03 Vector Algebra

Introduction:

Vectors is a useful tool to deal with the problems contained in mechanics, rotational motion and in applied mathematics.

Physical quantities are broadly divided in two categories viz **(a) Vector Quantities** & **(b) Scalar quantities.**

(a) Vector Quantities:

Any quantity, such as velocity, momentum, or force, that has both magnitude and direction and for which vector addition is defined and meaningful; is treated as vector quantities.

Note:

Quantities having magnitude and direction but not obeying the vector law of addition will not be treated as vectors.

For example, the rotations of a rigid body through finite angles have both magnitude & direction but do not satisfy the law of vector addition therefore not a vector.

(b) Scalar Quantities:

A quantity, such as mass, length, time, density or energy, that has size or magnitude but does not involve the concept of direction is called scalar quantity.

Mathematical Description of Vector & Scalar:

To understand vectors mathematically we will first understand directed line segment.

Directed line segment:

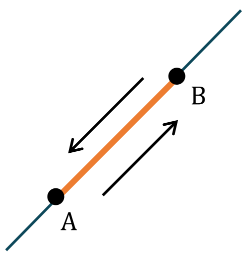
Any given portion of a given straight line where the two end points are distinguished as **Initial** and **Terminal** is called a **Directed Line Segment**.

The directed line segment with initial point A and terminal point B is denoted by the symbol $\overrightarrow{AB}$.

The two end points of a directed line segment are not interchangeable and the directed line segments. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ must be thought of as different.

(a) Vector:

A directed line segment is called vector. Every directed line segment have three essential characteristics.



(i) Length: The length of $\overrightarrow{AB}$ will be denoted by the symbol $|\overrightarrow{AB}|$

Clearly, we have $|\overrightarrow{AB}| = |\overrightarrow{BA}|$

(ii) Support: The line of unlimited length of which a directed line segment is a part is called its line of support or simply the Support.

(iii) Sense: The sense of $\overrightarrow{AB}$ is from A to B and that of $\overrightarrow{BA}$ from B to A so that the sense of a directed line segment is from its initial to the terminal point.

(b) Scalar:

Any real number is a scalar.

Some Special Vectors:

(a) Zero Vector (Null vector):

A vector of zero magnitude i.e. which has the same initial & terminal point, is called a **Zero Vector**. It is denoted by $\vec{O}$ and its direction is arbitrary.

Note:- zero vector has many properties similar to the number zero.

E.g. A boy throwing a ball up and catching it back in his hand, the displacement of the ball is a null vector.

(b) Unit Vector:

A vector of unit magnitude in the direction of vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

Illustration 1:

Find unit vector of

(1) $\vec{A} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

(2) $\vec{A} = a\hat{i} + (b+c)\hat{j} + c\hat{k}$

Solution:

(1) $|\vec{A}| = \sqrt{9+16+25} = \sqrt{50} \Rightarrow$ unit vector: $\dfrac{3\hat{i}+4\hat{j}-5\hat{k}}{\sqrt{50}}$

(2) $|\vec{A}| = \sqrt{a^2+(b+c)^2+c^2} \Rightarrow$ unit vector: $\dfrac{a\hat{i}+(b+c)\hat{j}+c\hat{k}}{\sqrt{a^2+(b+c)^2+c^2}}$

Illustration 2:

Find $\vec{B}$ in direction of $3\hat{i}+4\hat{j}+5\hat{k}$ & $\vec{B}$ has magnitude 3

Solution:

$\vec{B} = (3) \times$ unit vector in given direction

$= 3\left(\dfrac{3\hat{i}+4\hat{j}+5\hat{k}}{5\sqrt{2}}\right)$

(c) Equal Vectors:

Two vectors are said to be equal if they have the same magnitude, same direction & represent the same physical quantity with same or parallel line of support.

(d) Vectors can be Classified as:

(i) Free vectors: Vectors which when transformed into space from one point to another point without affecting their magnitude and direction, can be considered as free vectors i.e. the physical effect produced by them remains unaltered.

e.g. displacement, velocity. (Generally, **IIT-JEE** deals with free vectors only)

(ii) Localized Vector: Vectors which when transformed into space from one point to another point with affecting their magnitude and direction, can be considered as localized vectors i.e. the physical effect produced by them are changed

e.g. force in case of rotational motion (torque will be changed).

Note that:

(i) Number of distinct unit vectors in space perpendicular to a given plane is 2.
(one upward and one downward).

(ii) Number of unit vectors in space parallel to a given plane is infinite.

(iii) Number of distinct unit vectors perpendicular to given line in space. (Infinitely many, think of the line as perpendicular to the xy plane. The unit vector might make any angle θ with the x-axis.)

(iv) Number of distinct unit vectors parallel to a line in space is 2.

(v) Two vectors are equal if they have equal components in an arbitrary direction

(e) Multiplication of Vector by Scalars:

If $\vec{a}$ is a vector & m is a scalar, then $(m\vec{a})$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $|\vec{a}|$. This multiplication is called **Scalar Multiplication**. If $\vec{a}$ and $\vec{b}$ are vectors & m, n are scalars, then:

$$m(\vec{a}) = (\vec{a})m = m\vec{a}$$

$$m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$$

$$(m+n)\vec{a} = m\vec{a} + n\vec{a}$$

$$m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$$

(f) Coplanar Vectors:

A given number of vectors are called coplanar if their line of support are all parallel to the same plane. Note that "**Two Free Vectors Are Always Coplanar**".

(g) Co-initial Vector:

Vectors having the same initial point are called **Co-initial Vectors** and the vectors having the same final point are called **Coterminous Vectors**.

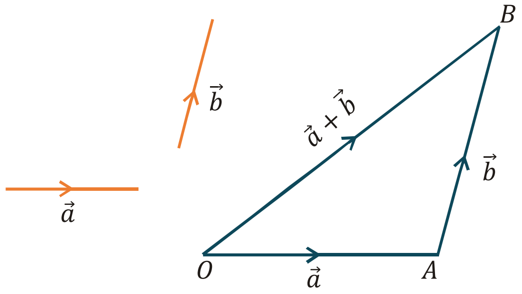
Vector Addition:

The vectors have magnitude as well as direction, therefore their addition is different than addition of real numbers.

Let $\vec{a}$ and $\vec{b}$ be two vectors in a plane, which are represented by $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Their addition can be performed in the following two ways.

(i) Triangle law of vectors: If two vectors are represented in magnitude & direction by two sides of a triangle taken in same order then their sum is represented by the third side taken in reverse order.

Addition of Two Vectors:

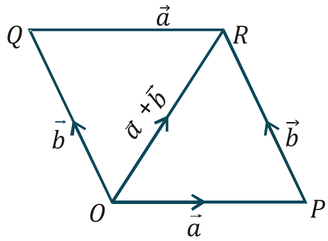


Let $\vec{a}$ and $\vec{b}$ be two given vectors. Take any point O. Let $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{AB}=\vec{b}$, so that the final point of $\vec{a}$ is the initial point of $\vec{b}$. Then the vector $\overrightarrow{OB}$ is called the sum of vectors $\vec{a}$ and $\vec{b}$

Thus $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=\vec{a}+\vec{b}$

This is known as triangle law of vector addition.

(ii) Parallelogram law of addition of vectors: If two vectors be represented in magnitude and direction by the two adjacent sides of a parallelogram then their sum will be represented by the diagonal through the co-initial point.



Let $\vec{a}$ and $\vec{b}$ be vectors drawn from point O denoted by line segments $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Now complete the parallelogram $OPRQ$. Then the vector represented by the diagonal OR will represent the sum of the vectors $\vec{a}$ and $\vec{b}$.

i.e. $\overrightarrow{OP}+\overrightarrow{OQ}=\overrightarrow{OR}$

or $\vec{a}+\vec{b}=\overrightarrow{OR}$

This method of addition of two vectors is called **Parallelogram law of addition of vectors.**

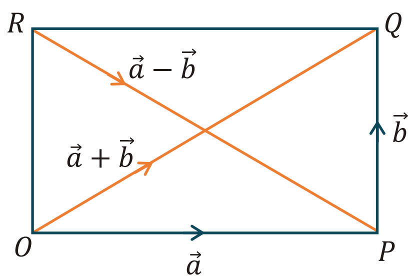
Illustration 3:

If $\vec{a}$, $\vec{b}$ are any two vectors, then give the geometrical interpretation of the relation $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$.

Solution:

Let $\overrightarrow{OP}=\vec{a}$ and $\overrightarrow{PQ}=\vec{b}$ complete the parallelogram $OPQR$. Then

$\overrightarrow{OR}=\vec{b}$ and $\overrightarrow{RQ}=\vec{a}$



From ΔOPQ $\quad \overrightarrow{OQ}=\vec{a}+\vec{b}$, by the vector law of addition

From ΔORP $\quad \overrightarrow{RP}=\vec{a}-\vec{b}$

But $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$

i.e., $|\overrightarrow{OQ}|=|\overrightarrow{RP}|$

i.e., $OQ = RP$

i.e., diagonals of parallelogram are equal.

i.e., $OPQR$ is a rectangle

$OP \perp^{r} OR \Rightarrow \vec{a} \perp^{r} \vec{b}$

Properties of Vector Addition:

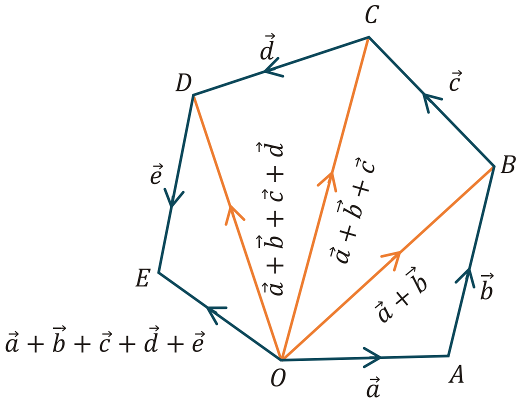
(1) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative)

(2) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associative)

(3) $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$ (additive identity)

(4) $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$ (additive inverse)

Polygon Law of Vector Addition (Addition of More Than Two Vectors):



Addition of more than two vectors is found to be by repetition of triangle law. Suppose we have to find the sum of five vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and $\vec{e}$. If these vectors be represented by line segment $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ and $\overrightarrow{DE}$ respectively, then their sum will be denoted by $\overrightarrow{OE}$. This is the vector represented by rest (last) side of the polygon $OABCDE$ in reverse order. We can also make it clear this way:

By triangle's law

$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$ or $\vec{a}+\vec{b}=\overrightarrow{OB}$

$\overrightarrow{OB}+\vec{c}=\overrightarrow{OC}$ or $(\vec{a}+\vec{b})+\vec{c}=\overrightarrow{OC}$

$\overrightarrow{OC}+\vec{d}=\overrightarrow{OD}$ or $(\vec{a}+\vec{b}+\vec{c})+\vec{d}=\overrightarrow{OD}$

$\overrightarrow{OD}+\vec{e}=\overrightarrow{OE}$ or $(\vec{a}+\vec{b}+\vec{c}+\vec{d})+\vec{e}=\overrightarrow{OE}$

Here, we see that $\overrightarrow{OE}$ is represented by the line segment joining the initial point O of the first vector $\vec{a}$ and the final point of the last vector $\vec{e}$.

In order to find the sum of more than two vectors by this method, a polygon is formed. Therefore, this method is known as the **polygon law of addition.**

Note: If the initial point of the first vector and the final point of the last vector are the same, then the sum of the vectors will be a null vector.

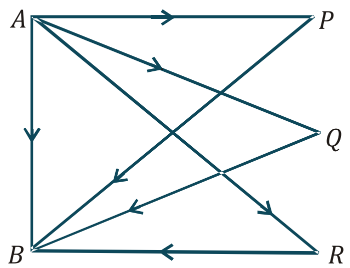
Illustration 4:

A, B, P, Q, R are five points in any plane. If forces $\overrightarrow{AP}, \overrightarrow{AQ}, \overrightarrow{AR}$ acts on point A and force $\overrightarrow{PB}, \overrightarrow{QB}, \overrightarrow{RB}$ acts on point B then resultant is:

(A) $3\overrightarrow{AB}$ (B) $3\overrightarrow{BA}$ (C) $3\overrightarrow{PQ}$ (D) $3\overrightarrow{PR}$

Ans. (A)

Solution:



From figure

$\overrightarrow{AP}+\overrightarrow{PB}=\overrightarrow{AB}$

$\overrightarrow{AQ}+\overrightarrow{QB}=\overrightarrow{AB}$

$\overrightarrow{AR}+\overrightarrow{RB}=\overrightarrow{AB}$

So $(\overrightarrow{AP}+\overrightarrow{AQ}+\overrightarrow{AR}) + (\overrightarrow{PB}+\overrightarrow{QB}+\overrightarrow{RB}) = 3\overrightarrow{AB}$

so required resultant = $3\overrightarrow{AB}$.

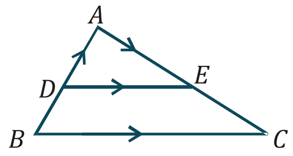
Illustration 5:

Prove that the line joining the middle points of two sides of a triangle is parallel to the third side and is of half its length.

Solution:

Let the middle points of side AB and AC of a ΔABC be D and E respectively.

$\Rightarrow \overrightarrow{BA}=2\overrightarrow{DA}$ and $\overrightarrow{AC}=2\overrightarrow{AE}$



Now in ΔABC, by triangle law of addition

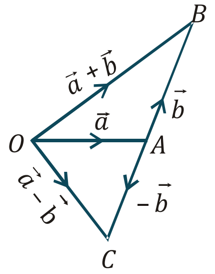
$\Rightarrow \overrightarrow{BA}+\overrightarrow{AC}=\overrightarrow{BC}$

$\Rightarrow 2\overrightarrow{DA}+2\overrightarrow{AE}=\overrightarrow{BC} \;\Rightarrow\; \overrightarrow{DA}+\overrightarrow{AE}=\frac{1}{2}\overrightarrow{BC}$

$\Rightarrow \overrightarrow{DE}=\frac{1}{2}\overrightarrow{BC}$

Hence, line DE is parallel to third side BC of triangle and half of it.

Subtraction of Vectors:



Vector $-\vec{b}$ has length equals to vector $\vec{b}$ but its direction is opposite. Subtraction of vector $\vec{a}$ and $\vec{b}$ is defined as addition of $\vec{a}$ and $(-\vec{b})$. It is written as follows:

$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$

Geometrical Representation:

In the given diagram, $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$. We extend the line AB in opposite direction upto C, where $AB = AC$. The line segment $\overrightarrow{AC}$ will represent the vector $-\vec{b}$. By joining the points O and C, the vector represented by $\overrightarrow{OC}$ is $\vec{a}+(-\vec{b})$. i.e. denotes the vector $\vec{a}-\vec{b}$.

Note:

(i) $\vec{a}-\vec{a}=\vec{a}+(-\vec{a})=\vec{0}$

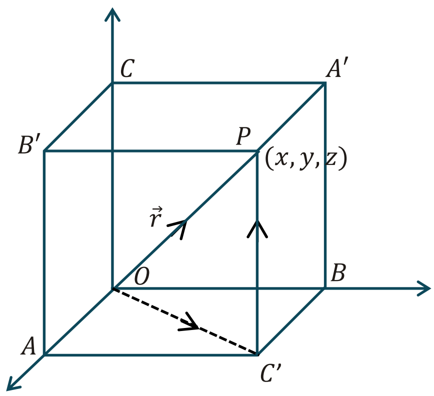
(ii) $\vec{a}-\vec{b}\neq\vec{b}-\vec{a}$

Hence subtraction of vectors does not obey the commutative law.

(iii) $\vec{a}-(\vec{b}-\vec{c})\neq(\vec{a}-\vec{b})-\vec{c}$

i.e. subtraction of vectors does not obey the associative law.

