04 Limits

1. Introduction:

The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry. We use limits to describe the way a function f varies. Some functions vary continuously; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump or vary erratically. We also use limits to define tangent lines to graphs of functions. This geometric application leads at once to the important concept of derivative of a function

2. Definition:

Let $f(x)$ be defined on an open interval about 'a' except possibly at 'a' itself. If $f(x)$ gets arbitrarily close to L (a finite number) for all x sufficiently close to 'a' we say that $f(x)$ approaches the limit L as x approaches 'a' and we write $\lim_{x\to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L".

This implies if we can make the value of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

3. Left Hand Limit and Right Hand Limit of A Function:

The value to which $f(x)$ approaches, as x tends to 'a' from the left hand side $(x \to a^-)$ is called left hand limit of $f(x)$ at $x = a$. Symbolically, LHL = $\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$.

The value to which $f(x)$ approaches, as x tends to 'a' from the right hand side $(x \to a^+)$ is called right hand limit of $f(x)$ at $x = a$. Symbolically, RHL = $\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$.

Limit of a function $f(x)$ is said to exist as, $x \to a$ when $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ = Finite quantity.

Example:

Graph of $y = f(x)$

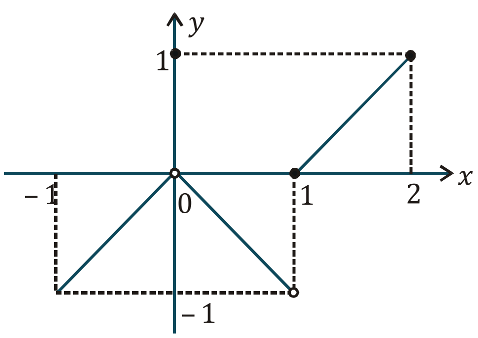


Fig. 1

$\lim_{x\to -1^+} f(x) = \lim_{h\to 0} f(-1+h) = f(-1^+) = -1$

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = f(0^-) = 0$

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = f(1^-) = -1$

$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = f(1^+) = 0$

$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = f(2^-) = 1$

$\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 1} f(x)$ does not exist.

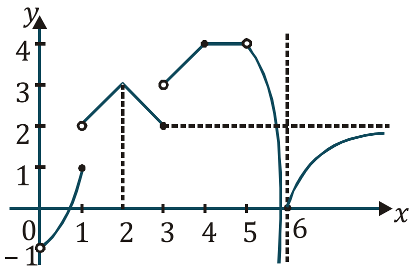
Important Note:

In $\lim\limits_{x\to a} f(x), x \neq a$ $\boldsymbol{x \to a}$ **necessarily implies**. That is while evaluating limit at $x = a$, we are not concerned with the value of the function at $x = a$. In fact, the function may or may not be defined at $x = a$. Also, it is necessary to note that if $f(x)$ is defined only on one side of '$\boldsymbol{x = a}$', one sided limit are good enough to establish the existence of limits, & if $f(x)$ is defined on either side of '$\boldsymbol{a}$' both sided limits are to be considered.

As in $\lim\limits_{x\to 1} \cos^{-1} x = 0$, though $f(x)$ is not defined for $x > 1$, even in its immediate vicinity.

Illustration 1:

Consider the adjacent graph of $y = f(x)$ Find the following:



(a) $\lim\limits_{x\to 0^-} f(x)$ (b) $\lim\limits_{x\to 0^+} f(x)$ (c) $\lim\limits_{x\to 1^-} f(x)$

(d) $\lim\limits_{x\to 1^+} f(x)$ (e) $\lim\limits_{x\to 2^-} f(x)$ (f) $\lim\limits_{x\to 2^+} f(x)$

(g) $\lim\limits_{x\to 3^-} f(x)$ (h) $\lim\limits_{x\to 3^+} f(x)$ (i) $\lim\limits_{x\to 4^-} f(x)$

(j) $\lim\limits_{x\to 4^+} f(x)$ (k) $\lim\limits_{x\to \infty} f(x) = 2$ (l) $\lim\limits_{x\to 6^-} f(x) = -\infty$

Solution:

(a) As $x \to 0^-$: limit does not exist (the function is not defined to the left of $x = 0$)

(b) As $x \to 0^+$: $f(x) \to -1 \Rightarrow \lim\limits_{x\to 0^+} f(x) = -1$.

