02 Functions

1. Function: Domain, Co-Domain & Range of a Function:

Function:

A relation R from set A to set B is called a function if each element of A is uniquely associated with some element of B. It is denoted by the symbol :

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

which reads 'f' is a function from A to B 'or' f maps A to B,

If an element $a \in A$ is associated with an element $b \in B$, then b is called "the f image of a" or "image of a under f" or "the value of the function f at a". Also a is called the "pre-image of b" or "argument of b under the function f". We write it as

$$b = f(a) \text{ or } f : a \to b \text{ or } f : (a, b)$$

Thus, a function 'f' from set A to set B is subset of $A \times B$ in which each a belonging to A appears in one and only one ordered pair belonging to f.

2. Representation of Function:

(a) Ordered pair: Every function from $A \to B$ satisfies the following conditions:

(i) $f \subset A \times B$

(ii) $\forall\ a \in A$ there exist $b \in B$ and

(iii) $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

(b) Formula based (uniformly/nonuniformly):

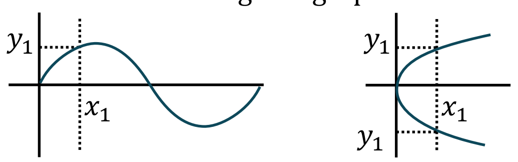
(i) $f : \mathbb{R} \to \mathbb{R}, y = f(x) = 4x$, $f(x) = x^2$ (uniformly defined)

(ii) $f(x) = \begin{cases} x+1 & -1 \le x < 4 \\ -x & 4 \le x < 7 \end{cases}$ (non-uniformly defined)

(iii) $f(x) = \begin{cases} x^2 & x \ge 0 \\ -x-1 & x < 0 \end{cases}$ (non-uniformly defined)

(c) Graphical representation:

If a vertical line cuts a given graph at more than one point then it cannot be the graph of a function.



Graph (1) Graph (2)

Graph (1) represent a function but graph (2) does not represent a function.

Every function is a relation but every relation is not necessarily a function.

Domain, Co-domain & Range of a Function:

Let $f : A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f. The set of f images of all the elements of A is known as the range of f.

Domain of $f = \{a | a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) | a \in A, f(a) \in B\}$

(a) If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined.

(b) For a continuous function, the interval from minimum to maximum value of a function gives the range

(c) It should be noted that range is a subset of co-domain.

Note:

(i) The complete set of all positive real numbers is denoted by $\mathbb{R}^+$ or $\mathbb{R} > 0$

(ii) The complete set of all negative real numbers is denoted by $\mathbb{R}^-$ or $\mathbb{R} < 0$

(iii) The complete set of all real numbers other then zero is denoted by $\mathbb{R}^*$ or $\mathbb{R} \neq 0$

(iv) The complete set of all integers is denoted by $\mathbb{R}$

Illustration 1:

Find the domain of function $y = \sqrt{5 - 2x}$.

Solution:

$5 - 2x \geq 0 \Rightarrow x \leq \frac{5}{2}$

$\therefore$ Domain is $(-\infty, 5/2]$

Illustration 2:

Find the domain of function $y = \frac{1}{\sqrt{x - |x|}}$.

Solution:

$x - |x| > 0 \Rightarrow |x| < x \Rightarrow x$ cannot take any real values

$\therefore$ Domain is ϕ

Illustration 3:

Find the range of function $f(x) = \log_{\sqrt{2}}\left((x-1)^2 + 4\right)$.

Solution:

$f(x) = \log_{\sqrt{2}}\left((x-1)^2 + 4\right)$

$4 \leq (x - 1)^2 + 4 < \infty$

$\Rightarrow \log_{\sqrt{2}} 4 \leq \log_{\sqrt{2}} (x-1)^2 + 4 < \infty$

$\Rightarrow 4 \leq \log_{\sqrt{2}} (x-1)^2 + 4 < \infty$

$\therefore$ Range of $f(x) = [4, \infty)$

Illustration 4:

Find the range of function $f(x) = 3 - \cos x$.

Solution:

$f(x) = 3 - \cos x$

$-1 \leq \cos x \leq 1$

$2 \leq 3 - \cos x \leq 4$

$\therefore$ Range of $f(x) = [2, 4]$

3. Some Important Functions:

(a) Polynomial Function:

If a function f is defined by $f(x) = a_0\, x^n + a_1\, x^{n-1} + a_2\, x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n. If n is odd, then polynomial is of odd degree, if n is even, then polynomial is of even degree.

Note: (i) Range of odd degree polynomial is always $\mathbb{R}$.

(ii) Range of even degree polynomial is never $\mathbb{R}$.

(iii) A polynomial of degree one with no constant term is called an odd linear function.
i.e. $f(x) = ax, a \neq 0$

(iv) $f(x) = ax + b, a \neq 0$ is a linear polynomial

(v) $f(x) = c$ is non-linear polynomial (its degree is zero)

(vi) $f(x) = 0$ is a polynomial but its degree is **not** defined

(vii) There are four polynomial functions, satisfying the relation; $f(x) . f(1/x) = f(x) + f(1/x)$.
They are : (a) $f(x) = 0$ (b) $f(x) = 2$
(c) $f(x) = x^n + 1$ & (d) $f(x) = 1 - x^n$, where n is a positive integer.

(b) Algebraic Function:

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0\,(x)\, y^n + P_1(x)\, y^{n-1} + \ldots\ldots + P_{n-1}\,(x)\, y + P_n\,(x) = 0$. Where n is a positive integer and $P_0\,(x), P_1\,(x)$ .... are Polynomials in x.

e.g. $x^3 + y^3 - 3xy = 0$.

In other words, a function 'f' is called an algebraic function if it can be constructed using finite number of algebraic operations (such as addition, subtraction, multiplication, division, and taking radicals) within polynomials.

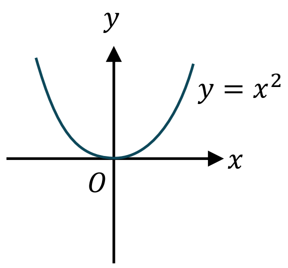
Example: $f(x) = \sqrt{x^2 + 1}$; $g(x) = \dfrac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x+1}$

Note: (i) All polynomial functions are Algebraic but not the converse.

(ii) A function that is not algebraic is called **Transcendental Function.**

Basic algebraic function:

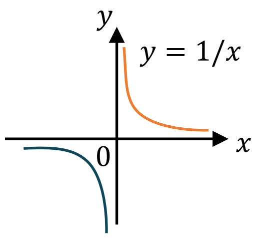
(i) $y = x^2$



Domain: $\mathbb{R}$

Range: $\mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$

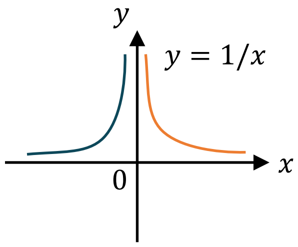
(ii) $y = \dfrac{1}{x}$



Domain: $\mathbb{R} - \{0\}$ or $\mathbb{R}_0$

Range: $\mathbb{R} - \{0\}$

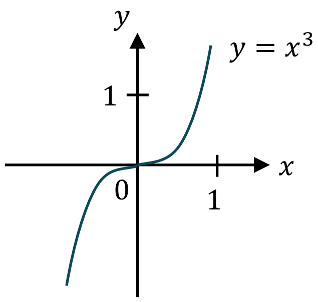
(iii) $y=\frac{1}{x^2}$



Domain: $\mathbb{R}_0$

Range: $\mathbb{R}^+$ or $\{0, \infty\}$

(iv) $y=x^3$



Domain: $\mathbb{R}$

Range: $\mathbb{R}$

(c) Rational Function:

A rational function is a function of the form $y = f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ & $h(x)$ are polynomials & $h(x) \neq 0$,

Domain : $\mathbb{R} - \{x \mid h(x) = 0\}$

Any rational function is automatically an algebraic function.

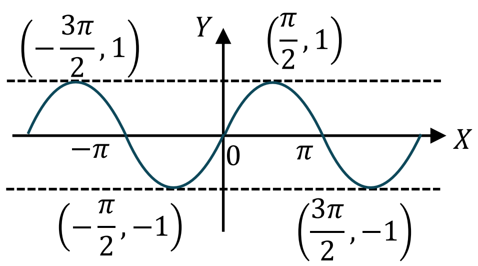
(d) Trigonometric Functions:

(i) Sine function

$f(x) = \sin x$

Domain : $\mathbb{R}$

Range : $[-1, 1]$, period 2π

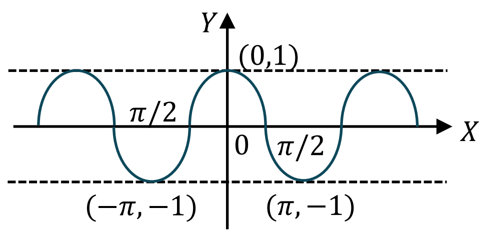


(ii) Cosine function

$f(x) = \cos x$

Domain : $\mathbb{R}$

Range : $[-1, 1]$, period 2π

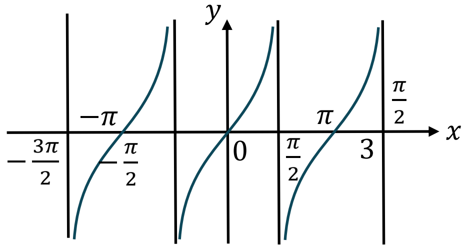


(iii) Tangent function

$f(x) = \tan x$

Domain : $\mathbb{R} - \left\{x \mid x = \frac{(2n+1)\pi}{2}, n \in I\right\}$

Range : $\mathbb{R}$, period π

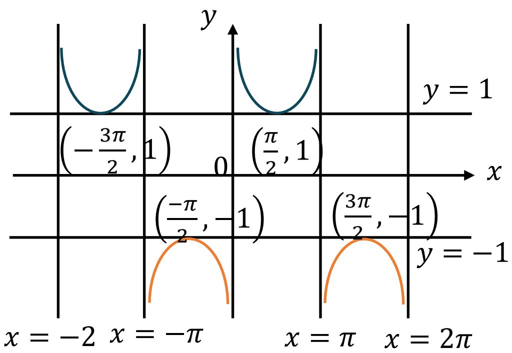


(iv) Cosecant function

$f(x) = \text{cosec}\, x$

Domain : $\mathbb{R} - \{x \mid x = n\pi, n \in I\}$

Range : $\mathbb{R} - (-1, 1)$, period 2π

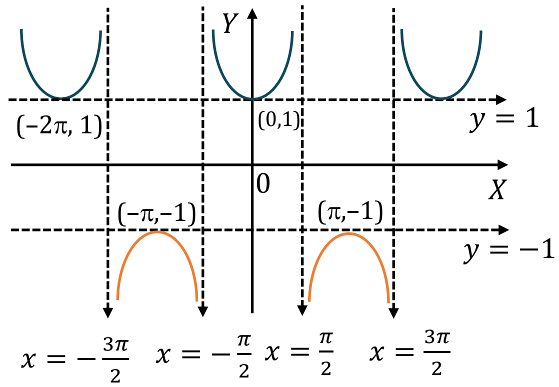


(v) Secant function

$f(x) = \sec x$

Domain : $\mathbb{R} - \{x \mid x = (2n+1)\,\pi/2 : n \quad I\}$

Range : $\mathbb{R} - (-1, 1)$, period 2π

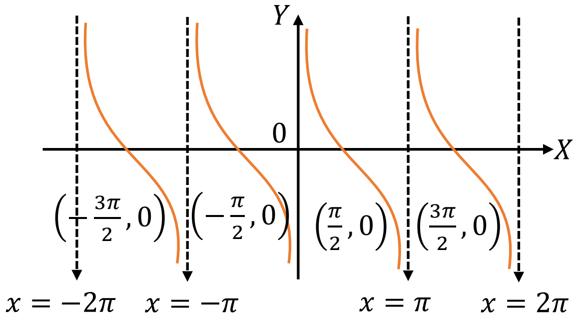


(vi) Cotangent function

$f(x) = \cot x$

Domain : $\mathbb{R} - \{x \mid x = n\pi, n \in I\}$

Range : $\mathbb{R}$, period π



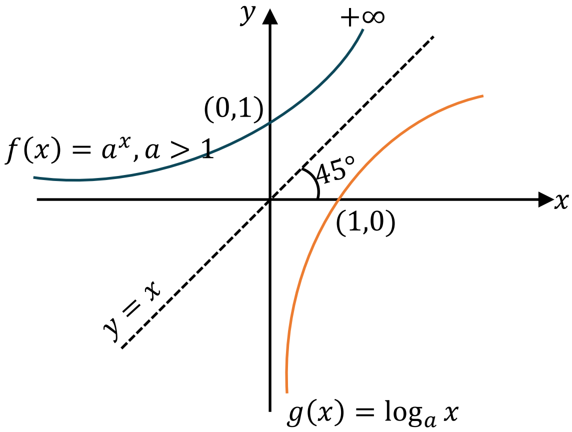
(e) Exponential and Logarithmic Function:

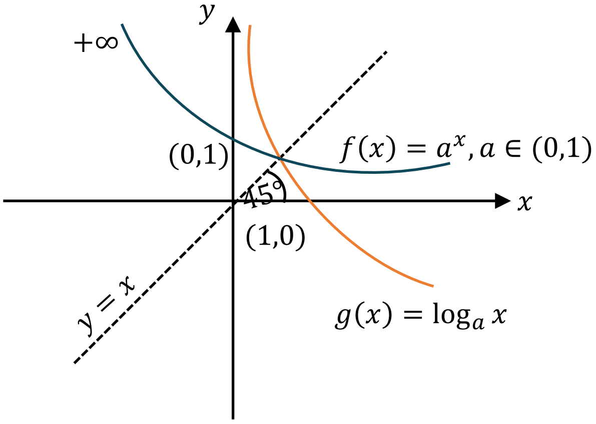
A function $f(x) = a^x (a > 0), a \in 1, x \in \mathbb{R}$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$.

