01 Fundamentals of Algebra

Useful Mathematical Symbols for Reference

Symbol	How it is read
$\forall$	For all...
$\exists$	There exists...
$\wedge$	And
$\vee$	or
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\nless$	is not less than
$\ngtr$	is not greater than
$\in$	belongs to
$\notin$	does not belong to
$\|$, : , s.t.	such that
$\Rightarrow$	implies (If... then...)
$\Leftrightarrow$	implies and implied by (if and only if / iff)
!	factorial
$\sqrt{\ }$	The square root of
$\sqrt[n]{\ }$	n^{th} root, $n \in N, n \geq 2$
$\sum$	The summation of
$\prod$	The product of

Greek Letters (Capital, Small) and there pronunciations

Symbol (Capital, Small)	How it is read	Symbol (Capital, Small)	How it is read
A,α	alpha	N,ν	nu
B,β	beta	I,ι	iota
Γ,γ	gamma	Λ,λ	lambda
Δ,δ	delta	Π,π	Pi
E,ε	epsilon	P,ρ	rho
Θ,θ	theta	M,μ	mu
Σ,σ	sigma	Φ,ϕ	phi
Ψ,ψ	psi	Ω,ω	omega

1. Indices:

Definition of Indices:

If 'a' is any non-zero real or imaginary number and 'm' is a positive integer, then $a^m = a.a.a....a$ (m times). Here a is called the base and m is the index, power or exponent.

Law of indices:

(i) $a^0 = 1, (a \neq 0)$

(ii) $a^{-m} = \frac{1}{a^m}, (a \neq 0)$

(iii) $a^m . a^n = a^{m+n}$

(iv) $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$

(v) $(a^m)^n = a^{mn} = (a^n)^m$

(vi) $\sqrt[q]{a^p} = a^{\left(\frac{p}{q}\right)}, q \in N$ and $q \geq 2$

(vii) If $x = y$, then $a^x = a^y$, but the converse may not be true. e.g. : $(1)^6 = (1)^8$, but $6 \neq 8$

For $a^x = a^y$ we have following possibilities

- If $a \neq \pm 1, 0$, then $x = y$
- If $a = 1$, then x, y may be any real number
- If $a = -1$, then x, y may be both even or both odd
- If $a = 0$, then x, y may be any positive real number

But if we have to solve the equations like $\left(f(x)\right)^{g(x)} = \left(f(x)\right)^{h(x)}$ (i.e. same base, different indices) then we have to solve :

(a) $f(x) = 1$ (b) $f(x) = -1$

(c) $f(x) = 0$ (d) $g(x) = h(x)$

Verification should be done in (b) and (c) cases

(viii) If $a^x = b^x$ then consider the following cases:

- If $a \neq \pm b$, then $x = 0$
- If $a = b \neq 0$, then x may have any real value for which a^x is well defined.
- If $a = -b \neq 0$, then x is even.
- If $a = b = 0$, then x can be any positive real.

If we have to solve the equation of the form $[f(x)]^{h(x)} = [g(x)]^{h(x)}$, i.e., same index, different bases, then we have to solve:

(a) $f(x) = g(x)$

(b) $f(x) = -g(x)$

(c) $h(x) = 0$

Verification should be done in (a), (b) and (c) cases.

Illustration 1:

$\left[(125)^{1/3}\right]^2$ =

(A) 25 (B) 5 (C) 125 (D) 625

Solution:

Ans. (A)

$\left[(5^3)^{1/3}\right]^2 \Rightarrow (5)^2 \Rightarrow 25$

Illustration 2:

$\dfrac{(2)^3(8)^{-1/3}(4)^2}{(64)^{-1/6}(8)^2}$ =

(A) 8 (B) 2 (C) 16 (D) 64

Solution:

Ans. (B)

$\dfrac{2^3.2^{-1}.2^4}{2^{-1}.2^6} \Rightarrow \dfrac{2^7}{2^6} = 2$

Illustration 3:

$\sqrt[2]{\sqrt[3]{729}}$ =

(A) 3 (B) 9 (C) 27 (D) 81

Ans. (A)

Solution:

$\left((3^6)^{1/3}\right)^{1/2} \Rightarrow 3$

Illustration 4:

If $a^x = b, b^y = c, c^z = a$, prove that $xyz = 1$, where a, b, c are distinct numbers.

Solution:

We have, $a^{xyz} = (a^x)^{yz}$

$\Rightarrow a^{xyz} = (b)^{yz}$ $[\because a^x = b]$

$\Rightarrow a^{xyz} = (b^y)^z$

$\Rightarrow a^{xyz} = c^z$ $[\because b^y = c]$

$\Rightarrow a^{xyz} = a$ $[\because c^z = a]$

$\therefore a^{xyz} = a^1$

$\Rightarrow xyz = 1$

Illustration 5:

Solve $(x^2 - 4)^{2x} = (x^2 + 2x)^{2x}$.

Solution:

Case-1 : When $x = 0$

Base: $x^2 + 2x = 0$, so $x = 0$ does not satisfy given equation

Case-2 : $x^2 - 4 = x^2 + 2x \Rightarrow x = -2$

$x = -2$ does not satisfy the given equation.

Case-3 : $x^2 - 4 = -x^2 - 2x \neq 0 \Rightarrow 2x^2 + 2x - 4 = 0$

$x^2 + x - 2 = 0 \Rightarrow x = 1$

satisfies the given equation

From all above cases, $x \in \{1\}$

2. Surds:

If 'x' is a rational number, which is not the n^{th} power ($n \in N \setminus \{1\}$) of any rational number, then the number $x^{1/n}$ usually denoted by $\sqrt[n]{x}$ is called surd. The sign '$\sqrt[n]{x}$' is called the radical sign. The number in the angular part of the sign, i.e., 'n' is called order of the surd. In case of $n = 2$ the expression $\sqrt[2]{x}$, simply written as $\sqrt{x}$.

Note :

- If $\sqrt[n]{x}$ is a surd then $-(\sqrt[n]{x})$ is also a surd.
- Every surd is an irrational number (but every irrational number is not a surd).
- To rationalize the denominator of a fraction of the form $\frac{a}{\sqrt{b}}$, multiply the numerator and denominator of the fraction by $\sqrt{b} \Rightarrow \frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}}.\frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b^2}} = \frac{a\sqrt{b}}{b}$.

Ex.

(a) $\sqrt{3}$ is a surd and $\sqrt{3}$ is an irrational number.

(b) $\sqrt[3]{5}$ is surd and $\sqrt[3]{5}$ is an irrational number.

(c) π is an irrational number, but it is not a surd.

Conjugate of a Surd

If two binomial surds (surds containing two terms such as $2 + \sqrt{3}, 2\sqrt{5} - \sqrt{7}$ etc.) are such that only the sign connecting the individual terms are different, then they are said to be conjugate of each other. If these surds are quadratic, then their product would always be rational. So in case of a binomial quadratic surd, we use its conjugate as its rationalizing factor.

Ex. Conjugate of $3\sqrt{2} + \sqrt{5}$ is $3\sqrt{2} - \sqrt{5}$ or $-3\sqrt{2} + \sqrt{5}$

Illustration 6:

Rationalize the denominator of $\frac{1}{3\sqrt{2}+\sqrt{5}}$.

Solution:

A conjugate of $3\sqrt{2} + \sqrt{5}$ is $3\sqrt{2} - \sqrt{5}$

Therefore multiplying the conjugate in the numerator and denominator of the given fraction.

$$\frac{3\sqrt{2}-\sqrt{5}}{(3\sqrt{2}+\sqrt{5})(3\sqrt{2}-\sqrt{5})}$$

$$=\frac{3\sqrt{2}-\sqrt{5}}{(3\sqrt{2})^2-(\sqrt{5})^2}=\frac{3\sqrt{2}-\sqrt{5}}{18-5}=\frac{3\sqrt{2}-\sqrt{5}}{13}$$

Illustration 7:

Rationalize the denominator of $\frac{1}{\sqrt{3}-\sqrt{2}-1}$.

Solution:

$$\frac{1}{\sqrt{3}-\sqrt{2}-1}=\frac{1}{\sqrt{3}-\sqrt{2}-1}\times\frac{\sqrt{3}+\sqrt{2}+1}{\sqrt{3}+\sqrt{2}+1}$$

$$=\frac{\sqrt{3}+\sqrt{2}+1}{\left(\sqrt{3}-\sqrt{2}-1\right)\left(\sqrt{3}+\sqrt{2}+1\right)}=\frac{\sqrt{3}+\sqrt{2}+1}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}+1\right)^2}=\frac{\sqrt{3}+\sqrt{2}+1}{-2\sqrt{2}}$$

$$=-\left(\frac{\sqrt{6}+\sqrt{2}+2}{4}\right)$$

Illustration 8:

Find rational numbers a and b, such that $\frac{4+3\sqrt{5}}{4-3\sqrt{5}}=a+b\sqrt{5}$.

Solution:

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}}\times\frac{4+3\sqrt{5}}{4+3\sqrt{5}}=a+b\sqrt{5}$$

$$\Rightarrow\frac{61+24\sqrt{5}}{-29}=a+b\sqrt{5}$$

$$\Rightarrow a=-\frac{61}{29},b=-\frac{24}{29}$$

Illustration 9:

If $x = 3 - 2\sqrt{2}$, find $x^2 + \frac{1}{x^2}$.