(c) As $x \to 1^-$: $f(x) \to 1 \Rightarrow \lim\limits_{x\to 1^-} f(x) = 1$.

(d) As $x \to 1^+$: $f(x) \to 2 \Rightarrow \lim\limits_{x\to 1^+} f(x) = 2$.

(e) As $x \to 2^-$: $f(x) \to 3 \Rightarrow \lim\limits_{x\to 2^-} f(x) = 3$.

(f) As $x \to 2^+$: $f(x) \to 3 \Rightarrow \lim\limits_{x\to 2^-} f(x) = 3$.

(g) As $x \to 3^-$: $f(x) \to 2 \Rightarrow \lim\limits_{x\to 3^-} f(x) = 2$.

(h) As $x \to 3^+$: $f(x) \to 3 \Rightarrow \lim\limits_{x\to 3^+} f(x) = 3$.

(i) As $x \to 4^-$: $f(x) \to 4 \Rightarrow \lim\limits_{x\to 4^-} f(x) = 4$.

(j) As $x \to 4^+$: $f(x) \to 4 \Rightarrow \lim\limits_{x\to 4^+} f(x) = 4$.

(k) As $x \to \infty$: $f(x) \to 2 \Rightarrow \lim\limits_{x\to \infty} f(x) = 2$.

(l) As $x \to 6^-$, $f(x) \to -\infty \Rightarrow \lim\limits_{x\to 6^-} f(x) = -\infty$ limit does not exist because it is not finite.

4. Fundamental Theorems on Limits:

Let $\lim\limits_{x\to a} f(x) = l$ & $\lim\limits_{x\to a} g(x) = m$. If l & m exist finitely then :

(a) Sum rule : $\lim\limits_{x\to a}\{f(x) + g(x)\} = l + m$

(b) Difference rule : $\lim\limits_{x\to a}\{f(x) - g(x)\} = l - m$

(c) Product rule : $\lim\limits_{x\to a} f(x)\cdot g(x) = l\cdot m$

(d) Quotient rule : $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{1}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x)$; where k is constant.

(f) Power rule : If m and n are integers then $\lim\limits_{x\to a}\ [f(x)]^{m/n} = l^{m/n}$ provided $l^{m/n}$ is a real number.

(g) $\lim\limits_{x\to a}\ f[g(x)] = f\left(\lim\limits_{x\to a}\ g(x)\right) = f(m)$; provided $f(x)$ is continuous at $x = m$.

For example : $\lim\limits_{x\to a}\ \ell n(g(x)) = \ell n[\lim\limits_{x\to a} g(x)]$

$= \ell n(m)$; provided ℓnx is continuous at $x = m, m = \lim\limits_{x\to a} g(x)$.

5. Indeterminate Forms:

$\dfrac{0}{0}, \dfrac{\infty}{\infty}, \infty-\infty, 0\times\infty, 1^{\infty}, 0^0, \infty^0$

Initially we will deal with first five forms only and the other two forms will come up after we have gone through differentiation.

Note :

(i) Here 0,1 are not exact, in fact both are approaching to their corresponding values.

(ii) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number It does not obey the laws of elementary algebra,

(a) $\infty+\infty\to\infty$ (b) $\infty\times\infty\to\infty$ (c) $\infty^{\infty}\to\infty$ (d) $0^{\infty}\to 0$

6. General Methods to be Used to Evaluate Limits:

(a) Factorization:

Important factors :

(i) $x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + ... + a^{n-1}), n \in N$

(ii) $x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + ... + a^{n-1})$, n is an odd natural number.

