05 Logarithm

1. DEFINITION :

The logarithm of a number N to a base 'a' is an exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$. (Read it "Log N on base a"). Here N is usually called argument of the Logarithm and 'a' is called base of the Logarithm.

Hence : $\log_a N = x \Leftrightarrow a^x = N$, $a > 0$, $a \neq 1$ and $N > 0$

By the definition of logarithm, $\log_2 16$ is the exponent indicating the power to which 2 must be raised in order to obtain 16.

As $2^4 = 16$, hence $\log_2 16 = 4$.

Similarly

Exponential Form		Logarithmic Form
$3^5 = 243$	$\Leftrightarrow$	$\log_3 243 = 5$
$5^4 = 625$	$\Leftrightarrow$	$\log_5 625 = 4$
$2^{-3} = \frac{1}{8}$	$\Leftrightarrow$	$\log_2 \frac{1}{8} = -3$
$7^0 = 1$	$\Leftrightarrow$	$\log_7 1 = 0$

Note that the expressions $\log_3(-27)$, $\log_1 16$, $\log_0 5$ and $\log_2 0$ has no sense in real numbers since the equations $3^x = -27$, $1^x = 6$, $0^x = 5$, $2^x = 0$ are absurd for any real x, the reason being obvious that no such exponent x in real number could be found.

In general, the expression $\log_a N$ is meaningful if and only if, $a > 0$, $a \neq 1$ and $N > 0$.

The existence and uniqueness of the number $\log_a N$ follows from the properties of exponential functions.

Illustration 1 :

If $\log_4 m = 1.5$, then find the value of m.

Solution :

$\log_4 m = 1.5 \Rightarrow m = 4^{3/2} \Rightarrow m = 8$

Illustration 2 :

If $\log_5 p = a$ and $\log_2 q = a$, then prove that $\frac{p^4q^4}{100} = 100^{2a-1}$

Solution :

$\log_5 p = a \Rightarrow p = 5^a$

$\Rightarrow \log_2 q = a \Rightarrow q = 2^a$

$$\Rightarrow \frac{p^4q^4}{100} = \frac{5^{4a}.2^{4a}}{100} = \frac{(10)^{4a}}{100} = \frac{(100)^{2a}}{100} = 100^{2a-1}$$

Illustration 3 :

The value of N, satisfying $\log_a[1 + \log_b\{1 + \log_c(1 + \log_p N)\}] = 0$ is –

(A) 4 (B) 3 (C) 2 (D) 1

Ans. (D)

Solution :

$1+\log_b\{1+\log_c(1+\log_p N)\}=a^0=1$

$\Rightarrow \log_b\{1+\log_c(1+\log_p N)\}=0$

$\Rightarrow 1+\log_c(1+\log_p N)=1$

$\Rightarrow \log_c(1+\log_p N)=0$

$\Rightarrow 1+\log_p N=1$

$\Rightarrow \log_p N=0$

$\Rightarrow N=1$

2. Fundamental logarithmic identity :

From the definition of the logarithm of the number N to the base 'a', we have an identity :

$a^{\log_a N}=N$, $a>0$, $a\neq 1$ and $N>0$

This is known as the **FUNDAMENTAL LOGARITHMIC IDENTITY.**

Note :

Using the basic definition of logarithm we have 3 important deductions :

(a) $\log_a 1=0$ i.e. logarithm of unity to any base is zero ($a>0;\ a\neq 1$).

(b) $\log_N N=1$ i.e. logarithm of a number to the same base is 1. ($N>0;\ N\neq 1$)

(c) $\log_{\frac{1}{N}} N=-1=\log_N \frac{1}{N}$ i.e. logarithm of a number to the base as its reciprocal is –1. ($N>0;\ N\neq 1$)

3. The principal properties of logarithms :

If m, n are arbitrary positive numbers where $a>0$, $a\neq 1$ and x is any real number, then

(a) $\log_a mn=\log_a m+\log_a n$ (b) $\log_a \frac{m}{n}=\log_a m-\log_a n$ (c) $\log_a m^x=x\log_a m$

Illustration 4 :

Find the value of $2\log\frac{2}{5}+3\log\frac{25}{8}-\log\frac{625}{128}$

Solution :

$2\log\frac{2}{5}+3\log\frac{25}{8}+\log\frac{128}{625}$

$=\log\frac{2^2}{5^2}+\log\left(\frac{5^2}{2^3}\right)^3+\log\frac{2^7}{5^4}$

$=\log\frac{2^2}{5^2}\cdot\frac{5^6}{2^9}\cdot\frac{2^7}{5^4}=\log 1=0$

Illustration 5 :

If $\log_e x-\log_e y=a, \log_e y-\log_e z=b$ & $\log_e z-\log_e x=c$, then find the value of $\left(\frac{x}{y}\right)^{b-c}\times\left(\frac{y}{z}\right)^{c-a}\times\left(\frac{z}{x}\right)^{a-b}$

Solution :

$\log_e x - \log_e y = a \Rightarrow \log_e \frac{x}{y} = a \Rightarrow \frac{x}{y} = e^a$

$\log_e y - \log_e z = b \Rightarrow \log_e \frac{y}{z} = b \Rightarrow \frac{y}{z} = e^b$

$\log_e z - \log_e x = c \Rightarrow \log_e \frac{z}{x} = c \Rightarrow \frac{z}{x} = e^c$

$\therefore \left(e^a\right)^{b-c} \times \left(e^b\right)^{c-a} \times \left(e^c\right)^{a-b}$

$= e^{a(b-c)+b(c-a)+c(a-b)} = e^0 = 1$

Illustration 6 :

If $a^2 + b^2 = 23ab$, then prove that $\log\frac{(a+b)}{5} = \frac{1}{2}(\log a + \log b)$.

