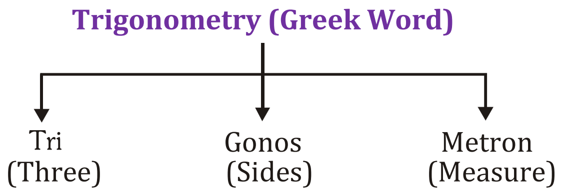
01 Trigonometric Ratios and Identities

Introduction:



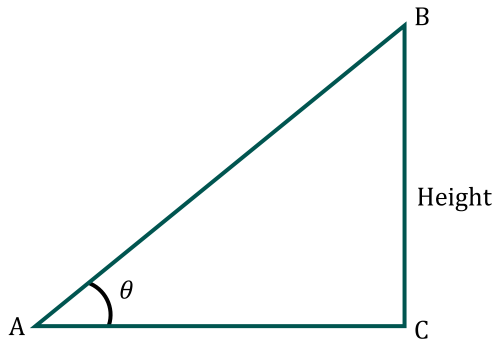
The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analyzing a musical tone and in many other areas.

Trigonometry is the branch of mathematics in which we study about triangle. Basically there are six parameters in a triangle (Three sides & three angles).

How they are related to each other is mainly discussed in trigonometry.

Trigonometry to measure height of tower building or mountain:

Trigonometry is used to in measuring the height of a building or a mountain. The distance of a building from the viewpoint and the elevation angle can easily determine the height of a building using the trigonometric functions.



It is used to find the distance of the shore from a point in the sea

It is also used in oceanography in calculating the height of tides in oceans.

Trigonometry in Aviation:

Aviation technology has been evolved in many up-gradations in the last few years. It has taken into account the speed, direction, and distance as well as have to consider the speed and direction of the wind. The wind plays a vital role in when and how a flight will travel. This equation cab is solved by using trigonometry.

Trigonometry in Criminology:

Trigonometry is even used in the investigation of a crime scene. The functions of trigonometry are helpful to calculate a trajectory of a projectile and to estimate the causes of a collision in a car accident. Further, it is used to identify how an object falls or in what angle the gun is shot.

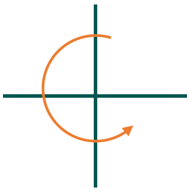
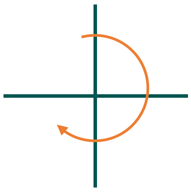
Trigonometry in Marine Biology:

Trigonometry is often used by marine biologists for measurements to figure out depth of sunlight that affects algae to photosynthesis. Using the trigonometric function and mathematical models, marine biologists estimate the size of larger animals like whales and also understand their behaviors.

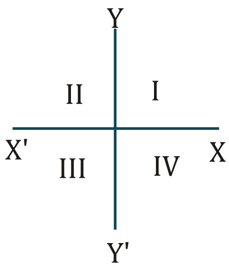
Trigonometry in Navigation:

Trigonometry is used in navigating directions; it estimates in what place the compass to get a straight direction. With the help of a compass and trigonometric functions in navigation, it will help to pinpoint a location and also to find distance as well as to see the horizon.

Measurement of Angle:

Sign convention :  Anticlockwise direction will indicate +ve angle ,  Clockwise turning will indicate –ve angle

If α is the angle measured from positive x-axis in anticlockwise direction.



$0° < \alpha < 90°$ I[st] quadrant

$90° < \alpha < 180°$ II[nd] quadrant

$180° < \alpha < 270°$ III[rd] quadrant

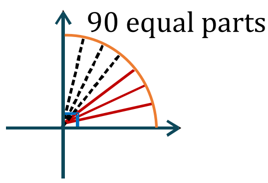
$270° < \alpha < 360°$ IV[th] quadrant

System of measurement of angle:

System of Measurement of angle

- Degree (English) (Sexagesimal system
- Radian (Circular system)
- French (Grade system) (Centisimal system)

Degree system : When we draw a line perpendicular to another and then divide the angle between them in 90 equal parts. Then each part will be equal to one degree.



$1° = 60'$ & $1' = 60''$

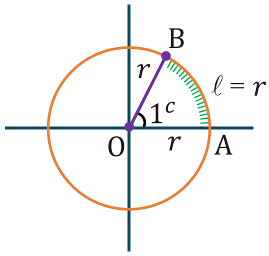
Hence $90° = 1$ right angle

or $180° = 2$ right angle.

Grade system : One right angle = 100^g grades, $1^g = 100'$ minutes, $1' = 100''$ seconds

Radian : If the length of arc (ℓ) is equal to r (radius), then the angle subtended by it at the center of the circle is equal to 1 radian. (1^c)

Commonly, $\boxed{\text{Angle} = \frac{\text{arc}}{\text{radius}}}$, i.e. $\ell = r\theta$, where θ is the angle subtended at the center by circular arc of length ℓ. If $\ell = r$ then $\theta = 1^c$



If $\ell = 2r$ then $\theta = 2^c$

If $\ell = 3r$ then $\theta = 3^c$

⋮

If $\ell = kr$ then $\theta = k^c$

⋮

If $\ell = \pi r$ then $\theta = \pi^c$

Hence $\pi^c = 180°$ or $\frac{\pi^c}{2} = 90° \Rightarrow$ 1 right angle $= \frac{\pi}{2}$ radians

$\Rightarrow 1° = \frac{\pi^c}{180} = 0.01746$ rad & $1^c = \frac{180^o}{\pi} = \frac{180}{22/7} = \frac{630}{11} = 57.2°$

Relation in three systems : $\frac{D}{90} = \frac{G}{100} = \frac{R}{\pi/2}$

Illustration 1:

Find the radian measures corresponding to the following degree measures:

(i) 25° **(ii)** – 47° 30′

Solution:

(i) 25°

Here 180° = π radians

It can be written as

$25° = \frac{\pi}{180} \times 25$ radian

So we get

$= \frac{5\pi}{36}$ radian

(ii) – 47°30′

Here 1° = 60

It can be written as

$-47°30' = -47\frac{1}{2}$ degree

So we get

$= \frac{-95}{2}$ degree

Here 180° = π radian

$= \frac{-95}{2}$ degree $= \frac{\pi}{180} \times \left(\frac{-95}{2}\right)$ radian

It can be written as

$= \left(\frac{-19}{36\times 2}\right)\pi$ radian $= \frac{-19}{72}\pi$ radian

We get

$-47°30' = \frac{-19}{72}\pi$ radian

Illustration 2:

Find the degree measures corresponding to the following radian measures (Use π = 22/7)

(i) 11/16 **(ii)** –4

Solution:

(i) 11/16

Here $\frac{1}{2}$ radian = 180°

$\frac{11}{16}$ radian = $\frac{180}{\pi} \times \frac{11}{16}$ degree

We can write it as

= $\frac{45\times11}{\pi\times4}$ degree

Se we get

= $\frac{45\times11\times7}{22\times4}$ degree

= $\frac{315}{8}$ degree

= $39\frac{3}{8}$ degree

Take 1° = 60'

= 39° + $\frac{3\times60}{8}$ minutes

We get

= 39° + 22' + $\frac{1}{2}$ minutes

Consider 1' = 60''

= 39° 22' 30''

(ii) – 4

Here π radian = 180°

– 4 radian = $\frac{180}{\pi}$ × (– 4) degree

We can write it as

= $\frac{180\times7(-4)}{22}$ degree

By further calculation

= $\frac{-2520}{11}$ degree = $-229\frac{1}{11}$ degree

Take 1° = 60'

– 229° + $\frac{1\times60}{11}$ minutes

So we get

= –229° + 5' + $\frac{5}{11}$ minutes

Again 1° = 60'

= – 229° 5' 27''

Illustration 3:

If the arcs of same length in two circles subtend angles of 60° and 75° at their centers. Find the ratio of their radii.

Solution:

Let r_1 and r_2 be the radii of the given circles and let their arcs of same length $'s'$ subtend angles of 60° and 75° at their centers.

Now, $60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)$ and $75° = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$

$\therefore \frac{\pi}{3} = \frac{s}{r_1}$ and $\frac{5\pi}{12} = \frac{s}{r_2}$

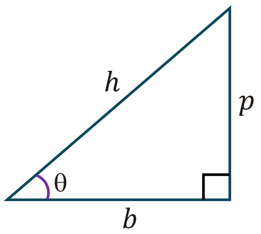
$\Rightarrow \frac{\pi}{3} r_1 = s$ and $\frac{5\pi}{12} r_2 = s$

$\Rightarrow \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2$

$\Rightarrow 4r_1 = 5r_2$

$\Rightarrow r_1 : r_2 = 5 : 4$

T-Ratios (or Trigonometric functions):



In a right angle triangle

$\sin\theta = \frac{p}{h}$; $\cos\theta = \frac{b}{h}$; $\tan\theta = \frac{p}{b}$; $\text{cosec}\,\theta = \frac{h}{p}$; $\sec\theta = \frac{h}{b}$ and $\cot\theta = \frac{b}{p}$

'p' is perpendicular; 'b' is base and 'h' is hypotenuse.

Basic Trigonometric Identities:

(1) $\sin\theta.\ \text{cosec}\theta = 1$

(2) $\cos\theta.\ \sec\theta = 1$

(3) $\tan\theta.\ \cot\theta = 1$

(4) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ & $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(5) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(6) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(7) $\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}$

(8) $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$

(9) $\text{cosec}\theta + \cot\theta = \frac{1}{\text{cosec}\theta - \cot\theta}$

(i) Expressing trigonometrical ratio in terms of each other :

	$sin\theta$	$cos\theta$	$tan\theta$	$cot\theta$	$sec\theta$	$cosec\theta$
$sin\theta$	$sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{sec\theta}$	$\frac{1}{cosec\theta}$
$cos\theta$	$\sqrt{1-\sin^2\theta}$	$cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{sec\theta}$	$\frac{\sqrt{\mathrm{cosec}^2\theta-1}}{cosec\theta}$
$tan\theta$	$\frac{sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{cos\theta}$	$tan\theta$	$\frac{1}{cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\mathrm{cosec}^2\theta-1}}$
$cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{sin\theta}$	$\frac{cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{tan\theta}$	$cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\mathrm{cosec}^2\theta-1}$
$sec\theta$	$\frac{1}{\sqrt{1+\sin^2\theta}}$	$\frac{1}{cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{cot\theta}$	$sec\theta$	$\frac{cosec\theta}{\sqrt{\mathrm{cosec}^2\theta-1}}$
$cosec\theta$	$\frac{1}{sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{sec\theta}{\sqrt{\sec^2\theta-1}}$	$cosec\theta$

(ii) Values of T-Ratios of some standard angles :

Angles / T-ratio	0°	30°	45°	60°	90°
	0	**π/6**	**π/4**	**π/3**	**π/2**
sinθ	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cosθ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	½	0
tanθ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	N.D.
cotθ	N.D.	$\sqrt{3}$	1	$1/\sqrt{3}$	0
secθ	1	$2/\sqrt{3}$	$\sqrt{2}$	2	N.D.
cosecθ	N.D.	2	$\sqrt{2}$	$2/\sqrt{3}$	1

Illustration 4:

If $\sin\theta + \sin^2\theta = 1$, then prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$

Solution:

Given that $\sin\theta = 1 - \sin^2\theta = \cos^2\theta$

L.H.S. $= \cos^6\theta(\cos^2\theta + 1)^3 - 1 = \sin^3\theta(1 + \sin\theta)^3 - 1 = (\sin\theta + \sin^2\theta)^3 - 1 = 1 - 1 = 0$

Illustration 5:

$4(\sin^6\theta + \cos^6\theta) - 6(\sin^4\theta + \cos^4\theta)$ is equal to

(A) 0 (B) 1 (C) –2 (D) none of these

Ans. (C)

Solution:

$4[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 6[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta]$

$= 4[1 - 3\sin^2\theta\cos^2\theta] - 6[1 - 2\sin^2\theta\cos^2\theta]$

$= 4 - 12\sin^2\theta\cos^2\theta - 6 + 12\sin^2\theta\cos^2\theta = -2$

Illustration 6:

Prove that : (i) $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$ (ii) $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$.

Solution:

(i) $\sin^4\theta+\cos^4\theta = (\sin^2\theta+\cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta = 1 - 2\sin^2\theta\cos^2\theta$

(ii) $\sin^6\theta+\cos^6\theta = (\sin^2\theta)^3+(\cos^2\theta)^3$

$= (\sin^2\theta+\cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta+\cos^2\theta)$ ($\because (a+b)^3 = a^3 + b^3 + 3ab(a+b)$)

$= 1 - 3\sin^2\theta\cos^2\theta$

Illustration 7:

Prove that $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \text{cosec}\theta - \cot\theta$

Solution:

$$\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1+\cos\theta)}} = \frac{1-\cos\theta}{\sin\theta} = \text{cosec}\theta - \cot\theta$$

Illustration 8:

Prove that $\sqrt{\sec^2 A + \text{cosec}^2 A} = \tan A + \cot A$, when $0° < A < 90°$

Solution:

$$\sqrt{\sec^2 A + \text{cosec}^2 A} = \sqrt{\frac{1}{\cos^2 A} + \frac{1}{\sin^2 A}} = \frac{1}{\sin A\cos A} = \frac{\sin^2 A+\cos^2 A}{\sin A\cos A} = \tan A + \cot A$$

OR $\sqrt{1 + \tan^2 A + 1 + \cot^2 A} = \sqrt{\tan^2 A + \cot^2 A + 2(\tan A.\cot A)} = \tan A + \cot A$

Illustration 9:

$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} =$

(A) $\frac{1-\sin\theta}{\cos\theta}$ (B) $\frac{1-\cos\theta}{\sin\theta}$ (C) $\frac{1+\sin\theta}{\cos\theta}$ (D) $\frac{1+\cos\theta}{\sin\theta}$

Ans. (C)

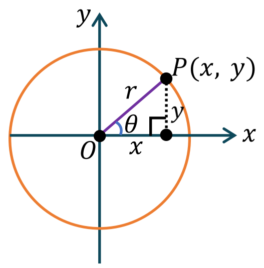
Solution:

$$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \frac{(\tan\theta+\sec\theta)-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}$$ [$\because \sec^2\theta - \tan^2\theta = 1$]

$$= \frac{(\sec\theta+\tan\theta)\{1-(\sec\theta-\tan\theta)\}}{\tan\theta-\sec\theta+1} = \frac{(\sec\theta+\tan\theta)(\tan\theta-\sec\theta+1)}{\tan\theta-\sec\theta+1}$$

$$= \sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} \Rightarrow \frac{1+\sin\theta}{\cos\theta}$$

Real definition of sine & cosine of a real number: (Circular functions)



$\sin\theta = \frac{\text{distance of point P from } x\text{-axis}}{\text{radius of the circle}}$

$\Rightarrow \sin\theta = \frac{y}{r}$

Similarly, $\cos\theta = \frac{\text{distance of point P from } y\text{-axis}}{\text{radius of the circle}}$

$\Rightarrow \cos\theta = \frac{x}{r}$

As $\theta \to 0, y$ decreases, x increases,

$y \to 0, x \to r,$

$\therefore$ sin0 = 0 & cos0 = 1

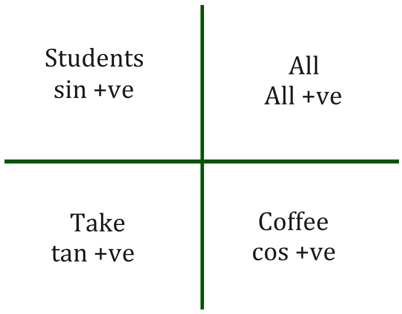
As $\theta \to 90°, y$ increases, x decreases

$y \to r, x \to 0$

$\therefore$ sin90° = 1 & cos90° = 0

Note : All other trigonometric ratios can be derived using sine and cos definition.

Sign of T-Ratios:



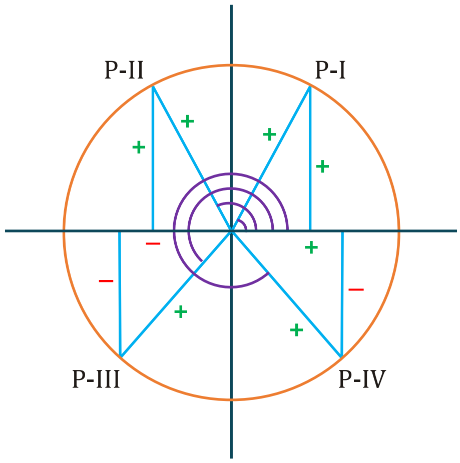


Illustration 10:

If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric functions.

Solution:

Since $\cos x = -\frac{3}{5}$, we have $\sec x = -\frac{5}{3}$

Now, $\sin^2 x + \cos^2 x = 1$, i.e., $\sin^2 x = 1 - \cos^2 x$

or $\sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Hence $\sin x = \pm\frac{4}{5}$

Since x lies in third quadrant, sin x is negative. Therefore

$\sin x = -\frac{4}{5}$

which also gives

$\text{cosec } x = -\frac{5}{4}$

Further, we have

$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3}$ and $\cot x = \frac{\cos x}{\sin x} = \frac{3}{4}$

Illustration 11:

If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigonometric functions.