Note : $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = na^{n-1}$

Illustration 2:

Evaluate: $\lim\limits_{x\to 2}\left[\dfrac{1}{x-2} - \dfrac{2(2x-3)}{x^3-3x^2+2x}\right]$

Solution:

We have

$$\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right] = \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x-3}{x(x-1)}\right] = -\frac{1}{2}$$

(b) Rationalization or Double Rationalization:

Illustration 3:

Evaluate: $\lim\limits_{x\to 1}\dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$

Solution:

$$\lim_{x\to 1}\frac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}=\lim_{x\to 1}\frac{(4-\sqrt{15x+1})(2+\sqrt{3x+1})(4+\sqrt{15x+1})}{(2-\sqrt{3x+1})(4+\sqrt{15x+1})(2+\sqrt{3x+1})}$$

$$=\lim_{x\to 1}\frac{(15-15x)}{(3-3x)}\times\frac{2+\sqrt{3x+1}}{4+\sqrt{15x+1}}=\frac{5}{2}$$

Illustration 4:

Evaluate: $\lim\limits_{x\to 1}\left(\dfrac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\right)$

Solution:

This is of the form $\dfrac{3-3}{2-2}=\dfrac{0}{0}$ if we put $x=1$

To eliminate the $\dfrac{0}{0}$ factor, multiply by the conjugate of numerator and the conjugate of the denominator

$$\therefore \text{ Limit} = \lim_{x\to 1}\left(\sqrt{x^2+8}-\sqrt{10-x^2}\right)\frac{(\sqrt{x^2+8}+\sqrt{10-x^2})}{(\sqrt{x^2+8}+\sqrt{10-x^2})}\times\frac{(\sqrt{x^2+3}+\sqrt{5-x^2})}{(\sqrt{x^2+3}+\sqrt{5-x^2})(\sqrt{x^2+3}-\sqrt{5-x^2})}$$

$$=\lim_{x\to 1}\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\times\frac{(x^2+8)-(10-x^2)}{(x^2+3)-(5-x^2)}=\lim_{x\to 1}\left(\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\right)\times 1=\frac{2+2}{3+3}=\frac{2}{3}$$

(c) Limit when $x\to\infty$:

(i) Divide by greatest power of x in numerator and denominator.

(ii) Put $x = 1/y$ and apply $y\to 0$

Illustration 5:

Evaluate: $\lim\limits_{x\to\infty}\dfrac{x^2+x+1}{3x^2+2x-5}$

Solution:

$\lim\limits_{x\to\infty}\dfrac{x^2+x+1}{3x^2+2x-5}$, $\qquad\qquad \left(\dfrac{\infty}{\infty}\text{ form}\right)$

Put $x=\dfrac{1}{y}$

Limit = $\lim\limits_{y\to 0}\dfrac{1+y+y^2}{3+2y-5y^2}=\dfrac{1}{3}$

Illustration 6:

If $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2$, then

(A) $a=1, b=1$ (B) $a=1, b=2$ (C) $a=1, b=-2$ (D) none of these

Ans. (C)

Solution:

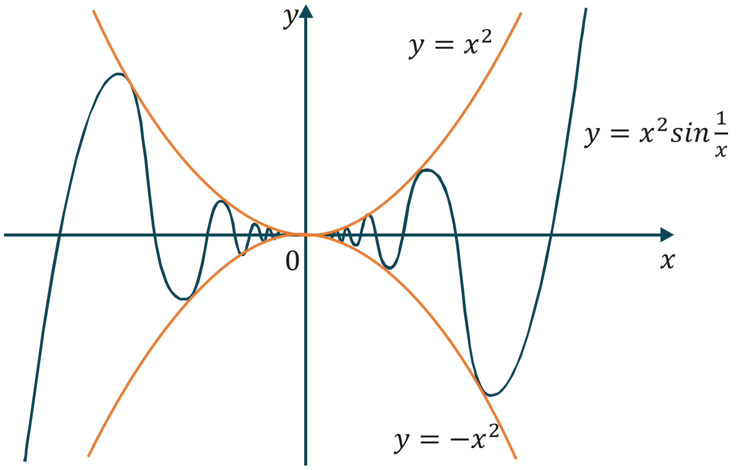
$$\lim_{x\to\infty}\left(\frac{x^3+1}{x^2+1}-(ax+b)\right)=2 \Rightarrow \lim_{x\to\infty}\frac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}=2$$

$$\Rightarrow \lim_{x\to\infty}\frac{x(1-a)-b-\frac{a}{x}+\frac{(1-b)}{x^2}}{1+\frac{1}{x^2}}=2 \Rightarrow\ 1-a=0, -b=2 \Rightarrow a=1, b=-2$$

