

01

Methods of Differentiation

Meaning of Derivative or $\frac{dy}{dx}$

Let $y = f(x)$; $y + \Delta y = f(x + \Delta x)$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ (average rate of change of function)}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \dots(i)$$

Physical Meaning:

It denotes the instantaneous rate of change of function

Geometrical Meaning:

It denotes the slope of tangent at any point on curve $y = f(x)$.

Note:

(i) If $y = f(x)$ then the symbols $\frac{dy}{dx} = Dy = f'(x) = y_1$ or y' have the same meaning.

(ii) $f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y_2 = D^2y$

(iii) If $y = f(x)$ then $f'(a)$ means value of $f'(x)$ at $x = a$

(iv) However, a dot, the symbol denotes the time derivative, e.g. $\dot{S} = \frac{dS}{dt}$; $\dot{\theta} = \frac{d\theta}{dt}$ etc.

Derivative by First Principle:

Finding the value of the limit given by (i) in respect of **variety** of functions is called finding the derivative by first principle/by delta method/by ab-initio method/by fundamental definition of calculus.

Illustration 1:

Find derivative by using 1st principle of $f(x) = x^n$; $n \neq 1$

Solution:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^n - (x)^n}{h}$$

$$\lim_{h \rightarrow 0} \frac{x^n}{h} \left[\left(1 + \frac{h}{x}\right)^n - 1 \right] = \lim_{h \rightarrow 0} \frac{x^n}{h} \left[1 + \frac{nh}{x} + \frac{n(n-1)h^2}{x^2 2!} + \dots - 1 \right]$$

$$\lim_{h \rightarrow 0} x^n \left[\frac{n}{x} + h(\dots \text{terms}) \right]$$

$$= n x^{n-1}$$

Illustration 2:

Find derivative by using 1st principle of $f(x) = a^x, a > 0, a \neq 1$

Solution:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} a^x \left(\frac{a^h - 1}{h} \right) = a^x \ell na$$

Illustration 3:

Find derivative by using 1st principle of $f(x) = \sin x$.

Solution:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{2 \cos\left(x + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos\left(x + \frac{h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{h/2}$$

$$= \cos x \cdot 1 = \cos x$$

Derivative of Standard Functions:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$\sin x$	$\cos x$
e^x	e^x	$\cos x$	$-\sin x$
a^x	$a^x \ell na, a > 0$	$\tan x$	$\sec^2 x$
$\ell nx / \log_e x$	$\frac{1}{x}$	$\sec x$	$\sec x \tan x$
$\log_a x$	$\left(\frac{1}{x}\right) \log_a e, a > 0, a \neq 1$	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cdot \cot x$
Constant	0	$\cot x$	$-\operatorname{cosec}^2 x$

Note:

The derivative values of some all co-trigonometric functions begin with a -ve sign.

Derivative of Inverse Trigonometric Function:

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}, x \in R$
$\cot^{-1} x$	$-\frac{1}{1+x^2}, x \in R$
$\sec^{-1} x$	$\frac{1}{ x \sqrt{x^2-1}}, x > 1$
$\operatorname{cosec}^{-1} x$	$-\frac{1}{ x \sqrt{x^2-1}}, x > 1$

Illustration 4:

If $D(y)$ denotes, derivative of y w. r. t. x then find:

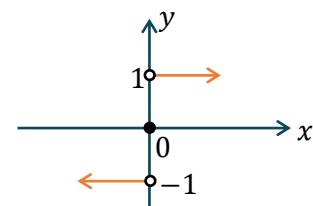
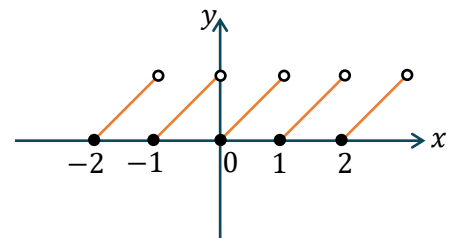
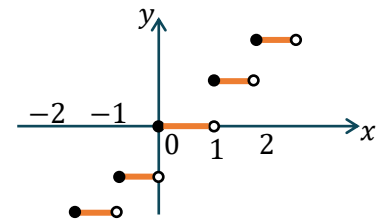
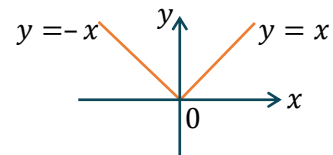
- (i) $D(e^\pi), D(x^e)$
- (ii) $D\left(\frac{1}{\sin x}\right)$
- (iii) $D\left(\frac{\sin 2x}{1+\cos 2x}\right)$
- (iv) $D(\tan(\tan^{-1} x))$
- (v) $y = \left(\frac{x^a}{x^b}\right)^{a+b} \left(\frac{x^b}{x^c}\right)^{b+c} \left(\frac{x^c}{x^a}\right)^{c+a}$, find $\frac{dy}{dx}$

Solution:

- (i) $D(e^\pi) = 0, D(x^e) = e \cdot x^{e-1}$
- (ii) $D\left(\frac{1}{\sin x}\right) = D(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
- (iii) $D\left(\frac{\sin 2x}{1+\cos 2x}\right) = D\left(\frac{2 \sin x \cos x}{1+2 \cos^2 x-1}\right) = D(\tan x) = \sec^2 x$
- (iv) $D(\tan(\tan^{-1} x)) = D(x) = 1$
- (v) $y = \left(\frac{x^a}{x^b}\right)^{a+b} \left(\frac{x^b}{x^c}\right)^{b+c} \left(\frac{x^c}{x^a}\right)^{c+a}$
 $\Rightarrow y = (x^{a-b})^{a+b} (x^{b-c})^{b+c} (x^{c-a})^{c+a}$
 $\Rightarrow y = x^{a^2-b^2+b^2-c^2+c^2-a^2}$
 $\Rightarrow y = x^0 = 1 \quad \therefore \frac{dy}{dx} = 0$

Note:

- (i) $\frac{d}{dx}(|x|) = \begin{cases} 1, x > 0 \\ -1, x < 0 \\ \text{not differentiable}, x = 0 \end{cases}$
- (ii) $\frac{d}{dx}([x]) = 0, x \notin I$ and non-differentiable at $x \in I$
- (iii) $\frac{d}{dx}(\{x\}) = 1, x \notin I$ and non-differentiable at $x \in I$
- (iv) $\frac{d}{dx}(\operatorname{sgn} x) = 0, x \neq 0$ and non-differentiable at $x = 0$



Supplementary Theorems/Results:

(i) Sum/difference rule:

$$D(f(x) \pm g(x)) = f'(x) \pm g'(x)$$

Illustration 5:

If $y = e^x + 3 \ln x - 4 \tan^{-1} x + \sin 3x + 4 \sin^3 x$, then $\frac{dy}{dx}$ is

Solution:

$$\begin{aligned} y &= e^x + 3 \ln x - 4 \tan^{-1} x + \sin 3x + 4 \sin^3 x \\ \Rightarrow y &= e^x + 3 \ln x - 4 \tan^{-1} x + 3 \sin x - 4 \sin^3 x + 4 \sin^3 x \\ \Rightarrow y &= e^x + 3 \ln x - 4 \tan^{-1} x + 3 \sin x \\ \Rightarrow \frac{dy}{dx} &= e^x + \frac{3}{x} - \frac{4}{(1+x^2)} + 3 \cos x \end{aligned}$$

(b) Product Rule:

$$D(f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Note:

If 3 functions are involved then remember

$$D(f(x) \cdot g(x) \cdot h(x)) = f(x) \cdot g(x) \cdot h'(x) + g(x) \cdot h(x) \cdot f'(x) + h(x) \cdot f(x) \cdot g'(x)$$

This result can be generalised to the product of n functions.

