04 Maxima and Minima

1. Introduction:

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter

- What is the shape of a vessel that can with-stand maximum pressure?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?

These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maxima and minima.

Maxima & Minima:

(a) Local Maxima/Relative maxima :

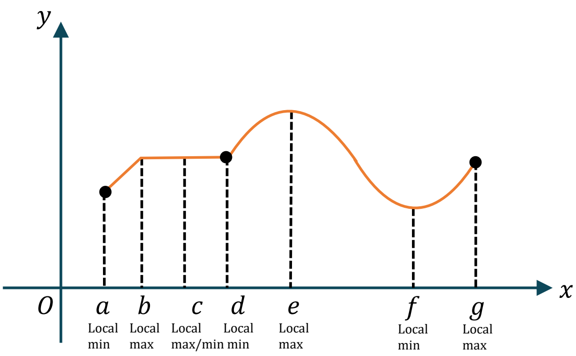
A function $f(x)$ is said to have a local maxima at $x = a$ if $f(a) \geq f(x)\ \forall\ x \in (a-h, a+h) \cap D_{f(x)}$

Where h is some positive real number.

(b) Local Minima/Relative minima :

A function $f(x)$ is said to have a local minima at $x = a$ if $f(a) \leq f(x)\ \forall\ x \in (a-h, a+h) \cap D_{f(x)}$

Where h is some positive real number.



(c) Absolute maxima (Global maxima) :

A function f has an absolute maxima (or global maxima) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the maximum value of f on D.

(d) Absolute minima (Global minima) :

A function f has an absolute minima at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the minimum value of f on D.

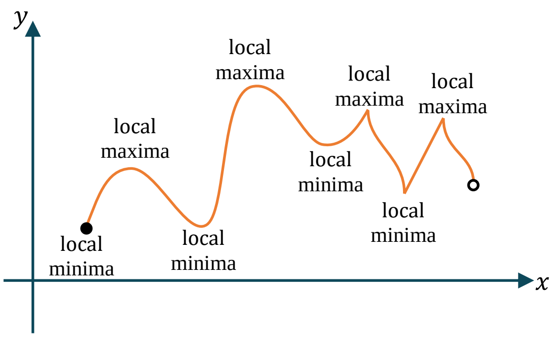
Note :

(i) The term 'extrema' is used for both maxima or minima.

(ii) A local maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iii) A function can have several extreme values such that local minimum value may be greater than a local maximum value.

(iv) It is not necessary that $f(x)$ always has local maxima/minima at end points of the given interval when they are included.



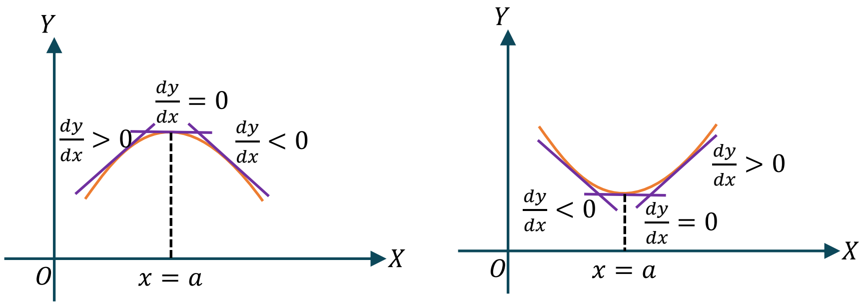
2. Derivative Test for Ascertaining Maxima/Minima:

(a) First derivative test :

If $f'(x) = 0$ at a point (say $x = a$) and

(i) If $f'(x)$ changes sign from positive to negative in the neighbourhood of $x = a$ then $x = a$ is said to be a point **local maxima.**

(ii) If $f'(x)$ changes sign from negative to positive in the neighbourhood of $x = a$ then $x = a$ is said to be a point **local minima.**

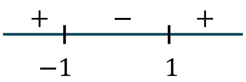


Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then $f(x)$ is either increasing or decreasing throughout this neighbourhood implying that $x = a$ is not a point of extremum of f.

