02 Matrices

Definition, Special Type of Matrices:

Introduction:

A rectangular array of mn numbers (which may be **real or complex**) in the form of $'m'$ horizontal lines (called **rows**) and $'n'$ vertical lines (called **columns**), is called a matrix of order m by n, written as $m \times n$ matrix.

Such an array is enclosed by [ ] or ( ) or ||. An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

In compact form, the above matrix is represented by $A = [a_{ij}]_{m\times n}$. The number $a_{11}, a_{12}, \ldots$..etc. are known as the elements of the matrix A, a_{ij} belongs to the i^{th} row and j^{th} column and is called the $(i,j)^{th}$ element of the matrix $A = [a_{ij}]$.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 9 \end{bmatrix}$ is a matrix having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements:

$a_{11} = 1, a_{12} = 2, a_{13} = 3, a_{21} = 0, a_{22} = -1, a_{23} = 9.$

Special Type of Matrices:

(a) Row Matrix (Row vector) : $A = [a_{11}, a_{12}, \ldots\ldots\ldots a_{1n}]$ i.e. row matrix has exactly one row.

(b) Column Matrix (Column vector) : $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ i.e. column matrix has exactly one column.

(c) Zero or Null Matrix : $(A = O_{m\times n})$ An $m \times n$ matrix whose all entries are zero.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 null matrix & $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3×3 null matrix

(d) Horizontal Matrix : A matrix of order $m \times n$ is a horizontal matrix if $n > m$ e.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$

(e) Vertical Matrix : A matrix of order $m \times n$ is a vertical matrix if $m > n$ e.g. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

(f) Square Matrix : If number of rows = number of columns $\Rightarrow$ matrix is a square matrix.
If number of rows = number of columns = n then, matrix is of the order $'n'$.

Note: The pair of elements a_{ij} & a_{ji} are called **Conjugate Elements.**

Trace of Matrix:

The sum of the elements of a **square matrix** A lying along the principal diagonal is called the trace of A i. e. $(tr(A))$. Thus, if $A = [a_{ij}]_{n\times n}$, then $tr(A)=\sum_{i=1}^{n} a_{ii} = a_{11}+a_{22}+........a_{nn}$

Properties of Trace of a Matrix:

Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ and λ be a scalar then

(i) $tr(\lambda A) = \lambda tr(A)$

(ii) $tr(A + B) = tr(A) + tr(B)$

Square Matrices:

Triangular Matrix

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix} \quad ; B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 4 & 3 & 3 \end{pmatrix}$$

Upper Triangular — $a_{ij} = 0 \ \forall\, i > j$

Lower Triangular — $a_{ij} = 0 \ \forall\, i < j$

Diagonal Matrix denoted as

$A = \text{diag}\,(a_{11}, a_{22} \dots \dots \dots . a_{nn})$

where $a_{ij} = 0$ for $i \neq j$

Scalar Matrix

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

If $a_{11} = a_{22} = a_{33} = a$

Unit or Identity Matrix

$$a_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

If $a_{11} = a_{22} = a_{33} = 1$

Note:

(i) Minimum number of zeros in triangular matrix of order $n = \frac{n(n-1)}{2}$.

(ii) Minimum number of zero in a diagonal matrix of order $n = n(n-1)$.

Illustration 1:

Choose the correct answer

(A) Every identity matrix is a scalar matrix

(B) Every scalar matrix is an identity matrix

(C) Every diagonal matrix is an identity matrix

(D) A square matrix whose each element is 1 is an identity matrix

Ans. (A)

Solution:

We know that every identity matrix is a scalar matrix.

Illustration 2:

$A = [a_{ij}]_{m\times n}$ is a square matrix, if

(A) $m < n$ (B) $m > n$ (C) $m = n$ (D) None of these

Ans. (C)

Solution:

For a square matrix number of columns = number of rows.

Illustration 3:

In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 7 & \sqrt{3} & \sqrt{5} \end{bmatrix}$

(i) The order of the matrix,

(ii) The number of elements,

(iii) Write the elements $a_{13}, a_{21}, a_{33}, a_{24}, a_{23}$.

Solution:

(i) 3×4

(ii) 12

(iii) $19, 1, \sqrt{3}, 1, 0$

Algebra of Matrices:

Equality of Matrices:

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if

(a) both have the same order.

(b) $a_{ij} = b_{ij}$ for each pair of i & j.

Illustration 4:

Find the value of x, y, z and w which satisfy the matrix equation $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4w-8 \end{bmatrix} = \begin{bmatrix} -x-1 & 0 \\ 3 & 2w \end{bmatrix}$.

Solution:

As the given matrices are equal so their corresponding elements are equal.

$\Rightarrow x + 3 = -x - 1 \Rightarrow 2x = -4$

$\therefore x = -2$ ...(i)

$\Rightarrow 2y + x = 0 \Rightarrow 2y - 2 = 0$ [from (i)]

$\Rightarrow y = 1$ ...(ii)

$\Rightarrow z - 1 = 3 \Rightarrow z = 4$ ...(iii)

$\Rightarrow 4w - 8 = 2w \Rightarrow 2w = 8$

$\therefore w = 4$ ...(iv)

Addition: $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(a) **Addition of matrices is commutative:** i.e. $A + B = B + A$

(b) **Matrix addition is associative:** $(A + B) + C = A + (B + C)$

(c) **Additive inverse:** If $A + B = 0 = B + A$, then B is called **additive inverse** of A.

(d) **Existence of additive identity:** Let $A = [a_{ij}]$ be an $m \times n$ matrix and **O** be an $m \times n$ zero matrix, then $A + O = O + A = A$. In other words, 0 is the **additive identity** for matrix addition.

(e) **Cancellation laws** hold good in case of addition of matrices. If A, B, C are matrices of the same order, then $A + B = A + C \Rightarrow B = C$ (**left cancellation law**) and $B + A = C + A \Rightarrow B = C$ (**right-cancellation law**)

Note: The zero matrix plays the same role in matrix addition as the number zero does in addition of numbers.