Note:

Differentiate one function at a time & keep remaining as it is and complete the process for all functions one by one.

$$\frac{d}{dx}(kf(x)) = k \cdot \frac{d}{dx} f(x), \text{ (where } k \text{ is any constant)} = kf'(x)$$

Illustration 6:

Find derivative w.r.t. x :

(i) $y = e^x + 3 \ln x - 4 \sin x$

(ii) $y = x \sin^{-1} x$

(iii) $y = (x^2 e^x \ln x)$

Solution:

(i) $y = e^x + 3 \ln x - 4 \sin x \Rightarrow \frac{dy}{dx} = e^x + \frac{3}{x} - 4 \cos x$

(ii) $y = x \sin^{-1} x \Rightarrow \frac{dy}{dx} = 1 \cdot \sin^{-1} x + x \cdot \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$

(iii) $y = (x^2 e^x \ln x) \Rightarrow \frac{dy}{dx} = 2x \cdot (e^x \ln x) + x^2 (e^x) \ln x + x^2 e^x \cdot \frac{1}{x}$
 $\Rightarrow \frac{dy}{dx} = 2xe^x \ln x + x^2 e^x \ln x + xe^x$

Illustration 7:

If $f(x) = (1+x)(3+x^2)^{1/2}(9+x^3)^{1/3}$, then $f'(-1)$ is equal to -

- (A) 0 (B) $2\sqrt{2}$ (C) 4 (D) 6

Ans. C

Solution:

$f'(x)|_{x=-1} = (x^2 + 3)^{1/2}(9 + x^3)^{1/3} = 2.2 = 4$
 (\because value of the other two expression in $f'(x)$ vanishes)

(d) Quotient rule : $y = \frac{f(x)}{g(x)}$ (prove by first principle)

$$D\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)} \quad \boxed{\text{to be remembered as}} \quad D\left(\frac{N^r}{D^r}\right) = \frac{D^r \frac{d}{dx}(N^r) - N^r \frac{d}{dx}(D^r)}{(D^r)^2}$$

Proof:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\text{L.H.S.} = d\left(f(x) \cdot \frac{1}{g(x)}\right)$$

$$\Rightarrow \frac{d(f(x))}{dx} \cdot \frac{1}{g(x)} + f(x) \cdot \frac{d}{dx}\left(\frac{1}{g(x)}\right)$$

$$\Rightarrow f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{(-1)}{(g(x))^2} \cdot g'(x) = \text{R.H.S.}$$

$$\text{Similarly, } \frac{d\left(\frac{f(x)h(x)}{g(x)}\right)}{dx} = \frac{[f(x)h'(x) + f'(x)h(x)]g(x) - f(x)h(x) \cdot g'(x)}{(g(x))^2}$$

Note:

While applying quotient rule, think twice if your function could be simplified prior to differentiation.

Consider the example

Illustration 8:

$$y = \frac{x^3 + x^2 + 1}{x^2 + 1}, \text{ then find } \frac{dy}{dx}$$

Solution:

$$\text{M-1} = \frac{(3x^2 + 2x + 0)(x^2 + 1) - (x^3 + x^2 + 1)(2x + 0)}{(x^2 + 1)^2}$$

$$\text{M-2 } f(x) = \frac{x^3}{x^2 + 1} + \frac{x^2 + 1}{x^2 + 1} \Rightarrow \frac{x^3}{x^2 + 1} + 1 \text{ (Now apply quotient rule)}$$

Illustration 9:

$$y = \frac{\sin^{-1}x - \cos^{-1}x}{\tan^{-1}x + \cot^{-1}x}, \text{ then find } \frac{dy}{dx}$$

Solution:

$$y = \frac{(\sin^{-1}x - \cos^{-1}x)}{\frac{\pi}{2}} = \frac{2}{\pi}(\sin^{-1}x - \cos^{-1}x)$$

$$\frac{dy}{dx} = \frac{2}{\pi} \left[\frac{1}{\sqrt{1-x^2}} - \left(\frac{-1}{\sqrt{1-x^2}} \right) \right] = \frac{4}{\pi\sqrt{1-x^2}}$$

General Note:

If $f'(x)$ is not defined on $x = c$ then it is wrong to conclude that $f(x)$ is not derivable at $x = c$. In such cases we find, LHD at $x = c$ and RHD at $x = c$

Illustration 10:

$$f(x) = x^{\frac{1}{3}} \sin x \text{ at } x = 0$$

Solution:

$$f'(x) = x^{1/3} \cos x + \frac{1}{3} x^{-2/3} \cdot \sin x = x^{1/3} \cos x + \frac{\sin x}{3x^{2/3}}$$

$f'(0)$ seems to be non-defined but $f'(0^+)$ and $f'(0^-)$ is 0.

$$\begin{aligned} f'(0^+) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^{\frac{1}{3}} \sin h}{h} \quad (\because f(0) = 0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f'(0^-) &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(-h)^{\frac{1}{3}} \sin(-h)}{-h} = 0 \end{aligned}$$

Chain Rule:

Proof:

If $y = f(u)$ and $u = g(x)$

$$\text{then } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x).$$

It can be extended to any number of chains.

Illustration 11:

$$\frac{d}{dx} \tan(\log x)$$

Solution:

$$\sec^2(\log x) \cdot \frac{1}{x}$$

Illustration 12:

$$\frac{d}{dx} (\sin x^2)$$

Solution:

$$\cos x^2 \cdot 2x$$

Illustration 13:

$$\frac{d}{dx} \log \sin x^2$$

Solution:

$$= \frac{1}{\sin x^2} \cdot \cos x^2 \cdot 2x$$

Illustration 14:

$$\frac{d}{dx} [\tan^{-1}(\log \sin x^2)]$$

Solution:

$$\frac{1}{1+(\log \sin x^2)^2} \cdot \frac{1}{\sin x^2} \cdot \cos x^2 \cdot 2x$$

Logarithmic Differentiation:

To find the derivative of:

- (i) a function which is the product or quotient of a number of functions
- (ii) a function of the form $[f(x)]^{g(x)}$ where f & g are both derivable functions, it will be found convenient to take the logarithm of the function first & then differentiate

OR

express $y = (f(x))^{g(x)} = e^{g(x)\ln(f(x))}$ and then differentiate.