Illustration 1:

Let $f(x) = x + \dfrac{1}{x}$; $x \neq 0$. Discuss the local maximum and local minimum values of $f(x)$.

Solution :



Here, $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2-1}{x^2} = \dfrac{(x-1)(x+1)}{x^2}$

Using number line rule, $f(x)$ will have local maximum at $x = -1$ and local minimum at $x = 1$

$\therefore$ local maximum value of $f(x) = -2$ at $x = -1$

and local minimum value of $f(x) = 2$ at $x = 1$

Illustration 2:

If $f(x) = \begin{cases} 3x^2+12x-1, & -1 \le x \le 2 \\ 37-x, & 2 < x \le 3, \end{cases}$, then

(A) $f(x)$ is increasing on [–1, 2]

(B) $f(x)$ is continuous on [–1, 3]

(C) $f'(x)$ does not exist at $x = 2$

(D) $f(x)$ has the maximum value at $x = 2$

Ans. (A,B,C,D)

Solution:

Given, $f(x) = \begin{cases} 3x^2+12x-1, & -1 \le x \le 2 \\ 37-x, & 2 < x \le 3 \end{cases}$

$\Rightarrow f'(x) = \begin{cases} 6x+12, & -1 \le x \le 2 \\ -1, & 2 < x \le 3 \end{cases}$

(A) which shows $f'(x) > 0$ for $x \in [-1, 2)$

So, $f(x)$ is increasing on [–1, 2)

Hence, (A) is correct.

(B) for continuity of $f(x)$. (check at $x = 2$)

RHL = 35, LHL = 35 and $f(2) = 35$

So, (B) is correct

(C) $Rf'(2) =-1$ and $Lf'(2) = 24$

so, not differentiable at $x = 2$.

Hence, (C) is correct.

(D) we know $f(x)$ is increasing on [–1, 2) and decreasing on (2, 3],

Thus maximum at $x = 2$,

Hence, (D) is correct.

$\therefore$ (A), (B), (C), (D) all are correct.

Illustration 3:

If $f(x) = 2x^3 - 3x^2 - 36x + 6$ has local maximum and minimum at $x = a$ and $x = b$ respectively, then ordered pair (a, b) is -

(A) (3, –2) (B) (2, –3) (C) (–2, 3) (D) (–3, 2)

Ans. (C)

Solution:

$f(x) = 2x^3 - 3x^2 - 36x + 6$

$f'(x) = 6x^2 - 6x - 36$ & $f''(x) = 12x - 6$

Now $f'(x) = 0 \Rightarrow 6(x^2 - x - 6) = 0$

$\Rightarrow (x - 3)\ (x + 2) = 0 \Rightarrow x = -2, 3$

$f''(-2) = -30$

$\therefore\ x = -2$ is a point of local maximum

$f''(3) = 30$

$\therefore\ x = 3$ is a point of local minimum

Hence, (–2, 3) is the required ordered pair.

Illustration 4:

Find the point of local maxima of $f(x) = \sin x\ (1 + \cos x)$ in $x \in (0, \pi/2)$.

Solution:

Let $f(x) = \sin x\ (1 + \cos x) = \sin x + \frac{1}{2}\sin 2x$

$\Rightarrow f'(x) = \cos x + \cos 2x$

$f''(x) = -\sin x - 2\sin 2x$

Now $f'(x) = 0\ \Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow \cos 2x = \cos(\pi - x) \Rightarrow x = \pi/3$

Also $f''(\pi/3) = -\sqrt{3}/2 - \sqrt{3} < 0$

$f(x)$ has a maxima at $x = \pi/3$ **Ans.**

Illustration 5:

Find the global maximum and global minimum of $f(x) = \dfrac{e^x + e^{-x}}{2}$ in $[-\log_e 2, \log_e 7]$.

Solution:

$f(x) = \dfrac{e^x + e^{-x}}{2}$ is differentiable at all x in its domain.