Solution:

Since $\cot x = -\frac{5}{12}$, we have $\tan x = -\frac{12}{5}$

Now $\sec^2 x = 1 + \tan^2 x = 1 + \frac{144}{25} = \frac{169}{25}$

Hence $\sec x = \pm \frac{13}{5}$

Since x lies in second quadrant, $\sec x$ will be negative. Therefore

$\sec x = -\frac{13}{5}$,

which also gives

$\cos x = -\frac{5}{13}$

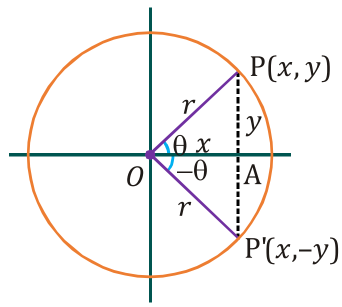
Further, we have

$\sin x = \tan x \cos x = \left(-\frac{12}{5}\right) \times \left(-\frac{5}{13}\right) = \frac{12}{13}$

and $\text{cosec}\, x = \frac{1}{\sin x} = \frac{13}{12}$

Trigonometric ratios of allied angles:

(i) T-ratios of angle (–θ):



In ΔAOP & ΔAOP', we have PA = P'A

Hence ∠POA = ∠P'OA

∴ If ∠POA measured anticlockwise is θ, then ∠P'OA measured clockwise is (–θ)

∴ $\sin(-\theta) = \frac{AP'}{OP} = -\frac{AP}{OP} = -\sin\theta$

$\cos(-\theta) = \frac{OA}{OP'} = \frac{OA}{OP} = \cos\theta$

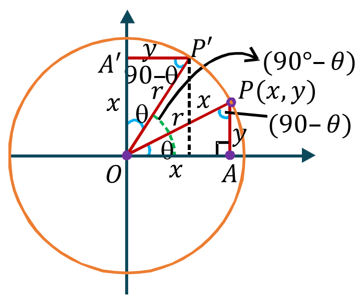
$\tan(-\theta) = \frac{AP'}{OA} = -\frac{AP}{OA} = -\tan\theta$

(ii) T-ratios of angle (90° – θ):

In Ist quadrant all trigonometric ratios are positive

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$.

ΔPOA & ΔP′OA′ are congruent



∴ $\sin(90° - \theta) = \frac{x}{r} = \cos\theta$

$\cos(90° - \theta) = \frac{y}{r} = \sin\theta$

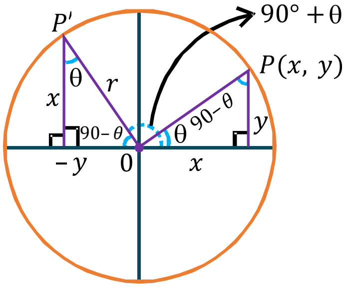
$$\Rightarrow \begin{cases} \sin(90° - \theta) = \cos\theta & ; \quad \cot(90° - \theta) = \tan\theta \\ \cos(90° - \theta) = \sin\theta & ; \quad \text{cosec}(90° - \theta) = \sec\theta \\ \tan(90° - \theta) = \cot\theta & ; \quad \sec(90° - \theta) = \text{cosec}\,\theta \end{cases}$$

Note:

θ and $90° - \theta$ are called complementary angles.

Ex. $\sin60° = \cos30°$, $\tan10° = \cot80°$

(iii) T-ratios of angle $(90° + \theta)$:



$\sin(90° + \theta) = \frac{x}{r} = \cos\theta$

$\cos(90° + \theta) = \frac{-y}{r} = -\sin\theta$

$\tan(90° + \theta) = -\cot\theta$ replace θ by $-\theta$ in (i)

$\sin(90° - (-\theta)) = \sin(90° + \theta) = \cos\theta$

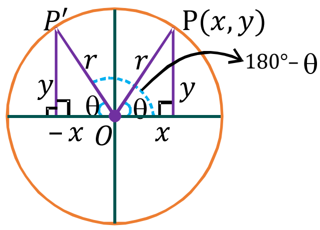
Ex. $\sin(120°) = \sin(90° + 30°) = \cos(30°) = \frac{\sqrt{3}}{2}$

$\cos(120°) = -\sin(30°) = -1/2$

$\tan(135°) = \tan(90° + 45°) = -\cot45° = -1$

$\cos(150°) = \cos(90° + 60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$

(iv) T-ratios of angle $(180° - \theta)$:



$\sin(180° - \theta) = \frac{y}{r} = \sin\theta$

OR Replace θ by $90° - \theta$ in $\sin(90° + \theta) = \cos\theta$

$\cos(180° - \theta) = \frac{-x}{r} = -\cos\theta$

$\tan(180° - \theta) = -\tan\theta$

Ex. $\sin120° = \sin(180° - 60°) = \sin60° = \frac{\sqrt{3}}{2}$

Illustration 12:

$\cos 10° + \cos 20° + \cos 30° + ... + \cos 80° + \cos 100° + \cos 150° + \cos 160° + \cos 170° = 0$.

Solution:

$\cos10° + \cos20° + \cos30° + ... + \cos170°$

We know that sum of cosine of supplementary angles is zero.

If $\alpha + \beta = 180°$; $\alpha = 180° - \beta \Rightarrow \cos\alpha = -\cos\beta$

$\cos\alpha + \cos\beta = 0$

$\cos10° + \cos170° + \cos20° + \cos160° + ... + \cos80° + \cos10° + \cos90° = 0$

Note: Similarly if $\alpha + \beta = 180°$ then $\tan\alpha + \tan\beta = 0$

Illustration 13:

$$\tan\frac{\pi}{11} + \tan\frac{2\pi}{11} + \tan\frac{4\pi}{11} + \tan\frac{7\pi}{11} + \tan\frac{9\pi}{11} + \tan\frac{10\pi}{11} = 0$$

Solution:

$$\tan\frac{\pi}{11} + \tan\frac{2\pi}{11} + \tan\frac{4\pi}{7} + \tan\frac{7\pi}{11} + \tan\frac{9\pi}{11} + \tan\frac{10\pi}{11}$$

$$\left(\tan\frac{\pi}{11} + \tan\frac{10\pi}{11}\right) + \left(\tan\frac{2\pi}{11} + \tan\frac{9\pi}{11}\right) + \tan\left(\frac{4\pi}{7} + \tan\frac{7\pi}{11}\right)$$

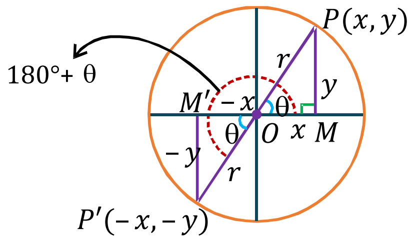
We know that if $\alpha + \beta = \pi$

$\tan\alpha + \tan\beta = 0$

$$\therefore \left(\tan\frac{\pi}{11} + \tan\frac{10\pi}{11}\right) + \left(\tan\frac{2\pi}{11} + \tan\frac{9\pi}{11}\right) + \tan\left(\frac{4\pi}{7} + \tan\frac{7\pi}{11}\right) = 0$$

(v) T-ratios of angle (180° + θ):

$\sin(180° + \theta) = \frac{-y}{r} = -\sin\theta$



OR Replace θ by 90° + θ in sin(90° + θ) = cos θ

$\cos(180° + \theta) = -\cos\theta$,

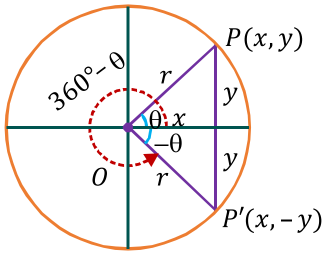
$\tan(180° + \theta) = \tan\theta$

Ex. $\sin(210°) = \sin(180° + 30°) = -\sin 30° = \frac{-1}{2}$

$\cos(240°) = \cos(180° + 60°) = -\cos 60° = -\frac{1}{2}$

(vi) T-ratios of angle (360°– θ):

$\sin(360° - \theta) = \frac{-y}{r} = -\sin\theta = \sin(-\theta)$



Replace θ by 270° – θ in sin(90° + θ) = cos θ

$\cos(360° - \theta) = \cos(-\theta) = \frac{x}{r} = \cos\theta$

$\tan(360° - \theta) = \tan(-\theta) = -\tan\theta$

Ex. $\sin(315°) = \sin(360° - 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$

$\cos(315°) = \cos 45° = \frac{1}{\sqrt{2}}$

$\tan(330°) = \tan(360° - 30°) = -\tan 30° = -\frac{1}{\sqrt{3}}$

$\tan(-120°) = -\tan 120° = -\tan(180° - 60°) = \tan 60° = \sqrt{3}$

Memory tips:

Rule-1:

(i) Reduction about 0° ± θ, 180° ± θ; 360° ± θ. No change in trigonometric ratio (sine remain sine)

(ii) Reduction about 90° ± θ, 270° ± θ. Change trigonometric ratio to its complementary.
(sine change to cos)

Rule-2:

Depending on the quadrant in which given angle lies, decide the sign as per trigonometric ratio being reduced.

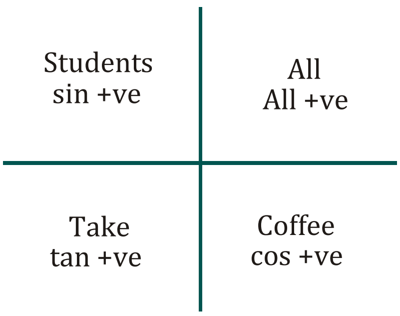


Illustration 14:

Find the value of $\sin\dfrac{31\pi}{3}$.

Solution:

We know that values of $\sin x$ repeats after an interval of 2π. Therefore

$$\sin\frac{31\pi}{3} = \sin\left(10\pi + \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Illustration 15:

Find the value of $\cos(-1710°)$.

Solution:

We know that values of $\cos x$ repeats after an interval of 2π or 360°.

Therefore

$\cos(-1710°) = \cos(-1710° + 5 \times 360°)$

$= \cos(-1710° + 1800°) = \cos 90° = 0.$

Illustration 16:

Find the values of the trigonometric functions in Qus. 1 to 5.

1. $\sin 765°$ **2.** $\operatorname{cosec}(-1410°)$ **3.** $\tan\dfrac{19\pi}{3}$

4. $\sin\left(-\dfrac{11\pi}{3}\right)$ **5.** $\cot\left(-\dfrac{15\pi}{4}\right)$

Solution:

1. $\sin 765°$

We know that values of $\sin x$ repeat after an interval of 2π or 360°

So we get

$\sin 765° = \sin(2 \times 360° + 45°)$

By further calculation

$= \sin 45° = 1/\sqrt{2}$

2. $\operatorname{cosec}(-1410°)$

We know that values of $cosecx$ repeat after an interval of 2π or 360°

So we get

$\operatorname{cosec}(-1410°) = \operatorname{cosec}(-1410° + 4 \times 360°)$

By further calculation

$= \operatorname{cosec}(-1410° + 1440°) = \operatorname{cosec} 30° = 2$

3. $\tan\dfrac{19\pi}{3}$

We know that values of $\tan x$ repeat after an interval of π or 180°

So we get

$\tan\dfrac{19\pi}{3} = \tan 6\dfrac{1}{3}\pi$

By further calculation , We get

$= \tan\left(6\pi + \dfrac{\pi}{3}\right) = \tan\dfrac{\pi}{3}$

$= \tan 60° = \sqrt{3}$

4. $\sin\left(-\frac{11\pi}{3}\right)$

We know that values of $\sin x$ repeat after an interval of 2π or 360°

So we get

$$\sin\left(-\frac{11\pi}{3}\right) = \sin\left(-\frac{11\pi}{3} + 2 \times 2\pi\right)$$

By further calculation

$$= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

5. $\cot\left(-\frac{15\pi}{4}\right)$

We know that values of tan x repeat after an interval of π or 180°

So we get

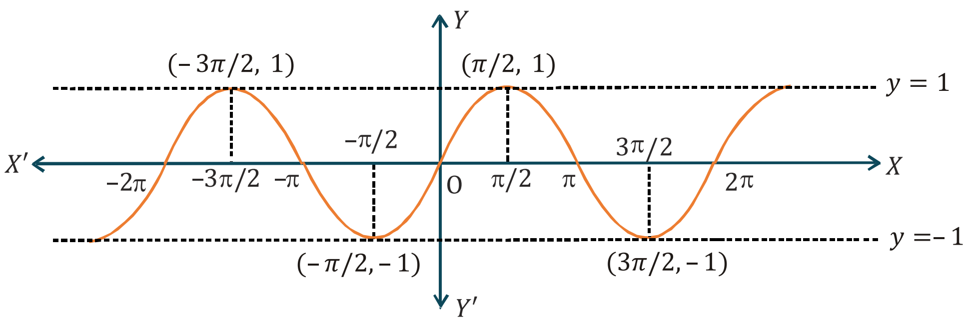
$$\cot\left(-\frac{15\pi}{4}\right) = \cot\left(-\frac{15\pi}{4} + 4\pi\right)$$

By further calculation

$$= \cot\frac{\pi}{4} = 1$$

Graphs of trigonometric ratios:

(1) $y = \sin x, x \in R, y \in [-1, 1]$



Periodic with period 2π

Note :

(i) $-1 \le \sin\theta \le 1$

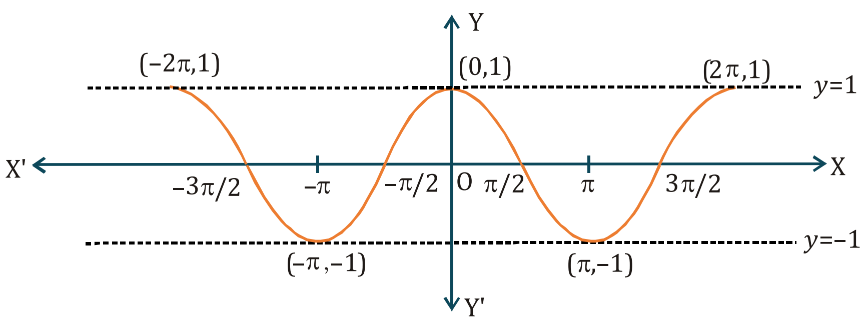
(ii) $\sin 0 = 0$; $\sin\pi = 0$; $\sin 2\pi = 0$

$\sin(n\pi) = 0, n \in I$

i.e. sine of integral multiple of $\pi = 0$

(iii) $\sin(2n\pi + \theta) = \sin\theta, n \in I$

(2) $y = \cos x, x \in R, y \in [-1,1]$



Periodic with period 2π

Note:

(i) $-1 \le \cos\theta \le 1$

(ii) $\cos\frac{\pi}{2} = 0; \cos\frac{3\pi}{2} = 0; \cos\frac{5\pi}{2} = 0$

$\cos(2n-1)\frac{\pi}{2} = 0, n \in I$

i.e. cosine of odd integral multiple of $\frac{\pi}{2}$ is zero.

(iii) $\cos 0 = 1; \cos 2\pi = 1 ; \cos 4\pi = 1$

$\cos 2m\pi = 1, m \in I$

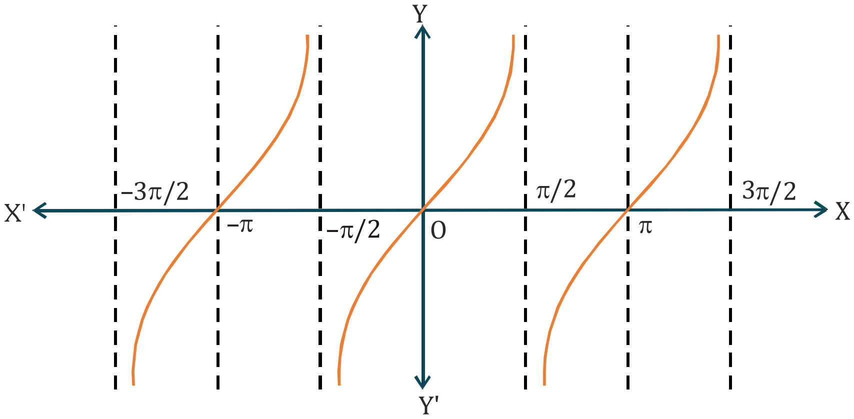
i.e. cos of even multiple of $\pi = 1$

(iv) $\cos\pi = -1; \cos 3\pi = -1; \cos 5\pi = -1$

$\cos(2m-1)\pi$ = cos of odd multiple of $\pi = -1$

$\cos(2n\pi + \theta) = \cos\theta,\ n \in I$

(3) $y = \tan x, x \neq \frac{\pi}{2}, \frac{3\pi}{2}$ ...... $\neq (2n-1)\frac{\pi}{2} : n \in I$



Periodic with period π

Note :

(i) $\tan(n\pi) = 0$

(ii) $\tan(n\pi + \theta) = \tan\theta,\ n \in I$

Domains, Ranges and Periodicity of Trigonometric Functions:

T-Ratio	Domain	Range	Period
$\sin x$	R	[−1,1]	2π
$\cos x$	R	[−1,1]	2π
$\tan x$	$R-\{(2n+1)\pi/2 ; n \in I\}$	R	π

Highlight of the table:

(1) $-1 \le \sin\theta \le 1; -1 \le \cos\theta \le 1$

(2) $\sin\theta = 0, \sin\pi = 0, \sin 2\pi = 0$

$\Rightarrow \sin n\pi = 0$ where $n \in I$

$\therefore$ sine of integral multiple of π is zero

(3) tan 0 = 0, tan π = 0

⇒ tan$n$π = 0, where n ∈ I

∴ tangent of integral multiple of π is zero.