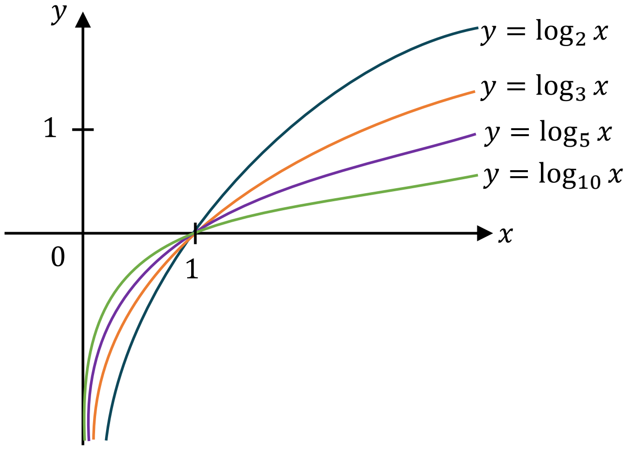
Note that $f(x)$ & $g(x)$ are inverse of each other & their graphs are as shown. (If functions are mirror image of each other about the line $y = x$)

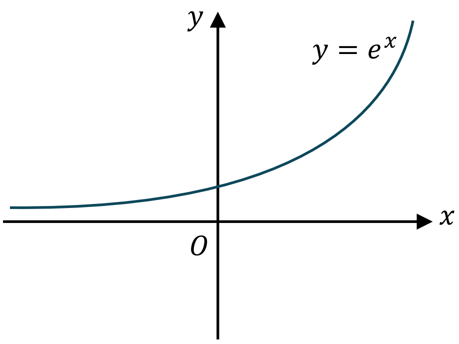
Domain of a^x is $\mathbb{R}$ **Range** $\mathbb{R}^+$

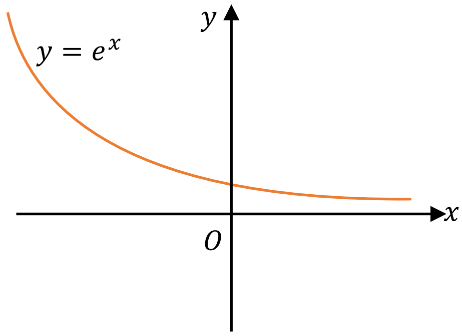
Domain of $\log_a x$ is $\mathbb{R}^+$ **Range** $\mathbb{R}$

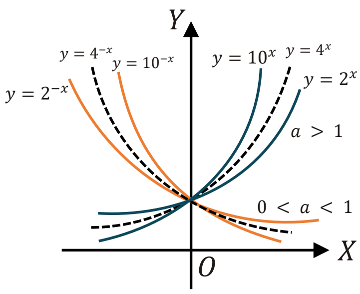












Domain : $\mathbb{R}$

Range : $\mathbb{R}^+$

Note-1 : $f(x) = a^{1/x}, a > 0$ **Domain :** $\mathbb{R} - \{0\}$ **Range :** $\mathbb{R}^+ - \{1\}$

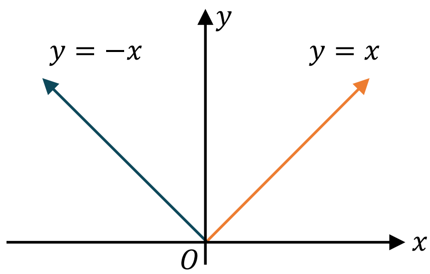
Note-2 : $f(x) = \log_x a = \dfrac{1}{\log_a x}$ **Domain :** $\mathbb{R}^+ - \{1\}$ **Range :** $\mathbb{R} - \{0\}$

$(a > 0)\ (a \neq 1)$

(f) Absolute Value Function:

A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as:

$$y = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$



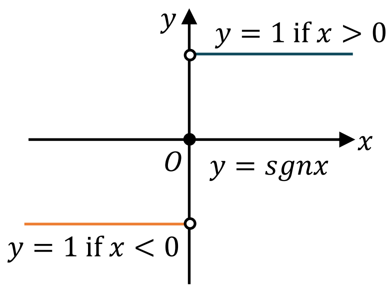
For $f(x) = |x|$, domain is $\mathbb{R}$ and range is $[0,\infty)$

For $f(x) = \frac{1}{|x|}$, domain is $\mathbb{R} - \{0\}$ and range is $\mathbb{R}^+$.

(g) Signum Function:

A function $y = f(x) = \text{Sgn}\,(x)$ is defined as follows:

$$y = f(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \\ -1 & \text{for} \quad x < 0 \end{cases}$$



It is also written as Sgn $x = \frac{|x|}{x}$; $x \neq 0$;

$= 0$; $x = 0$

Note : $f(x) = (\text{sgn}(x))x \Rightarrow f(x) = |x|$

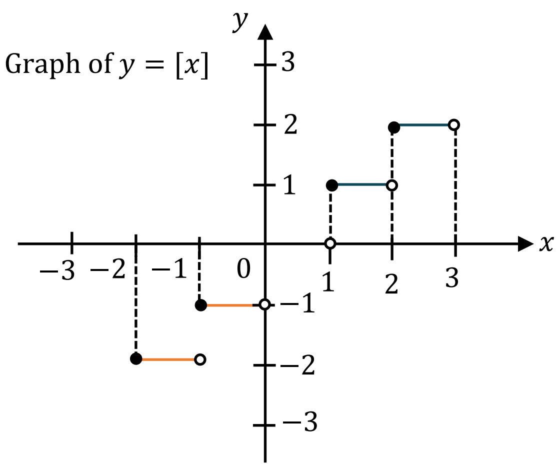
Domain : $\mathbb{R}$

Range : $\{-1, 0, 1\}$

(h) Greatest Integer or Step up Function:

The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x. Note that for:

x	$[x]$
$[-2,-1)$	-2
$[-1,0)$	-1
$[0,1)$	0
$[1,2)$	1



Domain : $\mathbb{R}$

Range : I

Properties of greatest integer function:

(i) $[x] \leq x < [x] + 1$ and $x - 1 < [x] \leq x, 0 \leq x - [x] < 1$

(ii) $[x + m] = [x] + m$, if m is an integer.

(iii) $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

Illustration 5:

If $y = 2[x] + 3$ & $y = 3[x - 2] + 5$, then find $[x + y]$ where [.] denotes greatest integer function.

Solution:

$y = 3[x-2] + 5 = 3[x] - 1$

So $3[x] - 1 = 2[x] + 3$

$[x] = 4 \Rightarrow 4 \le x < 5$

Then $y = 11$

so $x + y$ will lie in the interval $[15, 16)$

so $[x + y] = 15$

Illustration 6:

Find the value of $\left[\frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{1000}\right] + \ldots\ldots \left[\frac{1}{2} + \frac{2946}{1000}\right]$ where [.] denotes greatest integer function ?

Solution: $\left[\frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{1000}\right] + \ldots\ldots \left[\frac{1}{2} + \frac{499}{1000}\right] + \left[\frac{1}{2} + \frac{500}{1000}\right] + \ldots\ldots \left[\frac{1}{2} + \frac{1499}{1000}\right] + \left[\frac{1}{2} + \frac{1500}{1000}\right] + \ldots\ldots$

$\ldots + \left[\frac{1}{2} + \frac{2499}{1000}\right] + \left[\frac{1}{2} + \frac{2500}{1000}\right] + \ldots\ldots \left[\frac{1}{2} + \frac{2946}{1000}\right]$

$= 0 + 1 \times 1000 + 2 \times 1000 + 3 \times 447$

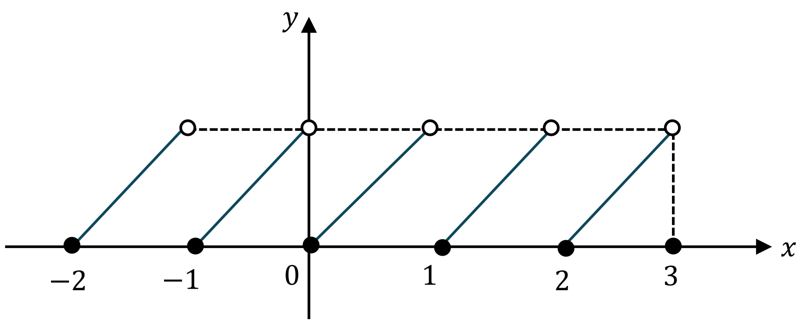
$= 3000 + 1341 = 4341$

(i) Fractional Part Function:

It is defined as : $g(x) = \{x\} = x - [x]$

e.g. the fractional part of the number 2.1 is 2.1–2 = 0.1 and the fractional part of –3.7 is 0.3 The period of this function is 1 and graph of this function is as shown.

x	$[x]$
$[-2, -1)$	$x + 2$
$[-1, 0)$	$x + 1$
$[0, 1)$	x
$[1, 2)$	1



Domain : $\mathbb{R}$

**Range : ** $[0,1)$

Properties of fractional part function:

(i) $0 \le \{x\} < 1$

(ii) $\{[x]\} = [\{x\}] = 0$

(iii) $\{\{x\}\} = \{x\}$

(iv) $\{x + m\} = \{x\}, m \in I$

(v) $\{x\} + \{-x\} = \begin{cases} 1, & x \notin I \\ 0, & x \in I \end{cases}$

Illustration 7:

Solve the equation $|2x - 1| = 3[x] + 2\{x\}$ where [.] denotes greatest integer and {.} denotes fractional part function.

Solution:

We are given that, $|2x-1| = 3[x] + 2\{x\}$

Let, $2x-1 \leq 0$ i.e. $x \leq \frac{1}{2}$. The given equation yields.

$1-2x = 3[x] + 2\{x\}$

$\Rightarrow 1-2[x]-2\{x\} = 3[x] + 2\{x\} \Rightarrow 1-5[x] = 4\{x\} \Rightarrow \{x\} = \frac{1-5[x]}{4}$

$\Rightarrow 0 \leq \frac{1-5[x]}{4} < 1 \Rightarrow 0 \leq 1-5[x] < 4 \Rightarrow -\frac{3}{5} < [x] \leq \frac{1}{5}$

Now, $[x] = 0$ as zero is the only integer lying between $-\frac{3}{5}$ and $\frac{1}{5}$

$\Rightarrow \{x\} = \frac{1}{4} \Rightarrow x = \frac{1}{4}$ which is less than $\frac{1}{2}$, Hence $\frac{1}{4}$ is one solution.

Now, let $2x - 1 > 0$ i.e. $x > \frac{1}{2}$

$\Rightarrow 2x - 1 = 3[x] + 2\{x\} \Rightarrow 2[x] + 2\{x\} - 1 = 3[x] + 2\{x\}$

$\Rightarrow [x] = -1 \Rightarrow -1 \leq x < 0$ which is not a solution as $x > \frac{1}{2}$

$\Rightarrow x = \frac{1}{4}$ is the only solution.

Illustration 8:

The domain of the function $f(x) = \frac{1}{\sqrt{x-[x]}}$ is, where [.] represent greatest integer function.

(A) R (B) $R - Z$ (C) Z (D) $R - N$

Ans. (C)

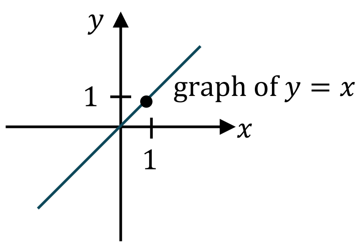
Solution:

$f(x) = \frac{1}{\sqrt{\{x\}}}$ $\{x\} \neq 0$

$x \in R-Z$

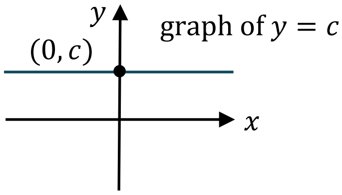
(j) Identity Function:

The function $f: A \to A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function defined on R is a bijection.



(k) Constant Function:

A function $f: A \to B$ is said to be a constant function if every element of A has the same f image in B. Thus $f: A \to B; f(x) = c, \forall\, x \in A, c \in B$ is a constant function. Note that the range of a constant function is a singleton.



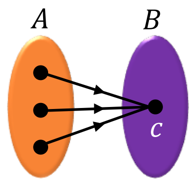


Illustration 9:

If $f(x) = ax^7 + bx^3 + cx - 5$; a, b, c are real constants and $f(-7) = 7$ then maximum value of $|f(7) + 17\cos x|$ is

Ans. (34)

Solution:

$f(-x) = -[ax^7 + bx^3 + cx] - 5$; $f(-x) = -[f(x) + 5] - 5$

$f(-x) = -f(x) - 10$ put $x = 7$

$f(7) = -17$ so, $f(7) + 17\cos x = -17(\cos x - 1) \in [-34, 0]$

4. Algebraic Operations on Functions:

If f & g are real valued functions of x with domain set A, B respectively, $f + g, f - g, (f.g)$ & (f/g) as follows:

(a) $(f \pm g)(x) = f(x) \pm g(x)$ domain in each case is $A \cap B$

(b) $(f.g)(x) = f(x).g(x)$ domain is $A \cap B$

(c) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ domain $A \cap B - \{x | g(x) = 0\}$

Illustration 10:

Find the domain of function.

$$f(x) = \frac{1}{\log(2-x)} + \sqrt{x+1}$$

Solution:

Domain of $\sqrt{x+1}$ is $x + 1 \geq 0 \Rightarrow x \geq -1 \Rightarrow x \in [-1, \infty)$

Domain of $\dfrac{1}{\log(2-x)}$ is

$2 - x > 0$ and $\log(2 - x) \neq 0$

$x < 2$ and $x \neq 1$

$x \in (-\infty, 1) \cup (1, 2)$

Hence the domain of $\dfrac{1}{\log(2-x)} + \sqrt{x+1}$ is

$[-1, \infty) \cap \{(-\infty, 1) \cup (1, 2)\}$

$[-1, 1) \cup (1, 2)$

Illustration 11:

Find the domain of function.