Illustration 15:

If $y = x^x$, then find $\frac{dy}{dx}$

Solution:

M-1 $\log y = \log x^x$

$$\Rightarrow \log y = x \log x$$

$$\Rightarrow \left[\frac{1}{y} \cdot \frac{dy}{dx} \right] = \left[1 \cdot \log x + x \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{dy}{dx} = y [\log x + 1]$$

$$\Rightarrow \frac{dy}{dx} = x^x [\log x + 1]$$

M-2 $y = e^{x \ln x}$

$$\Rightarrow \frac{dy}{dx} = e^{x \ln x} \left(\frac{d}{dx} x \ln x \right)$$

$$\Rightarrow x^x \left[x \cdot \frac{1}{x} + \ln x \right] = x^x [1 + \ln x]$$

Illustration 16:

If $y = (\sin x)^{\cos x}$, then find $\frac{dy}{dx}$

Solution:

M-1 $\log y = \log(\sin x)^{\cos x}$

$$\Rightarrow \log y = \cos x \cdot \log(\sin x)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \left[-\sin x \cdot \log(\sin x) + \frac{\cos x \cdot 1 \cdot \cos x}{\sin x} \right]$$

$$\Rightarrow \frac{dy}{dx} = y \left[-\sin x \cdot \log(\sin x) + \frac{\cos^2 x}{\sin x} \right]$$

M-2 $y = e^{\cos x \ln \sin x}$

$$\Rightarrow \frac{dy}{dx} = e^{\cos x \ln \sin x} \left(\frac{\cos x \cdot \cos x}{\sin x} + (-\sin x) \ln \sin x \right)$$

$$\Rightarrow y \left(\frac{\cos^2 x}{\sin x} - \sin x \cdot \ln \sin x \right)$$

Illustration 17:

Find equation of tangent at $x = 1$ of curve $y = x^x + x^e$

Solution:

$$\frac{dy}{dx} = x^x[\ln x + 1] + e \cdot x^{e-1}$$

$$\left. \frac{dy}{dx} \right|_{x=1} = \text{Slope at } x = 1$$

$$= (1)^1 [\ln 1 + 1] + e \cdot 1^{e-1} = 1(0 + 1) + e \cdot 1 = 1 + e$$

Point:

$$x = 1$$

$$\Rightarrow y = 1 + 1 = 2$$

Equation of tangent:

$$y - 2 = (1 + e)(x - 1)$$

$$\Rightarrow y - 2 = x - 1 + ex - e$$

$$\Rightarrow y = x - 1 + 2 + ex - e$$

$$\Rightarrow y = x(1 + e) + (1 - e)$$

Illustration 18:

If $f(x) = (x + 1)(x + 2)(x + 3) \dots (x + n)$ then $f'(0)$ is -

- (A) $n!$ (B) $\frac{n(n+1)}{2}$ (C) $(n!)(\ln n!)$ (D) $n! \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}\right)$

Ans. (D)

Solution:

$$f(x) = (x + 1)(x + 2)(x + 3) \dots (x + n)$$

$$\ln(f(x)) = \ln(x + 1) + \ln(x + 2) + \dots + \ln(x + n)$$

$$\frac{f'(x)}{f(x)} = \frac{1}{x + 1} + \frac{1}{x + 2} + \dots + \frac{1}{x + n}$$

$$\therefore f'(0) = (n!) \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$$

Derivative of $f(x)$ w. r. t. $g(x)$:

If $y = f(x)$ and $z = g(x)$, then derivative of $f(x)$ w. r. t. $g(x)$ is given by

$$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{f'(x)}{g'(x)}$$

$$\therefore \text{Differential coefficient of } f(x) \text{ w.r.t. } g(x) = \frac{\text{derivative of } f(x) \text{ w.r.t. } x}{\text{derivative of } g(x) \text{ w.r.t. } x} = \frac{f'(x)}{g'(x)}$$

Illustration 19:

Differentiate $\sin x$ w. r. t. x^3

Solution:

$$\frac{d(\sin x)}{d(x^3)} = \frac{\cos x}{3x^2}$$

Methods of Differentiation

Illustration 20:

Find differentiation of x^x w. r. t. $\cos x$

Solution:

$$\Rightarrow \frac{d(x^x)}{d(\cos x)} = \frac{x^x(\ln x + 1)}{-\sin x}$$

Illustration 21:

Differentiate $\sin^{-1} x$ w. r. t. e^x

Solution:

$$\frac{d(\sin^{-1} x)}{d(e^x)} = \left(\frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{e^x} \right)$$

Illustration 22:

Differentiation of $e^{\sin^{-1}(x)}$ w.r.t $e^{-\cos^{-1}(x)}$.

Solution:

$$\frac{d(e^{\sin^{-1} x})}{d(e^{-\cos^{-1} x})} \Rightarrow \frac{e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}}{e^{-\cos^{-1} x} \cdot \left(\frac{-1(-1)}{\sqrt{1-x^2}} \right)} \Rightarrow e^{\frac{\pi}{2}} \quad \left\{ \because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right\}$$

Alternate:

$$\frac{d(e^{\sin^{-1} x})}{d(e^{-\cos^{-1} x})} = \frac{d\left(e^{\frac{\pi}{2} - \cos^{-1} x}\right)}{d\left(e^{-\cos^{-1} x}\right)} \Rightarrow \frac{d\left(e^{\frac{\pi}{2}} \cdot e^{-\cos^{-1} x}\right)}{d\left(e^{-\cos^{-1} x}\right)} = e^{\frac{\pi}{2}} \cdot 1 = e^{\pi/2}$$

Illustration 23:

Differentiate $f(\tan x)$ w.r.t. to $g(\sec x)$ at $x = \frac{\pi}{4}$. Where $f'(1) = 3$ and $g'(\sqrt{2}) = 4$

Solution:

$$\frac{d(f(\tan x))}{d(g(\sec x))} = \frac{(f'(\tan x)) \cdot \sec^2 x}{(g'(\sec x)) \cdot \sec x \tan x} = \frac{f'\left(\tan \frac{\pi}{4}\right) \cdot \sec^2 \frac{\pi}{4}}{g'\left(\sec \frac{\pi}{4}\right) \cdot \tan \frac{\pi}{4} \cdot \sec \frac{\pi}{4}} = \frac{3 \cdot 2}{4 \times \sqrt{2}} = \frac{3}{2\sqrt{2}}$$

Parametric Differentiation:

If $x = f(t)$ & $y = g(t)$, then $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{dx/dt} = \frac{g'(t)}{f'(t)}$

Where $\frac{dx}{dt} = f'(t)$ and $\frac{dy}{dt} = g'(t)$

Illustration 24:

If $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2at}{1+t^2}$, then $\frac{dy}{dx} =$

Solution:

$$x = \frac{1-t^2}{1+t^2} \text{ and } y = \frac{2at}{1+t^2}$$

Differentiating with respect to t , we get

$$\frac{dx}{dt} = \frac{(1+t^2)(0-2t) - (1-t^2)(0+2t)}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$$

$$\text{and } \frac{dy}{dt} = \frac{(1+t^2)2a - 2at(2t)}{(1+t^2)^2} = \frac{2a(1-t^2)}{(1+t^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a(1-t^2)}{-2t}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a(t^2-1)}{2t}$$

