put in (3) $\Rightarrow {}^{8}C_7 \cdot (8y)^1 \cdot y^7 = \dfrac{1}{4}$

$\Rightarrow y^8 = \dfrac{1}{2^8}$

$\Rightarrow y = \dfrac{1}{2}$ and $x = 4$

Note : $\boxed{\sum_{r=0}^{n} {}^{n}C_r (a)^{n-r} (b)^r = (a+b)^n}$

Number of Terms in Expansion:

$$(x+a)^n = {}^{n}C_0 x^n + {}^{n}C_1 x^{n-1} a^1 + {}^{n}C_2 x^{n-2} a^2 + \ldots\ldots + {}^{n}C_n a^n$$

$$(x-a)^n = {}^{n}C_0 x^n - {}^{n}C_1 x^{n-1} a^1 + {}^{n}C_2 x^{n-2} a^2 - \ldots\ldots + {}^{n}C_n (-a)^n$$

Total Number of Terms in:

(a) $(x+a)^n$ is $n+1$

e.g. $(1+x)^{10}$: Total terms = 11

(b) $(x+a)^n + (x-a)^n$ –
- n is even → $\frac{n}{2}+1$
- n is odd → $\frac{n+1}{2}$

e.g. (i) $(x+2)^{10} + (x-2)^{10}$

Total term $= \frac{10}{2} + 1 = 6$

(ii) $(x+2)^{11} + (x-2)^{11}$

Total terms : $\frac{11+1}{2} = 6$

(c) $(x+a)^n - (x-a)^n$ –
- n is even → $\frac{n}{2}$
- n is odd → $\frac{n+1}{2}$

e.g. (i) $(x+2)^{10} - (x-2)^{10}$

Total terms : $\frac{10}{2} = 5$

(ii) $(x+2)^{11} - (x-2)^{11}$

Total terms : $\frac{11+1}{2} = 6$

(d) Number of terms in the expansion of $(x+y+z)^n$:

$(x+y+z)^n$ can be expanded as -

$(x+y+z)^n = (x+y)+z\}^n$

$= {}^nC_0\,(x+y)^n + {}^nC_1\,(x+y)^{n-1}.\,z + {}^nC_2\,(x+y)^{n-2}\,z^2 + ... + {}^nC_n\,z^n$

$= (n+1)$ terms $+\, n$ terms $+\, (n-1)$ terms + ... + 1 term

Total number of terms $= (n+1) + n + (n-1) + \cdots + 1$

$= \boxed{\frac{(n+1)(n+2)}{2}}$

Illustration 8:

The number of terms in the expansion of $(x+2y+z)^8$ are -

(A) 9 (B) 28 (C) 45 (D) None of these

Ans. (C)

Solution:

Since given expression is trinomial

$\therefore$ Number of terms $= \frac{(8+1)(8+2)}{2} = 45$

Illustration 9:

Find the number of terms in expansion of $(1+4x+6x^2+4x^3+x^4)^{10}$

(A) 40 (B) 41 (C) 42 (D) 43

Ans. (B)

Solution:

$[(1+x)^4]^{10} = (1+x)^{40}$

$\therefore$ number of terms = 41

Illustration 10:

Find sum of rational terms in the expansion of $(3^{\frac{1}{3}} + 5^{\frac{1}{2}})^9$

(A) 31527 (B) 26717 (C) 16517 (D) 40571

Ans. (A)

Solution:

$T_{(r+1)} = {}^9C_r\,(3^{\frac{1}{3}})^{9-r}\,(5^{\frac{1}{2}})^r$

$= {}^9C_r(3)^{3-\frac{r}{3}}(5)^{\frac{r}{2}}$

$\frac{r}{3} \in w \rightarrow r = 0, 3, 6, 9$

$\frac{r}{2} \in w \rightarrow r = 0, 2, 4, 6, 8$

$\therefore$ Common $'r' = 0,6$

$\therefore$ Rational terms are T_1 and T_7

Sum = $T_1 + T_7$

$= {}^9C_0(3^{\frac{1}{3}})^9(5^{\frac{1}{2}})^0 + {}^9C_6(3^{\frac{1}{3}})^3(5^{\frac{1}{2}})^6$

$= (3)^3 + \frac{9\times8\times7}{3\times2}\times3\times(5)^3$

$= 27 + (84 \times 375)$

$= 27 + 31500$

$= 31527$

Middle Term:

Middle term in the expansion of $(x+y)^n$ is $\begin{cases} T_{\frac{n}{2}+1} = T_{\frac{n+2}{2}} & \text{when n is even} \\ T_{\frac{n+1}{2}} \;\&\; T_{\frac{n+3}{2}} & \text{when n is odd} \end{cases}$

Explanation:

If n is even then number of terms are odd and, in this case, only one middle term

$\Rightarrow$ When $n = 2m$ then number of terms = $2m+1 \Rightarrow T_{(m+1)}$ is middle term

similarly, if n is odd then number of terms are even and in this case 2 middle terms.

$\Rightarrow n = 2m+1$ then number of terms $= 2m+2 \Rightarrow T_{(m+1)} \;\&\; T_{(m+2)}$ are 2 middle terms.

Illustration 11:

Find the middle term(s) in the expansion of

(i) $\left(1-\frac{x^2}{2}\right)^{14}$ (ii) $\left(3a-\frac{a^3}{6}\right)^9$

Solution:

(i) $\left(1-\frac{x^2}{2}\right)^{14}$

Here, n is even, therefore middle term is $\left(\frac{14+2}{2}\right)^{th}$ term.

It means T_8 is middle term, $T_8 = {}^{14}C_7\left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16}x^{14}$.

(ii) $\left(3a-\frac{a^3}{6}\right)^9$

Here, n is odd therefore, middle terms are $\left(\frac{9+1}{2}\right)^{th}$ & $\left(\frac{9+1}{2}+1\right)^{th}$.

It means T_5 & T_6 are middle terms

$T_5 = {}^9C_4\,(3a)^{9-4}\left(-\frac{a^3}{6}\right)^4 = \frac{189}{8}\,a^{17},\;\; T_6 = {}^9C_5\,(3a)^{9-5}\left(-\frac{a^3}{6}\right)^5 = -\frac{21}{16}\,a^{19}.$

Illustration 12:

Find the middle term in the expansion of $\left(2x^2-\frac{1}{x}\right)^7$.

Solution:

$n = 7$

$\therefore$ middle terms = $T_{\frac{7+1}{2}}$ & $T_{\frac{7+3}{2}}$

$T_4 = {}^7C_3(16)x^5, T_5 = {}^7C_4 8x^2$

Properties of Binomial Coefficient:

Property-1 ${}^nC_r \times r! = {}^nP_r$

Property-2 ${}^nC_r = {}^nC_{n-r}$

if ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

Property-3 $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-(r-1)}{r}$

Property-4 ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

Property-5 ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$

Property-6 $\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$

Proof : $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \cdots + {}^nC_nx^n = \sum_{r=0}^{n} {}^nC_r x^r\,; n \in N$ ...(i)

where ${}^nC_0, {}^nC_1, {}^nC_2, \ldots C_n$ are called combinatorial (binomial) coefficients.

Property-1 ${}^nC_r = \frac{n!}{r!(n-r)!}$

So, ${}^nC_r \times r! = \frac{n!}{r!(n-r)!} \times r! = \frac{n!}{(n-r)!} = {}^nP_r$

Property-2 $\quad {}^nC_r = \frac{n!}{r!(n-r)!}$

$${}^nC_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!(n-n+r))!} = \frac{n!}{(n-r)!\,r!} \Rightarrow {}^nC_r = {}^nC_{n-r}$$

Property-3 $\quad \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{n-(r-1)}{r}$

Property-4 $\quad {}^nC_r + {}^nC_{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-(r+1))!} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1).r!.(n-r-1)!}$

$$= \frac{n!}{r!\,(n-r).(n-r-1)!} + \frac{n!}{(r+1).\,r!.\,(n-r-1)!}$$

$$= \frac{n!}{r!\,(n-r-1)!}\left[\frac{1}{n-r} + \frac{1}{r+1}\right] = \frac{n!}{r!\,(n-r-1)!}\left[\frac{r+1+n-r}{(n-r).(r+1)}\right]$$

$$= \frac{(n+1)!}{(r+1)!\,(n-r)!}$$

$$= {}^{n+1}C_{r+1}$$

Property-5 & 6 $\quad {}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}.\frac{n-1}{r-1}\,{}^{n-2}C_{r-2} = ... = \frac{n(n-1)(n-2).......(n-r+1)}{r(r-1)(r-2)..........1}$

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

Properties of Binomial Coefficient:

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \cdots \ldots \ldots + {}^nC_nx^n = \sum_{r=0}^{n} {}^nC_rx^r \,;\, n \in N \qquad ...(i)$$

where ${}^nC_0, {}^nC_1, {}^nC_2, \ldots {}^nC_n$ are called combinatorial (binomial) coefficients.

(a) The sum of all the binomial coefficients is 2^n.