$f(x) = \sqrt{1-x} - \sin\frac{2x-1}{3}$

Solution:

Domain of $\sqrt{1-x}$ is $1 - x \geq 0 \Rightarrow x \leq 1 \Rightarrow x \in (-\infty, 1]$

Domain of $\sin\left(\frac{2x-1}{3}\right)$ is $x \in R$

Hence the Domain of $\sqrt{1-x} - \sin\left(\frac{2x-1}{3}\right)$ is

$(-\infty, 1] \cap R \Rightarrow (-\infty, 1]$

Illustration 12:

Find the domain of function.

$f(x) = \sqrt{x+3} - \sqrt{16-x^2}$

Solution:

$\sqrt{x+3}$ is real if $x + 3 \geq 0 \Leftrightarrow x \geq -3$

$\sqrt{16-x^2}$ is real if $16 - x^2 \geq 0 \Leftrightarrow -4 \leq x \leq 4$.

Thus, the domain of the given function is

$\{x: x \in [-3, \infty) \cap [-4, 4] = [-3, 4]$

Illustration 13:

let $f(x) = \frac{1}{x}, g(x) = \sqrt{x+1}$ then find the domain of $f(x).g(x)$?

Solution:

Domain of $f(x)$ is $x \in R - \{0\}$

domain of $g(x)$ is $x + 1 \geq 0 \Rightarrow x \in [-1, \infty)$

domain of $f(x).g(x)$ is

$R - (0) \cap [-1, \infty) \Rightarrow x \in [-1, \infty) - \{0\}$

5. Some Illustrations on Domain:

Illustration 14:

Find the domain of the following function:

(i) $y = \log_{(x-4)} x^2 - 11x + 24$

(ii) $f(x) = \log_2 \left(-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right)$

Solution:

(i) $y = \log_{(x-4)} x^2 - 11x + 24$

Here 'y' would assume real value if,

$x - 4 > 0$ and $\neq 1, x^2 - 11x + 24 > 0 \Rightarrow x > 4$ and $\neq 5, (x - 3)(x - 8) > 0$

$\Rightarrow x > 4$ and $\neq 5, x < 3$ or $x > 8 \Rightarrow x > 8 \Rightarrow$ Domain $(y) = (8, \infty)$

(ii) We have $f(x)=\log_2\left(-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right)$

$f(x)$ is defined if $-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1>0$

or if $\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)<-1$ or if $\left(1+\frac{1}{\sqrt[4]{x}}\right)>(1/2)^{-1}$

or if $1+\frac{1}{\sqrt[4]{x}}>2$ or if $\frac{1}{\sqrt[4]{x}}>1$ or if $x^{1/4}<1$ or if $0<x<1$

$\therefore \quad D(f)=(0,1)$

Illustration 15:

Find the domain $f(x)=\frac{1}{\sqrt{|[|x|-5]|-11}}$ where [.] denotes greatest integer function.

Solution:

$|[|x|-5]|>11$

so $[|x|-5]>11$ or $[|x|-5]<-11$

$[|x|]>16 \qquad [|x|]<-6$

$|x|\geq 17$ or $|x|<-6$ (Not Possible)

$\Rightarrow x\leq -17$ or $x\geq 17$

So $x\in(-\infty,-17]\cup[17,\infty)$

Illustration 16:

The domain of $f(x)=\sqrt{\frac{4-x^2}{2+[x]}}$ (where [.] denotes greatest integer function) is :

(A) [–2, 2] (B) (–1, 2) (C) (∞, –2) ∪ [–1, 2] (D) (–∞, –2)

Ans. (C)

Solution:

$\frac{4-x^2}{2+[x]}\geq 0 \Rightarrow 4-x^2\geq 0$ & $2+[x]>0$

$\Rightarrow x\in[-2,2]$ and $x\in[-1,\infty) \Rightarrow x\in[-1,2]$ ...(1)

on the other hand $4-x^2<0, 2+[x]<0$

$\Rightarrow x\in R-[-2,2]$ & $x\in(-\infty,-2) \Rightarrow x\in(-\infty,-2)$ ...(2)

Domain : $x\in(-\infty,-2)\cup[-1,2]$

Illustration 17:

Find the domain of following function:

$f(x)=\sqrt{x+3}-\sqrt{16-x^2}$

Solution:

$\sqrt{x+3}$ is real iff $x+3\geq 0 \Leftrightarrow x\geq -3$

$\sqrt{16-x^2}$ is real iff $16-x^2\geq 0 \Leftrightarrow -4\leq x\leq 4$.

Thus, the domain of the given function is

$\{x: x\in[-3,\infty)\cap[-4,4]=[-3,4]$

Illustration 18:

Find the domain of following function:

$$f(x) = \frac{3}{\sqrt{4-x^2}} \log(x^3 - x)$$

Solution:

Domain of $\sqrt{4-x^2}$ is $[-2, 2]$ but $\sqrt{4-x^2} = 0$ for $x = \pm 2 \Rightarrow x \in (-2, 2)$

$\log(x^3 - x)$ is defined for $x^3 - x > 0$ i.e. $x(x-1)(x+1) > 0$.

$\therefore$ domain of $\log(x^3 - x)$ is $(-1, 0) \cup (1, \infty)$.

Hence the domain of the given function is

$\{(-1, 0) \cup (1, \infty)\} \cap (-2, 2) \equiv (-1, 0) \cup (1, 2)$.

Illustration 19:

The domain of the function $f(x) = \dfrac{1}{\sqrt{\{\sin(x)\} + \{\sin(\pi + x)\}}}$ (where {.} denotes the fractional part) is:

(A) $[2n\pi, (2n+1)\pi], n \in z$

(B) $\left(0, \dfrac{\pi}{2}\right)$

(C) $\left\{(2n+1)\dfrac{\pi}{2}, n \in z\right\}$

(D) $R - \left\{\dfrac{n\pi}{2}, n \in z\right\}$

Ans. (D)

Solution:

$$\{\sin x\} + \{-\sin x\} \neq 0 \Rightarrow \sin x \neq \pm 1, 0 \Rightarrow x \neq \frac{n\pi}{2}\ n \in I$$

Illustration 20:

The domain of the function $f(x) = \log_{1/2}\left(-\log_2\left(1 + \dfrac{1}{\sqrt[4]{x}}\right) - 1\right)$ is:

(A) $0 < x < 1$ (B) $0 < x \le 1$ (C) $x \ge 1$ (D) null set

Ans. (D)

Solution:

$$f(x) = \log_{1/2}\left(-\log_2\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$$

$$\left(-\log_2\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right) > 0 \Rightarrow -\infty < \log_2\left(1 + \frac{1}{x^{1/4}}\right) < -1$$

$$\Rightarrow 0 < 1 + \frac{1}{x^{1/4}} < \frac{1}{2} \Rightarrow -1 < \frac{1}{x^{1/4}} < -\frac{1}{2} \Rightarrow x \in \phi \text{ (null set)} \Rightarrow x \in \phi$$

Illustration 21:

Find the domain of the function $f(x) = \sqrt{-\log_{\frac{x+4}{2}}\left(\log_2 \dfrac{2x-1}{3+x}\right)}$

Ans. $(-4, -3) \cup (4, \infty)$

Solution:

$f(x) = \sqrt{-\log_{\frac{x+4}{2}}\left(\log_2 \dfrac{2x-1}{3+x}\right)}$. For domain: $\log_{\frac{x+4}{2}}\left(\log_2 \dfrac{2x-1}{3+x}\right) \le 0$

Case-I $0 < \frac{x+4}{2} < 1 \Rightarrow -4 < x < -2$ ...A

then $\log_{\frac{x+4}{2}}\left(\log_2 \frac{2x-1}{3+x}\right) \le 0 \Rightarrow \log_2 \frac{2x-1}{3+x} \ge 1$

$\Rightarrow \frac{2x-1}{3+x} \ge 2 \Rightarrow x < -3$ ...B

$\Rightarrow$ on $A \cap B$ $x \in (-4, -3)$ ...(i)

Case-II $\frac{x+4}{2} > 1 \Rightarrow x > -2$ ...A

$\log_{\frac{x+4}{2}}\left(\log_2 \frac{2x-1}{3+x}\right) \le 0 \Rightarrow 0 < \log_2 \frac{2x-1}{3+x} \le 1 \Rightarrow 1 < \frac{2x-1}{3+x} \le 2 \Rightarrow x \in (4, \infty)$ ...(ii)

$\therefore$ (i) $\cup$ (ii) gives domain $x \in (-4, -3) \cup (4, \infty)$

Illustration 22:

Let $f(x) = (x^{12} - x^9 + x^4 - x + 1)^{-\frac{1}{2}}$. The domain of the function is :

Ans. $(-\infty, \infty)$

Solution:

$f(x) = (x^{12} - x^9 + x^4 - x + 1)^{-\frac{1}{2}}$

Dr: $x^{12} - x^9 + x^4 - x + 1 > 0$

For $x \le 0$ it is obvious that Dr > 0.

For $x \ge 1$, $(x^{12} - x^9) + (x^4 - x) + 1$ is positive

Since $x^{12} \ge x^9, x^4 \ge x$.

For $0 < x < 1$, Dr $= x^{12} + (x^4 - x^9) + (1 - x) > 0$

Since $x^4 > x^9, x < 1$.

Hence Dr > 0 for all $x \in R$

Domain is $x \in R$

Illustration 23:

If $f(x) = \frac{1}{x^2+1}$ and $g(x) = \sin\pi x + 8\left\{\frac{x}{2}\right\}$ where $\{\cdot\}$ denotes fractional part function then the find range of $f(g(x))$

Ans. $\left(\frac{1}{65}, 1\right]$

Solution:

As $g(x)$ is periodic with period 2 so $f(g(x))$ is periodic with period 2.

Now $g(x) = \sin \pi x + 8\left\{\frac{x}{2}\right\} = \sin \pi x + 8\frac{x}{2} \forall\, 0 \le x < 2$

$g(x) = \sin \pi x + 4x; 0 \le x < 2 \Rightarrow g'(x) = \pi \cos \pi x + 4 \uparrow$.

So $g(0) = 0$ and $g(2^-) = 8$so, $g(x) \in [0, 8)$

so range of $f(g(x))$ is $\left(\frac{1}{1+64}, \frac{1}{1}\right] \equiv \left(\frac{1}{64}, 1\right]$.

Illustration 24:

Domain of definition of the function $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$, is :

(A) (1,2)

(B) $(-1,0) \cup (1,2)$

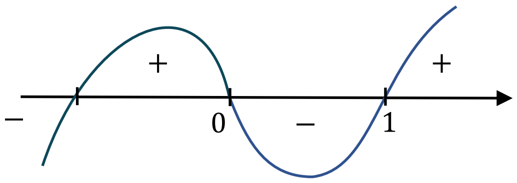
(C) $(1,2) \cup (2,\infty)$

(D) $(-1,0) \cup (1,2) \cup (2,\infty)$

Ans. (D)

Solution:

$4 - x^2 \neq 0, x^3 - x > 0 \Rightarrow x \neq \pm 2$ and $-1 < x < 0$ or ; $1 < x < \infty$



$\therefore D = (-1,0) \cup (1,\infty) - \{2\}$ or $D = (-1,0) \cup (1,2) \cup (2,\infty)$.

6. Some Illustrations on Range:

Observations:

(1) Range of $\sin x$ & $\cos x$ is $[-1,1]$, whereas range of $\sin 2x, \cos 2x, |\sin x|, |\cos x|$ is $[0,1]$

Note: same for $\sin(ax+b)$ & $\cos(ax+b)$

(2) Range of $\tan x$, $\cot x$ is $(-\infty, \infty)$ whereas range of $\tan 2x$, $\cot 2x$, $|\tan x|$, $|\cot x|$ is $[0,\infty)$

(3) Range of $y = \sec x$ & $y = \operatorname{cosec} x$ is $(-\infty,-1] \cup [1,\infty)$ whereas range of $\sec 2x$, $\operatorname{cosec} 2x$, $|\sec x|$, $|\operatorname{cosec} x|$ is $[1,\infty)$

(4) Range of $\frac{\text{linear}}{\text{linear}}$ Range of $y = \frac{ax+b}{cx+d}$ is $R - \left\{\frac{a}{c}\right\}$

(5) To find Range of $y = \frac{\text{quadratic}}{\text{quadratic}}, \frac{\text{linear}}{\text{quadratic}}, \frac{\text{quadratic}}{\text{linear}}$

cross multiply & apply $D \geq 0$ provided N^r & D^r have no common factor

(6) Range of $y = a\sin x + b\cos x$ is $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$

(7) Range of $y = \left(x + \frac{1}{x}\right)$ is $[2,\infty)$ if $x > 0$ & $(-\infty,-2]$ if $x < 0$

Illustration 25:

Find the range of following functions:

(i) $f(x) = \frac{1}{8 - 3\sin x}$

(ii) $f(x) = \log_{\sqrt{2}}\left(2 - \log_2(16\sin^2 x + 1)\right)$

Solution:

(i) $f(x) = \frac{1}{8 - 3\sin x}$

$-1 \leq \sin x \leq 1$

$\therefore$ Range of $f = \left[\frac{1}{11}, \frac{1}{5}\right]$

(ii) $f(x) = \log_{\sqrt{2}}\left(2-\log_2(16\sin^2 x+1)\right)$

$1 \le 16\sin^2 x + 1 \le 17$

$\therefore \quad 0 \le \log_2 16\sin^2 x + 1 \le \log_2 17$

$\therefore \quad 2 - \log_2 17 \le 2 - \log_2(16\sin^2 x + 1) \le 2$

Now consider $0 < 2 - \log_2(16\sin^2 x + 1) \le 2$

$\therefore \quad -\infty < \log_{\sqrt{2}}[2-\log_2(16\sin^2 x+1)] \le \log_{\sqrt{2}} 2 = 2$

$\therefore \quad$ the range is $(-\infty, 2]$

Illustration 26:

Find the range of $f(x) = \dfrac{x-[x]}{1+x-[x]}$, where [.] denotes greatest integer function.