Illustration 5:

If $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$ and $A + B - D = 0$ (zero matrix), then D matrix will be-

(A) $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 6 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ (C) $\begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & -2 \\ -3 & -7 \\ -5 & -6 \end{bmatrix}$

Ans. (C)

Solution:

Let $D = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$$\therefore A + B - D = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \Rightarrow \begin{bmatrix} 1-1-a & 3-2-b \\ 3+0-c & 2+5-d \\ 2+3-e & 5+1-f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow -a = 0 \Rightarrow a = 0, 1 - b = 0 \Rightarrow b = 1,$

$\Rightarrow 3 - c = 0 \Rightarrow c = 3, 7 - d = 0 \Rightarrow d = 7,$

$\Rightarrow 5 - e = 0 \Rightarrow e = 5, 6 - f = 0 \Rightarrow f = 6$

$$\therefore D = \begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$$

Multiplication of a Matrix by a Scalar:

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$; $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

Properties of Scalar Multiplication:

(a) If A and B are two matrices of the same order and $'k'$ be a scalar then $k(A + B) = kA + kB$.

(b) If k_1 and k_2 are two scalars and $'A'$ is a matrix, then $(k_1 + k_2)A = k_1A + k_2A$.

(c) If k_1 and k_2 are two scalars and $'A'$ is a matrix, then $(k_1k_2)A = k_1(k_2A) = k_2(k_1A)$

Multiplication of Matrices (Row by Column):

Let A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$, then the matrix multiplication AB is possible if and only if $n = p$ and matrices are said to be **conformable** for multiplication.

In the product AB, A is called pre-factor and B is called post factor.

$\Rightarrow$ AB is possible if and only if number of columns in pre-factor = number of rows in post-factor.

Let $A_{m\times n} = [a_{ij}]$ and $B_{n\times p} = [b_{ij}]$, then order of AB is $m \times p$ & $\boxed{(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}}$

e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}_{3\times 4}$

Then order of AB is 2×4.

$(AB)_{11} = a_{11}b_{11} + a_{12}\,b_{21} + a_{13}b_{31} = \sum_{r=1}^{3} a_{1r}b_{r1}$

$(AB)_{23} = a_{21}b_{13} + a_{22}\,b_{23} + a_{23}b_{33} = \sum_{r=1}^{3} a_{2r}b_{r3}$

In general, $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} = \sum_{r=1}^{3} a_{ir}b_{rj}$

Illustration 6:

If $[1\ x\ 2] \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ -1 \end{bmatrix} = O$, then the value of x is

(A) –1 (B) 0 (C) 1 (D) 2

Ans. (A)

Solution:

The LHS of the equation

$= [24x + 92x + 5] \begin{bmatrix} x \\ 1 \\ -1 \end{bmatrix} = [2x + 4x + 9 - 2x - 5] = 4x + 4$

Thus $4x + 4 = 0$; $x = -1$

Illustration 7:

If A, B are two matrices such that $A + B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $A - B = \begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}$, then find AB.

Solution:

Given $A + B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ...(i) &

$\Rightarrow A - B = \begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}$ ...(ii)

Adding (i) & (ii)

$\Rightarrow 2A = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$

Subtracting (ii) from (i)

$$\Rightarrow 2B = \begin{bmatrix} -2 & 0 \\ 4 & 4 \end{bmatrix} \Rightarrow B = \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$$

Now $AB = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}$

Properties of Matrix Multiplication:

(a) Matrix Multiplication is not Commutative: i.e. $AB \neq BA$

Here both AB & BA exist and also, they are of the same type but $\boldsymbol{AB \neq BA}$.

e.g. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$;

then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$AB \neq BA$ (in general)

(b) $AB = \mathbf{O} \not\Rightarrow A = \mathbf{O}$ or $B = \mathbf{O}$ (in general)

e.g. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, then $AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Note:

If A and B are two non-zero matrices such that $AB = \mathbf{O}$ then A and B are called the divisors of zero. If A and B are two matrices such that

(i) $AB = BA$ then A and B are said to commute

(ii) $AB = -BA$ then A and B are said to anti-commute

(c) Matrix Multiplication Is Associative:

If A, B & C are conformable for the product AB & BC, then $(AB)\,C = A(BC)$

(d) Distributivity:

$$\left.\begin{aligned} A(B+C) &= AB + AC \\ (A+B)C &= AC + BC \end{aligned}\right] \text{Provided } A, B \,\&\, C \text{ are conformable for respective products}$$

(e) $\boldsymbol{tr(AB) = tr(BA)}$

Illustration 8:

Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} \& C = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$ be the matrices then, prove that in matrix multiplication cancellation law does not hold.

Solution:

We have to show that $AB = AC$; though B is not equal to C.

We have $AB = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}_{3\times 4}$

Now, $AC = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}_{3\times 4}$

Here, $AB = AC$ though B is not equal to C. Thus, cancellation law does not hold in general.

Illustration 9:

If $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}3 & -1 & 3\\ -1 & 0 & 2\end{bmatrix}$, then find $2A-B$.

Solution:

We have

$$2A-B=2\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{bmatrix}-\begin{bmatrix}3 & -1 & 3\\ -1 & 0 & 2\end{bmatrix}$$

$$=\begin{bmatrix}2 & 4 & 6\\ 4 & 6 & 2\end{bmatrix}+\begin{bmatrix}-3 & -1 & -3\\ 1 & 0 & -2\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 5 & 3\\ 5 & 6 & 0\end{bmatrix}$$

Illustration 10:

Which of the following holds true for matrix multiplication?

(A) follows commutative property (B) follows distributive property

(C) follows associative property (D) None of these

Ans. (B) and (C)

Positive Integral Powers of a Square Matrix:

For a square matrix A, $A^n=\underbrace{A.A.A..............A}_{\text{upto } n \text{ times}}$, where $n \in N$

Note:

(i) $\boldsymbol{A^m.A^n = A^{m+n}}$

(ii) $\boldsymbol{(A^m)^n = A^{mn}}$, where $m, n \in N$

(iii) If A and B are square matrices of same order and $AB = BA$ then

$(A+B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + ...+ {}^nC_nB^n$

Note that for a unit matrix I of any order, $I^m = I$ for all $m \in N$.