Put $x = 1$, in (i) we get

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots \ldots \ldots \ldots . + {}^nC_n = 2^n \Rightarrow \sum_{r=0}^{n} {}^nC_r = 2^n \qquad ...(ii)$$

(b) Put $x = -1$ in (i) we get

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 \ldots \ldots \ldots . + (-1)^n \cdot {}^nC_n = 0 \Rightarrow \sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0 \qquad ...(iii)$$

(c) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots = 2^{n-1}$$

Proof :

Add (ii) & (iii), we get

${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = 2^{n-1}$

Proof :

Subtract (iii) from (ii), we get

${}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$

Summation of Series:

Type-1:

If ${}^nC_0, {}^nC_1, {}^nC_2, ... {}^nC_n$ are binomial coefficients in the expansion $(1+x)^n$ then -

Illustration 13:

If $T_r = {}^{10}C_r(2^r - 1)$, then find $\sum_{r=0}^{10} T_r = ?$

Solution:

$$\sum_{r=0}^{10} T_r = \sum_{r=0}^{10}\left[{}^{10}C_r(2^r - 1)\right]$$

$$= \sum_{r=0}^{10}\left[{}^{10}C_r(2)^r\right] - \sum_{r=0}^{10}\left[{}^{10}C_r\right] = (1+2)^{10} - (2)^{10} = 3^{10} - 2^{10}$$

Illustration 14:

Evaluate : $\sum_{r=0}^{11} {}^{23}C_r$

Solution:

$= {}^{23}C_0 + {}^{23}C_1 + ... + {}^{23}C_{11}$. Total 12 terms

In $(1+x)^{23}$ put $x = 1$

$$2^{23} = {}^{23}C_0 + {}^{23}C_1 + ... + {}^{23}C_{11} + {}^{23}C_{12} + {}^{23}C_{13} + ... + {}^{23}C_{23}$$

$$\Rightarrow \quad 2^{23} = ({}^{23}C_0 + {}^{23}C_1 + ... + {}^{23}C_{11}) + ({}^{23}C_{11} + {}^{23}C_{10} + ... + {}^{23}C_0)$$

$$\Rightarrow \quad 2^{23} = 2[{}^{23}C_0 + {}^{23}C_1 + ... + {}^{23}C_{11}]$$

$$\Rightarrow \quad \sum_{r=0}^{11} {}^{23}C_r = \frac{2^{23}}{2} = 2^{22}$$

Illustration 15:

$$\frac{{}^nC_1}{{}^nC_0} + 2.\frac{{}^nC_2}{{}^nC_1} + 3.\frac{{}^nC_3}{{}^nC_2} ...... + n.\frac{{}^nC_n}{{}^nC_{n-1}} = \frac{n(n+1)}{2}$$

Solution:

$$\sum_{r=1}^{n} r.\frac{{}^nC_r}{{}^nC_{r-1}} = \sum_{r=1}^{n} r\cdot\frac{n-r+1}{r}$$

$$= \sum_{r=1}^{n}(n-r+1) = n + n - 1 + \;....+ 2 + 1$$

$$= \frac{n(n+1)}{2}$$

Illustration 16:

$${}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + ... + {}^nC_r = ?$$

Solution:

$${}^{r+1}C_{r+1} + {}^{r+1}C_r + {}^{r+2}C_r + ... + {}^nC_r$$

$$\Downarrow$$

$${}^{r+2}C_{r+1} + {}^{r+2}C_r + ... + {}^nC_r \qquad \left[{}^rC_r = {}^{r+1}C_{r+1} = 1\right]$$

$$\Downarrow$$

$${}^{r+3}C_{r+1} .... + {}^nC_r$$

$${}^{n+1}C_{r+1} = {}^{n+1}C_{n-r}$$

Type-2:

Find the sum of the following series:

Illustration 17:

1. ${}^nC_1 + 2.{}^nC_2 + 3.{}^nC_3 + \cdots \ldots \ldots + n.\ {}^nC_n = n.2^{(n-1)}$

Solution:

Method-I : (Federal Carl Gauss Method)

$S = 0.\ {}^nC_0 + \quad 1.\ {}^nC_1 + \quad 2.{}^nC_2 \quad + \ldots\ldots\ldots + n.\ {}^nC_n$

$S = n.\ {}^nC_0 + (n-1).\ {}^nC_1 + (n-2).\ {}^nC_2 + \ldots\ldots\ldots + 0.{}^nC_n$

Add $2S = n(C_0 + C_1 + C_2 + \ldots\ldots\ldots\ldots C_n)$

$2S = n(C_0 + C_1 + C_2 + \ldots\ldots\ldots\ldots C_n)$

$= n.2^n \Rightarrow S = n.2^{n-1}$

Method-II :

$$\sum T_r = \sum_{r=1}^{n} r.{}^nC_r = \sum \frac{r.n!}{r!(n-r)!} = \sum \frac{n.(n-1)!}{(r-1)!(n-r)!} = n\sum_{r=1}^{n} {}^{n-1}C_{r-1}$$

$= n[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots\ldots {}^{n-1}C_{n-1}] = n.2^{n-1}$

or $\quad \sum_{r=1}^{n} r\left(\frac{n}{r}.{}^{n-1}C_{r-1}\right) = n.2^{n-1}$

Method-III (Calculus Method):

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots\ldots + {}^nC_nx^n$

Differentiate both the sides, we get

$n(1+x)^{n-1} = 0 + {}^nC_1 + 2.\ {}^nC_2x + 3.\ {}^nC_3x^2 + \ldots\ldots\ldots + n.\ {}^nC_n\, x^{n-1}$

Put $x = 1$

$n.2^{n-1} = {}^nC_1 + 2.\ {}^nC_2 + 3.\ {}^nC_3 + \ldots\ldots\ldots\ldots + n.\ {}^nC_n$

Illustration 18:

${}^nC_0 + 3\,{}^nC_1 + 5\,{}^nC_2 + \ldots\ldots\ldots + (2n+1)\,{}^nC_n$

Solution:

$S = {}^nC_0 + 3\ {}^nC_1 + 5\ {}^nC_2 + \ldots\ldots\ldots\ldots\ldots + (2n+1)\,{}^nC_n$

$S = (2n+1)\,{}^nC_0 + (2n-1)\,{}^nC_1 + (2n-3)\,{}^nC_2 + \ldots\ldots\ldots + 1\,{}^nC_n$

$2S = (2n+2)(C_0 + C_1 + C_2 + \ldots\ldots\ldots + C_n)$

$\Rightarrow S = (2n+2)2^{n-1} = (n+1)2^n$

OR

$x(1+x^2)^n = C_0x + C_1x^2.x + C_2x^4.x + \ldots\ldots\ldots + C_nx^{2n}.x$

$x(1+x^2)^n = C_0x + C_1x^2.x + C_2x^4.x + \ldots\ldots\ldots + C_nx^{2n}.x$

$x(1+x^2)^n = C_0x + C_1x^3 + C_2x^5 + \ldots\ldots\ldots + C_nx^{2n+1}$

Differentiate it w.r.t. x :

$(1+x^2)^n + x.n(1+x^2)^{n-1}.2x = C_0 + 3\,C_1x^2 + 5C_2x^4 \ldots\ldots$

Put $x = 1$

${}^nC_0 + 3\ {}^nC_1 + 5\,{}^nC_2 + \ldots\ldots\ldots + (2n+1)\,{}^nC_n = (n+1)2^n$

Short Trick :

(1) If $(a)\,{}^nC_0 + (a+d)\,{}^nC_1 + (a+2d)\,{}^nC_2 + \ldots\ldots + [a+nd]\,{}^nC_n$

= [Ist coefficient + last coefficient]2^{n-1}

= $[a + (a+nd)]2^{n-1}$

Coefficient of binomial coefficient are in A.P.

All the sign is +ve

${}^nC_0$ to ${}^nC_n$

(2) If $(a)\,{}^nC_0 - (a+d)\,{}^nC_1 + (a+2d)\,{}^nC_2 \ldots\ldots + (-1)^n(a+nd)\,{}^nC_n = 0$

Coefficient of binomial coefficient are in A.P.

Alternate sign $(+,-,+,-)$

${}^nC_0$ to ${}^nC_n$

Illustration 19:

$1.\,{}^nC_1 + 2.\,{}^nC_2 + 3.\,{}^nC_3 + \ldots\ldots + n\,{}^nC_n.$

Solution:

$0.\,{}^nC_0 + 1.\,{}^nC_1 + 2.\,{}^nC_2 + 3.\,{}^nC_3 + \ldots\ldots + n\,{}^nC_n$

$= [0+n].2^{n-1} = n(2)^{n-1}$

Illustration 20:

$1.{}^nC_0 + 3{}^nC_1 + 5nC_2 + \ldots\ldots\ldots + (2n+1)nC_n$

Solution:

$= [1 + (2n+1)].2^{n-1}$

$= (n+1).2^n$

Illustration 21:

$(1)^2.C_1 + (2)^2.C_2 + (3)^2.C_3 + \ldots\ldots + (n)^2.C_n = ?$

Solution:

$$\sum_{r=1}^{n}(r)^2\left({}^nC_r\right) = \sum_{r=1}^{n}\left[r^2\cdot\left(\frac{n}{r}\right)\left\{{}^{n-1}C_{r-1}\right\}\right] = n\sum_{r=1}^{n}\left[r\left\{{}^{n-1}C_{r-1}\right\}\right]$$

$= n[1.\,{}^{n-1}C_0 + 2.\,{}^{n-1}C_1 + 3.\,{}^{n-1}C_2 + \ldots\ldots + n.\,{}^{n-1}C_{n-1}]$

A.P.