(d) Squeeze Play Theorem (Sandwich Theorem) :

Statement : If $f(x) \le g(x) \le h(x)$; $\forall\ x$ in the neighbourhood at $x=a$ and $\lim_{x\to a} f(x) = \ell = \lim_{x\to a} h(x)$ then, $\lim_{x\to a} g(x) = \ell$

Ex.1 $\lim_{x\to 0} x^2 \sin\frac{1}{x}=0$,

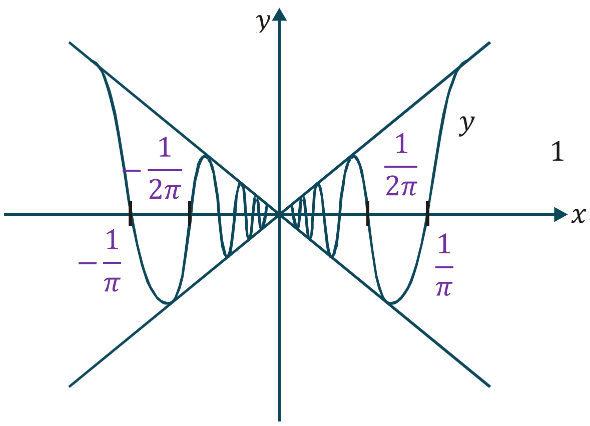


$\because\ \sin\left(\frac{1}{x}\right)$ lies between –1 & 1

$\Rightarrow -x^2 \le x^2\sin\frac{1}{x} \le x^2$

$\Rightarrow \lim_{x\to 0} x^2\sin\frac{1}{x}=0$ as $\lim_{x\to 0}(-x^2)=\lim_{x\to 0}x^2=0$

Ex.2 $\lim_{x\to 0} x\sin\frac{1}{x}=0$



$\because \ \sin\left(\frac{1}{x}\right)$ lies between –1 & 1

$\Rightarrow -x \le x\sin\frac{1}{x} \le x$

$\Rightarrow \lim_{x\to 0} x\sin\frac{1}{x} = 0$ as $\lim_{x\to 0}(-x) = \lim_{x\to 0} x = 0$

Illustration 7:

Evaluate: $\lim_{n\to\infty}\frac{[x]+[2x]+[3x]+.....[nx]}{n^2}$ (Where [.] denotes the greatest integer function.)

Solution:

We know that $x - 1 < [x] \le x$

$\Rightarrow x + 2x + ...nx - n < \sum_{r=1}^{n}[rx] \le x + 2x + ...+ nx$

$\Rightarrow \frac{xn}{2}(n+1) - n < \sum_{r=1}^{n}[rx] \le \frac{x.n(n+1)}{2}$

$\Rightarrow \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} < \frac{1}{n^2}\sum_{r=1}^{n}[rx] \le \frac{x}{2}\left(1+\frac{1}{n}\right)$

Now, $\lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right) = \frac{x}{2}$ and $\lim_{n\to\infty}\frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} = \frac{x}{2}$

Thus, $\lim_{n\to\infty}\frac{[x]+[2x]+......+[nx]}{n^2} = \frac{x}{2}$

7. Limit of Trigonometric Functions:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 = \lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\tan^{-1} x}{x} = \lim_{x\to 0}\frac{\sin^{-1} x}{x}$$ [where x is measured in radians]

If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a}\frac{\sin f(x)}{f(x)} = 1$, e.g., $\lim_{x\to 1}\frac{\sin(\ell nx)}{(\ell nx)} = 1$

Illustration 8:

Evaluate: $\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}$

Solution:

$\lim_{x\to 0}\frac{x^3\cos x}{\sin x(1-\cos x)} = \lim_{x\to 0}\frac{x^3\cos x(1+\cos x)}{\sin x.\sin^2 x} = \lim_{x\to 0}\frac{x^3}{\sin^3 x}.\cos x(1+\cos x) = 2$