Representation of a Vector in Space in Terms of 3 Orthonormal Triad of Unit Vectors:



Let $P(x,y,z)$ be a point in space with reference to OX, OY and OZ as the coordinate axes, then $OA = x$, $OB = y$ and $OC = z$

Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along OX, OY and OZ respectively, then

$\overrightarrow{OA} = x\hat{i}, \overrightarrow{OB} = y\hat{j}, \overrightarrow{OC} = z\hat{k}$

$\overrightarrow{OP} = \overrightarrow{OC'} + \overrightarrow{C'P} = \overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{OC}$ $\quad [\because \overrightarrow{C'P} = \overrightarrow{OC}]$

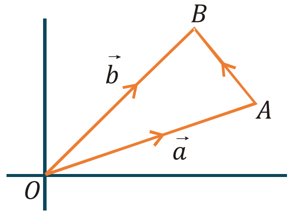
$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = x\hat{i} + y\hat{j} + z\hat{k}$

If $\overrightarrow{OP} = \vec{r}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = \overrightarrow{OP} = \sqrt{x^2 + y^2 + z^2}$

Position Vector:



Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ & $\vec{b}$ are position vectors of two point A and B, then $\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of $B - pv$ of A.

Illustration 6:

If $\vec{a}$ and $\vec{b}$ are two vectors of magnitude 1 inclined at 120°, then the angle between $\vec{b}$ and $\vec{b} - \vec{a}$ is-

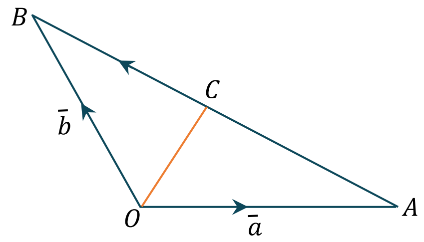
(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{5\pi}{6}$

Ans. (C)

Solution:

$\overrightarrow{OA}=\vec{a},\ \overrightarrow{OB}=\vec{b},\ \overrightarrow{AB}=\vec{b}-\vec{a}\ \angle AOB=120°$

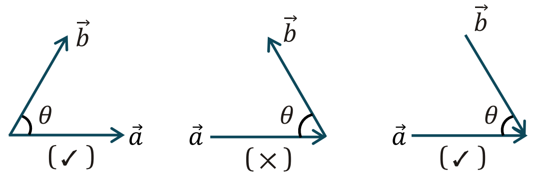
From the diagram, it is clear that



$\angle OBC = 30° = \frac{\pi}{6}$

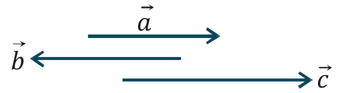
Angle Between Vectors:

Angle between two vectors is angle between head-head or tail-tail. Angle between $\vec{a}$ & $\vec{b}$ is represented by $\vec{a}\wedge\vec{b}=\theta$.



Collinear Vectors:

Two vectors are said to be collinear if their directed line segments are parallel disregard of their direction. Collinear vectors are also called **Parallel Vectors**. If they have the same direction they are named as like vectors otherwise unlike vectors. ($\vec{a}, \vec{b}, \vec{c}$ are collinear)



Note: Symbolically, two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a}=K\vec{b}$, where $K \in R$.

If $K > 0$, like parallel vectors,

$K < 0$, unlike parallel vectors.

Illustration 7:

If $\vec{a}$ and $\vec{b}$ and are non-collinear vectors, then find the value of x for which vectors:

$\vec{\alpha}=(x-2)\vec{a}+\vec{b}$ and $\vec{\beta}=(3+2x)\vec{a}-2\vec{b}$ are collinear.

Solution:

Since the vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear.

$\therefore$ there exists scalar λ such that $\vec{\alpha}=\lambda\ \vec{\beta}$

$\Rightarrow (x-2)\vec{a}+\vec{b}=\lambda\{(3+2x)\vec{a}-2\vec{b}\} \Rightarrow \big(x-2-\lambda(3+2x)\big)\ \vec{a}+(1+2\lambda)\vec{b}=0$

$\Rightarrow x-2-\lambda(3+2x)=0$ and $1+2\lambda=0$

$\Rightarrow x-2-\lambda(3+2x)=0$ and $\lambda=-\frac{1}{2}$

$\Rightarrow\ x-2+\frac{1}{2}\ (3+2x)=0 \Rightarrow 4x-1=0 \Rightarrow x\ =\ \frac{1}{4}$

Illustration 8:

If $A \equiv (2\hat{i}+3\hat{j}), B \equiv (p\hat{i}+9\hat{j})$ and $C \equiv (\hat{i}-\hat{j})$ are collinear, then the value of p is:

(A) 1/2 (B) 3/2 (C) 7/2 (D) 5/2

Ans. (C)

Solution:

$\overrightarrow{AB} = (p-2)\hat{\imath} + 6\hat{\jmath}, \overrightarrow{AC} = -\hat{\imath} - 4\hat{\jmath}$

Now A, B, C are collinear $\Leftrightarrow \overrightarrow{AB} || \overrightarrow{AC} \Leftrightarrow \frac{p-2}{-1} = \frac{6}{-4} \Leftrightarrow p = \frac{7}{2}$

Illustration 9:

The value of λ when $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = 8\hat{i} + \lambda\hat{j} + 4\hat{k}$ are parallel is:

(A) 4 (B) –6 (C) –12 (D) 1

Ans. (C)

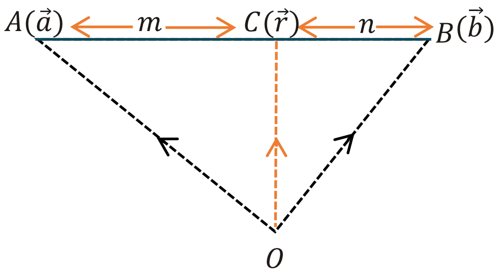
Solution:

Since $\vec{a}$ & $\vec{b}$ are parallel $\Rightarrow \frac{2}{8} = -\frac{3}{\lambda} = \frac{1}{4} \Rightarrow \lambda = -12$

Section Formula:

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point $C(\vec{c})$ which divides AB in the ratio $m : n$ is given by:

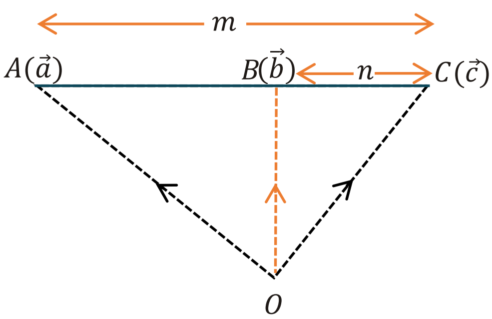
(a) Internal Division:



$$\overrightarrow{OC} = \vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Note: Position vector of mid-point of $AB = \frac{\vec{a}+\vec{b}}{2}$

(b) External division:



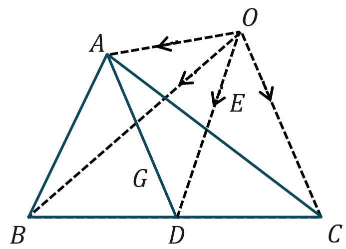
$$\overrightarrow{OC} = \vec{r} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Illustration 10:

Prove that the medians of a triangle are concurrent.

Solution:

Let ABC be a triangle and position vectors of three vertices A, B and C with respect to the origin O be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.



$\therefore\ \overrightarrow{OA}=\vec{a},\ \overrightarrow{OB}=\vec{b},\ \overrightarrow{OC}=\vec{c}$

Again, let D be the middle point of the side BC,

So the position vector of point D is $\overrightarrow{OD}=\dfrac{\vec{b}+\vec{c}}{2}$

Now take a point G, which divides the median AD in the ratio 2 : 1.

Position vector of point G is $\overrightarrow{OG}=\dfrac{1\cdot\overrightarrow{OA}+2\cdot\overrightarrow{OD}}{1+2}=\dfrac{1\cdot\vec{a}+2\cdot\frac{1}{2}\left(\vec{b}+\vec{c}\right)}{1+2}=\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

Similarly, the position vector of the middle points of the other two medians, which divide the medians in the ratio 2 : 1 will comes out to the same $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$, which is the position vector of G.

Hence, the medians of the triangles meet in G i.e. are concurrent.

Illustration 11:

If the middle points of sides BC, CA & AB of triangle ABC are respectively D, E, F then position vector of centroid of triangle DEF, when position vector of A, B, C are respectively $\hat{i}+\hat{j},\ \hat{j}+\hat{k},\ \hat{k}+\hat{i}$ is

(A) $\frac{1}{3}(\hat{i}+\hat{j}+\hat{k})$ (B) $(\hat{i}+\hat{j}+\hat{k})$ (C) $2(\hat{i}+\hat{j}+\hat{k})$ (D) $\frac{2}{3}(\hat{i}+\hat{j}+\hat{k})$

Ans. (D)

Solution:

The position vector of points D, E, F are respectively $\dfrac{\hat{i}+\hat{j}}{2}+\hat{k},\ \hat{i}+\dfrac{\hat{k}+\hat{j}}{2}$ and $\dfrac{\hat{i}+\hat{k}}{2}+\hat{j}$

So, position vector of centroid of ΔDEF = $\dfrac{1}{3}\left[\dfrac{\hat{i}+\hat{j}}{2}+\hat{k}+\hat{i}+\dfrac{\hat{k}+\hat{j}}{2}+\dfrac{\hat{i}+\hat{k}}{2}+\hat{j}\right]=\dfrac{2}{3}[\hat{i}+\hat{j}+\hat{k}]$.

Illustration 12:

$A\left(3\hat{i}-4\hat{j}-4\hat{k}\right)$ $B\left(2\hat{i}-\hat{j}+\hat{k}\right)$ and $C\left(\hat{i}-3\hat{j}-5\hat{k}\right)$. Show that the points are vertices of right-angled triangle and also find:

(i) ortho centre of ΔABC

(ii) circum centre of ΔABC

(iii) centroid of ΔABC

(iv) circumradius of ΔABC

Solution:

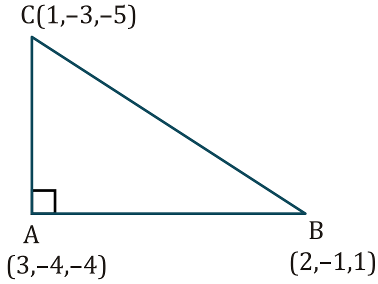
$AB=\sqrt{(1)^2+(3)^2+(5)^2}=\sqrt{35}$

$BC=\sqrt{(-1)^2+(-2)^2+(-6)^2}=\sqrt{41}$

$AC=\sqrt{(-2)^2+(1)^2+(-1)^2}=\sqrt{6}$

$AB^2 + AC^2 = BC^2$

$\therefore$ right angled Δ at A



(i) Orthocenter of $\Delta ABC = A = (3, -4, -4)$

(ii) Circumcenter of ABC : − mid − point of $BC = \left(\frac{3}{2}, -2, -2\right)$

(iii) Centroid $= \left(\frac{3+1+2}{3}, \frac{-4-3-1}{3}, \frac{-4-5+1}{3}\right)$

(iv) Circum radius = $\frac{\text{Hypotenuse}}{2} = \frac{\sqrt{41}}{2}$

Illustration 13:

If $P(\vec{p}), Q(\vec{q}), R(\vec{r})$ and $S(\vec{s})$ be four points such that $3\vec{p} + 8\vec{q} = 6\vec{r} + 5\vec{s}$, then the lines PQ and RS are -

(A) skew (B) intersecting (C) parallel (D) none of these

Ans. (B)

Solution:

Given $3\vec{p} + 8\vec{q} = 6\vec{r} + 5\vec{s}$

$\Rightarrow \frac{3\vec{p}+8\vec{q}}{8+3} = \frac{6\vec{r}+5\vec{s}}{5+6}$

$\Rightarrow$ The point which divides PQ in ratio 8 : 3 is the same as the point which divides RS in the ratio 5 : 6.

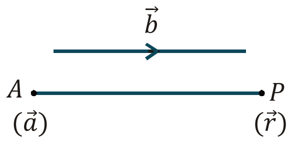
Hence, the line PQ and RS intersect.

Vector Equation of a Line:

(i) Vector Equation of a Straight Line Through a Given Point and Parallel to a Given Vector:

Let the line passes through the given point $A(\vec{a})$ and parallel to the given vector $\vec{b}$.

Taking any point $P(\vec{r})$ on the line.



Then, $P\left(\vec{r}\,\overrightarrow{AP} = t\vec{b}, t \in R\right)$ and is parameter

or, $\vec{r} - \vec{a} = t\vec{b}$

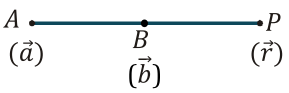
or, $\vec{r} = \vec{a} + t\vec{b}$

This represents the vector equation of the required straight line.

(ii) Vector Equation of a Straight Line Through Two Given Points:

Let $A(\vec{a})$ and $B(\vec{b})$ be two given points. Taking $P(\vec{r})$-be any point on the line.

Since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear, then $\overrightarrow{AP} = t\overrightarrow{AB}, t \in R$



or, $\vec{r} - \vec{a} = t\left(\vec{b} - \vec{a}\right)$

or, $\vec{r} = \vec{a} + t\left(\vec{b} - \vec{a}\right)$

or, $\vec{r} = (1-t)\vec{a} + t\vec{b}$

This represents the vector equation of the required straight line.

Important Note:

(i) Two lines in a plane are either intersecting or parallel, conversely two intersecting or parallel line must be in the same plane.

(ii) However, in space we can have two neither parallel nor intersecting lines. Such non-coplanar lines are known as skew lines.

(iii) If two lines are parallel and have a common point then they are coincident.

Illustration 14:

Find equation of line which is passing through (0, –1, 1) & parallel to the line $\vec{r} = (1, 2, 3) - \lambda(1, 2, -1)$

Solution:

Direction: $-\hat{i} - 2\hat{j} + \hat{k}$

Required line & given line are parallel lines implies they have same direction ratio.

Point: (0, –1, 1) direction: (–1, –2, 1)

equation: $\vec{r} = (0\hat{i} - \hat{j} + \hat{k}) + \lambda(-\hat{i} - 2\hat{j} + \hat{k})$

Illustration 15:

If a line has a vector equation $\vec{r} = 2\hat{\imath} + 6\hat{\jmath} + \lambda\,(\hat{\imath} - 3\hat{\jmath})$, then which of the following statements does not hold good?

(A) the line is parallel to $2\hat{\imath} + 6\hat{\jmath}$

(B) the line passes through the point $3\hat{\imath} + 3\hat{\jmath}$

(C) the line passes through the point $\hat{\imath} + 9\hat{\jmath}$

(D) the line is parallel to XY- plane

Ans. (A)

Solution:

For option (A)

Since $(\hat{\imath} - 3\hat{\jmath})$ is not parallel to $2\hat{\imath} + 6\hat{\jmath}$.

For option (B)

$3\hat{\imath} + 3\hat{\jmath} = 2\hat{\imath} + 6\hat{\jmath} + \lambda\,(\hat{\imath} - 3\hat{\jmath})$

$\Rightarrow \hat{\imath} - 3\hat{\jmath} = \lambda\,(\hat{\imath} - 3\hat{\jmath})$

$\Rightarrow \lambda = 1$

Hence, the line passes through the point $3\hat{\imath} + 3\hat{\jmath}$

For option (C)

$\hat{\imath} + 9\hat{\jmath} = 2\hat{\imath} + 6\hat{\jmath} + \lambda\,(\hat{\imath} - 3\hat{\jmath})$

$\Rightarrow -\hat{\imath} + 3\hat{\jmath} = \lambda\,(\hat{\imath} - 3\hat{\jmath})$

$\Rightarrow \lambda = -1$

Hence, the line passes through the point $\hat{\imath} + 9\hat{\jmath}$

For option (D)

Normal vector to xy plane $= \hat{k}$

Line is parallel to $\hat{\imath} - 3\hat{\jmath}$

Since $\hat{k}.(\hat{\imath} - 3\hat{\jmath}) = 0$

Hence, line is parallel to XY plane.