Solution :

$a^2 + b^2 = (a+b)^2 - 2ab = 23ab$

$\Rightarrow (a+b)^2 = 25ab \Rightarrow a + b = 5\sqrt{ab}$ ....(i)

Using (i)

L.H.S. $= \log\frac{(a+b)}{5} = \log\frac{5\sqrt{ab}}{5} = \frac{1}{2}\log ab = \frac{1}{2}(\log a + \log b)$ = R.H.S.

Illustration 7 :

If $\log_a x = p$ and $\log_b x^2 = q$, then $\log_x \sqrt{ab}$ is equal to (where $a, b, x \in \mathbb{R}^+ - \{1\}$)-

(A) $\frac{1}{p} + \frac{1}{q}$ (B) $\frac{1}{2p} + \frac{1}{q}$ (C) $\frac{1}{p} + \frac{1}{2q}$ (D) $\frac{1}{2p} + \frac{1}{2q}$

Solution :

$\log_a x = p \Rightarrow a^p = x \Rightarrow a = x^{\frac{1}{p}}$.

similarly $b^q = x^2 \Rightarrow b = x^{\frac{2}{q}}$

Now, $\log_x \sqrt{ab} = \log_x \sqrt{x^{1/p} x^{2/q}} = \log_x x^{\left(\frac{1}{p}+\frac{2}{q}\right)\cdot\frac{1}{2}} = \frac{1}{2p} + \frac{1}{q}$

4. Base changing theorem :

Can be stated as "quotient of the logarithm of two numbers is independent of their common base."

Symbolically, $\log_b m = \frac{\log_a m}{\log_a b}$, where $a > 0, a \neq 1, b > 0, b \neq 1$

Note :

(i) $\log_b a . \log_a b = \frac{\log a}{\log b} . \frac{\log b}{\log a} = 1$; hence $\boxed{\log_b a = \frac{1}{\log_a b}}$.

(ii) $a^{\log_b c} = c^{\log_b a}$

(iii) Base power formula: $\log_{a^k} m = \frac{1}{k}\log_a m$

(iv) The base of the logarithm can be any positive number other than 1, but in normal practice, only two bases are popular, these are 10 and e(=2.718 approx). Logarithms of numbers to the base 10 are named as 'common logarithm' and the logarithms of numbers to the base e are named as Natural or Napierian logarithm. **We will consider log x as $\log_e x$ or $\ell n\ x$.**

(v) Conversion of base e to base 10 & viceversa :

$$\log_e a = \frac{\log_{10} a}{\log_{10} e} = 2.303 \times \log_{10} a\ ;\ \log_{10} a = \frac{\log_e a}{\log_e 10} = \log_{10} e \times \log_e a = 0.434 \log_e a$$

(vi) Some important values : $\log_{10}2 \approx 0.3010$; $\log_{10}3 \approx 0.4771$; $\ell n2 \approx 0.693$, $\ell n10 \approx 2.303$

(vii) The positive real number 'n' is called the antilogarithm of a number 'm' to base 'a' if $\log_a n = m$

Thus, $\log_a n = m \Leftrightarrow n = \text{antilog}_a m$

Illustration 8 :

If a, b, c are distinct positive real numbers different from 1 such that

$(\log_b a . \log_c a - \log_a a) + (\log_a b . \log_c b - \log_b b) + (\log_a c . \log_b c - \log_c c) = 0$, then abc is equal to –

(A) 0 (B) e (C) 1 (D) none of these

Solution :

$(\log_b a \log_c a - 1) + (\log_a b . \log_c b - 1) + (\log_a c \log_b c - 1) = 0$

$\Rightarrow \frac{\log a}{\log b} . \frac{\log a}{\log c} + \frac{\log b}{\log a} . \frac{\log b}{\log c} + \frac{\log c}{\log a} . \frac{\log c}{\log b} = 3$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3 \log a \log b \log c$

$\Rightarrow (\log a + \log b + \log c) = 0$ [$\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c$]

$\Rightarrow \log abc = \log 1 \Rightarrow abc = 1$

Illustration 9 :

Evaluate : $81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$

Solution :

$81^{\log_3 5} + 3^{3\log_9 36} + 3^{4\log_9 7}$

$= 3^{4\log_3 5} + 3^{\log_3 (36)^{3/2}} + 3^{\log_3 7^2}$ = 625 + 216 + 49 = 890.

Illustration 10 :

Show that $\log_4 18$ is an irrational number.

Solution :

$\log_4 18 = \log_4(3^2 \times 2) = 2\log_4 3 + \log_4 2 = 2\frac{\log_2 3}{\log_2 4} + \frac{1}{\log_2 4} = \log_2 3 + \frac{1}{2}$

assume the contrary, that this number $\log_2 3$ is rational number.

$\Rightarrow \log_2 3 = \frac{p}{q}$. Since $\log_2 3 > 0$ both numbers p and q may be regarded as natural number

$\Rightarrow 3 = 2^{\frac{p}{q}} \Rightarrow 2^p = 3^q$

But this is not possible for any natural number p and q. The resulting contradiction completes the proof.

