# 06 Sequence and Series

## 1. Definition :

**Sequence :**

A succession of terms $a_1, a_2, a_3, a_4$........ formed according to some rule or law.

Examples are :1, 4, 9, 16, 25

$-1, 1, -1, 1,$........

$\frac{x}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}, ......$

A finite sequence has a finite (i.e. limited) number of terms, as in the first example above. An infinite sequence has an unlimited number of terms, i.e. there is no last term, as in the second and third examples.

**Series :**

The indicated sum of the terms of a sequence. In the case of a finite sequence $a_1, a_2, a_3, \ldots, a_n$ the corresponding series is $a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$. This series has a finite or limited number of terms and is called a finite series.

## 2. Arithmetic Progression (A.P.) :

A.P. is a sequence whose terms differ by a fixed number. This fixed number is called the common difference. If $a$ is the first term & $d$ the common difference, then A.P. can be written as

$a, a + d, a + 2d, ..., a + (n - 1)\ d$ , .....

(a) $n^{th}$ term of AP $T_n = a + (n-1)d$, where $d = t_n - t_{n-1}$

(b) The sum of the first $n$ terms : $S_n = \frac{n}{2}[a + \ell] = \frac{n}{2}[2a + (n-1)d]$

where $\ell$ is $n^{th}$ term.

**Note :**

(i) $n^{th}$ term of an A.P. is of the form $An + B$ i.e. a linear expression in '$n$', in such a case the coefficient of $n$ is the common difference of the A.P. i.e. $A$.

(ii) Sum of first '$n$' terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in '$n$', in such case the common difference is twice the coefficient of $n^2$. i.e. $2A$

(iii) Also $n^{th}$ term $T_n = S_n - S_{n-1}$

**Illustration 1:**

If $(x + 1)$, $3x$ and $(4x + 2)$ are first three terms of an A.P. then its $5^{th}$ term is -

(A) 14 (B) 19 (C) 24 (D) 28

**Ans. (C)**

**Solution:**

$(x+1), 3x, (4x+2)$ are in AP

$\Rightarrow 3x - (x+1) = (4x+2) - 3x \Rightarrow x = 3$

$\therefore a = 4, d = 9 - 4 = 5 \Rightarrow T_5 = 4 + (4)5 = 24$

**Illustration 2:**

The sum of first four terms of an A.P. is 56 and the sum of its last four terms is 112. If its first term is 11 then find the number of terms in the A.P.

**Solution :**

$a + a + d + a + 2d + a + 3d = 56$

$\Rightarrow 4a + 6d = 56$

$\Rightarrow 44 + 6d = 56$ (as $a = 11$)

$\Rightarrow 6d = 12$ hence $d = 2$

Let total number of terms = $n$

Now sum of last four terms.

$\Rightarrow a + (n-1)d + a + (n-2)d + a + (n-3)d + a + (n-4)d = 112$

$\Rightarrow 4a + (4n - 10)d = 112 \Rightarrow 44 + (4n - 10)2 = 112$

$\Rightarrow 4n - 10 = 34$

$\Rightarrow n = 11$ **Ans.**

**Illustration 3:**

The sum of first $n$ terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their $11^{th}$ terms.

**Solution :**

Let $a_1$ and $a_2$ be the first terms and $d_1$ and $d_2$ be the common differences of two A.P.s respectively then

$$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]} = \frac{7n+1}{4n+27} \Rightarrow \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_2+\left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of $11^{th}$ terms

$$\Rightarrow \frac{n-1}{2} = 10 \Rightarrow n = 21$$

so ratio of $11^{th}$ terms is $\frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$ **Ans.**

## 3. Properties of A.P. :

(a) If each term of an A.P. is increased, decreased, multiplied or divided by the some nonzero number, then the resulting sequence is also an A.P.

(b) Three numbers in A.P. : $a - d, a, a + d$

Four numbers in A.P. : $a - 3d, a - d, a + d, a + 3d$

Five numbers in A.P. : $a - 2d, a - d, a, a + d, a + 2d$

Six numbers in A.P. : $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(c) The common difference can be zero, positive or negative.

(d) $k^{th}$ term from the last = $(n - k + 1)^{th}$ term from the beginning (If total number of terms = $n$).

(e) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms. $\Rightarrow T_k + T_{n-k+1}$ = constant = $a + \ell$.

(f) Any term of an AP (except the first ) is equal to half the sum of terms which are equidistant from it. $a_n = (1/2)(a_{n-k} + a_{n+k}), k < n$

For $k=1,\ a_n = (1/2)(a_{n-1} + a_{n+1})$; For $k=2,\ a_n = (1/2)(a_{n-2} + a_{n+2})$ and so on.

(g) If $a, b, c$ are in AP, then $2b = a + c$.

