Methods of Differentiation

SOLUTIONS

EXERCISE - O

1. **Ans. (B)**

$$y=\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$$

$$\ell n\, y=\frac{1}{2}\ell n\left(\frac{1-x}{1+x}\right)$$

$$\ell n\, y=\frac{1}{2}\left(\ell n(1-x)-\ell n(1+x)\right)$$

$$\Rightarrow \frac{y'}{y}=\frac{1}{2}\left(\frac{-1}{1-x}-\frac{1}{1+x}\right)$$

$$\Rightarrow \frac{y'}{y}=\frac{-1}{2}\left(\frac{2}{1-x^2}\right)$$

$$\Rightarrow y'=\frac{y}{x^2-1}$$

2. **Ans. (B)**

$$x = y\ell n(xy) \Rightarrow \ell nxy = \frac{x}{y}$$

$$\frac{dx}{dy}=\ell n(xy)+\frac{y}{xy}(x+y\frac{dx}{dy})$$

$$\Rightarrow \frac{dx}{dy}=\frac{x}{y}+1+\frac{y}{x}\frac{dx}{dy} \qquad \left(\because \ell n(xy)=\frac{x}{y}\right)$$

$$\Rightarrow \frac{dx}{dy}\left(1-\frac{y}{x}\right)=\frac{x+y}{y}$$

$$\Rightarrow \frac{dx}{dy}=\frac{x}{y}\left(\frac{x+y}{x-y}\right)$$

3. **Ans. (B)**

$(\cos x)^y = (\sin y)^x$

Taking log on both sides:

$y\ \ell n\ (\cos x) = x\ \ell n\ (\sin y)$

$$\frac{y(-\sin x)}{\cos x}+y'\ell n(\cos x)=\frac{x(\cos y)}{\sin y}y'+\ell n(\sin y)$$

$$\Rightarrow y'=\frac{\ell n\sin y+y\tan x}{\ell n\cos x-x\cot y}$$

4. **Ans. (A)**

$$\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)=\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x.\sqrt{x}}\right)$$

$= \tan^{-1}\sqrt{x} - \tan^{-1} x$

$$\therefore \frac{d}{dx}\left(\tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)\right)=\frac{d}{dx}(\tan^{-1}\sqrt{x}-\tan^{-1}x)$$

$$=\left(\frac{1}{1+x}\right)\frac{1}{2\sqrt{x}}-\frac{1}{1+x^2}$$

5. **Ans. (B)**

$$\frac{d\Delta_1}{dx}=\begin{vmatrix}1 & 0 & 0\\ a & x & b\\ a & a & x\end{vmatrix}+\begin{vmatrix}x & b & b\\ 0 & 1 & 0\\ a & a & x\end{vmatrix}+\begin{vmatrix}x & b & b\\ a & x & b\\ 0 & 0 & 1\end{vmatrix},\Delta_2=x^2-ab$$

$$\frac{d\Delta_1}{dx} = x^2-ab+x^2-ab+x^2-ab$$

$= 3(x^2-ab)$

$$\therefore \frac{d}{dx}\Delta_1=3\Delta_2$$

6. **Ans. (A)**

$$\left(f(x)-f(2x)\right)'=f'(x)-2f'(2x)\Big|_{x=1}=5$$

$f'(1)-2f'(2)=5$ ...(1)

at $x=2 \Rightarrow f'(2)-2f'(4)=7$ ...(2)

$(f(x)-f(4x))'=f'(x)-4f'(4x)$

at $x=1 \Rightarrow f'(1)-4f'(4)$

eq^n (1) + 2 × eq^n (2)

$f'(1)-4f'(4)=19$

7. **Ans. (A)**

$$f'(0^+)=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0^+}\frac{h\tan^{-1}h+\sec^{-1}\frac{1}{h}-\frac{\pi}{2}}{h}$$

$$=\lim_{h\to 0^+}(\tan^{-1}h-\frac{\sin^{-1}h}{h}) = 0 - 1$$

$$f'(0^+)=-1$$

$$f'(0^-)=\lim_{h\to 0^+}\frac{f(0-h)-f(0)}{-h}$$

$$=\lim_{h\to 0^+}\frac{(-h)\tan^{-1}(-h)+\cos^{-1}(-h)-\frac{\pi}{2}}{-h}$$

$$= \lim_{h\to 0^+}\left(-\tan^{-1}(h)+\frac{\pi-\cos^{-1}h-\frac{\pi}{2}}{-h}\right)$$

$$= \lim_{h\to 0^+}\left(-\tan^{-1}(h)-\frac{\sin^{-1}h}{h}\right)$$

$f'(0^-)=-1$

$f'(0^+)=f'(0^-)=-1$

so $f'(0) = -1$

8. **Ans. (A)**

$x = t^3 + t + 5$

$\frac{dx}{dt} = 3t^2 + 1$

$y = \sin t$

$\frac{dy}{dt} = \cos t$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\cos t}{3t^2+1}$$

$$\frac{d^2y}{dx^2}=-\frac{((3t^2+1)\sin t+6t\cos t)}{(3t^2+1)^2}\cdot\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=-\left(\frac{(3t^2+1)\sin t+6t\cos t)}{(3t^2+1)^3}\right)$$

9. **Ans. (B)**

$y = x + e^x$

$\frac{dy}{dx} = 1 + e^x$

$\frac{dx}{dy} = \frac{1}{1+e^x}$

$$\frac{d^2x}{dy^2}=-\frac{1}{(1+e^x)^2}.e^x.\frac{dx}{dy}$$

$$\boxed{\frac{d^2x}{dy^2}=\frac{-e^x}{(1+e^x)^3}}$$

10. **Ans. (C)**

$f(x) = x + \sin x$ $\qquad f\left(\frac{\pi}{4}\right)=\frac{\pi}{4}+\frac{1}{\sqrt{2}}$

$$g'(f(x))=\frac{1}{f'(x)}=\frac{1}{1+\cos x}$$

Put $x = \frac{\pi}{4}$

$= g'\left(\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right) = \frac{1}{1+\cos\left(\frac{\pi}{4}\right)}$

$= \frac{1}{1+\cos\left(\frac{\pi}{4}\right)} \Rightarrow \frac{\sqrt{2}}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}$

$\boxed{2-\sqrt{2}}$

11. **Ans. (C)**

Differentiating with respect to x,

$x.\frac{1}{\ell nx}.\frac{1}{x}+\ell n(\ell nx)-2x+2y.\frac{dy}{dx}=0$

at $x = e$ we get

$1-2e+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{2e-1}{2y}$

$\Rightarrow \frac{dy}{dx}=\frac{2e-1}{2\sqrt{4+e^2}}$ as $y(e)=\sqrt{4+e^2}$

12. **Ans. (B)**

$y = f(f(f(x))) + (f(x))^2$

$\frac{dy}{dx} = f'(f(f(x)))\, f'(f(x))\, f'(x) + 2f(x)\, f'(x)$

$= f'(1)f'(1)f'(1) + 2f(1)f'(1)$

$= 3 \times 3 \times 3 + 2 \times 1 \times 3$

$= 27 + 6$

$= 33$

13. **Ans. (A)**

$$\lim_{x\to 4}\frac{\sqrt{f(x)}-\sqrt{g(x)}}{\sqrt{x}-2} = \lim_{x\to 4}\frac{\frac{1}{2}\frac{f'(x)}{\sqrt{f(x)}}-\frac{1}{2}\frac{g'(x)}{\sqrt{g(x)}}}{\frac{1}{2\sqrt{x}}} = \frac{\frac{9}{\sqrt{2}}-\frac{6}{\sqrt{2}}}{\frac{1}{2}} = 3\sqrt{2}$$

14. **Ans. (C)**

$8f(x) + 6f\left(\frac{1}{x}\right) = x + 5$ ... (i)

Replacing x by $1/x$ we get

$8f\left(\frac{1}{x}\right) + 6f(x) = \frac{1}{x} + 5$ ... (ii)

(i) × 8 $\Rightarrow 64f(x) + 48f\left(\frac{1}{x}\right) = 8x + 40$ ... (iii)

(ii) × 6 $\Rightarrow 36f(x) + 48f\left(\frac{1}{x}\right) = \frac{6}{x} + 30$ ... (iv)

(iii) – (iv) $\Rightarrow 28f(x) = 8x - \frac{6}{x} + 10$

Differentiating w.r.t. x $28f'(x) = 8 + \frac{6}{x^2}$

Now $y = x^2 f(x) \Rightarrow \frac{dy}{dx} = x^2 f'(x) + 2x f(x) \Rightarrow \left(\frac{dy}{dx}\right)_{x=-1} = f'(-1) - 2f(-1)$

$= \frac{14}{28} - 2\left(\frac{2}{7}\right) = \frac{-1}{14}\left\{\because\ f(-1) = \frac{2}{7}\right\}$

15. Ans. (C)

$y = \ell n\left(x+\sqrt{a^2+x^2}\right) - \ell n\, a$

$\Rightarrow y' = \frac{1+\frac{2x}{2\sqrt{a^2+x^2}}}{x+\sqrt{a^2+x^2}} - 0$

$\Rightarrow y' = \frac{\left(x+\sqrt{a^2+x^2}\right)}{\left(x+\sqrt{a^2+x^2}\right)} \cdot \frac{1}{\sqrt{a^2+x^2}}$

$\Rightarrow y' = \frac{1}{\sqrt{a^2+x^2}}$

16. Ans. (C)

$g(x) = f(-x + f(f(x)))$

$g'(x) = f'(-x + f(f(x))(-1 + f'(f(x))f'(x))$

$g'(0) = f'(f(f(0))(-1 + f'(f(0))f'(0))$

$= f'(0)(-1 + f'(0)f'(0))$

$= 2(-1 + 2^2)$

$g'(0) = 6$

17. Ans. (D)

we know that

$\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$

$\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2} = 0$

$k = 0$

18. Ans. (B,D)

If $f(x) = (2x - 3\pi)^5 + \frac{4}{3}x + \cos x$ and g is the inverse function of f, then find right answer:

$f(x) = (2x - 3\pi)^5 + \frac{4}{3}x + \cos x$

$f\left(\frac{3\pi}{2}\right) = 2\pi,\ g'(f(x)) = \frac{1}{f'(x)}$

$g'(f(x)) = \frac{1}{5(2x-3\pi)^4 2 + \frac{4}{3} - \sin x}$

Put $x = \frac{3\pi}{2}$

$$g'\left(f\left(\frac{3\pi}{2}\right)\right) = \frac{1}{5(2.\frac{3\pi}{2} - 3\pi)^4.2 + \frac{4}{3} - \sin\left(\frac{3\pi}{2}\right)}$$

$$g'(2\pi) = \frac{1}{\frac{4}{3}+1} = \frac{3}{7}$$

$$g'(f(x)) = \frac{1}{f'(x)}$$

$$g''(f(x))\, f'(x) = -\frac{1}{f'(x)^2} f''(x)$$

$$g''(f(x)) = \frac{-1}{(f'(x))^3}.f''(x)$$

$$g''(2\pi) = \frac{-1}{\left(\frac{4}{3}+1\right)^3}\left(20.2\left(2\frac{3\pi}{2} - 3\pi\right)^3.2 - \cos\frac{3\pi}{2}\right) \quad \left[\text{We have } f''(x) = 20.2(2x-3\pi)^3.2 - \cos x\right]$$

$$g''(2\pi) = 0$$

$\therefore$ B & D

19. **Ans. (A,B)**

$$f(x) = \frac{\sqrt{x - 2\sqrt{x-1}}}{\sqrt{x-1} - 1}.x$$

Add & subtract 1 in $x - 2\sqrt{x-1}$

$$= \frac{\sqrt{x-1+1-2\sqrt{x-1}}}{\sqrt{x-1}-1}.x$$

$$= \left(\frac{\sqrt{\left(\sqrt{x-1}-1\right)^2}}{\sqrt{x-1}-1}\right)x$$

$$= \left(\frac{\left|\left(\sqrt{x-1}-1\right)\right|}{\sqrt{x-1}-1}\right)x$$

Case 1: $\sqrt{x-1} > 1$ & $x - 1 \geq 0$	Case 1: $\sqrt{x-1} < 1$ & $x - 1 \geq 0$
$x - 1 > 1$ & $x \geq 1$	$x - 1 < 1$ & $x \geq 1$
$x > 2$ & $x \geq 1$	$x < 2$ & $x \geq 1$
$\therefore x \in (2, \infty)$	$\therefore x \in [1, 2)$
$\Rightarrow f(x) = x; x \in (2, \infty)$	$\Rightarrow f(x) = -x; x \in [1, 2)$

$$f(x) = \begin{cases} x & , \quad x > 2 \\ -x & , \quad 1 \leq x < 2 \end{cases}$$

$$f'(x)=\begin{cases}1 & , \quad x>2 \\ -1 & , \quad 1\le x<2\end{cases}$$

$f'(10) = 1$ and $f'\left(\frac{3}{2}\right)=-1$

Since $2 \notin$ domain so option (C) is incorrect.