(4) $\cos\frac{\pi}{2} = 0;\ \cos\frac{3\pi}{2} = 0;\ \cos\left(-\frac{\pi}{2}\right) = 0$

⇒ $\cos(2n-1)\frac{\pi}{2} = 0$, where n ∈ I

cosine of odd integral multiple of $\frac{\pi}{2}$ is zero.

(5) cos0 = 1, cos π = –1, cos 2π = 1

∴ cos2mπ = 1; cos(2m – 1)π = – 1 , where m ∈ I

cosine of odd integral multiple of π is – 1 and even integral multiple of π is 1. Same is the fate of sec.

(6) $\tan\frac{\pi}{2} = $ ND; $\tan\frac{3\pi}{2} = $ ND

$\tan(2n+1)\frac{\pi}{2} = $ ND, where n ∈ I

tangent of odd integral multiple of $\frac{\pi}{2}$ is ND

(7) $\sin\left(2n\pi + \frac{\pi}{2}\right) = 1, \sin\left(2n\pi - \frac{\pi}{2}\right) = -1 = \sin\left(2n\pi + \frac{3\pi}{2}\right)$

Illustration 17:

Number of solution of the equation (2sinθ – 1) (sinθ – 2) = 0 in (0, π)

Solution:

(2sinθ–1) (sinθ–2) = 0 in (0, π)

⇒ sinθ = $\frac{1}{2}$ OR sinθ = 2 (Not possible)

θ = 30° or 180° – 30° = 150°

⇒ 2 solution

Illustration 18:

Find values log(cos(12346π)) = 0

Solution:

log(cos(12346π)) = log(1) = 0 [∵ $\cos 2n\pi = 1$]

Illustration 19:

cos1 – sin1 > 0. True/False

[Ans. F]

Solution:

$1^c \approx 57.2°$

So if 45° < θ < 90°

sinθ > cosθ

So sin1 > cos1

Illustration 20:

tan(11π) = 0 True/False

[Ans. T]

Solution:

tan(11π) = tan(10π + π) = tan π = 0

[tan nπ = 0]

Illustration 21:

$\frac{\sin 4}{\sin 6}$ is negative. True/False

[Ans. F]

Solution:

$4^c \approx 4 \times 57° \approx 228°$

$6^c \approx 6 \times 57° \approx 342°$

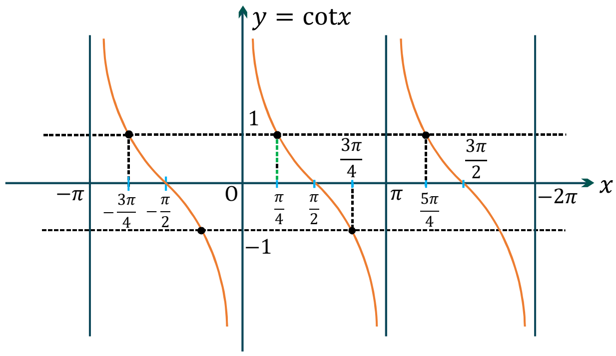
$\sin 228° = \sin(180° + 48°) = -\sin 48°$

$\sin 342° = \sin(360° - 18°) = -\sin 18°$

So $\frac{\sin 4}{\sin 6} = +ve$

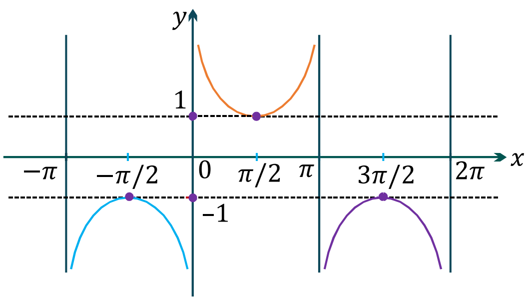
Graphs of Trigonometric Ratios:

(1) $y = \cot x, x \in R, y \in (-\infty, \infty), x \neq n\pi$ for $n \in I$

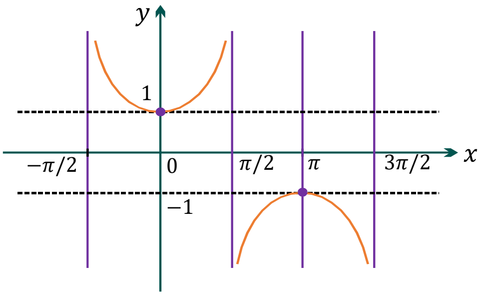


Note: $\cot (2n - 1)\frac{\pi}{2} = 0 : n \in I$

(2) $y = \text{cosec } x, y \in (-\infty, -1] \cup [1, \infty), x \in R - n\pi, \ x \neq n\pi$ for $n \in I$



(3) $y = \sec x, x \neq (2n + 1)\frac{\pi}{2}$ for $n \in I$



Domains, Ranges and Periodicity of Trigonometric Functions:

T-Ratio	Domain	Range	Period
$\cot x$	$R-\{n\pi : n \in I\}$	R	π
$\sec x$	$R-\{(2n+1)\,\pi/2 : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π
$\text{cosec}\, x$	$R-\{n\pi : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π

Highlight of the table:

(1) $\tan\theta$, $\cot\theta$ can attain all values from $(-\infty, \infty)$ i.e. any real value.

$\sec\theta$ & $\text{cosec}\theta \in (-\infty, -1]$ or $[1, \infty)$

(2) $\cot\frac{\pi}{2} = 0;\ \cot\frac{3\pi}{2} = 0$

$\therefore\ \cos(2n-1)\frac{\pi}{2} = 0,\ \text{where } n \in I$

cotangent of odd integral multiple of $\frac{\pi}{2}$ is zero.

(3) $\cot 0° = \text{ND}$, $\cot\pi = \text{ND}$, $\cot 2\pi = \text{ND}$

$\therefore\ \cot n\pi = \text{ND}$, where $n \in I$

cotangent of integral multiple of π is not defined.

Illustration 22:

If $\text{cosec}\theta = -2$ then find the two possible values of θ in $(0, 2\pi)$

Solution:

Given: $\text{cosec}\theta + 2 = 0$

$\Rightarrow \sin\theta = -\frac{1}{2}$

$\Rightarrow \theta = \left(\pi + \frac{\pi}{6}\right), \left(2\pi - \frac{\pi}{6}\right)$

$\Rightarrow \theta = 210°, 330°$

Illustration 23:

If $\sec\theta = 2$ then number of solution of equation in $(-2\pi, 2\pi)$ is

Solution:

Given : $\sec\theta = 2$

$\Rightarrow cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \left(\frac{\pi}{3}\right), \left(2\pi - \frac{\pi}{3}\right)$

$\Rightarrow \theta = 60°, 300°$

Illustration 24:

If $\cot\theta = 1$ then number of solution of equation in $(-2\pi, 2\pi)$ is

Solution:

Given: $\cot\theta = 1$

$\Rightarrow \theta = \left(\frac{\pi}{4}\right), \left(\pi + \frac{\pi}{4}\right)$

$\Rightarrow \theta = 45°, 225°$

Trigonometric Ratio of Compound angles:

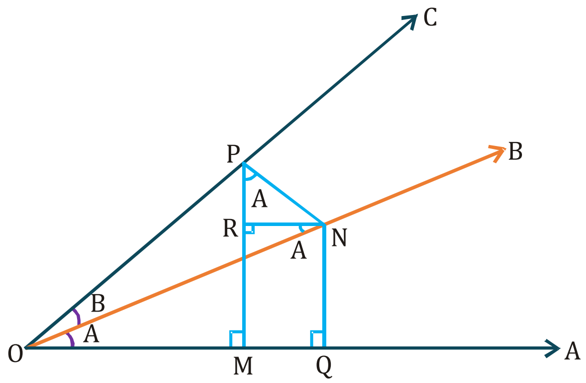
$A \pm B,\ A \pm B \pm C,\ \frac{A}{2} \pm \frac{B}{2}$ etc. are compound angles

$\sin(A+B) \neq \sin A + \sin B$

$\sin 90° \neq \sin 60° + \sin 30°$

$\mathbf{\sin(A+B) = \sin A\cos B + \cos A\sin B}$

Proof: From the figure



$\sin(A+B) = \frac{PM}{OP} = \frac{PR+RM}{OP}$

$= \frac{PR}{OP} + \frac{RM}{OP} = \frac{PR}{PN}\frac{PN}{OP} + \frac{NQ}{ON}\frac{ON}{OP}$

$= \cos A\sin B + \sin A\cos B$

$\Rightarrow \sin(A+B) = \sin A\cos B + \cos A\sin B$

Replace B by $-B$

$\sin(A+(-B)) = \sin A\cos(-B) + \cos A\sin(-B)$

$\therefore$ $\mathbf{\sin(A-B) = \sin A\cos B - \cos A\sin B}$

Replace A by $\frac{\pi}{2}+A$

$\text{in}\left(\frac{\pi}{2}+A+B\right) = \sin\left(\frac{\pi}{2}+A\right)\cos B + \cos\left(\frac{\pi}{2}+A\right)\sin B$

or $\mathbf{\cos(A+B) = \cos A\cos B - \sin A\sin B}$

$\mathbf{\cos(A-B) = \cos A\cos B + \sin A\sin B}$

Illustration 25:

$\sin 99°\cos 21° + \cos 99°\sin 21° =$

Solution:

$\sin(99° + 21°) = \sin 120° = \frac{\sqrt{3}}{2}$

Illustration 26:

The value of $\sin(n+1)A\sin(n+2)A + \cos(n+1)A\cos(n+2)A$ -

(A) $\sin(A+B)$ (B) $\cos(A+B)$ (C) $\cos A$ (D) $\sin A$

Ans. (C)

Solution:

We have $\sin(n+1)A\sin(n+2)A + \cos(n+1)A\cos(n+2)A$

$= \cos(n+1)A\cos(n+2)A + \sin(n+1)A\sin(n+2)A$

$= \cos[(n+1)A - (n+2)A] = \cos(-A) = \cos A$

Values of trigonometric ratios of some important angles:

$\sin 75° = \sin(45° + 30°) = \sin 45°\cos 30° + \cos 45°\sin 30° = \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2}$

or $\sin 75° = \cos 15° = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{1}{\sqrt{6}-\sqrt{2}}$;

Similarly $\sin 15° = \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{1}{\sqrt{6}+\sqrt{2}}$

Also $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3} = \cot 15°$

$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$

Note :

Remember value of sine & cosine of 15° & 75°

Illustration 27:

$$\cos\frac{2\pi}{3}\cos\frac{\pi}{4}-\sin\frac{2\pi}{3}\sin\frac{\pi}{4}=?$$

Solution:

$$\cos\frac{11\pi}{12}=\cos\left(\pi-\frac{\pi}{12}\right)=-\cos\frac{\pi}{12}=-\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

Illustration 28:

Prove that $\dfrac{\sin(x-y)}{\sin(x+y)}=\dfrac{\tan x-\tan y}{\tan x+\tan y}$

Solution:

$$\frac{\sin(x-y)}{\sin(x+y)}=\frac{\sin x\cos y-\cos x\sin y}{\sin x\cos y+\cos x\sin y}$$

$$=\frac{\cos x\cos y(\tan x-\tan y)}{\cos x\cos y(\tan x+\tan y)}=\frac{\tan x-\tan y}{(\tan x+\tan y)}$$

Illustration 29:

If $\tan A = k\tan B$ prove that $\sin(A+B)=\dfrac{k+1}{k-1}\sin(A-B)$.

Solution:

$\tan A = k\tan B \Rightarrow \dfrac{\tan A}{\tan B}=\dfrac{k}{1}$ using componendo & dividendo

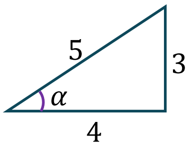
$$\frac{\tan A+\tan B}{\tan-\tan B}=\frac{k+1}{k-1}\Rightarrow\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{\frac{\sin A}{\cos A}-\frac{\cos B}{\cos B}}=\frac{k+1}{k-1}$$

$$\frac{\sin A\cos B+\cos A\sin B}{\sin A\cos B-\cos A\sin B}=\frac{k+1}{k-1}$$

$$\Rightarrow\sin(A+B)=\frac{(k+1)}{(k-1)}\sin(A-B)$$

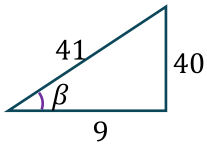
Illustration 30:

$\sin\alpha=\dfrac{3}{5}$ & $\cos\beta=\dfrac{9}{41}$ find $\sin(\alpha-\beta)$. All possible value.



$\sin\alpha=\frac{3}{5}$

$\cos\alpha=\frac{4}{5}$



$\sin\beta=\frac{40}{41}$

$\cos\beta=\frac{9}{41}$

Solution:

Proper solution

Case	1	2	3	4	
α	I	I	II	II	Quadrant
β	I	IV	I	IV	

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

Case (1) $=\dfrac{3}{5}\times\dfrac{9}{41}-\dfrac{4}{5}\times\dfrac{40}{41}=\dfrac{27-160}{205}=\dfrac{-133}{205}$

Case (2) $=\left(\dfrac{3}{5}\right)\times\dfrac{9}{41}-\left(\dfrac{4}{5}\right)\left(-\dfrac{40}{41}\right)=\dfrac{27+160}{205}=\dfrac{187}{205}$

Case (3) $=\left(\dfrac{3}{5}\right)\times\left(\dfrac{9}{41}\right)-\left(-\dfrac{4}{5}\right)\left(\dfrac{40}{41}\right)=\dfrac{187}{205}$

Case (4) $=\left(\dfrac{3}{5}\right)\times\left(\dfrac{9}{41}\right)-\left(-\dfrac{4}{5}\right)\left(-\dfrac{40}{41}\right)=\dfrac{-133}{205}$

Formulae & Identities:

(a) $\sin(A+B).\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

Proof : $(\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B = \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 B = (1-\cos^2 A) - (1-\cos^2 B) = \cos^2 B - \cos^2 A$

Illustration 31:

Prove that $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A$

Solution:

$\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right) = \sin\left(\frac{\pi}{8}+\frac{A}{2}+\frac{\pi}{8}-\frac{A}{2}\right).\sin\left(\frac{\pi}{8}+\frac{A}{2}-\frac{\pi}{8}+\frac{A}{2}\right)$

$= \sin\frac{\pi}{4}.\sin A = \frac{1}{\sqrt{2}}\sin A$

(b) $\cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B) = \cos^2 B - \sin^2 A$

Illustration 32:

$\cos^2\left(\frac{\pi}{4}+\theta\right) - \sin^2\left(\frac{\pi}{4}-\theta\right) = 0$

Solution:

$\cos\left(\frac{\pi}{4}+\theta+\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}+\theta-\frac{\pi}{4}+\theta\right) = \cos\frac{\pi}{2}\cdot\cos 2\theta = 0$

Formula to transform the product into sum or difference:

(1) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(2) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(3) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(4) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Proof :

(1) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B = 2\sin A\cos B$

Very Frequently used formulae

(i) $2\sin 3\theta\cos\theta = \sin(3\theta+\theta) + \sin(3\theta-\theta) = \sin 4\theta + \sin 2\theta$

(ii) $2\cos 11\theta\cos\theta = \cos(11\theta+\theta) + \cos(11\theta-\theta) = \cos 12\theta + \cos 10\theta$

(iii) $2\sin 5\theta\sin 3\theta = \cos(5\theta-3\theta) - \cos(5\theta+5\theta) = \cos 2\theta - \cos 8\theta$

Illustration 33:

If $\theta = 7.5°$. Find $\frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta}$

Solution:

Multiply & divide by 2

$\Rightarrow \frac{2\sin 8\theta\cos\theta - 2\sin 6\theta\cos 3\theta}{2\cos 2\theta\cos\theta - 2\sin 3\theta\sin 4\theta}$

$= \frac{\sin 9\theta + \sin 7\theta - (\sin 9\theta + \sin 3\theta)}{\cos 3\theta + \cos\theta - (\cos\theta - \cos 7\theta)} = \frac{\sin 7\theta - \sin 3\theta}{\cos 3\theta + \cos 7\theta}$

$= \frac{2\cos 5\theta\sin 2\theta}{2\cos 5\theta\cos 2\theta} = \tan 2\theta = \tan 15° = 2-\sqrt{3}$

Illustration 34:

Prove that $\cos^2 73 + \cos^2 47 + \cos 73 \cos 47 = \frac{3}{4}$

Solution:

$\cos^2 73° + 1 - \sin^2 47° + \frac{2\cos 73° \cos 47°}{2}$

$= (1 + \cos 120° \cos 26°) + \frac{\cos 120° + \cos 26°}{2}$

$= 1 - \frac{\cos 26}{2} + \frac{\cos 26}{2} - \frac{1}{4} = \boxed{\frac{3}{4}}$

Illustration 35:

$2\sin\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{12}\right) =$

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) $\frac{1}{6}$

Ans. (B)

Solution:

$\cos\left(\frac{5\pi}{12} - \frac{\pi}{12}\right) - \cos\left(\frac{5\pi}{12} + \frac{\pi}{12}\right)$

$= \cos\frac{\pi}{3} - \cos\frac{\pi}{2} = \frac{1}{2}$

Illustration 36:

Prove that $\sin(45 + A)\sin(45 - A) = \frac{1}{2}\cos 2A$

Solution:

$= \frac{1}{2}[2\sin(45 + A)\sin(45 - A)]$

$= \frac{1}{2}[\cos(45 + A - 45 + A) - \cos(90)]$

$= \frac{1}{2}[\cos 2A]$

Illustration 37:

$\frac{2\sin(A - C)\cos C - \sin(A - 2C)}{2\sin(B - C)\cos C - \sin(B - 2C)} = \frac{\sin A}{\sin B}$

Solution:

$\frac{\sin A + \sin(A - 2C) - \sin(A - 2C)}{\sin B + \sin(B - 2C) - \sin(B + 2C)} =$ R.H.S.

