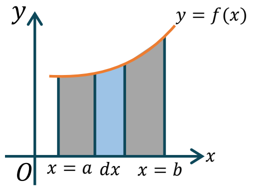
04 Area Under the Curve

1. Area Under the Curves and x-axis (Vertical Strips):

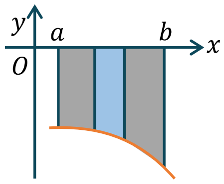
(a)



Area bounded by the curve, the x −axis and the ordinate at $x = a$ and $x = b$ is given by $A = \int_a^b y dx$, where $y = f(x)$ lies above the x-axis and $b > a$.

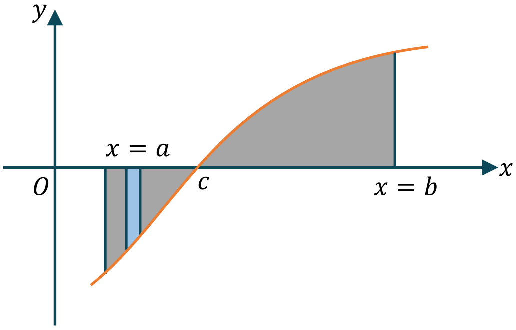
Here vertical strip of thickness dx is considered at distance x.

(b)



If $y = f(x)$ lies completely below the x-axis then $A = \left|\int_a^b y dx\right|$

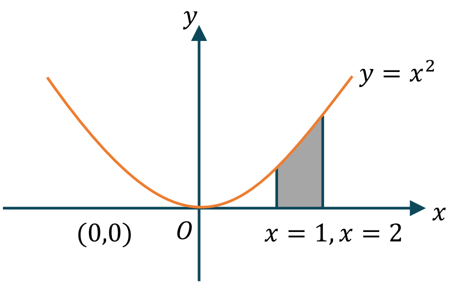
(c)



If curve crosses the x-axis at $x = c$, then $A = \left|\int_a^c y dx\right| + \int_c^b y dx$

Illustration 1:

Find area bounded by $x = 1, x = 2, y = x^2, y = 0$.



Solution:

$A = \int_1^2 (x^2 - 0)dx$

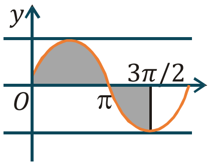
$= \frac{[x^3]_1^2}{3}$

$= \frac{8-1}{3} = \frac{7}{3}$

Illustration 2:

Find area enclosed between $y = \sin x$ & x-axis as x varies from 0 to $\frac{3\pi}{2}$.

Solution:



$A = \int_0^{\pi} \sin x\, dx + \left|\int_{\pi}^{3\pi/2} \sin x\, dx\right|$

$= [-\cos x]_0^{\pi} + \left|[-\cos x]_{\pi}^{3\pi/2}\right| = (1+1) + |(0-1)| = 3$

Illustration 3:

Find the area bounded by $y = \sec^2 x, x = \frac{\pi}{6}, x = \frac{\pi}{3}$ & x-axis

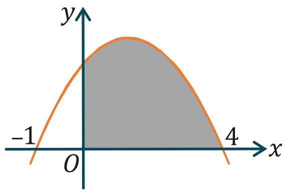
Solution:

Area bounded $= \int_{\pi/6}^{\pi/3} ydx = \int_{\pi/6}^{\pi/3} \sec^2 xdx = [\tan x]_{\pi/6}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{6} = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$ sq.units.

Illustration 4:

Compute the larger area bounded by $y = 4 + 3x - x^2$ and the coordinates axes

Solution:



For $y = 0$ & $x = -1, 4$

$A = \int_0^4 ydx = \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_0^4 = \frac{56}{3}$

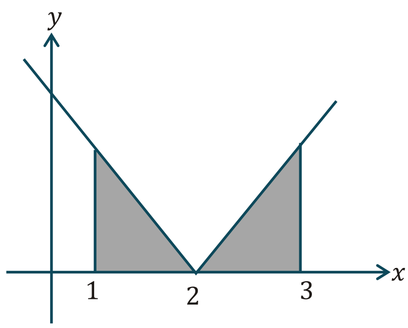
Illustration 5:

The area of the region bounded by the curves $y = |x-2|$, $x = 1$, $x = 3$ and the x-axis is

(A) 4 (B) 2 (C) 3 (D) 1

Ans. (D)

Solution:



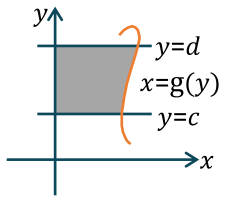
Required area $= \int_1^3 |x-2|\,dx$

$= \int_1^2 (2-x)dx + \int_2^3 (x-2)dx$

$= \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^3 = \frac{1}{2} + \frac{1}{2} = 1.$

2. Area Under the Curves and y-axis (Horizontal Strips):

(a) If $g(y) \geq 0$ for $y \in [c, d]$ then area bounded by curve $x = g(y)$ and y-axis between abscissa $y = c$ and $y = d$ is $\int_{y=c}^{d} g(y)dy$

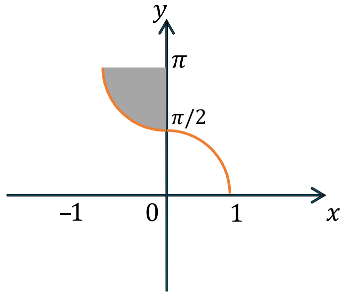


Graph of x=g(y)

Illustration 6:

Find area bounded between $y = \cos^{-1} x$ and y-axis between $y = \frac{\pi}{2}$ and $y = \pi$.

Solution:



Graph of $y = \cos^{-1} x$

$y = \cos^{-1} x$

$\Rightarrow x = \cos y$

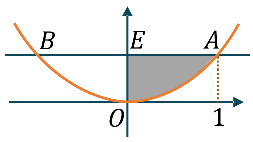
Required area = $-\int_{\frac{\pi}{2}}^{\pi} \cos y \, dy$

$= -\sin y\Big]_{\frac{\pi}{2}}^{\pi} = 1$

Illustration 7:

Find the area bounded by the parabola $x^2 = y$, y-axis and the line $y = 1$.

Solution:

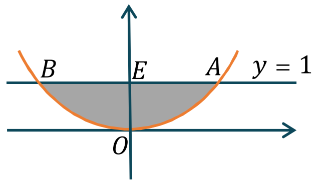


Area $OEBO$ = Area $OAEO = \int_0^1 |x| \, dy = \int_0^1 \sqrt{y} \, dy = \frac{2}{3}$

Illustration 8:

Find the area bounded by the parabola $x^2 = y$ and line $y = 1$.