20. **Ans. (B, D)**

If $y=(x)^{(\ell nx)^{\ell n(\ell nx)}}$, then $\frac{dy}{dx}$ is equal to

$$y=(x)^{(\ell nx)^{\ell n(\ell nx)}}$$

Taking log on base e

$$\ell ny=(\ell nx)^{\ell n(\ell nx)}.\ell nx$$

$$\ell ny = (\ell nx)^{\ell n(\ell nx)+1}$$

$$\ell ny = (\ell nx)^{\ell n(e\ell nx)}$$

Again, taking log on base e

$$\ell n(\ell ny) = \left(\ell n(e\ell nx)\right)\ell n(\ell nx)$$

$$\frac{1}{\ell ny}\frac{1}{y}y'=\frac{1}{e\ell nx}.\frac{e}{x}\ \ell n(\ell nx)\ +\left(\frac{\ell n(e\,\ell nx)}{\ell nx}\right)\left(\frac{1}{x}\right)$$

$$\frac{y'}{y.\ell ny}=\frac{\ell n(\ell nx)+\ell n(e\ell nx)}{x.\ell nx}$$

$$y' = y\ \ell ny\left(\frac{\ell n(\ell nx)+\ell n(\ell nx)+1}{x.\ell nx}\right)$$

$$y' = y\ \ell ny\left(\frac{2\ell n(\ell nx)+1}{x.\ell nx}\right)......(D)$$

$$=y(\ell nx)^{\ell n(\ell nx)}\frac{\ell nx}{x.\ell nx}(2\ell n(\ell nx)+1)$$

$$y' = y(\ell nx)^{\ell n(\ell nx)}\left(\frac{(2\ell n(\ell nx)+1)}{x}\right)......(B)$$

21. **Ans. (A,B,C,D)**

$2^x + 2^y = 2^{x+y}$ ...(i)

diff. both sides w.r.t. x

$$\Rightarrow 2^x.\ell n2 + 2^y\cdot\ell n2\frac{dy}{dx} = 2^{x+y}\cdot\ell n2\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow 2^x - 2^{x+y} = (2^{x+y} - 2^y)\ \frac{dy}{dx} \qquad ...(ii)$$

$$\Rightarrow \frac{2^x(1-2^y)}{2^y(2^x-1)}=\frac{dy}{dx}........(D)$$

From (i) & (ii)

$$2^x - 2^x - 2^y = (2^x + 2^y - 2^y)\ \frac{dy}{dx}\Rightarrow\frac{dy}{dx}=-\frac{2^y}{2^x}.........(A)$$

$$\Rightarrow\frac{dy}{dx}=\frac{2^x-2^{x+y}}{2^{x+y}-2^y}=\frac{-2^y}{2^y(2^x-1)}=\frac{1}{1-2^x}=1-2^y..........(B\&C)$$

22. Ans. (B,C,D)

$f(x) = \sin 2x\{\sin(x+x^2).\sin(x-x^2)+\cos(x+x^2)\cos(x-x^2)\} + \sin 2x^2\{\cos(x+x^2))$
$\cos(x-x^2) - \sin(x-x^2)\sin(x+x^2)\}$

$\Rightarrow f(x) = \sin 2x.\cos 2x^2 + \cos 2x.\sin 2x^2$

$\Rightarrow f(x) = \sin(2x+2x^2)$

$\Rightarrow f'(x) = (2+4x)\cos(2x+2x^2)$

Now $f'\left(-\frac{1}{2}\right) = (2-2)\cos\left(-1+\frac{1}{2}\right) = 0$

$f'(-1) = -2\cos 0 = -2$

$f''(x) = 4\cos(2x+2x^2) - (2+4x)^2\sin(2x+2x^2)$

$f''(0) = 4 - 0 = 4$

23. Ans. (A,B,C,D)

$u = e^x \sin x, v = e^x \cos x$

$v\frac{du}{dx} - u\frac{dv}{dx} = v(e^x \cos x + e^x \sin x) - u(e^x \cos x - e^x \sin x)$

$= e^x \sin x\,(v + u) + e^x \cos x\,(v - u)$

$= u(v + u) + v(v - u) = v^2 + u^2$

again $\frac{du}{dx} = e^x \sin x + e^x \cos x$

$\frac{d^2u}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$

$\frac{d^2u}{dx^2} = 2v$

similarly, other options can be checked.

24. Ans. (A)

$f(x) = (p_1a_1^x + p_2a_2^x + \ldots\ldots + p_na_n^x)^{\frac{1}{x}}$

$\ell n(f(x)) = \frac{1}{x}(p_1a_1^x + p_2a_2^x + \ldots\ldots + p_na_n^x)$

$\lim_{x\to 0^+} f(x) = e^{\lim_{x\to 0^+}\frac{1}{x}[p_1a_1^x + p_2a_2^x + p_3a_3^x + \ldots\ldots + p_na_n^x - 1)}$

By applying L' Hospital rule

$= e^{\lim_{x\to 0^+}\frac{p_1a_1^x\ell na_1 + p_2a_2^x\ell na_2 + \ldots\ldots\ldots + p_na_n^x\ell na_n}{1}}$

$= e^{(p_1\ell na_1 + p_2\ell na_2 + \ldots\ldots\ldots + p_n\ell na_n)}$

$= e^{\ell n(a_1^{p_1}a_2^{p_2}\ldots\ldots a_n^{p_n})}$

$= a_1^{p_1}.a_3^{p_3}\ldots\ldots a_n^{p_n}$

25. Ans. (B)

$\lim_{x\to\infty} f(x)$

$f(x) = [p_1a_1^x + p_2a_2^x + \ldots\ldots + p_na_n^x]^{\frac{1}{x}}$

$\ell n\, f(x) = \frac{\ell n[p_1a_1^x + p_2a_2^x + \ldots\ldots + p_na_n^x]}{x}$

$\Rightarrow \lim_{x\to\infty} \ell nf(x) = \lim_{x\to\infty}\frac{\ell n(p_1a_1^x + \ldots\ldots\ldots + p_na_n^x]}{x}$

By applying L' Hospital rule

$$\Rightarrow \lim_{x\to\infty} \ell nf(x) = \lim_{x\to\infty} \frac{[p_1a_1^x \ell na_1 + p_2a_2^x \ell na_2 + ... + p_na_n^x \ell na_n]}{(p_1a_1^x + p_2a_2^x + ... + p_na_n^x)}$$

$$\Rightarrow \lim_{x\to\infty} \ell n\, f(x) = \lim_{x\to\infty} \frac{a_1^x[p_1 \ell na_1 + p_2 \frac{a_2^x}{a_1^x} \ell na_2 + ... + p_n \left(\frac{a_n}{a_1}\right)^x \ell na_n]}{\left[a_1^x[p_1 + p_2\left(\frac{a_2}{a_1}\right)^x + ... + p_n\left(\frac{a_n}{a_1}\right)^x\right]}$$

$\because a_1 > a_2 > \dots\dots\dots\dots\dots a_n > 1$

There $\left(\frac{a_2}{a_1}\right)^x, \left(\frac{a_3}{a_1}\right)^x, ..., \left(\frac{a_n}{a_1}\right)^x$, all tends to zero as, $x \to \infty$ so

$$\Rightarrow \lim_{x\to\infty} \ell nf(x) = \lim_{x\to\infty} \frac{p_1 \ell na_1}{p_1}$$

$$\Rightarrow \lim_{x\to\infty} f(x) = a_1$$

26. Ans. (C)

$$\lim_{x\to-\infty} f(x)$$

$$\Rightarrow \lim_{x\to-\infty} \ell nf(x) = \lim_{x\to-\infty} \frac{\ell n(p_1a_1^x + p_2a_2^x + ... + p_na_n^x)}{x}$$

By applying L' Hospital rule

$$\Rightarrow \lim_{x\to-\infty} \frac{1\times(p_1a_1^x \ell na_1 + ... + p_na_n^x \ell na_n}{(p_1a_1^x + p_2a_2^x + ... + p_na_n^x)}$$

$$\lim_{x\to-\infty} \ell nf(x) = \lim_{x\to-\infty} \frac{a_n^x\left[p_1\left(\frac{a_1}{a_n}\right)^x \ell n\, a_1 + ... + p_n \ell n\, a_n\right]}{a_n^x\left[p_1\left(\frac{a_1}{a_n}\right)^x .... + p_n\right]}$$

$\because a_1 > a_2 > \dots\dots\dots\dots\dots a_n > 1$

there $\left(\frac{a_1}{a_n}\right)^x, \left(\frac{a_2}{a_n}\right)^x, ..., \left(\frac{a_{n-1}}{a_n}\right)^x$, all tends to zero when $x \to \infty$ so

$$\Rightarrow \lim_{x\to-\infty} \ell nf(x) = \frac{p_1 \ell na_n}{p_n}$$

$$\Rightarrow \boxed{\lim_{x\to-\infty} f(x) = a_n}$$

27. Ans. (A)

$\because\ y^2 + y - x = 2 \Rightarrow 2y\frac{dy}{dx} + \frac{dy}{dx} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y+1}$$

put $x = 4 \Rightarrow y = 2, -3$

$y \in \left(-\infty, -\frac{1}{2}\right) \Rightarrow y = -3$

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{1}{2y+1}\right|_{y=-3} = -\frac{1}{5}$

Now $(1+2y)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=4} = \frac{2}{125} \Rightarrow f''(4) = \frac{2}{125}$

28. **Ans. (D)**

$\because$ $h(x)$ is mirror image of $f(x)$ in $y = -\frac{1}{2}$

$\Rightarrow \frac{f(x)+h(x)}{2} = -\frac{1}{2} \Rightarrow f(x) + h(x) = -1$

$\Rightarrow f''(x) + h''(x) = 0 \;\forall\; x \in \left(-\frac{9}{4}, \infty\right)$

29. **Ans. (C)**

(P) $\Rightarrow f(x) = x^2 - 4x - 3, x > 2$

$f(x) = (x-2)^2 - 7$

$(x-2)^2 - 7 = 2 \rightarrow (x-2)^2 = 9$

$x = 5$ or 1 ($\because x > 2, x = 1$ is rejected)

g is the inverse of f

$\therefore g(f(x)) = x$

$g'(f(x)).f'(x) = 1$

$f'(x) = 2x - 4$

at $x = 5 \rightarrow g'(f(5)).f'(5) = 1$

$g'(2) = \frac{1}{f'(5)} = \frac{1}{6}$ (as $f(5) = 2$)