Illustration 25:

If $\tan y = \frac{2t}{1-t^2}$ and $\sin x = \frac{2t}{1+t^2}$, then find

Solution:

$$\tan y = \frac{2t}{1-t^2} \quad \dots(i)$$

$$\text{and } \sin x = \frac{2t}{1+t^2} \quad \dots(ii)$$

From (i), differentiating w.r.t. t of y , we get, $\sec^2 y \frac{dy}{dx} = \frac{2(1+t^2)}{(1-t^2)^2}$ and $\frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{1}{(1+\tan^2 y)}$

$$\text{or } \frac{dy}{dt} = \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{1}{\left[1+\left(\frac{2t}{1-t^2}\right)^2\right]} = \frac{2}{1+t^2} \quad \dots(iii)$$

and from (ii), differentiating w.r.t. t of x , we get

$$\cos x \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\text{or } \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{1}{\sqrt{1-\frac{(2t)^2}{(1+t^2)^2}}} = \frac{2}{1+t^2} \quad \dots(iv)$$

Hence $\frac{dy}{dx} = 1$.

Illustration 26:

If $x = at^2, y = 2at$, then find $\frac{d^2y}{dx^2}$

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} \Rightarrow \frac{dy}{dx} = \frac{1}{t} = \frac{2a}{y}$$

$$\Rightarrow y \frac{dy}{dx} = 2a \Rightarrow y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-(dy/dx)^2}{y} = -\frac{1}{2at^3}$$

Illustration 27:

If $x = 2 \cos t - \cos 2t, y = 2 \sin t - \sin 2t$, then at $t = \frac{\pi}{4}, \frac{dy}{dx} = ?$

Solution:

$$\frac{dx}{dt} = -2 \sin t + 2 \sin 2t \text{ and } \frac{dy}{dt} = 2 \cos t - 2 \cos 2t$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos t - \cos 2t}{\sin 2t - \sin t}$$

Put $t = \frac{\pi}{4}$, we have $\left[\frac{dy}{dx}\right]_{t=\pi/4}$

$$= \frac{\cos \pi/4 - \cos \pi/2}{\sin \pi/2 - \sin \pi/4} = \sqrt{2} + 1.$$

Derivative of IMPLICIT Functions:

$$\phi(x, y) = 0$$

(a) To find $\frac{dy}{dx}$ of implicit functions, we differentiate each term w.r.t. x regarding y as a function

of x then collect terms with $\frac{dy}{dx}$ together on one side.

(b) In the case of implicit functions, generally, both x & y are present in answers of $\frac{dy}{dx}$. Corresponding to every curve represented by an implicit equation, there exist one or more explicit functions representing that equation. It can be shown that $\frac{dy}{dx}$ at any point on the curve remains the same whether the process of differentiation is done explicitly or implicitly.

Illustration 28:

If $\sin y = x \sin(a + y)$ then prove that $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$. Also find $\frac{dy}{dx}$ explicitly.

Solution:

$$x = \frac{\sin y}{\sin(a + y)} \Rightarrow \frac{dx}{dy} = \frac{\sin(a + y) \cos y - \sin y \cos(a + y)}{\sin^2(a + y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin a}{\sin^2(a + y)}$$

$$\text{Explicitly } \frac{1}{x} = \frac{\sin(a + y)}{\sin y} = \sin a \cdot \cot y + \cos a \Rightarrow \sin a \cdot \cot y = \frac{1}{x} - \cos a$$

$$\Rightarrow \cot y = \frac{1 - x \cos a}{x \sin a}$$

$$\Rightarrow -\operatorname{cosec}^2 y \frac{dy}{dx} = -\frac{1}{x^2 \sin a}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x^2 \sin a (1 + \cot^2 y)}$$

(substitute the value of $\cot y$ to get $\frac{dy}{dx}$ explicitly)]

Illustration 29:

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$ (explicit form)

Solution:

$$(x\sqrt{1+y})^2 = (y\sqrt{1+x})^2 \Rightarrow xy(x-y) + (x^2 - y^2) = 0 \text{ or } (x-y)(xy+x+y) = 0$$

$$\therefore y = -\frac{x}{1+x} = \frac{1}{1+x} - 1 \text{ (as } y = x \text{ does not satisfy)}$$

$$\text{hence } y' = -\frac{1}{(1+x)^2}$$

Illustration 30:

If $\cos(x+y) = y \sin x$, then find $\frac{dy}{dx}$

Solution:

$$\cos(x+y) = (y \sin x)$$

$$\Rightarrow -\sin(x+y) \left(1 + \frac{dy}{dx}\right) = y \cos x + \sin x \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y \cos x + \sin(x+y)}{\sin(x+y) + \sin x}$$

Illustration 31:

If $\sin^2 x + 2 \cos y + xy = 0$, then find $\frac{dy}{dx}$

Solution:

$$\sin^2 x + 2 \cos y + xy = 0$$

$$\Rightarrow 2 \sin x \cos x - 2 \sin y \frac{dy}{dx} + y + x \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{y + \sin 2x}{2 \sin y - x}$$

Illustration 32:

If $x^3 + 8xy + y^3 = 64$, then find $\frac{dy}{dx}$

Solution:

$$x^3 + 8xy + y^3 = 64 \Rightarrow 3x^2 + 8 \left(y + x \frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{3x^2 + 8y}{8x + 3y^2}$$

Derivative of Infinite Series:

Illustration 33:

If $y = e^{x+e^{x+e^{x+\dots\infty}}}$, then $\frac{dy}{dx} = ?$

Solution:

$$y = e^{x+y}$$

$$\Rightarrow \log y = (x+y) \log e \Rightarrow \frac{1}{y} \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{1-y}$$

Illustration 34:

$$\text{If } y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots \infty}}}, \text{ then } \frac{dy}{dx} = ?$$

Solution:

$$\begin{aligned} y &= \sqrt{\log x + y} \\ \Rightarrow y^2 &= \log x + y \\ \Rightarrow 2y \frac{dy}{dx} &= \frac{1}{x} + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{x(2y - 1)} \end{aligned}$$

Illustration 35:

$$\text{If } y = (\sin x)^{(\sin x)^{(\sin x)^{\dots \infty}}}, \text{ then } \frac{dy}{dx} = ?$$

Solution:

$$\begin{aligned} y &= (\sin x)^{(\sin x)^{(\sin x)^{\dots \infty}}} \\ \Rightarrow y &= (\sin x)^y \Rightarrow \log_e y = y \log \sin x \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{dy}{dx} [\log \sin x + y \cot x] \\ \therefore \frac{dy}{dx} &= \frac{y^2 \cot x}{1 - y \log \sin x} \end{aligned}$$

Illustration 36:

The differential equation satisfied by the function $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots \infty}}}$, is