Solution:



Graph of $y = x^2$

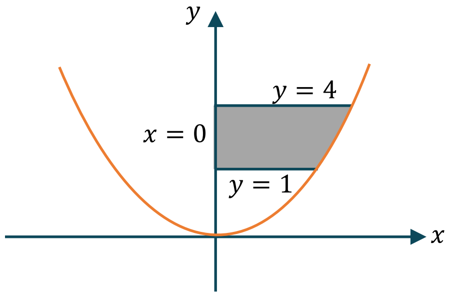
Required area is area $OABO$

= 2 area ($OAEO$) = $2\int_0^1 |x| \, dy = 2\int_0^1 \sqrt{y} dy = \frac{4}{3}$.

Illustration 9:

Find the area in the first quadrant bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$.

Solution:



Required area $= \int_1^4 x \, dy = \int_1^4 \frac{\sqrt{y}}{2} = dy = \frac{1}{2}\left[\frac{2}{3} y^{3/2}\right]_1^4$

$= \frac{1}{3}\,[4^{3/2} - 1] = \frac{1}{3}\,[8 - 1]$

$= \frac{7}{3} = 2\frac{1}{3}$ sq.units.

Illustration 10:

Find the area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ lying between the ordinates $x = 0$ and $x = 3$ and x-axis

Solution:

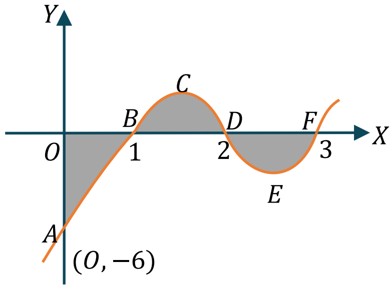
To determine the sign, we follow the usual rule of change of sign.

$y = +ve$ for $x > 3$

$y = -ve$ for $2 < x < 3$

$y = +ve$ for $1 < x < 2$

$y = -ve$ for $x < 1$.



$$\int_0^3 |y|\,dx = \int_0^1 |y|\,dx + \int_1^2 |y|\,dx + \int_2^3 |y|\,dx$$

$$= \int_0^1 -y\,dx + \int_1^2 y\,dx + \int_2^3 -y\,dx$$

Now let $F(x) = \int (x-1)(x-2)(x-3)\,dx = \int (x^3 - 6x^2 + 11x - 6)\,dx = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x$.

$\therefore F(0) = 0, F(1) = -\frac{9}{4}, F(2) = -2, F(3) = -\frac{9}{4}$.

Hence required Area $= -[F(1) - F(0)] + [F(2) - F(1)] - [F(3) - F(2)] = 2\frac{3}{4}$ sq.units.

(b) If $g(y) \le 0$ for $y \in [c, d]$ then area bounded by curve $x = g(y)$ and y-axis between abscissa $y = c$ and $y = d$ is $-\int_{y=c}^{d} g(y)dy$

Note:

General formula for area bounded by curve $x = g(y)$ and y-axis between abscissa $y = c$ and $y = d$ is $\int_{y=c}^{d} |g(y)|\,dy$.

3. Symmetric area:

If the curve is symmetric in all four quadrants, then

Total area = 4(Area in any one of the quadrants).

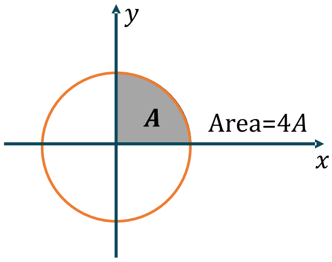
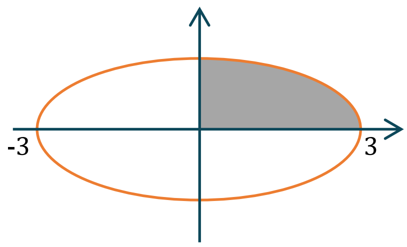


Illustration 11:

Find the area bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Solution:

Area bounded by ellipse in first quadrant $= \int_0^3 \frac{2}{3}\sqrt{9 - x^2}\,dx = \frac{3\pi}{2}$



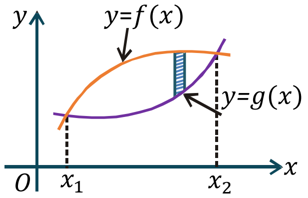
$\because$ Curve is symmetrical about all four quadrant

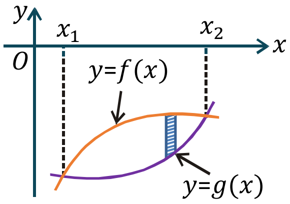
$\therefore$ Total area = 4 (Area in any one of the quadrants)

$= 4\left(\frac{3\pi}{2}\right) = 6\pi$

Area Enclosed Between Two Curves:

(a) Area bounded by two curves $y = f(x)$ & $y = g(x)$



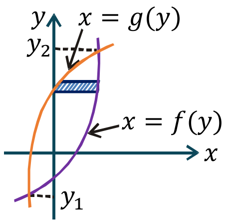


such that $f(x) > g(x)$ is

$$A = \int_{x_1}^{x_2} [f(x) - g(x)]\, dx$$

where x_1 and x_2 are roots of equation $f(x) = g(x)$

(b) In case horizontal strip is taken we have

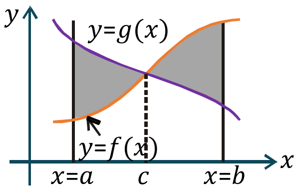


$$A = \int_{y_1}^{y_2} [f(y) - g(y)]\, dy$$

Where y_1&y_2 are roots of equation $f(y) = g(y)$

(c) If the curves $y = f(x)$ and $y = g(x)$ intersect at $x = c$, then required area

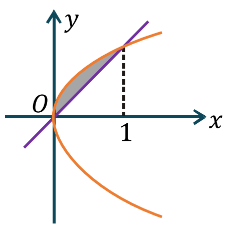
$$A = \int_a^c \big(g(x) - f(x)\big)dx + \int_c^b \big(f(x) - g(x)\big)dx = \int_a^b |f(x) - g(x)|dx$$



Note: Required area must have all the boundaries indicated in the problem.

Illustration 12:

Find the area enclosed between $y^2 = x$ and $y = x$.