(p) $\rightarrow$ (2)

(Q). $\Rightarrow f(x) = x^3 + 3x^2 + 6x - 5 + 4e^{2x}$

g is the inverse of $f, g'(-1) = ?$

$f(0) = -1$

$f'(x) = 3x^2 + 6x + 6 + 8e^{2x}$

$g(f(x)) = x \rightarrow g'(f(x)) = \frac{1}{f'(x)}$

at $x = 0, g'(-1) = \frac{1}{f'(x)} = \frac{1}{14}$

(Q $\rightarrow$ 4)

(R) let $h(x)$ be the inverse of $f(x)$

$$\therefore G(x)=\frac{1}{h(x)}$$

$$G'(x)=\frac{-h'(x)}{(h(x))^2}\rightarrow G'(2)=\frac{-h'(2)}{(h(2))^2}$$

$$h(f(x))=x\rightarrow h'(f(x))=\frac{1}{f'(x)}$$

$$f(3)=2, f'(3)=\frac{1}{9}\Rightarrow h(2)=3$$

at $x=3$

$$h'(f(3))=\frac{1}{f'(3)}\Rightarrow h'(2)=9$$

$$\therefore G'(2)=-\frac{h'(2)}{(h(2))^2}=\frac{-9}{(3)^2}=-1$$

$(\text{R}\rightarrow 1)$

(S) $e^{xy}+y cosx=2, y=y(x)$

at $x=0$

$1+y(0)=2\rightarrow \boxed{y(0)=1}$

Now on diff. the given eq^{n} w.r.t. x

$e^{xy}(y+xy')+y'cosx-y\sin x=0$

at $x=0$

$y(0)+y'(0)=0\rightarrow \boxed{y'(0)=-1}$

Again on diff. w.r.t x, we get

$e^{xy}(y+xy')^2+e^{xy}(y'+xy''+y')+y''cosx$

$-y'sinx-y'sinx-ycosx=0$

at $x=0$

$(y(0))^2+2y'(0)+y''(0)-y(0)=0$

$1+2(-1)+y''(0)-1=0$

$\boxed{y''(0)=2}$

$\therefore y''(0)+y'(0)=2-1=1$

$(\text{S})\rightarrow(3)$

Ans $\Rightarrow$ (C)

30. **Ans. (C)**

(P) $f'(x)=2x+g'(1), f''(x)=2$

$g'(x)=2x+f'(2), g''(x)=2$

at $x=1, f'(1)=2+g'(1)$ and $g'(1)=2+f'(2)$

$\Rightarrow f'(1)=4+f'(2)$

(Q) $2f(x)f(y)=f(x-y)+f(x+y)$ ....(1)

Replacing x by y and y by x, then

$2f(y)f(x)=f(y-x)+f(y+x)$ ....(2)

From equation (1) and (2), we get $f(x-y)=f(y-x)$

Put $y = 0$, then differentiate,

$f'(x) = -f'(-x)$

Put $x = 2$

$f'(2) + f'(-2) = 0$

(R) $f'(x) = 3x^2 + 2xf'(1) + f''(2)$

$\Rightarrow f'(1) = 3 + 2f'(1) + f''(2)$

$\Rightarrow f'(1) + f''(2) = -3$

and $f''(x) = 6x + 2f'(1)$

$\therefore f''(2) = 12 + 2f'(1)$

$\therefore -2f'(1) + f''(2) = 12$

$\Rightarrow f'(1) = -5, f''(2) = 2$

Now, $f(x) = x^3 - 5x^2 + 2x + 6$

$\Rightarrow f(0) = 6$

And $f(3) = -6$

$\Rightarrow f(0) + f(3) + 1 = 1$

(S) $n(a+b)^{n-1} = na^{n-1} + nb^{n-1}$

$(a+b)^{n-1} = a^{n-1} + b^{n-1}$

$\Rightarrow n - 1 = 1$

EXERCISE - S

1. **Ans. (0)**

$y\sqrt{x^2+1} = \log\left\{\sqrt{x^2+1} - x\right\}$

Differentiating both sides *w.r.t.* x, we get

$$\frac{dy}{dx}\sqrt{x^2+1} + y.\frac{1}{2\sqrt{x^2+1}}.2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1} - x} \times \left\{\frac{1}{2}\frac{2x}{\sqrt{x^2+1}} - 1\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}\left\{\frac{-1}{\sqrt{x^2+1}} - \frac{xy}{\sqrt{x^2+1}}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{(x^2+1)}\{1 + xy\}$$

$$\Rightarrow (x^2+1)\frac{dy}{dx} + xy + 1 = 0.$$

2. **Ans. (1)**

$f(x) = x + 2 \Rightarrow f'(x) = 1$

$\therefore f'(f(x)) = f'(x+2) = 1$ at $x = 4$.

3. Ans. (2)

$3f(x) - 2f(1/x) = x$ ...(i)

Let $1/x = y$, then $3f(1/y) - 2f(y) = 1/y$

$\Rightarrow -2f(y) + 3f(1/y) = 1/y$

$\Rightarrow -2f(x) + 3f(1/x) = 1/x$ ...(ii)

From 3 × (i) + 2 × (ii),

$9f(x) - 6f(1/x) - 4f(x) + 6f(1/x) = 3x + 2/x$

$$5f(x) = 3x + \frac{2}{x} \Rightarrow f(x) = \frac{1}{5}\left[3x + \frac{2}{x}\right]$$

$$\Rightarrow f'(x) = \frac{1}{5}\left[3 - \frac{2}{x^2}\right] \Rightarrow f'(2) = \frac{1}{5}\left[3 - \frac{2}{4}\right] = \frac{1}{2}.$$

$$\Rightarrow 4f'(2) = 2$$

4. Ans. (5)

$$\frac{d}{dx}\tan^{-1}(\sec x + \tan x) = \frac{d}{dx}\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$$

$$= \frac{d}{dx}\tan^{-1}\left(\frac{\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}\right) = \frac{d}{dx}\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{1}{2}.$$

$10\lambda = 10 \times \frac{1}{2} = 5$

5. Ans. (1)

$$f(x) = 2\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) = 2\tan^{-1}\left[\tan\frac{x}{2}\right] = x$$

$$\Rightarrow f'(x) = 1. \text{ Hence } f'\left(\frac{\pi}{3}\right) = 1.$$

6. Ans. (2)

Given $f(2) = 4, \quad f'(2) = 1$

$$\therefore \lim_{x\to 2}\frac{xf(2) - 2f(x)}{x-2} = \lim_{x\to 2}\frac{xf(2) - 2f(2) + 2f(2) - 2f(x)}{x-2}$$

$$= \lim_{x\to 2}\frac{(x-2)f(2)}{x-2} - \lim_{x\to 2}\frac{2f(x) - 2f(2)}{x-2}$$

$$= f(2) - 2\lim_{x\to 2}\frac{f(x) - f(2)}{x-2} = f(2) - 2f'(2) = 4 - 2(1) = 4 - 2 = 2$$

Aliter: Applying *L*-Hospital rule, we get $\lim_{x\to 2}\frac{f(2) - 2f'(2)}{1} = 2$

7. Ans. (1.00)

$$y = 1 + \frac{x_1}{x-x_1} + \frac{x_2.x}{(x-x_1)(x-x_2)} + \frac{x_3x^2}{(x-x_1)(x-x_2)(x-x_3)} + ...$$

$$= \frac{x}{x-x_1} + \frac{x_2x}{(x-x_1)(x-x_2)} + ...$$

$$= \frac{x}{x-x_1}\left[1 + \frac{x_2}{x-x_2}\right] + ...$$

$$= \frac{x^2}{(x-x_1)(x-x_2)} + \frac{x_3x^2}{(x-x_1)(x-x_2)(x-x_3)}........$$

$$y = \frac{x^n}{(x-x_1)(x-x_2)(x-x_3).......(x-x_n)}$$

$$lny = nlnx - ln(x-x_1) - ln(x-x_2) \ldots - ln(x-x_n)$$

$$= \frac{1}{y}y' = \frac{n}{x} - \frac{1}{x-x_1} - \frac{1}{x-x_2}..... - \frac{1}{x-x_n}$$

$$\therefore\ y' = y\left[\frac{1}{x} - \frac{1}{x-x_1} + \frac{1}{x} - \frac{1}{x-x_2} + ...\frac{1}{x} - \frac{1}{x-x_n}\right]$$

$$= y\left[\frac{-x_1}{x(x-x_1)} + \frac{-x_2}{x(x-x_2)} + ...\frac{-x_n}{x(x-x_n)}\right]$$

$$y' = \frac{y}{x}\left[\frac{x_1}{x_1-x} + \frac{x_2}{x_2-x} + \frac{x_3}{x_3-x} + ...\frac{x_n}{x_n-x}\right]$$

$$\therefore = \frac{x}{y}\frac{dy}{dx} = \left[\frac{x_1}{x_1-x} + \frac{x_2}{x_2-x} + \frac{x_3}{x_3-x} + ... + \frac{x_n}{x_n-x}\right]$$

Ans $\boxed{K=1}$

8. Ans. (36.00)

x = cosec θ – sin θ & y = cosec^n θ – sin^n θ

y = cosec^n θ – sin^n θ

$$\frac{dy}{d\theta} = -n\,\text{cosec}^{n-1}\theta\,(\text{cosec}\theta\cot\theta) - n\sin^{n-1}\theta.\cos\theta$$

$$\frac{dx}{d\theta} = -\text{cosec}\theta\ \cot\theta - \cos\theta$$

$$\frac{dy}{dx} = \frac{n\,\text{cosec}^n\theta\cot\theta + \sin^{n-1}\theta}{\text{cosec}\theta\cot\theta + \cos\theta}$$

$$\frac{dy}{dx} = \frac{n\,\text{cosec}^n\theta\frac{\cos\theta}{\sin\theta} + \sin^{n-1}\theta\cos\theta}{\text{cosec}\theta\frac{\cos\theta}{\sin\theta} + \cos\theta}$$

$$\frac{dy}{dx} = \frac{n(\text{cosec}^n\theta + \sin^n\theta)}{\text{cosec}\theta + \sin\theta}$$

Also, $x^2 + 4 = (\text{cosec}\theta - \sin\theta)^2 + 4$

$= (\text{cosec}\theta + \sin\theta)^2$

$(x^2 + 4)\left(\frac{dy}{dx}\right)^2 = (\text{cosec}\theta + \sin\theta)^2\left(\frac{n^2(\text{cosec}^n\theta + \sin^n\theta)^2}{(\text{cosec}\theta + \sin\theta)^2}\right)$

$(x^2 + 4) = n^2(\text{cosec}^n\theta + \sin^n\theta)^2$ .....(i)

$n^2(y^2 + 4)\left(\frac{dy}{dx}\right)^2 = n^2[((\text{cosec}^n\theta - \sin^n\theta)^2 + 4)]$

$n^2(y^2 + 4) = n^2(\text{cosec}^n\theta + \sin^n\theta)^2$ .....(ii)

By (i) & (ii)

$\boxed{(x^2+4)\left(\frac{dy}{dx}\right)^2 - n^2(y^2+4) = 0}$

In the above question we have been given

$n = 6$

$\therefore$ we will get, $(x^2 + 4)\left(\frac{dy}{dx}\right)^2 - 36(y^2 + 4) = 0$

Ans $\boxed{\therefore K = 36}$

9. **Ans. (3.00)**

$x = 2cost - cos2t, y = 2\,sint - sin2t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t - 2\cos 2t}{-2\sin t + 2\sin 2t}$