Illustration 16:

Vector equation of a line passing through points (1, 0, –1) and (2, 5, 1) is:

(A) $\hat{i}-\hat{k}+\lambda(\hat{i}+5\hat{j}+2\hat{k})$ (B) $2\hat{i}+5\hat{j}+\hat{k}+\lambda(\hat{i}-\hat{k})$

(C) $2\hat{i}+5\hat{j}+\hat{k}+\lambda(\hat{i}+5\hat{j}+2\hat{k})$ (D) $\hat{i}-\hat{k}+\lambda(2\hat{i}+5\hat{j}+\hat{k})$

Ans. (A, C)

Solution:

$\vec{r}=\vec{a}+\lambda\vec{b}$

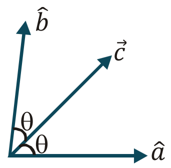
$\vec{b}=2\hat{i}+5\hat{j}+\hat{k}-(\hat{i}-\hat{k})$

$\vec{b}=\hat{i}+5\hat{j}+2\hat{k}$

$\vec{r}=\hat{i}-\hat{k}+\lambda(\hat{i}+5\hat{j}+2\hat{k})$ **or** $\vec{r}=2\hat{i}+5\hat{j}+\hat{k}+\lambda(\hat{i}+5\hat{j}+2\hat{k})$

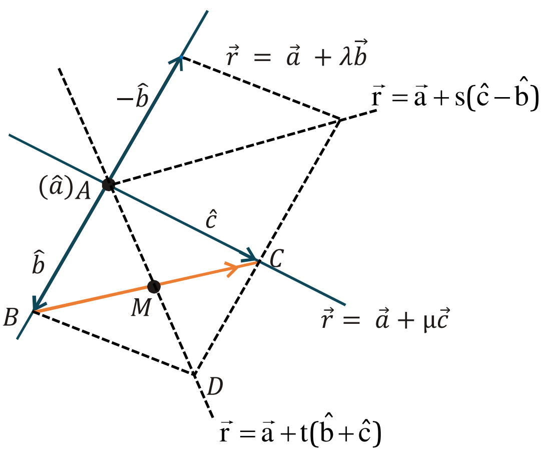
Vector Equation of The Bisectors of the Angles Between The Lines:

Note: **Angle Bisector of Two Vectors:** If $\vec{a}$ & $\vec{b}$ are two vectors & $\vec{c}$ is along their angle bisector then $\vec{c}=\lambda(\hat{a}+\hat{b})$ where $\lambda > 0$



Lines given by $\vec{r}=\vec{a}+\lambda\vec{b}$ and $\vec{r}=\vec{a}+\mu\vec{c}$ has angle bisectors along line given by $\vec{r}=\vec{a}+t(\hat{b}+\hat{c})$ and $\vec{r}=\vec{a}+s(\hat{c}-\hat{b})$, where t & s are parameters.

Proof:



Take $\overrightarrow{AB}=\hat{b}$ and $\overrightarrow{AC}=\hat{c}$ as unit vector and note that $ABDC$ is a rhombus so diagonal is bisecting the angle.

Hence $\overrightarrow{AM}=\dfrac{\hat{b}+\hat{c}}{2}$ $\therefore$ one bisector is $\vec{r}=\vec{a}+t(\hat{b}+\hat{c})$

Similarly, another bisector is $\vec{r}=\vec{a}+s(\hat{c}-\hat{b})$

Illustration 17:

Let O be the centre a regular hexagon $ABCDEF$. Then the magnitude of sum of the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}, \overrightarrow{OE}$ and $\overrightarrow{OF}$ is

(A) $\sqrt{3}$ (B) 0 (C) 2 (D) 1

Ans. (B)

Solution:

We know that the centre of a regular hexagon bisects all the diagonals passing through it.

$\therefore\ \overrightarrow{OA} = -\overrightarrow{OD}, \overrightarrow{OB} = -\overrightarrow{OE}$ and $\overrightarrow{OC} = -\overrightarrow{OF}$

$\Rightarrow\ \overrightarrow{OA} + \overrightarrow{OD} = 0,\ \overrightarrow{OB} + \overrightarrow{OE} = 0$ and $\overrightarrow{OC} + \overrightarrow{OF} = 0$

Hence, $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF}$

$(\overrightarrow{OA} + \overrightarrow{OD}) + (\overrightarrow{OB} + \overrightarrow{OE}) + (\overrightarrow{OC} + \overrightarrow{OF}) = \vec{0}$

Illustration 18:

If S is the circumcentre, O is the orthocentre of ΔABC, then $\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} =$

(A) $\overrightarrow{SO}$ (B) $2\overrightarrow{SO}$ (C) $\overrightarrow{OS}$ (D) $2\overrightarrow{OS}$

Ans. (A)

Solution:

Let $\vec{G}$ is centroid

$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} = \vec{A} - \vec{S} + \vec{B} - \vec{S} + \vec{C} - \vec{S} = \vec{A} + \vec{B} + \vec{C} - 3\vec{S}$

$\Rightarrow\ \overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} = 3\vec{G} - 3\vec{S}$

Now $\vec{G} = \dfrac{\vec{O} + 2\vec{S}}{3}$ (Here $\vec{S}$ is circumcentre and $\vec{O}$ is orthocenter)

$\Rightarrow\ \overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} = \vec{O} + 2\vec{S} - 3\vec{S} = \vec{O} - \vec{S} = \overrightarrow{SO}$

Test of Collinearity:

3 points A, B, C will be collinear if $AB = \lambda BC$,

Necessary and sufficient condition:

Three points A, B, C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that; $x\vec{a} + y\vec{b} + z\vec{c} = 0$, where $x + y + z = 0$.

Illustration 19:

	Vectors	Collinear/non-collinear
1.	$\hat{i} + \hat{j} + \hat{k},\ 3\hat{i} + 3\hat{j} + 3\hat{k}$	Collinear
2.	$2\hat{i} - \hat{j} + \hat{k},\ -4\hat{i} + 2\hat{j} - 2\hat{k}$	Collinear
3.	$2\hat{i} + 4\hat{j} + 6\hat{k},\ 3\hat{i} + 6\hat{j} + 8\hat{k}$	Non-Collinear
4.	$3\hat{i} + 6\hat{j} + 8\hat{k},\ 3\hat{i} - \frac{9}{2}\hat{j} + 6\hat{k}$	Collinear

Illustration 20:

Prove that the points with position vectors $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}+3\hat{j}-4\hat{k}$ & $-7\hat{j}+10\hat{k}$ are collinear.

Solution:

If we find, three scalars ℓ, m & n such that $\ell\vec{a}+m\vec{b}+n\vec{c}=\vec{0}$, where $\ell+m+n=0$ then points are collinear.

$\ell(\hat{i}-2\hat{j}+3\hat{k})+m(2\hat{i}+3\hat{j}-4\hat{k})+n(-7\hat{j}+10\hat{k})=\vec{0}$

$\Rightarrow (\ell+2m)\hat{i}+(-2\ell+3m-7n)\hat{j}+(3\ell-4m+10n)\hat{k}=\vec{0}$

$\Rightarrow \ell+2m = 0, -2\ell+3m-7n = 0, 3\ell-4m+10n = 0$

Solving, we get $\ell = 2, m = -1, n = -1$

since $\ell+m+n = 0$

Hence, the points are collinear.

Alter:

$\overrightarrow{AB}=\vec{b}-\vec{a}=\left(2\hat{i}+3\hat{j}-4\hat{k}\right)-\left(\hat{i}-2\hat{j}+3\hat{k}\right)=\hat{i}+5\hat{j}-7\hat{k}$

$\overrightarrow{BC}=\vec{c}-\vec{b}=\left(-7\hat{j}+10\hat{k}\right)-\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=-2\hat{i}-10\hat{j}+14\hat{k}=-2\left(\hat{i}+5\hat{j}-7\hat{k}\right)$

$\therefore \overrightarrow{AB}=-2\overrightarrow{BC}$

Hence $\vec{a}, \vec{b}$ & $\vec{c}$ are collinear.

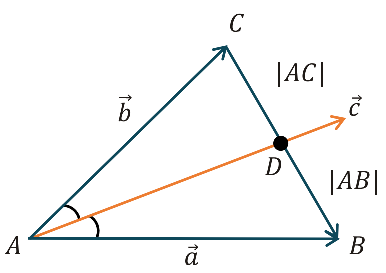
Illustration 21:

The vector $\vec{c}$, directed along the bisector of the angle between the vector $7i-4\hat{j}-4\hat{k}$ and $-2i-\hat{j}+2\hat{k}$ with $|\vec{c}|=5\sqrt{6}$ is -

(A) $\frac{5}{3}(\hat{i}-\hat{j}+2\hat{k})$ (B) $\frac{5}{3}(5\hat{i}+5\hat{j}+2\hat{k})$ (C) $\frac{5}{3}(\hat{i}+7\hat{j}+2\hat{k})$ (D) none of these

Ans. (A)

Solution:



Let $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$

and $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$

Angle bisector of A divides the BC in the ratio of $\left|\overrightarrow{AB}\right|:\left|\overrightarrow{AC}\right|$, $\left|\overrightarrow{AB}\right|=9$, $\left|\overrightarrow{AC}\right|=3$

$\overrightarrow{AD}=\left(\frac{9(-2\hat{i}-\hat{j}+2\hat{k})+3(7\hat{i}-4\hat{j}-4\hat{k})}{9+3}\right)=\frac{\hat{i}-7\hat{j}+2\hat{k}}{4}$

$\vec{c}=\left(\frac{\overrightarrow{AD}}{\left|\overrightarrow{AD}\right|}\right)5\sqrt{6}=\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$ -

Illustration 22:

Let $\vec{a}$ & $\vec{b}$ are non-collinear vectors (linearly independent) and $\left(\sin\theta-\frac{1}{2}\right)\vec{a}+\left(\cos\theta-\frac{\sqrt{3}}{2}\right)\vec{b}=\vec{0}$ then find general solution of θ.

Solution:

It is possible only when,

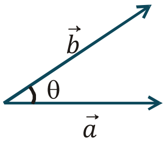
$$\sin\theta - \frac{1}{2} = 0 \quad \& \quad \cos\theta - \frac{\sqrt{3}}{2} = 0$$

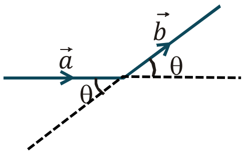
$$\sin\theta = \frac{1}{2} \qquad \cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = 2n\pi + \frac{\pi}{6}$$

Scalar Product of Two Vectors (Dot Product):

Definition: Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ. Then the scalar product of $\vec{a}$ with $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ and is defined as $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta;\ 0 \le \theta \le \pi$.





(a) $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta (0 \le \theta \le \pi)$

Note that if θ is acute then $\vec{a}\cdot\vec{b} > 0$ & if θ is obtuse then $\vec{a}\cdot\vec{b} < 0$

(b) (i) $\vec{a}\cdot\vec{a} = |\vec{a}|^2 = \vec{a}^2$ (ii) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)

(c) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (distributive)

(d) $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b};(\vec{a}, \vec{b} \ne \vec{0})$

(e) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1;\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Simple Identities to Remember Are:

(i) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}^2 - \vec{b}^2$

(ii) $(\vec{a}+\vec{b})^2 = \vec{a}^2 + 2\vec{a}\cdot\vec{b} + \vec{b}^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = (\vec{a})^2 + (\vec{b})^2 + 2|\vec{a}||\vec{b}|\cos\theta$

(iii) $(\vec{a}-\vec{b})^2 = \vec{a}^2 - 2\vec{a}\cdot\vec{b} + \vec{b}^2 = (\vec{a})^2 + (\vec{b})^2 - 2|\vec{a}||\vec{b}|\cos\theta$

(iv) $(\vec{a}+\vec{b})^2 = (\vec{a}-\vec{b})^2 + 4\vec{a}\cdot\vec{b}$

(v) $(\vec{a}+\vec{b}+\vec{c})^2 = \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2\sum(\vec{a}\cdot\vec{b})$

(vi) $\vec{a}\cdot\vec{b} = \frac{1}{4}((\vec{a}+\vec{b})^2 - (\vec{a}-\vec{b})^2)$

General Expression for Dot Product:

If = $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & = $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ &

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\cdot\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Illustration 23:

If $\vec{a}$ and $\vec{b}$ are unit vectors, then which of the following values of $\vec{a}\cdot\vec{b}$ is not possible?

(A) $\sqrt{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{-1}{2}$

Ans. (A)

Solution:

Since $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta=\cos\theta$ and $\cos\theta$ never be equal to $\sqrt{3}$

Illustration 24:

If p^{th}, q^{th}, r^{th} terms of a G.P. are the positive numbers a, b, c then angle between the vectors $\log a^2\hat{i}+\log b^2\hat{j}+\log c^2\hat{k}$ and $(q-r)\hat{i}+(r-p)\hat{j}+(p-q)\hat{k}$ is :

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$

(C) $\sin^{-1}\left(\frac{1}{\sqrt{a^2+b^2+c^2}}\right)$ (D) none of these

Ans. (B)

Solution:

Let x_0 be first term and x the common ratio of the G.P.

$\therefore\ a = x_0\, x^{p-1}, b = x_0\, x^{q-1}, c = x_0\, x^{r-1}$

$\Rightarrow\ \log a = \log x_0 + (p-1)\log x$;

$\log b = \log x_0 + (q-1)\log x$; $\log c = \log x_0 + (r-1)\log x$

If $\vec{a}=\log a^2\hat{i}+\log b^2\hat{j}+\log c^2\hat{k}$ and $\vec{b}=(q-r)\hat{i}+(r-p)\hat{j}+(p-q)\hat{k}$

$\therefore\ \vec{a}\cdot\vec{b}=\Sigma 2(\log a)(q-r)=2\sum(\log x_0+(p-1)\log x)(q-r)=0$

$\Rightarrow\ \vec{a}\wedge\vec{b}=\frac{\pi}{2}.$

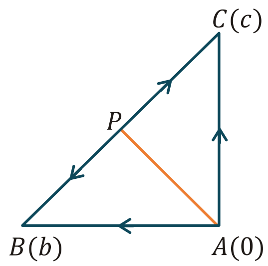
Illustration 25:

Prove that the medians to the hypotenuse of an isosceles triangle is perpendicular to the hypotenuse.

Solution:

The triangle being isosceles, we have

$AB = AC$ ...(i)



Now $\overrightarrow{AP}=\frac{\vec{b}+\vec{c}}{2}$ where P is mid-point of BC.

Also $\overrightarrow{BC}=\vec{c}-\vec{b}$

$\therefore\ \overrightarrow{AP}\cdot\overrightarrow{BC}=\frac{\vec{b}+\vec{c}}{2}\cdot(\vec{c}-\vec{b})=\frac{1}{2}(c^2-b^2)$

$=\frac{1}{2}(AC^2-AB^2)=0$ {by (i)}

$\therefore$ Median AP is perpendicular to base BC.

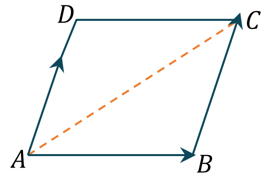
Illustration 26:

The vectors $\overrightarrow{AB}=3\hat{i}+2\hat{j}+2\hat{k}$ & $\overrightarrow{BC}=-\hat{i}-2\hat{k}$ are adjacent sides of a parallelogram then angle between diagonals is

(A) 60° (B) 45° (C) 90° (D) 135°

Ans. (B, D)

Solution:



$\because$ The diagonals $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$ = $2\hat{i}+2\hat{j}$

And $\overrightarrow{BD}=\overrightarrow{BC}-\overrightarrow{AB}=-4\hat{i}-2\hat{j}-4\hat{k}$

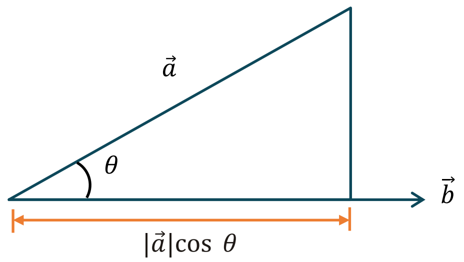
$\therefore$ $|\overrightarrow{AC}|=2\sqrt{2}, |\overrightarrow{BD}|=6$

$\therefore$ Angle between diagonals is

$$\Rightarrow \left|\frac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{|\overrightarrow{AC}||\overrightarrow{BD}|}\right|=\cos\theta \Rightarrow Q = 45° \text{ or } 135°$$

Geometrical Interpretation:

Projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\vec{a}\cdot\hat{b}$. It can be +ve, –ve or zero.



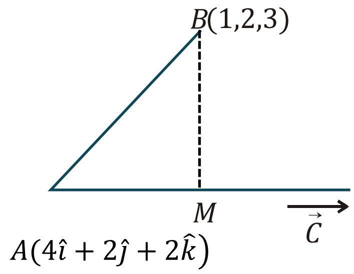
Note that vector component of $\vec{a}$ along $\vec{b}=\left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}=(\vec{a}\cdot\hat{b})\hat{b}$

and vector component of $\vec{a}$ perpendicular to $\vec{b}=\vec{a}-\left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\hat{b}$

Illustration 27:

Find the distance of the point $B(\hat{i}+2\hat{j}+3\hat{k})$ from the line which is passing through $A(4\hat{i}+2\hat{j}+2\hat{k})$ and which is parallel to the vector $\vec{C}=2\hat{i}+3\hat{j}+6\hat{k}$.