Illustration 38:

$2\cos\frac{\pi}{13}.\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$

Solution:

$\cos\frac{10\pi}{13} + \cos\frac{8\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$

Identities for Converting Sum to Product:

Let $A + B = C$ & $A - B = D$

$\therefore \quad A = \frac{C+D}{2}, B = \frac{C-D}{2}$

1. $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

2. $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

3. $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

4. $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Proof:

1. **2sinA cosB = sin(A + B) + sin(A – B)**

$= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$= \sin\left(\frac{C+D}{2} + \frac{C-D}{2}\right) + \sin\left(\frac{C+D}{2} - \frac{C-D}{2}\right) = \sin C + \sin D$

Illustration 39:

(i) $\frac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan\theta$ (ii) $\frac{\sin A + \sin 3A}{\cos A - \cos 3A} = \cot A$

Solution:

(i) $\text{LHS} = \frac{2\cos 6\theta \cdot \sin\theta}{2\cos 6\theta \cdot \cos\theta} = \tan\theta$

(ii) $\frac{2\sin 2A\cos A}{2\sin 2A\sin A} = \cot A$

Illustration 40:

$\frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} =$

(A) $\tan\theta$ (B) $\tan 5\theta$ (C) $\tan 3\theta$ (D) None of these

Ans. (C)

Solution:

$$\frac{(\sin\theta + \sin 5\theta) + \sin 3\theta}{(\cos\theta + \cos 5\theta) + \cos 3\theta} = \frac{2\sin\frac{\theta + 5\theta}{2}.\cos\frac{\theta - 5\theta}{2} + \sin 3\theta}{2\cos\frac{\theta + 5\theta}{2}.\cos\frac{\theta - 5\theta}{2} + \cos 3\theta}$$

$$= \frac{2\sin 3\theta\cos(-2\theta) + \sin 3\theta}{2\cos 3\theta\cos(-2\theta) + \cos 3\theta} = \frac{\sin 3\theta[2\cos 2\theta + 1]}{\cos 3\theta[2\cos\theta + 1]} = \frac{\sin 3\theta}{\cos 3\theta} \Rightarrow \tan 3\theta$$

Illustration 41:

$\frac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}$ is equal to -

(A) $\sin 4A$ (B) $\cos 4A$ (C) $\tan 4A$ (D) None of these

Ans. (C)

Solution:

$$\frac{(\sin A + \sin 7A) + (\sin 3A + \sin 5A)}{(\cos A + \cos 7A) + (\cos 3A + \cos 5A)} = \frac{2\sin 4A\cos 3A + 2\sin 4A\cos A}{2\cos 4A\cos 3A + 2\cos 4A\cos A}$$

$$= \frac{2\sin 4A(\cos 3A + \cos A)}{2\cos 4A(\cos 3A + \cos A)} = \tan 4A$$

Illustration 42:
Prove that sin50° – sin70° + sin10° = 0

Solution:

$2\cos 60°\sin(-10°) + \sin 10° = 2 \times \frac{1}{2}(-\sin 10°) + \sin 10° = 0.$

Illustration 43:
If three angles A,B,C are in A.P. then the value of $\frac{\sin A - \sin C}{\cos C - \cos A}$ is equal to -

(A) $\cot\left(\frac{A-C}{2}\right)$ (B) $\cot B$ (C) $\tan B$ (D) None of these

Ans. (B)

Solution:

Angles A, B, C are in A.P. then $2B = A + C$

Now $\frac{\sin A - \sin C}{\cos C - \cos A} = \frac{2\cos\frac{A+C}{2}\sin\frac{A-C}{2}}{2\sin\frac{A+C}{2}\sin\frac{A-C}{2}} = \cot\frac{A+C}{2} = \cot\frac{2B}{2} = cot\ B$

Illustration 44:
Prove that $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right)$

Solution:

$\underbrace{\boldsymbol{\cos\alpha + \cos\beta}} + \underbrace{\boldsymbol{\cos\gamma + \cos(\alpha+\beta+\gamma)}} = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right)$

L.H.S.$=2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\cdot\cos\left(\frac{\alpha+\beta}{2}\right)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left[\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right]$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)2\cos\left(\frac{\alpha-\beta+\alpha-\beta+2\gamma}{4}\right)\cos\left(\frac{\alpha+\beta+2\gamma-\alpha+\beta}{4}\right)$

$= 4\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\beta+\gamma}{2}\right)\cdot\cos\left(\frac{\gamma+\alpha}{2}\right)$

Illustration 45:
Prove that $\frac{\cos 20° + \sin 10°}{\cos 20° - \sin 10°} = \sqrt{3}\tan 40°$

Solution:

$\frac{\sin 70° + \sin 10°}{\sin 70° - \sin 10°}$

$= \frac{2\sin\frac{(70°+10°)}{2}.\cos\frac{(70°-10°)}{2}}{2\cos\frac{(70°+10°)}{2}.\sin\frac{(70°-10°)}{2}}$

$= \frac{2\sin 40°\cos 30°}{2\cos 40°\sin 30°} = \tan 40° \times \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}\tan 40°$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

Proof : $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$ Divide by $\cos A\cos B$

$\Rightarrow \tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

Similarly $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Also, $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot B + \cot A}$, $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$

Note : (i) $\tan\left(\dfrac{\pi}{4}+A\right) = \dfrac{1+\tan A}{1-\tan A}$ (ii) $\tan\left(\dfrac{\pi}{4}-A\right) = \dfrac{1-\tan A}{1+\tan A}$

Illustration 46:

If $\tan A = -\dfrac{1}{2}$ and $\tan B = -\dfrac{1}{3}$ then $A+B =$

(A) $\dfrac{\pi}{4}$ (B) $\dfrac{3\pi}{4}$ (C) $\dfrac{5\pi}{4}$ (D) None

Solution:

Using the $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B} = \dfrac{-\frac{1}{2}-\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = \dfrac{-\frac{5}{6}}{\frac{5}{6}} = -1 \Rightarrow A+B = \dfrac{3\pi}{4}$

Illustration 47:

Prove that $\dfrac{\cos 9°+\sin 9°}{\cos 9°-\sin 9°} = \tan 54°$

Divide by $\cos 9° = \dfrac{1+\tan 9°}{1-\tan 9°} = \tan(45°+9°)$

Illustration 48:

Prove that $\cot 16°\cot 44° + \cot 44°\cot 76° - \cot 76°\cot 16° = 3$

Solution:

$(\cot 16°\cot 44° - 1) + (\cot 44°\cot 76° - 1) - (\cot 76°\cot 16° + 1)$

$= \cot 60°(\cot 44° + \cot 16°) + \cot 120°(\cot 44° + \cot 76°) - \cot 60°(\cot 16° - \cot 76°)$

$= \dfrac{1}{\sqrt{3}}[\cot 44° + \cot 16° - \cot 44° - \cot 76° - \cot 16° + \cot 76°] = 0]$

Illustration 49:

Prove that $\tan 70° = \cot 70° + 2\cot 40°$.

Solution:

L.H.S. $= \tan 70° = \tan(20°+50°) = \dfrac{\tan 20°+\tan 50°}{1-\tan 20°\tan 50°}$

or $\tan 70° - \tan 20°\tan 50°\tan 70° = \tan 20° + \tan 50°$

or $\tan 70° = \tan 70°\tan 50°\tan 20° + \tan 20° + \tan 50° = 2\tan 50° + \tan 20°$

$= \cot 70° + 2\cot 40° =$ R.H.S.

Illustration 50:

$\tan 3A.\tan 2A\tan A = \tan 3A - \tan 2A - \tan A$

Solution:

$\because \tan 3A = \tan(A+2A) = \dfrac{\tan A + \tan 2A}{1 - \tan A\tan 2A}$

$\Rightarrow \tan 3A - \tan A\tan 2A\tan 3A = \tan A + \tan 2A$

$\Rightarrow \tan 3A - \tan 2A - \tan A = \tan A\tan 2A\tan 3A$

Illustration 51:

$\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

Solution:

$\cot x \cot 2x \cot 3x[\tan 3x - \tan x - \tan 2x]$

$\cot x \cot 2x \cot 3x[\tan x \tan 2x \tan 3x] = 1$

Illustration 52:

If $A + B = 45°$. Find value of $(1 + \tan A)(1 + \tan B)$

Solution:

$(1 + \tan A)(1 + \tan B) = 1 + \tan A + \tan B + \tan A \tan B$

$= 1 + \tan(A + B)(1 - \tan A \tan B) + \tan A \tan B$ $\left\{\begin{array}{l} \because \quad \tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B} \\ \Rightarrow \quad \tan A + \tan B = 1 - \tan A \tan B \end{array}\right.$

$= 1 + 1 - \tan A \tan B + \tan A \tan B = 2$

Trigonometrical ratios of the sum of more than 2 angles:

$\sin(A + B + C) = \sin(A + B) \cos C + \cos(A + B) \sin C$

$= [\sin A \cos B + \cos A \sin B] \cos C + [\cos A \cos B - \sin A \sin B] \sin C$

$= \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

$\cos(A + B + C) = \cos(A + B) \cos C - \sin(A + B) \sin C$

$= (\cos A \cos B - \sin A \sin B)\cos C - (\sin A \cos B + \cos A \sin B) \sin C$

$= \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C.$

Also, $\tan(A + B + C) = \dfrac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C} = \dfrac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \dfrac{\tan A + \tan B}{1 - \tan A \tan B}\tan C}$

$= \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B} = \dfrac{s_1 - s_3}{1 - s_2}$

$\tan(A_1 + A_2 + ... + A_n) = \dfrac{s_1 - s_3 + s_5 - s_7 + ...}{1 - s_2 + s_4 - s_6 + ...}$

where $s_1 = \tan A_1 + \tan A_2 + ... + \tan A_n$ = the sum of the tangent of the separate angles,

$s_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + ...$ = the sum of the tangents taken two at a time,

$s_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + ...$

= the sum of the tangents taken three at a time, and so on.

Trigonometric Ratios of Multiple Angles:

Trigonometrical ratios of an angle 2θ in terms of the angle:

(i) $\sin 2\theta = 2 \sin \theta \cos \theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(iii) $1 + \cos 2\theta = 2 \cos^2 \theta$

(iv) $1 - \cos 2\theta = 2 \sin^2 \theta$

(v) $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{\sin 2\theta}{1+\cos 2\theta}$

(vi) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

1. **$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$**

Proof :

$\sin 2A = \sin(A+A) = \sin A\cos A + \sin A\cos A = 2\sin A\cos A = \frac{2\sin A\cos A}{\sin^2 A+\cos^2 A} = \frac{2\tan A}{1+\tan^2 A}$

Ex. $\sin 4A = 2\sin 2A\cos 2A$

$\sin 100A = 2\sin 50A\cos 50A$

$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

2. **$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$**

Proof :

$\cos(A+A) = \cos A\cos A - \sin A\,.\,\sin A = \cos^2 A - \sin^2 A = \frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$

Also $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Illustration 53:

If $\sin A = \frac{4}{5}$. Find $\cos 2A$

Solution:

$\cos 2A = 1 - 2\sin^2 A = 1 - 2\times\frac{16}{25} = -\frac{7}{25}$

Example:

(i) $\frac{1+\cos 2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \cot\theta$ **(ii)** $\frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$ **(iii)** $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta$

Illustration 54:

$\frac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} =$

(A) $\frac{1}{2}\tan\theta$ (B) $\frac{1}{2}\cot\theta$ (C) $\tan\theta$ (D) $\cot\theta$

Ans. (D)

Solution:

$\frac{(1+\cos 2\theta)+\sin 2\theta}{(1-\cos 2\theta)+\sin 2\theta} = \frac{2\cos^2\theta+2\sin\theta\cos\theta}{2\sin^2\theta+2\sin\theta\cos\theta} = \frac{2\cos\theta[\cos\theta+\sin\theta]}{2\sin\theta[\sin\theta+\cos\theta]} = \cot\theta$

Illustration 55:

Prove that $\frac{\sec 8A-1}{\sec 4A-1} = \frac{\tan 8A}{\tan 2A}$

Solution:

$= \frac{1-\cos 8A}{1-\cos 4A}\times\frac{\cos 4A}{\cos 8A} = \frac{2\sin^2 4A}{2\sin^2 2A}\cdot\frac{\cos 4A}{\cos 8A} = \frac{\sin 8A}{\cos 8A}\cdot\frac{\sin 4A}{2\sin^2 2A}$

$= \tan 8A\cdot\frac{2\sin 2A\cos 2A}{2\sin^2 2A} = \frac{\tan 8A}{\tan 2A}$

3. **$\sin 3A = 3\sin A - 4\sin^3 A$**

Proof :

$\sin 3A = \sin(A+2A) = \sin A\cos 2A + \cos A\sin 2A = \sin A(1-2\sin^2 A) + 2\sin A(1-\sin^2 A)$

$= 3\sin A - 4\sin^3 A$

Illustration 56:

sin90° = 3sin30° – 4sin³30°

Solution:

$1 = 3\cdot\frac{1}{2} - 4\cdot\frac{1}{8} = 1$

Similarly, $\cos 3A = 4\cos^3 A - 3\cos A$

Example: $\cos 60 = 4\cos^3 20 - 3\cos 20$

$$\& \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Illustration 57:

$\frac{3\cos\theta + \cos 3\theta}{3\sin\theta - \sin 3\theta}$ is equal to-

(A) $1 + \cot^2\theta$ (B) $\cot^4\theta$ (C) $\cot^3\theta$ (D) $2\cot\theta$

Ans. (C)

Solution:

$$\frac{3\cos\theta + 4\cos^3\theta - 3\cos\theta}{3\sin\theta - 3\sin\theta + 4\sin^3\theta} = \cot^3\theta$$

Illustration 58:

Find the value of $-6\sin 40° + 8\sin^3 40°$

Solution:

$-2(3\sin 40° - 4\sin^3 40°) = -2\sin 120° = -\sqrt{3}$

Illustration 59:

Prove that $(4\cos^2 9° - 3)(4\cos^2 27° - 3) = \tan 9°$

Solution:

$$\frac{(4\cos^3 9° - 3\cos 9°)}{\cos 9°}\frac{(4\cos^3 27° - 3\cos 27°)}{\cos 27°} = \frac{\cos 27°}{\cos 9°}\times\frac{\cos 81°}{\cos 27°} = \frac{\sin 9°}{\cos 9°} = \tan 9°$$

Illustration 60:

Prove that $\tan 3A \,.\, \tan 2A \,.\, \tan A = \tan 3A - \tan 2A - \tan A$

Solution:

$\tan(3A) = \tan(A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$

$\tan 3A - \tan A \tan 2A \tan 3A = \tan A + \tan 2A$

or $\tan A \tan 2A \tan 3A = \tan 3A - \tan 2A - \tan A$

Trigonometric Ratios of Sub Multiple Angles:

Since the trigonometric relations are true for all values of angle θ, they will be true if instead of θ be substitute $\frac{\theta}{2}$

(i) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(ii) $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(iii) $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

(iv) $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

(v) $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$

(vi) $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(vii) $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

(viii) $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

(ix) $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

(x) $2\sin\frac{\theta}{2} = \pm\sqrt{1+\sin\theta} \pm \sqrt{1-\sin\theta}$

(xi) $2\cos\frac{\theta}{2} = \pm\sqrt{1+\sin\theta} \mp \sqrt{1-\sin\theta}$

(xii) $\tan\frac{\theta}{2} = \frac{\pm\sqrt{1+\tan^2\theta}-1}{\tan\theta}$

for (vii) to (xii), we decide the sign of ratio according to value of θ.

