01 Relations

1. Cartesian product of two sets:

Given two sets A & B. The cartesian product $A \times B$ is the set of all ordered pairs of the form (a, b) where the first entry comes from set A & second comes from set B.

$A \times B = \{(a, b) \mid a \in A, b \in B\}$

e.g. $A = \{1, 2, 3\}$

$B = \{p, q\}$

$A \times B = \{(1, p), (1, q), (2, p), (2, q), (3, p), (3, q)\}$

Note : (i) If either A or B is the null set, then $A \times B$ will also be empty set, i.e. $A \times B = \phi$

(ii) If $n(A) = p, n(B) = q$, then $n(A \times B) = pq$, where $n(X)$ denotes the number of elements in set X.

(iii) A Relation R from set A to B is any subset of $A \times B$. If ARB & $(a, b) \in R$ then b is image of a under R and a is preimage of b under R.

Note : If $n(A) = m, n(B) = n$, then number of relations defined from set A to B are 2^{mn}.

Illustration 1:

If $A \times B = \{(p, x);\ (p, y);\ (q, x);\ (q, y)\}$, find A and B.

Solution:

A is a set of all first entries in ordered pairs in $A \times B$. B is a set of all second entries in ordered pairs in $A \times B$. Thus $A = \{p, q\}$ and $B = \{x, y\}$

Illustration 2:

If A and B are two sets, and $A \times B$ consists of 6 elements: If three elements of $A \times B$ are (2,5) (3,7) (4,7) find $A \times B$.

Solution:

Since, (2, 5) (3, 7) and (4, 7) are elements of $A \times B$. So, we can say that 2, 3, 4 are the elements of A and 5, 7 are the elements of B

So, $A = \{2, 3, 4\}$ and $B = \{5, 7\}$

Now, $A \times B = \{(2, 5); (2, 7); (3, 5); (3, 7); (4, 5); (4, 7)\}$

Thus, $A \times B$ contain six ordered pairs.

2. Introduction:

Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.

Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

Ex. If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then $R = \{(1, b), (2, c), (1, a), (3, a)\}$ being a subset of $A \times B$, is a relation from A to B. Here $(1, b), (2, c), (1, a)$ and $(3, a) \in R$, so we write $1Rb, 2Rc, 1Ra$ and $3Ra$. But $(2, b) \notin R$, so we write $2 \not{R}\, b$.

(a) Total Number of Relations:

Let A and B be two finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}.

(b) Domain and Range of a relation:

Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Domain $(R) = \{a : (a, b) \in R\}$

and, Range $(R) = \{b : (a, b) \in R\}$

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

Ex. Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} be two sets and let R be a relation from A to B defined by the phrase "$(x, y) \in R \Leftrightarrow x > y$". Under this relation R, we have $3R2, 5R2, 5R4, 7R2, 7R4$ and $7R6$

i.e. $R = \{(3,2), (5,2), (5,4), (7,2), (7,4), (7,6)\}$

$\therefore$ Dom $(R) = \{3, 5, 7\}$ and Range $(R) = \{2, 4, 6\}$

(c) Inverse Relation:

Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$R^{-1} = \{(b, a) : (a, b) \in R\}$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Dom (R) = Range (R^{-1}) and Range (R) = Dom (R^{-1})

Illustration 3:

Let A be the set of first ten natural numbers and let R be a relation on A defined by $(x, y) \in R \Leftrightarrow x + 2y = 10$, i.e. $R = \{(x, y) : x \in A, y \in A$ and $x + 2y = 10\}$. Express R and R^{-1} as sets of ordered pairs. Determine also

(i) domain of R and R^{-1} (ii) range of R and R^{-1}

Solution:

We have $(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \dfrac{10 - x}{2}$, $x, y \in A$

where $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Now, $x = 1 \Rightarrow y = \dfrac{10-1}{2} = \dfrac{9}{2} \notin A$.