Illustration 11 :

If in a right angled triangle, a and b are the lengths of sides and c is the length of hypotenuse and $c - b \neq 1$, $c + b \neq 1$, then show that $\log_{c+b} a + \log_{c-b} a = 2 \log_{c+b} a . \log_{c-b} a$.

Solution :

We know that in a right angled triangle

$c^2 = a^2 + b^2$

$c^2 - b^2 = a^2$ ...(i)

LHS = $\dfrac{1}{\log_a(c+b)} + \dfrac{1}{\log_a(c-b)} = \dfrac{\log_a(c-b)+\log_a(c+b)}{\log_a(c+b).\log_a(c-b)}$

$= \dfrac{\log_a(c^2-b^2)}{\log_a(c+b).\log_a(c-b)} = \dfrac{\log_a a^2}{\log_a(c+b).\log_a(c-b)}$ (using (i))

$= \dfrac{2}{\log_a(c+b).\log_a(c-b)}$ = $2\log_{(c+b)} a \,.\, \log_{(c-b)} a$ = RHS

5. logarithmic equations :

Illustration 12 :

$\log_{\frac{1}{2}}\left(\log_2 \sqrt{2}x\right) = 1$, then find x ?

Solution :

$\log_{\frac{1}{2}}\left(\log_2 \sqrt{2}x\right) = 1$

$\Rightarrow \log_2\left(\sqrt{2}x\right) = \dfrac{1}{2}$

$\Rightarrow \sqrt{2}x = 2^{\frac{1}{2}}$

$\Rightarrow x = 1$

Illustration 13 :

Solve the equation $2\log_2(\log_2 x) + \log_{1/2}\left(\dfrac{3}{2} + \log_2 x\right) = 1$.

Solution :

Let $\log_2 x = t$

$\Rightarrow 2\log_2(t) + \log_{\frac{1}{2}}\left(\dfrac{3}{2} + t\right) = 1$

$\Rightarrow 2\log_2 t - \log_2\left(\dfrac{3}{2} + t\right) = 1$

$\Rightarrow \log_2\left(\dfrac{t^2}{\frac{3}{2} + t}\right) = 1$

$\Rightarrow \dfrac{2t^2}{3+2t} = 2$

$\Rightarrow t^2 - 2t - 3 = 0$

$\Rightarrow (t+1)(t-3) = 0$

$\Rightarrow t = 3 \because t > 0$

$\Rightarrow x = 8$

Illustration 14 :

Solve the equation log x^2 – log $(2x)$ = 3 log 3 – log 6.

Solution :

$\log x^2 - \log 2x = 3\log 3 - \log 6,\ x > 0$

$\Rightarrow 2\log x - \log 2 - \log x = 3\log 3 - \log 2 - \log 3 \Rightarrow \log x = 2\log 3 \Rightarrow \log x = \log 9$

$\Rightarrow x = 9$

Illustration 15 :

Solve the equation $(\log_5 x)^2 + \log_5 x + 1 = \dfrac{7}{\log_5 x - 1}$

Solution :

Put $\log_5 x = t$, we get $t^2 + t + 1 = \dfrac{7}{t-1}$

$(t-1)(t^2+t+1) = 7 \Rightarrow t^3 + t^2 + t - t^2 - t - 1 = 7 \Rightarrow t^3 - 8 = 0$

$\Rightarrow (t-2)(t^2+2t+4) = 0 \Rightarrow t - 2 = 0;\ t^2 + 2t + 4 \neq 0 \Rightarrow t = 2$

Now, $t = \log_5 x$, so $\log_5 x = 2$

$x = 5^2 \Rightarrow x = 25$

Illustration 16 :

Solve the equation $|x-1|^{\log_2 x^2 - 2\log_x 4} = (x-1)^7$

Solution :

Obviously $x = 2$ is a solution. Since, left side is positive, $x - 1 > 0$.

The equation reduces $\log_2 x^2 - 2\log_x 4 = 7$

$\Rightarrow 2t - \dfrac{4}{t} = 7, t = \log_2 x$

$\Rightarrow 2t^2 - 7t - 4 = 0 \Rightarrow t = 4, -\dfrac{1}{2}$

But $t > 0$ since $x > 1$. $\therefore\ t = 4$

$\Rightarrow x = 2^4 = 16$

$\therefore\ x = 2, 16$

Illustration 17 :

Solve the equation $x^{\log_3 x^2 + (\log_3 x)^2 - 10} = \dfrac{1}{x^2}$

Solution :

Taking $\log_3$ on both sides, we get

$(2t + t^2 - 10)t = -2t, \qquad t = \log_3 x$

$\Rightarrow t(t^2 + 2t - 8) = 0 \Rightarrow t = 0, 2, -4$

$\Rightarrow x = 1, 9, \dfrac{1}{81}$

Illustration 18 :

Solve the equation $4^{\log_2 \ell nx} = \ell nx - (\ell nx)^2 + 1$

Solution :

$4^{\log_2 \ell nx} = 2^{2\log_2(\ell nx)} = 2^{\log_2(\ell nx)^2} = (\ell nx)^2$

$\Rightarrow (\ell n x)^2 = \ell n x - (\ell n x)^2 + 1 \Rightarrow 2(\ell n x)^2 - \ell n x - 1 = 0 \Rightarrow \ell n x = 1, -\frac{1}{2}$

But $\ell nx > 0$

$\therefore\ \ell nx = 1 \Rightarrow x = e.$

Illustration 19 :