Solution:

Area $= \int_0^1 \left(\sqrt{x} - x\right)$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1 = \left(\frac{2}{3} - \frac{1}{2}\right) - (0 - 0) = \frac{1}{6}$$

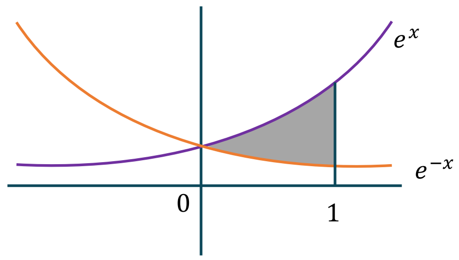
Illustration 13:

The area of figure bounded by $y = e^x, y = e^{-x}$ and the straight line $x = 1$ is

(A) $e + \frac{1}{e}$ (B) $e - \frac{1}{e}$ (C) $e + \frac{1}{e} - 2$ (D) $e + \frac{1}{e} + 2$

Ans. (C)

Solution:



Given equations of curves $y=e^x; y=e^{-x}$ and straight line $x=1$

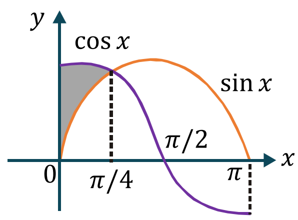
We know that area of the figure bounded by the curves and straight line

$$=\int_0^1 (e^x-e^{-x})dx=[e^x+e^{-x}]_0^1=e+\frac{1}{e}-2.$$

Illustration 14:

Find the area enclosed between $y = \sin x$; $y = \cos x$ and y-axis in the 1st quadrant

Solution:



$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx$$

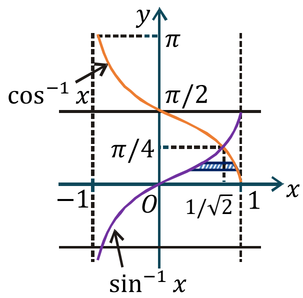
$$= [\sin x + \cos x]_0^{\pi/4} = \left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) - (0+1)$$

$$= \sqrt{2}-1$$

Illustration 15:

Find the area bounded by $y = \sin^{-1} x$; $y = \cos^{-1} x$ and the x-axis

Solution:



If vertical strip is used

$$A = \int_0^{1/\sqrt{2}} \sin^{-1} x\, dx + \int_{1/\sqrt{2}}^{1} \cos^{-1} x\, dx$$

If horizontal strip is used

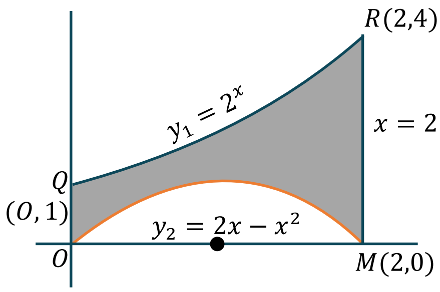
$$A = \int_0^{\pi/4} (\cos y - \sin y)dy$$

$$= [\sin y + \cos y]_0^{\frac{\pi}{4}} = \sqrt{2}-1$$

Illustration 16:

Compute the area of the figure bounded by the straight lines $x = 0, x = 2$ and the curves $y = 2^x, y = 2x - x^2$.

Solution:



The required area= $\int_0^2 (y_1 - y_2)\ dx$

where $y_1 = 2^x$and $y_2 = 2x - x^2$ = $\int_0^2 (2^x - 2x + x^2)dx$

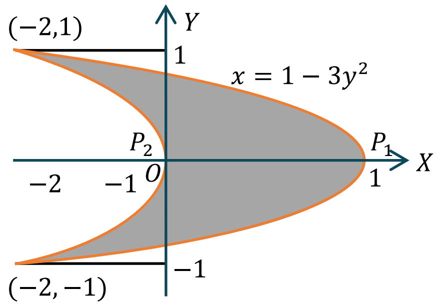
$$\left[\frac{2^x}{\ell n2} - x^2 + \frac{1}{3}x^3\right]_0^2 = \left(\frac{4}{\ell n2} - 4 + \frac{8}{3}\right) = \frac{1}{\ell n2} = \frac{3}{\ell n2} - \frac{4}{3}$$ = sq.units.

Illustration 17:

Compute the area of the figure bounded by the parabolas $x = -2y^2, x = 1 - 3y^2$.

Solution:

Solving the equations $x = -2y^2, x = 1 - 3y^2$,
we find that ordinates of the points of intersection of the two curves as $y_1 = -1, y_2 = 1$.
The points are (–2, –1) and (–2, 1).
The required area



$$2\int_0^1 (x_1 - x_2)\,dy = 2\int_0^1 [(1-3y^2)-(-2y^2)]dy = 2\int_0^1 (1-y^2)dy$$

$$= 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3} \text{ sq.units.}$$

Illustration 18:

Area bounded by curves $y = x^2$ and $y = 2 - x^2$ is

(A) 8/3 (B) 3/8 (C) 3/2 (D) None of these

Ans. (A)

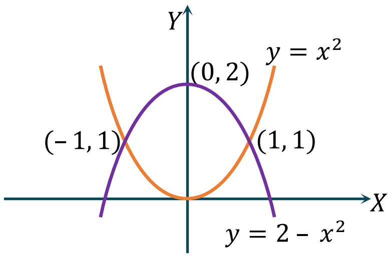
Solution:

$y = x^2$ ...(i)

$y = 2 - x^2$ ...(ii)

∴By equation (i) and (ii) , we get, $x = \pm 1$

∴ $y = 1$



∴ Required area $= 2\left[\int_0^1 (2-x^2)dx - \int_0^1 x^2 dx\right]$

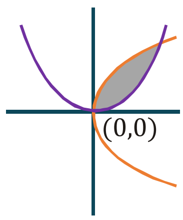
$$= 2\left[2x - \frac{2x^3}{3}\right]_0^1 = 4\left[x - \frac{x^3}{3}\right]_0^1 = 4\left(\frac{2}{3}\right) = \frac{8}{3}.$$

4. Standard Areas (To be Remembered):

(a) Area bounded by parabolas $y^2 = 4ax; x^2 = 4by, a > 0; b > 0$ is $\dfrac{16ab}{3}$

Intersection point:

$$\left.\begin{aligned} x^2 &= 4by \\ y^2 &= 4ax \end{aligned}\right\}$$



$x^2 = 4b\sqrt{4ax}$

$x^4 = 64b^2 ax$

x = 0 or $x = 4b^{2/3}a^{1/3}$

$$A = \int_0^{4b^{2/3}a^{1/3}} \left(\sqrt{4ax} - \frac{x^2}{4b}\right)dx = \frac{16ab}{3}$$

(b) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab sq.units.

(c) Area included between the parabola $y^2 = 4\,ax$ & the line $y = mx$ is $8\,a^2/3\,m^3$ sq.units.

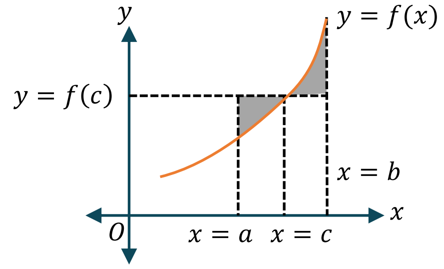
(d) The area of the region bounded by one arch of sin ax (or cos ax) and x-axis is $2/a$ sq.units.

(e) Average value of a function $y = f(x)$ over an interval $a \le x \le b$ is defined as :

$$y(av) = \frac{1}{b-a}\int_a^b f(x)dx$$

(f) If $y = f(x)$ is a monotonic function in (a, b), then the area bounded by the ordinates at $x = a$, $x = b$, $y = f(x)$ and $y = f(c)$ [where $c \in (a, b)$] is minimum when $c = \frac{a+b}{2}$.