Solution:

We have, $x = 3 - 2\sqrt{2}$.

$$\therefore\ \frac{1}{x}=\frac{1}{3-2\sqrt{2}}=\frac{1}{3-2\sqrt{2}}\times\frac{3+2\sqrt{2}}{3+2\sqrt{2}}=\frac{3+2\sqrt{2}}{(3)^2-(2\sqrt{2})^2}=\frac{3+2\sqrt{2}}{9-8}=3+2\sqrt{2}$$

Thus, $x^2 + \frac{1}{x^2} = (3 - 2\sqrt{2})^2 + (3 + 2\sqrt{2})^2$

$= 2((3)^2 + (2\sqrt{2})^2) = 2(9 + 8) = 34$

Illustration 10:

Simplify $\frac{1}{(5-2\sqrt{6})}-\frac{1}{(3-2\sqrt{2})}$

(A) $4(1+\sqrt{6}-\sqrt{2})$ (B) $2(1+\sqrt{5}+\sqrt{2})$ (C) $2(1+\sqrt{6}-\sqrt{2})$ (D) $2(1+\sqrt{6}+\sqrt{5})$

Solution:

Ans. (C)

$$\frac{(5+2\sqrt{6})}{(5+2\sqrt{6})(5-2\sqrt{6})}-\frac{(3+2\sqrt{2})}{(3-2\sqrt{2})(3+2\sqrt{2})}$$

$\Rightarrow (5 + 2\sqrt{6}) - (3 + 2\sqrt{2})$

$\Rightarrow 2(1+\sqrt{6}-\sqrt{2})$

3. Factorizations (Type 1 to 3)

3.1 $a^2 - b^2 = (a - b)(a + b)$

Illustration 11:

$(3x - y)^2 - (2x - 3y)^2$

Solution:

Use $a^2 - b^2 = (a - b)(a + b)$

$\Rightarrow (3x - y)^2 - (2x - 3y)^2$

$= (3x - y + 2x - 3y)(3x - y - 2x + 3y)$

$= (5x - 4y)(x + 2y)$

3.2 Factorising the Quadratic expression

Illustration 12:

$x^2 + 6x - 187$

Solution:

$x^2 + 6x - 187$

$= x^2 + 17x - 11x - 187$

$= x(x + 17) - 11(x + 17)$

$= (x + 17)(x - 11)$

3.3 Factorisation by converting the given expression into a perfect square.

Illustration 13:

$9x^4 - 10x^2 + 1$

Solution:

$9x^4 - 10x^2 + 1 = (3x^2)^2 - 2 \cdot 3x^2 + 1 - 4x^2$

$= (3x^2 - 1)^2 - (2x)^2$

$= (3x^2 - 1 - 2x)(3x^2 - 1 + 2x)$

$= (x - 1)(3x + 1)(x + 1)(3x - 1)$

Factorization (Type 4 to 5)

3.4 $a^3 \pm b^3 \equiv (a \pm b)(a^2 \mp ab + b^2)$

Illustration 14:

$a^6 - b^6$

Solution:

$a^6 - b^6 = (a^2)^3 - (b^2)^3$

$= (a^2 - b^2)(a^4 + a^2b^2 + b^4)$

$= (a - b)(a + b)(a^2 - ab + b^2)(a^2 + ab + b^2)$

3.5 Using Factor Theorem:

Illustration 15:

$x^3 - 13x - 12$

Solution:

As $x = -1$ makes given expression 0, $x + 1$ is a factor

$$\begin{array}{r|l} & x^2 - x - 12 \\ \hline x+1 & x^3 - 13x - 12 \\ & x^3 + x^2 \\ \hline & -x^2 - 13x - 12 \\ & -x^2 - x \\ \hline & -12x - 12 \\ & -12x - 12 \\ \hline & 0 \end{array}$$

$\Rightarrow x^3 - 13x - 12 = (x + 1)(x^2 - x - 12)$

$= (x + 1)(x - 4)(x + 3)$

Factorization (Type 6 to 7)

3.6 $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$

Illustration 16:

$8x^3 + y^3 + 27z^3 - 18xyz$

Solution:

$8x^3 + y^3 + 27z^3 - 18xyz = (2x)^3 + (y)^3 + (3z)^3 - 3(2x)(y)(3z)$

$= (2x + y + 3z)(4x^2 + y^2 + 9z^2 - 2xy - 6xz - 3yz)$

Illustration 17:

Factorize $(a - b)^3 + (b - c)^3 + (c - a)^3$

Solution:

Let $x = a - b, y = b - c, z = c - a$

$\Rightarrow x + y + z = 0$

$\Rightarrow x^3 + y^3 + z^3 = 3xyz$

$\Rightarrow (a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)(b - c)(c - a)$

Illustration 18:

Factorize

$27x^3 + y^3 + 64z^3 - 36xyz$

Solution:

$(3x)^3 + y^3 + (4z)^3 - 3(3x)y(4z)$

$= (3x + y + 4z)(9x^2 + y^2 + 16z^2 - 3xy - 4yz - 12zx)$

Illustration 19:

Factorize $x^2(y - z) + y^2(z - x) + z^2(x - y)$

Solution:

$x^2(y - z) + x(z^2 - y^2) + yz(y - z)$

$= (y - z)(x^2 - x(z + y) + yz)$

$= (y - z)(x^2 - xz - xy + yz)$

$= (y - z)(x(x - z) - y(x - z))$

$= (y - z)(x - z)(x - y)$

$= -(x - y)(y - z)(z - x)$

3.7 Important Algebraic Identities

- $xy + ay + bx + ab = (x + a)(y + b)$
- $x^2 + 2xy + y^2 = (x + y)^2$
- $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x + y + z)^2$
- $x^2 - y^2 = (x - y)(x + y)$
- $x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2 = (x^2 + x + 1)(x^2 - x + 1)$
- $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
- $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
- $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1}), n \in \mathbb{N}$
- $x^{2n+1} + y^{2n+1} = (x + y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - \ldots + y^{2n}), n \in \mathbb{N}$
- $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

  $$= \frac{1}{2}(x+y+z)\left\{(x-y)^2 + (y-z)^2 + (z-x)^2\right\}$$
- $x^3 + y^3 + z^3 = 3xyz$ if $x + y + z = 0$ or $x = y = z$
- $x^3 + y^3 + z^3 + 3(x + y)(y + z)(z + x) = (x + y + z)^3$

Problem Solving Strategies :

- When facing a problem with gigantic numbers, try replacing them with smaller numbers and look for a pattern. You can often prove your pattern works and solve the problem by substituting variable expressions for the numbers.

Illustration-20 :

Compute $\sqrt{2022 \times 2020 \times 2018 \times 2016 + 16}$ without using calculator.

Solution :

Let $x = 2019$

$\sqrt{2022 \times 2020 \times 2018 \times 2016 + 16}$

$= \sqrt{(x+3)(x+1)(x-1)(x-3) + 16}$

$= \sqrt{(x^2 - 9)(x^2 - 1) + 16}$

$= \sqrt{x^4 - 10x^2 + 25} = \sqrt{(x^2 - 5)^2} = x^2 - 5$

$= 2019^2 - 5 = (2000)^2 + (19)^2 + 2 \times 2000 \times 19 - 5$

$= 4000000 + 361 + 76000 - 5$

$= 4,076, 356$

- Focusing on what makes a problem tricky helps identify what strategies might be best to solve the problem.
- If you have a problem that involves expressions of the form $a + b$ and $a - b$, where a and/or b involve square roots, consider finding a way to multiply the expressions to get rid of the square roots.

Illustration-21 :

Suppose $x = z - \sqrt{z^2 - 5}$ and $5y = z + \sqrt{z^2 - 5}$. Find x when $y = 2/3$.

Solution :

$x = z - \sqrt{z^2 - 5}$ and $5y = z + \sqrt{z^2 - 5}$, multiplying both equations

we get

$5xy = z^2 - (z^2 - 5) = 5$

$\Rightarrow xy = 1$

$\Rightarrow x = \dfrac{3}{2}$.

- When making a substitution, take some time to look for the substitution that simplifies your work the most.
- If we have the product of two variables added to linear terms with both variables, such as $mn + 3m + 5n$, then there is a constant we can add that will allow us to factor. For example, adding 15 to $mn + 3m + 5n$ gives us $mn + 3m + 5n + 15 = (m + 5)(n + 3)$.

Illustration-22 :

Find all integral solutions of x and y when $xy - y - 2x = 3$.

Solution :

$xy - y - 2x = 3$

$\Rightarrow xy - y - 2x + 2 = 5$

$\Rightarrow y(x - 1) - 2(x - 1) = 5$

$\Rightarrow (x - 1)(y - 2) = 5$

$x-1$	$y-2$	(x,y)
5	1	(6,3)
1	5	(2,7)
–5	–1	(–4,1)
–1	–5	(0,–3)

All possible (x, y) and (6, 3), (2, 7), (0, –3) and (–4, 1)

- Guessing has a long and glorious history in mathematics and science. It is often a very important first step in many discoveries. Don't be afraid to guess! But make sure you test your guesses – a guess itself is only a first step.

Illustration-23 :

Find all pairs of positive integers m and n such that m^2 is 105 greater than n^2.

Solution :

Turning the words into math is easy :

$m^2 = n^2 + 105.$

$\Rightarrow (m - n)(m + n) = 105.$

$(m - n)(m + n) = 1.105 = 3.35 = 5.21 = 7.15$

Because m and n are positive, we know that $m-n$ is smaller than $m+n$, so we only have these four cases to consider :

$m-n=1$	$m-n=3$	$m-n=5$	$m-n=7$
$m+n=105$	$m+n=35$	$m+n=21$	$m+n=15$

Each of these systems of equations gives us a solution (m,n). Adding the equations in the first case gives us $2m=106$, so $m=53$. Substitution then gives $n=52$. Similarly, we can work through each of the other three cases to find the four solutions (m,n) = (53, 52); (19, 16); (13, 8); (11, 4).

Illustration-24 :

The number 7,999,999,999 has two prime factors. Find them.

Solution :

Let 7,999,999,999 = 8,000,000,000 – 1

$= (2000)^3 - 1^3 = (2000-1)(2000^2 + 2000 + 1)$

= (1999) (4002001)

According to question, 7,999,999,999 has two prime factors, they must be 1999 and 4002001.

Illustration-25 :

Factor $x^4 + 4y^4$.

Solution :

$x^4 + 4y^4 = (x^2)^2 + (2y^2)^2 - 2(x^2)(2y^2) + (2xy)^2$

$= (x^2 + 2y^2)^2 - (2xy)^2$

$= (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)$

Miscellaneous Algebraic Manipulations

Illustration 26:

Suppose $x+\frac{1}{x}=5$ find

(i) $x^2+\frac{1}{x^2}$ **(ii)** $x^4-\frac{1}{x^4}$

Solution:

(i) Given : $x+\frac{1}{x}=5$

By squaring both sides $\left(x+\frac{1}{x}\right)^2=(5)^2 \Rightarrow x^2+\frac{1}{x^2}+2=25 \Rightarrow x^2+\frac{1}{x^2}=23$

(ii) $\left(x-\frac{1}{x}\right)^2=\left(x+\frac{1}{x}\right)^2-4=21 \Rightarrow x-\frac{1}{x}=\pm\sqrt{21}$

$x^2-\frac{1}{x^2}=\pm5\sqrt{21}$

from (i) we have $x^2+\frac{1}{x^2}=23$

So $x^4-\frac{1}{x^4}=\pm23\times5\sqrt{21}=\pm115\sqrt{21}$

Illustration 27:

Simplify $\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}$

Solution:

Let $x=\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}$

By squaring both sides

$$x^2=\left(\sqrt{6+\sqrt{11}}\right)^2+\left(\sqrt{6-\sqrt{11}}\right)^2+2\sqrt{36-11}$$

= 12 + 10 = 22

Since x is positive, so $x=\sqrt{22}$.