This shows that 1 is not related to any element in A. Similarly, we can observe. that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation

Further we find that:

For $x = 2, y = \dfrac{10-2}{2} = 4 \in A$ $\therefore$ $(2, 4) \in R$

For $x = 4, y = \dfrac{10-4}{2} = 3 \in A$ $\therefore$ $(4, 3) \in R$

For $x = 6,\ y = \dfrac{10-6}{2} = 2 \in A$ $\therefore$ $(6, 2) \in R$

For $x = 8,\ y = \dfrac{10-8}{2} = 1 \in A$ $\therefore$ $(8, 1) \in R$

Thus, $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

$\Rightarrow R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, Dom$(R) = \{2, 4, 6, 8\}$= Range(R^{-1})

and, Range (R) =$\{4, 3, 2, 1\}$= Dom(R^{-1})

3. Types of Relation:

In this section we intend to define various types of relations on a given set A.

(a) Void Relation:

Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

(b) Universal Relation:

Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

(c) Identity Relation:

Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A. In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

Ex. The relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on set $A = \{1, 2, 3\}$. But relations $R_1 = \{(1,1), (2,2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R_1$ and in R_2 element 1 is related to elements 1 and 3.

(d) Reflexive Relation:

A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus, R on a set A is not reflexive if there exists an element $A \in A$ such that $(a, a) \notin R$.

Ex. Let $A = \{1, 2, 3\}$ be a set. Then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A. But $R_1 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \notin R_1$.

Note : Every Identity relation is reflexive but every reflexive relation is not identity.

(e) Symmetric Relation:

A relation R on a set A is said to be a symmetric relation iff

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e. $a\ R\ b \Rightarrow b\ R\ a$ for all $a, b, \in A$.

Ex. Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R \Leftrightarrow x$ is perpendicular to y. Then R is a symmetric relation on L, because $L_1 \perp L_2 \Rightarrow L_2 \perp L_1$

i.e. $(L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$.

Ex. Let $A = \{1, 2, 3, 4\}$ and let R_1 and R_2 be relation on A given by $R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$. Clearly, R_1 is a symmetric relation on A. However, R_2 is not so, because $(1,3) \in R_2$ but $(3, 1) \notin R_2$

(f) Transitive Relation:

Let A be any set. A relation R on A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. $a\ R\ b$ and $b\ R\ c \Rightarrow a\ R\ c$ for all $a, b, c \in A$

Ex. On the set N of natural numbers, the relation R defined by $x\ R\ y \Rightarrow x$ is less than y is transitive, because for any $x, y, z \in N$

$x < y$ and $y < z \Rightarrow x < z \Rightarrow x\ R\ y$ and $y\ R\ z \Rightarrow x\ R\ z$

Ex. Let L be the set of all straight lines in a plane. Then the relation 'is parallel to' on L is a transitive relation, because from any $\ell_1, \ell_2, \ell_3 \in L$.

$\ell_1 || \ell_2$ and $\ell_2 || \ell_3 \Rightarrow \ell_1 || \ell_3$

(g) Asymmetric Relation Definition

Asymmetric relation is the opposite of symmetric relation but not considered as equivalent to antisymmetric relation.

In Set theory, A relation R on a set A is known as asymmetric relation if no (b,a) ∈ R when (a,b) ∈ R or we can even say that relation R on set A is asymmetric if only if (a,b) ∈ R⟹(b, a) ∉R.

For example: If R is a relation on set A= (18,9) then (9,18) ∈ R indicates 18>9 but (9,18) R, Since 9 is not greater than 18.

Note- Asymmetric relation is the opposite of symmetric relation but not considered as equivalent to antisymmetric relation.

(h) Antisymmetric Relation:

Let A be any set. A relation R on set A is said to be an antisymmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

Ex. Let R be a relation on the set N of natural numbers defined by $x\ R\ y \Leftrightarrow$ 'x divides y' for all $x, y \in N$

This relation is an antisymmetric relation on N. Since for any two numbers $a, b \in N$

$a|b$ and $b|a \Rightarrow a = b$ i. e. $a\ R\ b$ and $b\ R\ a \Rightarrow a = b$

(i) Equivalence Relation:

A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Ex. Let R be a relation on the set of all lines in a plane defined by $(\ell_1, \ell_2) \in R \Leftrightarrow$ line ℓ_1 is parallel to line ℓ_2.

R is an equivalence relation.

Note : It is not necessary that every relation which is symmetric and transitive is also reflexive.