Solution:



$AB=\sqrt{3^2+1^2}=\sqrt{10}$

$AM=\overrightarrow{AB}\cdot\hat{c}=(-3\hat{i}+\hat{k})\cdot\frac{(2\hat{i}+3\hat{j}+6\hat{k})}{7}$

$= -6+6=0$

$BM^2 = AB^2 - AM^2$

So, $BM = AB = \sqrt{10}$

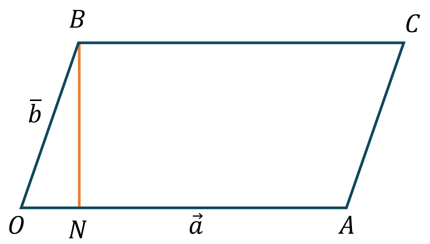
Illustration 28:

$\vec{a}$ and $\vec{b}$ are two given vectors. On these vectors as adjacent sides, a parallelogram is constructed. The vector which is the altitude of the parallelogram and perpendicular to $\vec{a}$ is given by

(A) $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}-\vec{b}$ (B) $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}-\vec{a}$ (C) $\frac{\vec{b}\times(\vec{a}\times\vec{b})}{|\vec{b}|^2}$ (D) $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}+\vec{b}$

Ans. (A)

Solution:



The altitude required is $\overrightarrow{BN}$

$\overrightarrow{BN} = \overrightarrow{ON} - \overrightarrow{OB}$ = projection of $\overrightarrow{OB}$ on $\overrightarrow{OA} - \vec{b}$

$(\vec{b}\cdot\hat{a})\hat{a} - \vec{b} = \left(\vec{b}\cdot\frac{\vec{a}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} - \vec{b} = \frac{(\vec{b}\cdot\vec{a})\vec{a}}{|\vec{a}|^2} - \vec{b}.$

Linear Combination of Vectors:

A vector $\vec{r}$ is said to be a linear combination of the vectors $\vec{a}, \vec{b}, \vec{c}, \ldots$ if there exist scalars $x, y, z\ldots$ such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots$

For example – $\vec{r} = 2\vec{a} + \vec{b} - 4\vec{c}$, $\vec{r} = \vec{a} + 2\vec{b} - 3\vec{c}$

are linear combination of the vectors $\vec{a}, \vec{b}, \vec{c}$.

Fundamental Theorem in Plane:

Let $\vec{a}, \vec{b}$ be non-zero, non-collinear vectors. then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. there exist some unique $x, y \in R$ such that $x\vec{a} + y\vec{b} = \vec{r}$.

Illustration 29:

Find a vector $\vec{c}$ in the plane of $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$ such that $\vec{c}$ is perpendicular to $\vec{b}$ and $\vec{c}\cdot(-2\hat{i} + 3\hat{j} - \hat{k}) = -1$

Solution:

Any vector in the plane of $\vec{a}$ & $\vec{b}$ can be written as $x\vec{a} + y\vec{b}$

let $\vec{c} = x\vec{a} + y\vec{b}$ [by fundamental theorem in plane]

Now, given that

$\vec{c}\cdot\vec{b} = 0 \Rightarrow (x\vec{a} + y\vec{b})\cdot\vec{b} = 0$

$x\vec{a}\cdot\vec{b} + y\vec{b}^2 = 0$

$\Rightarrow x(-2 + 1 - 1) + y(3) = 0$

$-2x + 3y = 0$ ...(i)

Also $(x\vec{a} + y\vec{b})\cdot(-2\hat{i} + 3\hat{j} - \hat{k}) = -1$

$\Rightarrow x\vec{a}\cdot(-2\hat{i} + 3\hat{j} - \hat{k}) + y\vec{b}\cdot(-2\hat{i} + 3\hat{j} - \hat{k}) = -1$

$\Rightarrow x(-4 + 3 + 1) + y(2 + 3 - 1) = -1$

$y = -\frac{1}{4}$

$x = \frac{3y}{2} = -\frac{3}{8}$

Hence the required vector $\vec{c} = -\frac{3}{8}(2\hat{i} + \hat{j} - \hat{k}) - \frac{1}{4}(-\hat{i} + \hat{j} + \hat{k})$

$= \frac{1}{8}[-6\hat{i} - 3\hat{j} + 3\hat{k} + 2\hat{i} - 2\hat{j} - 2\hat{k}] = \frac{1}{8}[-4\hat{i} - 5\hat{j} + \hat{k}]$

Illustration 30:

Let $\vec{a},\vec{b},\vec{c}$ be three vectors such that $\vec{a}\cdot\vec{a}=\vec{b}\cdot\vec{b}=\vec{c}\cdot\vec{c}=3$ and $\left|\vec{a}+\vec{b}-\vec{c}\right|^2+\left|\vec{b}+\vec{c}-\vec{a}\right|^2+\left|\vec{c}+\vec{a}-\vec{b}\right|^2=36$, then

(A) $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\frac{-9}{2}$ (B) $\vec{a},\vec{b},\vec{c}$ are coplanar vectors

(C) $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\frac{-27}{2}$ (D) None of these

Ans. (A,B)

Solution:

$\left|\vec{a}+\vec{b}-\vec{c}\right|^2+\left|\vec{b}+\vec{c}-\vec{a}\right|^2+\left|\vec{c}+\vec{a}-\vec{b}\right|^2=36$

$\Rightarrow 3\left(\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+\left|\vec{c}\right|^2\right)-2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)=36$

$\Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\frac{9}{2}$

Also $\left|\vec{a}+\vec{b}+\vec{c}\right|^2=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+\left|\vec{c}\right|^2+2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)=0$

$\Rightarrow \vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{c}=-\vec{a}-\vec{b}$

Hence $\vec{a},\vec{b},\vec{c}$ are coplanar and represents sides of triangle.

Illustration 31:

If $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$, $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$ and $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=-3(\hat{i}-\hat{k})$. Then the triplet (α,β,γ) is -

(A) (2, –1, –1) (B) (–2, 1, 1) (C) (–2, –1, 1) (D) (2, 1, –1)

Ans. (A)

Solution:

$\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}=-3(\hat{i}-\hat{k})$

$\alpha(1,2,3)+\beta(2,3,1)+\gamma(3,1,2)=-3(1,0,-1)$

$\Rightarrow \alpha+2\beta+3\gamma=-3$

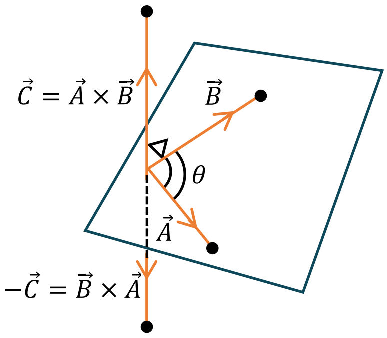
$\Rightarrow 2\alpha+3\beta+\gamma=0$

$\Rightarrow 3\alpha+\beta+2\gamma=3$

$\Rightarrow \alpha=2, \beta=-1, \gamma=-1$

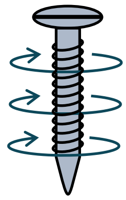
Vector Product of Two Vectors (Cross Product):

(a) If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them, then $\vec{a}\times\vec{b}=\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ forms a right handed screw system.



Sign Convention:

Right Handed Screw System:



$\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system it means that if we rotate vector $\vec{a}$ towards the direction of $\vec{b}$ through the angle θ, then $\hat{n}$ advances in the same direction as a right handed screw would, if turned in the same way.

(b) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ & a vector of magnitude r perpendicular to the plane of $\vec{a}$ & $\vec{b}$ $=\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

(c) If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

(d) Langrange's Identity:

$|\vec{a}\times\vec{b}|$ is very frequently needed for which Lagranges identity is used.

i.e. $\left|\vec{a}\times\vec{b}\right|^2 = a^2b^2-(\vec{a}\cdot\vec{b})^2 = \begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b}\end{vmatrix}$.

Proof:

$\left|\vec{a}\times\vec{b}\right|^2 = a^2\,b^2\sin^2\theta = a^2\,b^2\,(1-\cos^2\theta)$

$= a^2\,b^2 - a^2\,b^2\cos^2\theta = a^2\,b^2 - (\vec{a}.\vec{b})^2$

$= (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})-(\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{b})$

$= \begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$

(e) Properties of Cross Product:

(i) $\vec{a}\times\vec{b}=\vec{0}\Rightarrow\vec{a}=-\lambda\,\vec{b}(\vec{a}\neq 0;\vec{b}\neq 0)$ = i.e. $\vec{a}$ and $\vec{b}$ are Collinear/Linearly dependent however if

$\vec{a}\times\vec{b}=\vec{0}$ and $\vec{a}\cdot\vec{b}=0 \Rightarrow \vec{a}=0$ or $\vec{b}=0$

(ii) $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ (in general) (not commutative)

(iii) $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (distributive to be proved later using triple product)

(iv) $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$ (in general) (vector product is not associative).

(v) $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ and $\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$

(vi) $(\vec{a}\times\vec{b})\cdot\vec{a}=0, (\vec{a}\times\vec{b})\cdot\vec{b}=0$

(f) Expression for $\vec{a}\times\vec{b}$, where $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

Illustration 32:

A unit vector perpendicular to the plane determined by the points $(1,-1,2)$, $(2,0,-1)$ and $(0,2,1)$ is

(A) $\pm\frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k})$ (B) $\frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k})$ (C) $\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+\hat{k})$ (D) $\frac{1}{\sqrt{6}}(2\hat{i}-\hat{j}-\hat{k})$

Ans. (A)

Solution:

$\vec{a}=\hat{i}+\hat{j}-3\hat{k}$, $\vec{b}=-2\hat{i}+2\hat{j}+2\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -3\\ -2 & 2 & 2\end{vmatrix}=8\hat{i}+4\hat{j}+4\hat{k}$$

Hence unit vector $=\pm\frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$.

Illustration 33:

If $\vec{a}=2\hat{i}+3\hat{j}-5\hat{k}$, $\vec{b}=m\hat{i}+n\hat{j}+12\hat{k}$ and $\vec{a}\times\vec{b}=0$ then $(m,n)=$

(A) $\left(-\frac{24}{5},\frac{36}{5}\right)$ (B) $\left(\frac{24}{5},-\frac{36}{5}\right)$ (C) $\left(-\frac{24}{5},-\frac{36}{5}\right)$ (D) $\left(\frac{24}{5},\frac{36}{5}\right)$

Ans. (C)

Solution:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -5\\ m & n & 12\end{vmatrix}$$

$$=(36+5n)\hat{i}-(24+5m)\hat{j}+(2n-3m)\hat{k}=0$$

$$\Rightarrow m=\frac{-24}{5},\ n=\frac{-36}{5}.$$

Illustration 34:

The sine of the angle between the two vectors $3\hat{i}+2\hat{j}-\hat{k}$ and $12\hat{i}+5\hat{j}-5\hat{k}$ will be

(A) $\frac{\sqrt{115}}{\sqrt{14}\sqrt{194}}$ (B) $\frac{51}{\sqrt{14}\sqrt{144}}$ (C) $\frac{\sqrt{64}}{\sqrt{14}\sqrt{194}}$ (D) None of these

Ans. (A)

Solution:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 2 & -1\\ 12 & 5 & -5\end{vmatrix}=-5\hat{i}+3\hat{j}-9\hat{k}$$

$$\Rightarrow \sin\theta=\frac{\sqrt{25+9+81}}{\sqrt{14}\cdot\sqrt{194}}=\frac{\sqrt{115}}{\sqrt{14}\cdot\sqrt{194}}.$$

Illustration 35:

If $|\vec{a}\cdot\vec{b}|=3$ and $|\vec{a}\times\vec{b}|=4$, then the angle between $\vec{a}$ and $\vec{b}$ is –

(A) $\cos^{-1}\frac{3}{4}$ (B) $\cos^{-1}\frac{3}{5}$ (C) $\cos^{-1}\frac{4}{5}$ (D) $\frac{\pi}{4}$

Ans. (B)

Solution:

$|\vec{a}\cdot\vec{b}|=ab\cos\theta=3$ ...(i)

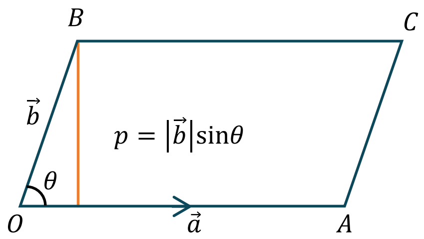
and $|\vec{a}\times\vec{b}|=ab\sin\theta=4$ ...(ii)

Dividing (ii) by (i),

we get $\tan\theta=\frac{4}{3}\Rightarrow\cos\theta=\frac{3}{5}\Rightarrow\theta=\cos^{-1}\frac{3}{5}$.

Geometrical Interpretation:

$|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$ denotes the area of parallelogram whose two adjacent sides are the vectors $\vec{a}$ & $\vec{b}$.



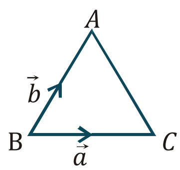
Note:

Area of a parallelogram / quad. if diagonal vectors $\vec{d}_1$ & $\vec{d}_2$ are known is given by $\frac{1}{2}\left|\vec{d}_1\times\vec{d}_2\right|$.

Vector Area of A Plane Triangle:

(i) Area of the triangle (in terms of sides)

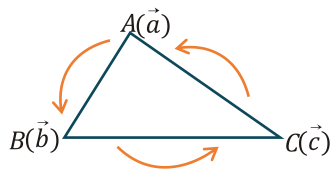
$$=\frac{|\vec{a}\times\vec{b}|}{2}=\frac{1}{2}\left|\overrightarrow{BA}\times\overrightarrow{BC}\right|=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$



(ii) If $\vec{a},\vec{b},\vec{c}$ are the position vectors then the vector area of ΔABC is

$$\vec{\Delta}=\frac{1}{2}[(\vec{c}-\vec{b})\times(\vec{a}-\vec{b})]$$

$$\vec{\Delta}=\frac{1}{2}[(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})]$$



Note: If 3 points with position vectors $\vec{a},\vec{b}$ and $\vec{c}$ are collinear then $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$.

Illustration 36:

Let $ABCD$ is a parallelogram where $\overrightarrow{AB}=\vec{a},\overrightarrow{AD}=\vec{b},|\vec{a}|=|\vec{b}|=2$ and $|\vec{a}\times\vec{b}|+\vec{a}\cdot\vec{b}=\sqrt{2}|\vec{a}||\vec{b}|$ $(\vec{a}\cdot\vec{b}>0)$, then area of this parallelogram, is (in square units)-

(A) $2\sqrt{2}$ (B) 2 (C) $\sqrt{2}$ (D) $8\sqrt{2}$

Ans. (A)

Solution:

$|\vec{a}||\vec{b}|\sin\theta+|\vec{a}||\vec{b}|\cos\theta=\sqrt{2}|\vec{a}||\vec{b}|$

$\Rightarrow \sin\theta+\cos\theta=\sqrt{2}$

$\Rightarrow \theta=\dfrac{\pi}{4}$

Area of $\boldsymbol{ABCD}$ = $|\vec{a}\times\vec{b}|=2\sqrt{2}$

Illustration 37:

If A, B, C, D are any four points in space, then $|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$ is equal to

(A) 2Δ (B) 4Δ (C) 3Δ (D) 5Δ

(Where Δ denotes the area of ΔABC)

Ans.(B)

Solution:

Let A be the origin and let the poisition vectors of B, C and D be $\vec{b},\vec{c}$ and $\vec{d}$ respectively.

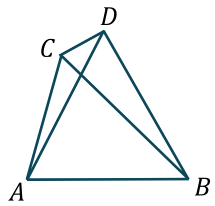
Then $\overrightarrow{AB}=\vec{b}$, $\overrightarrow{CD}=\vec{d}-\vec{c}$, $\overrightarrow{BC}=\vec{c}-\vec{b}$, $\overrightarrow{AD}=\vec{d}$, $\overrightarrow{CA}=-\vec{c}$ and $\overrightarrow{BD}=\vec{d}-\vec{b}$.