Proof : Let the function $y = f(x)$ be monotonically increasing.



Required area $A = \int_a^c [f(c) - f(x)]dx + \int_c^b [f(x) - f(c)]dx$

For minimum area, $\frac{dA}{dc} = 0$

$\Rightarrow [f'(c).c + f(c) - f'(c)a - f(c)] + [-f(c) - f'(c).b + f'(c).c + f(c)] = 0$

$\Rightarrow f'(c)\left\{c - \frac{a+b}{2}\right\} = 0 \qquad \Rightarrow c = \frac{a+b}{2} \qquad (\because f'(c) \neq 0)$

Illustration 19:

Area bounded by :

(a) $y^2 = 4x$ & $x^2 = 16y$

(b) $4y^2 = x$ & $x^2 = 4y$

Solution:

(a) $y^2 = 4ax \ \ x^2 = 4by$

$\therefore a = 1 \qquad \therefore b = 4$

Area = $\left|\frac{16.1.4}{3}\right| = \frac{64}{3}$

(b) $y^2 = \frac{x}{4}$, $x^2 = 4y$

$y^2 = 4ax$

$a = \frac{1}{16}$

$x^2 = 4by$

$b = 1$

Area = $\left|\frac{16.\frac{1}{16}.1}{3}\right| = \frac{1}{3}$

5. Area under various cases:

I. Where the curve sketching is very significant:

Curve Tracing:

The following procedure is to be applied in sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry : The symmetry of the curve is judged as follows :

(i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

(ii) If all the powers of x are even, the curve is symmetrical about the axis of y.

(iii) If powers of x & y both are even, the curve is symmetrical about the axis of x as well as y.

(iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

(v) If on interchanging the signs of x & y both, the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(b) Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

(c) Find the points where the curve crosses the x–axis & also the y–axis.

(d) Examine if possible the intervals when $f(x)$ is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

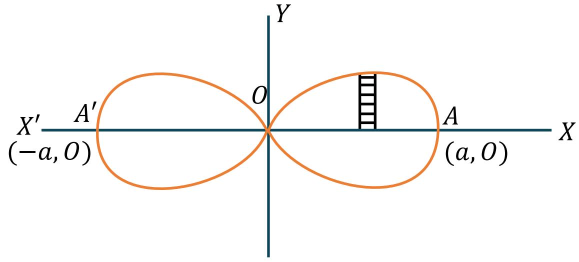
Illustration 20:

Find the area of a loop as well as the whole area of the curve $a^2y^2 = x^2(a^2 - x^2)$.

Solution:

The curve is symmetrical about both the axes. It cuts x-axis at $(0,0)$, $(-a,0)$, $(a,0)$

Area of a loop = $2\int_0^a y\,dx = 2\int_0^a \frac{x}{a}\sqrt{a^2-x^2}\,dx$



$= \frac{1}{a}\int_0^a \sqrt{a^2-x^2}(-2x)dx = \frac{1}{a}\left[\frac{2}{3}(a^2-x^2)^{3/2}\right]_0^a = \frac{2}{3}a^2$

Total area = $2 \times \frac{2}{3}a^2 = \frac{4}{3}a^2$ sq.units.

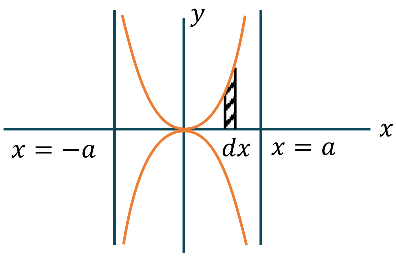
Illustration 21:

Find the whole area included between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

Solution:

(i) The curve is symmetric about both the axes (even powers of x & y)

(ii) Asymptotes are $x = \pm a$



$$A = 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{ax}{\sqrt{a^2-x^2}}dx$$

$$= 4a\left|-\sqrt{a^2-x^2}\right|_0^a$$

$$= 4a^2$$

Illustration 22:

Find the area bounded by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote.

Solution:

(i) The curve is symmetrical about the x-axis as it contains even powers of y.

(ii) It passes through $(2a, 0)$.

(iii) Its asymptote is $x = 0$, i.e., y-axis.

$$A = 2\int_0^{2a} y\,dx = 2\int_0^{2a} 2a\sqrt{\frac{2a-x}{x}}\,dx$$

Put $x = 2a\sin^2\theta$

$$A = 16a^2 \int_0^{\pi/2} \cos^2\theta\,d\theta$$

$= 4\pi a^2$

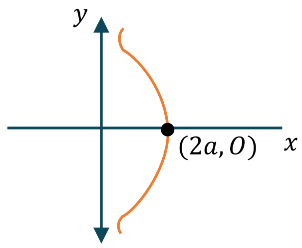


Illustration 23:

Area enclosed between the curves $y = ex.\ell nx$ and $y = \dfrac{\ell nx}{ex}$

Solution:

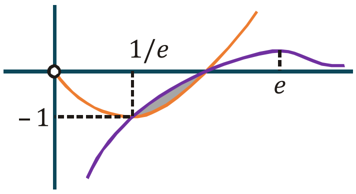
Solving $ex\ \ell nx = \dfrac{\ell nx}{ex}$

$\ell nx\left(e^2x^2 - 1\right) = 0$

$x = 1$ or $x = \dfrac{1}{e}$

Where $x = 1; y_1 = 0; y_2 = 0$

$x = \dfrac{1}{e}\ ;\ y_1 = -1\ ;\ y_2 - 1$



Also examine the increasing and decreasing behaviour of the curve

$$A = \int_{1/e}^{1}\left(\frac{\ell n\,x}{ex} - ex\ \ell n\,x\right)dx = \frac{e}{4} - \frac{5}{4e}$$

Illustration 24:

Area of the closed figure bounded by the curves $y = 2 - |2-x|$ and $y = \dfrac{3}{|x|}$

Solution:

$$y = \begin{cases} 2-(2-x) & \text{if } \quad x<2 \\ 2+(2-x) & \text{if } \quad x>2 \end{cases}$$

$$\text{Area } = \int_{\sqrt{3}}^{2}\left(x - \frac{3}{x}\right)dx + \int_{2}^{3}\left(4 - x - \frac{3}{x}\right)dx$$

$= \dfrac{4 - 3\ell n3}{2}$

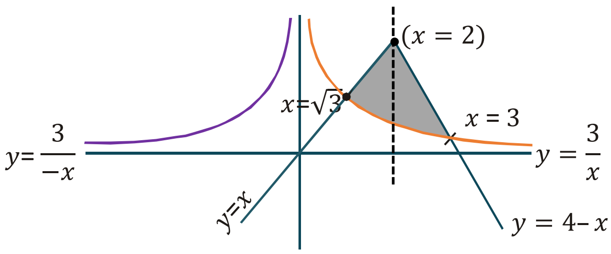


Illustration 25:

Area of the loop $ay^2 = x^2(a - x), a > 0$

Solution:

$ay^2 = x^2(a - x), a \geq 0$

Cuts x-axis at

$0 = x^2(a - x)$

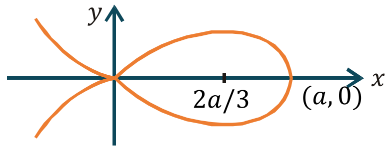
$x^2(a - x) = 0$

$\therefore x = 0; x = a$

cuts y-axis at

$ay^2 = 0$

$y = 0$



$$A = \int_0^a \left(\frac{x\sqrt{a-x}}{\sqrt{a}} - \left(\frac{-x\sqrt{a-x}}{\sqrt{a}} \right) \right) dx$$

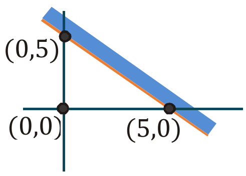
OR $A = 2\int_0^a \frac{x}{\sqrt{a}} \left(\sqrt{a-x} \right) dx = \frac{8}{15} a^2$

II. Problems based on inequalities:

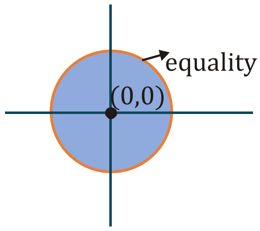
Illustration 26:

Shade the region.

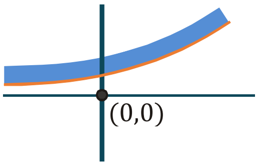
(i) $x + y > 5$



(ii) $x^2 + y^2 \leq 1$



(iii) $y > e^x$



(iv) $y^2 > 4x$

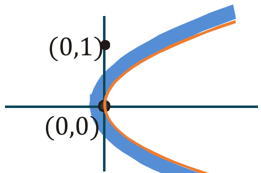
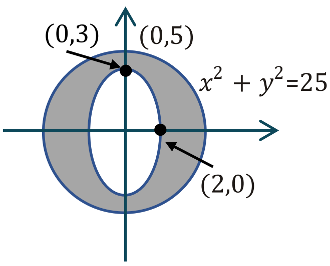


Illustration 27:

Area bounded by $\frac{x^2}{4} + \frac{y^2}{9} \geq 1$ & $x^2 + y^2 \leq 25$

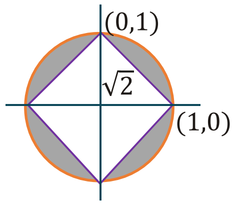
Solution:



Area = $\pi(5)^2 - \pi\,(2)(3)$

Illustration 28:

Area bounded by $|x| + |y| \geq 1$ & $x^2 + y^2 \leq 1$



Solution:

Area $= \pi(1)^2 - (\sqrt{2})^2$

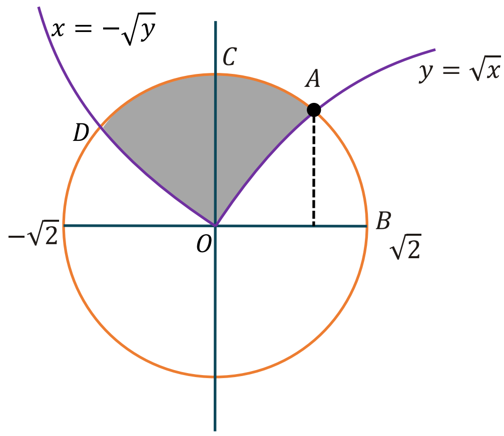
$= \pi - 2$

Illustration 29:

Find the area bounded by the regions $y^2 \geq x,\ x > -\sqrt{y}$ & curve $x^2 + y^2 \leq 2$.

Solution:

Common region is given by the diagram



If area of region $OAB = \lambda$

then area of $OCD = \lambda$

Because $y = \sqrt{x}$ & $x = -\sqrt{y}$

will bound same area with x & y axes respectively.

$y = \sqrt{x} \Rightarrow y^2 = x$

$x = -\sqrt{y} \Rightarrow x^2 = y$ and hence both the curves are

symmetric with respect to the line $y = x$

Area of first quadrant $OBC = \frac{\pi r^2}{4} = \frac{\pi}{2}$ $\quad (\because r = \sqrt{2})$

Area of region $OCA = \frac{\pi}{2} - \lambda$

Area of shaded region $= (\frac{\pi}{2} - \lambda) + \lambda = \frac{\pi}{2}$ sq.units.

III. Enclosed Area for curve represented parametrically:

If the equation of the curve be in parametric form then $A = \int_{t=\alpha}^{t=\beta} y.\frac{dx}{dt}.dt$ or $\int_{t=\gamma}^{t=\delta} x.\frac{dy}{dt}.dt$,

where α & β are values corresponding to values of x and γ & δ are values corresponding to values of y.

Illustration 30:

Find area bounded by curve $x = 2\cos t$ & $y = 3\sin t$ where t is a parameter.

$x = 2\cos t\ y = 3\sin t$

$\cos t = \frac{x}{2}$ $\qquad \sin t = \frac{y}{3}$

Solution:

Use: $\cos^2 t + \sin^2 t = 1$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

Area $= 6\pi$

Illustration 31:

Area bounded by curve $y = 2t$ & $x = t^2$ & $y = 3x$ where t is parameter.

Solution:

Since, $t = \frac{y}{2}$ so, $x = \frac{y^2}{4}$

$y^2 = 4x$; $y = 3x$

$4a = 4$; $m = 3$

$a = 1$

Area $= \left|\frac{8a^2}{3m^3}\right| = \frac{8.1}{3.27} = \frac{8}{81}$

IV. Area enclosed by inverse of a function:

The area bounded by a curve (say $y = f(x)$) on x axis is equal to the area bounded by the inverse of that curve $\left(f^{-1}(x)\right)$ on y axis.

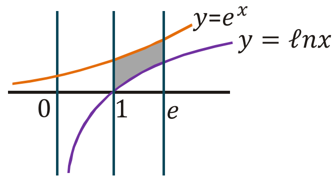
Illustration 32:

Find area bounded by $y = f(x)$ & $y = f^{-1}(x)$, $x = 1$ & $x = e$ where $f(x) = e^x$.

Solution:

$A = \int_1^e (e^x - \ell nx)dx$

$= \int_1^e e^x dx - \int_1^e \ell nx dx$



$= e^x\Big|_1^e - \int \underset{II}{1}.\underset{I}{\ell nx}$

$= \left(e^e - e\right) - \left(e - e - 0 + 1\right) = e^e - e - 1$.

Illustration 33:

Let $f(x) = x^3 + 3x + 2$ and $g(x)$ is the inverse of it. Find the area bounded by $g(x)$, the x-axis and the ordinate at $x = -2$ and $x = 6$.