Solution:

$$\begin{aligned} y &= \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots \infty}}} \\ \Rightarrow y &= \sqrt{\sin x + y} \\ \Rightarrow y^2 &= \sin x + y \\ \text{On differentiating both sides, we get} \\ 2y \frac{dy}{dx} &= \cos x + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} (2y - 1) = \cos x \end{aligned}$$

Derivative of Homogeneous Equation:

Illustration 37:

$$\text{If } ax^2 + 2hxy + by^2 = 0, \text{ then prove that } \frac{dy}{dx} = -\frac{ax + hy}{hx + by} = \frac{y}{x}$$

Solution:

$$\begin{aligned} 2ax + 2by y' + 2h(xy' + y) &= 0 \Rightarrow (ax + hy) + (by + hx)y' = 0 \\ \Rightarrow y' &= -\frac{ax + hy}{by + hx} \\ \therefore ax^2 + hxy + hxy + by^2 &= 0 \Rightarrow x(ax + hy) + y(by + hx) = 0 \\ \Rightarrow -\frac{ax + hy}{by + hx} &= \frac{y}{x} \end{aligned}$$

Illustration 38:

If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ then prove that

(i) $\frac{dy}{dx}$ is independent of a

(ii) $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

Solution:

Put $x = \sin \theta, y \in [-1, 1]$

And $y = \sin \phi, \theta \ \& \ \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos \theta + \cos \phi = a(\sin \theta - \sin \phi)$

$\therefore \tan\left(\frac{\theta-\phi}{2}\right) = \frac{1}{a} \Rightarrow \frac{\theta-\phi}{2} = \tan^{-1}\left(\frac{1}{a}\right)$

$\sin^{-1} x - \sin^{-1} y = 2 \tan^{-1}\left(\frac{1}{a}\right); \therefore \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Illustration 39:

If $x^m \cdot y^n = (x+y)^{m+n}$ then prove that $\frac{dy}{dx}$ is independent of m and n .

Solution:

$m \ln x + n \ln y = (m+n) \ln(x+y) \Rightarrow \frac{m}{x} + \frac{n}{y} \cdot y' = \frac{m+n}{x+y} (1+y')$

$\left(\frac{m}{x} - \frac{m+n}{x+y}\right) = y' \left(\frac{m+n}{x+y} - \frac{n}{y}\right)$ or $\frac{my-nx}{x} = \frac{dy}{dx} \left[\frac{my-nx}{y}\right]$

Hence $\frac{dy}{dx} = \frac{y}{x}$

Derivative of Inverse Function:

Theorem:

If the inverse functions f & g are defined by $y = f(x)$ & $x = g(y)$ & if $f'(x)$ exists & $f'(x) \neq 0$, then

$g'(y) = \frac{1}{f'(x)}$. This result can also be written as, if $\frac{dy}{dx}$ exists & $\frac{dy}{dx} \neq 0$, then $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ or $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$

Proof: $g(x) = f^{-1}(x)$

$\Rightarrow f(g(x)) = x$ differentiate w.r.t. x

$\Rightarrow f'(g(x)) \cdot g'(x) = 1 \Rightarrow \boxed{f'(g(x)) = \frac{1}{g'(x)}}$

Similarly, $\boxed{g'(f(x)) = \frac{1}{f'(x)}}$

Methods of Differentiation

Illustration 40:

Let $g(x) = f^{-1}(x)$ where $f(x) = x^3 + 3x$, then find $f^{-1}(4)$ or $g(4)$.

Solution:

$$g(4) = f^{-1}(4) = k$$

$$f^{-1}(4) = k$$

$$f(k) = 4$$

$$k^3 + 3k = 4$$

$$k^3 + 3k - 4 = 0$$

$$k = 1 \Rightarrow g(4) = 1$$

Illustration 41:

Let $g(x) = f^{-1}(x)$ where $f(x) = x + e^x$ then find $f^{-1}(1)$.

Solution:

$$g(1) = f^{-1}(1) = k$$

$$f(k) = 1$$

$$k + e^k = 1$$

$$k = 0$$

$$\therefore g(1) = 0$$

$$\therefore f^{-1}(1) = 0$$

Illustration 42:

Let $f(x) = \exp(x^3 + x^2 + x)$ for any real number x and let g be the inverse function for f . The value of $g'(e^3)$ is -

(A) $\frac{1}{6e^3}$

(B) $\frac{1}{6}$

(C) $\frac{1}{34e^{39}}$

(D) 6

Ans. (A)

Solution:

$$y = e^{x^3+x^2+x}$$

$$\Rightarrow \frac{dy}{dx} = e^{x^3+x^2+x} \cdot (3x^2 + 2x + 1) \Rightarrow g'(y) = \frac{dx}{dy} = \frac{1}{e^{x^3+x^2+x} \cdot (3x^2+2x+1)}$$

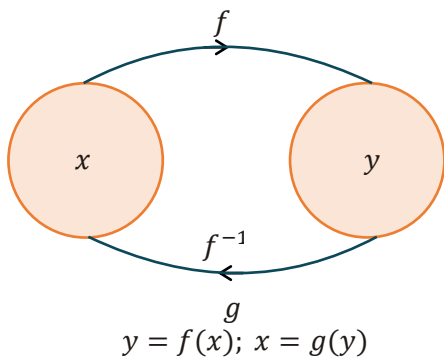
$$\Rightarrow y = e^3 \Rightarrow e^3 = e^{x^3+x^2+x} \Rightarrow x^3 + x^2 + x - 3 = 0$$

$$\Rightarrow (x - 1)(x^2 + 2x + 3) = 0 \Rightarrow x = 1$$

$$\therefore g'(e^3) = \frac{1}{6e^3}$$

Illustration 43:

If $y = f(x) = x^3 + x^5$ and g is the inverse of f find $g'(2)$



Solution:

$$\frac{dy}{dx} = 3x^2 + 5x^4 \Rightarrow g'(y) = \frac{dx}{dy} = \frac{1}{3x^2+5x^4}$$

when $y = 2$ then $2 = x^3 + x^5 \Rightarrow x = 1$

$$\therefore g'(2) = \left. \frac{dx}{dy} \right|_{\substack{x=1 \\ y=2}} = \frac{1}{3+5} = \frac{1}{8}$$

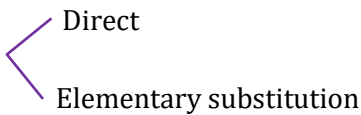
Alternatively: $(g \circ f)(x) = x$

$g'[f(x)] f'(x) = 1$; when $f(x) = 2$ then $x = 1$

$g'(2) \cdot f'(1) = 1$ (but $f'(1) = 8$)

$$\therefore g'(2) = \frac{1}{8}$$

Derivatives of Inverse Trigonometric Functions by Transforming Them into Simpler Functions:



Standard Substitution:

(a) For terms of the form $x^2 + a^2$ or $\sqrt{x^2 + a^2}$, put $x = a \tan \theta$ or $a \cot \theta$.