Solution:

$$y = \frac{x-[x]}{1+x-[x]} = \frac{\{x\}}{1+\{x\}}$$

$$\therefore \frac{1}{y} = \frac{1}{\{x\}} + 1 \Rightarrow \frac{1}{\{x\}} = \frac{1-y}{y} \Rightarrow \{x\} = \frac{y}{1-y}$$

$$0 \le \{x\} < 1 \Rightarrow 0 \le \frac{y}{1-y} < 1$$

Range = $[0, 1/2)$

Illustration 27:

Let $f(x) = (1+\sin^2 t)x^2 + (2\sin t)x + 1$ and $m(t)$ be the minimum value of $f(x)$. As $'t'$ varies, the range of $m(t)$ is :

(A) [–1, 1] (B) [–1, 0] (C) (0, 1/2] (D) [1/2, 1]

Ans. (D)

Solution:

$m(t) = f_{min} = \dfrac{1}{1+\sin^2 t}$ then desired range is $\left[\dfrac{1}{2}, 1\right]$

Illustration 28:

Find the range of $f(x) = \dfrac{x^2+x+1}{x^2+x-1}$

Solution:

$f(x) = \dfrac{x^2+x+1}{x^2+x-1}$ $\{x^2 + x + 1$ and $x^2 + x - 1$ have no common factor$\}$

$$y = \frac{x^2+x+1}{x^2+x-1}$$

$\Rightarrow yx^2 + yx - y = x^2 + x + 1$

$\Rightarrow (y-1)x^2 + (y-1)x - y - 1 = 0$

If $y = 1$, then the above equation reduces to $-2 = 0$. Which is not true.

Further if $y \ne 1$, then $(y-1)x^2 + (y-1)x - y - 1 = 0$ is a quadratic and has real roots if

$(y-1)^2 - 4(y-1)(-y-1) \ge 0$

i.e. if $y \le -3/5$ or $y \ge 1$ but $y \ne 1$

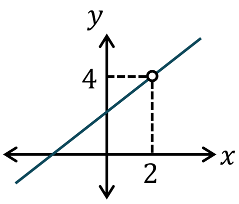
Thus, the range is $(-\infty, -3/5] \cup (1, \infty)$

Illustration 29:

Find the range of $f(x) = \dfrac{x^2-4}{x-2}$

Solution:

$f(x) = \dfrac{x^2-4}{x-2} = x + 2;\ x \neq 2$



$\therefore$ graph of $f(x)$ would be

Thus, the range of $f(x)$ is $R - \{4\}$ Further if $f(x)$ happens to be continuous in its domain then range of $f(x)$ is $[\min f(x), \max. f(x)]$. However, for sectionally continuous functions, range will be union of $[\min f(x), \max. f(x)]$ over all those intervals where $f(x)$ is continuous, as shown by

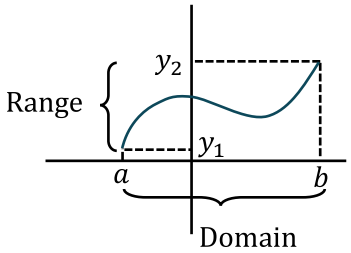


Illustration 30:

Find the range of function $y = \ell n\,(2x - x^2)$

Solution:

Domain of given function = (0,2)

$\because\ y = \ell n(2x - x^2) \qquad \therefore\ \dfrac{dy}{dx} = \dfrac{2(1-x)}{(2x-x^2)} \qquad \therefore\ \dfrac{dy}{dx} = 0$ at $x = 1$

$\therefore\ f(1) = 0$

$f(0^+) = -\infty$

$f(2^-) = -\infty \qquad \therefore$ Range = $(-\infty, 0)$

Illustration 31:

The range of $f(x) = \dfrac{1}{2-3\cos x}$ is

(A) $\left[-1, \dfrac{1}{5}\right]$ (B) $[-1, 0) \cup \left(0, \dfrac{1}{5}\right]$ (C) $(-\infty, -1] \cup \left[\dfrac{1}{5}, \infty\right)$ (D) $[-1, 5]$

Ans. (C)

Solution:

$-3 \leq -3\cos x \leq 3$

$-1 \leq 2 - 3\cos x \leq 5 \Rightarrow \dfrac{1}{2-3\cos x} \in (-\infty - 1] \cup \left[\dfrac{1}{5}, \infty\right)$

Illustration 32:

Let $f(x)=\ell n|x|$ and $g(x) = \sin x$. If A is the range of $f(g(x))$ and B is the range of $g(f(x))$ then

(A) $A \cup B = (-\infty, \infty)$ (B) $A \cup B = [-1, 0]$ (C) $A \cap B = [-1, 0]$ (D) $A \cap B = [0, 1]$

Ans. (C)

Solution:

A = range of $\ell n|\sin x| = (-\infty, 0]$

B = range of $\sin(\ell n|x|) = [-1, 1] \Rightarrow A \cup B = (-\infty, 1],\ A \cap B = [-1, 0]$

Illustration 33:

It is given that $f(x)$ is a function defined on N, satisfying $f(1) = 1$ and for any $x \in N$ $f(x + 5) \geq f(x) + 5$ and $f(x + 1) \leq f(x) + 1$. If $g(x) = f(x) + 1 - x$, then $g(2016)$ equals

Ans. (1)

Solution:

$f(x) + 5 \leq f(x + 5) \leq f(x + 4) + 1 \leq f(x + 3) + 2 \leq f(x + 2) + 3 \leq f(x + 1) + 4 \leq f(x) + 5$

Which is valid only if $f(x + 1) = f(x) + 1$

Now $f(1) = 1$

$\Rightarrow f(2) = 2$

$f(3) = 3$

$f(4) = 4$

$f(2016) = 2016$

$\Rightarrow g(2016) = 2016 + 1 - 2016 = 1$

Illustration 34:

Number of integers in the range of the function $f(x) = \dfrac{x^3+2x^2+3x+2}{x^3+2x^2+2x+1}$; $x \in R - \{0\}$ is:

Ans. (0)

Solution:

$f(x) = \dfrac{(x^2+x+2)(x+1)}{(x^2+x+1)(x+1)}$; $x \in R - \{0\}$

$f(x) = \dfrac{x^2+x+2}{x^2+x+1}$; $x \in R - \{0, -1\}$

$y = \dfrac{x^2+x+2}{x^2+x+1} \Rightarrow (y - 1)x^2 + (y - 1)x + y - 2 = 0$

$y \neq 1, D \geq 0$

$(y - 1)^2 - 4(y - 1)(y - 2) \geq 0 \Rightarrow 1 < y \leq 7/3$

At $x = 0$ we get $y = 2$

& $y = 2 \Rightarrow 2 = \dfrac{x^2+x+2}{x^2+x+1} \Rightarrow x(x + 1) = 0 \Rightarrow x = 0, -1$ but $x \neq 0, -1$ so $y \neq 2$

Range: $\left(1, \dfrac{7}{3}\right] - \{2\}$.

Hence, no integral values in this interval.

Illustration 35:

Range of the function $f(x) = |\sin x|\cos x| + \cos x|\sin x||$ is $[a, b]$ then $(a + b)$ is equal to

Ans. (1)

Solution:

$$f(x) = \begin{cases} |\sin 2x| & 0 \le x < \frac{\pi}{2} \quad \Rightarrow \quad 0 \le 2x < \pi \quad \Rightarrow \quad 0 \le \sin 2x \le 1 \\ 0 & \frac{\pi}{2} \le x < \pi \\ |-\sin 2x| & \pi \le x < \frac{3\pi}{2} \quad \Rightarrow \quad 2\pi \le 2x < 3\pi \quad \Rightarrow \quad 0 \le |-\sin 2x| \le 1 \\ 0 & \frac{3\pi}{2} \le x \le 2\pi \end{cases}$$

so range $[0, 1]$.

7. Equal or Identical Function:

Two function f & g are said to be equal if :

(a) The domain of f =the domain of g

(b) The co-domain of f = co −domain of g and

(c) $f(x) = g(x)$, for every x belonging to their common domain (i.e. should have the same graph)

Illustration 36:

The functions $f(x) = \log(x - 1) - \log(x - 2)$ and $g(x) = \log\left(\frac{x-1}{x-2}\right)$ are identical when x lies in the interval

(A) $[1, 2]$ (B) $[2, \infty)$ (C) $(2, \infty)$ (D) $(-\infty, \infty)$

Ans. (C)

Solution:

Since $f(x) = \log(x - 1) - \log(x - 2)$.

Domain of $f(x)$ is $x > 2$ or $x \in (2, \infty)$ ...(i)

$g(x) = \log\left(\frac{x-1}{x-2}\right)$ is defined if $\frac{x-1}{x-2} > 0 \Rightarrow x \in (-\infty, 1) \cup (2, \infty)$ ...(ii)

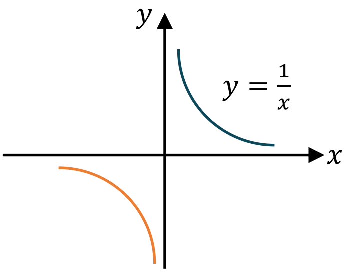
From (i) and (ii), $x \in (2, \infty)$.

Illustration 37:

$f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$ are identical functions or not?

Solution:

Clearly the graphs of $f(x)$ and $g(x)$ are exactly same so functions are identical.



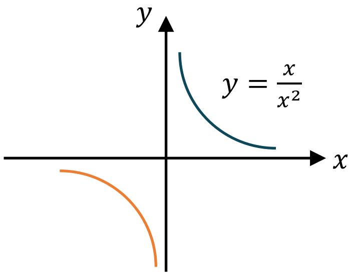
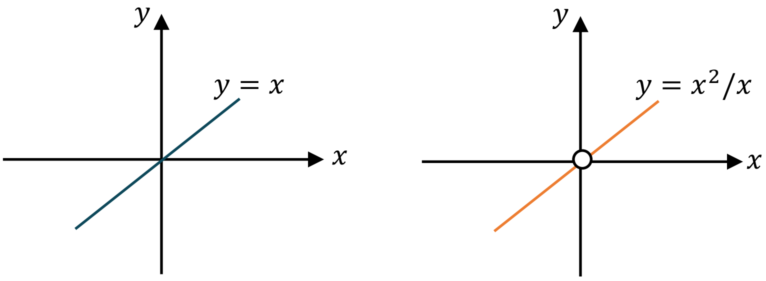


Illustration 38:

$f(x) = x$ and $g(x) = \frac{x^2}{x}$ are functions or not?

Solution:



Clearly the graphs of $f(x)$ and $g(x)$ are different at $x = 0$.

8. Homogeneous Functions:

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.
For examples $5x^2 + 3y^2 - xy$ is homogenous in x & y. Symbolically if,
$f(tx, ty) = t^n f(x, y)$ then $f(x, y)$ is homogeneous function of degree n.

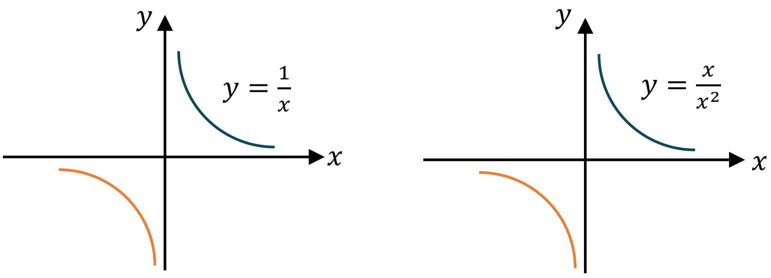


Illustration 39:

Which of the following function is not homogeneous?

(A) $x^3 + 8x^2y + 7y^3$ (B) $y^2 + x^2 + 5xy$ (C) $\frac{xy}{x^2+y^2}$ (D) $\frac{2x-y+1}{2y-x+1}$

Ans. (D)

Solution:

It is clear that (D) does not have the same degree in each term.

Illustration 40:

Find if the function $f(x, y) = x^3 + 2x^2y - 3xy^2 + y^3$ is a homogeneous function.

Solution:

The given expression is $f(x, y) = x^3 + 2x^2y - 3xy^2 + y^3$
Let us substitute $x = kx$, and $y = ky$ in the above expression.
$f(kx, ky) = (kx)^3 + 2(kx)^2(ky) - 3(kx)(ky)^2 + (ky)^3$
$f(kx, ky) = k^3x^3 + 2k^3x^2y - 3k^3xy^2 + k^3y^3$
$f(kx, ky) = k^3(x^3 + 2x^2y - 3xy^2 + y^3)$
$f(kx, ky) = k^3 f(x, y)$
Therefore, the above function is a homogeneous function.

9. Bounded Function:

A function is said to be bounded if there exists a finite M such that $|f(x)| \le M, \forall\, x \in D_f$

Illustration 41:

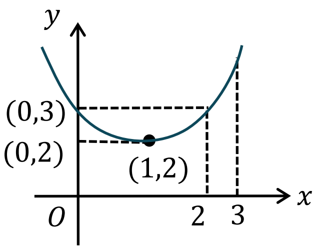
Range of quadratic expression $f(x) = x^2 - 2x + 3\ \forall\ x \in [0, 2]$ is

(A) $[0, 1]$ (B) $[2, 3]$ (C) $[1, 3]$ (D) $[2, \infty)$

Ans.(B)

Solution:

$f(x) = x^2 - 2x + 3\ \forall\ x \in [0, 2]$



$y \in [2, 3]\ \forall\ x \in [0, 2]$

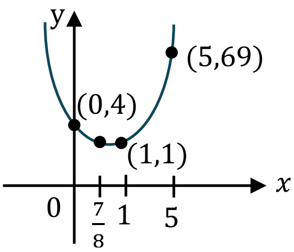
Illustration 42:

The minimum value of the quadratic expression $4x^2 - 7x + 4\ \forall\ x \in [1,5]$ is

(A) 1 (B) – 1 (C) 6 (D) $\frac{15}{16}$

Ans. (A)

Solution:



10. Implicit Explicit Function:

A function defined by an equation not solved for the dependent variable is called an **implicit function**. e.g. the equations $x^3 + y^3 = 1$ & $x^y = y^x$, defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit function.**

Illustration 43:

Which of the following function is implicit function?