Illustration 4:

Three relation R_1, R_2 and R_3 are defined on set $A = \{a, b, c\}$ as follows:

(i) $R_1 \{a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

(ii) $R_2 \{(a, b), (b, a), (a, c), (c, a)\}$

(iii) $R_3\{(a, b), (b, c), (c, a)\}$

Find whether each of R_1, R_2 and R_3 is reflexive, symmetric and transitive.

Solution:

(i) Reflexive: Clearly, $(a, a), (b, b), (c, c) \in R_1$. So, R_1 is reflexive on A.

Symmetric: We observe that $(a, b) \in R_1$ but $(b, a) \notin R_1$. So, R_1 is not symmetric on A.

Transitive: We find that $(b, c) \in R_1$ and $(c, a) \in R_1$ but $(b, a) \notin R_1$. So, R is not transitive on A.

(ii) Reflexive: Since $(a, a), (b, b)$ and (c, c) are not in R_2. So, it is not a reflexive relation on A.

Symmetric: We find that the ordered pairs obtained by interchanging the components of ordered pairs in R_2 are also in R_2. So, R_2 is a symmetric relation on A.

Transitive: Clearly $(c, a) \in R_2$ and $(a, b) \in R_2$ but $(c, b) \notin R_2$. So, it is not a transitive relation on R_2.

(iii) Reflexive: Since none of $(a, a), (b, b)$ and (c, c) is an element of R_3. So, R_3 is not reflexive on A.

Symmetric: Clearly, $(b, c) \in R_3$ but $(c, b) \notin R_3$. so, is not symmetric on A.

Transitive: Clearly, $(b, c) \in R_3$ and $(c, a) \in R_3$ but $(b, a) \notin R_3$. So, R_3 is not transitive on A.

Illustration 5:

Prove that the relation R on the set $\mathbb{Z}$ of all integers defined by $(x, y) \in R \Leftrightarrow x - y$ is divisible by n is an equivalence relation on $\mathbb{Z}$.

Solution:

We observe the following properties

Reflexivity: For any $a \in \mathbb{N}$, we have

$\Rightarrow$ $a - a = 0 \ = 0 \times n \Rightarrow a - a$ is divisible by $n \Rightarrow (a, a) \in R$

Thus, $(a, a) \in R$ for all $a \in \mathbb{Z}$

So, R is reflexive on $\mathbb{Z}$

symmetry: Let $(a, b) \in R$. Then,

$\Rightarrow$ $(a, b) \in R \Rightarrow (a - \ b)$ is divisible by n

$\Rightarrow$ $a - b = np$ for some $p \in \mathbb{Z}$

$\Rightarrow$ $b - a = n(-p)$

$\Rightarrow$ $b - \ a$ is divisible by n $\quad [\because\ p \in Z \Rightarrow -p \in Z]$

$\Rightarrow (b, a) \in R$

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b, \in \mathbb{Z}$

So, R is symmetric on $\mathbb{Z}$.

Transitivity: Let $a, b, c \in Z$ such that $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R \Rightarrow (a - b)$ is divisible by n

$\Rightarrow$ $a - \ b = np$ for some $p \in \mathbb{Z}$

$(b, c) \in R \Rightarrow (b - c)$ is divisible by n

$\Rightarrow b - c = nq$ for some $q \in \mathbb{Z}$

$\therefore$ $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow a - b = np$ and $b - c - nq$

$\Rightarrow (a - b) + (b - c) = np + nq$

$\Rightarrow$ $a - c = n(p + q)$

$\Rightarrow$ $a - c$ is divisible by n $\qquad [\because\ p, q \in \mathbb{Z} \Rightarrow p + q = \mathbb{Z}]$

$\Rightarrow (a, c) \in R$

thus, $(a, b) \in R$ and $(b, c) \in\ R \Rightarrow (a, c) \in R$ for all $a, b, c \in \mathbb{Z}$. so, R is transitive relation in $\mathbb{Z}$.

Illustration 6:

Show that the relation is congruent to' on the set of all triangles in a plane is an equivalence relation.

Solution:

Let S be the set of all triangles in a plane and let R be the relation on S defined by $(\Delta_1, \Delta_2) \in R \Leftrightarrow$ triangle Δ_1 is congruent to triangle Δ_2. We observe the following properties.