OR

$$x=\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}=\frac{1}{\sqrt{2}}\left(\sqrt{12+2\sqrt{11}}+\sqrt{12-2\sqrt{11}}\right)$$

$$=\frac{1}{\sqrt{2}}\left(\sqrt{\left(\sqrt{11}+1\right)^2}+\sqrt{\left(\sqrt{11}-1\right)^2}\right)$$

$$=\frac{1}{\sqrt{2}}\left(\sqrt{11}+1+\sqrt{11}-1\right)$$

$$\Rightarrow x=\sqrt{22}$$

Illustration 28:

If $\left(a+\frac{1}{a}\right)^2=3$, then $a^3+\frac{1}{a^3}$ equals :

(A) $6\sqrt{3}$ (B) $3\sqrt{3}$ (C) 0 (D) $6\sqrt{3}$

Ans. (C)

Solution:

$$a+\frac{1}{a}=\pm\sqrt{3}$$

$$a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)^3-3\left(a+\frac{1}{a}\right)=\pm3\sqrt{3}\mp3\sqrt{3}=0$$

Illustration 29:

If $x=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ and $y=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, then find $x^3 + y^3$.

Solution:

$$x=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\left(\sqrt{3}-\sqrt{2}\right)^2=5-2\sqrt{6}$$

$$y=5+2\sqrt{6}$$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$= (x + y)[(x + y)^2 - 3xy]$

$= 10 \times [100 - 3]$

$= 970$

4. Polynomial in One Variable

An algebraic expression of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is called a polynomial function in $'x'$ where $a_i (i = 0, 1, 2, \ldots, n)$ is a constant which belongs to the set of real numbers and sometimes to the set of complex numbers and n is a whole number.

- a_i is the coefficient of x^i for $i = 1, 2, 3, \ldots, n$ and a_0 is constant term of $p(x)$.
- If $a_n \neq 0$, then $a_n x^n$ is called leading term and a_n is called **leading co-efficient.**
- If $a_n = 1$, then polynomial is called **monic polynomial.**
- If $a_n \neq 0$, then **degree of the polynomial** is n.
- $f(x) = a_0$ is called **constant polynomial.** Its degree is 0, if $a_0 \neq 0$. If $a_0 = 0$, the polynomial $f(x)$ is called **ZERO polynomial.** Its degree is defined as ∞ to preserve following two properties listed below. Some people prefer not to define degree of zero polynomial.

If $f(x)$ is a polynomial of degree n and $g(x)$ is a polynomial of degree m then

1. $f(x) \pm g(x)$ is a polynomial of degree $\leq max\{n, m\}$
2. $f(x) \times g(x)$ is a polynomial of degree $m + n$.

(a) Types of Polynomials (w.r.t. Degree)

Degree of the polynomial in one variable is the largest exponent of the variable. For example, the degree of the polynomial $3x^7 - 4x^6 + x + 9$ is 7 and the degree of the polynomial $5x^6 - 4x^2 - 6$ is 6.

Polynomials classified by degree

Degree	Name	General form	Example
Undefined or ∞	Zero polynomial	0	0
0	(Non-zero) constant polynomial	a; $(a \neq 0)$	4
1	Linear polynomial	$ax + b$; $(a \neq 0)$	$x + 1$
2	Quadratic polynomial	$ax^2 + bx + c$; $(a \neq 0)$	$x^2 + 1$
3	Cubic polynomial	$ax^3 + bx^2 + cx + d$; $(a \neq 0)$	$x^3 + 1$

Usually, a polynomial of degree n, for n greater than 3, is called a polynomial of degree n, although the phrases quartic polynomial for degree 4 and quantic polynomial for degree 5 are sometimes used.

Note :

Polynomials having only one term are called monomials. E.g. $2, 2x, 7y^5, 12t^7$ etc. Polynomials having exactly two dissimilar terms are called binomials. E.g. $P(x) = 2x + 1, r(y) = 2y^7 + 5y^6$ etc. Polynomials having exactly three distinct terms are called trinomials.

E.g. $P(x) = 2x^2 + x + 6, Q(y) = 9y^6 + 4y^2 + 1$ etc.

(b) Division in Polynomials

Consider two polynomials $P(x)$ & $d(x)$ with $d(x)$ being not identically zero and degree of $d(x) \leq$ degree of $P(x)$ then there exists unique polynomials $Q(x)$ and $r(x)$ such that

$P(x) = Q(x) . d(x) + r(x)$

Here $P(x)$ is called as dividend,

$d(x)$ is called as divisor,

$Q(x)$ is called as quotient,

& $r(x)$ is called as remainder with degree of $r(x) <$ degree of $d(x)$

Note :
If $d(x)$ is a divisor of $P(x)$ then $kd(x)$ will also be a divisor of $P(x)$; $k \in R - \{0\}$ and $d(-x)$ will be a divisor of $P(-x)$.

Illustration 30:

Which of the following is polynomial?

(A) $x^2 + \frac{1}{x}$ (B) $\sqrt{x} + x^3 - x + 1$ (C) $x^5 + x^6 + \sqrt{3}$ (D) $\frac{3}{x} + 4$

Ans. (C)

Solution:

$x^5 + x^6 + \sqrt{3}$ is polynomial.

Illustration 31:

Degree of the polynomial $6x^2 - 9x^3 + \sqrt{3}x^5 + 8x^{10}$ is

(A) 3 (B) 10 (C) 8 (D) 5

Ans. (B)

Solution:

Largest exponent of the variable in polynomial $6x^2 - 9x^3 + \sqrt{3}x^5 + 8x^{10}$ is 10

So degree is 10.

Illustration 32:

Which of the following is not monomial?

(A) 2 (B) t^2 (C) $\sqrt{3}x$ (D) $1 + x^2$

Ans. (D)

Solution:

$1 + x^2$ is not monomial.

5. Remainder Theorem

Statement : Let $p(x)$ be a polynomial of degree ≥ 1 and $'a'$ is any real number. If $p(x)$ is divided by $(x - a)$, then the remainder is $p(a)$.

Illustration 33:

Let $P(x)$ *be* $x^3 - 7x^2 + 6x + 4$. Divide $p(x)$ with $(x - 6)$ and find the remainder

Solution:

Put $x = 6$ in $p(x)$ i. e. $p(6)$ will be the remainder.

$\therefore$ required remainder be

$p(6) = (6)^3 - 7.6^2 + 6.6 + 4 = 216 - 252 + 36 + 4 = 256 - 252 = 4$

$$\begin{array}{r|l} & x^2 - x \\ x-6 & x^3 - 7x^2 + 6x + 4 \\ & \underline{x^3 - 6x^2} \\ & \quad {}_{-}\quad{}_{+} \\ & -x^2 + 6x + 4 \\ & \underline{-x^2 + 6x} \\ & \quad {}_{+}\quad{}_{-} \\ & \text{Remainder} = 4 \end{array}$$

Thus, $p(a)$ is remainder on dividing $p(x)$ by $(x - a)$.

Remark :

(i) $p(-a)$ is remainder on dividing $p(x)$ by $(x+a)$ $[\because x+a=0 \Rightarrow x=-a]$

(ii) $p\left(\frac{b}{a}\right)$ is remainder on dividing $p(x)$ by $(ax-b)$ $[\because ax-b=0 \Rightarrow x=b/a]$

(iii) $p\left(\frac{-b}{a}\right)$ is remainder on dividing $p(x)$ by $(ax+b)$ $[\because ax+b=0 \Rightarrow x=-b/a]$

(iv) $p\left(\frac{b}{a}\right)$ is remainder on dividing $p(x)$ by $(b-ax)$ $[\because b-ax=0 \Rightarrow x=b/a]$

Illustration 34:

Find the remainder when

x^3-ax^2+6x-a is divided by $x-a$

Solution:

Let $p(x)=x^3-ax^2+6x-a$

$p(a)=a^3-a(a)^2+6(a)-a$

$=a^3-a^3+6a-a=5a$

So, by the Remainder theorem, remainder = $5a$.

Illustration 35:

Find remainder when $x^4-ax^3+bx^2+ax-b$ is divided by $x-1$.

Solution:

Let $p(x)=x^4-ax^3+bx^2+ax-b$

$p(1)=1-a+b+a-b=1$

So, by the remainder theorem, remainder = 1.

6. Factor Theorem

Statement:

Let $f(x)$ be a polynomial of degree ≥ 1 and a be any real constant such that $f(a)=0$, then $(x-a)$ is a factor of $f(x)$. Conversely, if $(x-a)$ is a factor of $f(x)$, then $f(a)=0$.

Proof :

By Remainder theorem, if $f(x)$ is divided by $(x-a)$, the remainder will be $f(a)$. Let $Q(x)$ be the quotient.

Then, we can write, $f(x)=(x-a)\times Q(x)+f(a)$ ($\because$ Dividend = Divisor × Quotient + Remainder)

If $f(a)=0$, then $f(x)=(x-a)\times Q(x)$

Thus, $(x-a)$ is a factor of $f(x)$.

Converse Let $(x-a)$ is a factor of $f(x)$.

Then we have a polynomial $Q(x)$ such that $f(x)=(x-a)\times Q(x)$

Replacing x by a, we get $f(a)=0$.

Hence, proved.

Note: A polynomial function of degree 'n' (≥ 1) has exactly n zeroes (real or imaginary).

i.e., $f(x)=a(x-r_1)(x-r_2)\ldots(x-r_n)$

Illustration 36:

Use the factor theorem to determine whether $(x - 1)$ is a factor of

$f(x) = 2\sqrt{2}x^3 + 5\sqrt{2}x^2 - 7\sqrt{2}$

Solution:

By using factor theorem, $(x - 1)$ is a factor of $f(x)$, only when $f(1) = 0$

$f(1) = 2\sqrt{2}(1)^3 + 5\sqrt{2}(1)^2 - 7\sqrt{2} = 2\sqrt{2} + 5\sqrt{2} - 7\sqrt{2} = 0$

Hence, $(x - 1)$ is a factor of $f(x)$.