$\therefore\ |\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$

$=|\vec{b}\times(\vec{d}-\vec{c})+(\vec{c}-\vec{b})\times\vec{d}-\vec{c}\times(\vec{d}-\vec{b})|$

$=|\vec{b}\times\vec{d}-\vec{b}\times\vec{c}+\vec{c}\times\vec{d}-\vec{b}\times\vec{d}-\vec{c}\times\vec{d}+\vec{c}\times\vec{b}|$

$=|-\vec{b}\times\vec{c}+\vec{c}\times\vec{b}|=|-2(\vec{b}\times\vec{c})|=2|\vec{b}\times\vec{c}|$



=4 (area of triangle ABC).

Illustration 38:

If vertices of a triangle are $A(1,-1,2), B(2,0,-1)$ and $C(0,2,1)$, then the area of a triangle is

(A) $\sqrt{6}$ (B) $2\sqrt{6}$ (C) $3\sqrt{6}$ (D) $4\sqrt{6}$

Ans. (B)

Solution:

Form two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\Delta=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{matrix}\right\|$$

$$=\frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{matrix}\right\|=\frac{1}{2}|8\hat{i}+4\hat{j}+4\hat{k}|$$

$$=\frac{1}{2}\sqrt{64+16+16}=\frac{\sqrt{96}}{2}=2\sqrt{6}.$$

Illustration 39:

The area of a parallelogram whose two adjacent sides are represented by the vector $3\hat{i}-\hat{k}$ and $\hat{i}+2\hat{j}$ is

(A) $\frac{1}{2}\sqrt{17}$ (B) $\frac{1}{2}\sqrt{14}$ (C) $\sqrt{41}$ (D) $\frac{1}{2}\sqrt{7}$

Ans. (C)

Solution:

The area of parallelogram is given by $=|\overrightarrow{AB}\times\overrightarrow{AD}|=\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$

Here we are given adjacent sides and so

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 0 & -1\\ 1 & 2 & 0\end{vmatrix}=2\hat{i}-\hat{j}+6\hat{k}$$

Hence required area is $=|2\hat{i}-\hat{j}+6\hat{k}|=\sqrt{41}$.

Illustration 40:

The area of the parallelogram whose diagonals are $\vec{a}=3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}-3\hat{j}+4\hat{k}$ is

(A) $10\sqrt{3}$ (B) $5\sqrt{3}$ (C) 8 (D) 4

Ans. (B)

Solution:

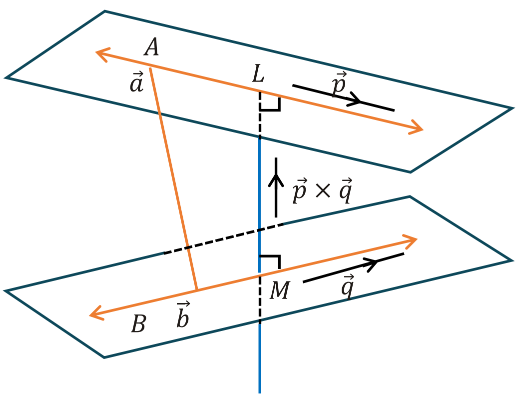
$$\Delta=\frac{1}{2}|\vec{a}\times\vec{b}|$$

But $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -2\\ 1 & -3 & 4\end{vmatrix}=-2\hat{i}-14\hat{j}-10\hat{k}$.

Hence $\Delta=\frac{1}{2}|\vec{a}\times\vec{b}|=\frac{1}{2}\sqrt{4+196+100}=5\sqrt{3}$.

Shortest Distance Between Two Lines:

If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect & are also not parallel are called **skew lines. In other words, the lines which are not coplanar are skew lines**. For Skew lines the direction of the shortest distance vector would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\overrightarrow{AB}$ along the direction of the line of shortest distance, $\overrightarrow{LM}$ is parallel to $\vec{p}\times\vec{q}$



i.e. $\overrightarrow{LM}$ = |Projection of $\overrightarrow{AB}$ on $\overrightarrow{LM}$ | = |Projection of $\overrightarrow{AB}$ on $\vec{p}\times\vec{q}$ |

$$=\left|\frac{\overrightarrow{AB}\cdot(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right|=\left|\frac{(\vec{b}-\vec{a})\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

If S.D. = 0 $\Rightarrow$ lines are intersecting and hence coplanar.

Illustration 41:

Find the shortest distance between the lines $\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$

Ans. $\frac{6}{\sqrt{5}}$ unit

Solution:

We know, the shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda\,\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \lambda\,\vec{b}_2$ is given by

$$d = \left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

Comparing the given equation with the equations $\vec{r} = \vec{a}_1 + \lambda\,\vec{b}_1$ and $r = \vec{a}_2 + \lambda\,\vec{b}_2$ respectively,

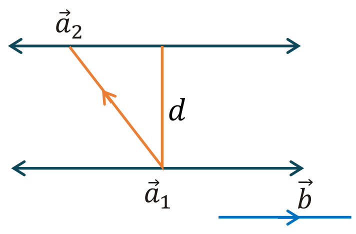
we have $\vec{a}_1 = 4\hat{i} - \hat{j}, \vec{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b}_2 = 2\hat{i} + 4\hat{j} - 5\hat{k}$

Now, $\vec{a}_2 - \vec{a}_1 = -3\hat{i} + 0\hat{j} + 2\hat{k}$ and $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\hat{i} - \hat{j} + 0\hat{k}$

$\therefore\ (\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = (-3\hat{i} + 0\hat{j} + 2\hat{k})\cdot(2\hat{i} - \hat{j} + 0\hat{k}) = -6$ and $|\vec{b}_1 \times \vec{b}_2| = \sqrt{4+1+0} = \sqrt{5}$

$\therefore$ Shortest distance $d = \left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \left|\frac{-6}{\sqrt{5}}\right| = \frac{6}{\sqrt{5}}$.

Shortest Distance Between Two Parallel Lines:



If two lines are given by $\vec{r}_1 = \vec{a}_1 + K_1\vec{b}$ & $\vec{r}_2 = \vec{a}_2 + K_2\vec{b}$ i.e. they are parallel then, $d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$

Illustration 42:

Find the shortest distance between lines $\vec{r} = (\hat{\imath} + 2\hat{\jmath} + \hat{k}) + \lambda(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$ and
$\vec{r} = (2\hat{\imath} - \hat{\jmath} - \hat{k}) + \mu(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$

Ans. $\left(\frac{\sqrt{101}}{3}\right)$

Solution:

Line (i) & (ii) are passing through the points $\vec{a}_1 = (\hat{\imath} + 2\hat{\jmath} + \hat{k})$ and $\vec{a}_2 = (2\hat{\imath} - \hat{\jmath} - \hat{k})$ respectively and are parallel to the vector $\vec{b} = (2\hat{\imath} + \hat{\jmath} + 2\hat{k})$. Hence the distance between the lines using the formula given by

$$\frac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|} = \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 1 & -3 & -2 \end{vmatrix}}{3} = \frac{|4\hat{i} - 6\hat{j} - 7\hat{k}|}{3} = \frac{\sqrt{16+36+49}}{3} = \frac{\sqrt{101}}{3}$$

Geometrical Results with Vectors & Problems:

(a) Tetrahedron:

(i) Lines joining the vertices of a tetrahedron (a pyramid on a triangular base) to the centroids of the opposite faces are concurrent and this point (P) of concurrecy with P.V. $\vec{g}=\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$ is called the centre of the tetrahedron.

(ii) In a tetrahedron, straight lines joining the mid points of each pair of opposite edges are also concurrent at the centre of the tetrahedron.

(iii) According to diagram AB, CD & AC, BD & AD, BC are pairs of opposite edges.

(b) Parallelepiped:

Four diagonals of any parallelepiped (A prism whose base is a $||^{gm}$) and the join of the mid point of each pair of opposite edges are concurrent and are bisected at the point of concurrence. (See the adjacent figure) P is called the center of the parallelepiped with P.V.

$$\frac{\vec{a}+\vec{b}+\vec{c}}{2}$$

i.e. $\dfrac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{2}$

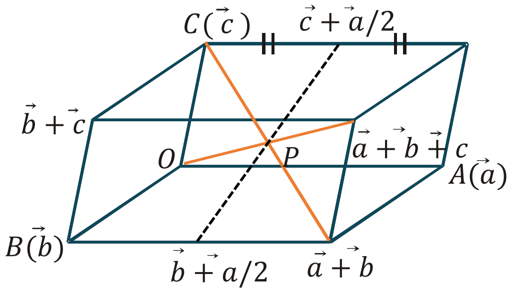
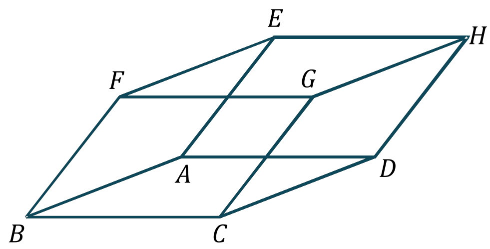


Illustration 43:

Figure shows a parallelepiped with $AB=3, AD=5, AE=7$ Distance between centre of parallelepiped & centre of the tetrahedron $FAHC$ is



Solution:

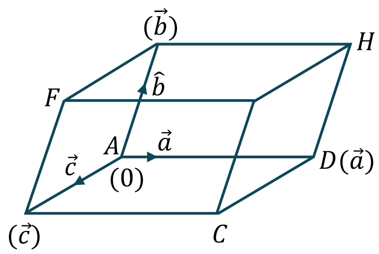
centre of || piped = $\dfrac{\vec{a}+\vec{b}+\vec{c}}{2}$

$F\left(\vec{b}+\vec{c}\right), A\left(\vec{0}\right), H\left(\vec{a}+\vec{b}\right), C\left(\vec{a}+\vec{c}\right)$

centre of tetrahedron

$$=\frac{\left(\vec{b}+\vec{c}\right)+\left(\vec{0}\right)+\left(\vec{a}+\vec{b}\right)+\left(\vec{a}+\vec{c}\right)}{4}$$

$$=\frac{\vec{a}+\vec{b}+\vec{c}}{2}$$



Since centre of parallelepiped and centre of tetrahedron is same therefore distance is 0.

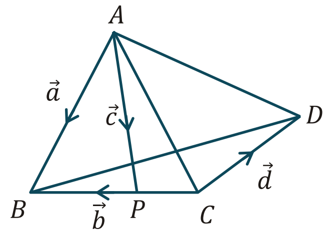
Illustration 44:

$ABCD$ is a tetrahedron with $\overrightarrow{AB}=\vec{a}$, $\overrightarrow{PB}=\vec{b}$, $\overrightarrow{AP}=\vec{c}$, $\overrightarrow{CD}=\vec{d}$ where P is the foot of perpendicular from A lying inside plane of ΔBCD. Which of the following statement(s) is/are always correct?

(A) $\vec{d}\times\vec{b}$ is parallel to $\vec{c}$.
(B) $\vec{d}\times\vec{b}$ is not parallel to $\vec{c}$.
(C) $(\vec{a}\times\vec{c})\cdot\vec{d}=0$
(D) $(\vec{a}\times\vec{c})\cdot\vec{d}\neq 0$

Ans. (A,D)

Solution:



$\overrightarrow{CD}\times\overrightarrow{PB}$ is normal to plane BCD

$\therefore\ \vec{d}\times\vec{b}\,||\,\vec{c}$

$\vec{a}\times\vec{c}$ is not perpendicular to $\vec{d}$

$\therefore\ (\vec{a}\times\vec{c})\cdot\vec{d}\neq 0$

Illustration 45:

Let $\overrightarrow{AB}=\hat{i}+\hat{j}-3\hat{k}$, $\overrightarrow{AC}=3\hat{i}+\hat{j}+4\hat{k}$ & $\overrightarrow{AD}=2\hat{j}-\hat{k}$ are three co-terminus edges of a tetrahedron $ABCD$. If position vector of center of tetrahedron is $\left(\hat{i}+2\hat{j}+3\hat{k}\right)$, then-

(A) Position vector of point A is $\left(\hat{j}+3\hat{k}\right)$

(B) Position vector of point A is $\left(\hat{i}+3\hat{k}\right)$

(C) Length of median drawn through vertex 'A' in ΔABC is $\frac{\sqrt{21}}{4}$

(D) Length of median drawn through vertex 'A' in ΔABC is $\frac{\sqrt{21}}{2}$

Ans. (A,D)

Solution:

$$\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}=\vec{g} \quad ...(1)$$

$\vec{b}-\vec{a}=\hat{i}+\hat{j}-3\hat{k}$

$\vec{c}-\vec{a}=3\hat{i}+\hat{j}+4\hat{k}$

$\vec{d}-\vec{a}=2\hat{j}-\hat{k}$

$$\vec{b}+\vec{c}+\vec{d}-3\vec{a}=4\hat{i}+4\hat{j} \quad ...(2)$$

putting value of eq. (2) in eq. (1);

$$\Rightarrow \frac{4\vec{a}+4\hat{i}+4\hat{j}}{4}=\hat{i}+2\hat{j}+3\hat{k}$$

$\Rightarrow A(\vec{a})=\hat{j}+3\hat{k}$

$B(\vec{b})=\hat{i}+2\hat{j}$

$C(\vec{c})=3\hat{i}+2\hat{j}+7\hat{k}$

$D(\vec{d})=3\hat{j}+2\hat{k}$

length of median $=\left|\frac{\overrightarrow{AB}+\overrightarrow{AC}}{2}\right|=\left|2\hat{i}+\hat{j}+\frac{\hat{k}}{2}\right| =\frac{\sqrt{21}}{2}$

Product of 3 vector:

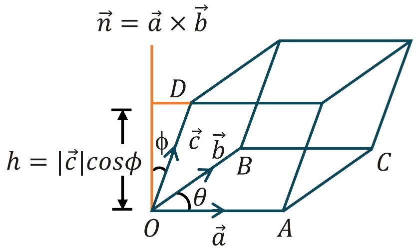
When 3 vectors are involved with a dot or a cross between them, then 6 different symbols are

(i) $(\vec{a}\cdot\vec{b})\vec{c}$ (ii) $(\vec{a}\cdot\vec{b})\cdot\vec{c}$ (iii) $(\vec{a}\cdot\vec{b})\times\vec{c}$ (iv) $(\vec{a}\times\vec{b})\vec{c}$ (v) $(\vec{a}\times\vec{b})\cdot\vec{c}$ (vi) $(\vec{a}\times\vec{b})\times\vec{c}$

First is scalar multiple of a vector; (ii), (iii), (iv) are meaningless (v) and (vi) are scalar and vector triple product respectively.

Scalar Triple Product / Box Product / Mixed Product:

(i) The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as : $(\vec{a}\times\vec{b})\cdot\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product.



(ii) Scalar triple product geometrically represents the volume of the parallelepiped whose three coterminous edges are represented by $\vec{a}$, $\vec{b}$ & $\vec{c}$ i.e. $V = [\vec{a}\,\vec{b}\,\vec{c}]$

(iii) In a scalar triple product the position of dot & cross can be interchanged i.e. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$

(iv) $\vec{a}\cdot(\vec{b}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$ If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$; $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ &

$\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ In general, if $\vec{a} = a_1\vec{l}+a_2\vec{m}+a_3\vec{n}$; $\vec{b} = b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$

$\vec{c} = c_1\vec{l}+c_2\vec{m}+c_3\vec{n}$ then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\vec{l}\ \vec{m}\ \vec{n}]$; where $\vec{l}, \vec{m}$ & $\vec{n}$ are non-coplanar vectors.