Illustration 11:

If $A=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$, then $A^5 =$

(A) 5A (B) 10A (C) 16A (D) 32A

Ans. (C)

Solution:

$$A=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} \Rightarrow A^5=\begin{bmatrix}2^5 & 0 & 0\\ 0 & 2^5 & 0\\ 0 & 0 & 2^5\end{bmatrix}=2^4\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}=16A.$$

Illustration 12:

A square matrix P satisfies $P^2 = I - P$, where I is the identify matrix. If $P^n = 5I - 8P (n \in N)$, then minimum value of n is equal to

(A) 4 (B) 5 (C) 6 (D) 7

Ans. (C)

Solution:

Since $P^2 = I - P$

$\Rightarrow P^3 = P(I - P) = P - P^2 = P - (I - P) = 2P - I$

Similarly, $P^4 = 2P^2 - P = 2I - 3P$ and

$\Rightarrow P^5 = 5P - 3I$

$\Rightarrow P^6 = 5P^2 - 3P = 5I - 8P$

So $n = 6$.

Illustration 13:

If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, show that $A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$, where k is any positive integer.

Solution:

We have,

$\Rightarrow A^2=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=\begin{bmatrix}5 & -8\\ 2 & -3\end{bmatrix}=\begin{bmatrix}1+2\times2 & -4\times2\\ 2 & 1-2\times2\end{bmatrix}$ and

$\Rightarrow A^3=\begin{bmatrix}5 & -8\\ 2 & -3\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=\begin{bmatrix}7 & -12\\ 3 & -5\end{bmatrix}=\begin{bmatrix}1+2\times3 & -4\times3\\ 3 & 1-2\times3\end{bmatrix}$

Thus, it is true for indices 2 and 3. Now assume

$\Rightarrow A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$

Then, $A^{k+1}=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=\begin{bmatrix}3+2k & -4(k+1)\\ k+1 & -1-2k\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1+2(k+1) & -4(k+1)\\ k+1 & 1-2(k+1)\end{bmatrix}$

Thus, if the law is true for A^k, it is also true for A^{k+1}. But it is true for $k = 2, 3$ etc. Hence, by induction, the required result follows.

The Transpose of a Matrix: (Changing rows & Columns)

Let A be any matrix of order $m \times n$. Then A^T or $A' = [a_{ji}]$ for $1 \le i \le m$ & $1 \le j \le n$ of order $n \times m$

Properties of Transpose:

If A^T & B^T denote the transpose of A and B,

(a) $(A + B)^T = A^T + B^T$; note that A and B have the same order.

(b) $(A\,B)^T = B^T A^T$ (Reversal law) A and B are conformable for matrix product AB

Note: In general: $(A_1.A_2,\ldots\ldots\ldots\ldots A_n)^T = A_n^T\ldots\ldots\ldots\ldots\ldots\ldots A_2^T.A_1^T$ (reversal law for transpose)

(c) $(A^T)^T = A$

(d) $(kA)^T = kA^T$, k is a scalar.

Illustration 14:

If A and B are matrices of order $m \times n$ and $n \times m$ respectively, then order of matrix $B^T(A^T)^T$ is -

(A) $m \times n$ (B) $m \times m$ (C) $n \times n$ (D) Not defined

Ans. (D)

Solution:

Order of B is $n \times m$ so order of B^T will be $m \times n$

Now $(A^T)^T = A$ & its order is $m \times n$. For the multiplication $B^T(A^T)^T$

Number of columns in pre-factor ≠ Number of rows in post-factor.

Hence this multiplication is not defined.

Illustration 15:

$A=\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$ then B^TA^T is

(A) a null matrix

(B) an identity matrix

(C) scalar, but not an identity matrix

(D) such that $Tr(B^T A^T) = 4$

Ans. (B)

Solution:

$A=\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}, B=\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$

$\Rightarrow A^T=\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}, B^T=\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$

B^TA^T = is an identity matrix.

Orthogonal Matrices:

A square matrix is said to be orthogonal matrix if $AA^T = I$

Note:

(i) The determinant value of orthogonal matrix is either 1 or −1.

(ii) Let $A=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \Rightarrow A^T=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$AA^T=\begin{bmatrix} a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3 \\ b_1a_1+b_2a_2+b_3a_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3 \\ c_1a_1+c_2a_2+c_3a_3 & c_1b_1+c_2b_2+c_3b_3 & c_1^2+c_2^2+c_3^2 \end{bmatrix}$$

If $AA^T = I$, then

$\sum_{i=1}^{3} a_i^2 = \sum b_i^2 = \sum c_i^2 = 1$ and $\sum_{i=1}^{3} a_ib_i = \sum_{i=1}^{3} b_ic_i = \sum_{i=1}^{3} c_ia_i = 0$

Illustration 16:

Determine the values of α, β, γ when $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.

Solution:

Let $A=\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

$\Rightarrow A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$

But given A is orthogonal.

$AA^T = I$

$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Equating the corresponding elements, we have

$\Rightarrow 4\beta^2 + \gamma^2 = 1$ ...(i)

$\Rightarrow 2\beta^2 - \gamma^2 = 0$ ...(ii)

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 1$ ...(iii)

From (i) and (ii), $6\beta^2 = 1$ $\beta^2 = \frac{1}{6}$ and $\gamma^2 = \frac{1}{3}$

From (iii) $\alpha^2 = 1 - \beta^2 - \gamma^2 = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$

Hence, $\alpha = \pm\frac{1}{\sqrt{2}}$, $\beta = \pm\sigma\pi\frac{1}{\sqrt{6}}$ and $\gamma = \pm\frac{1}{\sqrt{3}}$.

Illustration 17:

Let $A = \begin{pmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{pmatrix}$. If $AA^T = I^3$, then $|p|$ is :

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{5}}$ (C) $\frac{1}{\sqrt{6}}$ (D) $\frac{1}{\sqrt{3}}$

Ans. (A)

Solution:

A is orthogonal matrix

$\Rightarrow 0^2 + p^2 + p^2 = 1 \Rightarrow |p| = \frac{1}{\sqrt{2}}$

Symmetric & Skew Symmetric Matrix:

(a) Symmetric Matrix:

A square matrix $A = [a_{ij}]$ is said to be, symmetric if, $a_{ij} = a_{ji}$ $\forall$ i & j (conjugate elements are equal). Hence for symmetric matrix $\boldsymbol{A = A^T}$.