(b) For terms of the form $x^2 - a^2$ or $\sqrt{x^2 - a^2}$, put $x = a \sec \theta$ or $a \operatorname{cosec} \theta$.

For terms of the form $a^2 - x^2$ or $\sqrt{a^2 - x^2}$, put $x = a \sin \theta$ or $a \cos \theta$.

(d) If both $\sqrt{a+x}$, $\sqrt{a-x}$ are present, then put $x = a \cos 2\theta$.

For the type $\sqrt{(x-a)(b-x)}$ or $\sqrt{\frac{x-a}{b-x}}$ put $x = a \cos^2 \theta + b \sin^2 \theta$.

(f) For the type $\sqrt{(x-a)(x-b)}$ or $\sqrt{\frac{x-a}{x-b}}$ put $x = a \sec^2 \theta - b \tan^2 \theta$.

(g) For the type $(\sqrt{x^2 + a^2} \pm x)^n$ or $(x \pm \sqrt{x^2 - a^2})^n$, put the expression within the bracket = t .

Note that $\int \sqrt{a^2 + x^2} dx$ & $\int \sqrt{x^2 - a^2} dx$ to be evaluated by parts

Illustration 44:

Let $y = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right), x \in (0,1)$

Solution:

Let $\tan^{-1} x = \theta, \theta \in \left(0, \frac{\pi}{4}\right)$

$x = \tan \theta$

$$y = \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) = \cos^{-1} \cos 2\theta, 2\theta \in \left(0, \frac{\pi}{2}\right) = 2\theta$$

$y = 2 \tan^{-1} x$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

Illustration 45:

If $f(x) = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$ then find

- (i) $f'(2)$ (ii) $f' \left(\frac{1}{2} \right)$ (iii) $f'(1)$

Solution:

$x = \tan \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \Rightarrow y = \sin^{-1} \sin 2\theta$

$$y = \begin{cases} \pi - 2\theta & \frac{\pi}{2} < 2\theta < \pi \\ 2\theta & -\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2} \\ -(\pi + 2\theta) & -\pi < 2\theta < -\frac{\pi}{2} \end{cases} \Rightarrow f(x) = \begin{cases} \pi - 2 \tan^{-1} x & x > 1 \\ 2 \tan^{-1} x & -1 \leq x \leq 1 \\ -(\pi + 2 \tan^{-1} x) & x < -1 \end{cases} \Rightarrow f'(x) = \begin{cases} -\frac{2}{1+x^2} & x > 1 \\ \frac{2}{1+x^2} & -1 < x < 1 \\ \frac{-2}{1+x^2} & x < -1 \end{cases}$$

- (i) $f'(2) = -\frac{2}{5}$ (ii) $f' \left(\frac{1}{2} \right) = \frac{8}{5}$ (iii) $f'(1^+) = -1$ and $f'(1^-) = +1 \Rightarrow f'(1)$ does not exist

Illustration 46:

$$\frac{d}{dx} \left\{ \sin^2 \left(\cot^{-1} \sqrt{\frac{1+x}{1-x}} \right) \right\}$$

Solution:

Let $y = \sin^2 \left(\cot^{-1} \sqrt{\frac{1+x}{1-x}} \right)$. Put $x = \cos 2\theta$ $\theta \in \left(0, \frac{\pi}{2} \right]$

$$\therefore y = \sin^2 \cot^{-1} \left(\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}} \right) = \sin^2 \cot^{-1} (\cot \theta)$$

$$\therefore y = \sin^2 \theta = \frac{1-\cos 2\theta}{2} = \frac{1-x}{2} = \frac{1}{2} - \frac{x}{2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}$$

Successive Differentiation:

$y = f(x)$ the popular symbols used to denote the derivatives are $\frac{dy}{dx} = Dy = f'(x) = y_1 = y'$.

Higher order derivative are denoted as $\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = D^2y = f''(x) = y_2$ or y'' etc.

Similarly, n^{th} order derivatives are denoted by $\frac{d^n y}{dx^n}$ or $D^n y$ or y_n

Illustration 47:

$f(x) = be^{ax} + ae^{bx}$ find $f''(0)$

Solution:

$$f'(x) = be^{ax} \cdot a + ae^{bx} \cdot b \Rightarrow f''(x) = a^2 be^{ax} + ab^2 e^{bx}$$

$$f''(0) = a^2 b \cdot 1 + ab^2 \cdot 1$$

$$= a^2 b + ab^2$$

$$= ab(a + b)$$

Illustration 48:

$y = \log |x|$ find (i) $\frac{dy}{dx}$, (ii) $\frac{d^2y}{dx^2}$

Solution:

(i) $y = \log |x|$

$x > 0$ $y = \log x$ $\frac{dy}{dx} = \frac{1}{x}$	$x < 0$ $y = \log(-x)$ $\frac{dy}{dx} = \frac{1}{(-x)} \times (-1) = \frac{1}{x}$
--	---

(ii) $\frac{d^2y}{dx^2} = \frac{-1}{x^2}$

Illustration 49:

If $y = x + \sin x$, then find

(i) $\frac{d^2y}{dx^2}$ (ii) equation of tangent of curve $y = f'(x)$ at $x = \frac{\pi}{3}$

Solution:

(i) $\frac{dy}{dx} = 1 + \cos x \Rightarrow \frac{d^2y}{dx^2} = -\sin x$

(ii) at $x = \frac{\pi}{3}$

equation: $-y - \frac{3}{2} = -\frac{\sqrt{3}}{2}(x - \frac{\pi}{3})$

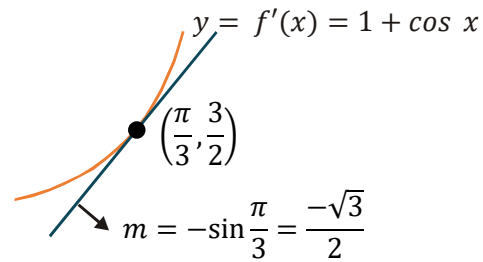


Illustration 50:

If $y = (\frac{1}{x})^x$, then prove $\frac{d^2y}{dx^2}$ at $x = 1$ is 0.

Solution:

$\Rightarrow \log y = x \log \frac{1}{x} \Rightarrow \log y = -x \log x \Rightarrow \frac{dy}{dx} = -y(1 + \ell n x)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-dy}{dx} (1 + \ell n x) - y (0 + \frac{1}{x}) = y(1 + \ell n x)^2 - y \cdot \frac{1}{x}$

{at $x = 1 \Rightarrow y = 1$ }

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=1} = 1(1 + \ell n 1)^2 - 1 \cdot \frac{1}{1} = 1(1) - 1 = 1 - 1 = 0$

Illustration 51:

$y = \sin^{-1} x$, show that $(1 - x^2) \frac{d^2y}{dx^2} - \frac{xdy}{dx} = 0$

Solution:

$y = \sin^{-1} x$
 $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2} \frac{dy}{dx} = 1$

$\frac{d}{dx} \left(\sqrt{1-x^2} \frac{dy}{dx} \right) = 0 \Rightarrow \sqrt{1-x^2} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \frac{d}{dx} (\sqrt{1-x^2})$

$\sqrt{1-x^2} \cdot \frac{d^2y}{dx^2} - \frac{dy}{dx} \frac{2x}{2\sqrt{1-x^2}} = 0$

$(1-x^2) \frac{d^2y}{dx^2} - \frac{xdy}{dx} = 0$

Hence proved.