**Illustration 4 :**

Four numbers are in A.P. If their sum is 20 and the sum of their squares is 120, then the middle terms are -

(A) 2, 4 (B) 4, 6 (C) 6, 8 (D) 8, 10

**Ans. (B)**

**Solution :**

Let the numbers are $a - 3d, a - d, a + d, a + 3d$

given, $a - 3d + a - d + a + d + a + 3d = 20 \Rightarrow 4a = 20 \Rightarrow a = 5$

and $(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 120 \Rightarrow 4a^2 + 20d^2 = 120$

$\Rightarrow 4 \times 5^2 + 20d^2 = 120 \Rightarrow d^2 = 1 \Rightarrow d = \pm 1$

Hence numbers are 2, 4, 6, 8 or 8, 6, 4, 2

**Illustration 5 :**

If $a_1, a_2, a_3, \ldots, a_n$ are in A.P. where $a_i > 0$ for all $i$, show that :

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{(n-1)}{\sqrt{a_1}+\sqrt{a_n}}$$

**Solution :**

$$\text{L.H.S.} = \frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

$$= \frac{1}{\sqrt{a_2}+\sqrt{a_1}}+\frac{1}{\sqrt{a_3}+\sqrt{a_2}}+\ldots\ldots+\frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{(a_2-a_1)}+\frac{\sqrt{a_3}-\sqrt{a_2}}{(a_3-a_2)}+\ldots\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

Let '$d$' is the common difference of this A.P.

then $a_2 - a_1 = a_3 - a_2 = \ldots\ldots = a_n - a_{n-1} = d$

Now L.H.S.

$$= \frac{1}{d}\left\{\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots\ldots+\sqrt{a_{n-1}}-\sqrt{a_{n-2}}+\sqrt{a_n}-\sqrt{a_{n-1}}\right\} = \frac{1}{d}\left\{\sqrt{a_n}-\sqrt{a_1}\right\}$$

$$= \frac{a_n-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{a_1+(n-1)d-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)} = \frac{1}{d}\frac{(n-1)d}{(\sqrt{a_n}+\sqrt{a_1})} = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}} = R.H.S.$$

**Illustration 6 :**

The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7, 8, 9) ....... and so on. Show that the sum of the numbers in $n^{th}$ group is $n^3 + (n - 1)^3$

**Solution :**

The groups are (1), (2, 3, 4), (5, 6, 7, 8, 9) .......

The number of terms in the groups are 1, 3, 5......

$\therefore$ The number of terms in the $n^{th}$ group = $(2n - 1)$

the last term of the $n^{th}$ group is $n^2$

If we count from last term common difference should be –1

So the sum of numbers in the $n^{th}$ group = $\left(\frac{2n-1}{2}\right)\left\{2n^2+(2n-2)(-1)\right\}$

= $(2n - 1)(n^2 - n + 1) = 2n^3 - 3n^2 + 3n - 1 = n^3 + (n - 1)^3$

## 4. GEOMETRIC PROGRESSION (G.P.) :

G.P. is a sequence of non-zero numbers in which, each of the succeeding term is equal to the preceding term multiplied by a constant. Thus, in a GP the ratio of successive terms is constant. This constant factor is called the COMMON RATIO of the sequence & is obtained by dividing any term by the immediately previous term. Therefore $a, ar, ar^2, ar^3, ar^4$ , .......... is a GP with '$a$' as the first term & '$r$' as common ratio.

(a) $n^{th}$ term ; $T_n = a\,r^{n-1}$

(b) Sum of the first $n$ terms; $S_n = \frac{a(r^n - 1)}{r-1}$, if $r \neq 1$

(c) Sum of infinite G.P. , $S_\infty = \frac{a}{1-r}$; $0 < |r| < 1$

## 5. Properties of GP :

(a) If each term of a G.P. be multiplied or divided by some non-zero quantity, then the resulting sequence is also a G.P.

(b) Three consecutive terms of a GP : $a/r,\ a, ar$ ;
Four consecutive terms of a GP : $a/r^3,\ a/r, ar, ar^3$ & so on.

(c) If $a, b, c$ are in G.P. then $b^2 = ac$.

(d) In a G.P, the product of two terms which are equidistant from the first and the last term is constant and is equal to the product of first and last term. $\Rightarrow T_k \,.\, T_{n-k+1}$ = constant = $a.\ell$

(e) If each term of a G.P. be raised to the same power, then resulting sequence is also a G.P.

(f) In a G.P., $T_r^2 = T_{r-k}.T_{r+k}$, $k < r, r \neq 1$

(g) If the terms of a given G.P. are chosen at regular intervals, then the new sequence is also a G.P.

(h) If $a_1, a_2, a_3 \ldots a_n$ is a G.P. of positive terms, then log $a_1$, log $a_2$,.....log $a_n$ are an A.P. and vice-versa.

(i) If $a_1, a_2, a_3$..... and $b_1, b_2, b_3$..... are two G.P.'s then $a_1b_1, a_2b_2, a_3b_3$..... & $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$.......... is also in G.P.