Note: Max. number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.

(b) Skew Symmetric Matrix:

Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} =- a_{ji}$ $\forall$ i & j (the pair of conjugate elements are additive inverse of each other). For a skew symmetric matrix, $\boldsymbol{A = -A^T}$.

Note:

(i) If A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \; \forall \; i$. Thus, the diagonal elements of a skew square matrix are all zero, but not the converse.

(ii) The determinant value of odd order skew symmetric matrix is zero.

Illustration 18:

If $A=\begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & -x & -3 \end{bmatrix}$ be symmetric matrix then find the value of x.

Solution:

$x = -4$

for symmetric matrix $A = A^T$

$\Rightarrow \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & -x & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & -x \\ 1 & 4 & -3 \end{bmatrix}$

$\therefore x = -4$

Illustration 19:

Given matrix $A=\begin{bmatrix} 1 & x & 1 \\ x & 2 & y \\ 1 & y & 3 \end{bmatrix}; B=\begin{bmatrix} 3 & -3 & z \\ -3 & 2 & -3 \\ z & -3 & 1 \end{bmatrix}$.

Obtain x, y, z if the matrix AB is symmetric.

Solution:

$$AB = \begin{bmatrix} 1 & x & 1 \\ x & 2 & y \\ 1 & y & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 & z \\ -3 & 2 & -3 \\ z & -3 & 1 \end{bmatrix} = \begin{bmatrix} 3-3x+z & -3+2x-3 & z-3x+1 \\ 3x-6+yz & -3x+4-3y & xz-6+y \\ 3-3y+3z & -3+2y-9 & z-3y+3 \end{bmatrix}$$

AB is symmetric matrix,

So, $-6 + 2x = 3x - 6 + yz$ ...(i)

$\Rightarrow 3 - 3y + 3z = z - 3x + 1$ ...(ii)

$\Rightarrow -3 + 2y - 9 = xz - 6 + y$ ...(iii)

So possible set of values of x, y, z

$\Rightarrow \left(-\frac{4\sqrt{2}}{3}, \frac{2}{3}, 2\sqrt{2}\right), \left(\frac{4\sqrt{2}}{3}, \frac{2}{3}, -2\sqrt{2}\right), (3,3,-1)$

Properties of Symmetric & Skew Symmetric Matrix:

(i) A is symmetric if $A^T = A$ and A is skew symmetric if $A^T = -A$

(ii) Let A be any square matrix then, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.

(iii) The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.

(iv) If A and B are symmetric matrices then,

(1) $AB + BA$ is a symmetric matrix

(2) $AB - BA$ is a skew symmetric matrix.

(v) Every square matrix can be uniquely expressed as a sum or difference of a symmetric and a skew symmetric matrix.

$$A=\underbrace{\frac{1}{2}(A+A^T)}_{\text{symmetric}}+\underbrace{\frac{1}{2}(A-A^T)}_{\text{skew symmetric}} \quad \text{and} \quad A=\frac{1}{2}(A^T+A)-\frac{1}{2}(A^T-A)$$

Illustration 20:

If A is symmetric as well as skew symmetric matrix, then A is -

(A) diagonal matrix
(B) null matrix
(C) triangular matrix
(D) none of these

Ans. (B)

Solution:

Let $A = [a_{ij}]$ Since A is skew symmetric $a_{ij} = -a_{ji}$

for $i = j, a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

for $i \neq j, a_{ij} = -a_{ji}$ $[\because A$ is skew symmetric$]$ and $a_{ij} = a_{ji}$ $[\because A$ is symmetric$]$

$\therefore a_{ij} = 0$ for all $i \neq j$

So, $a_{ij} = 0$ for all $'i'$ and $'j'$ i. e. A is null matrix.

Illustration 21:

Which one of the following is wrong?

(A) The elements on the main diagonal of a symmetric matrix are all zero

(B) The elements on the main diagonal of a skew - symmetric matrix are all zero

(C) For any square matrix A, $\frac{1}{2}(A + A')$ is symmetric matrix

(D) For any square matrix A, $\frac{1}{2}(A - A')$ is skew - symmetric matrix

Ans. (A)

Solution:

The elements of main diagonal of skew symmetric matrix are all zero but not necessarily for symmetric matrix.

$\Rightarrow \frac{A+A'}{2}$ is symmetric matrix.

$\Rightarrow \frac{A-A'}{2}$ is skew symmetric matrix.

Illustration 22:

Let $A = [a_{ij}]_{n\times n}$ where $a_{ij} = i^2 - j^2$. Then A is

(A) null matrix
(B) symmetric matrix
(C) skew-symmetric matrix
(D) unit matrix

Ans. (C)

Solution:

$$A = A=\begin{vmatrix} 1^2-1^2 & 1^2-2^2 & 1^2-3^2 \\ 2^2-1^2 & 2^2-2^2 & 2^2-3^2 \\ 3^3-1^2 & 3^2-2^2 & 3^2-3^2 \end{vmatrix}=\begin{vmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{vmatrix}$$

So A is skew symmetric matrix.

Adjoint of A Square Matrix:

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A then the adjoint of A, denoted by adj A, is defined as the transpose of the cofactor matrix.

Then, adj $A = [C_{ij}]^T \Rightarrow$ adj $A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

Theorem: $\boldsymbol{A(adj.A) = (adj.A).A = |A|\, I_n}$.