Solve the equation x : $\log_{x+1}(x^2 + x - 6)^2 = 4$

Solution :

We have,

$\log_{x+1}(x^2 + x - 6)^2 = 4$

$\Rightarrow (x^2 + x - 6)^2 = (x + 1)4 = (x^2 + 2x + 1)^2$

$\Rightarrow (x^2 + x - 6 - x^2 - 2x - 1)(x^2 + x - 6 + x^2 + 2x + 1) = 0$

$\Rightarrow (-x - 7)\ (2x^2 + 3x - 5) = 0$

$\Rightarrow (x + 7)\ (x - 1)\ (2x + 5) = 0$

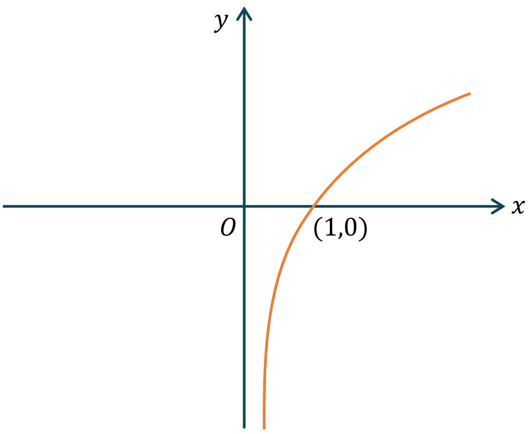
$\Rightarrow x\ = -7, -5/2, 1$

The values $x = -7$ and $x = -5/2$ are rejected because they make the base $x + 1$ negative

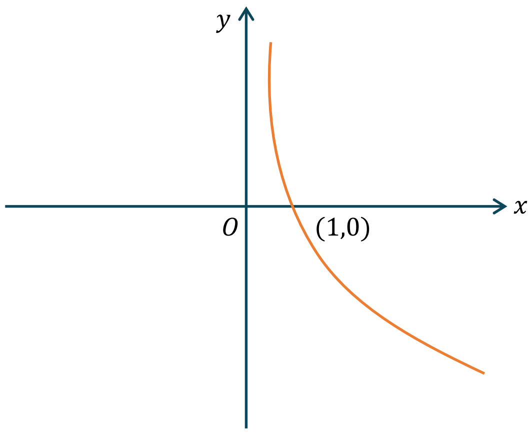
Hence, $x = 1$ is the only solution of the given equation.

6. GRAPHS OF LOGARITHMIC FUNCTION AND ITS INEQUALITIES :

Graph of $y = \log_a x$:



When $a > 1$



When $0 < a < 1$

(i) $\log_a x < \log_a y \Leftrightarrow \begin{cases} x < y & \text{if} \quad a > 1 \\ x > y & \text{if} \quad 0 < a < 1 \end{cases}$

(ii) If $a > 1$, then $\log_a x < p \Rightarrow 0 < x < a^p$ and $\log_a x > p \Rightarrow x > a^p$

(iii) If $0 < a < 1$, then $\log_a x < p \Rightarrow x > a^p$ and $\log_a x > p \Rightarrow 0 < x < a^p$

Note :

(i) If base of logarithm is greater than 1 then logarithm of greater number is greater. i.e.$\log_2 8 = 3$, $\log_2 4 = 2$ etc. and if base of logarithm is between 0 and 1 then on that base logarithm of greater number is smaller. i.e. $\log_{1/2} 8 = -3$, $\log_{1/2} 4 = -2$ etc.

(ii) It must be noted that whenever the number and the base are on the same side of unity then logarithm of that number to that base is positive, however if the number and the base are located on different side of unity then logarithm of that number to that base is negative.

e.g. $\log_{10}\sqrt[3]{10}=\frac{1}{3}$; $\log_{\sqrt{7}}49=4$; $\log_{\frac{1}{2}}\left(\frac{1}{8}\right)=3$; $\log_{2}\left(\frac{1}{32}\right)=-5$; $\log_{10}(0.001) = -3$

Illustration 20 :

Solve for x : $x^{\log_5 x}>5$

Solution :

as $x > 0$ (for existance)

now solving inequality

$x^{\log_5 x}>5$. Taking 'log' with base '5' we have $\log_5 x.\log_5 x > 1$

$\Rightarrow (\log_5 x - 1)(\log_5 x + 1) > 0 \Rightarrow \log_5 x > 1$ or $\log_5 x < -1$

$\Rightarrow x > 5$ or $x < 1/5$. Also we must have $x > 0$

Thus, $x \in (0,1/5)$ or $x \in (5,\infty)$

Illustration 21 :

Solve for x : $\log_3(2x + 1) < \log_3 5$.

Solution :

Checking existance

$2x + 1 > 0 \Rightarrow x>-\frac{1}{2}$

Now solving inequality we have $2x + 1 < 5$

$\Rightarrow\ 2x < 4$

$\Rightarrow\ x < 2$

$\Rightarrow\ x \in (-1/2,2)$

Illustration 22 :

Solve for x : $(\log_{10}100x)^2 + (\log_{10}10x)^2 + \log_{10}x \leq 14$.