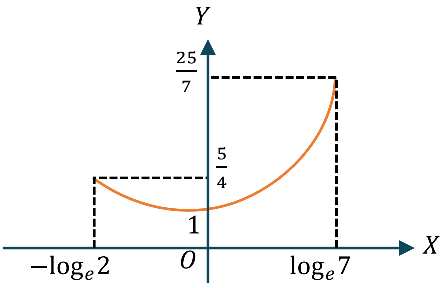
Then $f'(x) = \dfrac{e^x - e^{-x}}{2}, f''(x) = \dfrac{e^x + e^{-x}}{2}$

$f'(x) = 0 \Rightarrow \dfrac{e^x - e^{-x}}{2} = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$

$f''(0) = 1$ $\quad\quad\quad\quad$ $\therefore\ x = 0$ is a point of local minimum

Points $x = -\log_e 2$ and $x = \log_e 7$ are extreme points.

Now, check the value of $f(x)$ at all these three points $x = -\log_e 2, 0, \log_e 7$



$\Rightarrow f(-\log_e 2) = \dfrac{e^{-\log_e 2} + e^{+\log_e 2}}{2} = \dfrac{5}{4}$

$f(0) = \dfrac{e^0 + e^{-0}}{2} = 1$

$f(\log_e 7) = \dfrac{e^{\log_e 7} + e^{-\log_e 7}}{2} = \dfrac{25}{7}$

$\therefore\ x = 0$ is absolute minima & $x = \log_e 7$ is absolute maxima

Hence, absolute/global minimum value of $f(x)$ is 1 at $x = 0$

and absolute/global maximum value of $f(x)$ is $\dfrac{25}{7}$ at $x = \log_e 7$

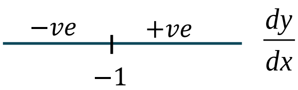
Illustration 6:

Identify a point of maxima/minima in $f(x) = (x+1)^4$.

Solution :

$f(x) = (x+1)^4$

$f'(x) = 4(x+1)^3$



at $x = -1$ $f(x)$ is having local minima

$\therefore$ at $x = -1 f(x)$ has point of minima.

Illustration 7:

Find point of local maxima and minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution:

$f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^4 - 20x^3 + 15x^2$

$= 5x^2(x^2 - 4x + 3)$

$= 5x^2(x-1)(x-3)$

$f'(x) = 0 \Rightarrow x = 0, 1, 3$

$f''(x) = 10x(2x^2 - 6x + 3)$

But at $x = 0$, derivative sign is positive in its neighbourhood.

Now $f''(1) < 0 \Rightarrow$ Maxima at $x = 1$

$f''(3) > 0 \Rightarrow$ Minima at $x = 3$

$\Rightarrow$ Neither maxima nor minima at $x = 0$.

3. Useful Formulae of Mensuration to Remember:

(a) Volume of a cuboid = ℓbh.

(b) Surface area of a cuboid = $2(\ell b + bh + h\ell)$.

(c) Volume of a prism = area of the base × height.

(d) Lateral surface area of prism = perimeter of the base × height.

(e) Total surface area of a prism = lateral surface area + 2 area of the base

(Note that lateral surfaces of a prism are all rectangles).

(f) Volume of a pyramid = $\frac{1}{3}$ area of the base x height.

(g) Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.

(Note that slant surfaces of a pyramid are triangles).

(h) Volume of a cone = $\frac{1}{3}\pi r^2 h$.

(i) Curved surface area of a cylinder = $2\pi rh$.

(j) Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.

(k) Volume of a sphere = $\frac{4}{3}\pi r^3$.

(l) Surface area of a sphere = $4\pi r^2$.

(m) Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

4. Summary of Working Rules for Solving Real Life Optimization Problem:

First: When possible, draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

Second: Write an equation for the quantity that is to be maximized or minimized. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

Third: If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

Fourth: Using derivative test, test each value of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither.

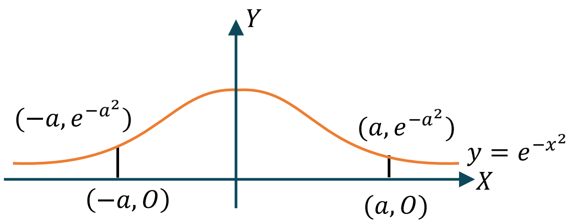
Fifth: If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Sixth : If the function $y = f(x)$ is defined only for $x \in [a, b]$ then examine $x = a$ & $x = b$ for possible extreme values.