(A) $y = \frac{x^2 + e^x + 5}{\sqrt{1 - \cos^{-1} x}}$ (B) $y = x^2$ (C) $xy - \sin(x + y) = 0$ (D) $y = \frac{x^2 \log x}{\sin x}$

Ans. (C)

Solution:

It is clear that in (C) y is not clearly expressed in x.

Illustration 44:

The domain of definition of the function, $y(x)$ is given by the equation, $2^x + 2^y = 2$ is

Solution:

$2^x + 2^y = 2$

$2^y = 2 - 2^x$

$\log_2 2^y = \log_2[2 - 2^x]$

$y = \log_2[2 - 2^x] > 0$

$2 > 2^x$

$2^1 > 2^x$

$x < 1$

11. Applications of Functional Rule:

Equation involving functions are called functional equation.

Illustration 45:

$2f(x) - 3f\left(\frac{1}{x}\right) = x^2$ where $x \neq 0$ then $f(2)$ =?

Solution:

$2f(x) - 3f\left(\frac{1}{x}\right) = x^2$ ...(i)

Replace $x \to \frac{1}{x}$

$2f\left(\frac{1}{x}\right) - 3f(x) = \frac{1}{x^2}$ ...(ii)

apply (i) × 2 + (ii) × 3

$\Rightarrow -5f(x) = 2x^2 + \frac{3}{x^2}$

$\Rightarrow f(x) = -\frac{1}{5}\left(2x^2 + \frac{3}{x^2}\right)$

$\Rightarrow f(2) = -\frac{7}{4}$

Illustration 46:

$f(x) + 2f\left(\frac{2002}{x}\right) = 3x$ for all $x > 0$

then find $f(2)$ = ?

Solution:

$f(x) + 2f\left(\frac{2002}{x}\right) = 3x$ ...(i)

Replace $x \to \frac{2002}{x}$

$f\left(\frac{2002}{x}\right) + 2f(x) = \frac{3 \times 2002}{x}$ ...(ii)

Apply (ii) × 2 – (i)

$\Rightarrow 3f(x) = \frac{6 \times 2002}{x} - 3x$

$\Rightarrow f(x) = \frac{1}{3}\left(\frac{12012}{x} - 3x\right)$

$\Rightarrow f(2) = \frac{1}{3}(6006 - 6) = 2000$

Illustration 47:

Determine all functions f satisfying the functional relation.

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)} \qquad \text{where } x \in \mathbb{R}-\{0,1\}$$

Solution:

Given $f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}=\frac{2}{x}-\frac{2}{1-x}$ ...(i)

Replacing x by $\frac{1}{1-x}$ we obtain

$$f\left(\frac{1}{1-x}\right)+f\left(\frac{1}{1-\frac{1}{1-x}}\right)=2(1-x)-\frac{2}{1-\frac{1}{1-x}}$$

$$\Rightarrow f\left(\frac{1}{1-x}\right)+f\left(1-\frac{1}{x}\right)=-2x+\frac{2}{x} \qquad \text{...(ii)}$$

Again, replacing x by $\left(1-\frac{1}{x}\right)$ in (i) we obtain

$$\Rightarrow f\left(1-\frac{1}{x}\right)+f\left(\frac{1}{1-\left(1-\frac{1}{x}\right)}\right)=\frac{2}{1-\frac{1}{x}}-\frac{2}{1-\left(1-\frac{1}{x}\right)}$$

$$\Rightarrow f\left(1-\frac{1}{x}\right)+f(x)=\frac{2x}{x-1}-2x \qquad \text{...(iii)}$$

subtracting (ii) from (i) then

$$f(x)-f\left(1-\frac{1}{x}\right)=2x-\frac{2}{1-x}$$

Now adding (iii) and (iv) we get

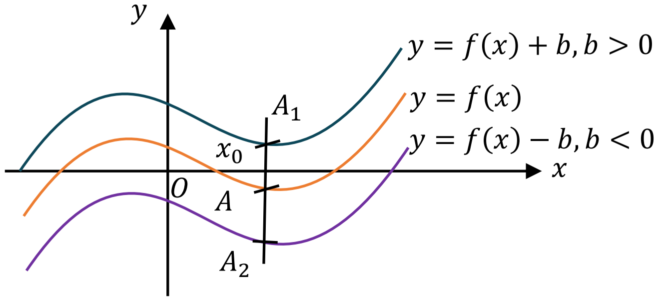
$$2f(x)=\frac{2x}{x-1}-\frac{2}{1-x}$$

$$\Rightarrow f(x)=\frac{x+1}{x-1}$$

12. Transformations of Graph:

12.1 Scaling and Origin Shifting:

(i) Drawing the graph of $y=f(x)+b, b\in\mathbb{R}$, from the known graph of $y=f(x)$



It is obvious that domain of $f(x)$ and $f(x)+b$ are the same. Let us take any point x_0 in the domain of $f(x)$. $y|_{x=x_0} = f(x_0)$.

The corresponding point on $f(x)+b$ would be $f(x_0)+b$.

For $b>0 \Rightarrow f(x_0)+b>f(x_0)$ it means that the corresponding point on $f(x)+b$ would be lying at a distance $'b'$ units above the point on $f(x)$.

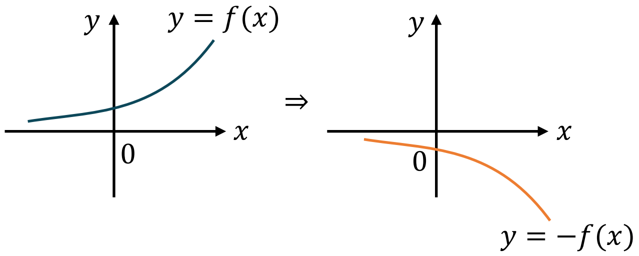
For $b<0 \Rightarrow f(x_0)+b<f(x_0)$ it means that the corresponding point on $f(x)+\mathrm{b}$ would be lying at a distance $'b'$ units below the point on $f(x)$.

Accordingly the graph of $f(x)+b$ can be obtained by translating the graph of $f(x)$ either in the positive

y –axis direction (if $b>0$) or in the negative y –axis direction (if $b<0$), through a distance $|b|$ units.

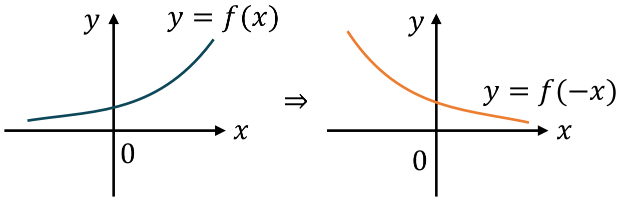
(ii) Drawing the graph of $y=-f(x)$ from the known graph of $y=f(x)$

To draw $y=-f(x)$, take the image of the curve $y=f(x)$ in the x-axis as plane mirror.

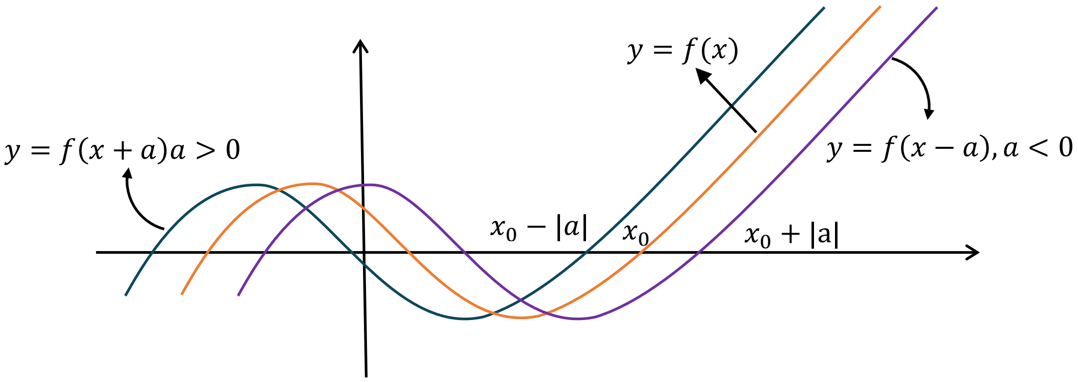


(iii) Drawing the graph of $y=f(-x)$ from the known graph of $y=f(x)$

To draw $y=f(-x)$, take the image of the curve $y=f(x)$ in the y-axis as plane mirror.



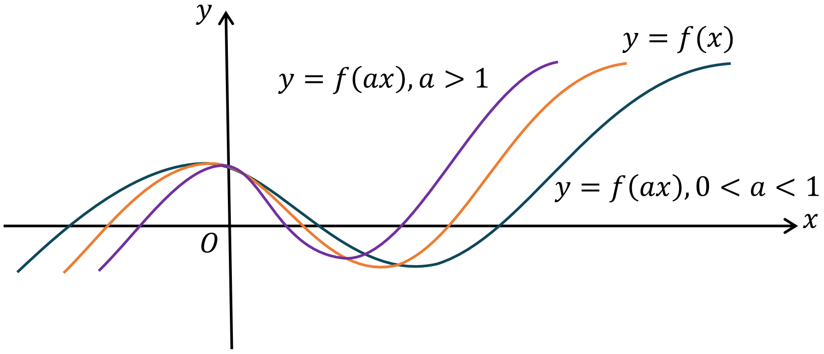
(iv) Drawing the graph of $y=f(x+a), a\in\mathbb{R}$ from the known graph of $y=f(x)$



(i) If $a>0$, shift the graph of $f(x)$ through $'a'$ units towards left of $f(x)$.

(ii) If $a<0$, shift the graph of $f(x)$ through $'a'$ units towards right of $f(x)$.

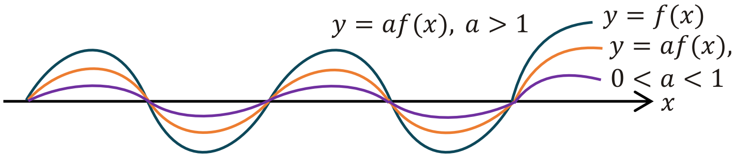
(v) Drawing the graph of $y = f(ax)$ from the known graph of $y = f(x)$.



Let us take any point $x_0 \in$ domain of $f(x)$. Let $ax = x_0$ or $x = \frac{x_0}{a}$.

Clearly if $0 < a < 1$, then $x > x_0$ and $f(x)$ will stretch by $\frac{1}{a}$ units along the x –axis and if $a > 1, x < x_0$, then $f(x)$ will compress by $'a'$ units along the x-axis.

(vi) Drawing the graph of $y = af(x)$ from the known graph of $y = f(x)$



It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of $1: a$.

Illustration 48:

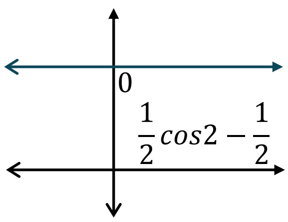
Draw the graph of $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$.

Solution:

$f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$

$= \frac{1}{2}[\cos(2x+2)+\cos 2] - \frac{1}{2}[\cos(2x+2)+1]$

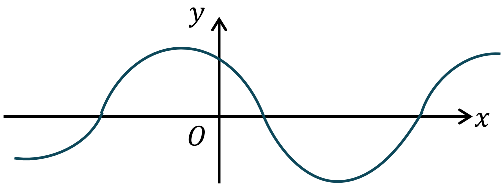
$= \frac{1}{2}\cos 2 - \frac{1}{2} < 0$



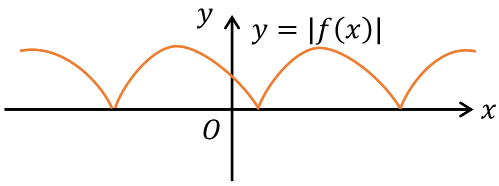
12.2 Transformation Due to Modulus:

(i) Drawing the graph of $y = |f(x)|$ from the known graph of $y = f(x)$

$|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) <$ 0. It means that the graph of $f(x)$ and $|f(x)|$ would coincide if $f(x) \geq 0$ and for the portions where $f(x) < 0$ graph of $|f(x)|$ would be image of $y = f(x)$ in x-axis.

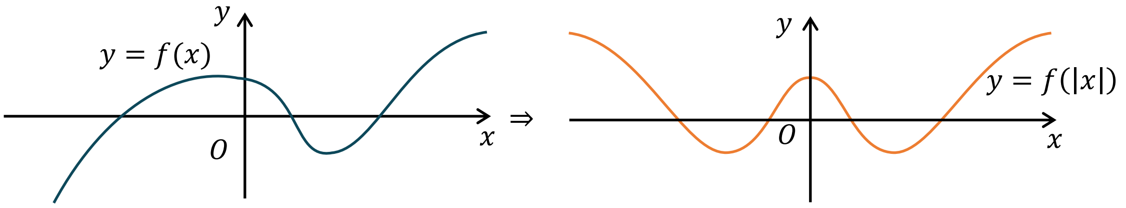


$\Rightarrow$



(ii) Drawing the graph of $y = f(|x|)$ from the known graph of $y = f(x)$

It is clear that, $f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$. Thus $f(|x|)$ would be a even function, graph of $f(|x|)$ and $f(x)$ would be identical in the first and the fourth quadrants ($as\ x \geq 0$) and as such the graph of $f(|x|)$ would be symmetric about the $y-$axis (as $(|x|)$ is even).



(iii) Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

Clearly $|y| \geq 0$. If $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$. Hence graph of $|y| = f(x)$ would exist only in the regions where $f(x)$ is non-negative and will be reflected about the x-axis only in those regions.