$= \frac{\cos t - \cos 2t}{-\sin t + \sin 2t}$

$= \frac{2\sin\left(\frac{3t}{2}\right)\sin\left(\frac{t}{2}\right)}{2\cos\left(\frac{3t}{2}\right)\sin\left(\frac{t}{2}\right)}$

$\frac{dy}{dx} = \tan\left(\frac{3t}{2}\right)$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$

$= \frac{3}{2}\sec^2\frac{3t}{2} \times \left(\frac{1}{-2\sin t + 2\sin 2t}\right)$

$\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$

$= \frac{3}{2} \times 2 \times \frac{1}{(-2+0)} = \frac{-3}{2}$

10. Ans. (1)

$x=\frac{1}{t^3}+\frac{1}{t^2}$

$\frac{dx}{dt}=\frac{-3}{t^4}-\frac{2}{t^3}\Rightarrow\frac{dy}{dt}=\frac{3}{2}\left(\frac{-2}{t^3}\right)-\frac{2}{t^2}$

$\frac{dy}{dx}=\frac{-\frac{3}{t^3}-\frac{2}{t^2}}{-\frac{3}{t^4}-\frac{2}{t^3}}\Rightarrow\frac{dy}{dx}=t$ so $x\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}=\left(\frac{1+t}{t^3}\right)t^3-t=1$

11. Ans (0)

$y=ae^{mx}+be^{-mx}$; $\therefore\frac{dy}{dx}=ame^{mx}-mbe^{-mx}$

Again $\frac{d^2y}{dx^2}=am^2e^{mx}+m^2be^{-mx}$

$\Rightarrow\frac{d^2y}{dx^2}=m^2(ae^{mx}+be^{-mx})\Rightarrow\frac{d^2y}{dx^2}=m^2y$

or $\frac{d^2y}{dx^2}-m^2y=0$.

12. Ans. (101)

$y=\tan^{-1}\left(\frac{1}{1+x(x+1)}\right)+\tan^{-1}\left(\frac{1}{1+(x+1)(x+2)}\right)+\tan^{-1}\left(\frac{1}{1+(x+2)(x+3)}\right)$ ---------

$y=\tan^{-1}\left(\frac{(x+1)-(x)}{1+(x+2)(x+3)}\right)+\tan^{-1}\left(\frac{(x+2)-(x+1)}{1+(x+1)(x+2)}\right)+\tan^{-1}\left(\frac{(x+3)-(x+2)}{1+(x+2)(x+3)}\right)$ ----------

$y=(\tan^{-1}(x+1)-\tan^{-1}x)+(\tan^{-1}(x+2)-\tan^{-1}(x+1))+(\tan^{-1}(x+3)-\tan^{-1}(x+2))$ ------

$y=\tan^{-1}(x+n)-\tan^{-1}x$

$\frac{dy}{dx}=\frac{1}{1+(x+n)^2}-\frac{1}{1+x^2}$

$n=10$, at $x=0$ we get

$f'(0)=\frac{1}{101}-1\Rightarrow f'(0)+1=\frac{1}{101}$

$\Rightarrow(f'(0)+1)^{-1}=101$ Ans.

13. Ans. (1.00)

$y=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\times\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$

$y=\cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$

$y=\cot^{-1}\left(\frac{1+|\cos x|}{\sin x}\right)$

if $x\in\left(0,\frac{\pi}{2}\right)\Rightarrow\frac{x}{2}\in\left(0,\frac{\pi}{4}\right)$

$y = \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$y = \cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\cos\frac{x}{2}\sin\frac{x}{2}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right)$

$y = \frac{x}{2}$

$\boxed{\frac{dy}{dx} = \frac{1}{2}}$

when $x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \frac{x}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$y = \cot^{-1}\left(\frac{1-\cos x}{\sin x}\right)$

$= \cot^{-1}\left(\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$

$= \cot^{-1}\left(\tan\frac{x}{2}\right)$

$= \frac{\pi}{2} - \tan^{-1}\left(\tan\frac{x}{2}\right)$

$y = \frac{\pi}{2} - \frac{x}{2}$

$\left|\frac{dy}{dx}\right| = \frac{1}{2}$

14. Ans. (2)

$g(x) = e^{ax} + f(x)$

$g'(x) = ae^{ax} + f'(x)$

$g'(0) = ae^0 + f'(0) = a + (-5)$

$= a - 5$

$g''(x) = a^2 e^{ax} + f''(x)$

$g''(0) = a^2 + f''(0) = a^2 + 3$

$g'(0) + g''(0) = 0$

$a - 5 + a^2 + 3 = 0$

$a^2 + a - 2 = 0$

$\boxed{a = -2, 1}$

$\therefore$ there are 2 possible values of 'a'.

Ans. = 2

15. Ans. (3.00)

Let $x^3 = \sin\alpha$; $y^3 = \sin\beta$

The equation become

$\cos\alpha + \cos\beta = a^3(\sin\alpha - \sin\beta)$

$$\Rightarrow 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 2a^3\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow a^3 = \cot\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow 2\cot^{-1}a^3 = \sin^{-1}x^3 - \sin^{-1}y^3$$

$$\Rightarrow 0 = \frac{3x^2}{\sqrt{1-x^6}} - \frac{3y^2}{\sqrt{1-y^6}} \times \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2}\cdot\sqrt{\frac{1-y^6}{1-x^6}}$$

$\Rightarrow \ell = 2, k = 6$

$$\Rightarrow \frac{k}{\ell} = 3$$

EXERCISE - JEE (Main) PYQ

1. Ans. (4)

$$\frac{dx}{dt} = 3\sec^2 t \;;\; \frac{dy}{dt} = 3\sec t\tan t$$

$$\frac{dy}{dx} = \frac{\tan t}{\sec t} = \sin t$$

$$\frac{d^2y}{dx^2} = \cos t\frac{dt}{dx} = \frac{\cos t}{3\sec^2 t} = \frac{\cos^3 t}{3} = \frac{1}{3.2\sqrt{2}} = \frac{1}{6\sqrt{2}}$$

2. Ans. (4)

$$|f(x)| = \begin{cases} 1 & , \; -2 \le x < 0 \\ 1-x^2 & , \; 0 \le x < 1 \\ x^2-1 & , \; 1 \le x \le 2 \end{cases}$$

and $f(|x|) = x^2 - 1, x \in [-2, 2]$

$$\text{Hence } g(x) = \begin{cases} x^2 & , \; x \in [-2,0) \\ 0 & , \; x \in [0,1) \\ 2(x^2-1) & , \; x \in [1,2] \end{cases}$$

It is not differentiable at $x = 1$

3. Ans. (4)

$(2x)^{2y} = 4e^{2x-2y}$

$\Rightarrow 2y\ell n2x = \ell n4 + 2x - 2y$

$$\Rightarrow y = \frac{x+\ell n2}{1+\ell n2x}$$

$$\Rightarrow y' = \frac{(1+\ell n2x) - (x+\ell n2)\frac{1}{x}}{(1+\ell n2x)^2}$$

$$\Rightarrow y'(1+\ell n2x)^2 = \left[\frac{x\ell n2x - \ell n2}{x}\right]$$

4. Ans. (1)

$e^y + xy = e$

differentiate w.r.t. x

$$e^y\frac{dy}{dx}+x\frac{dy}{dx}+y=0 \quad ...(1)$$

$$\frac{dy}{dx}(x+e^y)=-y \Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)}=-\frac{1}{e}$$

again differentiate w.r.t. x in (1)

$$e^y\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot e^y\cdot\frac{dy}{dx}+x\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{dy}{dx}=0$$

$$(x+e^y)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\cdot e^y+2\frac{dy}{dx}=0$$

$$\Rightarrow e\frac{d^2y}{dx^2}+\frac{1}{e^2}e+2\left(-\frac{1}{e}\right)=0 \Rightarrow \frac{d^2y}{dx^2}=\frac{1}{e^2}$$

5. Ans. (4)

$$f(x)=\tan^{-1}\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)$$

$$=\tan^{-1}\left(\frac{\tan x-1}{\tan x+1}\right)=\tan^{-1}\left(\tan\left(x-\frac{\pi}{4}\right)\right)$$

$$\because\ x-\frac{\pi}{4}\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$$

$$\therefore\ f(x)=x-\frac{\pi}{4}$$

$\Rightarrow$ its derivative w.r.t. $\frac{x}{2}$ is $\frac{1}{1/2}=2$

6. Ans. (2)

Put $x = \sin\theta, y = \sin\alpha$

$$y\sqrt{1-x^2}=k-x\sqrt{1-y^2}$$

$\Rightarrow \sin\alpha\cdot\cos\theta + \cos\alpha\cdot\sin\theta = k$

$\Rightarrow \sin(\alpha+\theta) = k \Rightarrow \alpha+\theta = \sin^{-1}k$

$\Rightarrow \sin^{-1}x + \sin^{-1}y = \sin^{-1}k$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-y^2}}\times\frac{dy}{dx}=0$$

at $x=\frac{1}{2}, y=\frac{-1}{4}$ $\quad \frac{dy}{dx}=\frac{-\sqrt{5}}{2}$

7. Ans. (1)

$$y(\alpha)=\sqrt{2\left(\frac{\tan\alpha+\cot\alpha}{1+\tan^2\alpha}\right)+\frac{1}{\sin^2\alpha}},\ \alpha\in\left(\frac{3\pi}{4},\pi\right)$$

$$=\frac{|\sin\alpha+\cos\alpha|}{|\sin\alpha|}=\frac{-(\sin\alpha+\cos\alpha)}{\sin\alpha}=-1-\cot\alpha$$

$y'(\alpha) = \operatorname{cosec}^2\alpha$

$$y'\left(\frac{5\pi}{6}\right)=4$$

8. Ans. (91)

Put $\cos\alpha = \frac{3}{5}, \sin\alpha = \frac{4}{5}, 0 < \alpha < \frac{\pi}{2}$

Now $\frac{3}{5}\cos kx - \frac{4}{5}\sin kx$

$= \cos\alpha.\cos kx - \sin\alpha.\sin kx = \cos(\alpha + kx)$

As we have to find derivate at $x = 0$

We have $\cos^{-1}(\cos(\alpha + kx)) = (\alpha + kx)$

$$\Rightarrow y = \sum_{k=1}^{6} k(\alpha + kx)$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{at\, x=0} = \sum_{k=x}^{6} k^2 = \frac{6\times 7\times 13}{6} = 91$$

9. Ans. (3)

$x^k + y^k = a^k, (a, K > 0)$

$$Kx^{k-1} + ky^{k-1}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} + \left(\frac{x}{y}\right)^{k-1} = 0 \Rightarrow k - 1 = -\frac{1}{3} \Rightarrow k = 2/3$$

10. Ans. (2)

$$\left(a + \sqrt{2}b\cos x\right)\left(a - \sqrt{2}b\cos y\right) = a^2 - b^2$$

$$\Rightarrow a^2 - \sqrt{2}\,ab\cos y + \sqrt{2}\,ab\cos x - 2b^2\cos x\cos y = a^2 - b^2$$

Differentiating both sides :

$$0 - \sqrt{2}\,ab\left(-\sin y\frac{dy}{dx}\right) + \sqrt{2}\,ab(-\sin x) - 2b^2\left[\cos x\left(-\sin y\frac{dy}{dx}\right) + \cos y(-\sin x)\right] = 0$$

At $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$:

$$ab\frac{dy}{dx} - ab - 2b^2\left(-\frac{1}{2}\frac{dy}{dx} - \frac{1}{2}\right) = 0$$

$$\Rightarrow \frac{dx}{dy} = \frac{ab + b^2}{ab - b^2} = \frac{a+b}{a-b}; a, b > 0$$