Illustration 8:

Determine the largest area of the rectangle whose base is on the x-axis and two of its vertices lie on the curve $y = e^{-x^2}$.

Solution:



Area of the rectangle will be $A = 2a\,e^{-a^2}$.

For max. area, $\frac{dA}{da} = \frac{d}{da}(2ae^{-a^2}) = e^{-a^2}[2-4a^2]$

$\frac{dA}{da} = 0 \Rightarrow a = \pm\frac{1}{\sqrt{2}}$

& sign of $\frac{dA}{da}$ changes from positive to negative at $a = +\frac{1}{\sqrt{2}}$

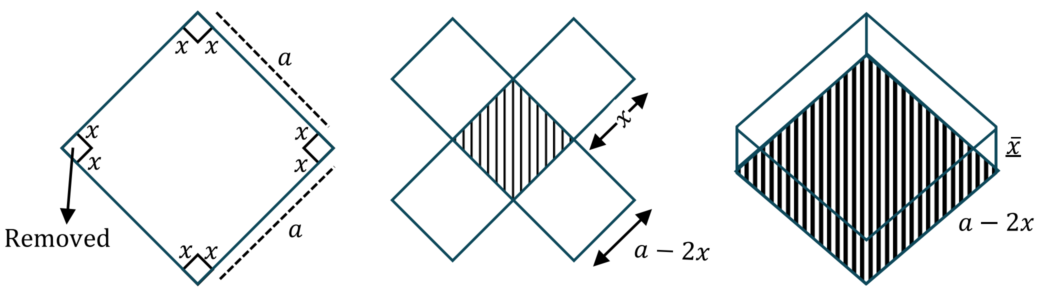
$\Rightarrow x = \frac{1}{\sqrt{2}}$ are points of maxima $\Rightarrow$ $A_{max} = \frac{2}{\sqrt{2}} \cdot e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{e^{1/2}}$ sq units . **Ans.**

Illustration 9:

A box of maximum volume with top open is to be made by cutting out four equal squares from four corners of a square tin sheet of side length a ft, and then folding up the flaps. Find the side of the square base cut off.

Solution:

Volume of the box is, $V = x(a - 2x)^2$ i.e., squares of side x are cut out then we will get a box with a square base of side $(a - 2x)$ and height x.



$\therefore \dfrac{dV}{dx} = (a-2x)^2 + x\cdot 2(a-2x)(-2)$

$\dfrac{dV}{dx} = (a-2x)(a-6x)$

For V to be extremum $\dfrac{dV}{dx} = 0 \Rightarrow x = a/2, a/6$

But when $x = a/2$; $V = 0$ (minimum) and we know minimum and maximum occurs alternately in a continuous function.

Hence, V is maximum when $x = a/6$. **Ans.**

Illustration 10:

If a right circular cylinder is inscribed in a given cone. Find the dimension of the cylinder such that its volume is maximum.

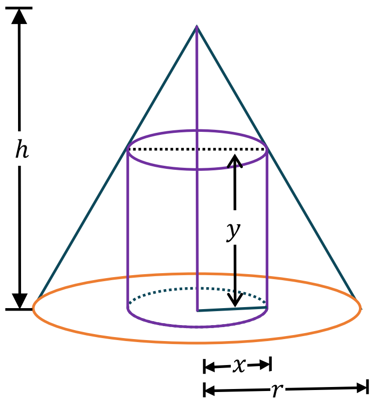
Solution:

Let x be the radius of cylinder and y be its height

$V = \pi x^2 y$

x, y can be related by using similar triangles

$\dfrac{y}{r-x} = \dfrac{h}{r} \Rightarrow y = \dfrac{h}{r}(r-x)$



$\Rightarrow V(x) = \pi x^2 \dfrac{h}{r}(r-x) \Rightarrow V(x) = \dfrac{\pi h}{r}\,(rx^2 - x^3)$

$V'(x) = \dfrac{\pi h}{r}\,(2rx - 3x^2)$

$V'(x) = 0 \Rightarrow x = 0, \dfrac{2r}{3}$

$V''(x) = \dfrac{\pi h}{r}(2r - 6x)$

$V''(0) = 2\pi h \Rightarrow x = 0$ is point of minima

$V''\left(\dfrac{2r}{3}\right) = -2\pi h \Rightarrow x = \dfrac{2r}{3}$ is point of maxima

Thus volume is maximum at $x = \left(\dfrac{2r}{3}\right)$ and $y = \dfrac{h}{3}$.