(j) Reflexivity:

For each triangle $\Delta \in S$, we have

$\Delta \cong \Delta \Rightarrow (\Delta, \Delta) \in R$ for all $\Delta \in S \Rightarrow R$ is reflexive on S

(k) Symmetry:

Let $\Delta_1, \Delta_2 \in S$ such that $(\Delta_1, \Delta_2) \in R$. Then, $(\Delta_1, \Delta_2) \in R \Rightarrow \Delta_1 \cong \Delta_2 \Rightarrow \Delta_2 \cong \Delta_1 \Rightarrow (\Delta_2, \Delta_1) \in R$

So, R is symmetric on S

(I) Transitivity:

Let $\Delta_1, \Delta_2, \Delta_3 \in S$ such that $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R$. Then,

$(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R \Rightarrow \Delta_1 \cong \Delta_2$ and $\Delta_2 \cong \Delta_3 \Rightarrow \Delta_1 \cong \Delta_3 \Rightarrow (\Delta_1, \Delta_3) \in R$

So, R is transitive on S.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on S.

Illustration 7:

Three relation R_1, R_2 and R_3 are defined on set $A = \{a, b, c\}$ as follows :

(i) $R_1 = \{a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

(ii) $R_2 = \{(a, b), (b, a), (a, c), (c, a)\}$

(iii)$R_3 = \{(a, b), (b, c), (c, a)\}$

Find whether each of R_1, R_2 and R_3 is reflexive, symmetric and transitive.

Solution:

(i) Reflexive: Clearly, $(a, a), (b, b), (c, c) \in R_1$. So, R_1 is reflexive on A.

Symmetric: We observe that $(a, b) \in R_1$ but $(b, a) \notin R_1$. So, R_1 is not symmetric on A.

Transitive: We find that $(b, c) \in R_1$ and $(c, a) \in R_1$ but $(b, a) \notin R_1$. So, R is not transitive on A.

(ii) Reflexive: Since (a, a), (b, b) and (c, c) are not in R_2. So, it is not a reflexive relation on A.

Symmetric: We find that the ordered pairs obtained by interchanging the components of ordered pairs in R_2 are also in R_2. So, R_2 is a symmetric relation on A.

Transitive: Clearly $(c, a) \in R_2$ and $(a, b) \in R_2$ but $(c, b) \notin R_2$. So, it is not a transitive relation on R_2.

(iii) Reflexive: Since non of (a, a), (b, b) and (c, c) is an element of R_3. So, R_3 is not reflexive on A.

Symmetric: Clearly, $(b, c) \in R_3$ but $(c, b) \notin R_3$. so, is not symmetric on A.

Transitive: Clearly, $(b, c) \in R_3$ and $(c, a) \in R_3$ but $(b, a) \notin R_3$. So, R_3 is not transitive on A.

Illustration 8:

Prove that the relation R in the set $\mathbb{N}$ of natural numbers defined by $xRy \Leftrightarrow '(x - y)$ is divisible by 5 is an equivalence relation.

Solution:

For proving the relation R an equivalence relation we shall prove that the relation R is reflexive, symmetric and transitive in the set $\mathbb{N}$.

(i) Reflexive :

Let $x \in \mathbb{N}$, then

$\Rightarrow x \in \mathbb{N} \Rightarrow x - x = 0 \Rightarrow x - x = 0 \times 5$

$\Rightarrow x - x$ is divisible by 5 $\Rightarrow x \, R \, x$

$\therefore R$ is reflexive.

(ii) Symmetric :

Let xRy & $x, y \in \mathbb{N}$, then

$xRy \Rightarrow x - y$ is divisible by 5

$\Rightarrow x - y = 5 \times p$ where $p \in I$

$\Rightarrow y - x = 5 \times (-p) \quad \Rightarrow y - x$ is divisible by 5 $\quad [\because p \in I \Rightarrow -p \in I]$

$\Rightarrow y\ R\ x$

$\therefore R$ is symmetric.