Deduction of New Identities by Differentiating a Given Identity:

Illustration 52:

If $\cos \frac{x}{2} \cdot \cos \frac{x}{2^2} \cdot \cos \frac{x}{2^3} \cdot \dots \cdot \cos \frac{x}{2^n} = \frac{\sin x}{2^n \sin(x/2^n)}$, then prove that $\sum_{r=1}^n \frac{1}{2^r} \tan \frac{x}{2^r} = \frac{1}{2^n} \cot \frac{x}{2^n} - \cot x$

Solution:

$$\cos \frac{x}{2} \cdot \cos \frac{x}{2^2} \cdot \cos \frac{x}{2^3} \cdot \dots \cdot \cos \frac{x}{2^n} = \frac{\sin x}{2^n \sin(x/2^n)}$$

Take log:

$$\log \cos \frac{x}{2} + \log \cos \frac{x}{2^2} + \log \cos \frac{x}{2^3} \cdot \dots \cdot \log \cos \frac{x}{2^n} = \log \sin x - \log 2^n - \log \sin \frac{x}{2^n}$$

Differentiate w.r.t x

$$\Rightarrow \frac{1}{2} \tan \frac{x}{2} + \frac{1}{2^2} \tan \frac{x}{2^2} + \dots + \frac{1}{2^n} \tan \frac{x}{2^n} = \frac{1}{2^n} \cot \frac{x}{2^n} - \cot x$$

$$\Rightarrow \boxed{\sum_{r=1}^n \frac{1}{2^r} \tan \frac{x}{2^r} = \frac{1}{2^n} \cot \frac{x}{2^n} - \cot x}$$

Illustration 53:

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, prove that

- (i) $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n2^{n-1}$
- (ii) $C_0 + 2C_1 + 3C_2 + \dots + (n + 1)C_n = (n + 2)2^{n-1}$

Solution:

(i) differentiate

$$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \dots + nC_n \cdot x^{n-1}$$

Put $x = 1$

$$n \cdot 2^{n-1} = C_1 + 2C_2 + 3C_3 + \dots + nC_n$$

(ii) Multiply both side by x

$$x(1 + x)^n = C_0x + C_1x^2 + C_n x^{n+1}$$

Now, differentiate.

$$(1 + x)^n + x \cdot n(1 + x)^{n-1} = c_0 + 2c_1x + \dots + (n + 1)c_nx^n$$

Put $x = 1$

$$2^n + n \cdot 2^{n-1} = c_0 + 2c_1 + 3c_2 + \dots + (n + 1)c_n$$

$$\Rightarrow 2^{n-1}(n + 2) = c_0 + 2c_1 + 3c_2 + \dots + (n + 1)c_n$$

Hence proved.

Derivative of Functions Expressed in the Determinant Form:

Let $F(x) = \begin{vmatrix} f & g & h \\ u & v & w \\ \ell & m & n \end{vmatrix}$ where all functions are differentiable then

$$F'(x) = \begin{vmatrix} f' & g' & h' \\ u & v & w \\ \ell & m & n \end{vmatrix} + \begin{vmatrix} f & g & h \\ u' & v' & w' \\ \ell & m & n \end{vmatrix} + \begin{vmatrix} f & g & h \\ u & v & w \\ \ell' & m' & n' \end{vmatrix}$$

The same operation can also be done column wise.

Proof: Differentiation of Determinant -

$$\Rightarrow P(x) = \begin{vmatrix} f(x) & g(x) \\ h(x) & r(x) \end{vmatrix} \Rightarrow P(x) = f(x)r(x) - g(x)h(x)$$

$$\Rightarrow P'(x) = f'(x)r(x) + f(x)r'(x) - g'(x)h(x) - g(x)h'(x)$$

$$\Rightarrow P'(x) = \begin{vmatrix} f'(x) & g'(x) \\ h(x) & r(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ h'(x) & r'(x) \end{vmatrix}$$

Illustration 54:

$$f(x) = \begin{vmatrix} e^x & x^2 + x + 1 & \sin x \\ x^2 & \cos x & 1 \\ x^3 + x & \tan x & 3x^2 + 1 \end{vmatrix}, \text{ find } f'(0)$$

Solution:

$$f'(x) = \begin{vmatrix} e^x & 2x+1 & \cos x \\ x^2 & \cos x & 1 \\ x^3+x & \tan x & 3x^2+1 \end{vmatrix} + \begin{vmatrix} e^x & x^2+x+1 & \sin x \\ 2x & -\sin x & 0 \\ x^3+x & \tan x & 3x^2+1 \end{vmatrix} + \begin{vmatrix} e^x & x^2+x+1 & \sin x \\ x^2 & \cos x & 1 \\ 3x^3+1 & \sec^2 x & 6x \end{vmatrix}$$

$$= 1 + 0 - 1 + 1 = 1$$

L' Hospital's Rule

(a) This rule is applicable for the indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function $f(x)$ and $g(x)$ are differentiable in certain neighborhood of the point 'a', except, may be, at the point 'a' itself and $g'(x) \neq 0$, and if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ or $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \infty$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ provided the limit $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists (L' Hospital's rule). The point 'a' may be either finite or improper ($+\infty$ or $-\infty$).

(b) Indeterminate forms of the type $0 \cdot \infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

(c) Indeterminate forms of the type $1^\infty, \infty^0$ or 0^0 are reduced to forms of the type $0 \times \infty$ by taking logarithms or by the transformation $[f(x)]^{\phi(x)} = e^{\phi(x) \cdot \ln f(x)}$

Illustration 55:

Evaluate $\lim_{x \rightarrow 0} |x|^{\sin x}$

Solution:

$$\lim_{x \rightarrow 0} |x|^{\sin x} = \lim_{x \rightarrow 0} e^{\sin x \log_e |x|} = e^{\lim_{x \rightarrow 0} \frac{\log_e |x|}{\cos x}}$$

$$= e^{\lim_{x \rightarrow 0} \frac{1/x}{-\operatorname{cosec} x \cot x}} \quad (\text{applying L'Hopital's rule})$$

$$= e^{\lim_{x \rightarrow 0} \frac{-\sin^2 x}{x \cos x}} = e^{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x}\right)^2 \left(\frac{x}{\cos x}\right)} = e^{-(1)^2 \cdot (0)} = e^0 = 1$$

Illustration 56:

Solve $\lim_{x \rightarrow 0^+} \log_{\sin x} \sin 2x$.