11. Ans. (39)

$$f(x) = \begin{cases} \frac{P(x)}{\sin(x-2)} & , \quad x \neq 2 \\ 7 & , \quad x = 2 \end{cases}$$

$P''(x)$ = const. $\Rightarrow P(x)$ is atmost 2nd degree polynomial

$f(x)$ is cont. at $x = 2$

$f(2^+) = f(2^-) = f(2)$

$$\lim_{x\to 2} \frac{P(x)}{\sin(x-2)} = 7$$

$\lim\limits_{x\to 2}\frac{(x-2)(ax+b)}{\sin(x-2)}=7 \Rightarrow 2a+b=7$ ...(1)

$P(x) = (x-2)(ax+b)$

$P(3) = (3-2)(3a+b) = 9 \Rightarrow 3a+b=9$ ...(2)

from (1) & (2) $a=2, b=3$

$P(5) = (5-2)(2(5)+3) = 3(13) = 39$

12. Ans. (3)

$f(x)=\cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$ $(0 < x < 1)$

$\cot^{-1}\sqrt{\frac{1-x}{x}}=\sin^{-1}\sqrt{x}$

$\therefore\ f(x)=\cos(2\tan^{-1}\sqrt{x}) = \cos\left(\tan^{-1}\left(\frac{2\sqrt{x}}{1-x}\right)\right)$

$f(x) = \frac{1-x}{1+x}$

Now $f'(x) = \frac{-2}{(1+x)^2}$

$\therefore f'(x)(1-x)^2 = -2\left(\frac{1-x}{1+x}\right)^2$

$\therefore (1-x)^2 f'(x) + 2\left(f(x)\right)^2 = 0.$

13. Ans. (1)

$y(x)=\cot^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right]$ $\left(\because \frac{x}{2}\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)\right)$

$y(x)=\cot^{-1}\left(\tan\frac{x}{2}\right)=\frac{\pi}{2}-\frac{x}{2}$

$y'(x)=\frac{-1}{2}$

14. Ans. (1)

$ln f(x+1) = ln(xf(x))$

$ln f(x+1) = lnx + ln f(x)$

$\Rightarrow g(x+1) = lnx + g(x)$

$\Rightarrow g(x+1) - g(x) = lnx$

$\Rightarrow g''(x+1) - g''(x) = -\frac{1}{x^2}$

Put x = 1, 2, 3, 4

$g''(2) - g''(1) = -\frac{1}{1^2}$ ...(1)

$g''(3) - g''(2) = -\frac{1}{2^2}$ ...(2)

$g''(4) - g''(3) = -\frac{1}{3^2}$ ...(3)

$g''(5) - g''(4) = -\frac{1}{4^2}$ ...(4)

Add all the equation we get

$g''(5) - g''(1) = -\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2}$

$|g''(5) - g''(1)| = \frac{205}{144}$

15. **Ans. (40)**

$\ell n(x+y) = 4xy$ (At $x = 0, y = 1$)

$x + y = e^{4xy}$

$\Rightarrow 1 + \frac{dy}{dx} = e^{4xy}\left(4x\frac{dy}{dx} + 4y\right)$

At $x = 0$ $\boxed{\frac{dy}{dx} = 3}$

$\frac{d^2y}{dx^2} = e^{4xy}\left(4x\frac{dy}{dx} + 4y\right)^2 + e^{4xy}\left(4x\frac{d^2y}{dx^2} + 4y\right)$

At $x = 0$, $\frac{d^2y}{dx^2} = e^0(4)^2 + e^0(24)$

$\Rightarrow \frac{d^2y}{dx^2} = 40$

16. **Ans. (4)**

$\log_e 2 \frac{d}{dx}$ $(\log_{\cos x} \operatorname{cosec} x)$

Let,

$y = \log_{\cos x} \operatorname{cosec} x = -\frac{\ell n(\sin x)}{\ell n(\cos x)}$

$\frac{dy}{dx} = -\frac{[\cot x \cdot \ell n(\cos x) + \tan x \cdot \ell n(\sin x)]}{(\ell n(\cos x))^2}$

$\left.\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \frac{4}{\ell n 2}$

Now, $\log_e 2 \cdot \frac{4}{\ell n 2} = 4$

17. Ans. (4)

$$x=2\sqrt{2}\cos t\sqrt{\sin 2t} \Rightarrow \frac{dx}{dt}=\frac{2\sqrt{2}\cos 3t}{\sqrt{\sin 2t}}$$

$$y(t)=2\sqrt{2}\sin t\sqrt{\sin 2t} \Rightarrow \frac{dy}{dt}=\frac{2\sqrt{2}\sin 3t}{\sqrt{\sin 2t}}$$

$$\therefore \frac{dy}{dx}=\tan 3t$$

$$\frac{dy}{dx}=-1 \text{ at } t=\frac{\pi}{4}$$

$$\frac{d^2y}{dx^2}=\frac{3}{2\sqrt{2}}\sec^3 3t\cdot\sqrt{\sin 2t}=-3 \text{ at } t=\frac{\pi}{4}$$

$$\therefore \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}=\frac{1+1}{-3}=-\frac{2}{3}$$

18. Ans. (2)

$y=\tan^{-1}(\sec x^3-\tan x^3)$

$$=\tan^{-1}\left(\frac{1-\sin x^3}{\cos x^3}\right)=\tan^{-1}\left(\frac{1-\cos\left(\frac{\pi}{2}-x^3\right)}{\sin\left(\frac{\pi}{2}-x^3\right)}\right)=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x^3}{2}\right)\right)$$

Since $\frac{\pi}{4}-\frac{x^3}{2}\in\left(-\frac{\pi}{2},0\right)$

$$y=\left(\frac{\pi}{4}-\frac{x^3}{2}\right)$$

$$y'=\frac{-3x^2}{2},\ y''=-3x$$

$4y=\pi-2x^3$

$$\Rightarrow 4y=\pi-2x^2\left(\frac{-y''}{3}\right)$$

$$\Rightarrow 12y=3\pi+2x^2y'' \Rightarrow x^2y''-6y+\frac{3\pi}{2}=0$$

19. Ans. (3)

$$f(x)=\begin{vmatrix} a & -1 & 0\\ ax & a & -1\\ ax^2 & ax & a\end{vmatrix}=a\begin{vmatrix} 1 & -1 & 0\\ x & a & -1\\ x^2 & ax & a\end{vmatrix}$$

$=a\,[1(a^2+ax)+1(ax+x^2)]$

$\Rightarrow f(x)=a(x+a)^2$

so, $f'(x) = 2a(x + a)$

as, $2f'(10) - f'(5) + 100 = 0$

$\Rightarrow 2 \times 2a(10 + a) - 2a(5 + a) + 100 = 0$

$\Rightarrow 40a + 4a^2 - 10a - 2a^2 + 100 = 0$

$2a^2 + 30a + 100 = 0$

$\Rightarrow a^2 + 15a + 50 = 0$

$(a + 10)\ (a + 5) = 0$

$a = -10$ or $a = -5$

Required = $(-10)^2 + (-5)^2 = 125$

20. **Ans. (16)**

$y(x) = (x^x)^x$

$\ell n\, y(x) = x^2 \cdot \ell n\, x$

$\frac{1}{y(x)} \cdot y'(x) = \frac{x^2}{x} + 2x \cdot \ell n\, x$

$y'(x) = y(x)[x + 2x\, \ell n\, x]$

$y(1) = 1; y'(1) = 1$

$y''(x) = y'(x)[x + 2x \cdot \ell n(x)] + y(x)\ [1 + 2(1 + \ell n\, x)]$

$y''(1) = 1[1 + 0] + 1(1 + 2) = 4$

$$\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^3 \cdot \frac{d^2x}{dy^2}$$

$$\Rightarrow 4 = -\frac{d^2x}{dy^2}$$

$$\frac{d^2x}{dy^2} = -4$$

Ans. $-4 + 20 = 16$

21. **Ans. (3)**

$y = x^x$

$\Rightarrow\ y' = x^x(1 + \ell nx)$

$\Rightarrow\ y'' = x^x\ (1 + \ell nx)^2 + x^x \cdot \frac{1}{x}$

$y''\ (2) = 4(1 + \ell n2)^2 + 2$

$y'\ (2) = 4(1 + \ell n2)$

$y''\ (2) - 2y'(2) = 4(1 + \ell n2)^2 + 2 - 8(1 + \ell n2)$

$= 4(1 + \ell n2)\ [1 + \ell n2 - 2] + 2 = 4((\ell n2)^2 - 1) + 2 = 4(\ell n2)^2 - 2$

22. **Ans. (2)**

$y = \sin^3(\pi/3\cos g(x))$

$$g(x) = \frac{\pi}{3\sqrt{2}}\left(-4x^3 + 5x^2 + 1\right)^{3/2}$$

$g(1) = 2\pi/3$

$$y'=3\sin^2\left(\frac{\pi}{3}\cos g(x)\right)\times\cos\left(\frac{\pi}{3}\cos g(x)\right)\times\frac{\pi}{3}(-\sin g(x))g'(x)$$

$$y'(1)=3\sin^2\left(-\frac{\pi}{6}\right).\cos\left(\frac{\pi}{6}\right).\frac{\pi}{3}\left(-\sin\frac{2\pi}{3}\right)g'(1)$$

$$g'(x)=\frac{\pi}{2\sqrt{2}}\left(-4x^3+5x^2+1\right)^{1/2}\left(-12x^2+10x\right)$$

$$g'(1)=\frac{\pi}{2\sqrt{2}}\left(\sqrt{2}\right)(-2)=-\pi$$

$$y'(1)=\frac{\cancel{3}}{4}.\frac{\sqrt{3}}{2}.\frac{\pi}{\cancel{3}}\left(\frac{-\sqrt{3}}{2}\right)(-\pi)=\frac{3\pi^2}{16}$$

$$y(1)=\sin^3\left(\pi/3\cos 2\pi/3\right)=-\frac{1}{8}$$

$$2y'(1)+3\pi^2y(1)=0$$

23. **Ans. (3)**

$$y=\frac{1-x^{32}}{1-x}\Rightarrow y-xy=1-x^{32}$$

$$y'-xy'-y=-32x^{31}$$

$$y''-xy''-y'-y'=-(32)(31)x^{30}$$

at $x=-1\Rightarrow y'-y''=496$

24. **Ans. (2)**

$$f(x)=\frac{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x-1}{\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\cos x}=\frac{\sin\left(x+\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{2}\right)}{\sin\left(x-\frac{\pi}{4}\right)}$$

$$=\frac{2\sin\left(\frac{x}{2}-\frac{\pi}{8}\right)\cos\left(\frac{x}{2}+\frac{3\pi}{8}\right)}{2\sin\left(\frac{x}{2}-\frac{\pi}{8}\right)\cos\left(\frac{x}{2}-\frac{\pi}{8}\right)}=\frac{\sin\left(\frac{\pi}{2}-\frac{x}{2}-\frac{3\pi}{8}\right)}{\cos\left(\frac{x}{2}-\frac{\pi}{8}\right)}=-\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)$$

$$f'(x)=-\frac{1}{2}\sec^2\left(\frac{x}{2}-\frac{\pi}{8}\right)$$

$$f''(x)=-\frac{1}{2}\sec^2\left(\frac{x}{2}-\frac{\pi}{8}\right).\tan\left(\frac{x}{2}-\frac{\pi}{8}\right)$$

$$f\left(\frac{7\pi}{12}\right)=-\tan\left(\frac{7\pi}{24}-\frac{\pi}{8}\right)$$

$$f''\left(\frac{7\pi}{12}\right)=\frac{-1}{2}\sec^2\frac{\pi}{6}\tan\frac{\pi}{6}$$

$$f\left(\frac{7\pi}{12}\right).f''\left(\frac{7\pi}{12}\right)=\frac{1}{2}\times\frac{4}{3}\times\frac{1}{3}=\frac{2}{9}$$