Illustration 37:

Use the factor theorem to determine whether $(x - a)$ is a factor of

$f(x) = x^3 - (5 + a)x^2 + (5a + 6)x - 6a$

Solution:

$(x - a)$ is a factor of $f(x)$, only when $f(a) = 0$

$f(a) = a^3 - 5a^2 - a^3 + 5a^2 + 6a - 6a = 0$

Hence, $(x - a)$ is a factor of $f(x)$.

Illustration 38:

Find the constants a, b, c such that $(2x^2 + 3x + 7)(ax^2 + bx + c) = 2x^4 + 11x^3 + 9x^2 + 13x - 35$

Solution :

Method : 1

$(2x^2 + 3x + 7)(ax^2 + bx + c) = 2x^4 + 11x^3 + 9x^2 + 13x - 35$

By comparing coefficient of x^4 from both sides

$\Rightarrow 2a = 2 \Rightarrow a = 1$

By comparing coefficient of x^3 from both sides

$\Rightarrow 2b + 3a = 11 \Rightarrow b = 4$

By comparing coefficient of x^2 from both sides

$\Rightarrow 2c + 3b + 7a = 9 \Rightarrow c = -5$

By comparing coefficient of x from both sides

$\Rightarrow 3c + 7b = 13$ $\quad$ (b & c satisfy)

By comparing constant

$\Rightarrow 7c = -35 \quad \Rightarrow c = -5$

So $a = 1, b = 4, c = -5$

Method : 2

Given that $(2x^2 + 3x + 7)(ax^2 + bx + c) = 2x^4 + 11x^3 + 9x^2 + 13x - 35$

$$\Rightarrow ax^2 + bx + c = \frac{2x^4 + 11x^3 + 9x^2 + 13x - 35}{2x^2 + 3x + 7}$$

$$= x^2 + 4x - 5$$

By comparing

$\Rightarrow a = 1, b = 4$ and $c = -5$.

Illustration 39 :

Show that $(x-3)$ is a factor of the polynomial $x^3-3x^2+4x-12$.

Solution :

Let $p(x)=x^3-3x^2+4x-12$ be the given polynomial. By factor theorem, $(x-a)$ is a factor of a polynomial $p(x)$ iff $p(a)=0$. Therefore, in order to prove that $x-3$ is a factor of $p(x)$, it is sufficient to show that $p(3)=0$.

Now, $p(x)=x^3-3x^2+4x-12$

$\Rightarrow p(3)=3^3-3\times 3^2+4\times 3-12=27-27+12-12=0$

Hence, $(x-3)$ is a factor of $p(x)=x^3-3x^2+4x-12$.

Illustration 40 :

The polynomials $P(x)=kx^3+3x^2-3$ and $Q(x)=2x^3-5x+k$, when divided by $(x-4)$ leave the same remainder. The value of k is

(A) 2 (B) 1 (C) 0 (D) –1

Solution :

$P(4)=64k+48-3=64k+45$

$\Rightarrow Q(4)=128-20+k=k+108$

given $P(4)=Q(4)$

$\Rightarrow 64k+45=k+108$

$\Rightarrow 63k=63\Rightarrow k=1$

Option **(B)** is correct

Illustration 41 :

If $f(x)$ is monic polynomial of degree 6 such that $f(0)=0, f(1)=-1, f(2)=-2, f(3)=-3, f(4)=-4$ and $f(5)=-5$, then find $f(x)$.

Solution :

According to question

$f(0)=0, f(1)=-1, f(2)=-2, \ldots, f(5)=-5$

$\Rightarrow f(x)+x=0$ has the roots $x=0, 1, 2, \ldots, 5$

$\Rightarrow f(x)+x=x(x-1)(x-2)(x-3)(x-4)(x-5)$ (By factor theorem)

$\Rightarrow f(x)=x(x-1)(x-2)(x-3)(x-4)(x-5)-x$.

Illustration 42 :

Let $P(x)=x^4+ax^3+bx^2+cx+d$, where a,b,c,d are constants. If $P(1)=10, P(2)=20, P(3)=30$, then compute $P(4)+P(0)$.

Solution :

$P(1)=10, P(2)=20$ and $P(3)=30$

we can write these information as

$\Rightarrow P(x)=10x$ for $x=1, 2, 3$

$\Rightarrow P(x)-10x=0$ has roots $x=1, 2$ and 3

By factor theorem

$\Rightarrow P(x)-10x=(x-\alpha)(x-1)(x-2)(x-3)$

$\Rightarrow P(x)=(x-\alpha)(x-1)(x-2)(x-3)+10x$

$\Rightarrow P(4)+P(0)=(4-\alpha)(3)(2)(1)+40+(-\alpha)(-1)(-2)(-3)+0$

$=24-6\alpha+40+6\alpha$

$=64$

Illustration 43 :

Let a, b and c are roots of $2x^3 + x^2 + x + 1 = 0$

find (i) $a + b + c$ (ii) abc

Solution :

By factor theorem

$2x^3 + x^2 + x + 1 = 2(x - a)(x - b)(x - c)$

$\Rightarrow 2x^3 + x^2 + x + 1 = 2(x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc)$

$\Rightarrow 2x^3 + x^2 + x + 1 = 2x^3 - 2(a + b + c)x^2 + 2(ab + bc + ca)x - 2abc$

By comparing coefficient of x^2 and constant term, we have

$-2(a + b + c) = 1$ and $-2abc = 1$

$\Rightarrow a + b + c = \frac{-1}{2}$ and $abc = \frac{-1}{2}$.

7. Equations Reducible to Quadratic Equations

There are certain equations which can be reduced to $ax^2 + bx + c = 0$ by some proper substitution.

(a) $a(f(x))^2 + b(f(x)) + c = 0$, where $f(x)$ is expression of x.

Method of solving : Put $f(x) = y$

Illustration 44:

(a) Solve $2^{x+3} + 2^{-x} - 6 = 0$

(b) Solve $\left(\frac{x}{x+1}\right)^2 + 6 - 5\left(\frac{x}{x+1}\right) = 0$

Solution:

(a) Put $2^x = y$

$\Rightarrow 8y + \frac{1}{y} - 6 = 0$

$\Rightarrow 8y^2 - 6y + 1 = 0$

$\Rightarrow (4y - 1)(2y - 1) = 0$

$\Rightarrow y = \frac{1}{4}, \frac{1}{2}$

$\Rightarrow 2^x = 2^{-2}$ and $2^x = 2^{-1}$

$\Rightarrow x = -2, x = -1$

(b) Put $\frac{x}{x+1} = y$

$\Rightarrow y^2 - 5y + 6 = 0$

$\Rightarrow (y - 2)(y - 3) = 0$

$\Rightarrow \frac{x}{x+1} = 2$ and $\frac{x}{x+1} = 3$

$\Rightarrow x = 2x + 2$ and $x = 3x + 3$

$\Rightarrow x = -2 \quad x = -\frac{3}{2}$

(b) $(x - a)^4 + (x - b)^4 = c,$

Method of Solving : Put $\frac{x-a+x-b}{2}=t \Rightarrow x=\frac{(a+b)}{2}+t$

Illustration-45 :

$(x - 1)^4 + (x - 7)^4 = 272$

Solution :

Put $x=\frac{7+1}{2}+t$

$\Rightarrow x = t + 4$

$\Rightarrow (t + 3)^4 + (t - 3)^4 = 272$

$\Rightarrow 2(t^4 + 6.9t^2 + 81) = 272$

$\Rightarrow t^4 + 54t^2 + 81 = 136$

$\Rightarrow t^4 + 54t^2 - 55 = 0$

$\Rightarrow (t^2 - 1)(t^2 + 55) = 0$

$\Rightarrow t^2 = 1$

$\Rightarrow t = \pm 1$

$\Rightarrow x = 5,3$

(c) $ma^{2x} + n(ab)^x + rb^{2x} = 0 \ (a \,\&\, b > 0)$

Method of Solving : Divide the equation by b^{2x} and put $\left(\frac{a}{b}\right)^x = t$ for $t > 0$

Illustration 46:

$3^{2x+2} + 5.6^x - 4^{x+1} = 0$

Solution:

$3^{2x+2} + 5.6^x - 4^{x+1} = 0$

$\Rightarrow 9 \times (3)^{2x} + 5 \times (2 \times 3)^x - 4(2)^{2x} = 0$

$\Rightarrow \ 9\left(\frac{3}{2}\right)^{2x} + 5\left(\frac{3}{2}\right)^x - 4 = 0$ ...(1)

Let $\left(\frac{3}{2}\right)^x = t$ for $t > 0$

Equation 1 becomes

$9t^2 + 5t - 4 = 0$

$\Rightarrow \ t=\frac{-5\pm\sqrt{25+144}}{18}=-1$ or $\frac{4}{9}$

$t = -1$ (rejected)

$t=\frac{4}{9} \Rightarrow \left(\frac{3}{2}\right)^x=\left(\frac{2}{3}\right)^2=\left(\frac{3}{2}\right)^{-2}$

$\Rightarrow x = -2$

Solution of the given equation is $x = -2$

(d) $m.a^{f(x)} + n.b^{f(x)} + r = 0$**, where** $ab = 1, a$ **&** $b > 0$ **and** $f(x)$ **is expression of** x

Method of Solving : put $a^{f(x)} = t$, then $b^{f(x)} = \frac{1}{t}$

Illustration 47:

Solve $\left(\sqrt{5+2\sqrt{6}}\right)^x + \left(\sqrt{5-2\sqrt{6}}\right)^x = 10$

Solution:

Let $a = \sqrt{5+2\sqrt{6}}$ and $b = \sqrt{5-2\sqrt{6}}$

$ab = 1$

Let $a^x = t$

Given equation become

$t + \frac{1}{t} = 10 \Rightarrow t^2 - 10t + 1 = 0$

$\Rightarrow t = \frac{10 \pm \sqrt{96}}{2} = 5 \pm 2\sqrt{6}$

$\Rightarrow \left(5+2\sqrt{6}\right)^{x/2} = \left(5+2\sqrt{6}\right) \Rightarrow \frac{x}{2} = 1 \Rightarrow x = 2$

or $\left(5+2\sqrt{6}\right)^{x/2} = \left(5-2\sqrt{6}\right) = \left(5+2\sqrt{6}\right)^{-1}$

$\Rightarrow \frac{x}{2} = -1 \Rightarrow x = -2$

Solutions of the given equation is $x = 2$ or -2.