(v) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar $[\vec{a}\,\vec{b}\,\vec{c}] = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

(vi) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Note: If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non- coplanar then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

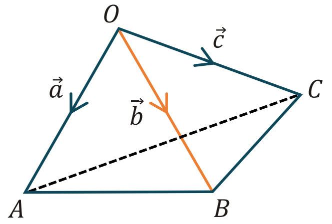
(vii) $[\hat{i}\ \hat{j}\ \hat{k}] = 1$

(viii) $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$

(ix) $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}]+[\vec{b}\ \vec{c}\ \vec{d}]$

(x) The Volume of the tetrahedron $OABC$ with O as origin & the pv's of A, B and C being $\vec{a}$, $\vec{b}$ & $\vec{c}$ are given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

The position vector of the centroid of a tetrahedron if the P.V. of its angular vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ given by $\frac{1}{4}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$



Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

(xi) $[\vec{a}-\vec{b} \quad \vec{b}-\vec{c} \quad \vec{c}-\vec{a}]=0$ & $[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}] = 2[\vec{a} \quad \vec{b} \quad \vec{c}]$

Illustration 46:

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors such that $\vec{b}\times\vec{c}=\vec{a}$, $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{c}\times\vec{a}=\vec{b}$ then -

(A) $[\vec{a}\,\vec{b}\,\vec{c}]=1$ (B) $[\vec{a}\,\vec{b}\,\vec{c}]\neq 1$ (C) $|\vec{a}| + |\vec{b}| + |\vec{c}| = 3$ (D) None of these

Ans. (A,C)

Solution:

$\vec{b} \times \vec{c} = \vec{a} \Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = \vec{a}\cdot\vec{a} = |\vec{a}|^2$

similarly, $\vec{a} \times \vec{b} = \vec{c} \Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = |\vec{c}|^2$

$\vec{c} \times \vec{a} = \vec{b} \Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = |\vec{b}|^2$

Also, $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\times\vec{c} \quad \vec{c}\times\vec{a} \quad \vec{a}\times\vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 1$

$\Rightarrow |\vec{a}| + |\vec{b}| + |\vec{c}| = 3$

Illustration 47:

Let $\vec{a}= a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$, $\vec{b}=b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}$ and $\vec{c}=c_1\hat{\imath}+c_2\hat{\jmath}+c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to:

Ans. $\frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

Solution:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = [\vec{a} \quad \vec{b} \quad \vec{c}]^2 = ((\vec{a}\times\vec{b})\cdot\vec{c})^2 = (|\vec{a}||\vec{b}|\sin\theta\,\vec{c}\cdot\vec{c})^2 = \frac{|\vec{a}|^2|\vec{b}|^2}{4}$$

$= \frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

Illustration 48:

If $\vec{a} = \hat{\imath}\,\hat{\jmath}\,\hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\vec{c} = 3\hat{\imath} + p\hat{\jmath} + 5\hat{k}$ are coplanar then the value of p will be

(A) –6 (B) –2 (C) 2 (D) 6

Ans. (A)

Solution:

Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors

$$\therefore \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & p & 5 \end{vmatrix} = 0 \Rightarrow p = -6.$$

Illustration 49:

Volume of the parallelepiped whose coterminous edges are $2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}, \hat{\imath} + 2\hat{\jmath} - 2\hat{k}, 3\hat{\imath} - \hat{\jmath} + \hat{k}$, is

(A) 5 cubic unit (B) 6 cubic unit (C) 7 cubic unit (D) 8 cubic unit

Ans. (C)

Solution:

$$V = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -2 \\ 3 & -1 & 1 \end{vmatrix} = |-7| = 7 \text{ cubic unit.}$$

Illustration 50:

If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $[\vec{a}\ \vec{b}\ \vec{c}] = 4$, then $\left[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\right]$ =

(A) 16 (B) 64 (C) 4 (D) 8

Ans. (A)

Solution:

$[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = (\vec{a}\times\vec{b})\cdot[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]$

$= (\vec{a}\times\vec{b})\cdot([\vec{b}\ \vec{c}\ \vec{a}]\vec{c} - [\vec{b}\ \vec{c}\ \vec{c}]\vec{a}) = (\vec{a}\times\vec{b})\cdot([\vec{b}\ \vec{c}\ \vec{a}]\vec{c} - 0)$

$= [\vec{b}\ \vec{c}\ \vec{a}][\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}] = 4.4 = 16.$

Illustration 51:

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors and λ is a real number, then the vectors

$\vec{a} + 2\vec{b} + 3\vec{c}, \lambda\vec{b} + 4\vec{c}$ and $(2\lambda - 1)\vec{c}$ are non-coplanar for

(A) No value of λ (B) All except one value of λ

(C) All except two values of λ (D) All values of λ

Ans. (C)

Solution:

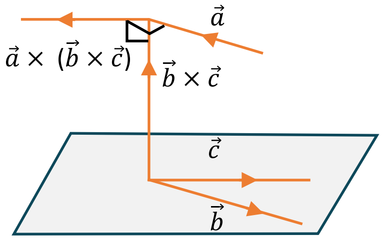
For coplanarity, $\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & 2\lambda - 1 \end{vmatrix} = 0 \Rightarrow \lambda = 0, \frac{1}{2}$

$\therefore$ All values except two values of $\lambda = 0, \frac{1}{2}$.

Vector Triple Product:

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then the expression $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector & is called **vector triple product**.

Geometrical interpretation of $\vec{a}\times(\vec{b}\times\vec{c})$



Consider the expression $\vec{a}\times(\vec{b}\times\vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ & $(\vec{b}\times\vec{c})$.

Now $\vec{a}\times(\vec{b}\times\vec{c})$ is vector perpendicular to the plane containing $\vec{a}$ & $(\vec{b}\times\vec{c})$ but $(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane $\vec{b}$ & $\vec{c}$, therefore $\vec{a}\times(\vec{b}\times\vec{c})$ is vector lies in the plane of $\vec{b}$ & $\vec{c}$ and perpendicular to $\vec{a}$. Hence, we can express $\vec{a}\times(\vec{b}\times\vec{c})$ in terms of $\vec{b}$ & $\vec{c}$ i.e. $\vec{a}\times(\vec{b}\times\vec{c})=x\vec{b}+y\vec{c}$ where x & y are scalars.

(a) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

(b) $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$

(c) $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$

Illustration 52:

If $\vec{a}=\hat{i}+2\hat{j}+4\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+\hat{k}$, $\vec{c}=\hat{i}+4\hat{j}-4\hat{k}$, then the vector $\vec{a}\times(\vec{b}\times\vec{c})$ is orthogonal to -

(A) $\vec{a}$ (B) $\vec{b}$ (C) $\vec{c}$ (D) $\vec{a}+\vec{b}+\vec{c}$

Ans. (A,D)

Solution:

$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$\vec{a}\times(\vec{b}\times\vec{c})=-7\vec{b}$ $(\because \vec{a}\cdot\vec{b}=0)$

and $\vec{b}$ is orthogonal to $\vec{a}$.

$\Rightarrow \vec{a}\times(\vec{b}\times\vec{c})$ is orthogonal to $\vec{a}$.

Also, $\vec{a}+\vec{b}+\vec{c}=4\hat{\imath}+3\hat{\jmath}+\hat{k}$ is also orthogonal to $\vec{b}$.

Hence, $\vec{a}+\vec{b}+\vec{c}$ is orthogonal to $\vec{a}\times(\vec{b}\times\vec{c})$.

Illustration 53:

If $|\hat{a}|=|\hat{b}|=|\hat{c}|=1$ such that $\hat{a}\times(\hat{b}\times\hat{c})=\frac{1}{2}\hat{b}$ ($\hat{b}$ and $\hat{c}$ are not collinear) then -

(A) $\vec{a}$ and $\vec{c}$ are perpendicular

(B) the angle between $\hat{a}$ and $\hat{c}$ is 30°

(C) the angle between $\hat{a}$ and $\hat{c}$ is 60°

(D) $\vec{a}$ and $\vec{b}$ are perpendicular

Ans. (C,D)

Solution:

$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$

$\Rightarrow (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c} = \frac{1}{2}\vec{b}$

$\Rightarrow (\vec{a} \cdot \vec{c} - \frac{1}{2})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = 0$

Since $\vec{b}$ and $\vec{c}$ are not collinear

$\Rightarrow \vec{a} \cdot \vec{c} - \frac{1}{2} = 0, \vec{a} \cdot \vec{b} = 0$

$\therefore$ Choices (C) and (D) become correct.

Illustration 54:

If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c}$ is a unit vector such that $\vec{c} \cdot \vec{a} = 0$, $[\vec{c}\,\vec{a}\,\vec{b}] = 0$ then a unit vector $\vec{d}$ perpendicular to both $\vec{a}$ and $\vec{c}$ is

(A) $\frac{1}{\sqrt{6}}\left(2\hat{i} - \hat{j} + \hat{k}\right)$ (B) $\frac{1}{\sqrt{2}}\left(\hat{j} + \hat{k}\right)$

(C) $\frac{1}{\sqrt{2}}\left(\hat{i} + \hat{j}\right)$ (D) $\frac{1}{\sqrt{2}}\left(\hat{i} + \hat{k}\right)$

Ans.(B)

Solution:

$\vec{c}$ is along the vector $\vec{a} \times (\vec{a} \times \vec{b})$

$\vec{a} \times (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b}$

$= (-1)(\hat{\imath} + \hat{\jmath} - \hat{k}) - 3(\hat{\imath} - \hat{\jmath} + \hat{k}) = -4\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

$\Rightarrow \vec{c} = \frac{-2\hat{i} + \hat{j} - \hat{k}}{\sqrt{6}} \Rightarrow \vec{d} = \frac{(\vec{a} \times \vec{c})}{|\vec{a} \times \vec{c}|}$; $\vec{a} \times \vec{c} = \begin{vmatrix} i & j & \hat{k} \\ 1 & 1 & -1 \\ -2 & 1 & -1 \end{vmatrix}$

$= -\hat{\jmath}(-3) + \hat{k} \cdot 3 = 3(\hat{\jmath} + \hat{k}) \Rightarrow \vec{d} = \frac{\hat{j} + \hat{k}}{\sqrt{2}}$

Illustration 55:

If $\vec{a}$, $\vec{b}$, $\vec{c}$ be the unit vectors such that $\vec{b}$ is not parallel to $\vec{c}$ and $\vec{a} \times \left(2\vec{b} \times \vec{c}\right) = \vec{b}$, then the angle that $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ are respectively:

(A) $\frac{\pi}{3}$ & $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ & $\frac{2\pi}{3}$ (C) $\frac{\pi}{2}$ & $\frac{2\pi}{3}$ (D) $\frac{\pi}{2}$ & $\frac{\pi}{3}$

Ans.(D)

Solution:

$\vec{a} \times (2\vec{b} \times \vec{c}) = \vec{b} \Rightarrow 2\left[(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}\right] = \vec{b}$

$\Rightarrow \vec{a} \cdot \vec{c} = \frac{1}{2}$ & $\vec{a} \cdot \vec{b} = 0$

Illustration 56:

If $\vec{a}$ & $\vec{b}$ are two non collinear vector $\vec{a}\cdot\vec{b}\neq 0$ and

$\underbrace{\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}.............\times(\vec{a}}_{2018\ \text{times}}\times(\vec{a}\times\vec{b})) = \lambda(\vec{a}\times\vec{b})$. Find λ.

Ans. $-|\vec{a}|^{2018}$

Solution:

$$\vec{a}\times(\vec{a}\times\vec{b}) = (\vec{a}\cdot\vec{b})\,\vec{a} - (\vec{a}\cdot\vec{a})\,\vec{b}$$

$$(\vec{a}\times(\vec{a}\times(\vec{a}\times\vec{b}))) = -|\vec{a}|^2\,(\vec{a}\times\vec{b})$$

$$(\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}\times\vec{b})))) = -|\vec{a}|^2\,((\vec{a}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{a})\vec{b})$$

$$(\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}\times\vec{b}))))) = |\vec{a}|^4\,(\vec{a}\times\vec{b})$$

Similarly, $\underbrace{\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}.............\times(\vec{a}}_{2018\ \text{times}}\times(\vec{a}\times\vec{b})) = -|\vec{a}|^{2018}\,(\vec{a}\times\vec{b})$

Product of More Than 3 Vectors:

(i) Scalar Product of Four vectors:

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}) = \begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}$$

(ii) Vector Product of Four vectors:

$\vec{V}=(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ is:

(1) Coplanar with $\vec{a},\vec{b}$

(2) Coplanar with $\vec{c},\vec{d}$

(3) parallel to the line of intersection of plane containing $\vec{a},\vec{b}$ and $\vec{c},\vec{d}$.

$\vec{V} = (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

Let $\vec{u} = \vec{a}\times\vec{b}$

then $\vec{u}\times(\vec{c}\times\vec{d}) = [\vec{a}\vec{b}\vec{d}]\vec{c} - [\vec{a}\vec{b}\vec{c}]\vec{d}$ ...(i)

again $\vec{V} = (\vec{a}\times\vec{b})\times(\underbrace{\vec{c}\times\vec{d}}_{\vec{v}}) = (\vec{a}\cdot\vec{v})\vec{b} - (\vec{b}\cdot\vec{v})\vec{a} = [\vec{a}\vec{c}\vec{d}]\vec{b} - [\vec{b}\vec{c}\vec{d}]\vec{a}$ ...(ii)

Nowfrom (i) and (ii)

$[\vec{a}\vec{b}\vec{d}]\vec{c} - [\vec{a}\vec{b}\vec{c}]\vec{d} = [\vec{a}\vec{c}\vec{d}]\vec{b} - [\vec{b}\vec{c}\vec{d}]\vec{a}$ ...(iii)

This shows that if there are 4 vectors, no 3 of which are coplanar, then any one of them can be expressed as linear combination of other 3 vectors.

Illustration 57:

If $\hat{i}\times[(\vec{a}-\hat{j})\times\hat{i}]+\hat{j}\times[(\vec{a}-\hat{k})\times\hat{j}]+\hat{k}\times[(\vec{a}-\hat{i})\times\hat{k}]=0$, then find vector $\vec{a}$.

Solution:

$$\hat{i}\times[(\vec{a}-\hat{j})\times\hat{i}] = (\hat{i}\cdot\hat{i})(\vec{a}-\hat{j})-(\hat{i}\cdot(\vec{a}-\hat{j}))\hat{i}$$

$$= \vec{a}-\hat{j}-(\hat{i}\cdot\vec{a})\hat{i}$$

Similarly, $\hat{j}\times[(\vec{a}-\hat{k})\times\hat{j}] = \vec{a}-\hat{k}-(\hat{j}\cdot\vec{a})\hat{j}$

And $\hat{k}\times[(\vec{a}-\hat{i})\times\hat{k}]=\vec{a}-\hat{i}-(\hat{k}\cdot\vec{a})\hat{k}$

$\therefore\ \hat{i}\times[(\vec{a}-\hat{j})\times\hat{i}]+\hat{j}\times[(\vec{a}-\hat{k})\times\hat{j}]+\hat{k}\times[(\vec{a}-\hat{i})\times\hat{k}]$

$=\vec{a}-\hat{j}-(\hat{i}\cdot\vec{a})\hat{i}+\vec{a}-\hat{k}-(\hat{j}\cdot\vec{a})\hat{j}+\vec{a}-\hat{i}-(\hat{k}\cdot\vec{a})\hat{k}=0$

$\therefore 3\vec{a}-(\hat{i}+\hat{j}+\hat{k})-\vec{a}=0$

or $\vec{a}=\dfrac{1}{2}(\hat{i}+\hat{j}+\hat{k})$

Illustration 58:

If $\vec{b}$ and $\vec{c}$ are two non-collinear vectors such that $\vec{a}\,||\,(\vec{b}\times\vec{c})$, then prove that $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})$ is equal to $|\vec{a}|^2(\vec{b}\cdot\vec{c})$.

Solution:

$\vec{a}\,||\,(\vec{b}\times\vec{c})$

$\Rightarrow\ \vec{a}=\lambda(\vec{b}\times\vec{c})$ and $\vec{a}=\vec{a}\perp\vec{b}$ and $\vec{a}\perp\vec{c}$

Now $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{c}\end{vmatrix}$

Necessary & Sufficient Condition for Coplanarity of Four Points:

4 points with pv's $\vec{a},\vec{b},\vec{c},\vec{d}$ are coplanar iff $\exists$ scalars x, y, z and t not all simultaneously zero and satisfying $x\vec{a}+y\vec{b}+z\vec{c}+t\vec{d}=0$, where $x+y+z+t=0$.

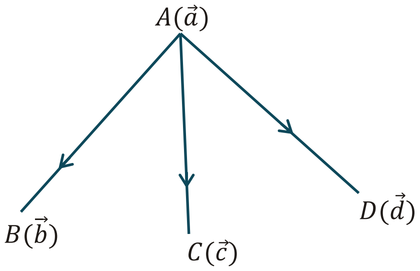


Illustration 59:

Examine for coplanarity of the following sets of points

(A) $4\hat{i}+8\hat{j}+12\hat{k}$, $2\hat{i}+4\hat{j}+6\hat{k}$, $3\hat{i}+5\hat{j}+4\hat{k}$, $5\hat{i}+8\hat{j}+5\hat{k}$

(B) $3\vec{a}+2\vec{b}-5\vec{c}$, $3\vec{a}+8\vec{b}+5\vec{c}$, $-3\vec{a}+2\vec{b}+\vec{c}$, $\vec{a}+4\vec{b}-3\vec{c}$.