Methods of Differentiation

Solution:

$$\begin{aligned} &\text{Here } \lim_{x \rightarrow 0^+} \log_{\sin x} \sin 2x \\ &= \lim_{x \rightarrow 0^+} \frac{\log \sin 2x}{\log \sin x} \quad \left(\frac{-\infty}{-\infty} \text{ form}\right) \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sin 2x} \cdot 2 \cos 2x}{\frac{1}{\sin x} \cdot \cos x} \quad \{\text{applying L'Hopital's rule}\} \\ &= \lim_{x \rightarrow 0^+} \frac{\left(\frac{2x}{\sin(2x)}\right) \cos 2x}{\left(\frac{x}{\sin x}\right) \cos x} = \lim_{x \rightarrow 0^+} \frac{\cos 2x}{\cos x} = 1 \end{aligned}$$

Illustration 57:

Evaluate $\lim_{n \rightarrow \infty} \left(\frac{e^n}{\pi}\right)^{1/n}$.

Solution:

$$\begin{aligned} &\text{Here, } A = \lim_{n \rightarrow \infty} \left(\frac{e^n}{\pi}\right)^{1/n} \quad (\infty^0 \text{ form}) \\ \therefore \log A &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\frac{e^n}{\pi}\right) = \lim_{n \rightarrow \infty} \frac{n \log e - \log \pi}{n} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \\ &= \lim_{n \rightarrow \infty} \frac{\log e - 0}{1} \quad \{\text{applying L'Hôpital's rule}\} \\ \log A &= 1 \Rightarrow A = e^1 \text{ or } \lim_{n \rightarrow \infty} \left(\frac{e^n}{\pi}\right)^{1/n} = e \end{aligned}$$

Illustration 58:

Let $y = \sin^{-1} \left(\frac{3x^4 + 6x^2 - 1}{(x^2 + 1)^3}\right)$, then $\frac{dy}{dx}$ at $x^2 = \tan 22 \frac{1}{2}^\circ$ is

- (A) $2 - \sqrt{2}$ (B) $3(2 - \sqrt{2})$ (C) $3(2 + \sqrt{2})$ (D) $3\sqrt{2\sqrt{2} - 2}$

Ans. (D)

Solution:

$$y = \sin^{-1} \left(\frac{3x^4 + 6x^2 - 1}{(x^2 + 1)^3}\right) = \sin^{-1} \left(\frac{3(x^2 + 1)^2 - 4}{(x^2 + 1)^3}\right) = \sin^{-1} \left(\frac{3}{x^2 + 1} - \frac{4}{(x^2 + 1)^3}\right)$$

Let $\frac{1}{x^2 + 1} = \sin \theta$

$y = \sin^{-1}(3\sin \theta - 4\sin^3 \theta)$

$$= \sin^{-1}(\sin 3\theta) = \begin{cases} 3\theta; & 3\theta \in \left[0, \frac{\pi}{2}\right] \Rightarrow \theta \in \left[0, \frac{\pi}{6}\right] \\ \pi - 3\theta; & 3\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \Rightarrow \theta \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \end{cases}$$

at $x^2 = \tan 22 \frac{1}{2}^\circ = \sqrt{2} - 1$

$\sin \theta = \frac{1}{\sqrt{2} - 1 + 1} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\therefore y = \pi - 3\theta = \pi - 3 \sin^{-1} \left(\frac{1}{x^2 + 1}\right)$

$$\frac{dy}{dx} = -3 \frac{1}{\sqrt{1 - \left(\frac{1}{x^2 + 1}\right)^2}} \left(-\frac{1}{x^2 + 1}\right) \cdot 2x = \frac{6x}{(x^2 + 1)\sqrt{(x^2 + 1)^2 - 1}}$$

$$\frac{6 \cdot \sqrt{\sqrt{2} - 1}}{(\sqrt{2} - 1 + 1) \cdot \sqrt{2} - 1} = \frac{6}{2} \cdot \sqrt{2} \sqrt{\sqrt{2} - 1} = 3 \cdot \sqrt{2\sqrt{2} - 1}$$

Illustration 59:

If $f'(5) = 7$ then $\lim_{t \rightarrow 0} \frac{f(5+t) - f(5-t)}{2t} =$

- (A) 0 (B) 3.5 (C) 7 (D) 14

Ans. (C)

Solution:

$$f'(5) = 7 \lim_{t \rightarrow 0} \frac{f(5+t) - f(5-t)}{2t}$$

$$\Rightarrow \lim_{t \rightarrow 0} \left[\frac{f(5+t) - f(5)}{2t} + \frac{f(5-t) - f(5)}{-2t} \right] = \frac{f'(5)}{2} + \frac{f'(5)}{2} = f'(5) = 7$$

Illustration 60:

If $f(x)$ be a polynomial in x . Then the second derivative of $f(e^x)$, is

- (A) $f''(e^x) \cdot e^x + f'(e^x)$ (B) $f''(e^x) \cdot e^{2x} + f'(e^x) \cdot e^{2x}$
 (C) $f''(e^x) \cdot e^{2x}$ (D) $f''(e^x) \cdot e^{2x} + f'(e^x) \cdot e^x$

Ans. (D)

Solution:

$$g = f(e^x) \Rightarrow g' = f'(e^x) e^x$$

$$\Rightarrow g'' = f''(e^x) e^{2x} + e^x f'(e^x)$$

Illustration 61:

$f'(x) = g(x)$ and $g'(x) = -f(x)$ for all real x and $f(5) = 2 = f'(5)$ then $f^2(10) + g^2(10)$ is

- (A) 2 (B) 4 (C) 8 (D) None of these

Ans. (C)

Solution:

$$f'(x) = g(x) \text{ and } g'(x) = -f(x)$$

$$\text{Now } \frac{d}{dx} [f^2(x) + g^2(x)] = 2f(x)f'(x) + 2g(x)g'(x) = 2f(x)g(x) - 2g(x)f(x) = 0$$

$$\therefore f^2(x) + g^2(x) = \text{constant}$$

$$f^2(5) + g^2(5) = 4 + 4 = 8$$

$$\therefore f^2(10) + g^2(10) = 8$$

Illustration 62:

Let $f(x) = x \sin x \forall x \in (0, \frac{\pi}{2})$ & g be the inverse of f in $(0, \frac{\pi}{2})$, then $(\sqrt{3}\pi + 6)g'(\frac{\pi}{12})$ is equal to

- (A) 6 (B) 12 (C) 4 (D) 3

Ans. (B)

Solution:

$$g'(f(x)) = \frac{1}{f'(x)}$$

$$f(x) = x \sin x$$

$$\text{When } f(x) = \frac{\pi}{12} \Rightarrow g'(\frac{\pi}{12}) = \frac{1}{f'(\frac{\pi}{6})} \quad \dots(i)$$

$$\text{Now } f(x) = x \sin x$$

$$f'(x) = x \cos x + \sin x$$

$$\Rightarrow f'(\frac{\pi}{6}) = \frac{\pi}{6} \times \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3}\pi + 6}{12} \quad \Rightarrow \text{from (i)}$$

$$g'(\frac{\pi}{12}) = \frac{12}{\sqrt{3}\pi + 6} \Rightarrow (\sqrt{3}\pi + 6)g'(\frac{\pi}{12}) = 12$$