25. **Ans. (2)**

Let $t = x^y$

$\ell nt = y\ell nx$

$$\frac{1}{t}t' = \frac{y}{x} + y\ell nx$$

$$\Rightarrow t' = x^y\left(\frac{y}{x} + y'\ell nx\right)$$

Let $y^x = \mu$

$x\ell ny = \ell n\mu$

$$\frac{x}{y}y' + \ell ny = \frac{1}{\mu}\cdot\mu'$$

$$\Rightarrow \mu' = y^x\left(\frac{y'x}{y} + \ell ny\right)$$

Now, $2x^y + 3y^x = 20$

$$2x^y\left[\frac{y}{x} + (\ell nx)\,y'\right] + 3y^x\left[\frac{xy'}{y} + \ell ny\right] = 0$$

$$y' = \frac{-(12\ell n2 + 8)}{12 + 8\ell n2} = -\left(\frac{2 + \log_e 8}{3 + \log_e 4}\right)$$

EXERCISE - JEE (Advanced) PYQ

1. **Ans. (A)**

$\because g(x) = \ell n(f(x))$

$\Rightarrow g(x+1) = \ell n(f(x+1))$

$\Rightarrow g(x+1) = \ell n(x.f(x))$

$\Rightarrow g(x+1) = \ell n(x) + \ell n(f(x))$

$\Rightarrow g(x+1) = \ell n(x) + g(x)$

$\Rightarrow g(x+1) - g(x) = \ln(x)$

Differ. w.r.t.(x) two times

$$\Rightarrow g''(x+1) - g''(x) = -\frac{1}{x^2}$$

put

$$\left.\begin{array}{ll} x = \frac{1}{2} & g''\left(1+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -\frac{4}{1} \\ x = 1+\frac{1}{2} & g''\left(2+\frac{1}{2}\right) - g''\left(1+\frac{1}{2}\right) = -\frac{4}{9} \\ x = 2+\frac{1}{2} & g''\left(3+\frac{1}{2}\right) - g''\left(2+\frac{1}{2}\right) = \frac{-4}{25} \\ x = (N-1)+\frac{1}{2} & g''\left(N+\frac{1}{2}\right) - g''\left(N-\frac{1}{2}\right) = -\frac{4}{(2N-1)^2} \end{array}\right\}$$

Add

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left[1 + \frac{1}{9} + \frac{1}{25} + \ldots + \frac{1}{(2N-1)^2}\right]$$

2. Ans. (A)

$f(x) = g(x)\sin x \Rightarrow f'(x) = g'(x)\sin x + g(x)\cos x$

At $x = 0$; $f'(0) = g'(0).0 + g(0).1$

$\Rightarrow f'(0) = g(0) \Rightarrow$ st. 2 is correct.

For Statement-1:

L.H.S. $\lim_{x\to 0}\frac{g(x)\cos x}{\sin x} - \frac{g(0)}{\sin x} = \lim_{x\to 0}\frac{g(x)\cos x - g(0)}{\sin x}$

Add and subtract $g'(x)\sin x$ on both sides:

$$\therefore\ L = \lim_{x\to 0}\frac{g(x)\cos x + g'(x)\sin x - g'(x)\sin x - g(0)}{\left(\frac{\sin x}{x}\right)x}$$

$$\Rightarrow L = \lim_{x\to 0}\left(\frac{f'(x) - f'(0) - g'(x)\sin x}{x}\right)\left(\frac{x}{\sin x}\right)\ (\because f'(0) = g(0))$$

$$\Rightarrow L = \lim_{x\to 0}\left(\frac{f'(0+x) - f'(0)}{x} - g'(x)\frac{\sin x}{x}\right)\left(\frac{x}{\sin x}\right)$$

$\Rightarrow L = f''(0) - g'(0) \Rightarrow L = f''(0)$

3. Ans. (2)

$\because\ g(x) = f^{-1}(x)$

$\therefore g(f(x)) = x$

$$\Rightarrow g'(f(x)) \cdot f'(x) = 1 \Rightarrow g'(f(x)) = \frac{1}{f'(x)} \Rightarrow g'(x^3 + e^{\frac{x}{2}}) = \frac{1}{3x^2 + \frac{1}{2}.e^{x/2}}$$

put $x = 0$

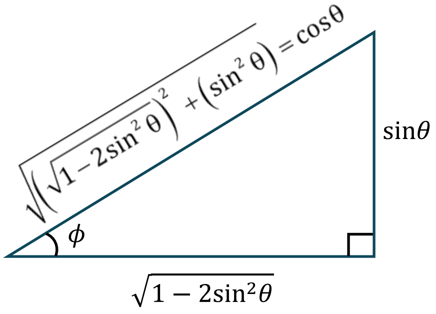
$\Rightarrow g'(1) = 2$

4. Ans. (1)

$$\because\ f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$$

Let $\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right) = \phi$

$$\Rightarrow \frac{\sin\theta}{\sqrt{1 - 2\sin^2\theta}} = \tan\phi$$



$\Rightarrow f(\theta) = \sin(\phi) = \frac{\sin\theta}{\cos\theta}$ {from Δ}

$\Rightarrow f(\theta) = \tan\theta$

Now, $\frac{d}{d(\tan\theta)}f(\theta) = \frac{d(\tan\theta)}{d(\tan\theta)} = 1$

5. **Ans. (B, C)**

Given

$\because \; g(f(x)) = x$

$\Rightarrow g'(f(x)) \cdot f'(x) = 1 \Rightarrow \; g'(f(x)) \; = \dfrac{1}{f'(x)}$

$\Rightarrow g'(x^3 + 3x + 2) = \dfrac{1}{3x^2+3}$

put $x = 0$ we get $g'(2) = \dfrac{1}{3}$

option A is wrong

* $h(g(g(x))) \; = \; x$; given

Put $x = f(x)$

$\Rightarrow h(g(g(f(x)))) = f(x)$; given $g(f(x)) = x$

$\Rightarrow \; h(g(x)) = f(x)$ ...(1)

put $x = 3$ $\; h(g(3)) = f(3) = 3^3 + 3 \times 3 + 2 \Rightarrow \; h(g(3)) = 38$

put $x = f(x)$ in eq. (1)

$\Rightarrow \; h(\underbrace{g(f(x))}_{x}) = f(f(x))$

$\Rightarrow \; h(x) = f(f(x))$

Now, put $x = 0$ $\; h(0) = f(\underbrace{f(0)}_{2}) = f(2) = 16 \Rightarrow \; h(0) = 16$

$\Rightarrow h'(x) = f'(f(x)) \cdot f'(x)$

$\Rightarrow h'(x) = f'(x^3 + 3x + 2) \cdot (3x^2 + 3)$

put $x = 1$

$\Rightarrow h'(1) = f'(6) \cdot 6$

$\Rightarrow h'(1) = 111 \times 6 \Rightarrow \; h'(1) = 666$

JEE (Main) Practice Paper

SECTION–A

1. **Ans. (4)**

$f(x) = \log_{x^2}(\ell nx) = \dfrac{\ell n\ell nx}{\ell nx^2} = \dfrac{\ell n\ell nx}{2\ell nx}$

$f'(x) = \dfrac{\dfrac{1}{2}\left[\dfrac{\ell nx}{\ell nx} \times \dfrac{1}{x} - \ell n\ell nx\dfrac{1}{x}\right]}{(\ell nx)^2}$

$f'(e) = \dfrac{1}{2}\left(\dfrac{1}{e} - \dfrac{1}{e}\ell n\ell ne\right) = \dfrac{1}{2e}$

2. Ans. (1)

$$f(x)=\frac{2\sin x.\cos x.\cos 2x.\cos 4x.\cos 8x.\cos 16x}{2\sin x}=\frac{\sin 32x}{2^5\sin x}$$

$$\therefore\ f'(x)=\frac{1}{32}.\frac{32\cos 32x.\sin x-\cos x.\sin 32x}{\sin^2 x}$$

$$\therefore\ f'\left(\frac{\pi}{4}\right)=\frac{32\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}.0}{32\left(\frac{1}{\sqrt{2}}\right)^2}=\sqrt{2}$$

3. Ans. (2)

$$\frac{dy}{dx}=\frac{1}{\log_e 10}\left[\frac{1}{\sin^{-1}x^2}\right].\frac{1}{\sqrt{1-x^4}}.2x$$

$$\left[\frac{dy}{dx}\right]_{x=\frac{1}{\sqrt{2}}}=\frac{1}{\log_e 10}\left[\frac{1}{\sin^{-1}\frac{1}{2}}\right].\frac{1}{\sqrt{1-\frac{1}{4}}}.2\times\frac{1}{\sqrt{2}}=\frac{\log_{10}e}{\pi}\times\frac{4\times\sqrt{3}}{2}\times\sqrt{2}=\frac{4\sqrt{6}}{\pi}\log_{10}e$$

4. Ans. (1)

$$u=f\left(x^3\right)\Rightarrow\frac{du}{dx}=f'\left(x^3\right).3x^2=\cos x^3.3x^2$$

$$v=g\left(x^2\right)=\frac{dv}{dx}=g'\left(x^2\right).2x=\sin x^2.2x$$

$$\frac{du}{dv}=\frac{3x}{2}\cos x^3.\cos ecx^2$$

5. Ans. (2)

$$y=\sin^n x\cos nx$$

$$\frac{dy}{dx}=-\sin^n x.n\sin nx+\cos nx.n\sin^{n-1}x\cos x$$

$$=n\sin^{n-1}x\left(-\sin x.\sin nx+\cos nx\cos x\right)$$

$$=n\sin^{n-1}x\cos\left(nx+x\right)$$

6. Ans. (3)

$$x\sin\alpha\cos y+x\cos\alpha\sin y=\sin y$$

$$x\sin\alpha\,\text{coty}+x\cos\alpha=1$$

$$\cot y=\frac{1-x\cos\alpha}{x\sin\alpha}$$

$$\tan y=\frac{x\sin\alpha}{1-x\cos\alpha}$$

$$\sec^2 y.\frac{dy}{dx}=\frac{\sin\alpha\left(1-x\cos\alpha\right)-x\sin\alpha\left(-\cos\alpha\right)}{\left(1-x\cos\alpha\right)^2}$$

$$\sec^2 y.\frac{dy}{dx}=\frac{\sin\alpha-\text{x}\sin\alpha\cos\alpha+x\sin\alpha\cos\alpha}{(1-x\cos\alpha)^2}$$

$$\sec^2 y\frac{dy}{dx}=\frac{\sin\alpha}{(1-x\cos\alpha)^2}$$

$m=\sin\alpha$

$n=-\cos\alpha$

$m^2+n^2=1$

7. **Ans. (3)**

$$y=f\left(\frac{2x-1}{x^2+1}\right)$$

$$\frac{dy}{dx}=f'\left(\frac{2x-1}{x^2+1}\right).\frac{(x^2+1).2-(2x-1).2x}{(x^2+1)^2}$$

$$=\sin\left(\frac{2x-1}{x^2+1}\right)^2.\frac{2x^2+2-4x^2+2x}{(x^2+1)^2}$$

8. **Ans. (2)**

$$5f(x)+3f\left(\frac{1}{x}\right)=x+2 \qquad \text{...(i)}$$

Replacing x by $\frac{1}{x}$

$$\therefore\ 5f\left(\frac{1}{x}\right)+3f(x)=\frac{1}{x}+2 \qquad \text{...(ii)}$$