(iii) Transitive :

Let xRy and yRz then

$xRy \Rightarrow x - y$ is divisible by $5 \Rightarrow x - y = 5 \times p, p \in I$

And $\quad yRz \Rightarrow y - z = 5q$; $(\because q \in I)$

$\Rightarrow (x - y) + (y - z) = 5p + 5q \Rightarrow x - z = 5(p + q)$

$\Rightarrow x - z$ is divisible by 5 $\quad [\because p \in I, q \in I \Rightarrow p + q \in I]$

$\Rightarrow xRz$

$\therefore R$ is transitive.

Thus R is reflexive, symmetric and transitive therefore R is an equivalence relation on the set N.

Illustration 9:

In the set $\mathbb{N} \times \mathbb{N}$ consider the relation R defined as $(a, b)R(c, d) \Leftrightarrow ad = bc; a, b, c, d \in \mathbb{N}$

Show that this is an equivalence relation.

Solution:

For proving the relation R on $N \times N$ to be an equivalence relation, we will prove that the relation R is reflexive, symmetric and transitive.

Reflexive : Let $x, y \in N$, then

$\Rightarrow (x, y) \in N \times N \Rightarrow xy = yx$, by commutativity in N

$\Rightarrow (x, y)R(x, y) \forall (x, y) \in N \times N$

$\therefore R$ is reflexive

Symmetric : Let $(x, y)R(p, q)$ where $x, y, p, q \in N$, then

$(x, y)R(p, q) \Rightarrow xq = yp \Rightarrow yp = xq$

$\Rightarrow py = qx \qquad$ [by commutativity in N]

$\Rightarrow (p, q)R(x, y)$

$\therefore R$ is symmetric.

Transitive :

Let $(x_1, y_1)R(x_2, y_2)(x_2, y_2)R(x_3, y_3)$ where

$\Rightarrow (x_1, y_1)\ (x_2, y_2), (x_3, y_3)\ N \times N$, then

$\Rightarrow (x_1, y_1)R(x_2, y_2) \Rightarrow x_1y_2 = y_1x_2 \qquad$ ...(1)

$\therefore (x_2, y_2)R(x_3, y_3) \Rightarrow x_2y_3 = y_2x_3 \qquad$ ...(2)

Now $\ x_1y_2 = y_1x_2$

$y_1 \dfrac{y_2x_3}{y_3}$ = putting the value of x_2 from (2).

$\Rightarrow x_1y_3 = y_1x_3 \qquad$ ...(3)

$\Rightarrow (x_1, y_1)R(x_3, y_3)$

Therefore, R is transitive.

Hence the relation defined on $N \times N$ is an equivalence relation.

Illustration 10:

Let the relation R be defined in the set of real number as follows :

(i) $aRb \Leftrightarrow a \geq b$ (ii) $aRb \Leftrightarrow |a| = |b|$

Which of the above defined relation are equivalence relation.

Solution:

Let A be the set of real numbers

Reflexive : If $a \in A$, then

$\Rightarrow a \in A \Rightarrow a \geq a \Rightarrow aRa$

$\therefore R$ is reflexive.

Symmetric : If aRb, then

$\Rightarrow aRb \Rightarrow a \geq b \Rightarrow b = a$ but $b \ngeq a$

does not imply $b \geq a$ does not imply bRa

$\therefore$ R is not symmetric.

Transitive : If aRb & bRc, then

aRb and $bRc \Rightarrow a \geq b$ and $b \geq c$

$\Rightarrow a \geq c \Rightarrow aRc$

$\therefore$ R is transitive.

Hence R is reflexive and transitive but not symmetric.

Thus, R is not an equivalence relation.

(ii) Reflexive: If $a \in A$, then

$\Rightarrow a = a \Rightarrow |a| = |a| \Rightarrow aRa \forall a \in A$

R is symmetric.

Symmetric : If aRb, then

$aRb \Rightarrow |a| = |b| \Rightarrow |b| = |a| \Rightarrow bRa$

$\therefore$ R is symmetric.

Transitive : If aRb and bRc, then

aRb and $bRc \Rightarrow |a| = |b|$ and $|b| = |c|$

$\Rightarrow |a| = |c| \Rightarrow aRc$

$\therefore$ R is transitive.

$\therefore$ R is reflexive, symmetric and transitive.

Hence R is an equivalence relation.