(e) $(x + a)(x + b)(x + c)(x + d) + e = 0$ **when** $b + c = a + d$

Illustration 48:

Solve $x(x + 1)(x + 2)(x + 3) - 8 = 0$

Solution:

$x(x + 1)(x + 2)(x + 3) - 8 = 0$

$\Rightarrow x(x + 3)(x + 1)(x + 2) - 8 = 0$

$\Rightarrow (x^2 + 3x)(x^2 + 3x + 2) - 8 = 0$

$\Rightarrow (x^2 + 3x)^2 + 2(x^2 + 3x) - 8 = 0$

$\Rightarrow (x^2 + 3x)^2 + 4(x^2 + 3x) - 2(x^2 + 3x) - 8 = 0$

$\Rightarrow (x^2 + 3x)(x^2 + 3x + 4) - 2(x^2 + 3x + 4) = 0$

$\Rightarrow (x^2 + 3x - 2)(x^2 + 3x + 4) = 0$

$\Rightarrow (x^2 + 3x - 2) = 0$ or $(x^2 + 3x + 4) = 0$

$\Rightarrow x = \frac{-3 \pm \sqrt{17}}{2}$ or $x = \frac{-3 \pm \sqrt{7}\, i}{2}$

(f) $(x + a)(x + b)(x + c)(x + d) + ex^2 = 0$**, where** $ad = bc$.

Method of solving : Divide given equation by x^2 and put $x+\frac{ad}{x}=t$

Illustration 49:

$(x + 1)(x + 2)(x + 3)(x + 6) = 3x^2$

Solution:

$(x + 1)(x + 6)(x + 2)(x + 3) - 3x^2 = 0$

$\Rightarrow (x^2 + 7x + 6)(x^2 + 5x + 6) - 3x^2 = 0$

$\Rightarrow \left(x+\frac{6}{x}+7\right)\left(x+\frac{6}{x}+5\right)-3=0$

Let $x+\frac{6}{x}=t$

$\Rightarrow (t + 7)(t + 5) - 3 = 0 \Rightarrow t^2 + 12t + 32 = 0 \Rightarrow t = -8$ or -4

when $x+\frac{6}{x}=-8 \Rightarrow x^2 + 8x + 6 = 0 \Rightarrow x=-4\pm\sqrt{10}$

$x+\frac{6}{x}=-4 \Rightarrow x^2 + 4x + 6 = 0 \Rightarrow x=-2\pm\sqrt{2}i$

Illustration 50:

Solve $(x^2 - 3x)(x^2 - 3x + 2) + 1 = 0$

Solution:

Let $x^2 - 3x = y$

$\Rightarrow y(y + 2) + 1 = 0$

$\Rightarrow y^2 + 2y + 1 = 0$

$\Rightarrow (y + 1)^2 = 0$

$\Rightarrow y = -1$

Putting $y = -1$ in $x^2 - 3x = y$

we have $x^2 - 3x = -1$

$\Rightarrow x^2 - 3x + 1 = 0$

$\Rightarrow x=\frac{3\pm\sqrt{5}}{2}$

(g) $ax^4 + bx^3 + cx^2 + dx + e = 0$**, where** $a = e$ & $b = \pm d$

Method of solving : Divide given equation by x^2 and put $x + \frac{1}{x} = t$ or $x-\frac{1}{x}=t$ whichever is applicable.

Illustration 51:

Solve $x^4 - 2x^3 + 3x^2 - 2x + 1 = 0$

Solution:

$x^4 - 2x^3 + 3x^2 - 2x + 1 = 0$

By dividing x^2 both sides we have

$x^2 - 2x + 3 - \frac{2}{x}+\frac{1}{x^2}=0$

$\Rightarrow x^2+\frac{1}{x^2}-2\left(x+\frac{1}{x}\right)+3=0$

Let $x+\frac{1}{x}=t$

Above equation become

$\Rightarrow t^2 - 2 - 2t + 3 = 0$

$\Rightarrow t^2 - 2t + 1 = 0 \Rightarrow (t-1)^2 = 0$

$\Rightarrow t = 1 \Rightarrow x + \frac{1}{x} = 1 \Rightarrow x^2 - x + 1 = 0 \Rightarrow x=\frac{1\pm\sqrt{3}\,i}{2}$

Roots are $\frac{1\pm\sqrt{3}\,i}{2}$

(h) By Guessing Rational Roots of Polynomial.

Illustration 52:

Solve : $x^4 + x^3 - 2x^2 - x + 1 = 0$

Solution:

Let $P(x) = x^4 + x^3 - 2x^2 - x + 1$

$P(1) = 0$ and $P(-1) = 0$

$\Rightarrow (x-1)\,(x+1)$ factor of $P(x)$

We can find other factor of $P(x)$ by dividing $x^2 - 1$ from $P(x)$.

$\Rightarrow P(x) = (x^2-1)\,(x^2+x-1) = 0$

$\Rightarrow x = \pm 1$ or $x^2 + x - 1 = 0$

$\Rightarrow x = \pm 1$ or $x = \frac{-1\pm\sqrt{5}}{2}$

solution of $P(x) = 0$ are $x = \pm 1$ or $\frac{-1\pm\sqrt{5}}{2}$.

Illustration 53:

Solve: $x^3 - 12x^2 + 41x - 30 = 0$

Solution:

Let $P(x) = x^3 - 12x^2 + 41x - 30$

$\Rightarrow P(1) = 0 \Rightarrow (x-1)$ is factor of $P(x)$

We can find other factor of $P(x)$ by dividing $(x-1)$ from $P(x)$

$\Rightarrow P(x) = (x-1)(x^2 - 11x + 30)$

$\Rightarrow x = 1$ or $x^2 - 11x + 30 = 0$

$\Rightarrow x = 1$ or $x = 5, 6$

Solution of $p(x)$ are $[x = 1, 5, 6]$.

Illustration 54:

$x^4 - 13x^2 + 36 = 0$

Solution:

Let $P(x) = x^4 - 13x^2 + 36$

$\Rightarrow P(2) = 0$ & $P(-2) = 0$

$\Rightarrow (x^2 - 4)$ is factor of $P(x)$

We can find other factor of $P(x)$ by dividing

$(x^2 - 4)$ from $P(x)$

$\Rightarrow P(x) = (x^2-4)(x^2-9)$

$\Rightarrow x = \pm 2$ or $x = \pm 3$

Ans. $[x = 2, -2, 3, -3]$

8. System of Equations

Illustration 55:

If $x - y = 2$ and $xy = 24$, find the value of $\frac{1}{x}+\frac{1}{y}$.

Solution:

$(x + y)^2 = (x - y)^2 + 4xy = 4 + 4(24)$

$\Rightarrow (x + y)^2 = 100$

$\Rightarrow x + y = 10, -10$

$\therefore \frac{x+y}{xy}=\frac{10}{24}=\frac{5}{12};\frac{x+y}{xy}=-\frac{10}{24}=-\frac{5}{12}$

Illustration 56:

If $2x - 3y - z = 0$ and $x + 3y - 14z = 0$, then find $\frac{x^2+3xy}{y^2+z^2}$.

Solution:

$\frac{2x}{z}-\frac{3y}{z}=1 \;\&\; \frac{x}{z}+\frac{3y}{z}=14$

Solving $\frac{x}{z}=5;\frac{y}{z}=3$

$$\Rightarrow \frac{\left(\frac{x}{z}\right)^2+\frac{3x}{z}.\frac{y}{z}}{\left(\frac{y}{z}\right)^2+1}=\frac{25+3(5)(3)}{(3)^2+1}=\frac{70}{10}=7$$

Illustration 57:

Given $a + b = 20$ and $a^3 + b^3 = 800$, find $a^2 + b^2$.

Solution:

$a + b = 20$

$\Rightarrow a^2 + b^2 + 2ab = 400$ ...(1)

$\Rightarrow a^3 + b^3 = 800$

$\Rightarrow (a + b)(a^2 - ab + b^2) = 800$

$\Rightarrow a^2 + b^2 - ab = 40$ ...(2)

By adding twice of second equation with first equation.

$\Rightarrow 3(a^2 + b^2) = 480$

$\Rightarrow a^2 + b^2 = 160.$

Illustration 58:

$x(y + z) = 29, y(z + x) = 26; z(x + y) = 51$ find x, y, z.

Solution:

$xy + zx = 29$ ...(1)

$yz + xy = 26$ ...(2)

$xz + yz = 51$ ...(3)

$\Rightarrow 2[xy + yz + zx] = 106$

$\Rightarrow xy + yz + zx = 53$

Now : $xy = 2, zx = 27; yz = 24$

$\Rightarrow x^2y^2z^2 = 24 \times 2 \times 27 = (36)^2$

$\Rightarrow xyz = \pm 36 \therefore (x, y, z) \equiv \left(\frac{3}{2}, \frac{4}{3}, 18\right)$ or $\left(-\frac{3}{2}, -\frac{4}{3}, -18\right)$

Illustration 59:

If $x^3 + y^3 = 35$; and $x^2y + xy^2 = 30$, then find (x, y).

Solution:

$$\frac{x^3+y^3}{x^2y+xy^2} = \frac{(x+y)(x^2-xy+y^2)}{xy(x+y)} = \frac{35}{30} = \frac{7}{6}$$

$\Rightarrow 6x^2 - 13xy + 6y^2 = 0 \Rightarrow (3x - 2y)(2x - 3y) = 0 \Rightarrow 3x = 2y$ or $2x = 3y$

Case-I : $3x = 2y,\ x^3 + \left(\frac{3x}{2}\right)^3 = 35 \Rightarrow 35x^3 = 8 \times 35 \Rightarrow x = 2, y = 3$

Case-II : $2x = 3y,\ x^3 + \left(\frac{2x}{3}\right)^3 = 35 \Rightarrow 35x^3 = 27 \times 35 \Rightarrow x = 3, y = 2$

9. Intervals

Intervals are basically subsets of $\mathbb{R}$ and are very important in mathematics as you will get to know shortly. If there are two numbers $a, b \in \mathbb{R}$ such that $a < b$, we can define four types of intervals as follows:

Closed Intervals

All numbers x between a and b including both numbers is written in closed interval. It is denoted by [ ].

i.e. $a \le x \le b$ or $x \in [a, b]$ or $\{x : x \in \mathbb{R}$ and $a \le x \le b\}$

Graphical Representation

Open Intervals

All numbers x between a and b excluding both numbers is written in open interval. It is denoted by][or ( ).

i.e. $a < x < b$ or $x \in]a, b[$ or $x \in (a, b)$ or $\{x : x \in \mathbb{R}$ and $a < x < b\}$

Graphical Representation

Open-Closed Intervals

All numbers x between a and b including b and excluding a is written in open - closed interval. It is denoted by $]a, b]$ or $(a, b]$ or $a < x \le b$ or $\{x : x \in \mathbb{R}$ and $a < x \le b\}$

Graphical Representation

Closed-Open Intervals

All numbers x between a and b including a but excluding b is written in closed-open interval. It is denoted by $[a, b[$ or $[a, b)$ or $a \le x < b$ or $\{x : x \in \mathbb{R}$ and $a \le x < b\}$.