Illustration 9:

Evaluate: $\lim_{x\to 0}\frac{(2+x)\sin(2+x) - 2\sin 2}{x}$

Solution:

$$\lim_{x\to 0}\frac{2(\sin(2+x)-\sin 2)+x\sin(2+x)}{x} = \lim_{x\to 0}\left(\frac{2.2.\cos\left(2+\frac{x}{2}\right)\sin\frac{x}{2}}{x} + \sin(2+x)\right)$$

$$= \lim_{x\to 0}\frac{2\cos\left(2+\frac{x}{2}\right)\sin\frac{x}{2}}{\frac{x}{2}} + \lim_{x\to 0}\sin(2+x) = 2\cos 2 + \sin 2$$

Illustration 10:

Evaluate: $\lim_{n\to\infty}\dfrac{\sin\frac{a}{n}}{\tan\frac{b}{n+1}}$

Solution:

As n $\to$ ∞, $\frac{1}{n}\to$ 0 and $\frac{a}{n}$ also tends to zero

$\sin\frac{a}{n}$ should be written as $\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}$ so that it looks like $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}$

The given limit $=\lim_{n\to\infty}\left(\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\dfrac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right).\dfrac{a(n+1)}{n.b}$

$=\lim_{n\to\infty}\left(\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\dfrac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right).\dfrac{a}{b}\left(1+\dfrac{1}{n}\right)=1\times 1\times\dfrac{a}{b}\times 1=\dfrac{a}{b}$

8. Limit of Exponential Functions:

(a) $\lim_{x\to 0}\dfrac{a^x-1}{x}=\ell n\,a(a>0)$ In particular $\lim_{x\to 0}\dfrac{e^x-1}{x}=1$.

In general, if $\lim_{x\to a}f(x)=0$,then $\lim_{x\to a}\dfrac{a^{f(x)}-1}{f(x)}=\ell na,\ a>0$

Illustration 11:

Evaluate: $\lim_{x\to 0}\dfrac{e^{\tan x}-e^x}{\tan x-x}$

Solution:

$\lim_{x\to 0}\dfrac{e^{\tan x}-e^x}{\tan x-x}=\lim_{x\to 0}\dfrac{e^x\times e^{(\tan x-x)}-e^x}{\tan x-x}$

$=\lim_{x\to 0}\dfrac{e^x(e^{\tan x-x}-1)}{\tan x-x}$ = $\lim_{x\to 0 y\to 0}\dfrac{e^x(e^y-1)}{y}$ where y = tan x − x and $\lim_{y\to 0}\dfrac{e^y-1}{y}=1$

= $e^0\times 1$ [as $x\to$ 0, tan x − $x\to$ 0]

= 1 × 1 = 1

(b) (i) $\lim_{x\to 0}(1+x)^{1/x}=e=\lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$ (**Note:** The base and exponent depend on the same variable.)

In general, if $\lim_{x\to a}f(x)=0$, then $\lim_{x\to a}(1+f(x))^{1/f(x)}=e$

(ii) $\lim_{x\to 0}\dfrac{\ell n(1+x)}{x}=1$

(iii) If $\lim_{x\to a}f(x)=1$ and $\lim_{x\to a}\phi(x)=\infty$, then; $\lim_{x\to a}\left[f(x)\right]^{\phi(x)}=e^k$

where $k=\lim_{x\to a}\phi(x)\ [f(x)-1]$

Illustration 12:

Evaluate: $\lim_{x\to1}(\log_3 3x)^{\log_x 3}$

Solution:

$\lim_{x\to1}(\log_3 3x)^{\log_x 3} = \lim_{x\to1}(\log_3 3+\log_3 x)^{\log_x 3}$

$= \lim_{x\to1}(1+\log_3 x)^{1/\log_3 x} = \text{e} \qquad \because \log_b a = \dfrac{1}{\log_a b}$

Illustration 13:

Evaluate: $\lim_{x\to0}\dfrac{x\ell n(1+2\tan x)}{1-\cos x}$

Solution:

$$\lim_{x\to0}\frac{x\ell n(1+2\tan x)}{1-\cos x} = \lim_{x\to0}\frac{x\ell n(1+2\tan x)}{\frac{1-\cos x}{x^2}.x^2}.\frac{2\tan x}{2\tan x} = 4$$

Illustration 14:

Evaluate: $\lim_{x\to\infty}\left(\dfrac{2x^2-1}{2x^2+3}\right)^{4x^2+2}$

Solution:

Since it is in the form of 1^∞

$$\lim_{x\to\infty}\left(\frac{2x^2-1}{2x^2+3}\right)^{4x^2+2} = e^{\lim_{x\to\infty}}\left(\frac{2x^2-1-2x^2-3}{2x^2+3}\right)(4x^2+2) = e^{-8}$$

Illustration 15:

Evaluate : $\lim_{x\to\infty}\left(\dfrac{7x^2+1}{5x^2-1}\right)^{\frac{x^5}{1-x^3}}$

Solution:

Here $f(x) = \dfrac{7x^2+1}{5x^2-1}$, $\phi(x) = \dfrac{x^5}{1-x^3} = \dfrac{x^2.x^3}{1-x^3} = \dfrac{x^2}{\frac{1}{x^3}-1}$

$\therefore \lim_{x\to\infty} f(x) = \dfrac{7}{5}$ & $\lim_{x\to\infty}\phi(x) \to -\infty$

$\Rightarrow \lim_{x\to\infty}(f(x))^{\phi(x)} = \left(\dfrac{7}{5}\right)^{-\infty} = 0$

Illustration 16:

$\lim_{x\to0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{\text{cosec}\,x}$ is equal to

(A) e (B) $\dfrac{1}{e}$ (C) 1 (D) None of these

Ans. (C)

Solution:

Given limit = $\lim_{x\to0}[(1+\tan x)^{\text{cosec}x} \times 1/(1+\sin x)^{\text{cosec}x}]$

$= \lim_{x\to0}[\{(1+\tan x)^{\cot x}\}^{\sec x} \times \{1/(1+\sin x)^{cosec\,x}\}] = e^{\sec 0}.\dfrac{1}{e} = e.\dfrac{1}{e} = 1$

Illustration 17:

If $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$ then the values of a and b are

(A) $a=1, b=2$ (B) $a=1, b\in R$ (C) $a\in R, b=2$ (D) $a\in R, b\in R$

Ans. (B)

Solution:

Since, $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)=1$

$$\therefore \lim_{x\to\infty}\left[\left(1+\frac{ax+b}{x^2}\right)^{\frac{x^2}{ax+b}}\right]^{\frac{2(ax+b)}{x}}=e^2\Rightarrow \lim_{x\to\infty}^{\frac{2(ax+b)}{x}}=e^2\Rightarrow \lim_{x\to\infty}\frac{2(ax+b)}{x}=2\Rightarrow 2a=2\Rightarrow a=1$$

Thus $a=1$ and $b\in R$.

Illustration 18:

Given $h(x)=\lim_{n\to\infty}\frac{x^{2n}f(x)+g(x)}{x^{2n}+1}, f(2)=5$ and $g\left(\frac{1}{2}\right)=-3$, then value of $h(2)-2h\left(\frac{1}{2}\right)$ is ....

(Given $f(x)$ and $g(x)$ are bounded functions)

Ans. (11)

Solution:

$$h(x)=\lim_{n\to\infty}\frac{x^{2n}f(x)+g(x)}{x^{2n}+1}=\begin{cases} g(x) & |x|<1 \\ \dfrac{f(x)+g(x)}{2} & |x|=1 \\ f(x) & |x|>1\end{cases}$$

$$\therefore h(2)-2h\left(\frac{1}{2}\right)=f(2)-2g\left(\frac{1}{2}\right)=5-2(-3)=11$$

9. Limit Using Series Expansion:

Expansion of function like binomial expansion, exponential & logarithmic expansion, expansion of $\sin x, \cos x, \tan x$ should be remembered by heart which are given below :