Solution :

Checking existance

$x > 0$

Now solving inequality,

Let $u = \log_{10}x$

$(2 + u)^2 + (1 + u)^2 + u \leq 14 \Rightarrow u^2 + 4u + 4 + u^2 + 2u + 1 + u \leq 14$

$\Rightarrow 2u^2 + 7u - 9 \leq 0 \Rightarrow 2u^2 + 9u - 2u - 9 \leq 0$

$\Rightarrow u(2u + 9) - 1(2u + 9) \leq 0 \Rightarrow (2u + 9)(u - 1) \leq 0$

$\Rightarrow\ \frac{-9}{2}\leq u\leq 1\Rightarrow\frac{-9}{2}\leq\log_{10}x\leq 1\Rightarrow 10^{-\frac{9}{2}}\leq x\leq 10$

Illustration 23 :

Solve for x : $\log_3((x + 2)(x + 4)) + \log_{1/3}(x + 2) < \frac{1}{2}\log_{\sqrt{3}}7$.

Solution :

Checking existance,

$(x + 2)(x + 4) > 0$ and $(x + 2) > 0$

$x < -4$ or $x > -2$ and $x > -2$

Now solving inequality.

$\log_3((x + 2)(x + 4)) + \log_{1/3}(x + 2) < \frac{1}{2}\log_{\sqrt{3}} 7$.

$\Rightarrow \log_3(x + 2)(x + 4) - \log_3(x + 2) < \log_3 7$

$\Rightarrow \log_3(x + 4) < \log_3 7$

$\Rightarrow x + 4 < 7 \Rightarrow x < 3$

$\Rightarrow -2 < x < 3 \Rightarrow x \in (-2,3)$

Illustration 24 :

Solve for x : $\log_{1/3}\log_4(x^2 - 5) > 0$

Solution :

Checking existence,

(i) $\log_4(x^2 - 5) > 0 \Rightarrow x^2 - 5 > 1$

$\Rightarrow \left(x-\sqrt{6}\right)\left(x+\sqrt{6}\right)>0$

$\Rightarrow x\in\left(-\infty,-\sqrt{6}\right)\cup\left(\sqrt{6},\infty\right)$

(ii) $x^2 - 5 > 0 \Rightarrow x\in\left(-\infty,-\sqrt{5}\right)\cup\left(\sqrt{5},\infty\right)$

solving inequality,

$\log_{\frac{1}{3}}\log_4\left(x^2-5\right)>0$

$\Rightarrow \log_4\left(x^2-5\right)<1$

$\Rightarrow x^2 - 5 < 4 \Rightarrow x^2 - 9 < 0$

$\Rightarrow (x - 3)(x + 3) < 0$

$\Rightarrow x \in (-3,3)$ ...(iii)

$\therefore$ Answer : (i) $\cap$ (ii) $\cap$ (iii)

$\Rightarrow x\in\left(-3,-\sqrt{6}\right)\cup\left(\sqrt{6},3\right)$

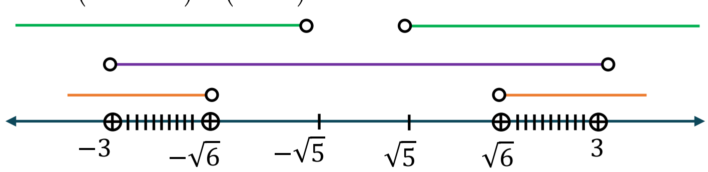


Illustration 25 :

Solve for x : $\dfrac{\log_2\left(4x^2-x-1\right)}{\log_2\left(x^2+1\right)}>1$

Solution :

Checking existence

(i) $x^2 + 1 \neq 1 \Rightarrow x \neq 0$

(ii) $4x^2 - x - 1 > 0$

Now solving inequality,

$$\frac{\log_2(4x^2 - x - 1)}{\log_2(x^2 + 1)} > 1$$

$\Rightarrow \log_2(4x^2 - x - 1) > \log_2(x^2 + 1) \Rightarrow \log_2(4x^2 - x - 1) - \log_2(x^2 + 1) > 0$

$$\Rightarrow \log_2 \frac{4x^2 - x - 1}{x^2 + 1} > 0$$

$\Rightarrow 4x^2 - x - 1 - x^2 - 1 > 0 \Rightarrow 3x^2 - x - 2 > 0 \Rightarrow 3x^2 - 3x + 2x - 2 > 0$

$\Rightarrow 3x(x - 1) + 2(x - 1) > 0 \Rightarrow (x - 1)(3x + 2) > 0 \Rightarrow x < -2/3$ or $x > 1$

$\Rightarrow x \in (-\infty, -2/3)$ or $x \in (1, \infty)$

Note : $x < -\frac{2}{3}$ and $x > 1$; $4x^2 - x - 1 > 0$

Illustration 26 :

Solve for x : $\log_4(3^x - 1)\log_{1/4}\left(\frac{3^x - 1}{16}\right) \le \frac{3}{4}$

Solution :

Checking existence $3^x - 1 > 0 \Rightarrow x > 0$,

Now solving inequality

$$\log_4(3^x - 1)\log_{1/4}\left(\frac{3^x - 1}{16}\right) \le \frac{3}{4}$$

$$\Rightarrow \log_4(3^x - 1).\left[-\log_4(3^x - 1) + \log_4 16\right] \le \frac{3}{4}$$

$$\Rightarrow \log_4(3^x - 1)\left[-\log_4(3^x - 1) + 2\right] \le \frac{3}{4} \Rightarrow -\left[\log_4(3^x - 1)\right]^2 + 2\left[\log_4(3^x - 1)\right] \le \frac{3}{4}$$