$= n[1+n]2^{n-1-1}$

$= n(n+1)2^{n-2}$

Type-3 :

Using Integration (With Suitable Limits as Per Requirement)

Illustration 22:

Find the sum of the following series:

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots\ldots + \frac{C_n}{n+1}$$

Solution:

Method-I : $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots\ldots + C_nx^n$

Integrating both the sides

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x = \left(C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + C_3\frac{x^4}{4} + \ldots\ldots + C_n\frac{x^{n+1}}{n+1}\right)_0^x$$

$$\frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + C_3\frac{x^4}{4} + \ldots + C_n\frac{x^{n+1}}{n+1} \qquad \ldots(i)$$

Put $x = 1$ in equation (i)

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \ldots\ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

Method-II :

$$S = \frac{1}{n+1}\left[(n+1)C_0 + \frac{(n+1)}{2}C_1 + \frac{(n+1)}{3}C_2 + \ldots + \frac{(n+1)}{n+1}C_n\right]$$

$$S = \frac{1}{n+1}\left[1 + {}^{n+1}C_1 + {}^{n+1}C_2 + \ldots\ldots\ldots + {}^{n+1}C_n\right] \Rightarrow S = \frac{1}{n+1}[2^{n+1}-1]$$

Method-III :

$$\sum_{r=0}^{n}\frac{{}^nC_r}{r+1} = \sum_{r=0}^{n}\frac{(n+1)}{r+1}\cdot{}^nC_r\left(\frac{1}{n+1}\right)$$

$$= \frac{1}{(n+1)}\sum_{r=0}^{n}{}^{n+1}C_{r+1}$$

$$= \frac{1}{(n+1)}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + \ldots\ldots + {}^{n+1}C_{n+1}\right]$$

$$= \frac{1}{(n+1)}[2^{n+1}-1]$$

Numerically Greatest Term:

Let numerically greatest term in the expansion of $(I + II)^n$ be T_{r+1}.

$\Rightarrow \begin{cases} |T_{r+1}| \geq |T_r| \\ |T_{r+1}| \geq |T_{r+2}| \end{cases}$ where $T_{r+1} = {}^nC_r(I)^{n-r}(II)^r$

Solving above inequalities we get $\dfrac{n+1}{1+\left|\frac{I}{II}\right|} - 1 \leq r \leq \dfrac{n+1}{1+\left|\frac{I}{II}\right|}$

Case-I : When $\dfrac{n+1}{1+\left|\frac{I}{II}\right|}$ is an integer equal to m, then T_m and T_{m+1} will be numerically greatest term.

Case-II : When $\dfrac{n+1}{1+\left|\frac{I}{II}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Proof :

Let numerically greatest term in the expansion of $(I+II)^n$ is T_{r+1}.

$\therefore \quad \boxed{|T_{r+1}| \geq |T_r|}$ and $\boxed{|T_{r+1}| \geq |T_{r+2}|}$

$$\frac{|T_{r+1}|}{|T_r|} \geq 1 \Rightarrow \frac{|{}^nC_r(I)^{n-r}(II)^r|}{|{}^nC_{r-1}(I)^{n-r+1}(II)^{r-1}|} \geq 1$$

$$\frac{n-r+1}{r}\left|\frac{II}{I}\right| \geq 1 \Rightarrow \frac{n+1}{r}-1 \geq \left|\frac{I}{II}\right| \Rightarrow \frac{n+1}{r} \geq 1+\left|\frac{I}{II}\right|$$

$$\boxed{r \leq \frac{n+1}{1+\left|\frac{I}{II}\right|}} \qquad \text{...(1)}$$

$$\therefore \quad |T_{r+1}| \geq |T_{r+2}| \Rightarrow \frac{|T_{r+1}|}{|T_{r+2}|} \geq 1$$

$$\Rightarrow \quad \left|\frac{{}^nC_r(I)^{n-r}(II)^r}{{}^nC_{r+1}(I)^{n-r-1}(II)^{r+1}}\right| \geq 1$$

$$\Rightarrow \quad \left(\frac{r+1}{n-r}\right)\left|\frac{I}{II}\right| \geq 1 \quad \Rightarrow \quad \left|\frac{I}{II}\right| \geq \frac{n-r}{r+1}$$

$$\Rightarrow \quad 1+\left|\frac{I}{II}\right| \geq \frac{n-r}{r+1}+1 \quad \Rightarrow \quad 1+\left|\frac{I}{II}\right| \geq \frac{n-r+r+1}{r+1}$$

$$\Rightarrow \quad \boxed{r \geq \frac{n+1}{\left(1+\left|\frac{I}{II}\right|\right)}-1} \qquad \text{...(2)}$$

From (1) & (2)

$$\boxed{\left(\frac{n+1}{1+\left|\frac{I}{II}\right|}\right)-1 \leq r \leq \left(\frac{n+1}{1+\left|\frac{I}{II}\right|}\right)}$$

Illustration 23:

If $x=\frac{1}{3}$, find the Numerically Greatest Term in the expansion of $(1+4x)^8$.

Solution:

Let $(r+1)^{th}$ term is numerically greatest. Now $I=1, II=4x|_{x=\frac{1}{3}}=\frac{4}{3}$

$$\frac{n+1}{1+\left|\frac{I}{II}\right|}-1 \leq r \leq \frac{n+1}{1+\left|\frac{I}{II}\right|}$$

$$\Rightarrow \frac{8+1}{1+\frac{3}{4}}-1 \leq r \leq \frac{8+1}{1+\frac{3}{4}} \quad \Rightarrow \frac{29}{7} \leq r \leq \frac{36}{7} \quad \Rightarrow r=5$$

$\Rightarrow T_6$ is Numerically Greatest Term.

Illustration 24:

If $x = 1$, then which term is numerically greatest in the expansion of $(3 - 2x)^9$.

Solution:

Let $(r + 1)^{th}$ term is numerically greatest

Now, $I = 3$, $II = -2x|_{x=1} = -2$

$$\frac{n+1}{1+\left|\frac{I}{II}\right|}-1 \le r \le \frac{n+1}{1+\left|\frac{I}{II}\right|}$$

$$\Rightarrow \frac{9+1}{1+\frac{3}{2}}-1 \le r \le \frac{9+1}{1+\frac{3}{2}} \Rightarrow 3 \le r \le 4.$$

Hence numerically greatest terms is T_4 and T_5

Divisibility Problems:

Lets consider a number $(7)^{13}$, which can be written as $(8-1)^{13}$

Now $(8-1)^{13} = \underbrace{{}^{13}C_0.(8)^{13} - {}^{13}C_1(8)^{12} + {}^{13}C_2(8)^{11} - {}^{13}C_3(8)^{10} + {}^{13}C_4(8)^9 - ..... + {}^{13}C_{12}(8)}_{\text{Multiple of 8}} - {}^{13}C_{13}$

$\therefore\ (8-1)^{13} = 8I - 1$

$= 8(I - 1) + 7$

Hence on dividing $(7)^{13}$ by 8, we get remainder 7.

Illustration 25:

Prove that for each $n \in N$, $2^{3n} - 1$ is divisible by 7

Solution:

$2^{3n} - 1 = (23)^n - 1 = (1 + 7)^n - 1$

$= [1 + {}^nC_1(7) + {}^nC_2(7)^2 \ldots\ldots\ldots\ldots + {}^nC_n\,(7)^n] - 1$

$= 7[\,{}^nC_1 + {}^nC_2\,7 + \ldots\ldots\ldots\ {}^nC_n\,7^{n-1}] = 7I$

Hence on dividing $2^{3n} - 1$ by 7, we get a reminder zero.

$\Rightarrow 2^{3n} - 1$ is divisible by 7 for all $n \in N$

Remainder Problems:

lustration 26:

Find the remainder when 5^{99} is divide by 8

Solution:

$5^{99} = 5(5^2)^{49} = 5(24+1)^{49} = 5(\,{}^{49}C_0\,24^{49} + {}^{49}C_1\,24^{48} + \cdots + {}^{49}C_{48}24 + 1) = 24(5)I + 5$

Hence remainder when 5^{99} is divided by 8 is 5.

Illustration 27:

Find the last two digits of the number $(17)^{10}$.

Solution:

$(17)^{10} = (289)^5 = (290 - 1)^5$

$= {}^5C_0(290)^5 - {}^5C_1(290)^4 + \ldots\ldots\ldots + {}^5C_4(290)^1 - {}^5C_5(290)^0$

$= {}^5C_0(290)^5 - {}^5C_1.(290)^4 + \ldots\ldots\ldots\ {}^5C_3(290)^2 + 5 \times 290 - 1$

= A multiple of 1000 + 1449

Hence, last two digits are 49.

Note: We can also conclude that last three digits are 449.