### Illustration 7 :

If $a, b, c, d$ and $p$ are distinct real numbers such that $\left(a^2+b^2+c^2\right)p^2-2p(ab+bc+cd)+\left(b^2+c^2+d^2\right)\le 0$ then $a, b, c, d$ are in

(A) A.P. (B) G.P. (C) H.P. (D) None of these

**Ans. (B)**

### Solution :

Here, the given condition $\left(a^2+b^2+c^2\right)p^2-2p(ab+bc+cd)+b^2+c^2+d^2\le 0$

$\Rightarrow (ap-b)^2+(bp-c)^2+(cp-d)^2\le 0$

$\because$ $a$ square can not be negative

$\therefore\ ap-b=0, bp-c=0, cp-d=0 \Rightarrow p=\frac{b}{a}=\frac{c}{b}=\frac{d}{c}\Rightarrow$ $a, b, c, d$ are in G.P.

### Illustration 8 :

If positive real numbers $a, b, c$ are in G.P., then the equations $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root if $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in -

(A) A.P. (B) G.P. (C) H.P. (D) None of these

**Ans. (A)**

### Solution :

$a, b, c$ are in G.P $\Rightarrow b^2=ac$

Now the equation $ax^2+2bx+c=0$ can be rewritten as $\mathrm{ax}^2+2\sqrt{\mathrm{ac}}\mathrm{x}+\mathrm{c}=0$

$\Rightarrow\left(\sqrt{a}x+\sqrt{c}\right)^2=0\Rightarrow x=-\sqrt{\frac{c}{a}},-\sqrt{\frac{c}{a}}$

If the two given equations have a common root, then this root must be $-\sqrt{\frac{c}{a}}$.

Thus $d\frac{c}{a}-2e\sqrt{\frac{c}{a}}+f=0 \Rightarrow \frac{d}{a}+\frac{f}{c}=\frac{2e}{c}\sqrt{\frac{c}{a}}=\frac{2e}{\sqrt{ac}}=\frac{2e}{b} \Rightarrow \frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in A.P.

### Illustration 9 :

A number consists of three digits which are in G.P. the sum of the right hand and left hand digits exceeds twice the middle digit by 1 and the sum of the left hand and middle digits is two third of the sum of the middle and right hand digits. Find the numbers.

### Solution :

Let the three digits be $a$, $ar$ and $ar^2$ then number is

$100a + 10ar + ar^2$ ...(i)

Given, $a + ar^2 = 2ar + 1$

or $a(r^2 - 2r + 1) = 1$

or $a(r - 1)^2 = 1$ ...(ii)

Also given $a + ar = \frac{2}{3}(ar + ar^2)$

$\Rightarrow$ $3 + 3r = 2r + 2r^2 \Rightarrow 2r^2 - r - 3 = 0 \Rightarrow (r + 1)(2r - 3) = 0$

$\therefore$ $r = -1, 3/2$

for $r = -1$, $a = \dfrac{1}{(r-1)^2} = \dfrac{1}{4} \notin I \qquad \therefore r \neq -1$

for $r = 3/2$, $a = \dfrac{1}{\left(\dfrac{3}{2}-1\right)^2} = 4 \qquad$ {from (ii)}

From (i), number is $400 + 10.4.\dfrac{3}{2} + 4.\dfrac{9}{4} = 469$ **Ans.**

**Illustration 10 :**

Find the value of $0.32\overline{58}$

**Solution :**

Let $R = 0.32\overline{58} \Rightarrow R = 0.32585858....$ ...(i)

Here number of figures which are not recurring is 2 and number of figures which are recurring is also 2.

then $100\,R = 32.585858......$ ...(ii)

and $10000\,R = 3258.5858.....$ ...(iii)

Subtracting (ii) from (iii) , we get

$9900\,R = 3226 \Rightarrow R = \dfrac{1613}{4950}$

**Aliter Method :** $R = .32 + .0058 + .000058 + .00000058 + ..........$

$$= .32 + \frac{58}{10^4}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + .........\infty\right) = .32 + \frac{58}{10^4}\left(\frac{1}{1-\frac{1}{100}}\right)$$

$$= \frac{32}{100} + \frac{58}{9900} = \frac{3168+58}{9900} = \frac{3226}{9900} = \frac{1613}{4950}$$

## 6. HARMONIC PROGRESSION (H.P.) :

A sequence is said to be in H.P. if the reciprocal of its terms are in AP.

If the sequence $a_1, a_2, a_3, ......., a_n$ is an HP then $1/a_1, 1/a_2, ......, 1/a_n$ is an AP . Here we do not have the formula for the sum of the $n$ terms of an HP. The general form of a harmonic progression is $\dfrac{1}{a}, \dfrac{1}{a+d}, \dfrac{1}{a+2d}, ........., \dfrac{1}{a+(n-1)d}$

**Note :**

No term of any H.P. can be zero.

(i) If $a, b, c$ are in HP, then $b = \dfrac{2ac}{a+c}$ or $\dfrac{a}{c} = \dfrac{a-b}{b-c}$

**Illustration 11 :**

The sum of three numbers in H.P. is 37 and the sum of their reciprocals is 1/4. Find the numbers.