Illustration 61:

$\sin 67\frac{1^o}{2} + \cos 67\frac{1^o}{2}$ is equal to

(A) $\frac{1}{2}\sqrt{4+2\sqrt{2}}$ (B) $\frac{1}{2}\sqrt{4-2\sqrt{2}}$ (C) $\frac{1}{4}\left(\sqrt{4+2\sqrt{2}}\right)$ (D) $\frac{1}{4}\left(\sqrt{4-2\sqrt{2}}\right)$

Ans. (A)

Solution:

$\sin 67\frac{1^o}{2} + \cos 67\frac{1^o}{2} = \sqrt{1+\sin 135^\circ} = \sqrt{1+\frac{1}{\sqrt{2}}}$ $\left(\text{using } \cos A + \sin A = \sqrt{1+\sin 2A}\right)$

$= \frac{1}{2}\sqrt{4+2\sqrt{2}}$

Illustration 62:

Prove that $\cos^4\frac{A}{2} - \sin^4\frac{A}{2} = \cos A$

Solution:

$= \left(\cos^2\frac{A}{2} - \sin^2\frac{A}{2}\right)\left(\cos^2\frac{A}{2} + \sin^2\frac{A}{2}\right) = \cos A$

Illustration 63:

Value of $\sqrt{2+\sqrt{2+2\cos 4\theta}} =$

(A) 2 sinθ (B) 2 cosθ (C) sinθ (D) cosθ

Ans. (B)

Solution:

$$\sqrt{2+\sqrt{2+2\cos 4\theta}} = \sqrt{2+\sqrt{2(1+\cos 4\theta)}} = \sqrt{2+\sqrt{2.2\cos^2 2\theta}}$$

$$= \sqrt{2+2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)} = \sqrt{2.2\cos^2\theta} = 2\cos\theta$$

Illustration 64:

Evaluate: (i) sin22.5° (ii) cos22.5° (iii) tan22.5°

Solution:

(i) $\sin^2(22.5°) = \dfrac{1-\cos 45°}{2} = \dfrac{1-\frac{1}{\sqrt{2}}}{2} = \dfrac{\sqrt{2}-1}{2\sqrt{2}}$

$= \dfrac{2-\sqrt{2}}{4} \Rightarrow \sin 22.5° = \dfrac{\sqrt{2-\sqrt{2}}}{2} = \cos 67\dfrac{1}{2}$

(ii) Similarly, $\cos(22.5°) = \dfrac{\sqrt{2+\sqrt{2}}}{2} = \sin 67\dfrac{1°}{2}$

(iii) $\tan 22\dfrac{1^o}{2} = \sqrt{\dfrac{1-\cos 45°}{1+\cos 45°}} = \dfrac{\sqrt{1-\frac{1}{\sqrt{2}}}}{\sqrt{1+\frac{1}{\sqrt{2}}}} = \sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}+1}} \times \sqrt{\dfrac{\sqrt{2}-1}{\sqrt{2}-1}} = \sqrt{2}-1 = \tan\dfrac{\pi}{8}$ (learn)

Also $\sqrt{2}+1 = \cot\dfrac{\pi}{8}$ (learn)

Illustration 65:

Find $\tan 7\dfrac{1^o}{2} = \sqrt{6}-\sqrt{4}-\sqrt{3}+\sqrt{2}$

Solution:

$$= \frac{2\sin 7\frac{1}{2}}{2\cos 7\frac{1}{2}} \cdot \frac{\sin 7\frac{1}{2}}{\sin 7\frac{1}{2}} = \frac{1-\cos 15}{\sin 15} = \frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \frac{2\sqrt{2}-(\sqrt{3}+1)}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \sqrt{6}-\sqrt{4}-\sqrt{3}+\sqrt{2}$$

Similarly $\cot 7\dfrac{1°}{2} = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

Some important results:

(i) $\sin\theta\sin(60°-\theta)\sin(60°+\theta) = \dfrac{1}{4}\sin 3\theta$

Proof:

$\dfrac{1}{2}\sin\theta[2\sin(60°-\theta)\sin(60°+\theta)]$

$= \dfrac{1}{2}\sin\theta[\cos 2\theta - \cos 120°]$

$= \dfrac{1}{2}\sin\theta\left[\cos 2\theta + \dfrac{1}{2}\right]$

$= \dfrac{1}{4}[2\sin\theta\cos 2\theta + \sin\theta]$

$= \frac{1}{4}[\sin 3\theta - \sin\theta + \sin\theta]$

$= \frac{1}{4}\sin 3\theta$

(ii) $\cos\theta\cos(60° - \theta)\cos(60° + \theta) = \frac{1}{4}\cos 3\theta$

(iii) $\tan\theta\tan(60° - \theta)\tan(60° + \theta) = \tan 3\theta$

Illustration 66:

$\sin 20°\sin 40°\sin 60°\sin 80° = \frac{3}{16}$

Solution:

Hint : $(\sin 20° \sin 40° \sin 80°)\sin 60° = \frac{1}{4} \sin 60° \cdot \sin 60° = \frac{3}{16}$

Illustration 67:

$\tan 6°\tan 42°\tan 66°\tan 78° = 1$

Solution:

$\frac{(\tan 6°\tan 54°\tan 66°)}{\tan 54°}\tan 42°\tan 78°$

$\frac{(\tan 18°)\tan 42°\tan 78°}{\tan 54°} = \frac{\tan 54°}{\tan 54°} = 1$

T-Ratio for Some Standard Angles:

18°, 36°, 54°, 72°

$$\sin 18° = \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} = \cos 72° = \cos\frac{2\pi}{5}$$

Proof: Let $\theta = 18° \Rightarrow 5\theta = 90° \Rightarrow 2\theta = 90° - 3\theta$

Now $\sin(2\theta) = \sin(90° - 3\theta) = \cos 3\theta$

$\Rightarrow 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta \Rightarrow 2\sin\theta = 4\cos^2\theta - 3$

$\Rightarrow 2\sin\theta = 1 - 4\sin^2\theta \Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$

$\Rightarrow \sin\theta = \frac{-2\pm 2\sqrt{5}}{8}$ $\therefore \sin 18° = \frac{\sqrt{5}-1}{4}$; $\frac{-\sqrt{5}-1}{4}$ is not possible as $\sin 18° > 0$.

Now $\cos 36° = 1 - 2\sin^2 18° = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2$

or $\cos 36° = \cos\pi/5 = \frac{\sqrt{5}+1}{4} = \sin 54° = \sin 3\pi/10$.

Now $\cos 18° = \sqrt{1 - \sin^2 18°} = \sqrt{1 - \left(\frac{\sqrt{5}-1}{4}\right)^2}$

or $\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4} = \cos\frac{\pi}{10} = \sin 72 = \sin^2\frac{\pi}{5}$

$\sin 36° = \sqrt{1 - \cos^2 36°} = \sqrt{1 - \left(\frac{\sqrt{5}+1}{4}\right)^2}$

$\Rightarrow \sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°$

Illustration 68:

Find the value of $\cos^2 48° - \sin^2 12°$

Solution:

$$\cos(60°)\cos(36°) = \frac{\sqrt{5}+1}{4} \times \frac{1}{2} = \frac{\sqrt{5}+1}{8}$$

Illustration 69:

Find the value of sin(132°) sin(12°)

Solution:

$$\frac{1}{2}[\cos(120°) - \cos(144°)] = \left[-\frac{1}{2} + \cos 36°\right] = \frac{1}{2}\left[\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right] = \frac{\sqrt{5}-1}{8}$$

Illustration 70:

Prove that $\tan 6° \tan 42° \tan 66° \tan 78° = 1$

Solution:

$$\frac{\sin 6° \sin 42° \sin 66° \sin 78°}{\cos 6° \cos 42° \cos 66° \cos 78°} = \frac{2\sin 6° \sin 66°}{2\cos 6° \cos 66°} \times \frac{2\sin 42° \sin 78°}{2\cos 42° \cos 78°}$$

$$= \frac{\cos 60° - \cos 72°}{\cos 60° + \cos 72°} \times \frac{\cos 36° - \cos 120°}{\cos 36° + \cos 120°} = \frac{\frac{1}{2} - \frac{\sqrt{5}-1}{4}}{\frac{1}{2} + \frac{\sqrt{5}-1}{4}} \times \frac{\frac{\sqrt{5}+1}{4} + \frac{1}{2}}{\frac{\sqrt{5}+1}{4} - \frac{1}{2}} = 1$$

Trigonometric Ratios for some standard Angles:

(i) $\sin 15° = \sin \frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75° = \cos \frac{5\pi}{12}$

(ii) $\cos 15° = \cos \frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \sin \frac{5\pi}{12}$

(iii) $\tan 15° = \tan \frac{\pi}{12} = 2 - \sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot 75° = \cot \frac{5\pi}{12}$

(iv) $\tan 75° = \tan \frac{5\pi}{12} = 2 + \sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15° = \cot \frac{\pi}{12}$

(v) $\tan(22.5°) = \tan \frac{\pi}{8} = \sqrt{2} - 1 = \cot(67.5°) = \cot \frac{3\pi}{8}$

(vi) $\tan(67.5°) = \tan \frac{3\pi}{8} = \sqrt{2} + 1 = \cot(22.5°) = \cot \frac{\pi}{8}$

Illustration 71:

Find the values of :

(i) $\sin 15°$

(ii) $\cos 15°$

(iii) $\sin 15° \cos 15°$

(iv) $\sin 15° \cos 75°$

Solution:

(i) $\sin 15° = \sin(45^o - 30^o) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(ii) $\cos 15° = \cos(45^o - 30^o) = \frac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\sin 15° \cos 15° = \frac{1}{2}(2 \sin 15° \cos 15°)$

$= \frac{1}{2} \sin 30° = \frac{1}{2} . \frac{1}{2} = \frac{1}{4}$

(iv) $\sin 15° \cos 75° = \sin 15° \sin 15° = \sin^2 15°$

$= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2 = \frac{4-2\sqrt{3}}{8}$

Illustration 72:

$\sin^2 72° - \sin^2 60° = \frac{\sqrt{5}-1}{8}$

Solution:

$$1 - \cos^2 72° - \frac{3}{4} = \frac{1}{4} - \left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}-1}{8}$$

Illustration 73:

Evaluate sin78° – sin66° – sin42° + sin6°.

Solution:

The expression = (sin78° – sin42°) – (sin66° – sin6°) = 2cos(60°) sin(18°) – 2cos36°. sin30°

$= \sin 18° - \cos 36° = \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2}$

Trigonometric identities in a triangle (Conditional identities):

If A, B, C are the angles of a triangle i.e. $A + B + C = \pi$, then

(i) $\sin(A + B) = \sin(\pi - C) = \sin C$

(ii) $\cos(A + B) = \cos(\pi - C) = -\cos C$

(iii) $\tan(A + B) = \tan(\pi - C) = -\tan C$

(iv) $\sin(2A + 2B) = \sin(2\pi - 2C) = -\sin 2C$

(v) $\cos(2A + 2B) = \cos(2\pi - 2C) = \cos 2C$

(vi) $\tan(2A + 2B) = \tan(2\pi - 2C) = -\tan 2C$

(vii) $\sin\left(\frac{A+B}{2}\right) = \sin\frac{\pi-C}{2} = \cos\frac{C}{2}$

(viii) $\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$

(ix) $\tan\left(\frac{A+B}{2}\right) = \cot\frac{C}{2}$

Illustration 74:

$\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4\sin A \sin B \sin C$ (where $A + B + C = \pi$)

Solution:

$\sin(\pi - 2A) + \sin(\pi - 2B) + \sin(\pi - 2C) = \sin 2A + \sin 2B + \sin 2C]$

$= 2\sin(A + B) \cos(A - B) + 2\sin C \cos C = 2\sin C[\cos(A - B) + \cos(\pi - A - B)]$

$= 2\sin C[\cos(A - B) - \cos(A + B)] = 4\sin A \sin B \sin C]$

Illustration 75:

If $A + B + C = \pi$, Prove that $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$

Solution:

$\sin^2 A + \sin(B + C)\sin(B - C) = \sin A[\sin(B + C) + \sin(B - C)] = 2\sin A \sin B \cos C]$

Illustration 76:

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} \text{(where } A + B + C = \pi)$$

Solution:

$$\sin^2\frac{A}{2} + \sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)$$

$$= 1 - \cos^2\frac{A}{2} + \cos\frac{A}{2}\sin\left(\frac{B-C}{2}\right)$$

$$= 1 - \cos\frac{A}{2}\left[\cos\frac{A}{2} - \sin\left(\frac{B-C}{2}\right)\right] = 1 - \cos\frac{A}{2}\left[\sin\left(\frac{B+C}{2}\right) - \sin\left(\frac{B-C}{2}\right)\right]$$

$$= 1 - 2\,cos\frac{A}{2}\,cos\frac{B}{2}\,sin\frac{C}{2}$$

Trigonometric identities in a triangle (conditional identities):

Illustration 77:

If A, B, C are the angles of a triangle, then

sin2A + sin2B + sin2C = 4sin$A$ sin$B$ sin$C$ $\Rightarrow (\sum sin\,2\,A = 4\prod sin\,A)$

Solution:

LHS = $2\sin(A + B)\cos(A - B) + 2\sin C\cos C = 2\sin C[\cos(A - B) + \cos(\pi - A - B)]$

$= 2\sin C[\cos(A - B) - \cos(A + B)] = 4\sin A\sin B\sin C]$

Illustration 78:

cos2A + cos2B + cos2C = – 1 – 4 cos$A$ cos$B$ cos$C$ (where $A + B + C = \pi$)

Solution:

$2\cos(A + B)\cos(A - B) + 2\cos^2 C - 1 = 2\cos(\pi - C)\cos(A - B) + 2\cos^2 C - 1]$

$= 2\cos C\,[\cos C - \cos(A - B)] - 1 = 2\cos C\,[\cos(\pi - \overline{A + B}) - \cos(A - B)] - 1$

$= 2\cos C\,[-\cos(A + B) - \cos(A - B)] - 1 = -4\cos A\cos B\cos C - 1$

$$\Rightarrow \left(\sum cos\,2\,A = -1 - 4\prod cos\,A\right)$$

Illustration 79:

$\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$

$\Sigma\tan A = \Pi\tan A$

Solution:

$\tan(A+B) = \tan(\pi - C) = -\tan C \quad \Rightarrow \dfrac{tan\,A + tan\,B}{1 - tan\,A\,tan\,B} = -\tan C$

$\Rightarrow \tan A + \tan B + \tan C = \tan A.\tan B.\tan C$

If A, B, C are the angles of a triangle i.e. $A + B + C = \pi$, then

(i) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

$\sum \cot A \cot B = 1$

(ii) $\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$

(iii) $\sum \cot\frac{A}{2} = \Pi \cot\frac{A}{2}$

(iv) $\sin A + \sin B + \sin C = 2\sin\frac{(A+B)}{2}\cos\frac{(A-B)}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\left[\cos\frac{(A-B)}{2} + \sin\frac{C}{2}\right]$

$= 2\cos\frac{C}{2}\left[\cos\frac{(A-B)}{2} + \cos\frac{(A+B)}{2}\right]$

$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(v) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Illustration 80:

If $x + y + z = xyz$ prove that $\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2}.\frac{2y}{1-y^2}.\frac{2z}{1-z^2}$

Solution:

$x = \tan A; y = \tan B; z = \tan C$

Hence $\frac{2\tan A}{1-\tan^2 A} + \frac{2\tan B}{1-\tan^2 B} + \frac{2\tan C}{1-\tan^2 C} = \tan 2A + \tan 2B + \tan 2C$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow \frac{\tan A + \tan B}{1-\tan A \tan B} = -\tan C \Rightarrow \tan(A+B) = \tan(\pi - C)$

or $A + B = \pi - C \Rightarrow A + B + C = \pi$

$\therefore$ $2A + 2B = 2\pi - 2C$

$\tan(2A + 2B) = -\tan 2C \Rightarrow \frac{\tan 2A + \tan 2B}{1-\tan 2A \tan 2B} = -\tan 2C$

so $\tan 2A + \tan 2B + \tan 2C = \tan 2A.\tan 2B.\tan 2C = \frac{2\tan A}{1-\tan^2 A}\frac{2\tan B}{1-\tan^2 B}\frac{2\tan C}{1-\tan^2 C}$

$= \frac{2x}{1-x^2}.\frac{2y}{1-y^2}.\frac{2z}{1-z^2}$

Illustration 81:

Find the value of $\frac{\sin 50° + \sin 100° + \sin 210°}{\sin 25° \sin 50° \sin 105°}$

Solution:

Let $A = 25°, B = 50°, C = 105° \Rightarrow A + B + C = \pi$

$\frac{4\sin 25° \sin 50° \sin 105°}{\sin 25° \sin 50° \sin 105°} = 4$

Illustration 82:

If $A+B+C=2S$ prove that $\sin(S-A)\sin(S-B)+\sin S\sin(S-C)=\sin A\sin B$.