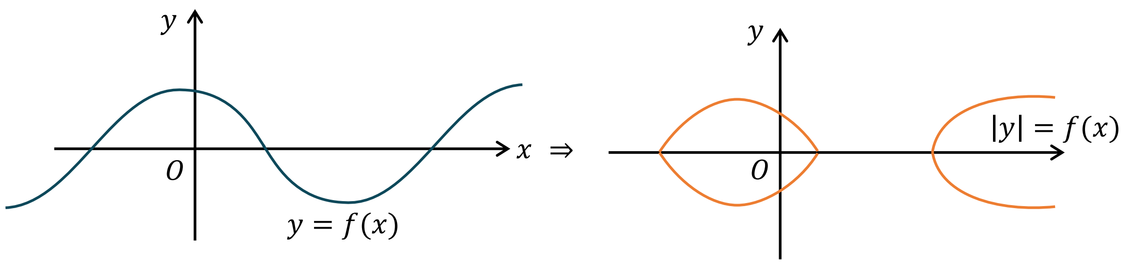


Illustration 49:

Draw the graph of $y = |2 - |x - 1||$.

Solution:

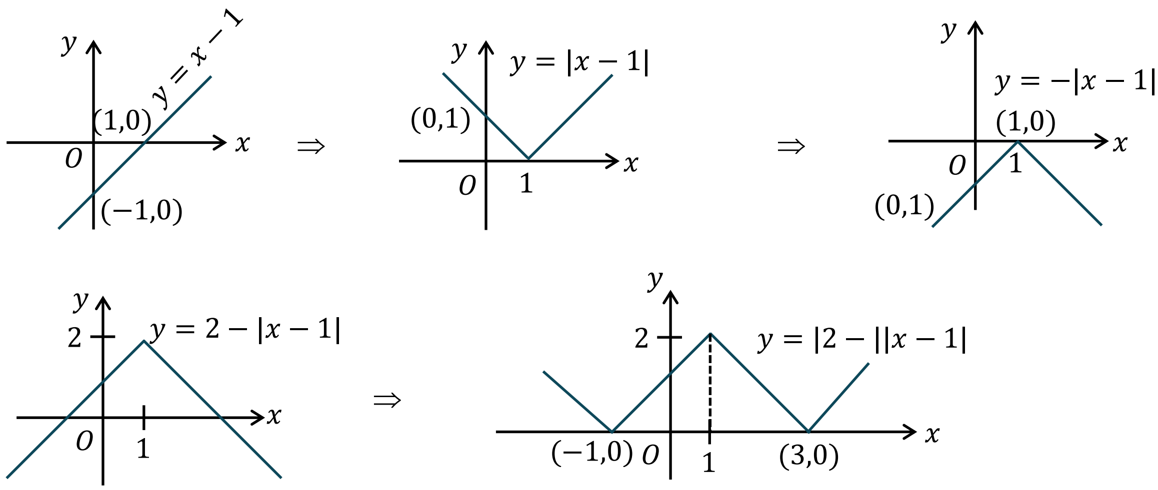


Illustration 50:

Draw the graph of $y = 2 - \dfrac{4}{|x-1|}$

Solution:

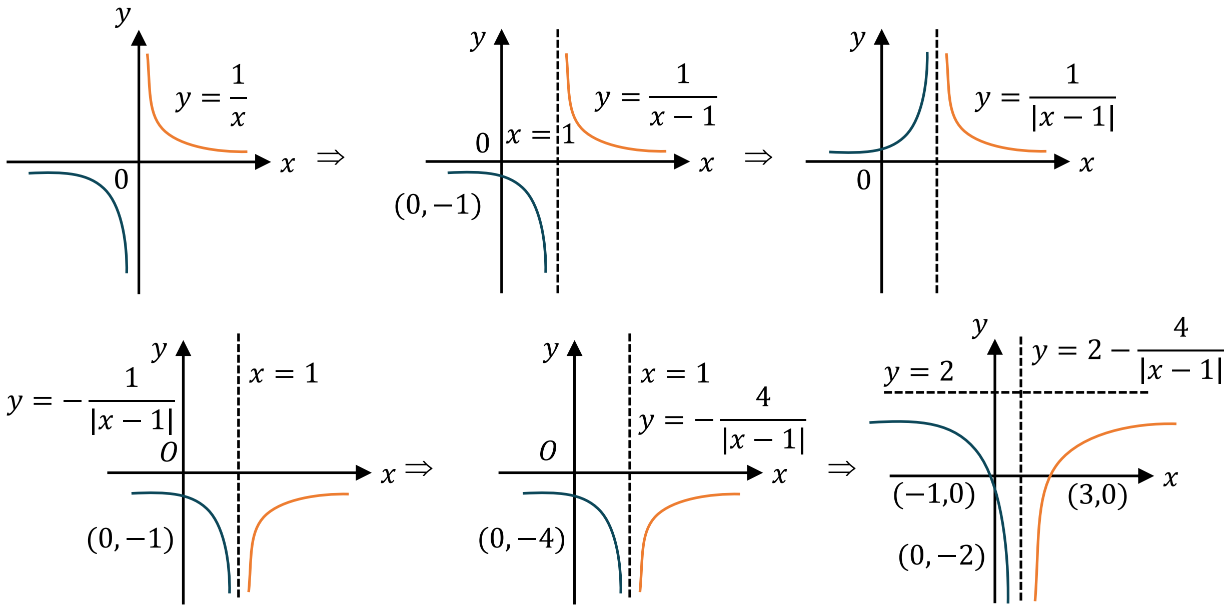


Illustration 51:

Draw the graph of $y = \left|e^{|x|} - 2\right|$

Solution:

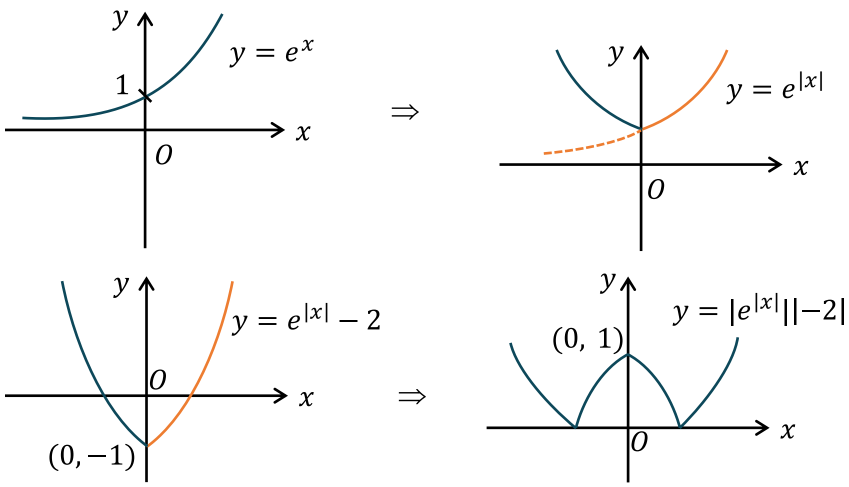
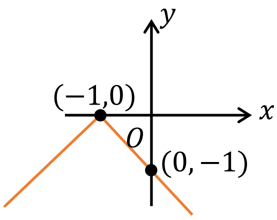
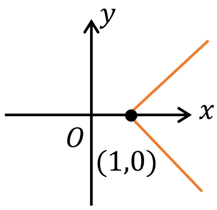
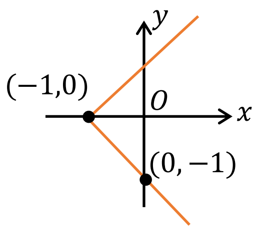


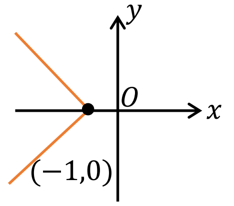
Illustration 52:

The graph of $|y| + x + 1 = 0$ is

(A) 

(B) 

(C) 

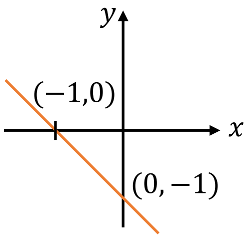
(D) 

Ans. (D)

Solution:

$|y| = -x - 1$

$y = -x - 1 |y| = -x - 1$



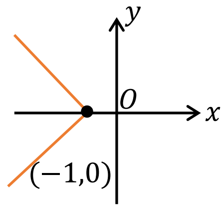
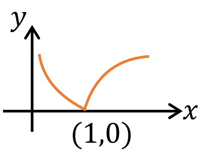
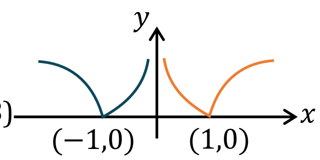
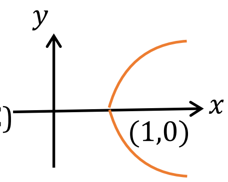


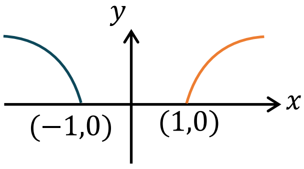
Illustration 53:

Graph of $|y| = \ell nx$

(A) 

(B) 

(C) 

(D) 

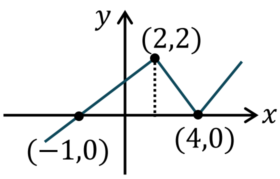
Ans. (C)

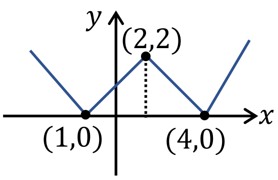
Solution:

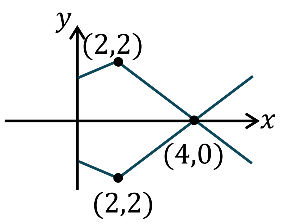
$y = \pm \ell nx$

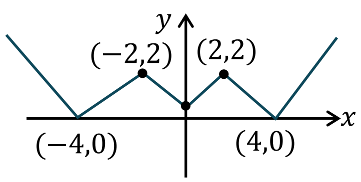
Illustration 54:

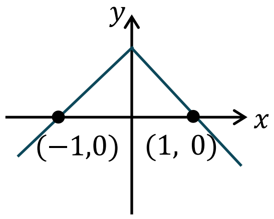
Given the graph of $y = f(x)$ is, which of the following is graph of $y = f(|x|)$



(A) 

(B) 

(C) 

(D) 

Ans. (C)

Solution:

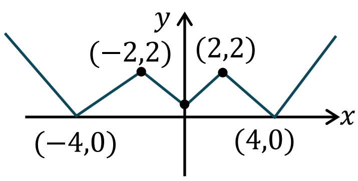
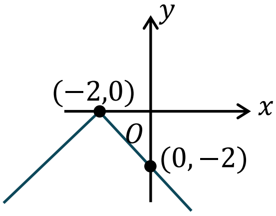
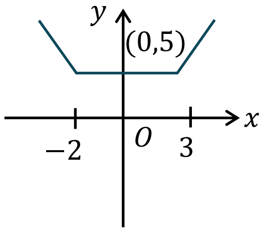
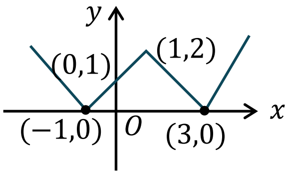


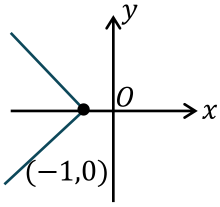
Illustration 55:

The graphs of the functions $f_1(x) = -|x+2|, f_2(x) = ||x-1|-2|, f_3(x) = |x+2| + |x-3|$

(A) 

(B) 

(C) 

(D) 

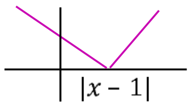
The correct order of graphs of functions $f_1(x), f_2(x), f_3(x)$ is

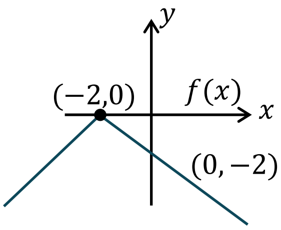
(A) BCD (B) ABC (C) ACB (D) ACD

Ans. (C)

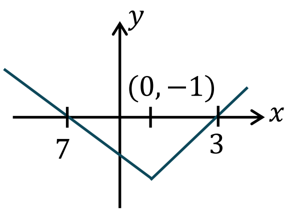
Solution:

$f_1(x) = -|x+2| =$





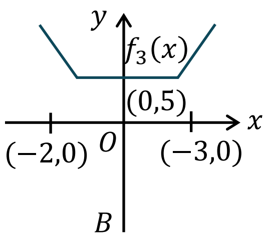
$f_2(x) = ||x-1|-2|$



$||x-1|-2|$

$f_3(x) = |x+2| + |x-3|$

$x \geq 3\ 2x-1$



$f_3(x) = |x+2| + |x-3|$

$x \geq 3, f_3(x) = 2x-1$

$-2 \leq x < 3, \quad f_3(x) = 5$

$x < -2, f_3(x) = -2x+1$

13. Max Function and Min Function:

Note:

(i) A function $h(x)$ is defined as

$h(x) = \max.\{f(x), g(x)\}$ then

$$h(x)=\begin{cases} f(x) & f(x)\ge g(x) \\ g(x) & g(x)> f(x) \end{cases}$$

(ii) A function $h(x)$ is defined as

$h(x) = \min.\{f(x), g(x)\}$ then

$$h(x)=\begin{cases} f(x) & f(x)\le g(x) \\ g(x) & g(x)< f(x) \end{cases}$$

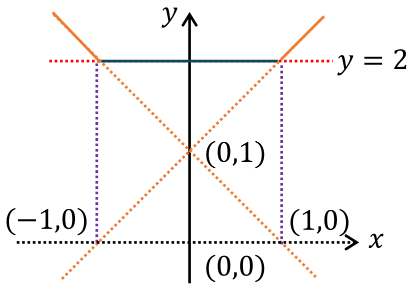
Illustration 56:

Find $f(x) = \max\{1+x, 1-x, 2\}$.