(a) $a^x=1+\frac{x\,\ell na}{1!}+\frac{x^2\ell n^2 a}{2!}+\frac{x^3\ell n^3 a}{3!}+\ldots, x\in\mathbb{R}, a>0, a\neq 1$

(b) $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots, x\in\mathbb{R}$

(c) $\ell n(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ for $-1<x\le 1$

(d) $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots, x\in\mathbb{R}$

(e) $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots, x\in\mathbb{R}$

(f) $\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots, -\frac{\pi}{2}<x<\frac{\pi}{2}$

(g) $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots, x\in(-1,1)$

(h) $\sin^{-1}x=x+\frac{1^2}{3!}x^3+\frac{1^2.3^2}{5!}x^5+\frac{1^2.3^2.5^2}{7!}x^7+\ldots, x\in(-1,1)$

(i) $\sec^{-1}x=1+\frac{x^2}{2!}+\frac{5x^4}{4!}+\frac{61x^6}{6!}+\ldots, x\in(-\infty,-1)\cup(1,\infty)$

(j) $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots, n\in\mathbb{R}, x\in(-1,1)$

Illustration 19:

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$$

Solution:

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}\Rightarrow \lim_{x\to 0}\frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+......-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+.....\right)-2x}{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}.....\right)}$$

$$\Rightarrow \lim_{x\to 0}\frac{2.\frac{x^3}{6}+2.\frac{x^5}{5!}+......}{\frac{x^3}{6}+\frac{x^5}{5!}.....}\Rightarrow \lim_{x\to 0}\frac{x^3\left(\frac{1}{3}+\frac{1}{60}x^2+.....\right)}{x^3\left(\frac{1}{6}+\frac{1}{120}x^2+.....\right)}=\frac{1/3}{1/6}=2$$

Illustration 20:

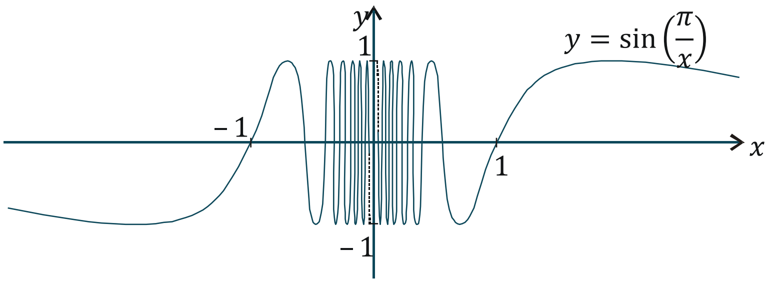
Evaluate $\lim_{x\to 0}\sin\frac{\pi}{x}$.

Solution:

Again, the function $f(x)=\sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get $f(1)=\sin\pi=0$, $f\left(\frac{1}{2}\right)=\sin 2\pi=0$,

$f(0.1)=\sin 10\pi=0$, $f(0.01)=\sin 100\pi=0$.

Based on this information, we might be tempted to guess that $\lim_{x\to 0}\sin\frac{\pi}{x}=0$ but this time our guess is wrong. Note that although $f(1/n)=\sin n\pi=0$ for any integer n, it is also true that $f(x)=1$ for infinitely many values of x that approach 0. [In fact, $\sin(\pi/x)=1$ when $\frac{\pi}{x}=\frac{\pi}{2}+2n\pi$ and solving for x, we get $x=2/(4n+1)$]. The graph of f is given in following figure



The dashed line indicates that the values of $\sin(\pi/x)$ oscillate between 1 and −1 infinitely often as x approaches 0. Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$\Rightarrow \lim_{x\to 0}\sin\frac{\pi}{x}$ does not exist.

Illustration 21:

$$\lim_{x\to 0}\left\{\frac{\sin x-x+\frac{x^3}{6}}{x^5}\right\}=$$

(A) 1/120 (B) −1/120 (C) 1/20 (D) None of these

Ans. (A)

Solution:

Expand $\sin x$ and then solve.

$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+......$ $\qquad \lim_{x\to 0}\frac{x-\frac{x^3}{6}+\frac{x^5}{120}-.....\ -x+\frac{x^3}{6}}{x^5}=\frac{1}{120}$