Put $\log_4(3^x - 1) = t \Rightarrow -t^2 + 2t \le \frac{3}{4}$

$\Rightarrow -4t^2 + 8t - 3 \le 0 \Rightarrow 4t^2 - 8t + 3 \ge 0 \Rightarrow 4t^2 - 6t - 2t + 3 \ge 0$

$\Rightarrow 2t(2t - 3) - 1(2t - 3) \ge 0 \Rightarrow (2t - 3)(2t - 1) \ge 0$

$$\Rightarrow \log_4(3^x - 1) \le \frac{1}{2} \text{ or } \log_4(3^x - 1) \ge \frac{3}{2}$$

$\Rightarrow 0 < 3^x - 1 \le 4^{1/2}$ or $3^x - 1 \ge 4^{3/2}$

$\Rightarrow 1 < 3^x \le 3$ or $3^x \ge 9$

$\Rightarrow 0 < x \le 1$ or $x \ge 2$

$\Rightarrow x \in (0, 1] \cup [2, \infty)$

Illustration 27 :

Solve for x : $\log_{1/3}(x^2 - 6x + 18) - 2\log_{1/3}(x - 4) < 0$

Solution :

Checking existence

(1) $x^2 - 6x + 18 > 0 \Rightarrow x \in \mathbb{R}$

(2) $x - 4 > 0 \Rightarrow x > 4$

Now solving inequality

$\Rightarrow \log_{1/3}(x^2 - 6x + 18) - \log_{1/3}(x - 4)^2 < 0$

$\Rightarrow \log_{1/3}\frac{(x^2-6x+18)}{(x-4)^2} < 0$ and $2\log_{1/3}(x - 4) = \log_{1/3}(x - 4)^2$

only when $x - 4 > 0$, so we get

$\Rightarrow \log_{1/3}\left(\frac{x^2-6x+18}{(x-4)^2}\right) < 0$ and $x - 4 > 0 \Rightarrow x > 4$ ...(i)

$\Rightarrow x^2 - 6x + 18 > (x - 4)^2 \Rightarrow x^2 - 6x + 18 > x^2 - 8x + 16$

$2x + 2 > 0 \Rightarrow x > -1 \Rightarrow x \in (-1,\infty)$ ...(ii)

from equation (i) and (ii), we get $x \in (4,\infty)$

Illustration 28 :

Solve for x : $\log_e(x^2 - 2x - 2) \le 0$

Solution :

The values of x satisfying the inequality $\log_e(x^2 - 2x - 2) \le 0$ must be such that

$0 < x^2 - 2x - 2 \le 1$

we have, $x^2 - 2x - 2 > 0 \Rightarrow (x - 1)^2 > 3$

$\Rightarrow |x - 1|^2 > 3 \Rightarrow |x-1| > \sqrt{3} \quad \Rightarrow x-1 > \sqrt{3} \quad$ or $\; x-1 < -\sqrt{3}$

$\Rightarrow x > 1+\sqrt{3}$ or $x < 1-\sqrt{3}$ ...(1)

Again $x^2 - 2x - 2 \le 1 \Rightarrow x^2 - 2x \le 3 \Rightarrow (x - 1)^2 \le 4 \Rightarrow |x - 1|^2 \le 4$

$\Rightarrow |x - 1| \le 2 \Rightarrow -2 \le x - 1 \le 2 \Rightarrow -1 \le x \le 3$ ...(2)

The value of x satisfying both the inequalities equation (1) and (2) are given by;

Hence, $x \in \left[-1, 1-\sqrt{3}\right) \cup \left(1+\sqrt{3}, 3\right]$

Illustration 29 :

Solve for x : $\log_x\left(2x - \frac{3}{4}\right) > 2$

Solution :

For existence of logarithm

$2x - \frac{3}{4} > 0$ and $x > 0$ and $x \ne 1$

so, $x \in \left(\frac{3}{8}, \infty\right) - \{1\}$

To find the value of x satisfying the inequality $\log_x[2x - (3/4)] > 2$

Case I. Let $0 < x < 1$

Then, $\log_x[2x - (3/4)] > 2 \Rightarrow [2x - (3/4)] < x^2$

$\Rightarrow x^2 - 2x + (3/4) > 0 \Rightarrow 4x^2 - 8x + 3 > 0 \Rightarrow (2x - 1)(2x - 3) > 0$

$\Rightarrow \left(x - \frac{1}{2}\right)\left[x - (3/2)\right] > 0 \Rightarrow x > 3/2$ or $x < 1/2$

$\Rightarrow x < 1/2$ because we have $0 < x < 1$.

$\therefore$ But for $\log[2x - (3/4)]$ to be meaningful, we must have

$2x - (3/4) > 0 \Rightarrow x > 3/8$

Therefore, if $0 < x < 1$, the values of x satisfying the given inequality are given by :

$3/8 < x < 1/2$

Case II. Let $x > 1$

Then, $\log_x[2x - (3/4)] > 2 \Rightarrow [2x - (3/4)] > x^2$

$\Rightarrow x^2 - 2x + (3/4) < 0 \Rightarrow 4x^2 - 8x + 3 < 0 \Rightarrow (2x - 1)(2x - 3) < 0$

$\Rightarrow \left(x - \frac{1}{2}\right)[x - (3/2)] < 0$

$\Rightarrow$ **$1/2 < x < 3/2$**

But we have $x > 1$

$\therefore$ We must have $1 < x < 3/2$ and obviously these values of x make $2x - (3/4) > 0$