**Solution :**

Three number in H.P. can be taken as

$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

then $\frac{1}{a-d}+\frac{1}{a}+\frac{1}{a+d} = 37$ ...(i)

and $a - d + a + a + d = \frac{1}{4} \Rightarrow a = \frac{1}{12}$

from (i), $\frac{12}{1-12d}+12+\frac{12}{1+12d}=37 \Rightarrow \frac{12}{1-12d}+\frac{12}{1+12d}=25$

$\Rightarrow \frac{24}{1-144d^2}=25$

$\Rightarrow 1-144d^2=\frac{24}{25}$

$\Rightarrow d^2 = \frac{1}{25\times 144}$

$\therefore d = \pm\frac{1}{60}$

$\therefore a - d, a, a + d$ are $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$ or $\frac{1}{10}, \frac{1}{12}, \frac{1}{15}$

Hence, three numbers in H.P. are 15, 12, 10 or 10, 12, 15 **Ans.**

**Illustration 12 :**

Suppose $a$ is a fixed real number such that $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$

If $p, q, r$ are in A.P., then prove that $x, y, z$ are in H.P.

**Solution :**

$\because p, q, r$ are in A.P.

$\therefore q - p = r - q$ ...(i)

$\Rightarrow p - q = q - r = k$ (let)

given $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz} \Rightarrow \frac{\frac{a}{x}-1}{p}=\frac{\frac{a}{y}-1}{q}=\frac{\frac{a}{z}-1}{r}$

$\Rightarrow \frac{\left(\frac{a}{x}-1\right)-\left(\frac{a}{y}-1\right)}{p-q}=\frac{\left(\frac{a}{y}-1\right)-\left(\frac{a}{z}-1\right)}{q-r}$ (by law of proportion)

$\Rightarrow \frac{\frac{a}{x}-\frac{a}{y}}{k}=\frac{\frac{a}{y}-\frac{a}{z}}{k}$ {from (i)}

$\Rightarrow a\left(\frac{1}{x}-\frac{1}{y}\right)=a\left(\frac{1}{y}-\frac{1}{z}\right) \Rightarrow \frac{1}{x}-\frac{1}{y}=\frac{1}{y}-\frac{1}{z}$

$\therefore \frac{2}{y}=\frac{1}{x}+\frac{1}{z}$

$\therefore \frac{1}{x},\frac{1}{y},\frac{1}{z}$ are in A.P.

Hence $x, y, z$ are in H.P.

## 7. MEANS

### (a) Arithmetic Mean :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if $a$, $b$, $c$ are in A.P., $b$ is A.M. of $a$ & $c$. So A.M. of $a$ and $c$ = $\frac{a+c}{2}=b$.

**Insertion of $n$-Arithmetic Means Between Two Numbers :**

If $a$, $b$ be any two given numbers & $a, A_1, A_2, \ldots, A_n, b$ are in AP, then $A_1, A_2, \ldots A_n$ are the '$n$' A.M's between $a$ & $b$ then. $A_1 = a + d$, $A_2 = a + 2d$ ,......, $A_n = a + nd$ or $b - d$, where $d=\frac{b-a}{n+1}$

$\Rightarrow A_1=a+\frac{b-a}{n+1},\ A_2=a+\frac{2(b-a)}{n+1}$ ,.......

**Note:** Sum of $n$ A.M's inserted between $a$ & $b$ is equal to $n$ times the single A.M. between $a$ & $b$

i.e. $\sum_{r=1}^{n} A_r = nA$ where $A$ is the single A.M. between $a$ & $b$.

### (b) Geometric Mean :

If $a, b, c$ are in G.P., then $b$ is the G.M. between $a$ & $c$, $b^2 = ac$. So G.M. of $a$ and $c$ = $\sqrt{ac}=b$

**Insertion of $n$-Geometric Means Between Two Numbers :**

If $a$, $b$ are two given positive numbers & $a, G_1, G_2, \ldots, G_n, b$ are in G.P. Then $G_1, G_2, G_3, \ldots G_n$ are '$n$' G.Ms between $a$ & $b$. where $b = ar^{n+1} \Rightarrow r = (b/a)^{1/n+1}$

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$ ................, $G_n = a(b/a)^{n/n+1}$

$= ar$, $= ar^2$, ................ $= ar^n = b/r$

**Note :** The product of $n$ G.Ms between $a$ & $b$ is equal to $n^{th}$ power of the single G.M. between $a$ & $b$ i.e., $\prod_{r=1}^{n} G_r = (G)^n$ where $G$ is the single G.M. between $a$ & $b$

### (c) Harmonic Mean :

If $a, b, c$ are in H.P., then $b$ is H.M. between $a$ & $c$. So, H.M. of $a$ and $c$ = $\frac{2ac}{a+c}$ = $b$.