Solution:

$$\frac{1}{2}[\cos(A-B)-\cos(2S-A-B)]+\frac{1}{2}[\cos C-\cos(2S-C)]$$

$$=\frac{1}{2}[\cos(A-B)-\cos C+\cos C-\cos(A+B)] = \sin A\sin B$$

Illustration 83:

In any triangle ABC, $\sin A - \cos B = \cos C$, then angle B is

(A) $\pi/2$ (B) $\pi/3$ (C) $\pi/4$ (D) $\pi/6$

Ans. (A)

Solution:

We have, $\sin A - \cos B = \cos C$

$\sin A = \cos B + \cos C$

$$\Rightarrow 2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow 2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right) \because \quad A+B+C=\pi$$

$$\Rightarrow 2\sin\frac{A}{2}\cos\frac{A}{2}=2\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \cos\frac{A}{2}=\cos\frac{B-C}{2} \text{ or } A=B-C \text{ ;But } A+B+C=\pi$$

Therefore $2B=\pi \Rightarrow B=\pi/2$

Illustration 84:

If $A+B+C=\frac{3\pi}{2}$, then $\cos 2A+\cos 2B+\cos 2C$ is equal to-

(A) $1-4\cos A\cos B\cos C$ (B) $4\sin A\sin B\sin C$

(C) $1+2\cos A\cos B\cos C$ (D) $1-4\sin A\sin B\sin C$

Ans. (D)

Solution:

$\cos 2A+\cos 2B+\cos 2C = 2\cos(A+B)\cos(A-B)+\cos 2C$

$$=2\cos\left(\frac{3\pi}{2}-C\right)\cos(A-B)+\cos 2C \quad \because A+B+C=\frac{3\pi}{2}$$

$$=-2\sin C\cos(A-B)+1-2\sin^2 C=1-2\sin C[\cos(A-B)+\sin C]$$

$$=1-2\sin C\left[\cos(A-B)+\sin\left(\frac{3\pi}{2}-(A+B)\right)\right]$$

$$=1-2\sin C[\cos(A-B)-\cos(A+B)]=1-4\sin A\sin B\sin C$$

Illustration 85:

Find $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}$ (where $A+B+C=\pi$)

Solution:

$$\frac{4\sin A\sin B\sin C}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Inequalities in a triangle:

Illustration 86:

In any ΔABC show that $\cot^2 A + \cot^2 B + \cot^2 C \geq 1$

Solution:

Let $x = \cot A$, $y = \cot B$ & $z = \cot C$

We know that $\Sigma \cot A \cot B = 1 \Rightarrow \Sigma xy = 1$

Also $(x - y)^2 \geq 0 \quad \Rightarrow x^2 + y^2 \geq 2xy$

$$y^2 + z^2 \geq 2yz$$

$$z^2 + x^2 \geq 2zx$$

$$\therefore \quad 2(x^2 + y^2 + y^2) \geq 2\Sigma xy$$

or $x^2 + y^2 + z^2 \geq 1$

$\Rightarrow \cot^2 A + \cot^2 B + \cot^2 C \geq 1$

Application of Trigonometry in Maximising & Minimising:

Type-I : By property of boundness of trig. function

$\sin x, \cos x \in [-1, -1]$; $-\infty < \tan x, \cot x < \infty$; $\sec x, \operatorname{cosec} x \in (-\infty, -1] \cup [1, \infty)$

$0 \leq \sin^2 x, \cos^2 x \leq 1$; $\sec^2 x, \operatorname{cosec}^2 x \geq 1$; $0 \leq \tan^2 x, \cot^2 x < \infty$

Illustration 87:

Find the range

(i) $\sin x, \sin(2x + 3), \sin \frac{x}{2}$

(ii) $4\sin x$, $4\cos(3x + 2)$

(iii) $3\sec^2 x$

Solution:

(i) $[-1, 1]$

(ii) $[-4, 4]$

(iii) $[3, \infty]$

Illustration 88:

Find range of y given by $y = \cos^4 x - \sin^4 x$

Solution:

$\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x = \cos 2x \in [-1, 1]$

Illustration 89:

$y = 4\tan x . \cos x$

Solution:

$\frac{4\sin x}{\cos x} . \cos x \ \forall \cos x \neq 0$

$= 4\sin x \ \forall \ x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\ldots \in (-4, 4)$

Illustration 90:

$y = \tan^2\left(x - \frac{\pi}{4}\right)$

Solution:

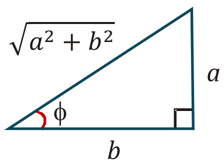
It is square. ∴ minimum = 0

$y \in [0, \infty]$

Type-II : When argument of sine & cosine are same

$-\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}$

Proof : $y = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right]$



$= \sqrt{a^2+b^2}\,[\sin x \sin\phi + \cos x \cos\phi]$

$\Rightarrow y = \sqrt{a^2+b^2}\cos(x-\phi)$

$y_{max} = \sqrt{a^2+b^2}$; $y_{min} = -\sqrt{a^2+b^2}$

Illustration 91:

Range of $y = 3\sin x + 4\cos x + 5$

[**Ans.** $y \in [0, 10]$]

Solution:

$-5 \le 3\sin x + 4\cos x \le 5$

$\Rightarrow -5 + 5 \le 3\sin x + 4\cos x + 5 \le 5 + 5$

$\Rightarrow 0 \le 3\sin x + 4\cos x + 5 \le 10$

Illustration 92:

If $b \le 3\sin^2 x + 6\cos^2 x - 4\sin x\cos x + 5 \le a$, find a & b.

Solution:

Let $E = 3 + 3\cos^2 x - 2\sin 2x + 5$

$\Rightarrow E = \frac{3}{2}(1+\cos 2x) - 2\sin 2x + 8 \Rightarrow E = \frac{3}{2}\cos 2x - 2\sin 2x + \frac{19}{2}$

$\Rightarrow E_{max} = \sqrt{\frac{9}{4}+4} + \frac{19}{2} = 12$ & $E_{min} = -\sqrt{\frac{9}{4}+4} + \frac{19}{2} = 7$

Illustration 93:

Find range of $y = 5\sin\left(x + \frac{\pi}{6}\right) + 3\cos x$

Solution:

$y = 5\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right) + 3\cos x \Rightarrow y = \frac{5\sqrt{3}}{2}\sin x + \frac{11}{2}\cos x$

$y_{max} = \sqrt{\frac{75}{4} + \frac{121}{4}} = \sqrt{\frac{196}{4}} = 7$ & $y_{min} = -7$

Hence range is $[-7, 7]$

Illustration 94:

Find maximum & minimum value of $y = \frac{17+(5\sin x+12\cos x)}{17-(5\sin x+12\cos x)}$

Solution:

$y_{max} = \frac{17+13}{17-13} = \frac{15}{2}, y_{min} = \frac{17-13}{17+13} = \frac{2}{15}$

Type-III: Argument of sine & cosine are different or a quadratic in sine/cosine is given then we make a perfect square in sine/cosine & then interpret.

Illustration 95:

$y = \cos 2x + 3\sin x$

Solution:

$y = 1-2\sin^2 x + 3\sin x = 1-2\left[\sin^2 x - \frac{3}{2}\sin x\right]$

$y = 1-2\left[\left(\sin x - \frac{3}{4}\right)^2 - \frac{9}{16}\right] = \frac{17}{8} - 2\left(\sin x - \frac{3}{4}\right)^2$

$\Rightarrow y_{min} = \frac{17}{8} - 2\left(-1-\frac{3}{4}\right)^2 = -\frac{32}{8} = -4$ & $y_{max} = \frac{17}{8} - 2\left(\frac{3}{4}-\frac{3}{4}\right)^2 = \frac{17}{8}$

Illustration 96:

Find the range of $y = \sin^2 x - 20\cos x + 1$

Solution:

$y = 1-\cos^2 x - 20\cos x + 1 \Rightarrow y = 102 - (\cos x + 10)^2$

$\Rightarrow y_{max} = 21$ & $y_{min} = -19$

Illustration 97:

Find maximum and minimum values of $y = \cos^2 x - 4\cos x + 13$

Solution:

$y = (\cos x - 2)^2 + 9 \Rightarrow y_{max} = 18$ & $y_{min} = 10$

Type-IV : Making use of reciprocal relationship between tan/cot, sin/cosec, sec/cos.

Illustration 98:

$y = a^2\tan^2\theta + b^2\cot 2\theta \ (a, b \geq 0)$

Solution:

Max. value is not meaningful we will take about its minimum value.

$y = (a\tan\theta - b\cot\theta)^2 + 2ab$

This is minimum when $a\tan\theta - b\cot\theta = 0$

$\Rightarrow y = 2ab$, when $a\tan\theta - b\cot\theta = 0$ i.e. $\tan\theta = \sqrt{\frac{b}{a}}$

Note: We can not take $y = (a\tan\theta + b\cot\theta)^2 - 2ab$

$\because$ If $a\tan\theta = -b\cot\theta$

$\Rightarrow \tan^2\theta = \frac{-b}{a}$ Not possible.

Illustration 99:

$y = a^2\sec^2\theta + b^2\text{cosec}^2\theta$

Solution:

$y = a^2 + b^2 + a^2\tan^2\theta + b^2\cot^2\theta = a^2 + b^2 + (a\tan\theta - b\cot\theta)^2 + 2ab$

$y = a^2 + b^2 + 2ab + (a\tan\theta - b\cot\theta)^2$

$\Rightarrow\ y_{min} = (a+b)^2$, when $\tan\theta = \sqrt{\dfrac{b}{a}}$

Illustration 100:

$y = 8\sec^2\theta + 18\cos^2\theta$

Solution:

$y = \left(2\sqrt{2}\sec\theta - 3\sqrt{2}\cos\theta\right)^2 + 24$

$\Rightarrow\ y_{min} = 24$ when $\cos^2\theta = \dfrac{2\sqrt{2}}{3\sqrt{2}} = \dfrac{2}{3}$

Illustration 101:

$y = 4\sin^2\theta + \text{cosec}^2\theta$

Solution:

$y = (2\sin\theta - \text{cosec}\theta)^2 + 4$

$\Rightarrow y_{min} = 4$, when $2\sin\theta = \text{cosec}\theta$ i.e. $\sin^2\theta = \dfrac{1}{2}$

Summation of trigonometric series:

Type-I:

Sum of sine/cosine of n angles which are in A.P. i.e. successive arguments of sine or cosine have the same difference.

Illustration 102:

$S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots\ldots\ldots + \sin(\alpha + \overline{n-1}\beta)$

Solution:

$2\sin\dfrac{\beta}{2}.S = 2\sin\dfrac{\beta}{2}(\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin(\alpha + \overline{n-1}\beta))$

$= \cos(\alpha - \dfrac{\beta}{2}) - \cos(\alpha + \dfrac{\beta}{2}) + \cos(\alpha + \dfrac{\beta}{2}) - \cos(\alpha + \dfrac{\beta}{2}) + \ldots\ldots\ \cos(\alpha + \overline{n - \tfrac{3}{2}}\beta)$

$-\ \cos(\alpha + \overline{n - \tfrac{1}{2}}\beta)$

$= \cos(\alpha - \dfrac{\beta}{2}) - \cos(\alpha + \overline{n - \tfrac{1}{2}}\beta)$

$\Rightarrow S = \sin\dfrac{n\beta}{2}\sin\left[\alpha + (n-1)\dfrac{\beta}{2}\right] / \sin\dfrac{\beta}{2} = \dfrac{\sin\left(\frac{\text{Firstangle} + \text{Last angle}}{2}\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$

Here $\dfrac{\beta}{2}$ is half of common difference.

Illustration 103:

$S = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots\ldots\ldots + \cos(\alpha + \overline{n-1}\beta)$

Solution:

$2\sin\frac{\beta}{2}.S = 2\sin\frac{\beta}{2}\ (cos\alpha + cos(\alpha+\beta) + cos(\alpha+2\beta) + \ldots\ldots\ldots\ldots + cos(\alpha + \overline{n-1}\beta))$

$\Rightarrow S = sin\frac{n\beta}{2}\cos\left[\alpha + (n-1)\ \frac{\beta}{2}\right] / \sin\frac{\beta}{2}$

Illustration 104:

$S = \sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \ldots\ldots\ldots + \sin n\theta.$

Solution:

$2\sin\left(\frac{\theta}{2}\right)S = \sin\frac{\theta}{2}[sin\,\theta + sin\,2\,\theta + sin\,3\,\theta \ldots\ldots\ldots + sin\,n\,\theta]$

$= \cos\frac{\theta}{2} - \cos\frac{3\theta}{2} + \cos\frac{3\theta}{2} - \frac{\cos 5\theta}{2} + \frac{\cos 5\theta}{2} - \cos\frac{7\theta}{2} \ldots\ldots + \cos\left(n-\frac{1}{2}\right)\theta - \cos\left(n+\frac{1}{2}\right)\theta$

$\Rightarrow 2\sin\frac{\theta}{2}S = \cos\frac{\theta}{2} - \cos\left(n+\frac{1}{2}\right)\theta$

$$\Rightarrow S = \frac{2\sin\left(\frac{n+1}{2}\right)\theta\sin\left(\frac{n\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

or $S = \dfrac{\sin\left(\frac{\theta+n\theta}{2}\right)\sin\frac{n\theta}{2}}{\sin\theta}$ by standard result.

If $\theta = \dfrac{2\pi}{n}$ then $S = 0$

Illustration 105:

$S = \cos\frac{\pi}{19} + \cos\frac{3\pi}{19} + \cos\frac{5\pi}{19} + \ldots\ldots + \cos\frac{17\pi}{19}$

Solution:

$S = \cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots\ldots + \cos 17\theta$, where $\theta = \frac{\pi}{19}$.

$2\sin\theta\, S = 2\sin\theta\,(\cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots\ldots + \cos 17\theta)$

$= \sin 2\theta + \sin 4\theta - \sin 2\theta + \sin 6\theta - \sin 4\theta + \ldots\ldots\ldots + \sin 18\theta - \sin 16\theta$

$S = \dfrac{\sin 18\theta}{2\sin\theta} = \dfrac{\sin(\pi-\theta)}{2\sin\theta}$ $\quad\left(\because \theta = \frac{\pi}{19}\right)$

or $S = \dfrac{1}{2}$

or by standard result

$$S = \frac{\cos\left(\frac{\frac{\pi}{19}+\frac{17\pi}{19}}{2}\right)\sin\frac{9\cdot 2\pi}{19\times 2}}{\sin\frac{2\pi}{19\times 2}} = \frac{2\cos\frac{9\pi}{19}\sin\frac{9\pi}{19}}{2\sin\frac{\pi}{19}} = \frac{\sin\frac{18\pi}{19}}{2\sin\frac{\pi}{19}} = \frac{1}{2}$$

Illustration 106:

$S = \sin\theta - \sin 2\theta + \sin 3\theta - \sin 4\theta + \ldots + n$ terms

Solution:

$S = \sin\theta + \sin(\pi + 2\theta) + \sin(2\pi + 3\theta) + \sin(3\pi + 4\theta) + \ldots\ldots + n$ terms

Let $\pi + \theta = d$

$S = \sin\theta + \sin(\theta + d) + \sin(\theta + 2d) + \sin(\theta + 3d) + \ldots\ldots + \sin(\theta + \overline{n-1}d)$

$2\sin\frac{d}{2}.S = 2\sin\frac{d}{2}[\sin\theta + \sin(\theta + d) + \sin(\theta + 2d) + \ldots\ldots \sin(\theta + \overline{n-1}\,d)]$

$= \cos\left(\frac{d}{2} - \theta\right) - \cos\left(\frac{d}{2} + \theta\right) + \cos\left(\frac{d}{2} + \theta\right) - \cos\left(\frac{3d}{2} + \theta\right) + \cos\left(\frac{3d}{2} + \theta\right) - \cos\left(\frac{5d}{2} + \theta\right) + \ldots\ldots$

$+ \cos\left(\frac{2n-3}{2}d + \theta\right) - \cos\left(\frac{2n-1}{2}d + \theta\right) = \cos\left(\frac{d}{2} - \theta\right) - \cos\left(\frac{2n-1}{2}d + \theta\right)$

$2\sin\frac{\theta}{2}.S = 2\sin(nd/2)\sin\frac{\left(\frac{2n-1}{2}d + \theta - \frac{d}{2} + \theta\right)}{2} \Rightarrow S = \frac{\sin(nd/2)\sin(\theta + (n-1)d/2)}{\sin\frac{\theta}{2}}$

Illustration 107:

$S = \cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots\ldots\ldots + \cos n\theta$

Solution:

$2\sin\frac{\theta}{2}\ S = 2\sin\frac{\theta}{2}(\cos\theta + \cos 2\theta + \ldots\ldots + \cos n\theta)$

$= \sin\frac{3\theta}{2} - \sin\frac{\theta}{2} + \sin\frac{5\theta}{2} - \sin\frac{3\theta}{2} + \sin\frac{7\theta}{2} - \sin\frac{5\theta}{2} + \ldots\ldots + \sin\left(n + \frac{1}{2}\right)\theta - \sin\left(n - \frac{1}{2}\right)\theta$

$S = \frac{\sin\left(n + \frac{1}{2}\right)\theta - \sin\frac{\theta}{2}}{2\sin\left(\frac{\theta}{2}\right)} \Rightarrow S = \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\frac{\theta}{2}}.\cos\left(\frac{n+1}{2}\right)\theta$

Type II: Splitting the sum of series as difference of 2 terms

Illustration 108:

$\text{cosec}x + \text{cosec}2x + \text{cosec}4x + \ldots + \text{cosec}2^n x = \cot\frac{x}{2} - \cot 2^n x$

Solution:

$T_1 = \frac{1}{\sin x} = \frac{\sin\frac{x}{2}}{\sin\frac{x}{2}\sin x} = \frac{\sin\left(x - \frac{x}{2}\right)}{\sin\frac{x}{2}\sin x} = \frac{\sin x\cos\frac{x}{2} - \cos x\sin\frac{x}{2}}{\sin\frac{x}{2}\sin x} = \cot\frac{x}{2} - \cot x$

$T_2 = \frac{1}{\sin 2x} \times \frac{\sin x}{\sin x} = \frac{\sin(2x - x)}{\sin 2x\sin x} = \cot x - \cot 2x$

Similarly,

$T_3 = \cot 2x - \cot 2^2 x$, $T_4 = \cot 2^2 x - \cot 2^3 x$,... $T_n = \cot 2^{n-2}x - \cot 2^{n-1}x$, $T_{n+1} = \cot 2^{n-1}x - \cot 2^n x$

$\Rightarrow S = \sum T_n = \cot\frac{x}{2} - \cot 2^n x$

Continued Product of Sine & Cosine Series:

Σ denotes continued sum & Π denotes continued product

$\prod_{r=1}^{n}\sin r\theta = \sin\theta\sin 2\theta\sin 3\theta \ldots\ldots\ldots\ldots\ldots \sin n\theta$

$\sum_{r=1}^{n}\sin r\theta = \sin\theta + \sin 2\theta + \sin 3\theta + \ldots\ldots + \sin n\theta$

If continued product of cosine series is given such that each angle is double or half of previous angle, then multiply & divide the series by 2 (sine of smallest angle).