Proof: $A.(adj\ A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

$$\begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix} = |A| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A.(adj.A) = |A|I$$

(whatever may be the value only $|A|$ will come out as a common element)

If $|A| \neq 0$, then $\dfrac{A.(adj.A)}{|A|} = I$ = unit matrix of the same order as that of A

Illustration 23:

If $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \\ 5 & 1 & 3 \end{bmatrix}$, then adj A is equal to -

(A) $\begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$

(B) $\begin{bmatrix} -14 & 4 & 22 \\ 4 & 22 & -14 \\ 22 & -14 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 14 & 4 & -22 \\ 4 & -22 & -14 \\ -22 & -14 & -4 \end{bmatrix}$

(D) none of these

Ans. (A)

Solution:

adj. $A = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}^T = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$

Illustration 24:

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then adj A =

(A) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$

Ans. (A)

Solution:

$A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$

Matrix formed by cofactors of $A=C=\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}$

Transpose of Matrix $C = C^T = \begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}$

Adjoint of matrix $A = C^T =$ adj $A=\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}$

Illustration 25:

If $A = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$, then adj A=

(A) A^2 (B) I (C) O (D) A'

Ans. (D)

Solution:

$A=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$

Matrix formed by Cofactors of $A=C=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$

$\therefore$ Adj $A = C^T = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} = A'$

Properties of Adjoint:

If $\boldsymbol{A}$ be a square matrix of order $\boldsymbol{n}$, then

(i) $|adj\ A| = |A|^{n-1}$

(ii) $adj\ (adj\ A) = |A|^{n-2} A$, where $|A| \neq 0$

(iii) $|\text{adj}(\text{adj}\ A)| = |A|^{(n-1)^2}$, where $|A| \neq 0$

(iv) $adj(AB) = (adj\ B)\ (adj\ A)$

(v) $adj(KA) = K^{n-1}\ (adj\ A)$, K is a scalar

(vi) $adj\ A^T = (adj\ A)^T$

Method to Find Adjoint of a 2 × 2 Square Matrix, Directly:

Let A be a 2×2 square matrix. In order to find the adjoint simply interchange the diagonal elements and reverse the sign of off diagonal elements (rest of the elements).

e.g. If $A = \begin{bmatrix}p & q\\ r & s\end{bmatrix} \Rightarrow adjA = \begin{bmatrix}s & -q\\ -r & p\end{bmatrix}$

Illustration 26:

If $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$, then $adj\ (adj\ A)$ is equal to -

(A) $8\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (B) $16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (C) $64\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (D) none of these

Ans. (B)

Solution:

$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 8$

Now $adj\ (adj\ A) = |A|^{3-2}A$

$= 8\begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Illustration 27:

If A is a square matrix of order $n \times n$ and k is a scalar, then $adj\ (kA)$ is equal to

(A) k adj A (B) k^n adj A (C) k^{n-1} adj A (D) k^{n+1} adj A

Ans. (C)

Solution:

$kA \text{ adj } (kA) = |kA|I_n$

$\Rightarrow kA \text{ adj } (kA) = k^n |A|I_n$

$\Rightarrow kA \text{ adj } (kA) = k^n A \text{ adj } A$

Pre-multiplying A^{-1}

$\Rightarrow \text{adj } (kA) = k^{n-1} \text{ adj } A$

Illustration 28:

If A is square matrix of order n then adj(adj A) =

(A) $|A|^{n-1} A$ (B) $|A|^{(n-2)}A$ (C) $|A|^{n-2}$ (D) $|A|^n A$

Ans. (B)

Solution:

We know that $A.\text{adj } A = |A|.I$ and $|\text{adj } A| = |A|^{n-1}$

$\therefore \text{adj } A.(\text{adj adj } A) = |\text{adj } A|I$

Pre-multiplying A

$\Rightarrow A.\text{adj } A.(\text{adj adj } A) = A\,|A|^{n-1}\,I$

$\Rightarrow |A|.I\,(\text{adj adj } A) = A\,|A|^{n-1}$

$\Rightarrow (\text{adj adj } A) = A\,|A|^{n-2}$

Illustration 29:

If A is square matrix of order 3 then $|\text{adj}(\text{adj } A)| =$

(A) $|A|$ (B) $|A|^2$ (C) $|A|^3$ (D) $|A|^4$

Ans. (D)

Solution:

$\because$ A is square matrix of order 3

$\therefore$ $|\text{adj } A| = |A|^2$

$\therefore$ $|\text{adj}(\text{adj } A)| = |\text{adj } A|^2 = (|A|^2)^2 = |A|^4$

Inverse of a Matrix (Reciprocal Matrix):

A square matrix A said to be invertible if and only if it is non-singular (i.e. $|A| \neq 0$) and there exists a matrix B such that, $AB = I = BA$.

B is called the **inverse** (reciprocal) of A and is denoted by A^{-1}. Thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

We have, $A.(\text{adj } A) = |A|I_n$

$A^{-1}.A(\text{adj } A) = A^{-1}I_n|A|$

$I_n(adjA) = A^{-1}|A|I_n$

$\therefore A^{-1} = \dfrac{(adjA)}{|A|}$

Note: The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$

Properties of Inverse:

(i) If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Note: If $A_1, A_2, \ldots\ldots\ldots\ldots\ldots A_n$ are all invertible square matrices of order n

then $(A_1A_2\ldots\ldots\ldots\ldots\ldots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1}\ldots\ldots\ldots\ldots A_2^{-1} A_1^{-1}$

(ii) If A be an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

(iii) If A is invertible, (a) $(A^{-1})^{-1} = A$ (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}; k \in N$

(iv) If A is non-singular matrix, then $|A^{-1}| = |A|^{-1}$

(v) If idempotent matrix is invertible then its inverse will be identity matrix.

(vi) A nilpotent matrix will not be invertible because its determinant value is zero.

(vii) Orthogonal matrix A is always invertible and $A^{-1} = A^T$.

(viii) $A = A^{-1}$ for an involutory matrix.

Cancellation Law:

Let A, B, C be square matrices of the same order $'n'$.

If A is a non-singular matrix, then

(a) $AB = AC \Rightarrow B = C$ (Left cancellation law)

(b) $BA = CA \Rightarrow B = C$ (Right cancellation law)

Note that these cancellation laws hold only if the matrix $'A'$ is **non – singular** (i.e. $|A| \neq 0$).

Illustration 30:

Prove that if A is non − singular matrix such that A is symmetric then A^{-1} is also symmetric.