Illustration 11:

Let a cuboid having square base has area 6. Then find its maximum volume.

Solution:

Total area = $2a^2 + 4ah = 6$

$h = \dfrac{(6-2a^2)}{4a} \Rightarrow V = a^2 h = \dfrac{a^2(6-2a^2)}{4a}$

$\dfrac{dv}{da} = 6 - 6a^2 = 0$

$a = 1$ & $\dfrac{d^2v}{da^2} = -ve$

Maximum $V = 1$

Important Note :

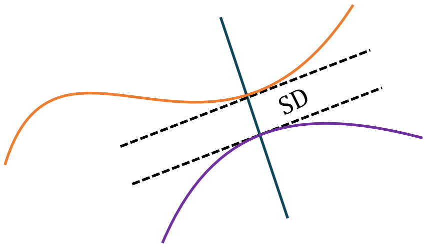
(i) If the sum of two real numbers x and y is constant then their product is maximum if they are equal.

i.e. $xy \frac{1}{4}[(x+y)^2-(x-y)^2]$

(ii) If the product of two positive numbers is constant then their sum is least if they are equal.

i.e. $(x+y)^2=(x-y)^2+4xy$

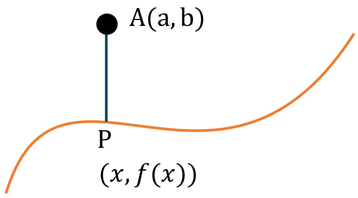
5. LEAST/GREATEST Distance Between Two Curves:



Least/Greatest distance between two non-intersecting curves usually lies along the common normal. (Wherever defined)

Note : Given a fixed point $A(a, b)$ and a moving point $P(x, f(x))$ on the curve $y = f(x)$. Then AP will be maximum or minimum if it is normal to the curve at P.

Proof : $F(x)=(x-a)^2+(f(x)-b)^2 \Rightarrow F'(x)=2(x-a)+2(f(x)-b).f'(x)$



$\therefore\ f'(x)=-\left(\frac{x-a}{f(x)-b}\right)$. Also $m_{AP}=\frac{f(x)-b}{x-a}$. Hence $f'(x)\,.\,m_{AP}=1$.

Illustration 12:

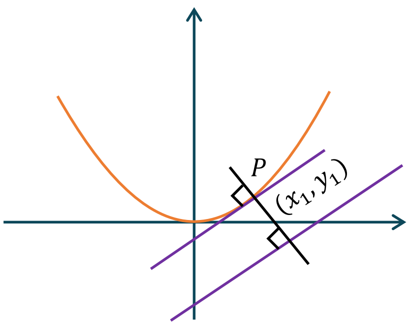
Find the co-ordinates of the point on the curve $x^2=4y$, which is at least distance from the line $y=x-4$.

Solution:

Let $P(x_1y_1)$ be a point on the curve $x^2=4y$

at which normal is also a perpendicular to the line $y=x-4$.

Slope of the tangent at (x_1, y_1) is $2x=4\frac{dy}{dx} \Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,y_1)}=\frac{x_1}{2}$



$\therefore \frac{x_1}{2} = 1 \Rightarrow x_1 = 2$

$\because x_1^2 = 4y_1 \Rightarrow y_1 = 1$

Hence required point is (2, 1)

6. Some Special Points on a Curve :

(a) Stationary points: The stationary points are the points of domain where $f'(x) = 0$.