Graphical Representation

The Infinite Intervals are Defined as follows:

- $(a, \infty) = \{x : x > a\}$
- $[a, \infty) = \{x : x \geq a\}$
- $(-\infty, b] = \{x : x \leq b\}$
- $(-\infty, b) = \{x : x < b\}$

Note:

$x \in \{1, 2\}$ denotes some particular values of x, i.e. $x = 1, 2$

If there is no value of x, then we can say $x \in \phi$ (Null set)

Intervals are particularly important in solving inequalities.

Union: $(A \cup B)$ / $(A$ *or* $B)$

If A contains some elements and B contains some elements, then $(A \cup B)$ contains elements which are in A or B.

Intersection: $(A \cap B)$ / $(A$ and $B)$

If A contains some elements and B contains some elements, then $(A \cap B)$ contains elements which are common in both A and B.

Difference : $(A - B)$

If A contains some elements and B contains some elements, then $(A - B)$ contains elements which are in A only but not in B.

Illustration 60:

Represent following sets on the number line

(i) $(-\infty, 2) \cup [7, \infty)$

(ii) $x \leq 4$ or $x > 8$

(iii) $(-\infty, -1) \cup (-5, 6]$

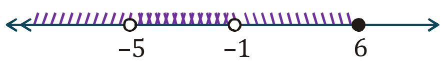
(iv) $(0, 5] \cap [-3, 4)$

(v) $[-2, 10) - (1, 5]$

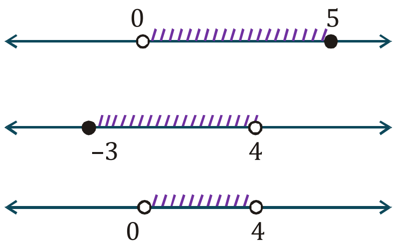
Solution:

(i) 

(ii) 

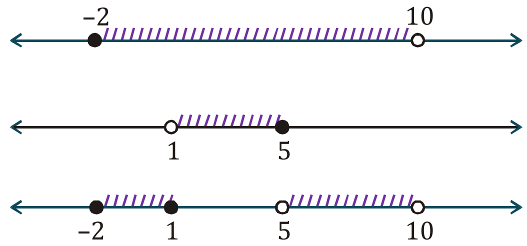
(iii) 

Union of given two sets is $(-\infty, 6]$

(iv) 

The intersection of given two sets is $(0, 4)$.

(v)



Difference of given two sets is $[-2, 1] \cup (5, 10)$

10. Inequalities

Basic Rules:

- If $a > b$ and $b > c$, then $a > c$.
- If $x > y$, then $x + c > y + c$ for any real number c. Additionally, if $a > b$, then $x + a > y + b$.
- If $x > y$ and $a > 0$, then $xa > ya$.
- If we multiply or divide an inequality by a negative number, we must reverse the sign.
  For example, if $x > y$ and $a < 0$, then $xa < ya$.
- If $x > y > 0$ and $a > b > 0$, then $xa > yb$.
- If $x > y$ and x and y have the same sign (positive or negative), then $\frac{1}{x} < \frac{1}{y}$.
- If $x > y \geq 0$, then for any positive real number a, we have $x^a > y^a$.

In particular, if $0 < a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$ for all positive integral values of $n > 1$.

E.g. $\sqrt[4]{2} < \sqrt[4]{7}, \sqrt[3]{3} < \sqrt[3]{5}, \sqrt[5]{10} < \sqrt[5]{13}$ etc.

If two simple surds of different orders viz. $\sqrt[n]{a}$ and $\sqrt[m]{b}$ have to be compared, they have to be expressed as surds of the same order i.e. LCM of n and m.

E.g. Compare $\sqrt[4]{6}$ and $\sqrt[3]{5}$,

we express both as the surds of 12th order.

$\therefore\ \sqrt[4]{6} = \sqrt[12]{6^3}$ and $\sqrt[3]{5} = \sqrt[12]{5^4}$. As $6^3 < 5^4 \Rightarrow\ \sqrt[4]{6} < \sqrt[3]{5}$

Illustration 61:

Which of the following is greater $5+\sqrt{3}, 3+\sqrt{14}$?

(Without calculating the values of $\sqrt{3}, \sqrt{14}$)

Solution:

Lets assume $3+\sqrt{14}$ is greater than $5+\sqrt{3}$.

$\Rightarrow\ 3+\sqrt{14} > 5+\sqrt{3}$

$\Rightarrow \sqrt{14}-\sqrt{3} > 2$

Squaring both sides

$\Rightarrow\ 14+3-2\sqrt{42} > 4$

$\Rightarrow 13 > 2\sqrt{42}$

$\Rightarrow \frac{13}{2} > \sqrt{42}$

Squaring again

$\Rightarrow \frac{169}{4} > 42$

$\Rightarrow 42.25 > 42$

Which is true so we can say that

What we assumed initially is true.

So, $(3+\sqrt{14})$ is greater than $5+\sqrt{3}$.

Illustration 62:

Solve $\frac{8-x}{7} \geq 4$

Solution:

$\frac{8-x}{7} \geq 4 \Rightarrow 8 - x \geq 28 \Rightarrow -x \geq 20$

$\Rightarrow x \leq -20 \Rightarrow x \in (-\infty, -20]$

Illustration 63:

Solve $\frac{1}{x} \geq 5$

Solution:

$\frac{1}{x} \geq 5 \Rightarrow \infty > \frac{1}{x} \geq 5 \Rightarrow 0 < x \leq \frac{1}{5} \Rightarrow x \in \left(0, \frac{1}{5}\right]$

Illustration 64:

Which fraction is larger?

(a) $\frac{13}{17}$ or $\frac{17}{21}$ (b) $\frac{31}{35}$ or $\frac{37}{41}$

Solution:

(a) $\frac{13}{17}$ or $\frac{17}{21}$

Making Numerator same: $\frac{13}{17} \times \frac{17}{17} = \frac{221}{289}$

$\frac{17}{21} \times \frac{13}{13} = \frac{221}{273}$

Now number which will have high denominator will be smaller

So $\frac{13}{17} < \frac{17}{21}$

(b) $\frac{31}{35}$ or $\frac{37}{41}$

Making denominator same: $\frac{31}{35} \times \frac{41}{41} = \frac{1271}{1435}$

$\frac{37}{41} \times \frac{35}{35} = \frac{1295}{1435}$

Now number which will have high Numerator will be larger

So $\frac{31}{35} < \frac{37}{41}$

Illustration 65:

What values of x satisfy the inequality, $7 - 3x < x - 1 \leq 2x + 9$?

Solution:

$7 - 3x < x - 1 \leq 2x + 9$

Equation can be written as:-

$\Rightarrow 7 - 3x < x - 1$ ...(1)

& $x - 1 \leq 2x + 9$ ...(2)

Solving (1)

$\Rightarrow 7 - 3x < x - 1$

$\Rightarrow 8 < 4x$

$\Rightarrow x > 2$ ...(3)

solving (2)

$\Rightarrow x - 1 \leq 2x + 9$

$\Rightarrow -10 \leq x$

$\Rightarrow x \geq -10$ ...(4)

intersection of (3) & (4) $x \in (2, \infty)$ Ans.

11. Trivial and Sum of squares (SOS) Inequality

The square of any real number is non-negative. So if x is real, then $x^2 \geq 0$. This is known as Trivial inequality. Equality holds only if $x = 0$.

Sum of squares (SOS) of real numbers is non negative. That is $\sum_{i=1}^{i=n} x_i^2 \geq 0$. This is known as SOS inequality.

Equality holds if $x_i = 0$

Ex. $x, y, z \in R$ and $x^2 + y^2 + z^2 = 0 \Rightarrow x = y = z = 0$.

Note :

- $f(x) = [g(x)]^{2n}$ where $n \in N \Rightarrow f(x) \geq 0$
- $f(x) = [g(x)]^{\frac{1}{2n}}, n \in N, g(x) \geq 0 \Rightarrow f(x) \geq 0$

Illustration 66:

If $a, b, c \in R$ and $a^2 + b^2 + c^2 - ab - bc - ca = 0$, prove that $a = b = c$.

Solution:

$a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

$\Rightarrow (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ac + a^2) = 0$

$\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow a - b = b - c = c - a = 0$

$\Rightarrow a = b = c$.

Illustration 67:

For $x, y \in \mathbb{R}$, find the all possible values (range) of expression $4x^2 + 9y^2 - 12x + 6y$.

Solution:

If $E(x, y) = 4x^2 + 9y^2 - 12x + 6y$

$= (2x)^2 - 2(2x) \times 3 + (3)^2 + (3y)^2 + 2(3y) + (1)^2 - 10$

$= (2x - 3)^2 + (3y + 1)^2 - 10$

By sum of square (SOS), $(2x - 3)^2 + (3y + 1)^2 \geq 0$

$\Rightarrow E(x, y) = (2x - 3)^2 + (3y + 1)^2 - 10 \geq 0 - 10$

So, Range of $E(x, y) = [-10, \infty)$

Illustration 68:

If $b^2 - 4ac < 0$ and $a > 0$, then show that $ax^2 + bx + c > 0 \ \forall\, x \in \mathbb{R}$

Solution:

$$\Rightarrow ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x\right] + c$$

$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right] + c$$

$$= a\left[x + \frac{b}{2a}\right]^2 - \left(\frac{b^2 - 4ac}{4a}\right) > 0 \ \forall\, x \in \mathbb{R}.$$ Hence proved.