**Insertion of '$n$' HM's between $a$ and $b$ :**

$a, H_1, H_2, H_3, \ldots, H_n, b \rightarrow$ H.P

$\frac{1}{a},\frac{1}{H_1},\frac{1}{H_2},\frac{1}{H_3},\ldots\ldots\frac{1}{H_n},\frac{1}{b} \rightarrow$ A.P.

$$\frac{1}{b}=\frac{1}{a}+(n+1)D \Rightarrow D=\frac{\frac{1}{b}-\frac{1}{a}}{n+1}$$

$$\frac{1}{H_n}=\frac{1}{a}+n\left(\frac{\frac{1}{b}-\frac{1}{a}}{n+1}\right)$$

**Important note :**

(i) If $A, G, H$, are respectively A.M., G.M., H.M. between two positive number $a$ & $b$ then

(a) $G^2$ = AH ($A, G, H$ constitute a GP) (b) $A \geq G \geq H$ (c) $A = G = H \Leftrightarrow a = b$

(ii) Let $a_1, a_2, \ldots\ldots, a_n$ be $n$ positive real numbers, then we define their arithmetic mean ($A$), geometric mean ($G$) and harmonic mean ($H$) as $A = \frac{a_1+a_2+\ldots+a_n}{n}$

$G = (a_1 a_2 \ldots a_n)^{1/n}$ and $H=\frac{n}{\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_n}\right)}$

It can be shown that $A \geq G \geq H$. Moreover, equality holds at either place if and only if $a_1 = a_2 = \ldots\ldots = a_n$

**Illustration 13 :**

If $2x^3 + ax^2 + bx + 4 = 0$ ($a$ and $b$ are positive real numbers) has 3 real roots, then prove that $a + b \geq 6(2^{1/3} + 4^{1/3})$.

**Solution :**

Let $\alpha, \beta, \gamma$ be the roots of $2x^3 + ax^2 + bx + 4 = 0$. Given that all the coefficients are positive, so all the roots will be negative.

Let $\alpha_1 = -\alpha$, $\alpha_2 = -\beta$, $\alpha_3 = -\gamma$ $\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = \frac{a}{2}$

$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = \frac{b}{2}$

$\alpha_1\alpha_2\alpha_3 = 2$

Applying AM $\geq$ GM, we have

$$\frac{\alpha_1+\alpha_2+\alpha_3}{3} \geq (\alpha_1\alpha_2\alpha_3)^{1/3} \Rightarrow a \geq 6\times 2^{1/3}$$

Also $\frac{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3}{3} > (\alpha_1\alpha_2\alpha_3)^{2/3} \Rightarrow b \geq 6 \times 4^{1/3}$

Therefore $a + b \geq 6(2^{1/3} + 4^{1/3})$.

**Illustration 14 :**

If $a_i > 0\ \forall\ i \in N$ such that $\prod_{i=1}^{n} a_i = 1$, then prove that $(1 + a_1)(1 + a_2)(1 + a_3)\ldots(1 + a_n) \geq 2^n$

**Solution :**

Using A.M. ≥ G.M.

$1 + a_1 \geq 2\sqrt{a_1}$

$1 + a_2 \geq 2\sqrt{a_2}$

⋮

$1 + a_n \geq 2\sqrt{a_n} \Rightarrow (1 + a_1)(1 + a_2)........(1 + a_n) \geq 2^n (a_1 a_2 a_3 .....a_n)^{1/2}$

As $a_1 a_2 a_3 .....a_n = 1$

Hence $(1 + a_1)(1 + a_2)..........(1 + a_n) \geq 2^n$.

**Illustration 15 :**

If $a, b, x, y$ are positive natural numbers such that $\frac{1}{x}+\frac{1}{y}=1$ then prove that $\frac{a^x}{x}+\frac{b^y}{y}\geq ab$.

**Solution :**

Consider the positive numbers $a^x, a^x,.......y$ times and $b^y, b^y,......x$ times

For all these numbers,

$$AM = \frac{\{a^x+a^x+......y\ time\}+\{b^y+b^y+......x\ times\}}{x+y} = \frac{ya^x+xa^y}{(x+y)}$$

$$GM = \left\{\left(a^x.a^x......y\ times\right)\left(b^y.b^y......x\ times\right)\right\}^{\frac{1}{(x+y)}} = \left[\left(a^{xy}\right).\left(b^{xy}\right)\right]^{\frac{1}{(x+y)}} = (ab)^{\frac{xy}{(x+y)}}$$

As $\frac{1}{x}+\frac{1}{y}=1, \frac{x+y}{xy}=1$, i.e, $x + y = xy$

So, using AM ≥ GM $\frac{ya^x+xb^y}{x+y}\geq(ab)^{\frac{xy}{x+y}}$

$\therefore \frac{ya^x+xb^y}{xy}\geq ab$ or $\frac{a^x}{x}+\frac{b^y}{y}\geq ab.$

## 8. ARITHMETICO - GEOMETRIC PROGRESSION :

A series, each term of which is formed by multiplying the corresponding term of an A.P. & G.P. is called the Arithmetico-Geometric Series , e.g. $1+ 3x + 5x^2 + 7x^3 + .........$

Here 1, 3, 5, ........ are in A.P. & 1, $x$, $x^2, x^3$ ............. are in G.P.