Where $\vec{a},\vec{b},\vec{c}$ are non-coplanar.

Ans. (A) Coplanar (B) Non-coplanar

Solution:

(A) Let $A\equiv 4\hat{i}+8\hat{j}+12\hat{k}$; $B\equiv 2\hat{i}+4\hat{j}+6\hat{k}$; $C\equiv 3\hat{i}+5\hat{j}+4\hat{k}$; $D\equiv 5\hat{i}+8\hat{j}+5\hat{k}$

$\overrightarrow{AB}=-2\hat{i}-4\hat{j}-6\hat{k}$; $\overrightarrow{BC}=\hat{i}+\hat{j}-2\hat{k}$; $\overrightarrow{CD}=2\hat{i}+3\hat{j}+\hat{k}$

$[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}]=\begin{vmatrix}-2 & -4 & -6\\ 1 & 1 & -2\\ 2 & 3 & 1\end{vmatrix}=[-2(1+6)+4(1+4)-6(3-2)]=-14+20-6=0$

Hence coplanar.

(B) Using these 4 points let us create any 3 vectors $\vec{v}_1 = 6\vec{b} + 10\vec{c}$, $\vec{v}_2 = -6a + 6\vec{c}$

$\vec{v}_3 = -2\vec{a} + 2\vec{b} + 2\vec{c}$ check coplanarity of these vectors

$$[\vec{v}_1 \ \ \vec{v}_2 \ \ \vec{v}_3] = \begin{vmatrix} 0 & 6 & 10 \\ -6 & 0 & 6 \\ -2 & 2 & 2 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}] \neq 0$$

hence non coplanar

Linear Independence and Dependence of Vectors:

(a) If $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are n non-zero vectors, & $k_1, k_2, ...k_n$ are n scalars & if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + ...k_n\vec{x}_n = \vec{0} \Rightarrow k_1 = 0, \ k_2 = 0....k_n = 0$, then we say that vectors $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are **linearly independent vectors.**

(b) If $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are not **linearly independent** then they are said to be **linearly dependent vectors**.

i.e. if $k_1\vec{x}_1 + k_2\vec{x}_2 + ...k_n\vec{x}_n = \vec{0}$ & if there exists at least one $k_r \neq 0$ then $\vec{x}_1, \vec{x}_2, ...\vec{x}_n$ are said to be **linearly dependent.**

Illustration 60:

Show that points with position vectors $\vec{a} - 2\vec{b} + 3\vec{c}, -2\vec{a} + 3\vec{b} - \vec{c}, 4\vec{a} - 7\vec{b} + 7\vec{c}$ are collinear. It is given that vectors $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar.

Solution:

The three points are collinear, if we can find λ_1, λ_2 and λ_3, such that

$\lambda_1(\vec{a} - 2\vec{b} + 3\vec{c}) + \lambda_2(-2\vec{a} + 3\vec{b} - \vec{c}) + \lambda_3(4\vec{a} - 7\vec{b} + 7\vec{c}) = \vec{0}$

with $\lambda_1 + \lambda_2 + \lambda_3 = 0$

equating the coefficients $\vec{a}, \vec{b}$ and $\vec{c}$ separately to zero, we get

$\lambda_1 - 2\lambda_2 + 4\lambda_3 = 0, -2\lambda_1 + 3\lambda_2 - 7\lambda_3 = 0$ and $3\lambda_1 - \lambda_2 + 7\lambda_3 = 0$

on solving we get, $\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 1$

So that $\lambda_1 + \lambda_2 + \lambda_3 = 0$

Hence the given vectors are collinear.

Illustration 61:

$\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors such that $\vec{P} = \vec{a} + \vec{b} + \vec{c}, \vec{Q} = 4\vec{a} + 3\vec{b} + 4\vec{c}$ and $\vec{R} = \vec{a} + \alpha\vec{b} + \beta\vec{c}$ are linearly dependent vectors, then number of possible values of α is

(A) 0 (B) 1 (C) 2 (D) infinite

Ans. (D)

Solution:

For linearly dependent vectors

$[\vec{P}\vec{Q}\vec{R}] = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} [\vec{a}\vec{b}\vec{c}] = 0$$

But, $[\vec{a}\vec{b}\vec{c}] \neq 0$, ($\because$ $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors)

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} = 0$$

$\Rightarrow \beta = 1, \ \alpha$ can be any real number.

Illustration 62:

Find value of $x \in R$ for which the vectors $\vec{a} = (1, -2, 3), \vec{b} = (-2, 3, -4), \vec{c} = (1, -1, x)$ form a linearly dependent system.

Ans. $\boldsymbol{x = 1}$

Solution:

For linearly dependent vectors

$\ell(\hat{\imath} - 2\hat{\jmath} + 3\hat{k}) + m(-2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}) + n(\hat{\imath} - \hat{\jmath} + x\hat{k}) = 0$

$\Rightarrow \ell - 2m + n = 0, -2\ell + 3m - n = 0, 3\ell - 4m + nx = 0$

$$\therefore \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & -1 \\ 3 & -4 & x \end{vmatrix} = 0$$

$\Rightarrow x = 1$

Illustration 63:

If $\vec{a}, \vec{b}, \vec{c}$ are linearly independent vectors, then which one of the following set of vectors is linearly dependent?

(A) $\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}$

(B) $\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}$

(C) $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}$

(D) $\vec{a} + 2\vec{b} + 3\vec{c}, \vec{b} - \vec{c} + \vec{a}, \vec{a} + \vec{c}$

Ans. (B)

Solution:

$[\vec{a} \times \vec{b} \ \ \vec{b} \times \vec{c} \ \ \vec{c} \times \vec{a}] = [\vec{a} \ \vec{b} \ \vec{c}]^2 \neq 0$

$$[\vec{a} - \vec{b} \ \ \vec{b} - \vec{c} \ \ \vec{c} - \vec{a}] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} [\vec{a}\vec{b}\vec{c}] = 0[\vec{a}\vec{b}\vec{c}] = 0$$

$[\vec{a} + \vec{b} \ \ \vec{b} + \vec{c} \ \ \vec{c} + \vec{a}] = 2[\vec{a}\vec{b}\vec{c}] \neq 0$

$$[\vec{a} + 2\vec{b} + 3\vec{c} \ \ \vec{b} - \vec{c} + \vec{a} \ \ \vec{a} + \vec{c}] = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} [\vec{a}\vec{b}\vec{c}] = -6[\vec{a}\vec{b}\vec{c}] \neq 0$$

Fundamental Theorem in Space:

If $\vec{a}, \vec{b}, \vec{c}$ are 3 non-zero non-coplanar vectors then any vector $\vec{r}$ can be expressed as a linear combination of these 3 vectors as $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$

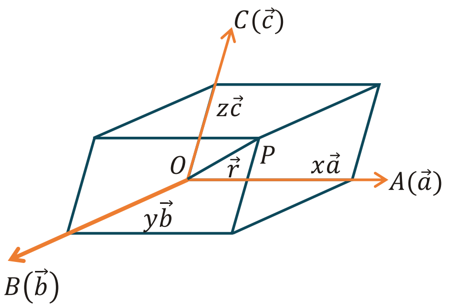


Illustration 64:

Let $\vec{a}$ and $\vec{b}$ be two non-zero perpendicular vectors. A vector $\vec{r}$ satisfying the equation $\vec{r} \times \vec{b} = \vec{a}$ can be-

(A) $\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$ (B) $2\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$ (C) $|\vec{a}|\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$ (D) $|\vec{b}|\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$

Ans. (A,B,C,D)

Solution:

$\vec{a},\vec{b},\vec{a}\times\vec{b}$ are non-coplanar

$\Rightarrow \vec{r} = x\vec{a} + y\vec{b} + z\left(\vec{a}\times\vec{b}\right)$

$\Rightarrow \vec{r}\times\vec{b} = \vec{a}$

$\Rightarrow x\vec{a}\times\vec{b} + z\left\{\left(\vec{a}\cdot\vec{b}\right)\vec{b} - \left(\vec{b}\cdot\vec{b}\right)\vec{a}\right\} = \vec{a}$

$\Rightarrow -\left(1+z\left|\vec{b}\right|^2\right)\vec{a} + x\vec{a}\times\vec{b} = 0$

$\Rightarrow x = 0$ and $z = -\dfrac{1}{\left|\vec{b}\right|^2}$

$\Rightarrow \vec{r} = y\vec{b} - \dfrac{\vec{a}\times\vec{b}}{\left|\vec{b}\right|^2}$

Solving Vector Equation:

Thereis no general method for solving such equations, however dot or cross with known or unknown vectors or dot with $\vec{a}\times\vec{b}$, generally isolates the unknown vector. Use of linear combination also proves to be advantageous.

Illustration 65:

Let there exist a vector $\vec{x}$ satisfying the conditions $\vec{x}\times\vec{a} = \vec{c}\times\vec{d}$ and $\vec{x} + 2\vec{d} = (\vec{v}\times\vec{d})$. Find $\vec{x}$ in terms of $\vec{a}$, $\vec{c}$ and $\vec{d}$

Ans. $\vec{x} = \dfrac{\vec{d}\times(\vec{c}\times\vec{d}) - 2|\vec{d}|^2\,\vec{a}}{\vec{d}\cdot\vec{a}}$

Solution:

$\vec{d}\times\left(\vec{x}\times\vec{a}\right) = \vec{d}\times\left(\vec{c}\times\vec{d}\right)$

$\Rightarrow \left(\vec{d}\cdot\vec{a}\right)\vec{x} - \left(\vec{d}\cdot\vec{x}\right)\vec{a} = \vec{d}\times\left(\vec{c}\times\vec{d}\right)$ ...(i)

Also, $\vec{x} + 2\vec{d} = (\vec{v}\times\vec{d})$

Now $\vec{d}\cdot\vec{x} + 2\vec{d}\cdot\vec{d} = \vec{d}\cdot(\vec{v}\times\vec{d})$

$\vec{d}\cdot\vec{x} + 2\vec{d}\cdot\vec{d} = 0$

$\Rightarrow \vec{d}\cdot\vec{x} = -2\left|\vec{d}\right|^2$ ...(ii)

From (i) and (ii) we get $\vec{x} = \dfrac{\vec{d}\times(\vec{c}\times\vec{d}) - 2\left|\vec{d}\right|^2\vec{a}}{\vec{d}\cdot\vec{a}}$

Illustration 66:

Vector $\vec{x}$ satisfying the relation $\vec{A}\cdot\vec{x} = c$ and $\vec{A}\times\vec{x} = \vec{B}$ is

(A) $\dfrac{c\vec{A} - (\vec{A}\times\vec{B})}{|\vec{A}|}$ (B) $\dfrac{c\vec{A} - (\vec{A}\times\vec{B})}{|\vec{A}|^2}$ (C) $\dfrac{c\vec{A} + (\vec{A}\times\vec{B})}{|\vec{A}|^2}$ (D) $\dfrac{c\vec{A} - 2(\vec{A}\times\vec{B})}{|\vec{A}|^2}$

Ans. (B)

Solution:

$\vec{A}\times\vec{x}=\vec{B}$ take cross product by $\vec{A}$ to get $\vec{A}\times(\vec{A}\times\vec{x}) = \vec{A}\times\vec{B} \Rightarrow (\vec{A}\cdot\vec{x})\vec{A}-(\vec{A}\cdot\vec{A})\vec{x}=\vec{A}\times\vec{B}$

$\Rightarrow \vec{x}=\dfrac{c\vec{A}-(\vec{A}\times\vec{B})}{|\vec{A}|^2}$

Illustration 67:

The value of $\vec{r}$ if exist where $\vec{r} = \vec{a}+\lambda\vec{b}$ and $\vec{r}\times\vec{c}=\vec{d}$ is

(A) $\vec{a}+\left(\dfrac{\vec{a}\cdot\vec{d}}{\vec{b}\cdot\vec{d}}\right)\vec{b}$ (B) $\vec{a}-\left(\dfrac{\vec{a}\cdot\vec{d}}{\vec{b}\cdot\vec{d}}\right)\vec{b}$ (C) $\left(\dfrac{\vec{a}\cdot\vec{d}}{\vec{b}\cdot\vec{d}}\right)\vec{a}-\vec{b}$ (D) $\left(\dfrac{\vec{a}\cdot\vec{d}}{\vec{b}\cdot\vec{d}}\right)\vec{a}+\vec{b}$

Ans. (B)

Solution:

$\vec{r}\cdot\vec{d} = \vec{a}\cdot\vec{d} + \lambda(\vec{b}\cdot\vec{d})$

Since , $\vec{r}\times\vec{c}=\vec{d}$

Therefore, $\vec{r}\perp\vec{d}$

$\Rightarrow 0 = \vec{a}\cdot\vec{d} + \lambda(\vec{b}\cdot\vec{d})$

$\Rightarrow \lambda = -\left(\dfrac{\vec{a}\cdot\vec{d}}{\vec{b}\cdot\vec{d}}\right)$

$\Rightarrow \vec{r} = \vec{a}-\left(\dfrac{\vec{a}\cdot\vec{d}}{\vec{b}\cdot\vec{d}}\right)\vec{b}$

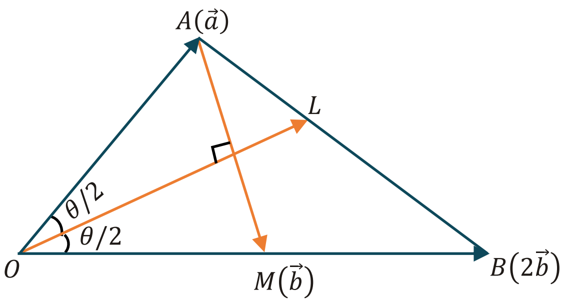
Illustration 68:

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be two sides of a triangle. The median $\overrightarrow{AM}$ is perpendicular to angle bisector $\overrightarrow{OL}$ and $|\overrightarrow{AM}|:|\overrightarrow{OL}|=1:2$. The angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is

(A) $\cos^{-1}(4/5)$ (B) $\cos^{-1}1/2$ (C) $\cos^{-1}3/5$ (D) $\cos^{-1}(1/\sqrt{2})$

Ans. (A)

Solution:



$\overrightarrow{AM}=\vec{b}-\vec{a}$

$\overrightarrow{OL}=\dfrac{2b\vec{a}+a(2\vec{b})}{2b+a}$

$\therefore\ 2b^2\,a\cos\theta - 2a^2\,b + 2ab^2 - 2a^2b\cos\theta = 0$

$\cos\theta\,(2b^2a - 2a^2b) - (2a^2b - 2ab^2) = 0$

$\therefore\ 2ab(b-a)\,(1+\cos\theta) = 0$

$\Rightarrow 0<\theta<\pi \Rightarrow b=a$

$\therefore$ OAM is an isosceles triangle

also $\overrightarrow{OL} = \frac{2}{3}(\vec{a}+\vec{b})$; $\overrightarrow{AM} = \vec{b}-\vec{a}$

$\therefore$ $|\overrightarrow{OL}|^2 : |\overrightarrow{AM}|^2 = \frac{4}{9}\left(\frac{1+\cos\theta}{1-\cos\theta}\right) = 4$

$\Rightarrow \frac{1-\cos\theta}{1+\cos\theta} = \frac{1}{9} \Rightarrow \cos\theta = \frac{4}{5}$

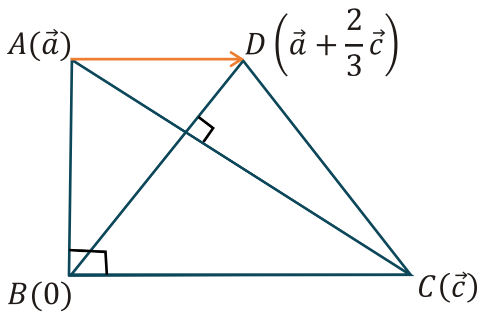
Illustration 69:

In a right trapezium $ABCD$, the diagonals are perpendicular, and the ratio of the length of bases $AD : BC = 2 : 3$. Then the ratio of length of diagonals is

(A) 3 : 2 (B) 1 : 3 (C) $2 : \sqrt{3}$ (D) $\sqrt{3} : \sqrt{2}$

Ans. (D)

Solution:



$\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$

$\Rightarrow (\vec{c}-\vec{a}) \cdot \left(\vec{a}+\frac{2}{3}\vec{c}\right) = 0$

$\Rightarrow \frac{2}{3}|\vec{c}|^2 - |\vec{a}|^2 = 0 \quad (\because \vec{a}\cdot\vec{c} = 0)$

$\Rightarrow |\vec{a}|^2 = \frac{2}{3}|\vec{c}|^2$

Now $|\overrightarrow{AC}| : |\overrightarrow{BD}| = |\vec{c}-\vec{a}| : \left|\vec{a}+\frac{2}{3}\vec{c}\right|$

$= \sqrt{c^2+a^2} : \sqrt{a^2+\frac{4}{9}c^2} = \sqrt{3} : \sqrt{2}$

Illustration 70:

If $\vec{a} = 2\hat{i}+3\hat{j}+\hat{k}, \vec{b} = \hat{i}-\hat{j}+\hat{k}, \vec{c} = \hat{i}+\hat{j}+\hat{k}$ and let $\vec{d}$ be such that $\vec{a}\times\vec{b} = \vec{d}\times\vec{b}$, $\vec{d}\cdot\vec{c} = 8$, then value of $\vec{d}\cdot\vec{b}$ is.