When Two Binomial Coefficients are in Multiplication:

(i) When Sum of the Suffixes is Constant:

Illustration 28:

If ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots.. {}^nC_n$ are binomial coefficients in the expansion $(1+x)^n$ then prove that -

(i) ${}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2} \ldots\ldots + {}^nC_n\,{}^nC_0 = {}^{2n}C_n$

(ii) ${}^nC_0\,{}^nC_{n-1} + {}^nC_1\,{}^nC_{n-2} + {}^nC_2\,{}^nC_{n-3} + \ldots\ldots + {}^nC_{n-1}\,{}^nC_0 = {}^{2n}C_{n-1}$

(iii) ${}^{m+n}C_r = {}^mC_r\,{}^nC_0 + {}^mC_{r-1}.\,{}^nC_1 + {}^mC_{r-2}.\,{}^nC_2 + \ldots\ldots + {}^mC_0.\,{}^nC_r$

Solution:

$(1+x)^n.(1+x)^n$

$= ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 \ldots.. + {}^nC_{n-1}.x^{n-1} + {}^nC_n.x^n)\,({}^nC_0 + {}^nC_1x + {}^nC_2x^2 \ldots$
$\ldots + {}^nC_{n-1}.x^{n-1} + {}^nC_n.x^n)$

For (i) Compare coefficient of x^n

For (ii) Compare coefficient of x^{n-1}

For (iii) Compare coefficient of x^r in $(1+x)^n.(1+x)^m$

Illustration 29:

Prove that : $\binom{20}{10}\binom{15}{0}+\binom{20}{9}\binom{15}{1}+\ldots\ldots\ldots+\binom{20}{0}\binom{15}{10}={}^{35}C_{25}$

Solution:

Compare coefficient of x^{25} in $(1+x)^{20}.(1+x)^{15}$

(It is similar to selecting 10 fruits from 20 apples and 15 mangoes)

(ii) When Difference of the Suffixes is Constant:

Illustration 30:

If ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots.. {}^nC_n$ are binomial coefficients in the expansion $(1+x)^n$ then prove that :

(i) ${}^nC_0\,{}^nC_1 + {}^nC_1\,{}^nC_2 + {}^nC_2\,{}^nC_3 + \ldots\ldots..\,{}^nC_{n-1}\,{}^nC_n = {}^{2n}C_{n-1}$ or ${}^{2n}C_{n+1}$

$= \dfrac{2n!}{(n-1)!(n+1)!}$

(ii) ${}^nC_0\,{}^nC_r + {}^nC_1\,{}^nC_{r+1} + {}^nC_2\,{}^nC_{r+2} + \ldots\ldots..\,{}^nC_{n-r}\,{}^nC_n = {}^{2n}C_{n-r}$ or ${}^{2n}C_{n+r}$

$= \dfrac{2n!}{(n-r)!(n+r)!}$

(iii) $C_0^2 + C_1^2 + C_2^2 + C_3^2 + \ldots.. + C_n^2 = {}^{2n}C_n = \dfrac{(2n)!}{n!n!}$

Solution:

$(1+x)^{2n} = (1+x)^n.(x+1)^n$

$= ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 \ldots.. + {}^nC_{n-1}.x^{n-1} + {}^nC_n.x^n)\,({}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} \ldots$
$\ldots + {}^nC_{n-1}.x + {}^nC_n)$

Now, compare the coefficient in both side

(i) of x^{n-1}

(ii) of x^{n-r}

(iii) of x^n

Illustration 31:

Find the value of $\sum_{j=0}^{n}\sum_{i=0}^{n} C_iC_j$

Solution:

Since $({}^nC_0 + {}^nC_1 + {}^nC_2 + \dots\dots\dots + {}^nC_n)^2 = (2^n)^2$

$= {}^nC_0^2 + {}^nC_1^2 + \dots\dots\dots + {}^nC_n^2 + 2[C_0C_1 + C_0C_2 \dots\dots\dots + C_{n-1}C_n]$

$\Rightarrow 2\sum_{0\le i<j\le n}\sum C_iC_j = (2^n)^2 - {}^{2n}C_n$

Miscellaneous Problems on Summation:

Illustration 32:

Prove that $\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left\{\left(\frac{1}{2}\right)^r+\left(\frac{3}{4}\right)^r+\left(\frac{7}{8}\right)^r+\left(\frac{15}{16}\right)^r+\dots\dots+m\text{ terms}\right\}$

Solution:

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{1}{2}\right)^r+\sum_{r=1}^{n}(-1)^r\,{}^nC_r\left(\frac{3}{4}\right)^r+\dots\text{ up to } m \text{ terms}$$

$$=\sum_{r=0}^{n}{}^nC_0(-1)^r\,{}^nC_r\left[\left(1-\frac{1}{2}\right)^r+\left(1-\frac{1}{2^2}\right)^r+\dots+\left(1-\frac{1}{2^m}\right)^r\right]$$

$$=\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\dots\dots+\left(1-\left(\frac{2^m-1}{2^m}\right)\right)^n=\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n+\left(\frac{1}{8}\right)^n+\dots\dots+\left(\frac{1}{2^m}\right)^n$$

$$=\left(\frac{1}{2}\right)^n+\left(\left(\frac{1}{2}\right)^n\right)^2+\left(\left(\frac{1}{2}\right)^n\right)^3+\dots\dots+=\left(\frac{1}{2^n}\right)^m=\left(\frac{1}{2}\right)^n\frac{1-\left(\left(\frac{1}{2}\right)^n\right)^m}{1-\left(\frac{1}{2}\right)^n}=\frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

Illustration 33:

Prove that : $\sum_{P=1}^{n}\left(\sum_{m=P}^{n}{}^nC_m\,{}^mC_P\right)=3^n-2^n$

Solution:

$$\sum_{P=1}^{n}{}^nC_P.{}^PC_P+{}^nC_{P+1}.{}^{P+1}C_P+\dots+{}^nC_n.{}^nC_P=\sum_{P=1}^{n}\text{cofficient of } x^P \text{ in}\left\{{}^nC_P(1+x)^P+\dots+{}^nC_n(1+x)^n\right\}$$

$$=\sum_{P=1}^{n}\text{cofficient of } x^P\text{in}\left\{{}^nC_0+{}^nC_1(1+x)+\dots\dots+{}^nC_P(1+x)^P+\dots\dots+{}^nC_n(1+x)^n\right\}$$

$$=\sum_{P=1}^{n}\text{cofficient of } x^P\text{in}\,(1+(1+x))^n=\sum_{P=1}^{n}{}^nC_P2^{n-P}=(2+1)^n-2^n=3^n-2^n$$

Illustration 34:

Prove that ${}^nC_0.\,{}^{2n}C_n - {}^nC_1.\,{}^{2n-2}C_n + {}^nC_2.\,{}^{2n-n}C_n - \dots\dots\dots\dots = 2^n$.

Solution:

Coefficient of x^n in $\{{}^nC_0(1+x)^{2n} - {}^nC_1(1+x)^{2n-2}\dots\dots\dots\}$

Coefficient of x^n in $(-1+(1+x)^2)^n$

$\Rightarrow$ Coefficient of x^n in $(x^2+2x)^n = 2^n$

Multinomial Theorem:

Using binomial theorem, we have $(x + a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$, $n \in N$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} x^{n-r} a^r = \sum_{r+s=n} \frac{n!}{r!s!} x^s a^r \text{ , where } s + r = n$$

This result can be generalized in the following form.

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{r_1+r_2+.....+r_k=n} \frac{n!}{r_1!r_2!.....r_k!} x_1^{r_1} x_2^{r_2} ......x_k^{r_k}$$

The general term in the above expansion $\frac{n!}{r_1!r_2!r_3!.....r_k!} . x_1^{r_1} x_2^{r_2} x_3^{r_3} ......x_k^{r_k}$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + ... + r_k = n$ because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$

Particular Cases:

(i) $(x + y + z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

(ii) $(x + y + z + u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

Illustration 35:

Find the coefficient of $x^2 y^3 z^5$ in $(x + y + z)^{10}$

Solution:

Ans. $\left[\frac{10!}{2!\,3!\,5!}\right]$

Illustration 36:

Coefficient of x^4 in the expansion of $(2 - x + 3x^2)^6$

Solution:

$(2 - x + 3x^2)^6 = [2 + x(3x - 1)^6$

$T_{r+1} = {}^6C_r 2^{6-r} x^r (3x - 1)^r$

$= {}^6C_0 2^6 + {}^6C_1 2^5 x(3x - 1) + {}^6C_2 2^4 x^2 (3x - 1)^2 + {}^6C_3 2^3 x^3 (3x - 1)^3 + {}^6C_4 2^2 x^4 (3x - 1)^4 + ...$

Coefficient of x^4 in ${}^6C_4 2^2 x^4 (3x - 1)^4$ is ${}^6C_4 2^2 = 60$

Coefficient of x^4 in ${}^6C_3 2^3 x^3 (3x - 1)^3$ is ${}^6C_3 2^3 \times 9 = 1440$

Coefficient of x^4 in ${}^6C_2 2^4 x^2 (3x - 1)^2$ is ${}^6C_2 2^4 \times 9 = 2160$

Total 3660

OR Use **Multinomial theorem. Make cases**

Solution:

$\frac{\lfloor 6}{\lfloor p \lfloor q \lfloor r}$ $(3x2)^p(-x)^q\ (2)^r$ Now $2p+q=4$

$\Rightarrow p, q, r\ p+q+r=6$

2, 0, 4
0, 4, 2
1, 2, 3

$$\Rightarrow \frac{\lfloor 6}{\lfloor 2 \lfloor 0 \lfloor 4}\cdot 3^2(-1)^0(2)^4+\frac{\lfloor 6}{\lfloor 0 \lfloor 4 \lfloor 2}(3)^0(-1)^4(2)^2+\frac{\lfloor 6}{\lfloor 0 \lfloor 4 \lfloor 2}(3)^1(-1)^2(2)^3=3660$$

Binomial Theorem for Negative and Fractional Indices:

If $n \in R$, then $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+........+\frac{n(n-1)(n-2).....(n-r+1)}{r!}x^r+...$ ∞.