**(a) Sum of $n$ terms of an Arithmetico-Geometric Progression :**

Let $S_n = a+(a+d)r+(a+2d)r^2+..........+[a+(n-1)d]r^{n-1}$

then $S_n = \frac{a}{1-r}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{[a+(n-1)d]\ r^n}{1-r},\ r\neq 1$

**(b) Sum to infinity :**

If $0<|r|<1$ & $n\to\infty$, then $\lim_{n\to\infty} r^n = 0, S_\infty = \frac{a}{1-r}+\frac{dr}{(1-r)^2}$

**Illustration 16 :**

Find the sum of series $4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + ......... \infty, |x| < 1$

**Solution :**

Let $S = 4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots\ldots\ldots \infty$

$-Sx = -4x + 9x^2 - 16x^3 + 25x^4 - 36x^5 + \ldots\ldots\ldots \infty$

On subtraction, we get

$S(1 + x) = 4 - 5x + 7x^2 - 9x^3 + 11x^4 - 13x^5 + \ldots\ldots \infty$

$-S(1 + x)x = -4x + 5x^2 - 7x^3 + 9x^4 - 11x^5 + \ldots\ldots \infty$

On subtraction, we get

$S(1 + x)^2 = 4 - x + 2x^2 - 2x^3 + 2x^4 - 2x^5 + \ldots\ldots \infty$

$$= 4 - x + 2x^2 (1 - x + x^2 - \ldots\ldots\ldots\infty) = 4 - x + \frac{2x^2}{1+x} = \frac{4+3x+x^2}{1+x}$$

$S = \dfrac{4+3x+x^2}{(1+x)^3}$ **Ans.**

**Illustration 17 :**

Find the sum of series up to $n$ terms $\left(\frac{2n+1}{2n-1}\right)+3\left(\frac{2n+1}{2n-1}\right)^2+5\left(\frac{2n+1}{2n-1}\right)^3+\ldots\ldots\ldots$.

**Solution :**

Let $x=\left(\frac{2n+1}{2n-1}\right)$

For $x \neq 1$, let

$S = x + 3x^2 + 5x^3 + \ldots\ldots + (2n-3)x^{n-1} + (2n-1)x^n$ ...(i)

$\Rightarrow xS = x^2 + 3x^3 + \ldots\ldots + (2n-5)x^{n-1} + (2n-3)x^n + (2n-1)x^{n+1}$ ...(ii)

Subtracting (ii) from (i), we get

$(1 - x)S = x + 2x^2 + 2x^3 + \ldots\ldots\ldots + 2x^{n-1} + 2x^n - (2n-1)x^{n+1}$

$= x + \dfrac{2x^2(1-x^{n-1})}{1-x} - (2n-1)x^{n+1}$

$= \dfrac{x}{1-x}[1 - x + 2x - 2x^n - (2n-1)x^n + (2n-1)x^{n+1}]$

$\Rightarrow S = \dfrac{x}{(1-x)^2}[(2n-1)x^{n+1} - (2n+1)x^n + 1 + x]$

Thus $\left(\frac{2n+1}{2n-1}\right)+3\left(\frac{2n+1}{2n-1}\right)^2+\ldots\ldots+(2n-1)\left(\frac{2n+1}{2n-1}\right)^n$

$$= \left(\frac{2n+1}{2n-1}\right)\left(\frac{2n-1}{2}\right)^2\left[(2n-1)\left(\frac{2n+1}{2n-1}\right)^{n+1}-(2n+1)\left(\frac{2n+1}{2n-1}\right)^n+1+\frac{2n+1}{2n-1}\right]$$

$$= \frac{4n^2-1}{4}\cdot\frac{4n}{2n-1} = n(2n+1) \textbf{ Ans.}$$

## 9. SIGMA NOTATIONS ( $\Sigma$ )

**THEOREMS :**

**(a)** $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$ **(b)** $\sum_{r=1}^{n} k\,a_r = k\sum_{r=1}^{n} a_r$ **(c)** $\sum_{r=1}^{n} k = nk$ ; where $k$ is a constant.

## 10. RESULTS

**(a)** $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first $n$ natural numbers)

**(b)** $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first $n$ natural numbers)

**(c)** $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first $n$ natural numbers)

**(d)** $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

**(e)** $\sum_{r=1}^{n}(2r-1) = n^2$ (sum of first $n$ odd natural numbers)

**(f)** $\sum_{r=1}^{n} 2r = n(n+1)$ (sum of first $n$ even natural numbers)

**Note :**

If $n^{th}$ term of a sequence is given by $T_n = an^3 + bn^2 + cn + d$ where $a, b, c, d$ are constants,

then sum of $n$ terms $S_n = \Sigma T_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + \Sigma d$

This can be evaluated using the above results.

**Illustration 18 :**

Find the sum of the series to $n$ terms whose $n^{th}$ terms is $3n + 2$.

**Solution :**

$$\Rightarrow S_n = \Sigma T_n = \Sigma(3n+2) = 3\Sigma n + \Sigma 2 = \frac{3(n+1)n}{2} + 2n = \frac{n}{2}(3n+7)$$

**Illustration 19 :**

If $T_k = k^3 + 3^k$, then find $\sum_{k=1}^{n} T_k$.