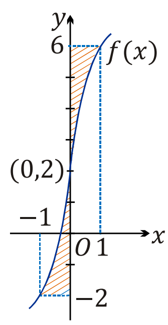
Ans. 9/2

Solution:

The required area will be equal to area enclosed by $y = f(x)$, the y-axis between the abscissa at $y = -2$ and $y = 6$

For $y = -2, x = -1$

and $y = 6, x = 1$



Hence $A = \int_0^1 \left(6 - f(x)\right)dx + \int_{-1}^0 \left(f(x) - (-2)\right)dx$

$= \int_0^1 \left(4 - x^2 - 3x\right)dx + \int_{-1}^0 \left(x^3 + 3x + 4\right)dx = 9/2$

Illustration 34:

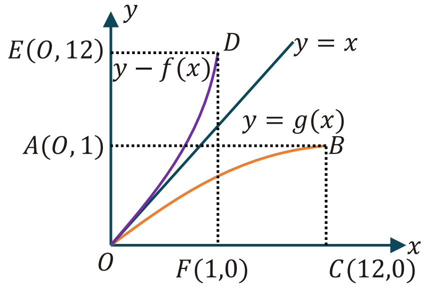
If $y = g(x)$ is the inverse of a bijective mapping $f: R \to R$, $f(x) = 6x^5 + 4x^3 + 2x$, find the area bounded by $g(x)$, the x-axis and the ordinate at $x = 12$.

Solution:

$f(x) = 12 \Rightarrow 6x^5 + 4x^3 + 2x = 12 \Rightarrow x = 1$

$\int_0^{12} g(x)dx$ = area of rectangle $OEDF$ – $\int_0^1 f(x)dx$

$= 1 \times 12 - \int_0^1 (6x^5 + 4x^3 + 2x)dx$ = 12 – 3 = 9 sq. units.



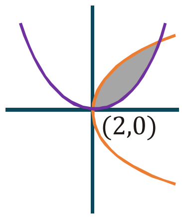
V. Shifting of origin:

Since area remains invariant even if the coordinate axes are shifted, hence shifting of origin in many cases prove to be very convenient in computing the areas.

Illustration 35:

Find area enclosed between the parabolas $y^2 = 4(x-2)$ and $(x-2)^2 = 16y$

Solution:



Let $x - 2 = X$

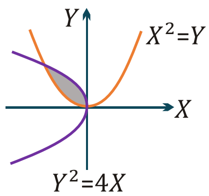
$Y^2 = 4X$; $X^2 = 16Y$

$$A = \left|\frac{16ab}{3}\right| = \left|\frac{16(1)(4)}{3}\right| = \frac{64}{3}$$

Illustration 36:

Area enclosed between the parabolas $y^2 - 2y + 4x + 5 = 0$ and $x^2 + 2x - y + 2 = 0$.

Solution:



$(y-1)^2 = -4(x+1)$; $(x+1)^2 = y-1$

Put $y - 1 = Y$ and $x + 1 = X$

$Y^2 = -4X$; $X^2 = Y$

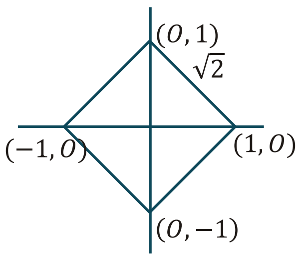
$$A = \left|\frac{16ab}{3}\right| = \left|\frac{16(-1)(1/4)}{3}\right| = \frac{4}{3}$$

lustration 37:

Find the area enclosed by $|x - 1| + |y + 1| = 1$.

Solution:

Shift the origin to $(1, -1)$.



$X = x - 1$ $\qquad Y = y + 1$

$|X| + |Y| = 1$

Area = $\sqrt{2} \times \sqrt{2}$ = 2 sq. units

Illustration 38:

Find the area of the region common to the circle $x^2 + y^2 + 4x + 6y - 3 = 0$ and the parabola $x^2 + 4x = 6y + 14$.

Solution:

Circle is $x^2 + y^2 + 4x + 6y - 3 = 0$

$\Rightarrow (x+2)^2 + (y+3)^2 = 16$

Shifting origin to (–2,–3).

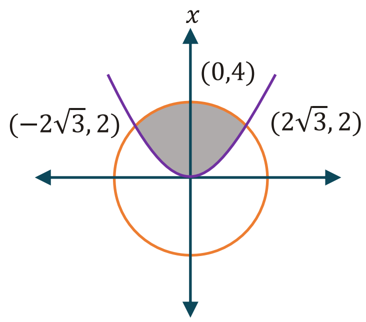
$X^2 + Y^2 = 16$

equation of parabola $\rightarrow (x+2)^2 = 6(y+3)$

$\Rightarrow X^2 = 6Y$

Solving circle & parabola, we get $X = \pm 2\sqrt{3}$

Hence they intersect at $\left(-2\sqrt{3},2\right)$ & $\left(2\sqrt{3},2\right)$



$$A = 2\left[\int_0^2 \sqrt{6Y}\, dY + \int_2^4 \sqrt{16 - Y^2}\, dY\right]$$

$$= 2\left[\frac{2}{3}\sqrt{6}\left[Y^{3/2}\right]_0^2 + \left[\frac{1}{2}Y\sqrt{16-Y^2} + \frac{16}{2}\sin^{-1}\frac{Y}{4}\right]_2^4\right] = \left(\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right) \text{ sq. units}$$

6. Determination of Unknown Parameters:

Illustration 39:

Let $y = f(x)$ is non-negative function such that area bounded by $y = f(x)$, x-axis, $x = 0$ & $x = t$, $t \in \left(0, \frac{\pi}{2}\right)$ is $\sin t - t\cos t\ \forall t \in \left(0, \frac{\pi}{2}\right)$ then find $f\left(\frac{\pi}{4}\right)$

Solution:

$$\sin t - t\cos t = \int_0^t f(x)\,dx$$

On differentiating

$$\cos t - (\cos t - t\sin t) = f(t)$$

$$\cos t - \cos t + t\sin t = f(t)$$

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\sin\frac{\pi}{4} = \frac{\pi}{4}.\frac{1}{\sqrt{2}}$$

Illustration 40:

Find the value of c for which the area of the figure bounded by the curves $y = \frac{4}{x^2}; x = 1$ and $y = c$ is equal to $\frac{9}{4}$.