(A) 6 (B) –6 (C) 3 (D) –3

Ans. (A)

Solution:

$(\vec{a}-\vec{d})\times\vec{b} = 0 \Rightarrow \vec{a} = \vec{d}+\lambda\,\vec{b}$

Dot with $\vec{c}$

$2+3+1 = 8+\lambda(1-1+1)$

$\Rightarrow \lambda = -2$

$\therefore \vec{d} = \vec{a}+2\vec{b}$

$= 4\hat{i}+\hat{j}+3\hat{k}$

$\therefore \vec{d}\cdot\vec{b} = 4-1+3 = 6$

Illustration 71:

If $a^2+b^2+c^2=1$, then maximum passible value of $3a+4b+12c$ is equal to (where $a, b, c \in R$)-

(A) 10 (B) 11 (C) 12 (D) 13

Ans. (D)

Solution:

$(3\hat{\imath}+4\hat{\jmath}+12\hat{k})\cdot(a\hat{\imath}+b\hat{\jmath}+c\hat{k}) \le |3\hat{\imath}+4\hat{\jmath}+12\hat{k}|\,|a\hat{\imath}+b\hat{\jmath}+c\hat{k}| = 13\sqrt{a^2+b^2+c^2}$

Illustration 72:

If $\hat{a},\hat{b},\hat{c}$ are unit vectors, then least value of $|\hat{a}+\hat{b}|^2+|\hat{b}+\hat{c}|^2+|\hat{c}+\hat{a}|^2$ will be-

(A) 1 (B) 3 (C) 9 (D) 12

Ans. (B)

Solution:

$1+1+2\hat{a}\cdot\hat{b}+1+1+2\hat{b}\cdot\hat{c}+1+1+2\hat{c}\cdot\hat{a}$

$=6+2\underbrace{\left(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a}\right)}_{\ge -\frac{3}{2}}$

(Since $|\hat{a}+\hat{b}+\hat{c}|^2 \ge 0$

$\Rightarrow 1+1+1+2\left(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a}\right) \ge 0$

$\Rightarrow \left(\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a}\right) \ge -\frac{3}{2}$)

$\Rightarrow$ Minimum value = 3

Illustration 73:

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $|\vec{a}+2\vec{b}|=|\vec{a}-2\vec{b}|$, then angle between $\vec{a}$ and $\vec{b}$ is -

(A) 120° (B) 60° (C) 90° (D) 30°

Ans. (C)

Solution:

$a^2+4b^2+2\vec{a}\cdot 2\vec{b} = a^2+4b^2-2\vec{a}\cdot 2\vec{b}$

$\Rightarrow \vec{a}\cdot\vec{b}=0 \Rightarrow \vec{a}\wedge\vec{b}=\frac{\pi}{2}$

Illustration 74:

Let $\vec{b}$ and $\vec{c}$ be non-collinear vector satisfying $\vec{a}\times(\vec{b}\times\vec{c})+(\vec{a}.\vec{b})\vec{b} = (4-2x-\sin y)\,\vec{b} + (x^2-1)\,\vec{c}$ and $(\vec{c}\cdot\vec{c})\vec{a}=\vec{c}$, then x is equal to

(A) 1 (B) 2 (C) 4 (D) 6

Ans. (A)

Solution:

$(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}+(\vec{a}\cdot\vec{b})\vec{b}$

$= (4-2x-\sin y)\vec{b} + (x^2-1)\,\vec{c}$

$\Rightarrow \vec{a}\cdot\vec{c}+\vec{a}\cdot\vec{b} = 4-2x-\sin y,\ \vec{a}\cdot\vec{b} = 1-x^2$

Also, $(\vec{c}\cdot\vec{c})\vec{a}=\vec{c}$

$\Rightarrow (\vec{c}\cdot\vec{c})\vec{a}\cdot\vec{c}=\vec{c}\cdot\vec{c}$

$|\vec{c}|^2\,\vec{a}\cdot\vec{c}=|\vec{c}|^2$

$\vec{a}\cdot\vec{c}=1$

Now, $1+\vec{a}\cdot\vec{b} = 4 - 2x - \sin y$

$\Rightarrow x^2 - 2x + 1 = \sin y - 1 \le 0$

$\Rightarrow x = 1, y = \pi/2$

Illustration 75:

Match List-I with List-II and select the correct answer using the code given below the list.

List-I | **List-II**

(P) If $\vec{A}=\hat{i}-3\hat{j}+4\hat{k}, \vec{B}=6\hat{i}+4\hat{j}-8\hat{k}$, $\vec{C}=5\hat{i}+2\hat{j}+5\hat{k}$ and a vector $\vec{r}$ satisfies $\vec{r}\times\vec{B}=\vec{C}\times\vec{B}$ and $\vec{r}\cdot\vec{A}=0$, then $\dfrac{|\vec{B}|}{|\vec{r}-\vec{C}|}$ is equal to — (1) $\dfrac{2}{3}$

(Q) If $\vec{a}$ and $\vec{b}$ are two orthogonal vectors of equal magnitude such that $|3\vec{a}+4\vec{b}|+|4\vec{a}-3\vec{b}|=20$, then $|(\vec{a}\times\vec{b})\times\vec{a}|$ is equal to — (2) 2

(R) If $\vec{a}$ and $\vec{b}$ are non-zero and non-collinear vectors, then the value of α for which the vectors $\vec{v}_1=(\alpha-2)\vec{a}+\vec{b}$ and $\vec{v}_2=(2+3\alpha)\vec{a}-3\vec{b}$ are collinear, is — (3) 8

Codes:

	P	**Q**	**R**
(A)	2	1	3
(B)	3	2	1
(C)	1	2	3
(D)	2	3	1

Ans. (D)

Solution:

(P) $\vec{r}\times\vec{B}=\vec{C}\times\vec{B}$

$\Rightarrow \vec{A}\times(\vec{r}\times\vec{B})=\vec{A}\times(\vec{C}\times\vec{B})$

$\Rightarrow (\vec{A}\cdot\vec{B})\vec{r}-(\vec{A}\cdot\vec{r})\vec{B}=(\vec{A}\cdot\vec{B})\vec{C}-(\vec{A}\cdot\vec{C})\vec{B}$

$\Rightarrow -38\vec{r}-0=-38\vec{C}-19\vec{B}$

$\Rightarrow \dfrac{|\vec{B}|}{|\vec{r}-\vec{C}|}=2.$

(Q) $|3\vec{a}+4\vec{b}|^2=25\lambda^2$ & $|4\vec{a}-3\vec{b}|^2=25\lambda^2$, where $\lambda=|\vec{a}|=|\vec{b}|=2$

$\therefore 10\lambda = 20 \Rightarrow |\vec{a}|=|\vec{b}|=2$

Now, $|(\vec{a}\times\vec{b})\times\vec{a}|=4|\vec{b}|=8$

(R) $\vec{v}_1=\lambda\vec{v}_2 \Rightarrow \alpha=\dfrac{2}{3}$

Illustration 76:

If $\left[\vec{a}+2\vec{b}+3\vec{c} \quad 2\vec{a}+3\vec{b}+\vec{c} \quad 3\vec{a}+\vec{b}+2\vec{c}\right]=-18$, where $\vec{a}$, $\vec{b}$, $\vec{c}$ are 3 non-coplanar vectors, then

$\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{c} \end{vmatrix}$ is equal

Ans. (1)

Solution:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}\left[\bar{a}\ \bar{b}\ \bar{c}\right]=-18 \Rightarrow [\bar{a}\,\bar{b}\,\bar{c}]=1$$

Hence, $\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{c} \end{vmatrix}=[\bar{a}\,\bar{b}\,\bar{c}]^2=1$

Illustration 77:

If $\vec{w}=\alpha(\vec{a}\times\vec{b})+\beta(\vec{b}\times\vec{c})+\gamma(\vec{c}\times\vec{a})$, $[\vec{a}\ \vec{b}\ \vec{c}]=2$ and $\vec{w}\cdot\left(\vec{a}+\vec{b}+\vec{c}\right)=8$, then $\alpha+\beta+\gamma=$

(A) 64 (B) 4 (C) 32 (D) 8

Ans. (B)

Solution:

$\vec{w}\cdot\vec{c}=\alpha[\vec{a}\ \vec{b}\ \vec{c}]$

$\therefore\ \alpha=\frac{1}{2}(\vec{w}\cdot\vec{c})$

Similarly, $\beta=\frac{1}{2}(\vec{w}\cdot\vec{a})$, $\gamma=\frac{1}{2}(\vec{w}\cdot\vec{b})$

$\therefore\ \alpha+\beta+\gamma=\frac{1}{2}(\vec{w}\cdot\vec{a}+\vec{w}\cdot\vec{b}+\vec{w}\cdot\vec{c})=\frac{1}{2}(8)=4$

Illustration 78:

Volume of a parallelepiped with coterminous edges $\vec{a}, \vec{b}, \vec{c}$ is 12 *cu* units. Volume of a tetrahedron with coterminous edges $\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{a}+\vec{b}-\vec{c}$ will be -

(A) 2 cu units (B) 3 cu units (C) 6 cu units (D) 12 cu units

Ans. (A)

Solution:

$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=12$

$\frac{1}{6}\left(\left[\vec{a}-\vec{b} \quad \vec{b}-\vec{c} \quad \vec{a}-\vec{c}+\vec{b}\right]\right)=\frac{1}{6}\left[\vec{a}\,\vec{b}\,\vec{c}\right]=2$

Illustration 79:

If $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+\hat{j}-2\hat{k}$ and $\vec{c}=\hat{i}+3\hat{j}-(\lambda^2+3\lambda)\hat{k}$ (where λ is a constant) and $\vec{a}$ is perpendicular to $\vec{c}-\lambda\vec{b}$, then sum of different values of λ is

(A) –1 (B) 1 (C) 4 (D) –4

Ans. (A)

Solution:

$(2\hat{i}-\hat{j}+\hat{k})\cdot((1-\lambda)\hat{i}+(3-\lambda)\hat{j}+(2\lambda-\lambda^2-3\lambda)\hat{k})=0$

$\Rightarrow 2-2\lambda-3+\lambda+2\lambda-\lambda^2-3\lambda=0$

$\Rightarrow -\lambda^2-2\lambda-1=0 \Rightarrow (\lambda+1)^2=0$

$\Rightarrow \lambda=-1$

Illustration 80:

$\vec{r}=\lambda(\vec{a}\times\vec{b})+\mu(\vec{b}\times\vec{c})+\nu(\vec{c}\times\vec{a})$ & $[abc]=1$, then $\lambda+\mu+\nu$ is :

(A) $\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ (B) $\vec{r}\cdot\dfrac{(\vec{a}+\vec{b}+\vec{c})}{3}$ (C) $3\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ (D) None of these

Ans. (A)

Solution:

$\vec{r}\cdot\vec{a}=\mu$, $\vec{r}\cdot\vec{b}=\nu$ and $\vec{r}\cdot\vec{c}=\lambda$

$\therefore \lambda+\mu+\nu=\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$

Illustration 81:

Let $x^2+y^2+z^2=5, x, y, z\in R$, then maximum value of $(x-y)^2+(y-z)^2+(z-x)^2$ is-

(A) 15 (B) 20 (C) 25 (D) 30

Ans. (A)

Solution:

$x^2+y^2+z^2=5$

Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$

$\vec{b}=x\hat{i}+y\hat{j}+z\hat{k}$

$\vec{a}\times\vec{b}=ab\sin\theta\,\hat{n}$

$(z-y)\hat{i}+(x-z)\hat{j}+(y-x)\hat{k}=ab\sin\theta\,\hat{n}$

$=\sqrt{3}\sqrt{x^2+y^2+z^2}\sin\theta\,\hat{n}$

Compare modulus

$\sqrt{(z-y)^2+(x-z)^2+(y-x)^2}=\sqrt{3(x^2+y^2+z^2)}\sin\theta$

$(x-y)^2+(y-z)^2+(z-x)^2=3(x^2+y^2+z^2)\sin\theta\le 15$

Illustration 82:

The constant value $(\lambda+\mu)$ for which the lines $\vec{r}=2\hat{i}+\hat{j}+\hat{k}+\lambda(\hat{i}-2\hat{j})$ and $\vec{r}=\hat{i}+\hat{j}-3\hat{k}+\mu(\hat{j}+2\hat{k})$ intersect each other, is equal to (where λ & μ are parameters) -

(A) 2 (B) –1 (C) 0 (D) 1

Ans. (D)

Solution:

If lines $\vec{r}=(2+\lambda)\hat{i}+(1-2\lambda)\hat{j}+\hat{k}$ & $\vec{r}=\hat{i}+(1+\mu)\hat{j}+(-3+2\mu)\hat{k}$

intersects each other, then

$2+\lambda=1, 1-2\lambda=1+\mu$ & $1=-3+2\mu$

$\Rightarrow \lambda=-1, \mu=2$

Hence, $\lambda+\mu=1$

Illustration 83:

Consider two lines L_1 : $\vec{r}=(\hat{i}-2\hat{k})+\lambda(2\hat{i}-\hat{j}+2\hat{k})$ and L_2 : $\vec{r}=\left(\frac{1}{2}\hat{i}-\hat{j}+2\hat{k}\right)+\mu(\hat{i}+m\hat{j})$ (where $\lambda, m \in R$).

Find sum of possible values of $'m'$, for which shortest distance between L_1 & L_2 is $\sqrt{10}$ units, is-

(A) –4 (B) –46 (C) 4 (D) 46

Ans. (C)

Solution:

$$S\cdot D=\left(\frac{1}{2}\hat{i}+\hat{j}-4\hat{k}\right)\cdot\frac{\left(2\hat{i}-\hat{j}+2\hat{k}\right)\times\left(\hat{i}+m\hat{j}\right)}{\left|\left(2\hat{i}-\hat{j}+2\hat{k}\right)\times\left(\hat{i}+m\hat{j}\right)\right|}=\sqrt{10}$$

$(9m+2^2=10(8m^2+4m+5)$

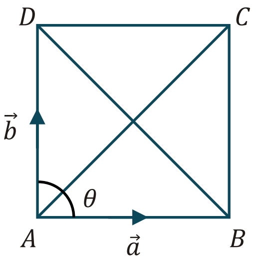
$\Rightarrow m^2-4m-46=0$

Illustration 84:

Given in a parallelogram $ABCD, AB=2, AD=3$ and M, m denotes the maximum and minimum integral value of product $\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{BD}\right|$, then $(M-m)$ is equal to

Ans. (7)

Solution:



$|\vec{a}|=2, |\vec{b}|=3$

$\therefore \left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{BD}\right|=\left|\vec{a}+\vec{b}\right|\cdot\left|\vec{a}-\vec{b}\right|$

$=\sqrt{4+9+12\cos\theta}\cdot\sqrt{4+9-12\cos\theta}$

$=\sqrt{169-144\cos^2\theta}$ $\because 0 \le \cos^2\theta < 1$

$\therefore \left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{BD}\right|\in(5,13]$

$\Rightarrow m=6, M=13 \Rightarrow M-m=7.$