Let $P = \cos\theta \cos 2\theta \cos 2^2\theta$ .... $\cos 2^{n-1}\theta$

multiply Nr & D^r by $\sin\theta$

$$\Rightarrow\ P = \frac{\sin\theta\cos\theta\cos 2\theta....\cos 2^{n-1}\theta}{\sin\theta} = \frac{2\sin\theta\cos\theta\cos 2\theta......\cos 2^{n-1}\theta}{2\sin\theta} = \frac{\sin 2\theta\cos 2\theta......\cos 2^{n-1}\theta}{2\sin\theta}$$

$$= \frac{2\sin 2\theta\cos 2\theta......\cos 2^{n-1}\theta}{2^2\sin\theta} = \frac{2^{n-1}2.\sin\theta\cos\theta\cos 2\theta......\cos 2^{n-1}\theta}{2^n\sin\theta} = \frac{\sin 2^n\theta}{2^n\sin\theta}$$

Illustration 109:

Prove that $\cos 20\cos 40\cos 80 = \frac{1}{8}$ or $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$

Solution:

$$\frac{2\sin 20}{2\sin 20}\times\cos 20\cos 40\cos 80 = \frac{2\sin 40\cdot\cos 40\cos 80}{2^2\sin 20} = \frac{2\sin 80\cos 80}{2^3\sin 20} = \frac{\sin 160}{2^2\sin 20} = \frac{\sin 20}{2^3\sin 20} = \frac{1}{8}$$

Direct: $= \frac{\sin 2^3(20)}{2^3\sin(20)} = \frac{\sin 160}{8\sin 20} = \frac{1}{8}$

Illustration 110:

$$\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = -\frac{1}{16}$$

Solution:

$$\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)\cos\left(\pi - \frac{\pi}{15}\right) = -\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$$

Multiply & divide by $2\sin\left(\frac{\pi}{15}\right)$ & solve

Direct method:

$$= -\frac{\sin 2^4\left(\frac{\pi}{15}\right)}{2^4\sin\frac{\pi}{15}} = -\frac{\sin\frac{16\pi}{15}}{16\cdot\sin\frac{\pi}{15}} = -\frac{\sin\left(\pi + \frac{\pi}{15}\right)}{16\cdot\sin\frac{\pi}{15}} = \frac{1}{16}$$

Illustration 111:

$$P = \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$$

Solution:

$$P = \sin\frac{\pi}{14}\ \sin\frac{3\pi}{14}\ \sin\frac{5\pi}{14}\times 1\times\sin\left(\pi - \frac{5\pi}{14}\right)\ \sin\left(\pi - \frac{3\pi}{14}\right)\ \sin\left(\pi - \frac{\pi}{14}\right)$$

$$P = \left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2 = q^2 \text{ (let)}$$

$$\Rightarrow q = \sin\left(\frac{7\pi - 6\pi}{14}\right)\sin\left(\frac{7\pi - 4\pi}{14}\right)\sin\left(\frac{7\pi - 2\pi}{14}\right)$$

$$\text{or } q = \cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}$$

$$\Rightarrow\ q = -\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}} = -\frac{\sin\left(\pi + \frac{\pi}{7}\right)}{8\sin\frac{\pi}{7}} = \frac{1}{8}. \qquad \therefore\ P = \frac{1}{64}$$

Illustration 112:

Find the value of $\sin\frac{\pi}{16}\sin\frac{3\pi}{16}\sin\frac{5\pi}{16}\sin\frac{7\pi}{16}$

Solution:

$\sin\frac{\pi}{16}\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}\cos\frac{\pi}{16} = \frac{1}{4}\left[\sin\frac{2\pi}{16}\sin\frac{6\pi}{16}\right] = \frac{1}{8}\left[2\sin\frac{\pi}{8}\sin\frac{3\pi}{8}\right]$

$= \frac{1}{8}\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right) = \frac{1}{8}\sin\frac{\pi}{4} = \frac{1}{8\sqrt{2}}$

Illustration 113:

$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{3\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{16}$

Solution:

$= \frac{1}{2}\left[\cos\frac{\pi}{9}\cdot\cos\frac{2\pi}{9}\cdot\cos\frac{4\pi}{9}\right] = \left(\frac{1}{2}\right)\frac{\sin 2^3\frac{\pi}{9}}{2^3\sin\frac{\pi}{9}} = \frac{1}{16}\frac{\sin\frac{8\pi}{9}}{\sin\frac{\pi}{9}} = \frac{1}{16}$

Illustration 114:

Show that $2^{\sin x} + 2^{\cos x} \geq 2^{1-\frac{1}{\sqrt{2}}}$

Solution:

Since A.M. of two positive quantities ≥ Their G.M. we have

$\frac{2^{\sin x} + 2^{\cos x}}{2} \geq \sqrt{2^{\sin x}2^{\cos x}} = \sqrt{2^{\sin x}2^{\cos x}} = \sqrt{2^{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}} \geq \sqrt{2^{-\sqrt{2}}}$

$\Rightarrow 2^{\sin x} + 2^{\cos x} \geq 2 \times 2^{-\frac{1}{\sqrt{2}}} = 2^{1-\frac{1}{\sqrt{2}}}$

Illustration 115:

If $\tan 6\theta = p/q$, find the value of $\frac{1}{2}(p\operatorname{cosec}2\theta - q\sec 2\theta)$ in terms of p and q.

Solution:

Here, we have $\tan 6\theta = p/q$

$\Rightarrow \frac{\sin 6\theta}{\cos 6\theta} = \frac{p}{q}$

$\Rightarrow \frac{p}{\sin 6\theta} = \frac{q}{\cos 6\theta} = \frac{\sqrt{p^2+q^2}}{\sqrt{1}} = \sqrt{p^2+q^2}$

Now $y = \frac{1}{2}(\text{p cosec } 2\theta - \text{q sec } 2\theta) = \frac{1}{2}\left(\frac{p}{\sin 2\theta} - \frac{q}{\cos 2\theta}\right)$

$\Rightarrow y = \frac{1}{2}\left[\frac{p\cos 2\theta - q\sin 2\theta}{\sin 2\theta\cos 2\theta}\right]$

$= \left[\frac{2k\sin 6\theta\cos 2\theta - 2k\cos 6\theta\sin 2\theta}{4\sin 2\theta\cos 2\theta}\right]$

$= k\frac{\sin(6\theta - 2\theta)}{\sin 4\theta} = k = \sqrt{p^2+q^2}$

Illustration 116:

The value of $\sqrt{3}\left|\dfrac{\dfrac{2\sin(140°)\sec(280°)}{\sec(220°)}+\dfrac{\sec(340°)}{cosec(20°)}}{\dfrac{\cot(200°)-\tan(280°)}{\cot(200°)}}\right|$ is ______

Solution:

$$\sqrt{3}\left|\frac{\dfrac{2\sin(140°)\sec(280°)}{\sec(220°)}+\dfrac{\sec(340°)}{\text{cosec}(20°)}}{\dfrac{\cot(200°)-\tan(280°)}{\cot(200°)}}\right|$$

$$=\sqrt{3}\left|\frac{\tan(20°)-\tan(80°)}{1+\tan 20°\tan 80°}\right|=\sqrt{3}\tan(60°)=3$$

Illustration 117:

The maximum value of $y=\dfrac{1}{\sin^6 x+\cos^6 x}$ is______

Solution:

$\sin^6 x+\cos^6 x=(\sin^2 x+\cos^2 x)(\sin^4 x+\cos^4 x-\sin^2 x\cos^2 x)$

$1-3\sin^2 x\cos^2 x=1-\dfrac{3(\sin 2x)^2}{4}$

$\Rightarrow y=\dfrac{4}{4-3(\sin 2x)^2}$

$\Rightarrow y_{max}=\dfrac{4}{4-3(1)}=4$

Illustration 118:

The value of cosec10°+cosec50°–cosec70° is ___________

Solution:

$\dfrac{1}{\sin 10°}+\dfrac{1}{\sin 50°}-\dfrac{1}{\sin 70°}=\dfrac{1}{\cos 80°}+\dfrac{1}{\cos 40°}-\dfrac{1}{\cos 20°}$

$=\dfrac{\cos 40°\cos 20°+\cos 80°\cos 20°-\cos 40°\cos 80°}{\cos 20°\cos 40°\cos 80°}$

$=8[\cos 20°(\cos 40°+\cos 80°)-\cos 40°\cos 80°]$

$=8[2\cos 20°\cos 60°\cos 20°-\cos 40°\cos 80°]$

$=4[2\cos^2 20°-2\cos 40°\cos 80°]$

$=4[1+\cos 40°-(\cos 120°+\cos 40°)]$

$=4\times\dfrac{3}{2}=6$

Illustration 119:

If $\log_{10}\sin x+\log_{10}\cos x=-1$ and $\log_{10}(\sin x+\cos x)=\dfrac{(\log_{10}n)-1}{2}$. Then the value of $\dfrac{n}{3}$ is______

Solution:

Given $\log_{10}\left(\dfrac{\sin 2x}{2}\right)=-1$

$\Rightarrow \dfrac{\sin 2x}{2}=\dfrac{1}{10}$

$\Rightarrow \sin 2x=\dfrac{1}{5}$

Also $\log_{10}(\sin x+\cos x)=\dfrac{\log_{10}\left(\frac{n}{10}\right)}{2}$

$\Rightarrow \log_{10}(\sin x + \cos x)^2 = \log_{10}\left(\frac{n}{10}\right)$

$\Rightarrow 1 + \sin 2x = \frac{n}{10}$

$\Rightarrow 1 + \frac{1}{5} = \frac{n}{10}$

$\Rightarrow \frac{6}{5} = \frac{n}{10}$

$\Rightarrow \frac{n}{3} = 4$

Illustration 120:

If $\sin^3 x \cos 3x + \cos^3 x \sin 3x = \frac{3}{8}$, Then the value of $8\sin 4x$ is _______

Solution:

$4\sin^3 x \cos 3x + 4\cos^3 x \sin 3x = \frac{3}{2}$

$\Rightarrow (3\sin x - \sin 3x)\cos 3x + (3\cos x + \cos 3x)\sin 3x = \frac{3}{2}$

$\Rightarrow 3[\sin x \cos 3x + \cos x \sin 3x] = \frac{3}{2}$

$\Rightarrow \sin 4x = \frac{1}{2}$

$\Rightarrow 8\sin 4x = 4$

Illustration 121:

If $\cos\theta - \sin\theta = \frac{1}{5}$ where $0 < \theta < \frac{\pi}{2}$

Column I	Column II
a. $(\cos\theta + \sin\theta)/2$	**p.** $\frac{4}{5}$
b. $\sin 2\theta$	**q.** $\frac{7}{10}$
c. $\cos 2\theta$	**r.** $\frac{24}{25}$
d. $\cos\theta$	**s.** $\frac{7}{25}$

Ans. $[a \rightarrow q; b \rightarrow r; c \rightarrow s; d \rightarrow p]$

Solution:

$\cos\theta - \sin\theta = \frac{1}{5}$ Where $0 < \theta < \frac{\pi}{2}$ ...(i)

Squaring both side of Eq. (i), we get

$1 - \sin 2\theta = \frac{1}{25}$

$\Rightarrow \sin 2\theta = \frac{24}{25} \Rightarrow \cos 2\theta = \frac{7}{25}$

Also , $(\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2 + 4\cos\theta\sin\theta = \frac{1}{25} + 2\sin 2\theta = \frac{1}{25} + \frac{48}{25} = \frac{49}{25}$

$\Rightarrow \cos\theta + \sin\theta = \frac{7}{5}$ ...(ii)

$\Rightarrow (\cos\theta + \sin\theta)/2 = \frac{7}{10}$

Also solving Eqs. (i) and (ii), we get $\cos\theta = \frac{4}{5}$

Illustration 122:

For all real values of θ

Column I	Column II
a. $A = \sin^2\theta + \cos^4\theta$	**p.** $A \in [-1,1]$
b. $A = 3\cos^2\theta + \sin^4\theta$	**q.** $A \in \left[\frac{3}{4}, 1\right]$
c. $A = \sin^2\theta - \cos^4\theta$	**r.** $A \in \left[2\sqrt{2}, \infty\right]$
d. $A = \tan^2\theta + 2\cot^2\theta$	**s.** $A \in [1,3]$

Ans. $[a \to q;\ b \to s;\ c \to p;\ d \to r]$

Solution:

a. $A = \sin^2\theta + \cos^4\theta$

$$= \frac{1-\cos 2\theta}{2} + \left(\frac{1+\cos 2\theta}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{2}\cos 2\theta + \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2\theta$$

$$= \frac{3}{4} + \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2}\right) = \frac{3}{4} + \frac{1}{8} + \frac{1}{8}\cos^4\theta$$

Now $-1 \le \cos\theta \le 1$

$$\frac{-1}{8} \le \frac{\cos 4\theta}{8} \le \frac{1}{8}$$

$$\Rightarrow \frac{3}{4} + \frac{1}{8} - \frac{1}{8} \le \frac{3}{4} + \frac{1}{4}\left(\frac{1+\cos 4\theta}{2}\right) \le \frac{3}{4} + \frac{1}{8} + \frac{1}{8}$$

$$\Rightarrow \frac{3}{4} \le A \le 1$$

b. $A = 3\cos^2\theta + \sin^4\theta = 3\,\frac{1+\cos 2\theta}{2} + \left(\frac{1-\cos 2\theta}{2}\right)^2$

$$= \frac{3+3\cos 2\theta}{2} + \frac{1-2\cos 2\theta + \cos^2 2\theta}{4}$$

$$= \frac{7+4\cos 2\theta + \cos^2 2\theta}{4} = \frac{(\cos 2\theta + 2)^2 + 3}{4}$$

Now $1 \le \cos 2\theta + 2 \le 3$

$$\Rightarrow 1 \le \frac{(\cos 2\theta + 2)^2 + 3}{4} \le 3$$

c. $A = \sin^2\theta - \cos^4\theta$

$$= \frac{1-\cos 2\theta}{2} - \left(\frac{1+\cos 2\theta}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{2}\cos 2\theta - \frac{1}{4} - \frac{1}{2}\cos 2\theta - \frac{1}{4}\cos^2 2\theta$$

$$= \frac{1}{4} - \cos 2\theta - \frac{1}{4}\cos^2 2\theta$$

$$= -\left(\frac{1}{4}\cos^2 2\theta + \cos 2\theta - \frac{1}{4}\right)$$

$$= \frac{5}{4} - \left(\frac{1}{2}\cos 2\theta + 1\right)^2$$

Now $-\frac{1}{2} \le \frac{1}{2}\cos 2\theta \le \frac{1}{2}$

$\Rightarrow \frac{1}{2} \leq \frac{1}{2}\cos 2\theta + 1 \leq \frac{3}{2}$

$\Rightarrow \frac{1}{4} \leq \left(\frac{1}{2}\cos 2\theta + 1\right)^2 \leq \frac{9}{4}$

$\Rightarrow -\frac{9}{4} \leq -\left(\frac{1}{2}\cos 2\theta + 1\right)^2 \leq -\frac{1}{4}$

$\Rightarrow -1 \leq \frac{5}{4} - \left(\frac{1}{2}\cos 2\theta + 1\right)^2 \leq 1$

d. $\tan^2\theta + 2\cot^2\theta = \left(\tan\theta - \sqrt{2}\cot\theta\right)^2 + 2\sqrt{2} \geq 2\sqrt{2}$

Illustration 123:

If $\cos\alpha + \cos\beta = 1/2$ and $\sin\alpha + \sin\beta = 1/3$

Column I	Column II
a. $\cos\left(\frac{\alpha+\beta}{2}\right)$	**p.** $\pm\frac{\sqrt{13}}{12}$
b. $\cos\left(\frac{\alpha-\beta}{2}\right)$	**q.** $\frac{2}{3}$
c. $\tan\left(\frac{\alpha+\beta}{2}\right)$	**r.** $\pm\frac{3}{\sqrt{13}}$
d. $\tan\left(\frac{\alpha-\beta}{2}\right)$	**s.** $\pm\sqrt{\frac{131}{13}}$

Ans. [$a \to r; b \to p; c \to q; d \to s$]

Solution:

$\cos\alpha + \cos\beta = \frac{1}{2}$

$\Rightarrow 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2}$ ...(i)

$\sin\alpha + \sin\beta = \frac{1}{3}$

$\Rightarrow 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{3}$ ...(ii)

Dividing Eq. (ii) by Eq.(i). we get

$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{2}{3}$

$\Rightarrow \cos\left(\frac{\alpha+\beta}{2}\right) = \pm\frac{3}{\sqrt{13}}$

Squaring and adding the given result, we have

$2 + 2\cos(\alpha-\beta) = \frac{13}{36}$

$\Rightarrow \cos(\alpha-\beta) = -\frac{59}{72}$

$\Rightarrow 2\cos^2\left(\frac{\alpha-\beta}{2}\right) = 1 - \frac{59}{72} = \frac{13}{72}$

$\Rightarrow \cos\left(\frac{\alpha-\beta}{2}\right) = \pm\frac{\sqrt{13}}{12}$

$\Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right) = \pm\sqrt{\frac{131}{13}}$

Illustration 124:

Column-I	Column-II
(a) sin(410°–A)cos(410°+A)+cos(410°–A)sin(410°+A) has the value equal to	**(p)** – 1
(b) $\frac{\cos^2 1°-\cos^2 2°}{2\sin 3°\sin 1°}$ is equal to	**(q)** 0
(c) sin(–870°)+cosec(–660°)+tan(–885°)+2cot(840°)+cos(480°)+sec(900°)	**(r)** $\frac{1}{2}$
(d) if $\cos\theta = \frac{4}{5}$ where $\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$ and $\cos\phi = \frac{3}{5}$ where $\phi \in \left(0, \frac{\pi}{2}\right)$ then $\cos(\theta - \phi)$ has the value equal to	**(s)** 1