Illustration 69:

Find the values of expression $E = x^2 + y^2 - 4x - 6y + 15;\ x, y \in \mathbb{R}$.

(A) 1 (B) 2 (C) e (D) π

Ans. (B, C, D)

Solution:

$E = x^2 - 4x + 4 + y^2 - 6y + 9 + 2$

$E = (x-2)^2 + (y-3)^2 + 2$

By SOS $(x-2)^2 + (y-3)^2 \geq 0$

Add 2 on both sides

$(x-2)^2 + (y-3)^2 + 2 \geq 2$

$E \geq 2$

So E can take values 2, e and π from the options.

12. Mean:

In any collection of data a specific value between two extremes (minimum/maximum) is called a mean of the data.

For Two Variables :

Let x, y be two positive real numbers with $x \leq y$.

- The **Arithmetic Mean** (A) of x and y is $\frac{x+y}{2}$

  Observe $x \leq \frac{x+y}{2} \leq y$

- The **Geometric Mean** (**G**) of x and y is $\sqrt{xy}$

Observe $x \le \sqrt{xy} \le y$

We have $x \le G \le A \le y$

Equality holds only when $x = y$.

Proof: For two positive numbers x and y, the Trivial inequality gives us $(\sqrt{x}-\sqrt{y})^2 \ge 0$

$\Rightarrow x+y-2\sqrt{xy} \ge 0$

$\Rightarrow \frac{x+y}{2} \ge \sqrt{xy} \quad \Rightarrow A \ge G$

Note :

- $x+\frac{1}{x} \ge 2 \ \forall x > 0$ and $x+\frac{1}{x} \le -2 \ \forall x < 0$

Illustration 70:

Find all possible values (range) of the expression $x+\frac{4}{x}$, when $x \in R - \{0\}$.

Solution:

When $x > 0$, $\frac{x+\frac{4}{x}}{2} \ge \sqrt{x \times \frac{4}{x}} = 2$ (By $A \ge G$)

$\Rightarrow x+\frac{4}{x} \ge 4$,

So, when $x > 0$, range $= [4, \infty)$ ...(1)

for $x < 0$, $\frac{x+\frac{4}{x}}{2} \le -\sqrt{x \times \frac{4}{x}}$

$\Rightarrow x+\frac{4}{x} \le -4$ ...(2)

From (1) and (2), range = $(-\infty, -4] \cup [4, \infty)$

Illustration 71:

Find the minimum value of $\frac{x^4+8}{x^2}$

Solution:

$\frac{x^4+8}{x^2} = x^2 + \frac{8}{x^2}$

$\because$ AM $\ge$ GM

$\Rightarrow \frac{x^2+\frac{8}{x^2}}{2} \ge \left(x^2 . \frac{8}{x^2}\right)^{\frac{1}{2}}$

$\Rightarrow x^2 + \frac{8}{x^2} \ge 2(8)^{\frac{1}{2}} \Rightarrow \frac{x^4+8}{x^2} \ge 4\sqrt{2}$

Minimum value of $\frac{x^4+8}{x^2}$ is $4\sqrt{2}$.

Illustration 72:

For $x < 0$, find the maximum value of $\frac{3x^2+12}{x}$.

Solution:

$\frac{3x^2+12}{x} = 3x + \frac{12}{x}$

As x is negative, Applying AM ≥ GM on $-3x$ & $-\frac{12}{x}$

$$\Rightarrow \frac{-3x - \frac{12}{x}}{2} \geq \left((-3x)\left(-\frac{12}{x}\right)\right)^{\frac{1}{2}} \Rightarrow -3x - \frac{12}{x} \geq 2 \times (36)^{\frac{1}{2}}$$

$$\Rightarrow -3x - \frac{12}{x} \geq 12 \Rightarrow 3x + \frac{12}{x} \leq -12$$

Maximum value of $\frac{3x^2+12}{x}$ is -12.

13. Ratio and Proportion

If a and b be two quantities of the same kind, then their ratio is $a:b$; which is denoted by the fraction $\frac{a}{b}$ (This may be an integer or fraction).

A ratio may be represented in a number of ways e.g. $\frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb} = \ldots$ where $m, n, \ldots$ are non-zero numbers.

Let a, b, c, d be positive integers now to compare two ratios $a : b$ and $c : d$ we use following :

- $(a : b) > (c : d)$ if $ad > bc$
- $(a : b) = (c : d)$ if $ad = bc$
- $(a : b) < (c : d)$ if $ad < bc$

To compare two or more ratio, reduce them to common denominator.

Illustration 73:

What term must be added to each term of the ratio 5 : 37 to make it equal to 1 : 3 ?

Solution:

Let x be added to each term of the ratio 5 : 37.

Then $\frac{x+5}{x+37} = \frac{1}{3} \Rightarrow 3x + 15 = x + 37$ i.e. $x = 11$

Illustration 74:

If $x : y = 3 : 4$; find the ratio of $7x - 4y : 3x + y$

Solution:

$\frac{x}{y} = \frac{3}{4} \Rightarrow x = \frac{3}{4}y$

Now $\frac{7x-4y}{3x+y} = \frac{7.\frac{3}{4}y - 4y}{3.\frac{3}{4}y + y}$ (putting the value of x)

$= \frac{5y}{13y} = \frac{5}{13}$ i.e. 5 : 13

Proportion:

When two ratios are equal, then the four quantities compositing them are said to be proportional.

So, if $\frac{a}{b}=\frac{c}{d}$, then it is written as $a : b = c : d$ or $a : b :: c : d$

Where 'a' and 'd' are known as extremes and 'b and c' are known as means.

(i) An important property of proportion : Product of extremes = product of means.

(ii) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ then each is equal to $\frac{a+c+e}{b+d+f}$

(iii) If $a : b = c : d$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Componendo and dividendo)

Illustration 75:

If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}$, prove that $(ab + bc + cd + de)^2 = (a^2 + b^2 + c^2 + d^2)(b^2 + c^2 + d^2 + e^2)$

Solution:

Let, $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=k$, then we have

(say)

i.e. $a = bk \quad \therefore ab = b^2k$

$b = ck \quad \therefore bc = c^2k$

$c = dk \quad \therefore cd = d^2k$

$d = ek \quad \therefore de = e^2k$

so, $(a^2 + b^2 + c^2 + d^2) = k^2(b^2 + c^2 + d^2 + e^2)$ ...(i)

Now L.H.S. $= (ab + bc + cd + de)^2$

$= (kb^2 + kc^2 + kd^2 + ke^2)^2$

$= k^2(b^2 + c^2 + d^2 + e^2)^2$

$= k^2(b^2 + c^2 + d^2 + e^2)(b^2 + c^2 + d^2 + e^2)$

$= (a^2 + b^2 + c^2 + d^2)(b^2 + c^2 + d^2 + e^2)$ = R.H.S

Illustration 76:

If $\frac{a}{b}=\frac{2}{3}$ and $\frac{b}{c}=\frac{4}{5}$, then find value of $\frac{a+b}{b+c}$

Solution:

$\frac{a}{b}=\frac{2}{3}$

Componendo $\Rightarrow \frac{a+b}{b}=\frac{5}{3}$ ...(1)

$\frac{b}{c}=\frac{4}{5}\Rightarrow\frac{c}{b}=\frac{5}{4}$

Componendo $\Rightarrow \frac{b+c}{b}=\frac{9}{4}$ ...(2)

(1) ÷ (2) $\Rightarrow \frac{a+b}{b+c}=\frac{5}{3}\times\frac{4}{9}=\frac{20}{27}$

14. Sign-Scheme (Wavy Curve) Method

Given $f(x)$ and $g(x)$ are polynomials.

To solve the inequalities of the type $\frac{f(x)}{g(x)} * 0$, where '*' can be $>, \geq, <$ or $\leq$, we take the following steps:

(i) Find all the roots of $f(x) = 0$ and $g(x) = 0$

(ii) Write all these roots on the real line in increasing order of values.

(iii) Check the sign of the expression $\frac{f(x)}{g(x)}$ at some x greater than the largest root. If it is positive, put + sign in rightmost interval. In case of negative, put $-ve$ sign in rightmost interval and while moving from right to left change sign in accordance with step (iv).

(iv) If a root occurs even number of times, then sign of expression will be same on both sides of the root and if a root occurs odd number of times, then sign of the expression will be different on both sides of the root.

(v) Write the answer according to need of the question.

Note:

- We don't give equality sign on '$\pm\infty$' in the solution as they are two improper points of number line.
- We can't take zeroes of denominator in the final answer as at these points expression is not defined (because division by '0' is not defined).
- In case of ≥ 0 or ≤ 0, zeroes of numerator will be part of the answer provided they are not appearing in denominator also.
- Do not cross multiply the terms in the inequalities.

Illustration 77:

Find the solution of $-x^2 + 6x + 7 \geq 0$

Solution:

$-x^2 + 6x + 7 \geq 0$

$\Rightarrow x^2 - 6x - 7 \leq 0$

$\Rightarrow (x + 1)(x - 7) \leq 0$

$\Rightarrow x \in [-1, 7]$

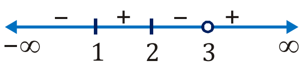
Illustration 78:

Find the solution of the inequalities $\frac{(x-1)(x-2)}{(x-3)} \geq 0$

Solution:

$x - 1 = 0, x - 2 = 0, x - 3 = 0 \Rightarrow x = 1, 2, 3$

Since $x - 3 \neq 0, x \neq 3$



So, $x \in [1, 2] \cup (3, \infty)$

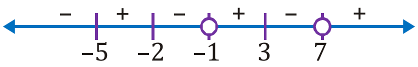
Illustration 79:

If $f(x)=\dfrac{(x-3)(x+2)(x+5)}{(x+1)(x-7)}$, then find x such that

(i) $f(x) > 0$ **(ii)** $f(x) < 0$.