(b) Critical points: There are three kinds of critical points as follows :

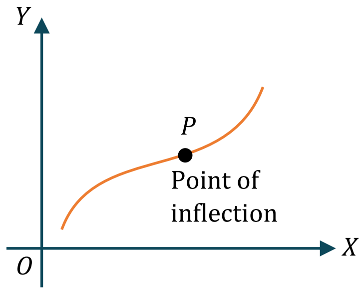
(i) The point at which $f'(x) = 0$

(ii) The point at which $f'(x)$ does not exists

(iii) The end points of interval (if included)

These points belongs to domain of the function.

Note: Local maxima and local minima occurs at critical points only but not all critical points will correspond to local maxima/local minima.



(c) Point of inflection: A point where the graph of a function has a tangent line and where the strict concavity changes is called a point of inflection. For finding point of inflection of any function, compute the points (x-coordinate) where $\frac{d^2y}{dx^2} = 0$ $\frac{d^2y}{dx^2}$ or does not exist. Let the solution is $x = a$, if $\frac{d^2y}{dx^2} = 0$ at $x = a$ and sign of $\frac{d^2y}{dx^2}$ changes about this point then it is called point of inflection.

If $\frac{d^2y}{dx^2}$ does not exist at $x = a$ and sign of $\frac{d^2y}{dx^2}$ changes about this point and tangent exist at this point then it is called point of inflection.

Illustration 13:

Find the critical point(s) & stationary point(s) of the function $f(x) = (x-2)^{\frac{2}{3}}(2x+1)$

Solution:

$f(x) = (x-2)^{\frac{2}{3}}(2x+1)$

$f'(x) = (x-2)^{\frac{2}{3}}.2 + (2x+1)\ \frac{2}{3}\ (x-2)^{-\frac{1}{3}}$

$= 2(x-2)^{\frac{2}{3}} + \frac{2}{3}(2x+1)\ \frac{1}{(x-2)^{1/3}}$

$= \left[2(x-2)+\frac{2}{3}(2x+1)\right]\frac{1}{(x-2)^{1/3}} = \frac{2(5x-5)}{3(x-2)^{1/3}}$

$f'(x)$ does not exist at $x = 2$ and $f'(x) = 0$ at $x = 1$

$\therefore x = 1, 2$ are critical points and $x = 1$ is stationary point.

Illustration 14:

The point of inflection for the curve $y = x^{\frac{5}{3}}$ is -

(A) (1, 1) (B) (0, 0) (C) (1, 0) (D) (0, 1)

Ans. (B)

Solution:

Here $\dfrac{d^2y}{dx^2} = \dfrac{10}{9x^{1/3}}$

From the given points we find that (0, 0) is the point of the curve where

$\dfrac{d^2y}{dx^2}$ does not exist but sign of $\dfrac{d^2y}{dx^2}$ changes about this point.

$\therefore$ (0, 0) is the required point

Illustration 15:

Find the inflection point of $f(x) = 3x^4 - 4x^3$. Also draw the graph of $f(x)$ giving due importance to maxima, minima and concavity.

Solution:

$f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

$f'(x) = 12x^2(x-1)$

$-\quad -\quad +$

$0 \quad 1$

$f'(x) = 0 \Rightarrow x = 0, 1$

examining sign change of $f'(x)$

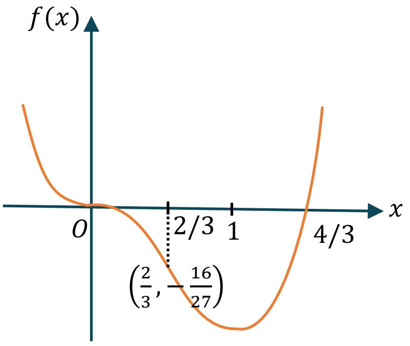
thus $x = 1$ is a point of local minima

$+\quad -\quad +$

$0 \quad 2/3$

$f''(x) = 12(3x^2 - 2x)$

$f''(x) = 12x(3x-2)$



$f''(x) = 0 \Rightarrow x = 0, \dfrac{2}{3}$

Again examining sign of $f''(x)$

thus $x = 0, \dfrac{2}{3}$ are the inflection points

Hence the graph of $f(x)$ is