Therefore, if $x > 1$, the values of x satisfying the given inequality are given by, $1 < x < 3/2$

$x \in \left(\frac{3}{8}, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)$

Illustration 30 :

Solve for x : $\log_{0.5}(x^2 - 5x + 6) \geq -1$

Solution :

Checking existence $x^2 - 5x + 6 > 0$,

$\Rightarrow x \in (-\infty, 2) \cup (3, \infty)$

Now solving inequality

$\log_{0.5}(x^2 - 5x + 6) \geq -1 \Rightarrow 0 < x^2 - 5x + 6 \leq (0.5)^{-1}$

$\Rightarrow x^2 - 5x + 6 \leq 2$

$$\begin{cases} x^2 - 5x + 6 > 0 \\ x^2 - 5x + 6 \leq 2 \end{cases} \Rightarrow x \in [1,2) \cup (3,4]$$

Hence, solution set of original inequation : $x \in [1,2) \cup (3,4]$

Illustration 31 :

Solve for x : $\log_2 x \leq \frac{2}{\log_2 x - 1}$.

Solution :

Let $\log_2 x = t$

$t \leq \frac{2}{t-1} \Rightarrow t - \frac{2}{t-1} \leq 0$

$\Rightarrow \frac{t^2 - t - 2}{t-1} \leq 0 \Rightarrow \frac{(t-2)(t+1)}{(t-1)} \leq 0$

$\Rightarrow t \in (-\infty, -1] \cup (1,2]$

or $\log_2 x \in (-\infty, -1] \cup (1,2]$

or $x \in \left(0, \frac{1}{2}\right] \cup (2,4]$

Illustration 32 :

Find all x such that $\log_{1/2} x > \log_{1/3} x$.

Solution :

We have $\log_{1/2} x > \log_{1/3} x$.

$\Rightarrow -\log_2 x > -\log_3 x \Rightarrow \log_2 x < \log_3 x$

$\Rightarrow \log_2 x < \dfrac{\log_2 x}{\log_2 3}$

$\Rightarrow \log_2 3 \log_2 x < \log_2 x$ (as $\log_2 3 > 0$)

$\Rightarrow \log_2 x (\log_2 3 - 1) < 0$

Since $\log_2 3 - 1 > 0$, from the latter inequality we obtain $\log_2 x < 0$, hence $x < 1$. But the original inequality is meaningful only when $x > 0$. Therefore all x that satisfy the original inequality lie in the interval $0 < x < 1$.

Answer : $x \in (0, 1)$

Illustration 33 :

Solve the inequation : $\log_{2x+3} x^2 < \log_{2x+3}(2x+3)$

Solution :

For existence of logarithm

(i) $x^2 > 0$ (ii) $2x + 3 > 0$ (iii) $2x + 3 \neq 1 \Rightarrow x \in \left(-\frac{3}{2}, \infty\right) - \{-1, 0\}$ ...(i)

Now solving inequality

Case I : $0 < 2x + 3 < 1$

$\Rightarrow -\frac{3}{2} < x < -1$

$\because \log_{2x+3} x^2 < \log_{2x+3} 2x+3$

$\Rightarrow x^2 > 2x+3$

$\Rightarrow (x-3)(x+1) > 0$

$\Rightarrow x \in (-\infty, -1) \cup (3, \infty)$; $but -\frac{3}{2} < x < -1$

$\Rightarrow x \in \left(-\frac{3}{2}, -1\right)$ intersection with (i) $\Rightarrow x \in \left(-\frac{3}{2}, -1\right)$

Case II : $2x + 3 > 1 \Rightarrow x > -1$

$\because \log_{2x+3} x^2 < \log_{2x+3} 2x+3$

$\Rightarrow x^2 < 2x+3$

$\Rightarrow (x-3)(x+1) < 0$

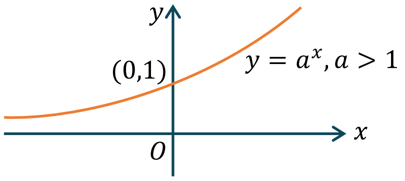
$\Rightarrow x \in (-1, 3)$; but $x > -1$

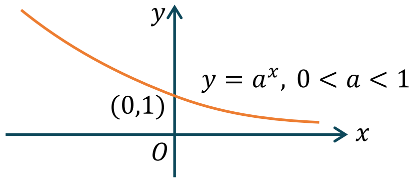
$\Rightarrow x \in (-1, 3)$ intersection with (i) $\Rightarrow x \in (-1, 3) - \{0\}$

$\therefore x \in$ case I $\cup$ case II

$\Rightarrow x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 0) \cup (0, 3)$

7. Graph of Exponential Function, its Equation and Inequalities





If $a^{f(x)} > b \Rightarrow \left[\begin{array}{ll} f(x) > \log_a b & \text{when } a > 1 \\ f(x) < \log_a b & \text{when } 0 < a < 1 \end{array}\right.$

Note : Domain of a^x is $\mathbb{R}$ and Range is $(0, \infty)$.

Illustration 34 :

Solve the equation $3^x.8^{\frac{x}{x+2}} = 6$.

Solution :

Some students solved this equation thus : rewriting it as

$3^x.2^{\frac{3x}{x+2}} = 3^1.2^1$

They chose a root x so that the exponents of the respective bases were the same :

$x = 1, \dfrac{3x}{x+2} = 1$

hence the "answer" $x = 1$.