Solution:

Given $f(x)=\dfrac{(x-3)(x+2)(x+5)}{(x+1)(x-7)}$



(i) $f(x) > 0 \Rightarrow x \in (-5,-2) \cup (-1,3) \cup (7,\infty)$

(ii) $f(x) < 0 \Rightarrow x \in (-\infty,-5) \cup (-2,-1) \cup (3,7)$

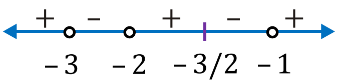
Illustration 80:

Solve for real $x: \dfrac{1}{x+1}+\dfrac{2}{x+2}>\dfrac{3}{x+3}$

Solution:

$$\frac{3x+4}{(x+1)(x+2)}>\frac{3}{x+3}$$

$$\Rightarrow \frac{3x+4}{(x+1)(x+2)}-\frac{3}{x+3}>0 \Rightarrow \frac{4x+6}{(x+1)(x+2)(x+3)}>0$$



So, $x \in (-\infty,-3) \cup \left(-2,-\dfrac{3}{2}\right) \cup (-1,\infty)$

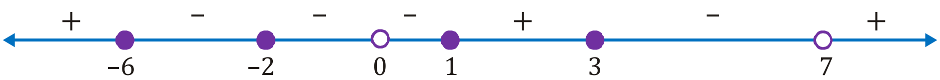
Illustration 81:

Let $f(x) = \dfrac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$. Solve the following inequality

(i) $f(x) > 0$ (ii) $f(x) \geq 0$ (iii) $f(x) < 0$ (iv) $f(x) \leq 0$

Solution:

We mark on the number line zeroes of numerator of expression : 1, – 2, 3 and – 6 (with black circles) and the zeroes of denominator 0 and 7 (with white circles), isolate the double points : – 2 and 0 and draw the wavy curve :



From graph, we get

(i) $x \in (-\infty,-6) \cup (1,3) \cup (7,\infty)$

(ii) $x \in (-\infty,-6] \cup \{-2\} \cup [1,3] \cup (7,\infty)$

(iii) $x \in (-6,-2) \cup (-2,0) \cup (0,1) \cup (3,7)$

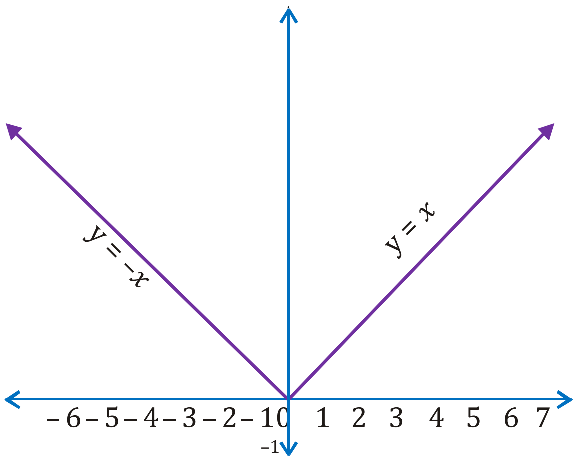
(iv) $x \in [-6,0) \cup (0,1] \cup [3,7)$

15. Modulus & Its graph

For any real number x, modulus or absolute value of x is denoted by $|x|$ and is defined as

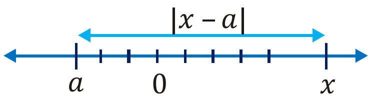
$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

- Graph of $y = |x|$



Note:

- $|x| = |-x| \geq 0$
- Geometrically $|x|$ is distance of real number x from zero along the real number line
- More generally $|x - a|$ is distance between 'x' and 'a' on the number line.



- $|x| = \sqrt{x^2}$
- $|xy| = |x|\,|y|$

Illustration 82:

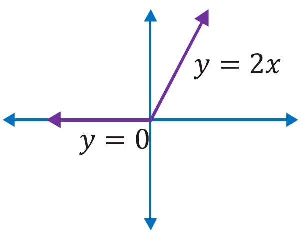
Sketch the graph of following equation and also find all possible values (Range) of y,

$y = |x| + x$

Solution:

$$y = |x| + x = \begin{cases} x + x & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$

$$= \begin{cases} 2x & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$



From graph we can find all
possible values (range) of y which is $[0, \infty)$

Modulus Equations

Illustration 83:

Solve for x

(a) $|2x + 5| = 2$ (b) $|x - 3| = -1$

Solution:

(a) $|2x + 5| = 2$

$\Rightarrow 2x + 5 = 2$ or $2x + 5 = -2$

$\Rightarrow x = -\frac{3}{2}$ or $x = -\frac{7}{2}$

Answer $\Rightarrow x \in \left\{-\frac{3}{2}, -\frac{7}{2}\right\}$

(b) $|x - 3| = -1$

A modulus quantity cannot be negative

So $x \in \phi$ is our answer.

Illustration 84:

Solve for x; $|2x - 3| = |3x + 5|$

Solution:

Case I: $x < -\frac{5}{3}$

$\Rightarrow -(2x - 3) = -(3x + 5)$

$\Rightarrow -2x + 3 = -3x - 5$

$\Rightarrow x = -8$ (Accepted as $x < -\frac{5}{3}$)

Case II: $-\frac{5}{3} \le x < \frac{3}{2}$

$\Rightarrow -(2x - 3) = (3x + 5)$

$\Rightarrow 5x = -2$

$\Rightarrow x = -\frac{2}{5}$ (Accepted as $-\frac{5}{3} \le x < \frac{3}{2}$)

Case III: $x \ge \frac{3}{2}$

$\Rightarrow 2x - 3 = 3x + 5$

$\Rightarrow x = -8$ (Rejected as $x \ge \frac{3}{2}$)

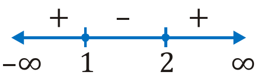
Final answer: $x \in \left\{-8, -\frac{2}{5}\right\}$

Illustration 85:

If $|x - 1||x - 2| = -(x^2 - 3x + 2)$, then find the interval in which x lies?

Solution:

$|(x - 1)(x - 2)| = -(x - 2)(x - 1)$



$\Rightarrow (x - 1)(x - 2) \leq 0$

$\Rightarrow 1 \leq x \leq 2$

Illustration 86:

Solve for x; $x^2 + 3|x| + 2 = 0$

Solution:

$x^2 + 3|x| + 2 = 0$

$\Rightarrow |x|^2 + 3|x| + 2 = 0$

$\Rightarrow |x|^2 + 2|x| + |x| + 2 = 0$

$\Rightarrow |x|\,(|x|+2) + 1(|x|+2) = 0$

$\Rightarrow (|x|+2)\,(|x|+1) = 0$

Either $|x| + 2 = 0$ or $|x| + 1 = 0$

$\Rightarrow |x| = -2 \qquad |x| = -1$

Modulus cannot be negative

So $x \in \phi$

Illustration 87:

Solve for x; $|x| - 2|x + 1| + 3|x + 2| = 0$

Solution:

Case 1: when $x < -2$

$\Rightarrow -x + 2(x + 1) - 3(x + 2) = 0$

$\Rightarrow -2x - 4 = 0$

$\Rightarrow x = -2$ (Rejected as $x < -2$)

Case 2: when $-2 \leq x < -1$

$\Rightarrow -x + 2(x + 1) + 3(x + 2) = 0$

$\Rightarrow 4x + 8 = 0 \Rightarrow x = -2$ (Accepted as $-2 \leq x < -1$)

Case 3: when $-1 \leq x < 0$

$\Rightarrow -x - 2(x + 1) + 3\,(x + 2) = 0$

$\Rightarrow -x - 2x - 2 + 3x + 6 = 0$

$\Rightarrow 4 = 0$

Not possible

So no solution in this interval

Case 4: $x \geq 0$

$\Rightarrow x - 2x - 2 + 3x + 6 = 0$

$\Rightarrow 2x + 4 = 0$

$\Rightarrow x = -2$ (Rejected as $x \geq 0$)

Final answer:- $x = -2$

16. Determinants

Introduction:

If the equations $a_1x + b_1 = 0, a_2x + b_2 = 0$ are satisfied by the same value of x, then $a_1b_2 - a_2b_1 = 0$. The expression $a_1b_2 - a_2b_1$ is called a determinant of the second order, and is denoted by:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

A determinant of second order consists of two rows and two columns.

Next consider the system of equations $a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0, a_3x + b_3y + c_3 = 0$

If these equations are satisfied by the same values of x and y, then on eliminating x and y we get.

$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$

The expression on the left is called a determinant of the third order, and is denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

A determinant of third order consists of three rows and three columns.

Value of a determinant:

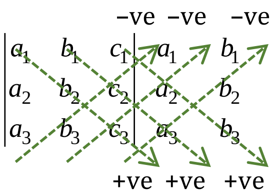
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

Note :

Sarus diagram to get the value of determinant of order three :

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$



$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)$

Note that the product of the terms in first bracket (i.e. $a_1a_2a_3b_1b_2b_3c_1c_2c_3$) is same as the product of the terms in second bracket.

Illustration 88:

The value of $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ is -

(A) 213 (B) –231 (C) 231 (D) 39

Ans. (C)

Solution:

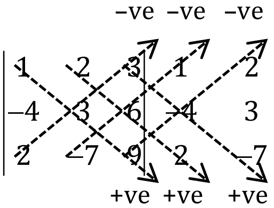
$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = 1 \begin{vmatrix} 3 & 6 \\ -7 & 9 \end{vmatrix} - 2 \begin{vmatrix} -4 & 6 \\ 2 & 9 \end{vmatrix} + 3 \begin{vmatrix} -4 & 3 \\ 2 & -7 \end{vmatrix}$$

$= (27 + 42) - 2(-36 - 12) + 3(28 - 6) = 231$

Alternative :

By sarrus diagram

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} =$$



$= (27 + 24 + 84) - (18 - 42 - 72) = 135 - (18 - 114) = 231$

Minors & Cofactors:

The minor of a given element of determinant is the determinant obtained by deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have "9 minors".

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element is given by: $C_{ij} = (-1)^{i+j}.M_{ij}$

Illustration 89:

Find the minors and cofactors of elements '– 3', '5', '– 1' & '7' in the determinant $\begin{vmatrix} 2 & -3 & 1 \\ 4 & 0 & 5 \\ -1 & 6 & 7 \end{vmatrix}$

Solution:

Minor of $-3 = \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix} = 33$; Cofactor of $-3 = -33$

Minor of $5 = \begin{vmatrix} 2 & -3 \\ -1 & 6 \end{vmatrix} = 9$; Cofactor of $5 = -9$

Minor of $-1 = \begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix} = -15$; Cofactor of $-1 = -15$

Minor of $7 = \begin{vmatrix} 2 & -3 \\ 4 & 0 \end{vmatrix} = 12$; Cofactor of $7 = 12$