**Solution :**

$$\Rightarrow \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 3^k = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3(3^n-1)}{3-1} = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3}{2}(3^n-1)$$

**Illustration 20 :**

Find the value of the expression $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$

**Solution :**

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j = \sum_{i=1}^{n}\frac{i(i+1)}{2} = \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right] = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{12}[2n+1+3] = \frac{n(n+1)(n+2)}{6}.$$

**Illustration 21 :**

Sum up to 16 terms of the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+.....$ is

(A) 450 (B) 456 (C) 446 (D) none of these

**Ans. (C)**

**Solution :**

$$t_n=\frac{1^3+2^3+3^3+....+n^3}{1+3+5+....(2n-1)}=\frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\{2+2(n-1)\}}=\frac{\frac{n^2(n+1)^2}{4}}{n^2}=\frac{(n+1)^2}{4}==\frac{n^2}{4}+\frac{n}{2}+\frac{1}{4}$$

$$\therefore\ S_n=\Sigma t_n=\frac{1}{4}\Sigma n^2+\frac{1}{2}\Sigma n+\frac{1}{4}\Sigma 1\ =\frac{1}{4}.\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}.\frac{n(n+1)}{2}+\frac{1}{4}.n$$

$$\therefore\ S_{16}=\frac{16.17.33}{24}+\frac{16.17}{4}+\frac{16}{4}=446$$

## 11. METHOD OF DIFFERENCE :

Sometimes the $n^{th}$ term of a sequence or a series cannot be determined by the method, we have discussed earlier. So, we compute the difference between the successive terms of given sequence to obtain the $n^{th}$ term.

If $T_1, T_2, T_3,.....,T_n$ are the terms of a sequence then sometimes the terms $T_2 - T_1, T_3 - T_2,......$ constitute an AP/GP. $n^{th}$ term of the series is determined & the sum to $n$ terms of the sequence can easily be obtained.

**Case 1 :**

(a) If difference series are in A.P., then

Let $T_n = an^2 + bn + c$, where $a, b, c$ are constant

(b) If difference of difference series is in A.P., then

Let $T_n = an^3 + bn^2 + cn + d$, where $a, b, c, d$ are constant

**Case 2 :**

(a) If difference is in G.P., then

Let $T_n = ar^n + b$, where $r$ is common ratio & $a, b$ are constant

(b) If difference of difference is in G.P., then

Let $T_n = ar^n + bn + c$, where $r$ is common ratio & $a, b, c$ are constant

Determine constant by putting $n$ = 1, 2, 3 ....... $n$ and putting the value of $T_1, T_2, T_3$ ...... and sum of series $(S_n) = \sum T_n$

**Illustration 22 :**

Find the sum of $n$ terms of the series 3 + 7 + 14 + 24 + 37 + ..........

**Solution :**

Clearly here the differences between the successive terms are

7 – 3, 14 – 7, 24 – 14, ........ i.e. 4, 7, 10, 13,........., which are in A.P.

Let $S = 3 + 7 + 14 + 24 + ........ + T_n$

$S = \quad 3 + 7 + 14 + ....... + T_{n-1} + T_n$

Subtracting, we get

$0 = 3 + [4 + 7 + 10 + 13 + ........ (n - 1) \text{ terms}] - T_n$

$\therefore \quad T_n = 3 + S_{n-1}$ of an A.P. whose $a = 4$ and $d = 3$.

$\therefore \quad T_n = 3 + \left(\frac{n-1}{2}\right)(2.4+(n-2)3) = \frac{6+(n-1)(3n+2)}{2}$ or, $T_n = \frac{1}{2}(3n^2 - n + 4)$

Now putting $n = 1, 2, 3, ......, n$ and adding

$\therefore \ S_n = \frac{1}{2}[3\sum n^2 - \sum n + 4n] = \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2+n+4)$ **Ans.**

**Aliter Method :**

Let $T_n = an^2 + bn + c$

Now, $T_1 = 3 = a + b + c$ ...(i)

$T_2 = 7 = 4a + 2b + c$ ...(ii)

$T_3 = 14 = 8a + 3b + c$ ...(iii)

Solving (i), (ii) & (iii) we get

$a = \frac{3}{2}, b = -\frac{1}{2}$ & $c = 2 \quad \therefore \quad T_n = \frac{1}{2}(3n^2 - n + 4)$

$\Rightarrow S_n = \Sigma T_n = \frac{1}{2}[3\sum n^2 - \sum n + 4n] = \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2+n+4)$ **Ans.**

### Illustration 23 :

Find the sum of $n$-terms of the series 1 + 4 + 10 + 22 + .....