Solution:

$$A=\int_{2/\sqrt{c}}^{1}\left(c-\frac{4}{x^2}\right)dx$$

$$\frac{9}{4}=\left[cx+\frac{4}{x}\right]_{2/\sqrt{c}}^{1}$$

$$\frac{9}{4}=c+4-2\sqrt{c}-2\sqrt{c}$$

$$0=c-4\sqrt{c}+\frac{7}{4}$$

$$c=\frac{1}{4},\frac{49}{4}$$

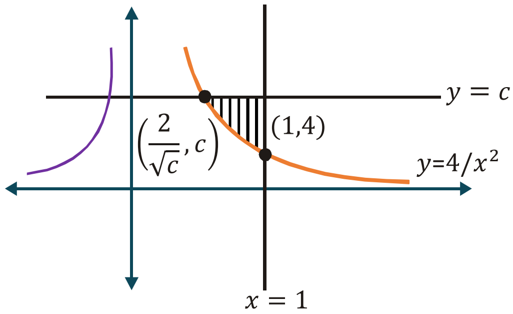


Illustration 41:

Find the area of the figure bounded by the parabola $y=ax^2+12x-14$ and the straight line $y=9x-32$ if the tangent drawn to the parabola at the point $x=3$ is known to make an angle $\pi-\tan^{-1}6$ with the x-axis.

Solution:

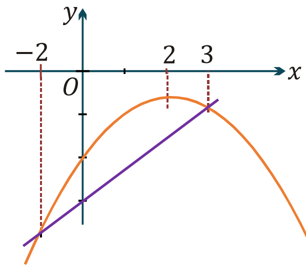
$y=ax^2+12x-14$

$$\frac{dy}{dx}=2ax+12\ ;\ \left.\frac{dy}{dx}\right|_{x=3}=6a+12$$

hence $\tan(\pi-\tan^{-1}6)=6a+12\Rightarrow -6=6a+12\ \Rightarrow a=-3$

hence $y=-3x^2+12x-14$ (note that $D<0, y<0\ \forall\ x\in R$)

point of intersection of the line with parabola are $x=-2$ or 3



Hence $A=\int_{-2}^{3}\left[\left[-3x^2+12x-14\right]-\left(9x-32\right)\right]dx$

$$A=\left(-x^3+6x^2-14x-\frac{9}{2}x^2+32x\right)_{-2}^{3}$$

$$A=\left(-x^3+\frac{3x^2}{2}+18x\right)_{-2}^{3}$$

$$A=-27+\frac{27}{2}+54-(8+6-36)=\frac{125}{2}$$

Illustration 42:

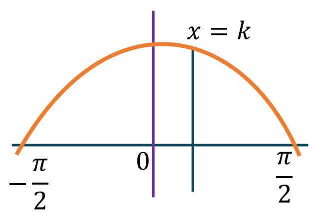
The area bounded by the x-axis and the part of graph of $y=\cos x$, between $x=-\pi/2$ and $x=\pi/2$ separated into two regions by the line $x=k$. If the area of the region for $-\pi/2\le x<k$ is three times the area of the region for $k\le x\le\pi/2$, then $k=$

Solution:

$$\int_{-\pi/2}^{K}\cos x\,dx=3\int_{K}^{\pi/2}\cos x\,dx$$

$$\sin k-\sin\left(-\frac{\pi}{2}\right)=3\left(\sin\frac{\pi}{2}-\sin k\right)$$

$$4\sin k=2\ \Rightarrow\ k=\frac{\pi}{6}$$



Miscellaneous Questions

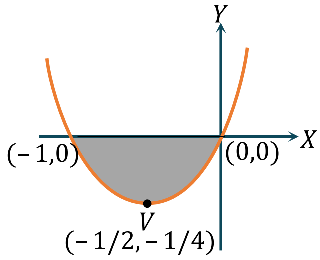
Illustration 43:

Let y be the function which passes through $(1,2)$ having slope $(2x+1)$. The area bounded between the curve and x-axis is

(A) 6 sq. unit (B) 5/6 sq. unit (C) 1/6 sq. unit (D) None of these

Ans. (C)

Solution:



$\frac{dy}{dx}=2x+1 \Rightarrow y = x^2 + x + c$

$\Rightarrow y = x^2 + x$, $[\because c = 0$ by putting $x = 1, y = 2)$

$\Rightarrow \left(x+\frac{1}{2}\right)^2 = y+\frac{1}{4}$, which is a equation of parabola, whose vertices is,

$V\left(\frac{-1}{2},\frac{-1}{4}\right)$

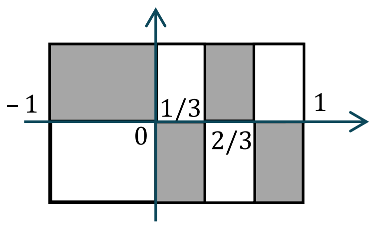
Area bounded between the curve and x-axis

$$=\left|\int_{-1}^{0} y dx\right|=\left|\int_{-1}^{0}(x^2+x)dx\right|=\left|\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{0}\right|=\left|0-\left(\frac{-1}{3}+\frac{1}{2}\right)\right|=\left|-\frac{1}{6}\right|=\frac{1}{6}$$

Illustration 44:

Let $S=\left\{(x,y);\frac{y(3x-1)}{x(3x-2)}<0\right\}$ and $S' = \{(x,y) \in A,B\,; -1 \le A \le 1$ and $-1 \le B \le 1\}$. Then area of $S \cap S'$ is

Solution:



Case-I when $y < 0$, then $\frac{3x-1}{x(3x-2)}>0 \Rightarrow x\in\left(0,\frac{1}{3}\right)\cup\left(\frac{2}{3},\infty\right)$

Case-II : when $y > 0$, then $\frac{(3x-1)}{x(3x-2)}<0 \Rightarrow x\in(-\infty,0)\cup\left(\frac{1}{3},\frac{2}{3}\right)$

Also, $x \in [-1, 1], y \in [-1, 1]$

Area of $S \cap S'$ = Area of shaded region = $1+\frac{1}{3}\times 3=2$

Illustration 45:

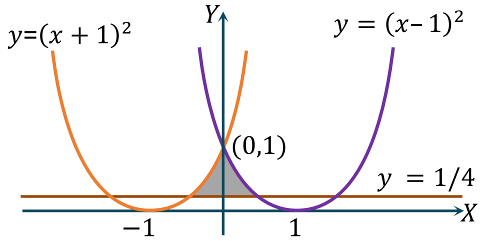
The area bounded by the curve $y = (x+1)^2, y = (x-1)^2$ and the line $y=\frac{1}{4}$ is

(A) 1/6 (B) 2/3 (C) 1/4 (D) 1/3

Ans. (D)

Solution:

Required area $=2\left|\int_{1/4}^{1}(\sqrt{y}-1)\,dy\right|$,(From the symmetry)



On solving, we get required area $=\frac{1}{3}$ sq. unit.

Illustration 46:

Find the smaller of the areas bounded by the parabola $4y^2-3x-8y+7=0$ and the ellipse $x^2+4y^2-2x-8y+1=0$.