Remarks

(i) The above expansion is valid for any n belongs to rational number other than a whole number if $|x| < 1$.

(ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1+x)^n$ is infinite, and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) The first term must be unity in the expansion, when index 'n' is a negative integer or fraction

$$(x+y)^n = \begin{cases} x^n\left(1+\frac{y}{x}\right)^n = x^n\left\{1+n\left(\frac{y}{x}\right)+\frac{n\ (n-1)}{2\ !}\left(\frac{y}{x}\right)^2+.....\right\} & \text{if } \left|\frac{y}{x}\right|<1 \\ y^n\left(1+\frac{x}{y}\right)^n = y^n\left\{1+n\left(\frac{x}{y}\right)+\frac{n\ (n-1)}{2\ !}\left(\frac{x}{y}\right)^2+.....\right\} & \text{if } \left|\frac{x}{y}\right|<1 \end{cases}$$

(iv) The general term in the expansion of $(1+x)^n$ is $T_{r+1} = \frac{n(n-1)(n-2).......(n-r+1)}{r\ !}x^r$

(v) When 'n' is any rational number other than whole number and $|x| < 1$ then approximate value of $(1+x)^n$ is $1+nx$ (x^2 and higher powers of x can be neglected)

(vi) Expansions to be remembered ($|x| < 1$)

(a) $(1+x)^{-1} = 1-x+x^2-x^3+..........+(-1)^r x^r+.........\infty$

(b) $(1-x)^{-1} = 1+x+x^2+x^3+..........+x^r+.........\infty$

Illustration 37:

Prove that the coefficient of x^r in $(1-x)^{-n}$ is ${}^{n+r-1}C_r$

Solution:

$(r+1)^{\text{th}}$ term in the expansion of $(1-x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)......(-n-r+1)}{r\ !}(-x)^r$$

$$= (-1)^r\ \frac{n(n+1)(n+2)......(n+r-1)}{r\ !}(-x)^r = \frac{n(n+1)(n+2)......(n+r-1)}{r\ !}x^r$$

$$= \frac{(n-1)!\ n(n+1)......(n+r-1)}{(n-1)\ !\ r\ !}x^r$$

Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)!\ r!} = {}^{n+r-1}C_r$ Proved.

Illustration 38:

If x is so small such that its square and higher powers may be neglected, then find the value of

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$$

Solution:

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$$

$$=\frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}}$$

$$=\frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2}$$

$$=\frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right)$$

$$=\frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right)=1-\frac{x}{8}-\frac{19}{12}x=1-\frac{41}{24}x$$

An Important Concept Using Pseudo Function:

Illustration 39:

If n is a natural number, then show that the integral part of the number $N=\left(3+\sqrt{7}\right)^n$ is an odd integer and $N(1-F)=2^n$ where F is fractional part of N.

Solution:

Let I be integral part of N

$\Rightarrow I+F=N\,;0<F<1$

Step-I :

Consider a positive proper fraction $F'=\left(3-\sqrt{7}\right)^n;\ 0<F'<1$

Step-II :

Now consider the value of $I+F+F'$ (sign should be adjusted s.t. irrational terms vanishes)

$I+F+F'=2\left[nC_0(3)^n+{}^nC_2(3)^{n-2}(7)+{}^nC_4(3)^{n-4}(7)^2+\ldots..+{}^nC_n(\sqrt{7})^n\right]$

$\Rightarrow I+F+F'$ is an even integer

Since I is an integer

$\Rightarrow F+F'$ must be an integer

Now $0<F+F'<2$

$\Rightarrow F+F'=1$ (Since $F+F'$ is an integer)

$\Rightarrow I+1=$ Even integer

$\Rightarrow I=$ odd integer

Now,

$\because N(1-F)=NF'$

$=\left(3+\sqrt{7}\right)^n\left(3-\sqrt{7}\right)^n=2^n]$

Illustration 40:

Show that the integral part of the number $N = \left(3\sqrt{3}+5\right)^{2n+1}$ is an even integer $NF = 2^{2n+1}$ where F is fractional part of N.

Solution:

Let I be integral part of N.

$\Rightarrow\ I + F = N; 0 < F < 1$

Step-I :

Consider a positive proper fraction $F' = \left(3\sqrt{3}-5\right)^{2n+1}$ $0 < F' < 1$

Step-II :

Now consider the value of $I + F - F'$ (sign should be adjusted s.t. irrational terms vanishes)

$\Rightarrow I + F - F' = \left(3\sqrt{3}+5\right)^{2n+1} - (3\sqrt{3}-5)^{2n+1} = 2[{}^{2n+1}C_1\left(3\sqrt{3}\right)^{2n}.5 + {}^{2n+1}C_3\left(3\sqrt{3}\right)^{2n-2}.5^3.....]$

$\Rightarrow I + F - F'$ must be an even integer

$\therefore\ F - F'$ should be an integer, but $-1 < F - F' < 1$

$\Rightarrow F - F' = 0$

$\Rightarrow F = F'$

$\Rightarrow I$ is an even integer

$\Rightarrow NF = NF' = 2^{2n+1}$

Sum of Binomial Coefficients

Illustration 41:

If $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$, then show that

(i) $\quad {}^nC_0 + 3\,{}^nC_1 + 3\,{}^2C_2 + ...+ 3\,{}^nC_n = 4^n$.

(ii) $\quad {}^nC_0 + 2\,{}^nC_1 + 3.\,{}^nC_2 + ...+ (n+1)\,{}^nC_n = 2^{n-1}\,(n+2)$.

(iii) $\quad {}^nC_0 - \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} - \frac{{}^nC_3}{4} + ... + (-1)^n\frac{{}^nC_n}{n+1} = \frac{1}{n+1}$.

Solution:

(i) $\quad (1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$

Put $x = 3$

${}^nC_0 + 3.{}^nC_1 + 3^2.\,{}^nC_2 + ...+ 3^n.\,{}^nC_n = 4^n$

(ii) **<u>I Method : By Summation</u>**

L. H.S. = ${}^nC_0 + 2.\,{}^nC_1 + 3.\,{}^nC_2 + ...+(n+1).\,{}^nC_n$.

$= \sum_{r=0}^{n}(r+1)\ .\ {}^nC_r = \sum_{r=0}^{n} r.\ {}^nC_r + \sum_{r=0}^{n} {}^nC_r$

$= n\sum_{r=1}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r = n.2^{n-1} + 2^n = 2^{n-1}\,(n+2)$ R.H.S.

<u>II Method : By Differentiation</u>

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$

Multiplying both sides by x,

$x(1+x)^n = {}^nC_0x + {}^nC_1x^2 + {}^nC_2x^3 + ...+ {}^nC_nx^{n+1}$.

Differentiating both sides

$(1+x)^n + x\,n(1+x)^{n-1} = {}^nC_0 + 2.\,{}^nC_1x + 3.{}^nC_2x^2 + ...+(n+1)\,{}^nC_nx^n$.

putting $x = 1$, we get

$\Rightarrow\ {}^nC_0 + 2.\ {}^nC_1 + 3.\ {}^nC_2 + \ldots\ldots + (n+1)\,{}^nC_n = 2^n + n\,.\,2^{n-1}$

$\Rightarrow\ {}^nC_0 + 2.\ {}^nC_1 + 3.\ {}^nC_2 + \ldots\ldots + (n+1)\,{}^nC_n = 2^{n-1}\,(n + 2)$ Proved.