**Solution :**

Let $S = 1 + 4 + 10 + 22 + ....... + T_n$ ...(i)

$S = \quad 1 + 4 + 10 + ....... + T_{n-1} + T_n$ ...(ii)

(i) – (ii) $\Rightarrow T_n = 1 + (3 + 6 + 12 + ........ + T_n - T_{n-1})$

$T_n = 1 + 3\left(\frac{2^{n-1}-1}{2-1}\right)$

$T_n = 3 \,.\, 2^{n-1} - 2$

So $S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2$

$= 3\left(\frac{2^n - 1}{2-1}\right) - 2n = 3.2^n - 2n - 3$ **Ans.**

**Aliter Method :**

Let $T_n = ar^n + b$, where $r = 2$

Now $T_1 = 1 = ar + b$ ...(i)

$T_2 = 4 = ar^2 + b$ ...(ii)

Solving (i) & (ii), we get

$a = \frac{3}{2}, b = -2$

$\therefore T_n = 3.2^{n-1} - 2$

$\Rightarrow S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2$

$= 3\left(\frac{2^n-1}{2-1}\right) - 2n = 3.2^n - 2n - 3$ **Ans.**

**Illustration 24 :**

Find the general term and sum of $n$ terms of the series 1 + 5 + 19 + 49 + 101 + 181 + 295 + ......

**Solution :**

The sequence of difference between successive term 4, 14, 30, 52, 80 ......

The sequence of the second order difference is 10, 16, 22, 28, ...... clearly it is an A.P.

So, let $n^{th}$ term

$\Rightarrow T_n = an^3 + bn^2 + cn + d$

$\Rightarrow a + b + c + d = 1$ ...(i)

$\Rightarrow 8a + 4b + 2c + d = 5$ ...(ii)

$\Rightarrow 27a + 9b + 3c + d = 19$ ...(iii)

$\Rightarrow 64a + 16b + 4c + d = 49$ ...(iv)

from (i), (ii), (iii) & (iv)

$\Rightarrow a = 1, b = -1, c = 0, d = 1 \Rightarrow \quad T_n = n^3 - n^2 + 1$

$$\therefore S_n = \sum(n^3 - n^2 + 1) = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} + n = \frac{n(n^2-1)(3n+2)}{12} + n$$

**Illustration 25 :**

If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n} \frac{1}{T_r}$.

**Solution :**

$\because T_n = S_n - S_{n-1}$

$$= \sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r = \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)n(n+1)(n+2)}{8} = \frac{n(n+1)(n+2)}{8}[(n+3)-(n-1)]$$

$$T_n = \frac{n(n+1)(n+2)}{8}(4) = \frac{n(n+1)(n+2)}{2}$$

$$\Rightarrow \frac{1}{T_n} = \frac{2}{n(n+1)(n+2)} = \frac{(n+2)-n}{n(n+1)(n+2)} = \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \quad \text{...(i)}$$

Let $V_n = \frac{1}{n(n+1)}$

$\therefore \frac{1}{T_n} = V_n - V_{n+1}$

Putting $n = 1, 2, 3, .... n$

$$\Rightarrow \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} + ....... + \frac{1}{T_n} = (V_1 - V_{n+1}) \Rightarrow \sum_{r=1}^{n} \frac{1}{T_r} = \frac{n^2+3n}{2(n+1)(n+2)}$$

**Illustration 26 :**

Find the sum of $n$ terms of the series 1 . 3 . 5 + 3 . 5 . 7 + 5 . 7 . 9 + .......

**Solution :**

The $n^{th}$ term is $(2n - 1)(2n + 1)(2n + 3)$

$T_n = (2n - 1)\,(2n + 1)\,(2n + 3)$

$T_n = \frac{1}{8}(2n - 1)\,(2n + 1)\,(2n + 3)\,\{(2n + 5) - (2n - 3)\}$

$= \frac{1}{8}(V_n - V_{n-1})$ [Let $V_n$ = (2$n$ – 1) (2$n$ + 1) (2$n$ + 3) (2$n$ + 5)]

$S_n = \sum T_n = \frac{1}{8}[V_n - V_0]$

$\therefore S_n = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{8} + \frac{15}{8} = n\,(2n^3 + 8n^2 + 7n - 2)$ **Ans.**

**Illustration 27 :**

Find the natural number '$a$' for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function $f$ satisfied $f(x+y) = f(x).f(y)$ for all natural number $x, y$ and further $f(1) = 2$.

**Solution :**

It is given that

$f(x + y) = f(x)\, f(y)$ and $f(1) = 2$

$f(1 + 1) = f(1)\, f(1) \Rightarrow f(2) = 2^2$, $f(1 + 2) = f(1)\, f(2) \Rightarrow f(3) = 2^3$, $f(2 + 2) = f(2)\, f(2) \Rightarrow f(4) = 2^4$

Similarly $f(k) = 2^k$ and $f(a) = 2^a$

Hence , = $\sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a)f(k) = f(a)\sum_{k=1}^{n} f(k) = 2^a \sum_{k=1}^{n} 2^k$ $2^a = (2^1 + 2^2 + \ldots\ldots\ldots + 2^n)$

$= 2^a \left\{\frac{2(2^n - 1)}{2-1}\right\} = 2^{a+1}(2^n - 1)$

But $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$

$2^{a+1}(2^n - 1) = 16\,(2^n - 1)$

$\therefore 2^{a+1} = 2^4$

$\therefore a + 1 = 4 \Rightarrow a = 3$ **Ans.**