Solution:

From the graph it is clear that

$$f(x)=\begin{cases} 1-x\ ; & x<-1 \\ 2\ ; & -1\le x\le 1 \\ 1+x\ ; & x>1 \end{cases}$$



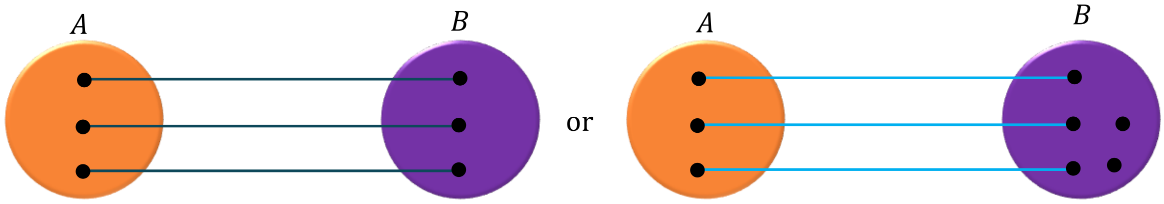
14. Classification of Functions:

One-One Function (Injective Mapping):

A function $f: A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus there exist $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B, f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagrammatically an injective mapping can be shown as

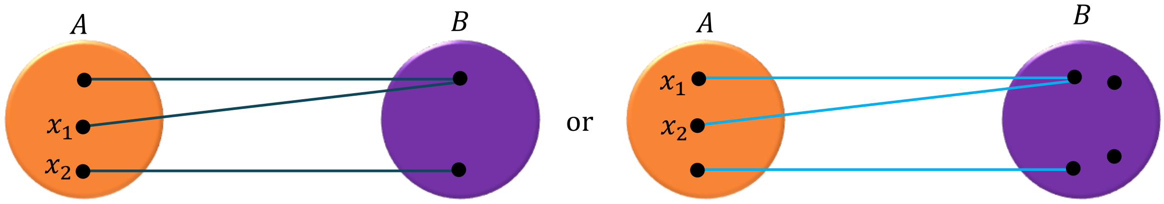


Many-one function (Not injective):

A function $f: A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f: A \to B$ is many one there exist $x_1, x_2 \in A, f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagrammatically a many one mapping can be shown as



Note:

(i) If a line parallel to x-axis cuts the graph of the function atmost at one point, then the function is one-one.

(ii) If any line parallel to x-axis cuts the graph of the function atleast at two points, then f is many-one.

(iii) If continuous function $f(x)$ is always increasing or decreasing in whole domain, then $f(x)$ is one-one.

(iv) All linear functions are one-one.

(v) All trigonometric functions in their domain are many one

(vi) All even degree polynomials are many one

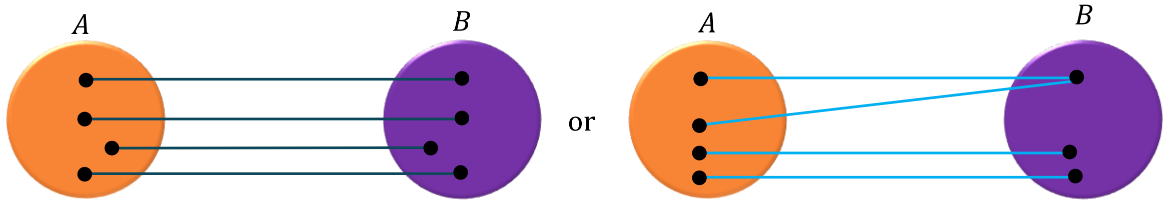
(vii) Linear by Linear is one-one

(viii) Quadratic by quadratic with no common factor is many one.

Onto function (Surjective mapping):

If the function $f: A \to B$ is such that each element in B (co-domain) is the f image of atleast one element in A, then we say that f is a function from A 'onto' B. Thus $f: A \to B$ is surjective iff $\forall b \in B, \exists$ some $a \in A$ such that $f(a) = b$.

Diagrammatically surjective mapping can be shown as

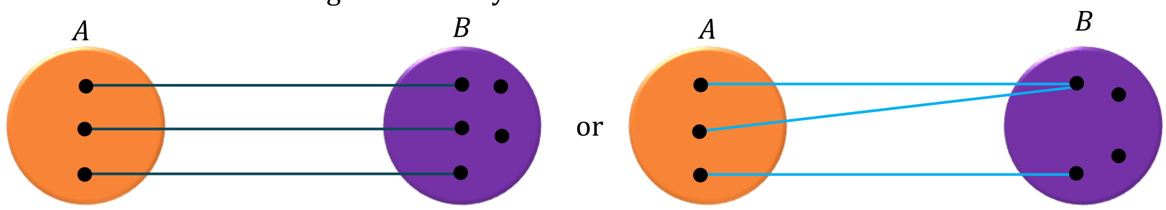


Note: If range is same as co-domain, then $f(x)$ is onto.

Into function:

If $f: A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then $f(x)$ is into.

Diagrammatically into function can be shown as



Note:

(i) A polynomial function of degree even defined from $\mathbb{R} \to \mathbb{R}$ will always be into.

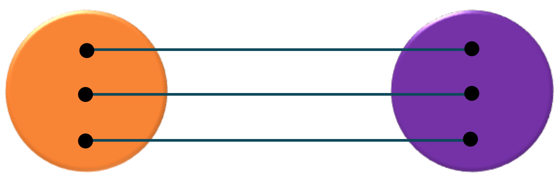
(ii) A polynomial function of degree odd defined from $\mathbb{R} \to \mathbb{R}$ will always be onto.

(iii) Quadratic by quadratic without any common factor define from $\mathbb{R} \to \mathbb{R}$ is always an into function.

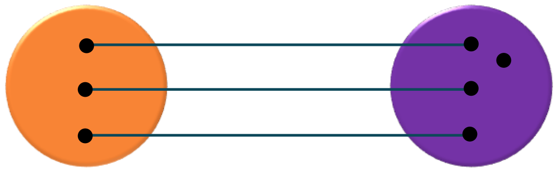
Thus a function can be one of these four types:

(i) one-one onto (injective & surjective)

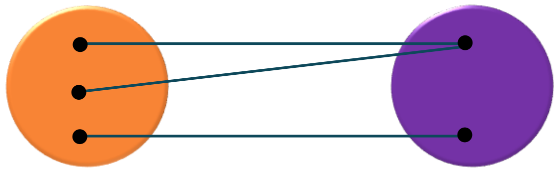
(also known as Bijective mapping)



(ii) one-one into (injective but not surjective)



(iii) many-one onto (surjective but not injective)



(iv) many-one into (neither surjective nor injective)

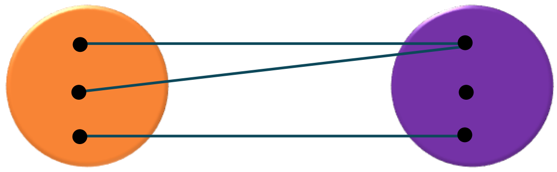


Illustration 57:

Let $A = \{x : -1 \le x \le 1\} = B$ be a mapping $f : A \to B$. For each of the following functions from A to B, find whether it is bijective or non-bijective.

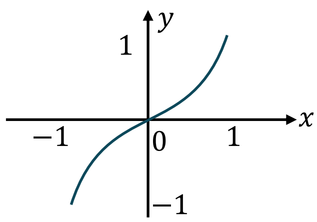
(A) $f(x) = x|x|$ (B) $f(x) = x^3$ (C) $f(x) = \sin\frac{\pi x}{2}$

Solution:

(A) $f(x) = x|x| = \begin{Bmatrix} -x^2 , & -1 < x < 0 \\ x^2 , & 0 \le x < 1 \end{Bmatrix}$,

Graphically,

The graph shows $f(x)$ is one-one, as the straight line parallel to x-axis cuts only at one point.



Here, range

$f(x) \in [-1,1]$

Thus, range = co-domain

Hence, onto.

Therefore, $f(x)$ is one-one onto or (Bijective).

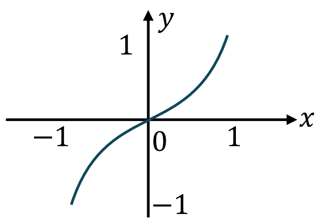
(B) $f(x) = x^3$,

Graphically;

Graph shows $f(x)$ is one-one onto

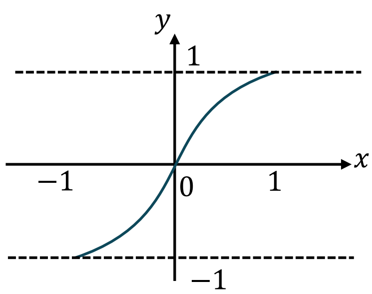
(i.e. Bijective)

[as explained in above example]



(C) $f(x) = \sin\frac{\pi x}{2}$

Graphically;



Which shows $f(x)$ is one-one and onto as range = co-domain.

Therefore, $f(x)$ is bijective.

Illustration 58:

Let $f:\mathbb{N}\to\mathbb{R}$ be a function defined as $f(x)=x-1000$. Show that $f(x)$ is an into function.

Solution:

Let $f(x)=y=x-1000 \Rightarrow x=y+1000=g(y)$ (say)

here $g(y)$ is defined for each $y\in\mathbb{Z}$, but $g(y)\notin\mathbb{N}$ for $y\leq -1000$

Hence $f(x)$ is into.

Illustration 59:

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=x+\sqrt{x^2}$, then f is

(A) injective (B) surjective (C) bijective (D) None of these

Ans. (D)

Solution:

We have, $f(x) = x + \sqrt{x^2} = x + |x|$

Clearly, f is not one – one as $f(-1)=f(-2)=0$ and $-1\neq -2$

Also, f is not onto as $f(x)\geq 0\ \forall x\in\mathbb{R}$

$\therefore$ range of $f=(0,\infty)\subset\mathbb{R}$

Illustration 60:

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined as $f(x)= 2x^3-6x^2+12x+3\cos x-4\sin x$; then f is -

(A) Injective (B) Surjective (C) Bijective (D) Not Surjective

Ans. (C)

Solution:

We have $f(x)= 2x^3-6x^2+12x+3\cos x-4\sin x$

$\Rightarrow f'(x)=6x^2-12x+12-3\sin x-4\cos x$

$$f'(x)=\underbrace{6(x-1)^2+6}_{g(x)}-\underbrace{(3\sin x+4\cos x)}_{h(x)}$$

range of $g(x)=[6,\infty)$

range of $h(x)=[-5,5]$

hence $f'(x)$ always lies in the interval $[1,\infty)$

$\Rightarrow f'(x)>0$

Hence $f(x)$ is increasing i.e. one-one function

Now $x\to\infty \Rightarrow f\to\infty$ & $x\to-\infty\Rightarrow f\to-\infty$ & $f(x)$ is continuous

hence its range is $\mathbb{R}\Rightarrow f$ is onto so f is bijective.

Illustration 61:

Let $f(x)=\dfrac{x^2+3x+a}{x^2+x+1}$, where $f:\mathbb{R}\to\mathbb{R}$. Find the value of parameter $'a'$ so that the given function is one-one.

Solution:

$$f(x)=\frac{x^2+3x+a}{x^2+x+1}$$

$$f'(x)=\frac{(x^2+x+1)(2x+3)-(x^2+3x+a)(2x+1)}{(x^2+x+1)^2}=\frac{-2x^2+2x(1-a)+(3-a)}{(x^2+x+1)^2}$$

Let, $g(x)=-2x^2+2x(1-a)+(3-a)$

$g(x)$ will be negative if $4(1-a)^2+8(3-a)<0$

$\Rightarrow 1+a^2-2a+6-2a<0\Rightarrow (a-2)^2+3<0$

which is not possible. Therefore function is not monotonic.

Hence, no value of a is possible.

Important Points:

(1) A polynomial of **degree odd** defined from $R \to R$ will always be onto $\boldsymbol{E.g.\, f(x) = x^3 + x + 1}$

(2) A polynomial of **degree even** defined from $R \to R$ will always many one & into. **E.g.** $\boldsymbol{y = x^4 + x + 1}$

(3) $\frac{\text{Quadratic}}{\text{Quadratic}}$ defined from $R \to R$ is always many one & into function $\boldsymbol{E.g.\, y = \frac{x^2 - x + 1}{x^2 + x + 1}}$

(4) $\frac{\textbf{Linear}}{\textbf{linear}}$ is always one - one

Illustration 62:

Find whether $f(x) = x + \cos x$ is one-one.

Solution:

The domain of $f(x)$ is $R. f'(x) = 1 - sinx.$

$\therefore$ $f'(x) \geq 0 \; \forall \; x \in$ complete domain and equality holds at discrete points only

$\therefore$ $f(x)$ is strictly increasing on R. Hence $f(x)$ is one-one

Illustration 63:

Identify whether the function $f(x) = -x^3 + 3x^2 - 2x + 4$ for $f: R \to R$ is ONTO or INTO

Solution:

As range $\equiv$ codomain, therefore given function is ONTO

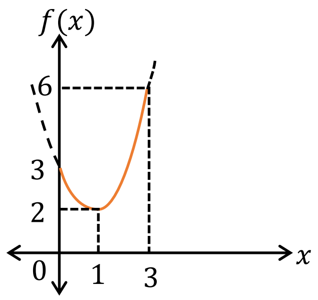
Illustration 64:

$f(x) = x^2 - 2x + 3; [0, 3] \to A$. Find whether $f(x)$ is injective or not. Also find the set A, if $f(x)$ is surjective.

Solution:

$f'(x) = 2(x-1); 0 \leq x \leq 3$

$$\therefore \; f'(x) = \begin{cases} -ve & ; \quad 0 \leq x < 1 \\ +ve & ; \quad 1 < x < 3 \end{cases}$$



$\therefore$ $f(x)$ is non monotonic. Hence it is not injective.

For $f(x)$ to be surjective, A should be equal to its range.