From (i),

$$25f(x)+15f\left(\frac{1}{x}\right)=5x+10 \qquad \text{...(iii)}$$

And from (ii),

$$9f(x)+15f\left(\frac{1}{x}\right)=\frac{3}{x}+6 \qquad \text{...(iv)}$$

Subtracting (iv) form (iii),

$$\therefore\ 16f(x)=5x-\frac{3}{x}+4$$

$$\therefore\ xf(x)=\frac{5x^2-3+4x}{16}=y$$

$$\therefore\ \frac{dy}{dx}=\frac{10x+4}{16}$$

$$\left.\frac{dy}{dx}\right|_{x=1}=\frac{10+4}{16}=\frac{7}{8}$$

9. Ans. (1)

$$y=\frac{(1-x)(1+x)(1+x^2).....(1+x^{2n})}{1-x}=\frac{1-x^{4n}}{1-x}$$

$$=\frac{dy}{dx}=\frac{-4n(x^{4n-1})\cdot(1-x)+(1-x^{4n})}{(1-x)^2}$$

$$\therefore \text{ at } x=0,\frac{dy}{dx}=\frac{-4n(0)(1-0)+(1-0)}{(1-0)^2}=1$$

10. Ans. (1)

$$y=x^2+\frac{1}{y}\Rightarrow y^2=x^2y+1\Rightarrow y^2-x^2y-1=0$$

$$\frac{dy}{dx}=\left(\frac{2xy}{2y-x^2}\right)=\left(\frac{2xy}{2y-\left(y-\frac{1}{y}\right)}\right)=\frac{2xy^2}{y^2+1}$$

11. Ans. (2)

$$x=\frac{\sin y}{\sin(a+y)}$$

$$\frac{dx}{dy}=\frac{\sin(a+y)\cos y-\sin y\cos(a+y)}{\sin^2(a+y)}$$

$$=\frac{\sin((a+y)-y)}{\sin^2(a+y)}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$$

12. Ans. (3)

$$y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+.....}}}$$

$$\Rightarrow y=\sqrt{\sin x+y}\Rightarrow y^2=y+\sin x$$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

13. Ans. (3)

$$f(x)=\sqrt{1-\sin 2x}$$

$$=\sqrt{(\sin x-\cos x)^2}=|\sin x-\cos x|$$

$$\text{If } x\in\left(0,\frac{\pi}{4}\right)\Rightarrow f(x)=-(\sin x-\cos x)$$

$$\Rightarrow f'(x)=-\cos x-\sin x=-(\sin x+\cos x) \text{ if } x\in\left(0,\frac{\pi}{4}\right)$$

14. **Ans. (1)**

$f(x) = e^x + 2x$

$f'(x) = e^x + 2 \Rightarrow f'(\ell n3) = 5$

$g(f(x)) = x \Rightarrow g'(f(x)) = \dfrac{1}{f'(x)}$

Put $x = \ell n3, g'(f(\ell n3)) = \dfrac{1}{5}$

15. **Ans. (2)**

$4^x - 8^y = 4 \Rightarrow 4^x \ell n4 - 8^y \ell n8.y' = 0$

$\Rightarrow y' = \dfrac{2}{3}\dfrac{4^x}{8^y} = \dfrac{2}{3}\left(\dfrac{4^x}{4^x - 4}\right)$

$\Rightarrow \lim\limits_{x\to\infty} y' = \dfrac{2}{3}$

16. **Ans. (2)**

We have $e^{xy^2} + y\cos\left(x^2\right) = 5$ ...(i)

Put $x = 0$, we get

$1 + y = 5 \Rightarrow y = 4 \Rightarrow (0,4)$ lies on the given curve.

Now, differentiating (i) with respect to x, we get

$\Rightarrow e^{xy^2}\left[x.2y.\dfrac{dy}{dx} + y^2\right] - y.2x.\sin\left(x^2\right) + \cos\left(x^2\right)\dfrac{dy}{dx} = 0$

As (0,4) satisfy it, we get $16 + \left.\dfrac{dy}{dx}\right]_{(0,4)} = 0 \Rightarrow \left.\dfrac{dy}{dx}\right]_{(0,4)} = -16$

17. **Ans. (3)**

$f'\left(0^+\right) = \lim\limits_{h\to 0}\dfrac{h^m \sin\frac{1}{h}}{h}$ must exist, $\Rightarrow m > 1$ and $f'(0) = 0$

For $m > 1$ $f'(x) = \begin{cases} mx^{m-1}\sin\dfrac{1}{x} - x^{m-2}\cos\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0 \quad , & \text{if } x = 0 \end{cases}$

Now $\lim\limits_{h\to 0} f'(0+h) = \lim\limits_{h\to 0} mh^{m-1}\sin\dfrac{1}{h} - h^{m-2}\cos\dfrac{1}{h}$

Limit exist if $m > 2$

$\therefore m \in N \Rightarrow m = 3$

18. **Ans. (3)**

$\because g(x) = \left(f\left(3f(x)+6\right)\right)^3 \Rightarrow g'(x) = 3\left(f\left(3f(x)+6\right)\right)^2 f'\left(3f(x)+6\right).3f'(x)$

$\therefore g'(0) = 3\left(f\left(3f(0)+6\right)\right)^2.f'\left(3f(0)+6\right).3f'(0)$

$= 9\left(f(-6+6)\right)^2 f'(-6+6)f'(0)$

$= 9\left(f(0)\right)^2\left(f'(0)\right)^2 = 9\times 4\times 1 = 36$

19. **Ans. (2)**

$y = 10^{10^x}$

$\ell ny = 10^x \ell n10$

d.w.r. to x

$\frac{1}{y}\frac{dy}{dx} = (\ell n10)^2 10^x$

$d = (\ell n10)^2$

20. **Ans. (4)**

Let degree of $f(x)$ be n

$n - 1 = n - 2 + n - 3$

$n = 4$

Let $f(x)$ be = $ax^4 + bx^3 + cx^2 + dx + e$

$(32ax^3 + 12bx^2$ ......)

= $(108ax^2 + 18bx + 2c).(24ax + 6b)$

coefficient of x^3

$32a = 108a.24a$ $\quad (a \neq 0)$

$a = \frac{1}{81}$

SECTION–B

1. **Ans. (9)**

We have, $x = \sec\theta - \cos\theta$

$\Rightarrow \frac{dx}{d\theta} = \sec\theta.\tan\theta + \sin\theta$

$\therefore \frac{dx}{d\theta} = \tan\theta.(\sec\theta + \cos\theta)$

Also $y = \sec^3\theta - \cos^3\theta$

$\Rightarrow \frac{dy}{d\theta} = 3\sec^2\theta.\sec\theta.\tan\theta + 3\cos^2\theta.\sin\theta = 3\tan\theta(\sec^3\theta + \cos^3\theta)$

So, $\left(\frac{dy}{dx}\right)^2 = \left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)^2 = \frac{9\tan^2\theta(\sec^3\theta + \cos^3\theta)^2}{\tan^2\theta(\sec\theta + \cos\theta)^2}$

So, $\left(\frac{dy}{dx}\right)^2_{x=0} \Rightarrow \left(\frac{dy}{dx}\right)^2_{at\ \theta = n\pi(n \in I)} = 9$

2. **Ans. (15)**

$\because \left(\frac{fg}{f-g}\right)' = \frac{(f-g)(f'g + fg') - fg(f' - g')}{(f-g)^2}$

$\left(\frac{fg}{f-g}\right)(6) = \frac{2(6+12) - 3(2)}{2^2} = \frac{30}{4} = \frac{15}{2}$

3. **Ans. (2)**

Let degree of $P(x)$ be n

$\therefore$ degree of $P'(x)$ be $n-1$

Degree of L.H.S. = degree of R.H.S.

$\therefore n = 2$

Let $P(x) = ax^2 + bx + c$

$\therefore ax^2 + bx + C - (2ax + b) = x^2 + 2x + 1$

$\therefore a = 1, b - 2a = 2 \Rightarrow b = 4$

And $c - b = 1 \Rightarrow c = 5$

$\therefore P(x) = x^2 + 4x + 5$

$\therefore P(-1) = 2$

4. **Ans. (8)**

$f(x) = 2\tan^{-1}x$ & $g(x) = x + 2 \Rightarrow f(g(x)) = 2\tan^{-1}(x+2)$ solution of inequality

$f^2(g(x)) - 5f(g(x)) + 4 > 0$ is

$f(g(x)) < 1$ or $f(g(x)) > 4 \Rightarrow \tan^{-1}(x+2) < \frac{1}{2}$ or $\tan^{-1}(x+2) > 2$

$\Rightarrow \tan^{-1}(x+2) < \frac{1}{2} \quad \left[As \tan^{-1}(x+2) < \frac{\pi}{2}\right]$

Or $x + 2 < \tan\left(\frac{1}{2}\right)$

$\Rightarrow x \in \left(-10, \tan\left(\frac{1}{2}\right) - 2\right)$

As $\frac{1}{2} < \frac{\pi}{6} \Rightarrow \tan\frac{1}{2} < \frac{1}{\sqrt{3}} \Rightarrow \tan\frac{1}{2} - 2 < \frac{1}{\sqrt{3}} - 2$

Hence total integer in the range are $\{-9, -8, -7, -6, -5, -4, -3, -2\} \Rightarrow 8$ integer

5. **Ans. (3)**

$$f(x) = \frac{x^2 + mx + 3}{x - 2}$$

$$f'(x) = \frac{(x-2)(2x+m) - (x^2 + mx + 3)}{(x-2)^2} = \frac{(2x^2 + mx - 4x - 2m) - (x^2 + mx + 3)}{(x-2)^2}$$

$$f'(x) = \frac{x^2 - 4x - 2m - 3}{(x-2)^2} > 0 \forall x > 3$$

Hence, $g(x) = x^2 - 4x - 2m - 3 > 0 \ \forall\ x > 3$

Now, $g(3) = 9 - 12 - 2m - 3 \geq 0$

$\Rightarrow 2m + 6 \leq 0$

Hence, $m \leq -3$

$\therefore a = 3$

6. Ans. (6)

$f''(0)=2;\ \lim_{x\to 0}\frac{f(5x)-3f(3x)+2f(2x)}{x^2}$

Applying L.hospital twice

$\lim_{x\to 0}\frac{25f''(5x)-27f''(3x)+8f''(2x)}{2}$

$=\frac{6f''(0)}{2}=6$

7. Ans. (5)

Give $\lim_{h\to 0}\frac{3f(h)-1}{2h}=2$

$\lim_{h\to 0}\frac{f(h)-\frac{1}{3}}{h}=\frac{4}{3}$

$\therefore$ for limit to exist $f(0)=\frac{1}{3}$ and $f'(0)=\frac{4}{3}$

Put $x = y = 0$ in functional relations we get, $\mu=f(0)=\frac{1}{3}$

Now, $f(x + y) = f(x) + f(y) + \lambda y(x^2 + xy)$

Differentiate w.r.t. x keeping y constant

$f'(x + y) = f'(x) + \lambda y(2x + y)$

put $x = 0$

$f'(y) = f'(0) + \lambda y^2 \quad \therefore f'(x)=2.\ \frac{2}{3} + \lambda x^2$

$f(x)=\frac{4}{3}x+\frac{\lambda x^3}{3}+C,\ f(0)=\frac{1}{3}\Rightarrow C=\frac{1}{3}$