Solution:

$A^T = A$ [$\because$ A is a symmetric matrix]

$\Rightarrow (A^T)^{-1} = A^{-1}$ [since A is non-singular matrix]

$\Rightarrow (A^{-1})^T = A^{-1}$ Hence proved

Illustration 31:

$\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$ is equal to -

(A) $\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ (B) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(C) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (D) none of these

Ans. (C)

Solution:

$$\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1} = \frac{1}{\sec^2\theta/2}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$$

$$\therefore \text{ Product} = \frac{1}{\sec^2\theta/2}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$$

$$= \frac{1}{\sec^2\theta/2}\begin{bmatrix} 1-\tan^2\theta/2 & -2\tan\theta/2 \\ 2\tan\theta/2 & 1-\tan^2\theta/2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta/2\ \sin^2\theta/2 & -2\sin\theta/2\cos\theta/2 \\ 2\sin\theta/2\ \cos\theta/2 & \cos^2\theta/2-\sin^2\theta/2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Illustration 32:

If $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $M = AB$, then M^{-1} is equal to

(A) $\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/6 \end{bmatrix}$

(C) $\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ (D) $\begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/6 \end{bmatrix}$

Ans. (C)

Solution:

$$M=\begin{bmatrix}0 & -1 & 2\\ 2 & -2 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 2\\ -2 & 2\end{bmatrix}$$

$$\Rightarrow |M| = 6,\ \text{adj } M=\begin{bmatrix}2 & -2\\ 2 & 1\end{bmatrix}$$

$$\therefore\ M^{-1}=\frac{1}{6}\begin{bmatrix}2 & -2\\ 2 & 1\end{bmatrix}=\begin{bmatrix}1/3 & -1/3\\ 1/3 & 1/6\end{bmatrix}$$

System of Equation & Criteria for Consistency:

Gauss - Jordan Method:

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\Rightarrow \begin{bmatrix}a_1x+b_1y+c_1z\\ a_2x+b_2y+c_2z\\ a_3x+b_3y+c_3z\end{bmatrix}=\begin{bmatrix}d_1\\ d_2\\ d_3\end{bmatrix} \Rightarrow \begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}d_1\\ d_2\\ d_3\end{bmatrix}$$

$\Rightarrow AX = B$ ...(i)

Multiplying $adjA$ on both the sides of (i)

$\Rightarrow (adj\ A)AX = (adj\ A)B \Rightarrow |A|X = (adj\ A)B$

$|A|X = (\text{adj}A)B$

- If $|A| \neq 0$
  - $X = \dfrac{(adjA)B}{|A|}$ or $X = A^{-1}B$ (unique solution)
- If $|A| = 0$
  - $(\text{adj } A)B$ = null matrix → Infinite solution or no solution
  - If $(\text{adj } A)B \neq 0$ → Inconsistent (no solution)

Illustration 33:

Solve the system $\begin{matrix}x+y+z=6\\ x-y+z=2\\ 2x+y-z=1\end{matrix}$ using matrix method.

Solution:

Let $A=\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 2 & 1 & -1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ & $B=\begin{bmatrix}6\\ 2\\ 1\end{bmatrix}$

Then the system is $AX = B$.

$= 6$, hence A is nonsingular,

Cofactor $A=\begin{bmatrix}0 & 3 & 3\\ 2 & -3 & 1\\ 2 & 0 & -2\end{bmatrix}$

$$\Rightarrow \text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$\Rightarrow X = A^{-1}B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \text{ i.e. } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 1, y = 2, z = 3$

Illustration 34:

The system of equations $x_1 - x_2 + x_3 = 2, 3x_1 - x_2 + 2x_3 = -6$ and $3x_1 + x_2 + x_3 = -18$ has

(A) No solution
(B) Exactly one solution
(C) Infinite solutions
(D) None of these

Ans. (C)

Solution:

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 1[-1-2] - 1[6-3] + 1[3+3] = 0$$

And $D_1 = \begin{vmatrix} 2 & -1 & 1 \\ -6 & -1 & 2 \\ -18 & 1 & 1 \end{vmatrix} = 2(-1-2) - 1(-36+6) + 1(-6-18)$

$= -6 + 30 - 24 = 0$

Also, $D_2 = 0 ; D_3 = 0$

So, the system is consistent $(D = D_1 = D_2 = D_3 = 0)$

i.e. system has infinite solution.

Finding Inverse Using Elementary Row Operation:

Let X, A and B be matrices of, the same order such that $X = AB$. In order to apply a sequence of elementary row operations on the matrix equation $X = AB$, we will apply these row operations simultaneously on X and on the first matrix A of the product AB on RHS.

Similarly, in order to apply a sequence of elementary column operations on the matrix equation $X = AB$, we will apply, these operations simultaneously on X and on the second matrix B of the product AB on RHS.

In view of the above discussion, we conclude that if A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations, write $A = IA$ and apply a sequence of row operation on $A = IA$ till we get, $I = BA$. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, then, write $A = AI$ and apply a sequence of column operations on $A = AI$ till we get, $I = AB$.

Note:

In case, after applying one or more elementary row (column) operations on $A = IA$ $(A = AI)$, if we obtain all zeros in one or more rows of the matrix A on L.H.S., then A^{-1} does not exist.

Illustration 35:

Obtain the inverse of the following matrix using elementary operations $A=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$.

Solution:

Write $A = I\,A$, i.e., $\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A$

$$\Rightarrow \begin{bmatrix}1&2&3\\0&1&2\\3&1&1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}A \text{ (applying } R_1\leftrightarrow R_2)$$

$$\Rightarrow \begin{bmatrix}1&2&3\\0&1&2\\0&-5&-8\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&-3&1\end{bmatrix}A \text{ (applying } R_3\to R_3-3R_1)$$

$$\Rightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&-5&-8\end{bmatrix}=\begin{bmatrix}-2&1&0\\1&0&0\\0&-3&1\end{bmatrix}A \text{ (applying } R_1\to R_1-2R_2)$$

$$\Rightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&0&2\end{bmatrix}=\begin{bmatrix}-2&1&0\\1&0&0\\5&-3&1\end{bmatrix}A \text{ (applying } R_3\to R_3+5R_2)$$

$$\Rightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&0&1\end{bmatrix}=\begin{bmatrix}-2&1&0\\1&0&0\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}A \text{ (applying } R_3\to\tfrac{1}{2}R_3)$$

$$\Rightarrow \begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{-1}{2}&\frac{1}{2}\\1&0&0\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}A \text{ (applying } R_1\to R_1-R_3)$$

$$\Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{-1}{2}&\frac{1}{2}\\-4&3&-1\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}A \text{ (applying } R_2\to R_2-R_3)$$