Solution:

C_1 is $4(y^2-2y)=3x-7$

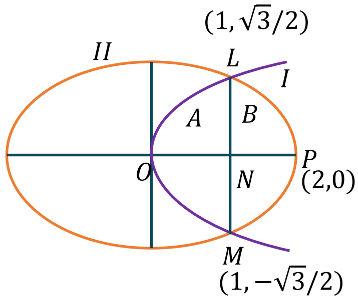
or $4(y-1)^2=3x-3=3(x-1)$ ...(i)

Above is parabola with vertex at (1, 1)

C_2 is $(x^2-2x)+4(y^2-2y)=-1$

or $(x-1)^2+4(y-1)^2=-1+1+4$

or $\frac{(x-1)^2}{2^2}+\frac{(y-1)^2}{1^2}=1$ ...(ii)



Above represents an ellipse with centre at (1, 1). Shift the origin to (1, 1) and this will not affect the magnitude of required area but will make the calculation simpler.

Thus the two curves are

$4Y^2=3X$ and $\frac{X^2}{2^2}+\frac{Y^2}{1}=1$ They meet at $\left(1,\pm\frac{\sqrt{3}}{2}\right)$

Required area = $2(A+B)=2\left[\int Y_1dX+\int Y_2dX\right]$

$=2\left[\frac{\sqrt{3}}{2}\int_0^1\sqrt{X}\,dX+\int_1^2\frac{\sqrt{4-X^2}}{2}dX\right]=\left[\frac{\sqrt{3}}{6}+\frac{2\pi}{3}\right]$ sq.units.

Illustration 47:

Find the equation of line passing through the origin & dividing the curvilinear triangle with vertex at the origin, bounded by the curves $y=2x-x^2$, $y=0$ & $x=1$ in two parts of equal areas.

Solution:

Area of region $OBA=\int_0^1(2x-x^2)dx$

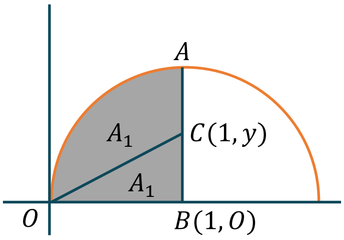
$=\left[x^2-\frac{x^3}{3}\right]_0^1=\frac{2}{3}$

$\frac{2}{3}=A_1+A_1\Rightarrow A_1=\frac{1}{3}$

Let pt. C has coordinates $(1, y)$

Area of $\Delta OCB = \frac{1}{2} \times 1 \times y = \frac{1}{3}$

$\Rightarrow y = \frac{2}{3}$



C has coordinates $\left(1, \frac{2}{3}\right)$

Line OC has slope $m = \frac{\frac{2}{3}-0}{1-0} = \frac{2}{3}$

Equation of line OC is $y = mx \Rightarrow y = \frac{2}{3}x$.

Illustration 48:

The area bounded by $y = x^2 + 1$ and the tangents to it drawn from the origin is :-

(A) 8/3 sq. units (B) 1/3 sq. units (C) 2/3 sq. units (D) none of these

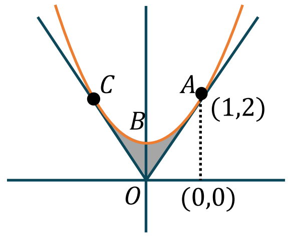
Solution:

The parabola is even function & let the equation of tangent is $y = mx$

Now we calculate the point of intersection of parabola & tangent

$mx = x^2 + 1$

$x^2 - mx + 1 = 0 \Rightarrow D = 0 \Rightarrow m^2 - 4 = 0 \Rightarrow m = \pm 2$



Two tangents are possible $y = 2x$ & $y = -2x$

Intersection of $y = x^2 + 1$ & $y = 2x$ is $x = 1$ & $y = 2$

Area of shaded region

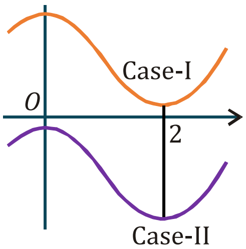
$OAB = \int_0^1 (y_2 - y_1)dx = \int_0^1 ((x^2+1) - 2x)dx = \frac{1}{3}$ sq. units

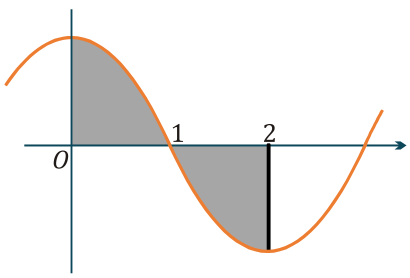
Area of total shaded region = $2\left(\frac{1}{3}\right) = \frac{2}{3}$ sq. units

Illustration 49:

If the area bounded by $f(x) = \frac{x^3}{3} - x^2 + a$ and the straight lines $x = 0$; $x = 2$ and the x-axis is minimum then find the value of a.

Solution:





$f'(x) = x^2 - 2x = x(x-2) = 0$

(note that $f(x)$ is monotonic in (0,2))

Hence for the minimum area and $f(x)$

must cross the x-axis are $\frac{0+2}{2}=1$

Hence $f(1)=\frac{1}{3}-1+a=0$

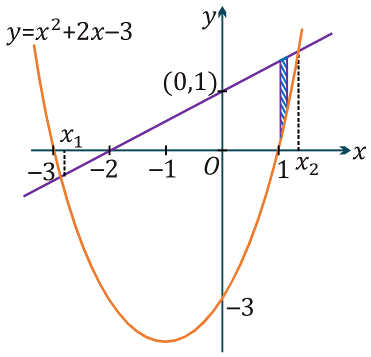
$\Rightarrow a=\frac{2}{3}$

Illustration 50:

If the area bounded by $y = x^2 + 2x - 3$ and the line $y = kx + 1$ is least. Find k and also the least area.

Solution:

x_1 and x_2 are the roots of the equation



$x^2 + 2x - 3 = kx + 1$

$x^2 + (2-k)x - 4 = 0$

$$\left.\begin{aligned} x_1 + x_2 &= k-2 \\ x_1 x_2 &= -4 \end{aligned}\right\}$$

$$A=\int_{x_1}^{x_2}\left[(kx+1)-\left(x^2+2x-3\right)\right]dx$$

$$=\left[(k-2)\frac{x^2}{2}-\frac{x^3}{3}+4x\right]_{x_1}^{x_2}=\left[(k-2)\frac{x_2^2-x_1^2}{2}.\frac{1}{3}\left(x_2^3-x_1^3\right)+4\left(x_2-x_1\right)\right]$$

$$=\left(x_2-x_1\right)\left[\frac{(k-2)^2}{2}-\frac{1}{3}\left(\left(x_2+x_1\right)^2-x_1x_2\right)+4\right]$$

$$=\sqrt{\left(x_2+x_1\right)^2\cdot 4x_1x_2}\left[\frac{(k-2)^2}{2}-\frac{1}{3}\left((k-2)^2+4\right)+4\right]$$

$$=\frac{\sqrt{(k-2)^2+16}}{6}\left[\frac{1}{6}(k-2)^2+\frac{16}{3}\right]=\frac{\left[(k-2)^2+16\right]^{3/2}}{6}$$

A is minimum if $k = 2$. Hence $A_{\min}=\frac{32}{3}$