(iii) **I Method : By Summation**

$$\text{L.H.S.} = {}^nC_0 - \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} - \frac{{}^nC_3}{4} + \ldots\ldots + (-1)^n.\frac{{}^nC_n}{n+1}$$

$$= \sum_{r=0}^{n}(-1)^r.\frac{{}^nC_r}{r+1}$$

$$= \frac{1}{n+1}.\sum_{r=0}^{n}(-1)^r\,{}^{n+1}C_{r+1} \qquad \left\{\text{using } \frac{n+1}{r+1}\,.\,{}^nC_r = {}^{n+1}C_{r+1}\right\}$$

$$= \frac{1}{n+1}\left[{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - \ldots\ldots + (-1)^n.\ {}^{n+1}C_{n+1}\right]$$

$$= \frac{1}{n+1}\left[-\,{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + \ldots\ldots + (-1)^n.\ {}^{n+1}C_{n+1} + {}^{n+1}C_0\right]$$

$$= \frac{1}{n+1}\left[(-1)\left({}^{n+1}C_0 - {}^{n+1}C_1 + {}^{n+1}C_2 + \ldots\ldots + (-1)^{n+1}.\ {}^{n+1}C_{n+1}\right) + {}^{n+1}C_0\right]$$

$$= \frac{1}{n+1} = \text{R.H.S., since } \left\{-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + \ldots + (-1)^n\ {}^{n+1}C_{n+1} = 0\right\}$$

II Method : By Integration

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_nx^n.$

Integrating both sides, within the limits – 1 to 0.

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{0} = \left[{}^nC_0x + {}^nC_1\frac{x^2}{2} + {}^nC_2\frac{x^3}{3} + \ldots\ldots + {}^nC_n\frac{x^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\Rightarrow \frac{1}{n+1} - 0 = 0 - \left[-{}^nC_0 + \frac{{}^nC_1}{2} - \frac{{}^nC_2}{3} + \ldots\ldots + (-1)^{n+1}\frac{{}^nC_n}{n+1}\right]$$

$$\Rightarrow {}^nC_0 - \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} - \ldots\ldots + (-1)^n\frac{{}^nC_n}{n+1} = \frac{1}{n+1} \quad \text{Proved}$$

Illustration 42:

If $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_nx^n$, then prove that

(i) ${}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \ldots\ldots + {}^nC_n^2 = {}^{2n}C_n$

(ii) ${}^nC_0\,{}^nC_2 + {}^nC_1\,{}^nC_3 + {}^nC_2\,{}^nC_4 + \ldots\ldots + {}^nC_{n-2}\ {}^nC_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$

(iii) $1.\ {}^nC_0^2 + 3.\ {}^nC_1^2 + 5.\ {}^nC_2^2 + \ldots\ldots + (2n+1).\ {}^nC_n^2 = 2n.\ {}^{2n-1}C_n + {}^{2n}C_n.$

Solution:

(i) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_nx^n.$ ...(i)

$(x+1)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + \ldots\ldots + {}^nC_nx^0$ ...(ii)

Multiplying (i) and (ii)

$({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_nx^n)({}^nC_0x^n + {}^nC_1x^{n-1} + \ldots\ldots + {}^nC_nx^0) = (1+x)^{2n}$

Comparing coefficient of x^n,

${}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \ldots\ldots + {}^nC_n^2 = {}^{2n}C_n$

(ii) From the product of (i) and (ii) comparing coefficients of x^{n-2} or x^{n+2} both sides,

${}^nC_0\,{}^nC_2 + {}^nC_1\,{}^nC_3 + {}^nC_2\,{}^nC_4 + \ldots\ldots + {}^nC_{n-2}\ {}^nC_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$.

(iii) **I Method : By Summation**

L.H.S. $= 1.\ {}^nC_0^2 + 3.\ {}^nC_1^2 + 5.\ {}^nC_2^2 + \ldots\ldots\ldots + (2n+1)\ {}^nC_n^2.$

$= \sum_{r=0}^{n}(2r+1)\ {}^nC_r^2 = \sum_{r=0}^{n} 2.r\,.\,({}^nC_r)^2 + \sum_{r=0}^{n}({}^nC_r)^2$

$= \left\{2\sum_{r=1}^{n} n\ {}^{n-1}C_{r-1}\ .{}^nC_r\right\} + {}^{2n}C_n$

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots\ldots\ldots\ {}^nC_nx^n$ ...(i)

$(1+x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + \ldots\ldots\ldots\ldots {}^nC_n\,x^n$

$(x+1)^{n-1} = {}^{n-1}C_0\,x^{n-1} + {}^{n-1}C_1\,x^{n-2} + \ldots\ldots\ldots + {}^{n-1}C_{n-1}x^0$ ...(ii)

Multiplying (i) and (ii) and comparing coefficients of x^n.

${}^{n-1}C_0\,.\,{}^nC_1 + {}^{n-1}C_1\,.\,{}^nC_2 + \ldots\ldots\ldots\ldots + {}^{n-1}C_{n-1}\,.\,{}^nC_n = {}^{2n-1}C_n$

$\sum_{r=0}^{n} {}^{n-1}C_{r-1}\,.\,{}^nC_r = {}^{2n-1}C_n$

Hence, required summation is $2n.\ {}^{2n-1}C_n + {}^{2n}C_n$ = R.H.S.

II Method : By Differentiation

$(1+x^2)^n = {}^nC_0 + {}^nC_1x^2 + {}^nC_2x^4 + {}^nC_3x^6 + \ldots\ldots + {}^nC_n\,x^{2n}$

Multiplying both sides by x

$x(1+x^2)^n = {}^nC_0x + {}^nC_1x^3 + {}^nC_2x^5 + \ldots\ldots\ldots + {}^nC_nx^{2n+1}.$

Differentiating both sides

$x.\,n(1+x^2)^{n-1}\,.\,2x + (1+x^2)^n = {}^nC_0 + 3.\ {}^nC_1x^2 + 5.\ {}^nC_2\,x^4 + \ldots\ldots + (2n+1)\ {}^nC_n\,x^{2n}$ ...(i)

$(x^2+1)^n = {}^nC_0\,x^{2n} + {}^nC_1\,x^{2n-2} + {}^nC_2\,x^{2n-4} + \ldots\ldots\ldots + {}^nC_n$ ...(ii)

Multiplying (i) & (ii)

$({}^nC_0 + 3{}^nC_1x^2 + 5{}^nC_2x^4 + \ldots\ldots\ldots + (2n+1)\ {}^nC_n\,x^{2n})\ ({}^nC_0\,x^{2n} + {}^nC_1x^{2n-2} + \ldots\ldots + {}^nC_n)$

$= 2n\,x^2\,(1+x^2)^{2n-1} + (1+x^2)^{2n}$

Comparing coefficient of x^{2n},

${}^nC_0{}^2 + 3{}^nC_1{}^2 + 5{}^nC_2{}^2 + \ldots\ldots\ldots + (2n+1)\ {}^nC_n{}^2 = 2n\,.\,{}^{2n-1}C_{n-1} + {}^{2n}C_n.$

${}^nC_0{}^2 + 3{}^nC_1{}^2 + 5{}^nC_2{}^2 + \ldots\ldots\ldots + (2n+1)\ {}^nC_n{}^2 = 2n\,.\,{}^{2n-1}C_n + {}^{2n}C_n.$ Proved.

Illustration 43:

${}^{10}C_0{}^2 - {}^{10}C_1{}^2 + {}^{10}C_2{}^2 - \ldots - \left({}^{10}C_9\right)^2 + \left({}^{10}C_{10}\right)^2 =$

(A) 0 (B) $\left({}^{10}C_5\right)^2$ (C) $-{}^{10}C_5$ (D) $2\,{}^9C_5$

Solution:

Ans. (C)

$(1+x)^{10} = {}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + \ldots + {}^{10}C_9x^9 + {}^{10}C_{10}x^{10}$

$(x-1)^{10} = {}^{10}C_0x^{10} - {}^{10}C_1x^9 + {}^{10}C_2x^8 - \ldots - {}^{10}C_9.x + {}^{10}C_{10}$

S = coff. of x^{10} in $\left(x^2-1\right)^{10} = -{}^{10}C_5$

Illustration 44:

The sum $\sum_{r=0}^{n}(r+1)C_r^2$ is equal to :

(A) $\frac{(n+2)(2n-1)!}{n!\,(n-1)!}$ (B) $\frac{(n+2)(2n+1)!}{n!\,(n-1)!}$ (C) $\frac{(n+2)(2n+1)!}{n!\,(n+1)!}$ (D) $\frac{(n+2)(2n-1)!}{n!\,(n+1)!}$

Solution:

Ans. (A)

$\because (1+x)^n = C_0 + C_1x + \ldots\ldots\ldots + C_nx^n$

Multiply by x & then differentiate

$(1+x)^n + x.n(1+x)^{n-1} = C_0 + 2C_1x + \ldots\ldots\ldots\ldots + (n+1)C_nx^n$ ....(i)

and $(x+1)^n = C_0x^n + C_1x^{n-1} + \ldots\ldots\ldots\ldots + C_n$ ....(ii)

Multiply (i) & (ii) & equate the coefficient of x^n on both side

$C_0^2 + 2C_1^2 + \cdots \ldots\ldots + (n+1)C_n^2 = {}^{2n}C_n + n.{}^{2n-1}C_{n-1} = \frac{(2n)!}{(n!)^2} + n\frac{(2n-1)!}{n!(n-1)!} = (n+2)\frac{(2n-1)!}{n!(n-1)!}$

Illustration 45:

If $(1+x+x^2+x^3)^5 = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{15}x^{15}$, then a_{10} equals to :

(A) 99 (B) 101 (C) 100 (D) 110

Ans. (B)

Solution:

$(x^4 - 1)^5(x-1)^{-5} = {}^5C_0(x-1)^{-5} - {}^5C_1x^4(x-1)^{-5} - {}^5C_2x^8(x-1)^{-5} \ldots\ldots$

Coefficient of $x^{10} = {}^5C_0 \times {}^{14}C_4 - {}^5C_1 \times {}^{10}C_6 - {}^5C_2 \times {}^6C_2 = 101$

Illustration 46:

The value of p, for which coefficient of x^{50} in the expression

$(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \ldots\ldots + 1001\,x^{1000}$ is equal to ${}^{1002}C_p$, is :

Ans. (50)

Solution:

Co-efficient of x^{50}

$S = (1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \ldots\ldots + 1001x^{1000}$ ....(i)

$\frac{xS}{1+x} = x(1+x)^{999} + 2x^2(1+x)^{998} \ldots\ldots + 1000x^{1000} + \frac{1001\;\;x^{1001}}{(1+x)}$ ....(ii)

$\frac{S}{1+x} = (1+x)^{1000} + x(1+x)^{999} + \ldots\ldots\ldots + x^{1000} - \frac{1001\;\;x^{1001}}{1+x}$

$\Rightarrow \frac{S}{1+x} = (1+x)^{1000}\left[\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{1-\frac{x}{1+x}}\right] - \frac{1001\;\;x^{1001}}{(1+x)}$

$\Rightarrow S = (1+x)^{1002} - x^{1001}(1+x) - 1001\,x^{1001}$

Co-efficient of $x^{50} = {}^{1002}c_{50}$

Illustration 47:

If $\{x\}$ denotes the fractional part of 'x', then $82\left\{\frac{3^{1001}}{82}\right\} =$

Ans. (3)

Solution:

$$82\left\{\frac{3^{1001}}{82}\right\} = 82\left\{\frac{3.(82-1)^{250}}{82}\right\} = 82\left\{\frac{3.[{}^{250}C_0(82)^{250} + {}^{250}C_1(82)^{249}(-1) + ... + {}^{250}C_{250}]}{82}\right\} = 3$$

Illustration 48:

The sum of $3.{}^nC_0 - 8.{}^nC_1 + 13.{}^nC_2 - 18.{}^nC_3 + ...$ upto $(n+1)$ terms is $(n \geq 2)$:

(A) zero (B) 1 (C) 2 (D) none of these

Ans. (A)

Solution:

$3.{}^nC_0 - 8\,.\,{}^nC_1 + 13\,.\,{}^nC_2 - 18\,{}^nC_3 + ...$ up to $(n+1)$ terms

$(1 + x^5)^n = C_0 + C_1x^5 + C_2x^{10} + ... + C_nx^{5n}$

Multiplying by x^3 and differentiating w.r.t. x

$x^3.n(1+x^5)^{n-1}.5x^4 + 3x^2(1+x^5)^n = 3C_0x^2 + 8C_1x^7 + 13C_2x^{12} + ... + (5n+3)C_nx^{5n+2}$

Now put $x = -1$

$3C_0 - 8C_1 + 13C_2 + ...+ (n+1)$ terms $= 0$

Illustration 49:

Let the co-efficients of x^n in $(1+x)^{2^n}$ & $(1+x)^{2^{n-1}}$ be P & Q respectively, then $\left(\frac{P+Q}{Q}\right)^5 =$

Ans. (3^5)

Solution:

$$P = {}^{2n}C_n \text{ and } Q = {}^{2n-1}C_n \Rightarrow \frac{P}{Q} = 2\,;\ \left(1+\frac{P}{Q}\right)^5 = (1+2)^5 = 3^5$$

Illustration 50:

In the expansion of $\left(3^{\frac{-x}{4}} + 3^{\frac{5x}{4}}\right)^n$, the sum of the binomial coefficients is 256 and four times the term with greatest binomial coefficient exceeds the square of the third term by $21n$, then find $4x$.

Ans. (2)

Solution:

$2^n = 256 = 2^8$

$n = 8$

$$\left(3^{\frac{-x}{4}} + 3^{\frac{5x}{4}}\right)^8$$

$4T_5 = T_3^2 + 21n$

$$4 \times {}^8C_4 \times 3^{\frac{-x}{4}\times 4} \times 3^{\frac{5x}{4}\times 4} = \left({}^8C_2 \times 3^{\frac{-x}{4}\times 6} \times 3^{\frac{5x}{4}\times 2}\right)^2 + 21n$$

$280 \times 3^{4x} = 28^2 \times 3^{2x} + 21 \times 8$

$$\Rightarrow\ x = \frac{1}{2}$$

Illustration 51:

The coefficient of x^8 in the expression $(2+x)^2(3+x)^3\ (4+x)^4$ must be

(A) 26 (B) 27 (C) 28 (D) 29

Ans. (D)

Solution:

The expression $(2+x)^2\ (3+x)^3\ (4+x)^4 = (x+2)(x+2)\ (x+3)(x+3)(x+3)(x+4)(x+4)(x+4)(x+4)$

$= x^9 + (2+2+3+3+3+4+4+4+4)\ x^8 + \ldots\ldots\ldots\ldots$

$\Rightarrow$ Co-efficient of $x^8 = 29$

Illustration 52:

The coefficient of x^{203} in the expression $(x-1)\ (x^2-2)\ (x^3-3) \ldots\ldots\ldots (x^{20}-20)$ must be

(A) 11 (B) 12 (C) 13 (D) 15

Ans. (C)

Solution:

Expression $= x.x^2.\ x^3 \ldots\ldots\ldots x^{20}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right)\ldots\ldots\left(1-\frac{20}{x^{20}}\right)$

Let $E=\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right)\ldots\ldots\left(1-\frac{20}{x^{20}}\right)$

Now Co-efficient of x^{203} in original expression

$\Rightarrow$ Co-efficient of x^{-7} in E.

But

$E=1-\left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+\ldots\ldots\right)+\left(\frac{1}{x}.\frac{6}{x^6}+\frac{2}{x^2}.\frac{5}{x^5}+\frac{3}{x^3}.\frac{4}{x^4}+\ldots\ldots\right)-\left(\frac{1}{x}.\frac{2}{x^2}.\frac{4}{x^4}+\ldots\ldots\right)$

= Co-efficient of $x^{-7} = -7+6+10+12-8 = 13$

Illustration 53:

The coefficient of x^{98} in the expression of $(x-1)(x-2)\ldots\ldots\ldots(x-100)$ must be

(A) $1^2+2^2+3^2+\ldots\ldots.+100^2$

(B) $(1+2+3+\ldots\ldots.+100)^2-(1^2+2^2+3^2+\ldots\ldots.+100^2)$

(C) $\frac{1}{2}\ [(1+2+3+\ldots\ldots.+100)^2-(1^2+2^2+3^2+\ldots\ldots.+100^2)]$

(D) None of these

Ans. (C)

Solution:

The Co-efficient of $x^{98} = (1.2 + 2.3 + \ldots\ldots\ldots.99.100)$

= Sum of product of first 100 natural numbers taken two at a time

$= \frac{1}{2}\ [(1+2+3+\ldots\ldots.+100)^2-(1^2+2^2+3^2+\ldots\ldots\ldots.+100^2)]$

Illustration 54:

The number of values of $'r'$ satisfying the equation, ${}^{39}C_{3r-1} - {}^{39}C_{r^2} = 39C_{r^2-1} - {}^{39}C_{3r}$ is :

Ans. (0, 3)

Solution:

${}^{39}C_{3r} + {}^{39}C_{3r-1} = {}^{39}C_{r^2} + {}^{39}C_{r^2-1}$

$= {}^{40}C_{3r} = {}^{40}C_{r^2} \Rightarrow r^2 = 3r \ \ or\ r = 0, 3$

Illustration 55:

Find the value of

$^6C_0.\ ^{12}C_6. - {}^6C_1\ ^{11}C_6 + {}^6C_2\ ^{10}C_6 - {}^6C_3.\ ^9C_6 + {}^6C_4.\ ^8C_6 - {}^6C_5.\ ^7C_6 + {}^6C_6.\ ^6C_6$

Ans. (1)

Solution:

Coeff of x^6: $^6C_0.(1+x)^{12} - {}^6C_1(1+x)^{11} + {}^6C_2(1+x)^{10} - {}^6C_3(1+x)^9 + {}^6C_4(1+x)^8 - {}^6C_5(1+x)^7 + {}^6C_6(1+x)^6$

$$=(1+x)^{12}\left[{}^6C_0 - {}^6C_1\left(\frac{1}{1+x}\right) + {}^6C_2\left(\frac{1}{1+x}\right)^2 - {}^6C_3\left(\frac{1}{1+x}\right)^3 + {}^6C_4\left(\frac{1}{1+x}\right)^4 - {}^6C_5\left(\frac{1}{1+x}\right)^5 + {}^6C_6\left(\frac{1}{1+x}\right)^6\right]$$

$$=(1+x)^{12}.\left(1-\frac{1}{1+x}\right)^6 = (1+x)^6.x^6 = 1 \times \text{coeff of } x^6 = 1$$

Illustration 56:

If n is a positive integer & $C_k = {}^nC_k$, find the value of $\left(\sum_{k=1}^{n}\frac{k^3}{n(n+1)^2.(n+2)}\left(\frac{C_k}{C_{k-1}}\right)^2\right)^{-1}$ is :

Ans. (12)

Solution:

$$\sum_{k=1}^{n} k^3\left(\frac{n-k+1}{k}\right)^2 = \sum_{k=1}^{n} k(n-k+1)^2 = \sum_{k=1}^{n}(n^2k + k^3 + k - 2nk^2 + 2nk - 2k^2)$$

$$= \frac{(n+1)^2.n(n+1)}{2} + \left[\frac{n(n+1)}{2}\right]^2 - \frac{2(n+1)n(n+1)(2n+1)}{6} = \frac{n(n+1)^2(n+2)}{12}$$

Illustration 57:

The value of λ if $\sum_{m=97}^{100} {}^{100}C_m\ .\ {}^mC_{97} = 2^{\lambda}\ .\ {}^{100}C_{97}$, is :

Ans. (3)

Solution:

$$\sum_{m=p}^{n} {}^nC_m.{}^mC_p = \sum_{m=p}^{n}\frac{n!}{m!(n-m)!}\times\frac{m!}{p!(m-p)!} = \sum_{m=p}^{n} {}^nC_p.{}^{n-p}C_{m-p}$$

$= {}^nC_p[{}^{n-p}C_0 + {}^{n-p}C_1 + \ldots\ldots + {}^{n-p}C_{n-p}] = {}^nC_p 2^{n-p}$; where $n = 100$ and $p = 97$.

Illustration 58:

If $(1+x+x^2+\ldots x^p)^6 = a_0 + a_1x + a_2x^2 + \cdots + a_{6p}\,x^{6p}$, then the value of :

$\frac{1}{p(p+1)^6}[a_1 + 2a_2 + 3a_3 + \ldots. + 6pa_{6p}]$ is

Ans. (3)

Solution:

$(1+x+x^2+\ldots\ldots+x^p)^n = a_0 + a_1x + \ldots.. + a_{np}x^{np}$

Differentiating both side w.r.t. x

$n(1+x+x^2+\ldots\ldots..+x^p)^{n-1}(1+2x+\ldots..+px^{p-1}) = a_1 + 2a_2x + \ldots.. + np\,a_{np}x^{np-1}$

Now put $x = 1$

$a_1 + 2a_2 + \ldots.. + np\,a_{np} = n(p+1)^{n-1}(1+2+\ldots.+p) = \frac{n(p+1)^n.p}{2}$, where $n = 6$

Illustration 59:

If $\sum_{k=1}^{19}\frac{(-2)^k}{k!(19-k)!}=\frac{-\lambda}{19!}$ then find λ.

Ans. (2)

Solution:

$\frac{1}{19!}\sum_{k=1}^{19}(-2)^k.{}^{19}C_k=\frac{1}{19!}\sum_{k=1}^{19}(-1)^k.2^k.{}^{19}C_k$

$=\frac{1}{19!}\left[-{}^{19}C_1.2+{}^{19}C_2.2^2-{}^{19}C_3.2^3+...-2^{19}.{}^{19}C_{19}\right]$

$=\frac{1}{19!}\left[{}^{19}C_0-{}^{19}C_1.2+{}^{19}C_2.2^2-...2^{19}.{}^{19}C_{19}-1\right]=\frac{1}{19!}\left((1-2)^{19}-1\right)=\frac{-2}{19!}$

$\therefore\ \lambda=2$

Illustration 60:

The index 'n' of the binomial $\left(\frac{x}{5}+\frac{2}{5}\right)^n$ if the only 9^{th} term of the expansion has numerically the greatest coefficient ($n\in\mathbb{R}$), is :

Ans. (n = 12)

Solution:

For T_9 to be the numerically greatest term, $r=\left[\frac{n+1}{1+\left|\frac{x}{a}\right|}\right]=\left[\frac{n+1}{1+\left|\frac{1}{2}\right|}\right]=8$

$\Rightarrow\ 8<\frac{2(n+1)}{3}<9\Rightarrow 11<n<12.5\Rightarrow n=12$

Illustration 61:

The value of the expression $\left(\sum_{r=0}^{10}{}^{10}C_r\right)\left(\sum_{K=0}^{10}(-1)^k\frac{{}^{10}C_k}{2^k}\right)$ is :

Ans. (1)

Solution:

$\left(\sum_{r=0}^{10}{}^{10}C_r\right)\left(\sum_{k=0}^{10}(-1)^k\frac{{}^{10}C_k}{2^k}\right)=({}^{10}C_0+........+{}^{10}C_{10})\left({}^{10}C_0-\frac{{}^{10}C_1}{2}+\frac{{}^{10}C_2}{2^2}.......+\frac{{}^{10}C_{10}}{2^{10}}\right)=2^{10}\times\left(1-\frac{1}{2}\right)^{10}=1$

Illustration 62:

If $(2\,{}^nC_1)^2+2.(2\,{}^nC_2)^2+3.(2\,{}^nC_3)^2+...+2n.(2\,{}^nC_{2n})^2=18\,.\,{}^{4n-1}C_{2n-1}$, then n is :

Ans. (9)

Solution:

$\because\ (1+x)^{2n}={}^{2n}C_0+{}^{2n}C_1x+.......+{}^{2n}C_{2n}x^{2n}$

differentiating it

$2n(1+x)^{2n-1}={}^{2n}C_1+2.{}^{2n}C_2x+........+2n\,{}^{2n}C_{2n}x^{2n-1}$

Again $(x+)^{2n}={}^{2n}C_0x^{2n}+{}^{2n}C_1x^{2n-1}+{}^{2n}C_2x^{2n-2}+...........+{}^{2n}C_{2n}$

Required expression = coefficient of x^{2n-1} in $2n(1+x)^{4n-1}$

$=2n.\,{}^{4n-1}C_{2n-1}$

$\Rightarrow 2n.\,{}^{4n-1}C_{2n-1}=18.\,{}^{4n-1}C_{2n-1}$ or $n=9$

Illustration 63:

If $\sum_{r=0}^{n} \frac{2r+3}{r+1}.{}^{n}C_r = \frac{(n+k).2^{n+1}-1}{n+1}$ then 'k' is

Ans. (2)

Solution:

$$\sum_{r=0}^{n} \frac{2r+3}{r+1}.{}^{n}C_r = \sum_{r=0}^{n} 2.{}^{n}C_r + \sum_{r=0}^{n} \frac{1}{r+1}.{}^{n}C_r = 2.2^n + \frac{1}{n+1} \cdot \sum_{r=0}^{n} .{}^{n+1}C_{r+1}$$

$$= 2^n + 1 + \frac{1}{n+1}.(2^n + 1 - 1) = \frac{(n+2).2^{n+1} - 1}{n+1} \Rightarrow k = 2$$

Illustration 64:

If $\sum_{r=0}^{n} \frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{1}{a(n+b)}$, then $a+b$ is

Ans. (5)

Solution:

$$\sum_{r=0}^{n} \frac{(-1)^r.C_r}{(r+1)(r+2)(r+3)} = \frac{1}{(n+1)(n+2)(n+3)} \sum_{r=0}^{n} (-1)^r.{}^{n+3}C_{r+3}$$

$$= \frac{-1}{(n+1)(n+2)(n+3)}\left[(1-1)^{n+3} - \left\{{}^{n+3}C_0(-1)^0 + {}^{n+3}C_1(-1)^1 + {}^{n+3}C_2(-1)^2\right\}\right]$$

$$= \frac{(n+2)(n+1)}{2(n+2)(n+1)(n+3)} = \frac{1}{2(n+3)} \quad \Rightarrow a = 2,\ b = 3 \text{ or } a + b = 5$$

Illustration 65:

$\sum_{k=1}^{3n} {}^{6n}C_{2k-1}(-3)^k$ is equal to :

Ans. (0)

Solution:

$$S = \sum_{k=1}^{3n} {}^{6n}C_{2k-1}(-3)^k \Rightarrow \qquad S = {}^{6n}C_1(-3) + {}^{6n}C_3(-3)^2 + \ldots\ldots + {}^{6n}C_{6n-1}(-3)^3$$

$$\Rightarrow S = (\sqrt{3}i)\sum_{k=1}^{3n} {}^{6n}C_{2k-1}(\sqrt{3}\ i)^{2k-1} \Rightarrow S = (\sqrt{3}i)\left[\frac{(1+\sqrt{3}\ i)^{6n} - (1-\sqrt{3}\ i)^{6n}}{2}\right] = 0$$

Illustration 66:

If x is very large as compare to y, then the value of k in $\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = 1 + \frac{y^2}{kx^2}$

Ans. (2)

Solution:

$$\sqrt{\frac{x}{x+y}}\sqrt{\frac{x}{x-y}} = \left(\frac{1}{1+\frac{y}{x}}\right)^{1/2}\left(\frac{1}{1-\frac{y}{x}}\right)^{1/2} = \left(1 - \frac{y^2}{x^2}\right)^{-1/2} = 1 + \frac{1}{2} \cdot \frac{y^2}{x^2} \Rightarrow k = 2$$