Hence $A^{-1}=\begin{bmatrix}\frac{1}{2}&\frac{-1}{2}&\frac{1}{2}\\-4&3&-1\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}$

JEE-Advanced (Part-1):

Illustration 36:

If $A=\begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is a unit matrix of order 2. Show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Solution:

We have, $I + A = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$ and $I - A = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}$

Now, $(I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$

$$=\begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}\begin{bmatrix} \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \frac{-2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \\ \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \end{bmatrix} = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\begin{bmatrix} \frac{1-t^2}{1+t^2} & -\frac{2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{bmatrix}, \text{ where } t = \tan\frac{\alpha}{2}.$$

$$=\begin{bmatrix} \frac{1-t^2+2t^2}{1+t^2} & \frac{-2t+t-t^3}{1+t^2} \\ \frac{-t+t^3+2t}{1+t^2} & \frac{2t^2+1-t^2}{1+t^2} \end{bmatrix} = \begin{bmatrix} \frac{1+t^2}{1+t^2} & \frac{-t\ (1+t^2)}{1+t^2} \\ \frac{t\ (1+t^2)}{1+t^2} & \frac{1+t^2}{1+t^2} \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix} = I + A$$

Illustration 37:

Determine the matrix B and C with integral element such that $A=\begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix} = B^3 + C^3$, where k is any integer.

(A) $B = A + I, C = -I$ (B) $C = A + I, B = -I$

(C) $B = A - I, C = -I$ (D) $B = A + I, C = +I$

Ans. (A)

Solution:

$A^2 = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 0 \\ -3k & 4 \end{bmatrix}$

$\Rightarrow A^2 + 3A = \begin{bmatrix} 1 & 0 \\ -3k & 4 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 3k & -6 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$\Rightarrow A^2 + 3A + 2I = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} + 2\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}.$

$\therefore A^3 + 3A^2 + 2A = O.$

$\Rightarrow (A + I)^3 - A = I^3 \Rightarrow A = (A + I)^3 + (-I)^3$

$\Rightarrow B^3 + C^3 \Rightarrow B = A + I$ and $C = -I$

Illustration 38:

Let $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then

(A) $[F(x)]^{-1} = F(x)$

(B) $[F(x)]^{-1} = -[F(-x)]$

(C) $[F(x)]^{-1} = -F(x)$

(D) $[F(x)]^{-1} = F(-x)$

Ans. (D)

Solution:

We have, $F(x)F(-x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$\Rightarrow F(-x)$ is the inverse of matrix $F(x)$ i.e. $[F(x)]^{-1} = F(-x)$.

JEE-Advanced (Part-2):

Illustration 39:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then prove that value of f & g satisfying the matrix equation $A^2 + fA + gI = O$ are equal to $-t_r(A)$ and determinant of A respectively. Given a, b, c, d are non-zero reals and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad A^2 + fA + gI = O$

$\Rightarrow A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cd+d^2 \end{bmatrix}$

$\Rightarrow f = (A)$

$= -(a+d)$

$\Rightarrow g = (ad - bc)$

$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cd+d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Illustration 40:

For the matrix $A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$ find A^{-2}.

Solution:

$A^{-2} \times A^{-1} \times A^{-1}$

$$\Rightarrow A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}; \ [C_{ij}] = \begin{bmatrix} 3 & -1 & -3 \\ 1 & 1 & 0 \\ -3 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow adj\ A = [C_{ij}]^T = \begin{bmatrix} 3 & 1 & 3 \\ -1 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow |A| = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$$

$= 4(3) + 4 - 15$

$= 16 - 15 = 1$ $\qquad A = \dfrac{Adj(A)}{|A|}$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & 1 & 3 \\ -1 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow A^{-1}A^{-1} = \begin{bmatrix} 3 & 1 & 3 \\ -1 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & 3 \\ -1 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 4 & -19 \\ -10 & 0 & 13 \\ -21 & -3 & 25 \end{bmatrix}$$

Illustration 41:

Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution:

$$BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} P \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{2\times3}$$

order of P will be 2×3

$$\Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$$

JEE-Advanced (Part-3):

Illustration 42:

$A=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & -2 & 4\end{bmatrix}$, $I=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ and $A^{-1}=\frac{1}{6}(A^2+cA+dI)$, then the value of c and d are -

(A) −6, −11 (B) 6, 11 (C) −6, 11 (D) 6, −11

Ans. (C)

Solution:

$A=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & -2 & 4\end{bmatrix} \Rightarrow |A| = 6$

$\Rightarrow A^{-1} \Rightarrow \frac{adjA}{|A|}=\frac{1}{6}\begin{bmatrix}6 & 0 & 0\\0 & 4 & -1\\0 & 2 & 1\end{bmatrix}$

$\Rightarrow A^2=\begin{bmatrix}1 & 0 & 0\\0 & -1 & 5\\0 & -10 & 14\end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{6}[A^2 + cA + dI]$

$\Rightarrow \frac{1}{6}\begin{bmatrix}6 & 0 & 0\\0 & 4 & -1\\0 & 2 & 1\end{bmatrix}=\frac{1}{6}\left\{\begin{bmatrix}1 & 0 & 0\\0 & -1 & 5\\0 & -10 & 14\end{bmatrix}+\begin{bmatrix}c & 0 & 0\\0 & c & c\\0 & -2c & 4c\end{bmatrix}+\begin{bmatrix}d & 0 & 0\\0 & d & 0\\0 & 0 & d\end{bmatrix}\right\}$

on comparing we get

$\Rightarrow -1 = 5 + c \Rightarrow c = -6$

$\Rightarrow 1 = 14 + 4c + d \Rightarrow 1 = 14 - 24 + d$

$\Rightarrow d = 11$

Illustration 43:

If $P=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\-\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$, $A=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ and $Q = PAP^T$ and $x = P^T Q^{2005} P$, then x is equal to -