But this "answer" is incorrect in the sense that only one root of the equation is found and nothing has been said about any other roots. Actually, if the exponents on the appropriate bases are equal, then the products of these powers are equal, however the converse is not in any way implied and is simply incorrect. For instance, the equation

$3^1.\ 2^1 = 3^2.\ 2^{\log_2(2/3)}$

is valid, but $1 \neq 2$ and $1 \neq \log_2(2/3)$. Therefore, the foregoing reasoning may lead to a loss of roots, and this is exactly what occurred in the equation at hand.

Taking logarithms of both members of the original to the base 10, we get

$x\log_{10}3 + \dfrac{3x}{x+2}\ \log_{10}2 = \log_{10}6$

or $\;x^2\log_{10} 3 + x\ (3 \log_{10} 2 + 2\log_{10} 3 - \log_{10}6) - 2 \log_{10}6 = 0$

We now have to solve this quadratic equation. This can be done using a familiar formula, but we will try to simplify the solution by an ingenious device, since we have alreadyseen, by trial and error, that $x_1 = 1$ is a root of the original equation and, consequently, satisfies the equivalent quadratic equation. For this reason, by Viete's theorem the second root of the quadratic equation is $x_2 = (-\ 2\log_{10}6)/\log_{10} 3 = -\ 2\log_3 6$ and so the original equation has two roots; $x_1 = 1$, $x_2 = -2 \log_3 6$.

Thus, it is useful to be able to guess a root, but never consider the guessing as the whole solution.

Illustration 35 :

$$\begin{cases} 2^{y-x}(x+y) = 1, \\ (x+y)^{x-y} = 2. \end{cases}$$

Solution :

The domain of definition is x + y > 0.

$$\begin{cases} x+y=\dfrac{1}{2^{y-x}}=2^{x-y}, \\ (x+y)^{x-y}=2. \end{cases}$$

From the first equation we find $x + y = 2^{x-y}$ and substitute it into the second equation.

Then $(2^{x-y})^{x-y} = 2 \Rightarrow 2^{(x-y)^2} = 2 \Rightarrow (x-y)^2 = 1$.

The solution of the given system is the solution of the collection of systems

$$\begin{cases} x-y=1, \\ x+y=2, \end{cases} \text{ or } \begin{cases} x-y=-1, \\ x+y=\dfrac{1}{2}. \end{cases}$$

Answer : $\left(\dfrac{3}{2},\dfrac{1}{2}\right)$, $\left(-\dfrac{1}{4},\dfrac{3}{4}\right)$.

Illustration 36 :

Solve for x : $2^{x+2} > \left(\dfrac{1}{4}\right)^{\frac{1}{x}}$

Solution :

We have $2^{x+2} > 2^{-\frac{2}{x}}$. Since the base 2 > 1, we have $x + 2 > -\dfrac{2}{x}$

(the sign of the inequality is retained).

Now $x+2+\dfrac{2}{x}>0 \quad \Rightarrow \quad \dfrac{x^2+2x+2}{x}>0$

$\Rightarrow \dfrac{(x+1)^2+1}{x}>0 \quad \Rightarrow \quad x\in(0, \infty)$

Illustration 37 :

Solve for x : $(1.25)^{1-x} < (0.64)^{2(1+\sqrt{x})}$

Solution :

We have $\left(\dfrac{5}{4}\right)^{1-x} < \left(\dfrac{16}{25}\right)^{2(1+\sqrt{x})}$ or $\left(\dfrac{4}{5}\right)^{x-1} < \left(\dfrac{4}{5}\right)^{4(1+\sqrt{x})}$

Since the base $0<\dfrac{4}{5}<1$, the inequality is equivalent to the inequality $x - 1 > 4(1+\sqrt{x})$

$\Rightarrow \dfrac{x-5}{4} > \sqrt{x}$

Now, R.H.S. is positive

$\Rightarrow \dfrac{x-5}{4}>0 \Rightarrow x > 5$ ...(i)

we have $\dfrac{x-5}{4}>\sqrt{x}$

both sides are positive, so squaring both sides

$\Rightarrow \frac{(x-5)^2}{16} > x$ or $\frac{(x-5)^2}{16} - x > 0$

Or $x^2 - 26x + 25 > 0$ or $(x - 25)(x - 1) > 0$

$\Rightarrow x \in (-\infty, 1) \cup (25, \infty)$ ...(ii)

intersection (i) & (ii) gives $x \in (25, \infty)$

8. Characteristic and Mantissa :

For any given number N, logarithm can be expressed as $\log_a N$ = Integer + Fraction. The integer part is called characteristic and the fractional part (always taken non negative) is called mantissa. When the value of $\log_{10} N$ is given, then to find digits of 'N' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $N \geq 1$) or the number of zeros after decimal & before first non-zero digit in the number (if $0 < N < 1$).

Note :

(i) The mantissa part of logarithm of a number is always non-negative $(0 \leq m < 1)$

(ii) If the characteristic of $\log_{10}N$ be 'C' and $C \geq 0$, then the number of digits in N is $(C + 1)$

(iii) If the characteristic of $\log_{10}N$ be '$-C$' and $C > 0$, then there exist $(C - 1)$ zeros after decimal in N.

In summary, if characteristic of $\log_{10}N$ is 'C' then number of digits ($N \geq 1$, $N \in \mathbb{N}$) or number of zeros after decimal in N ($0 < N < 1$) = $|C + 1|$.