$f(x)=\frac{4}{3}x+\frac{\lambda x^3}{3}+\frac{1}{3}$

$f(2)=\frac{25}{3}$

$8+8\lambda+1=25$

$8\lambda=16\Rightarrow\lambda=2\ \therefore\lambda+\frac{1}{\mu}=5$

8. Ans. (1)

When $x = \pi$, $y = 1$

$\frac{-4\sin x-3\cos x+y'\sin x+y\cos x}{2\sqrt{4\cos x-3\sin x+5+y\sin x}}$

$=y'\cos 2x-2\sin 2x.y$

Put $x = \pi$ & $y = 1 \Rightarrow\ y'(\pi) = 1 \Rightarrow y'(\pi) = 1$

9. Ans. (4)

$y = x^{n-1} \ell n\, x$

$$y_1 = \frac{x^{n-1}}{x} + (n-1)x^{n-2}\ln x$$

$$y_1 = x^{n-2} + (n-1)\cdot\frac{y}{x} \qquad ...(1)$$

$xy_1 = x^{n-1} + (n-1)y$

$xy_2 + y_1 = (n-1)\, x^{n-2} + (n-1)y_1 \qquad ....(2)$

$$xy_2 + y_1 = (n-1)(y_1 - (n-1)\frac{y}{x}) + (n-1)y_1$$

$$xy_2 + y_1 = (n-1)y_1 - (n-1)^2\frac{y}{x} + (n-1)y_1$$

$x^2y_2 + xy_1 = (n-1)xy_1 - (n-1)^2y + (n-1)xy_1$

$x^2y_2 + xy_1(1-n+1-n+1) + (n-1)^2 y = 0$

$x^2y_2 + xy_1(3-2n) + (n-1)^2 y = 0$

$f(n) = 3 - 2n$

$g(n) = (n-1)^2$

$f(4) + g(4) = -5 + 9 = 4$

10. Ans. (7)

$|f(x)| \le |e^x - e^2|$

$x = 2$

$|f(2)| \le 0 \Rightarrow \quad f(2) = 0$

$f(x) = ax^3 + bx^2 + cx + 5$

$f'(2) = 12a + 4b + c$

$$\lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = 12a + 4b + c$$

$$\lim_{x\to 2}\left|\frac{f(x)}{x-2}\right| = |12a+4b+c| \le \lim_{x\to 2}\left|\frac{e^x - e^2}{x-2}\right|$$

use L'Hôpital

$|12a + 4b + c| \le |e^2|$

$\ell = e^2, \quad [\ell] = 7$

JEE (Advanced) Practice Paper

1. Ans. (C)

$$g'(x)=f'\left(x-\sqrt{1-x^2}\right).\left(1+\frac{2x}{2\sqrt{1-x^2}}\right) \quad f'(x)=1-x^2$$

$$=\left[1-\left(x^2+1-x^2-2x\sqrt{1-x^2}\right)\right]\frac{(x+\sqrt{1-x^2})}{\sqrt{1-x^2}}$$

$$=2x\left(x+\sqrt{1-x^2}\right)$$

2. Ans. (B)

$$g(x)=f^{-1}(x) \qquad f(x)=(2x-3\pi)^5+\frac{4}{3}x+\cos x$$

$$g(f(x))=x \qquad x=\frac{3\pi}{2}$$

$$g'(f(x))f'(x)=1 \qquad f\left(\frac{3\pi}{2}\right)=2\pi$$

$$f'(x)=5(2x-3\pi)^4\cdot 2+\frac{4}{3}-\sin x$$

$$f'\left(\frac{3\pi}{2}\right)=0+\frac{4}{3}-\sin\frac{3\pi}{2}=\frac{7}{3}$$

$$g'(2\pi)=\frac{1}{f'\left(\frac{3\pi}{2}\right)}=\frac{3}{7}$$

3. Ans. (A)

$$f(x)=\frac{3x-1}{x^2-1}=\frac{1}{x-1}+\frac{2}{x+1}$$

$$f_1(x)=-\frac{1}{(x-1)^2}-\frac{2}{(x+1)^2}$$

$$f_2(x)=\frac{2!}{(x-1)^3}+\frac{2.2!}{(x+1)^3}$$

$$f_3(x)=\frac{-3!}{(x-1)^4}-\frac{2(3!)}{(x+1)^4}$$

$$\therefore\ f_{50}(x)=\frac{50!}{(x-1)^{51}}+\frac{2.50!}{(x+1)^3}$$

$$\Rightarrow f_{50}(0)=50!$$

4. **Ans. (B)**

$$y=\frac{(\cos 6x+6\cos 4x+15\cos 2x+10)2\cos x}{2\cos 5x\cos x+10\cos 3x\cos x+20\cos^2 x}$$

$$=\frac{(\cos 6x+6\cos 4x+15\cos 2x+10)2\cos x}{\cos 6x+\cos 4x+5\cos 4x+5\cos 2x+10+10\cos 2x}$$

$y = 2\cos x$

$$\frac{dy}{dx}=-2\sin x$$

5. **Ans. (C)**

$(f'(x))^2 - f(x)f''(x) = 0 \qquad \Rightarrow \dfrac{d}{dx}\left(\dfrac{f(x)}{f'(x)}\right)=0$

$f(x)=kf'(x)$, where k is constant

$x = 0 \Rightarrow f(0) = kf'(0) \Rightarrow \quad k=\dfrac{1}{2}$

$2=\dfrac{f'(x)}{f(x)} \Rightarrow 2x+c=\ell n f(x)$

$f(x) = e^{2x}$ as $f(0) = 1$

6. **Ans. (D)**

$F'(x) = f'(x)g(x) + f(x)g'(x)$

$F''(x) = 2c + f''(x)g(x) + f(x)g''(x) \qquad (\because f'(x)g'(x) = c)$

$F'''(x) = f'''(x)g(x) + g'''(x)f(x) + f''(x)g'(x) + f'(x)g''(x)$

$\quad = f'''(x)g(x) + g'''(x)f(x)$

$\Rightarrow \dfrac{F'''(x)}{F(x)}=\dfrac{f'''(x)}{f(x)}+\dfrac{g'''(x)}{g(x)}$

7. **Ans. (A,B,D)**

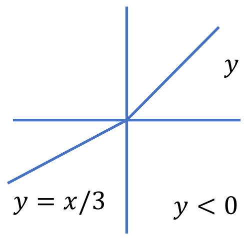
$x + |y| = 2y$

$3y = x$ if $y < 0$

$y = x$ is $y \geq 0$

$y = 1/3x$

$y = x \qquad\qquad y > 0$



(A) Domain and range of function is set of real numbers so (A) is true

(B) $f(0)$= L.H.L. = R.H.L.

So (B) is true (D) L.H.D. = 1/3 and R.H.D = 1 so (D) is true

8. **Ans. (B,C)**

$y^2=\sin x+y$

$$2yy'=\cos x+y' \Rightarrow y'=\frac{\cos x}{2y-1}=\frac{y\cos x}{2y^2-y}=\frac{y\cos x}{2\sin x+y}=\frac{y}{2\tan x+y\sec x}$$

9. **Ans. (B,C)**

Let $y = Ae^{-3x}\cos x + Be^{-3x}\sin x = e^{-3x}(A\cos x + B\sin x)$

$\frac{dy}{dx} = e^{-3x}(B\cos x - A\sin x) - 3(A\cos x + B\sin x)e^{-3x}$ ....(i)

Hence $e^{-3x}[(B-3A)\cos x - (A+3B)\sin x] = e^{-3x}\cos x$

$\therefore\ B - 3A = 1$ ...(ii)

And $A + 3B = 0$ ...(iii)

Squaring and adding (ii) and (iii)

$10(B^2 + A^2) = 1 \Rightarrow A^2 + B^2 = \frac{1}{10}$

10. **Ans. (B,C)**

$2yy_1 - 2(xy_1 + y) = 0 \Rightarrow y_1(y - x) = y \Rightarrow \frac{dy}{dx} = \frac{y}{y-x}$

11. **Ans. (A,C)**

$f(x) = (x^2 + bx + c)e^x$

$\therefore\ f'(x) = (x^2 + (b+2)x + (b+c))e^x$

$f(x) > 0$ iff $D = b^2 - 4c < 0$

Now $f'(x) > 0$ iff $D' = (b+2)^2 - 4(b+c) < 0 = D + 4 < 0$

Thus for $f'(x) > 0$ $D + 4 < 0$ holds. $\Rightarrow D < 0 \Rightarrow f(x) > 0$

Note that the converse need not be true, e.g. $b = c = 1, f(x) > 0$ but $f'(-1) = 0$

12. **Ans. (C,D)**

$$y = \frac{x^4 + 4x^2 + 4 - 3x^2}{x^2 + \sqrt{3}x + 2} = \frac{(x^2+2)^2 - (\sqrt{3}x)^2}{x^2 + \sqrt{3}x + 2} = \frac{(x^2 + \sqrt{3}x + 2)(x^2 - \sqrt{3}x + 2)}{(x^2 + \sqrt{3}x + 2)}$$

$\frac{dy}{dx} = 2x - \sqrt{3} \Rightarrow a = 2, b = -\sqrt{3}$

13. **Ans. (A,B,C)**

$f(-x) = -f(x),\ f(x) = f(4-x),\ f'(2+x) = f'(2-x)$ ...(i)

$x \to x + 2$ ...(iii)

$f(2+x) = f(2-x)$

$f'(2+x) = -f'(2-x)$ ...(ii)

From (ii) and (iii),

$f'(2+x) = f'(2-x) = 0 \Rightarrow f'(x) = 0, \quad \forall x \in R$

$\Rightarrow f(x)$ constant

$\Rightarrow f(x) = 0$ $\{\because f(0) = 0\}$

14. **Ans. (D)**

$g'(f(x)) = \left.\frac{1}{f'(x)}\right)_{(0,0)} = \left.\frac{1}{1+\cos x}\right)_{x=0} = \frac{1}{2}$

15. Ans. (D)

$$g'(y)=\frac{1}{1+\cos x}=\infty$$

$\cos x=-1$

16. Ans. (B)

17. Ans. (B)

$$f^{-1}(x)=xe^{x}=g(x)\ (\text{say})\Rightarrow f(x)=g^{-1}(x)$$

clearly, $g(e)=e^{(e+1)}\Rightarrow g^{-1}\left(e^{(e+1)}\right)=e$

$$\Rightarrow f\left(e^{(e+1)}\right)=e\Rightarrow f\left(f\left(e^{(e+1)}\right)\right)=f(e)$$

$$=g^{-1}(e)=1$$

Now, $ye^{y}=x$; $y=f(x)$

So, $\ell ny+y=\ell nx\Rightarrow \frac{y'}{y}+y'=\frac{1}{x}$

$$\Rightarrow f'(x)=\frac{y}{x(y+1)}$$

$$\therefore \lim_{x\to\infty}\frac{f(x)}{\ell nx}=\lim_{x\to\infty}xf'(x)=\lim_{x\to\infty}\frac{f(x)}{f(x)+1}=1$$

(as $f(x)\to\infty$ when $x\to\infty$)

18. Ans. (B)

$y=2\tan^{-1}x$

$$\frac{dy}{dx}=\frac{2}{1+x^{2}}$$

$$\frac{d^{2}y}{dx^{2}}=-\frac{4x}{\left(1+x^{2}\right)^{2}}$$

19. Ans. (A)

$x^{3}+y^{3}=3x^{2}y$

$$\frac{dy}{dx}=\frac{y}{x}\text{ and }\frac{d^{2}y}{dx^{2}}=0$$

20. Ans. (A)

$y=-2\tan^{-1}x$

$$\frac{dy}{dx}=-\frac{2}{1+x^{2}}$$

$$\frac{d^{2}y}{dx^{2}}=\frac{4x}{\left(1+x^{2}\right)^{2}}$$