(A) $\begin{bmatrix}1 & 2005\\0 & 1\end{bmatrix}$

(B) $\begin{bmatrix}4+2005\sqrt{3} & 6015\\2005 & 4-2005\sqrt{3}\end{bmatrix}$

(C) $\frac{1}{4}\begin{bmatrix}2+\sqrt{3} & 1\\-1 & 2-\sqrt{3}\end{bmatrix}$

(D) $\frac{1}{4}\begin{bmatrix}2005 & 2-\sqrt{3}\\2+\sqrt{3} & 2005\end{bmatrix}$

Ans. (A)

Solution:

$PP^T = I$

$\Rightarrow A=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$

and so, on

$\Rightarrow Q = PAP^T$

$\Rightarrow Q^2 = (PAP^T)\,(PAP^T) = PA^2P^T$

$\Rightarrow Q^{2005} = PA^{2005}P^T$

$\Rightarrow x = P^T(PA^{2005}P^T)P$

$\Rightarrow x = A^{2005} = \begin{bmatrix}1 & 2005\\0 & 1\end{bmatrix}$

Illustration 44:

Let p be an odd prime number and T_P be the following set of 2×2 matrices:

$T_p = \left\{A=\begin{bmatrix}a & b\\c & a\end{bmatrix} : a,b,c \in \{0,1,2,\ldots\ldots,p-1\}\right\}$

The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det(A) divisible by p is -

(A) $(p-1)^2$ (B) $2(p-1)$ (C) $(p-1)^2+1$ (D) $2p-1$

Ans. (D)

Solution:

If A is symmetric, $A^T = A$

$\Rightarrow \begin{bmatrix}a & b\\c & a\end{bmatrix}=\begin{bmatrix}a & c\\b & a\end{bmatrix} \Rightarrow b = c$

If A is skew symmetric, $A^T = -A$

$\Rightarrow \begin{bmatrix}a & b\\c & a\end{bmatrix}=\begin{bmatrix}-a & -c\\-b & -a\end{bmatrix} \Rightarrow a = 0, b + c = 0$

$\because b,c \geq 0 \Rightarrow a = 0,\ b = 0,\ c = 0$

Now, $\det(A) = a^2 - bc$

$= a^2 - b^2$ ($\because b = c$ for A being symmetric or skew symmetric or both)

$= (a-b)(a+b)$ is divisible by p.

Let $(a-b)(a+b) = \lambda p, \lambda \in I$

Range of $(a+b)$ is 0 to $2p-2$ which includes only one multiple of p i.e. p

$\therefore a + b = p$ ανδ $a - b \in I$

$\Rightarrow$ possible number of pairs of a and b will be $p-1$.

Also, range of $(a-b)$ is $1-p$ to $p-1$ which includes only one multiple of p i.e. 0

$\therefore a - b = 0$ and $a + b \in I$

$\Rightarrow$ Possible number of pairs of a and b will be p.

Hence total number of A in T_p will be $p + p - 1 = 2p - 1$.

JEE-Advanced (Part-4):

For illustration 1 to 3

Let A is matrix of order 2×2 such that $A^2 = o$.

Illustration 45:

$A^2 - (a+d)A + (ad - bc)I$ is equal to

(A) I (B) O (C) $-I$ (D) none of these

Ans. (B)

Solution:

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\Rightarrow A^2 - (a+d)A + (ad - bc)I$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad - bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = 0$$

Illustration 46:

$\mathrm{tr}(A)$ is equal to

(A) 1 (B) 0 (C) – 1 (D) none of these

Ans. (B)

Solution:

If $A = O, \mathrm{tr}(A) = 0$. suppose $A \neq O$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow |A| = 0$ and $A^2 - (a+d)A + (ad - bc)I = 0$

$\Rightarrow a + d = 0$

Illustration 47:

$(I + A)^{100} =$

(A) $100A$ (B) $100(I + A)$ (C) $100I + A$ (D) $I + 100A$

Ans. (D)

Solution:

$(I + A)^{100} = {}^{100}C_0 I^{100} + {}^{100}C_1 I^{99} A + {}^{100}C_2 I^{98} A^2 + \ldots\ldots + {}^{100}C_{100} A^{100}$

$= I + 100A + O + O + \ldots + O$

$= I + 100A$

For illustration 48 to 50

If A and B are two square matrices of order 3×3 which satisfy $AB = A$ and $BA = B$

Illustration 48:

Which of the following is true?

(A) If matrix A is singular then matrix B is non-singular

(B) If matrix A is non-singular then matrix B is singular

(C) If matrix A is singular then matrix B is also singular

(D) Cannot say anything.

Ans. (C)

Solution:

$AB = A \Rightarrow |AB| = |A|$ ...(1)

$\Rightarrow |A| = 0$ or $|B| = 1$

$\Rightarrow BA = B \Rightarrow |BA| = |B|$ ...(2)

$\Rightarrow |A| = 1$or $|B| = 0$

If $|A| = 0$, then from Eq. (2), $|B| = 0$

If $|B| = 0$, then from Eq. (1), $|A| = 0$

Illustration 49:

$(A + B)^9$ is equal to

(A) $9(A + B)$ (B) $9.I_{3\times 3}$ (C) $256(A + B)$ (D) $256\ I$

Ans. (C)

Solution:

$AB = A, BA = B$

$\Rightarrow ABA = A^2 \Rightarrow A(BA) = A^2 \Rightarrow AB = A^2 \Rightarrow A = A^2$

Similarly, $B^2 = B$

$\Rightarrow (A + B)^2 = A^2 + B^2 + AB + BA$

$\Rightarrow (A + B)^2 = A + B + A + B = 2(A + B)$

$\Rightarrow (A + B)^3 = (A + B)^2(A + B) = 2(A + B)^2 = 2^2(A + B)$

$\Rightarrow (A + B)^9 = 2^8(A + B) = 256(A + B)$

Illustration 50:

$(A + I)^5$ is equal to (where I is identity matrix)

(A) $I + 60I$ (B) $I + 16A$ (C) $I + 31A$ (D) none of these

Solution:

Ans. (C)

$(A + I)^5 = I + 5A + 10A^2 + 10A^3 + 5A^4 + A^5$

$= I + 5A + 10A + 10A + 5A + A$

$(\because A^2 = A \Rightarrow A^3 = A^4 = A^5 = \cdots = A)$

$= I + 31A$