Ans. [$a \to s$; $b \to r$; $c \to p$; $d \to q$]

Solution:

(a) sin(410°+400°)=sin810°=sin(720°+90°) = sin90°=1

(b) $\frac{\sin^2 2°-\sin^2 1°}{2\sin 3°\sin 1°} = \frac{\sin 3°\sin 1°}{2\sin 3°\sin 1°} = \frac{1}{2}$

(c) sin(–870°)+cosec(–660°)+tan(–885°)+2cot(840°)+cos(480°)+sec(900°)

= –sin(810°+60°)– cosec(720°–60°)–tan(810°+45°)+2cot120°+cos120°+sec180°

$= -\frac{1}{2} + \frac{2}{\sqrt{3}} + 1 - \frac{2}{\sqrt{3}} - \frac{1}{2} - 1 = -1$

(d) $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi = \frac{4}{5} \times \frac{3}{5} - \frac{3}{5} \times \frac{4}{5}$

Illustration 125:

Column-I	Column-II
(a) The maximum value of {cos(2A+θ)+cos(2B +θ), Where A, B are constant is	**(p)** $2\sin(A+B)$
(b) The maximum value of {cos2A+cos2B) Where (A+B) is constant and $A, B, \in \left(0, \frac{\pi}{2}\right)$, is	**(q)** $2\sec(A+B)$
(c) The minimum value of {sec2A+sec2B) Where $(A+B)$ is constant and $A, B, \in \left(0, \frac{\pi}{4}\right)$ is	**(r)** $2\cos(A+B)$
(d) The minimum value of $\sqrt{\{\tan\theta + \cot\theta - 2\cos 2(A+B)\}}$. Where $(A+B)$ is constant and $A, B, \in \left(0, \frac{\pi}{2}\right)$, is	**(s)** $2\cos(A-B)$

Ans. [$a \to s$; $b \to r$; $c \to q$; $d \to p$]

Solution:

(a) {cos(2A+θ)+ cos(2B+θ)} = 2cos(A–B)cos(A+B+θ)

Maximum value is $2\cos(A-B)$ When $\cos(A+B+\theta) = 1$

(b) {cos2A + cos2B} = 2cos(A + B) cos(A – B)

Maximum value is $2\cos(A-B)$ when $\cos(A+B) = 1$

(c) For $y = \sec x$, $x \in \left(0, \frac{\pi}{2}\right)$, tangent drawn to it at any point lies completely below the graph of $y = \sec x$, thus $\frac{\sec 2A + \sec 2B}{2} \geq \sec(A+B)$

$\Rightarrow \sec 2A + \sec 2B \geq 2\sec(A+B)$

Hence, the minimum value is 2 sec (A + B).

(d) $\sqrt{\{\tan\theta + \cot\theta - 2\cos 2(A+B)\}} = \sqrt{\left(\sqrt{\tan\theta} - \sqrt{\cot\theta}\right)^2 + 2 - 2\cos 2(A+B)}$

$= \sqrt{\left(\sqrt{\tan\theta} - \sqrt{\cot\theta}\right)^2 + 4\sin^2(A+B)}$

Minimum value occurs when $\sqrt{\tan\theta} = \sqrt{\cot\theta}$ and

Minimum value is $\sqrt{4\sin^2(A+B)} = 2\sin(A+B)$

Illustration 126:

Column-I	Column-II
(a) $\cos 20° + \cos 80° - \sqrt{3}\cos 50°$	**(p)** -1
(b) $\cos 0° + \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7}$	**(q)** $-\frac{3}{4}$
(c) $\cos 20° + \cos 40° + \cos 60° - 4\cos 10°\cos 20°\cos 30°$	**(r)** 1
(d) $\cos 20°\cos 100° + \cos 100°\cos 140° - \cos 140°\cos 200°$	**(s)** 0

Ans. $[a \to s; b \to r; c \to p; d \to q]$

Solution:

(a) $\cos 20° + \cos 80° - \sqrt{3}\cos 50° = 2\cos 30°\cos 50° - \sqrt{3}\cos 50°$

$= \sqrt{3}\cos 50° - \sqrt{3}\cos 50° = 0$

(b) $\cos 0° + \cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7}$

$= 1 + \left(\cos\frac{\pi}{7} + \cos\frac{6\pi}{7}\right) + \left(\cos\frac{2\pi}{7} + \cos\frac{5\pi}{7}\right) + \left(\cos\frac{3\pi}{7} + \cos\frac{4\pi}{7}\right)$

$= 1 + \left(\cos\frac{\pi}{7} + \cos\left(\pi - \frac{\pi}{7}\right)\right) + \left(\cos\frac{2\pi}{7} + \cos\left(\pi - \frac{2\pi}{7}\right)\right) + \left(\cos\frac{3\pi}{7} + \cos\left(\pi - \frac{3\pi}{7}\right)\right)$

$= 1 + 0 + 0 + 0 = 1$

(c) $\cos 20° + \cos 40° + \cos 60° - 4\cos 10°\cos 20°\cos 30°$

$= \cos 20° + \cos 40° + \cos 60° - 4\cos 10°0\cos 20°\cos 30°$

$= 2\cos 30°\cos 10° + 2\cos^2 30° - 1 - 4\cos 10°\cos 20°\cos 30°$

$= 2\cos 30°\,(2\cos 10°\cos 20°) - 1 - 4\cos 10°\cos 20°\cos 30° = -1$

(d) $\cos 20°\cos 100° + \cos 100°\cos 140° - \cos 140°\cos 200°$

$= \frac{1}{2}(\cos 120° + \cos 80° + \cos 240° + \cos 40° - \cos 340° - \cos 60°)$

$= \frac{1}{2}\left(-\frac{1}{2} + \cos 80° - \frac{1}{2} + \cos 40° - \cos 340° - \frac{1}{2}\right) = \frac{1}{2}\left(-\frac{3}{2} + \cos 80° + \cos 40° - \cos 20°\right)$

$= \frac{1}{2}\left(-\frac{3}{2} + 2\cos 60°\cos 20° - \cos 20°\right) = \frac{1}{2}\left(-\frac{3}{2}\right) = -\frac{3}{4}$

Illustration 127:

If $\sin\alpha = A\sin(\alpha + \beta)$, $A \neq 0$, then the value of $\tan\alpha$ is

(A) $\frac{A\sin\beta}{1 - A\cos\beta}$ (B) $\frac{A\sin\beta}{1 + A\cos\beta}$ (C) $\frac{A\cos\beta}{1 - A\sin\beta}$ (D) $\frac{A\cos\beta}{1 + A\sin\beta}$

Ans. (A)

Solution:

$\sin\alpha = A\sin(\alpha + \beta) = A(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$

$\Rightarrow \sin\alpha\,(1 - A\cos\beta) = A\sin\beta\cos\alpha$ ...(i)

$\Rightarrow \tan\alpha = \frac{A\sin\beta}{(1 - A\cos\beta)}$ ...(ii)

Illustration 128:

If $\sin\alpha = A\sin(\alpha+\beta)$, $A \neq 0$, the value of $\tan\beta$ is

(A) $\frac{\sin\alpha(1+A\cos\beta)}{A\cos\alpha\cos\beta}$ (B) $\frac{\sin\alpha(1-A\cos\beta)}{A\cos\alpha\cos\beta}$ (C) $\frac{\cos\alpha(1-A\sin\beta)}{A\cos\alpha\cos\beta}$ (D) $\frac{\cos\alpha(1+A\sin\beta)}{A\cos\alpha\cos\beta}$

Ans. (B)

Solution:

$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{(1-A\cos\beta)\tan\alpha}{A\cos\beta} = \frac{(1-A\cos\beta)\sin\alpha}{A\cos\alpha\cos\beta}$ [from Eqs. (i) and (ii)]

Illustration 129:

If $\sin\alpha = A\sin(\alpha+\beta)$, $A \neq 0$, which of the following is not the value of $\tan(\alpha+\beta)$?

(A) $\frac{\sin\beta}{\cos\beta - A}$ (B) $\frac{\sin\alpha\cos\alpha}{A\cos\beta-\sin^2\alpha}$ (C) $\frac{\sin\alpha\cos\alpha}{A\cos\beta+\sin^2\alpha}$ (D) None of these

Ans. (B)

Solution:

$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$= \frac{\frac{A\sin\beta}{1-A\cos\beta}+\frac{\sin\beta}{\cos\beta}}{1-\frac{A\sin\beta\sin\beta}{(1-A\cos\beta)\cos\beta}} = \frac{\sin\beta}{\cos\beta - A}$$

Also $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$= \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\alpha(1-A\cos\beta)}{A\cos\alpha\cos\beta}}{1-\frac{\sin^2\alpha(1-A\cos\beta)}{A\cos^2\alpha\cos\beta}}$$

$$= \frac{[A\sin\alpha\cos\beta+\sin\alpha-A\sin\alpha\cos\beta]\cos\alpha}{A\cos^2\alpha\cos\beta-\sin^2\alpha+A\sin^2\alpha\cos\beta} = \frac{\sin\alpha\cos\alpha}{A\cos\beta-\sin^2\alpha}$$

Illustration 130:

Column-I	**Column-II**
(a) Suppose ABC is a triangle with three acute angle A, B and C. The point whose coordinates are $(\cos B - \sin A, \sin B - \cos A)$ can be in the	$(\boldsymbol{p})$ I^{st} quadrant
(b) If $2^{\sin\theta} > 1$ and $3^{\cos\theta} < 1$, then $\theta \in$	$(\boldsymbol{q})$ 2^{nd} quadrant
(c) $\lvert\cos x + \sin x\rvert = \lvert\sin x\rvert + \lvert\cos x\rvert$	$(\boldsymbol{r})$ 3^{rd} quadrant
(d) If $\sqrt{\frac{1-\sin A}{1+\sin A}} + \frac{\sin A}{\cos A} = \frac{1}{\cos A}$, for all permission values of A, then A can belong to	$(\boldsymbol{s})$ 4^{th} quadrant

Ans. $[\boldsymbol{a} \rightarrow \boldsymbol{q}; \boldsymbol{b} \rightarrow \boldsymbol{q}; \boldsymbol{c} \rightarrow \boldsymbol{p}, \boldsymbol{r}; \boldsymbol{d} \rightarrow \boldsymbol{p}, \boldsymbol{s}]$

Solution:

(a) Since angle A, B and C are acute angle

$\therefore A + B > \frac{\pi}{2}$

$A > \frac{\pi}{2} - B$

$\sin A - \cos B > 0$

$\Rightarrow$ $\cos B - \sin A < 0$ ...(i)

Again $B > \frac{\pi}{2} - A$

$\sin B > \cos A$

$\sin B - \cos A > 0$ ...(ii)

from Eq. (i) and (ii), we get that x-coordinates is –ve and y-coordinates is +ve.

Therefore, point lies in 2^{nd} quadrant only

(b) $2^{\sin\theta} > 1 \Rightarrow \sin\theta > 0 \Rightarrow \theta \in 1^{st}$ or 2^{nd} quadrant

$3^{\cos\theta} < 1 \Rightarrow \cos\theta < 0 \Rightarrow \theta \in 2^{nd}$ or 3^{rd} quadrant

Hence $\theta \in 2^{nd}$ quadrant

(c) $|\cos x + \sin x| = |\sin x| + |\cos x|$

$\Rightarrow$ $\cos x$ and $\sin x$ must have same sign or at least one in zero

$\Rightarrow$ $x \in 1^{st}$ or 3^{rd} quadrant

(d) L.H.S = $\frac{1-\sin A}{|\cos A|} + \frac{\sin A}{\cos A} = \frac{1}{\cos A}$ which is truly only if $|\cos A| = \cos A$

$\Rightarrow$ $A \in 1^{st}$ or 4^{th} quadrant

Illustration 131:

Column-I	Column-II
(a) If $x^2 + y^2 = 1$ and $p = (3x - 4x^3)^2 + (3y - 4y^3)^2$, then p is equal to	**(p)** 1
(b) If $a + b = 3 - \cos 4\theta$ and $a - b = 4\sin 2\theta$, then the maximum value of $(ab)$is	**(q)** 4
(c) The least positive integral value of x for which $3\cos\theta = x^2 - 8x + 19$	**(r)** 5
(d) If $x = \frac{4\lambda}{1+\lambda^2}$ and $y = \frac{2-2\lambda^2}{1+\lambda^2}$, where λ is a real parameter, then $x^2 - xy + y^2$ lies between $[a, b]$ then $(a + b)$ is	**(s)** 8

Ans. $[a \rightarrow p; b \rightarrow p; c \rightarrow q; d \rightarrow s]$

Solution:

(a) $x = \sin\theta, y = \cos\theta$

$p = (3\sin\theta - 4\sin^3\theta)^2 + (3\cos\theta - 4\cos^3\theta)^2 = \sin^2 3\theta + \cos^2 3\theta = 1$

(b) on adding, we get $a = \frac{3-\cos 4\theta + 4\sin 2\theta}{2} = (1 + \sin 2\theta)^2$

on subtracting, we get $b = (1 - \sin 2\theta)^2 \Rightarrow ab = \cos^4 2\theta \le 1$

(c) $3\cos\theta = x^2 - 8x + 19$

$\Rightarrow 3\cos\theta = (x - 4)^2 + 3$

Now L.H.S. = $3\cos\theta \le 3$ or L.H.S. has the greatest value 3.

But R.H.S. $(x - 4)^2 + 3 \ge 3$ OR R.H.S. has the least value 3.

Hence, L.H.S. = R.H.S. when $3\cos\theta = (x - 4)^2 + 3 = 3$

$\Rightarrow \cos\theta = 1$ and $x - 4 = 0$

$\Rightarrow \theta = 2n\pi$ and $x = 4$, where $n \in Z$.

(d) $\lambda = \tan\theta$

$x = 2\sin 2\theta$ and $y = 2\cos 2\theta$

$E = x^2 - xy + y^2 = 4 - 4\sin 2\theta\cos 2\theta = 4 - 2\sin 4\theta$

$E \in [2,6] \Rightarrow a + b = 8$

Illustration 132:

	Column-I	Column-II
(a)	In triangle ABC, $3\sin A + 4\cos B = 6$ $3\cos A + 4\sin B = 1$, then $\angle C$ can be	**(p)** 60°
(b)	In any triangle, is $(\sin A + \sin B + \sin C)$ $(\sin A + \sin B - \sin C) = 3\sin A \sin B$, then the angle C	**(q)** 30°
(c)	If $8\sin x\cos^5 x - 8\sin^5 x\cos x = 1$, then $x =$	**(r)** 165°
(d)	'O'is the center of inscribed circle in a 30°– 60°– 90° triangle ABC with right angled at C. if the circle is tangent to AB at D, then the angle $\angle COD$ is	**(s)** 7.5°

Ans. $[a \to q; b \to p;\ c \to s; d \to r]$

Solution:

(a) $9 + 16 + 24\sin(A+B) = 37$ (on squaring and adding)

$24\sin(A+B) = 12$

$\sin(A+B) = \frac{1}{2} \Rightarrow \sin C = \frac{1}{2}$

$C = 30°$ or $150°$

$\Rightarrow C = 30°$ ($\because$ for $C = 150°$)

(b) $(\sin A + \sin B)^2 - \sin^2 C = 3\sin A\sin B$

$\Rightarrow \sin^2 A - \sin^2 C + \sin^2 B = \sin A\sin B$

$\Rightarrow \sin(A+C)\sin(A-C) + \sin^2 B = \sin A\sin B$

$\Rightarrow \sin B[\sin(A-C) + \sin(A+C)] = \sin A\sin B$

$\Rightarrow 2\sin A\cos C = \sin A$ (as $\sin B \neq 0$)

$\Rightarrow \cos C = \frac{1}{2}$

$\Rightarrow C = 60°$

(c) $2\sin x\cos x\,[4\cos^4 - 4\sin^4 x] = 1$

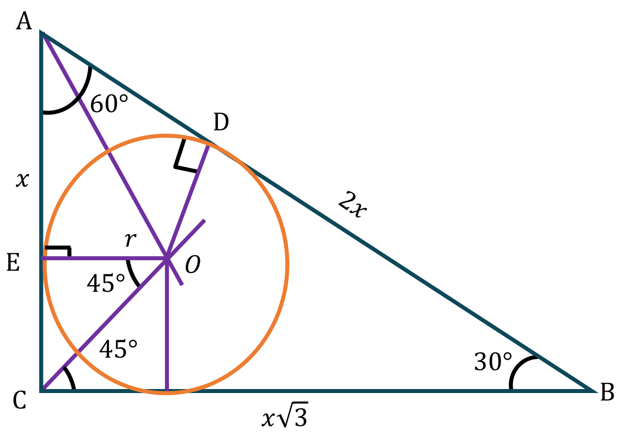
$\Rightarrow (\sin 2x)\,[2(\cos^2 x + \sin^2 x)][2(\cos^2 x - \sin^2 x)] = 1$

$\Rightarrow (\sin 2x)2\times 2\cos 2x = 1$

$\Rightarrow 2\sin 4x = 1$

$\Rightarrow \sin 4x = \frac{1}{2} \Rightarrow 4x = 30° \Rightarrow x = 7.5°$

(d)



Obviously $AEOD$ is a cyclic quadrilateral we have

$\angle COD = 120° + 45° = 165°$