By graph range is $[2, 6]$

$\therefore$ $A \equiv [2,6]$

Illustration 65:

Let $f: R \to [0,2]$ be defined by $f(x) = \log_{\sqrt{13}/2}\left(\sin^2 x + \sin x + a\right)$ then the set of values of a for which f is onto is:

(A) $\left(0, \frac{1}{4}\right]$ (B) $\left[\frac{1}{4}, \infty\right)$ (C) $\left\{\frac{3}{4}\right\}$ (D) $\left\{\frac{5}{4}\right\}$

Ans. (D)

Solution:

$$\sin^2 x + \sin x + a = \left(\sin x + \frac{1}{2}\right)^2 - \frac{1}{4} + a$$

$$-\frac{1}{4} \leq \left(\sin x + \frac{1}{2}\right)^2 - \frac{1}{4} \leq 2$$

$$-\frac{1}{4} + a \leq \left(\sin x + \frac{1}{2}\right)^2 + a - \frac{1}{4} \leq 2 + a \;\Rightarrow\; a = \frac{5}{4}$$

Illustration 66:

If $f(x) = 2[x] + \cos x$, then $f: R \to R$ is: (where [.] denotes greatest integer function)

(A) one–one and onto

(B) one–one and into

(C) many–one and into

(D) many one and onto

Ans. (C)

Solution:

$f(x) = 2[x] + \cos x$

$f(x) = \cos x \; x \in [0,1)$

$= 2 + \cos x \; x \in [1,2)$

$= 4 + \cos x x \in [2,3)$

$= 6 + \cos x x \in [3,4)$

For

$x \in [0,1) f'(x) = -ve$

$x \in [1,2) f'(x) = -ve$

$x \in [2,3) f'(x) = -ve$

$x \in [3,4) f'(x) = +ve$

$\Rightarrow$ graph is not one-one

if $x \in [0,1)$range: $[\cos 1, 1]$

$x \in [1,2)$range: $[2 + \cos 2, 2 + \cos 1)$

not onto function

Illustration 67:

The function $f : [0,3] \to [1,29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is

(A) one-one and onto

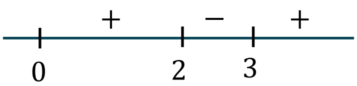
(B) onto but not one-one

(C) one-one but not onto

(D) neither one-one nor onto

Ans. (B)

Solution:



$F : [0,3] \to [1,29]$

$f(x) = 2x^3 - 15x^2 + 36x + 1$

$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3)$

in given domain function has local maxima, it is many-one

Now at $x = 0 \; f(0) = 1$

$x = 2 \; f(2) = 16 - 60 + 72 + 1 = 29$

$x = 3 \; f(3) = 54 - 135 + 108 + 1 = 163 - 135 = 28$

Has range = $[1, 29]$

Hence given function is onto

Illustration 68:

Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto function form Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is ___ .

Ans. (119)

Solution:

$n(X) = 5$

$n(Y) = 7$

$\alpha \to$ Number of one-one function X to Y = ${}^7C_5 \times 5! = 21 \times 120 = 2520$

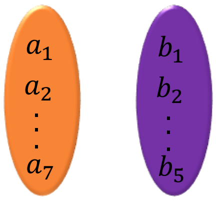
$\beta \to$ Number of onto function Y to X

Here two cases arrive

(i) one of $b_1, b_2, \ldots .. b_5$ gets 3 and remaining get 1 each.

(ii) two of $b_1, b_2, \ldots .. b_5$ get 2 and remaining get 1 each.

1, 1, 1, 1, 31, 1, 1, 2, 2



$$\frac{7!}{3!\,4!} \times 5! + \frac{7!}{(2!)^3 3!} \times 5! = ({}^7C_3 + 3\,.\,{}^7C_3)\,5! = 4 \times {}^7C_3 \times 5!$$

$$\frac{\beta - \alpha}{5!} = 4 \times {}^7C_3 - {}^7C_5 = 4 \times 35 - 21 = 119$$

Illustration 69:

The function $f: X \to Y$, defined by $f(x) = x^2 - 4x + 5$ is both one–one and onto if

(A) $X = [2, \infty)$ & $Y = [1, \infty)$ (B) $X = (-\infty, 2]$ & $Y = [1, \infty)$

(C) $X = [3, \infty)$ & $Y = [2, \infty)$ (D) $X = (-\infty, 2]$ & $Y = (1, \infty)$

Ans. (A,B,C)

Solution:

$f : [2, \infty) \to Y$

$f(x) = x^2 - 4x + 5$

$f(x) = (x - 2)^2 + 1$

For given domain by graph range is $[1, \infty)$

For function to be onto codomain $y = [1, \infty)$

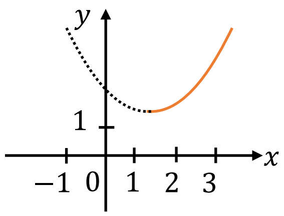


Illustration 70:

Let $f(x) = x^{135} + x^{125} - x^{115} + x^5 + 1$. If $f(x)$ divided by $x^3 - x$, then the remainder is some function of x say $g(x)$. Then $g(x)$ is an :

(A) one-one function (B) many one function

(C) into function (D) onto function

Ans. (A,D)

Solution:

$f(x) = (x^3 - x)\,Q(x) + ax^2 + bx + c$

$f(0) = 1 = c$

$f(1) = a + b + c = 3$

$f(-1) = a - b + c = -1$

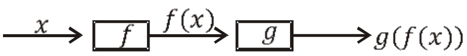
$\Rightarrow a = 0,\ b = 2,\ c = 1 \Rightarrow g(x) = 2x + 1$

15. Composite of Uniformly and Non-Uniformly Defined Functions:

Let $f: A \to B$ & $g: B \to C$ be two functions. Then the function $gof: A \to C$ defined by

$(gof)(x) = g(f(x))\ \forall\ x \in A$ is

Called the composite of the two functions f & g.

Diagrammatically 

Thus the image of every $x \in A$ under the function gof is the g – image of f – image of x.

Note: gof is defined only if $\forall\ x \in A$, $f(x)$ is an element of the domain of 'g' so that we can take its g-image.

Properties of Composite Functions:

(a) In general composite of functions is not commutative i.e. $gof \neq fog$.

(b) The composition of functions is associative i.e. if f, g, h are three functions such that $fo(goh)$ & $(fog)oh$ are defined, then $fo(goh) = (fog)oh$.

(c) The composition of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

Illustration 71:

If $f(x) = x^2 + 1, g(x) = \frac{1}{x-1}$, then find $(fog)(x)$ and $(gof)(x)$.

Solution:

Given, $f(x) = x^2 + 1$ ...(1)

$g(x)=\frac{1}{x-1}$ ...(2)

Now $(fog)(x) = f(g(x)) = f\left(\frac{1}{x-1}\right) = f(z)$, where $z=\frac{1}{x-1}$

$= z^2 + 1 [\because f(x) = x^2 + 1]$

$=\left(\frac{1}{x-1}\right)^2+1=\frac{1}{(x-1)^2}+1$

Note: Domain of $fog(x)$ is $x \in \mathbb{R}-\{1\}$

$(gof)(x) = g(f(x)) = g(x^2 + 1) = g(u)$, where $u = x^2 + 1$

$=\frac{1}{u-1}=\frac{1}{x^2+1-1}=\frac{1}{x^2}$

Note: Domain of $gof(x)$ is $x \in \mathbb{R}-\{0\}$

Illustration 72:

If f be the greatest integer function and g be the modulus function, then $(gof)\left(-\frac{5}{3}\right)-(fog)\left(-\frac{5}{3}\right)=$

(A) 1 (B) – 1 (C) 2 (D) 4

Ans.(A)

Solution:

Given $(gof)\left(-\frac{5}{3}\right)-(fog)\left(-\frac{5}{3}\right) = g\left\{f\left(\frac{-5}{3}\right)\right\}-f\left\{g\left(\frac{-5}{3}\right)\right\} = g(-2)-f\left(\frac{5}{3}\right)=2-1=1$

Illustration 73:

Let $f(x) = \begin{cases} x+1, & x\le 1 \\ 2x+1, & 1<x\le 2 \end{cases}$ and $g(x) = \begin{cases} x^2, & -1\le x<2 \\ x+2, & 2\le x\le 3 \end{cases}$, find (fog)

Solution:

$f(g(x))=\begin{cases} g(x)+1, & g(x)\le 1 \\ 2g(x)+1, & 1<g(x)\le 2 \end{cases}$

Here, $g(x)$ becomes the variable that means we should draw the graph.

It is clear that $g(x) \le 1;\ \forall x \in [-1,1]$

and $1 < g(x) \le 2;\ \forall x \in \left(1,\sqrt{2}\right]$

$\Rightarrow f(g(x))=\begin{cases} x^2+1, & -1\le x\le 1 \\ 2x^2+1, & 1<x\le \sqrt{2} \end{cases}$

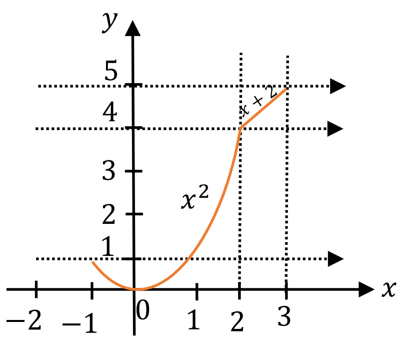


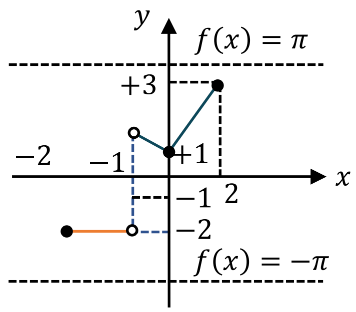
Illustration 74:

Find the domain and range of $h(x) = g(f(x))$, where

$f(x)=\begin{cases}[x], & -2\le x\le -1\\ |x|+1, & -1<x\le 2\end{cases}$ and $g(x)=\begin{cases}[x], & -\pi\le x\le 0\\ \sin x, & 0\le x\le \pi\end{cases}$, [.] denotes the greatest integer function.

Solution:

$$h(x)=g(f(x))=\begin{cases}[f(x)], & -\pi\le f(x)<0\\ \sin(f(x)), & 0\le f(x)\le \pi\end{cases}$$



From graph of $f(x)$, we get

$$h(x)=\begin{cases}[[x]], & -2\le x\le -1\\ \sin(|x|+1), & -1<x\le 2\end{cases}$$

$\Rightarrow$ Domain of $h(x)$ is $[-2,2]$

and Range of $h(x)$ is $\{-2,-1\}\cup[\sin 3,1]$

Illustration 75:

If $f(x) = ||x-3|-2| \quad 0\le x\le 4, g(x) = 4-|2-x| -1\le x\le 3$ then find $fog(2)$?

Solution:

$Fog(2) = f(4)$ ($\because g(2) = 4$)

$\therefore fog(2) = 1$

Illustration 76:

Define $fog(x)$ and $gof(x)$. Also find their domain and range.

$f(x) = [x], g(x) = \sin x$

Solution:

$gof = \sin[x]$domain: Rrange $\{\sin a : a \in I\}$

$fog = [\sin x]$domain: Rrange: $\{-1, 0, 1\}$

Illustration 77:

If $f(x)=\begin{cases}-x & ,x<0\\ x^2 & ,\ x\ge 0\end{cases}$ and $g(x)=\begin{cases}e^x, x>0\\ -e^x, x\le 0\end{cases}$ then $g(f(x))$ =?

Solution:

$$g[f(x)]=\begin{cases}e^{f(x)}, f(x)>0\\ -e^{f(x)}, f(x)\le 0\end{cases}=\begin{cases}e^{-x}, & -x>0 \ \& \ x<0\\ -e^{-x}, & -x\le 0 \ \& \ x<0\\ e^{x^2}, & x^2>0 \ \& \ x\ge 0\\ -e^{x^2}, & x^2\le 0 \ \& \ x\ge 0\end{cases}=\begin{cases}e^{-x}, & x<0\\ e^{x2}, & x>0\\ -e^{x^2}, & x=0\end{cases}$$

Illustration 78:

Let $f(x) = \ell n|x|$ and $g(x) = \sin x$. If A is the range of $f(g(x))$ and B is the range of $g(f(x))$ then

Solution:

A = range of $\ell n|\sin x| = (-\infty, 0]$

B = range of $\sin(\ell n|x|) = [-1,1] \Rightarrow A\cup B = (-\infty, 1], A\cap B = [-1, 0]$

Illustration 79:

Let $f(x)=\dfrac{\alpha x}{x+1}, x\ne -1$. Then for what value of α is $f(f(x)) = x$?

Solution:

$f(x) =, x \neq -1$

$$f(f(x)) = x \Rightarrow \frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1}+1} = x$$

$\Rightarrow \alpha^2 x = \alpha x^2 + x^2 + x$

on comparing

$\alpha^2 = 1$and $\alpha =-1 \Rightarrow \alpha =-1$

Illustration 80:

If a function is defined as $f(x) = \sqrt{\log_{h(x)} g(x)}$, where $g(x) = |\sin x| + \sin x, h(x) = \sin x + \cos x, 0 \leq x \leq \pi$. Then find the domain of $f(x)$.

Ans. $\left[\frac{\pi}{6}, \frac{\pi}{2}\right)$

Solution:

(i) $g(x) > 0 \Rightarrow |\sin x| + \sin x > 0 \Rightarrow 0 < x < \pi$ ...(A)

(ii) $0 < h(x) < 1$ or $h(x) > 1$

(a) $0 < \sin x + \cos x < 1 \Rightarrow \frac{\pi}{2} < x < \frac{3\pi}{4}$ ...(B)

(b) $\sin x + \cos x > 1 \Rightarrow 0 < x < \frac{\pi}{2}$ ...(C)

(iii) $\log_{h(x)} g(x) \geq 0$

for $h(x) > 1, g(x) \geq 1$

i.e. $|\sin x| + \sin x \geq 1 \Rightarrow \sin x \geq \frac{1}{2}, (\because \sin x > 0) \Rightarrow \frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ ...(D)

From (C) and (D) $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$

(b) $0 < h(x) < 1$ then $0 < g(x) \leq 1$

$0 < |\sin x| + \sin x \leq 1 \Rightarrow 0 < \sin x \leq \frac{1}{2}$

i.e. $0 < x \leq \frac{\pi}{6}$ & $\frac{5\pi}{6} \leq x < \pi$ ...(E)

From (B) & (E) $x \in \phi$ so final domain is $\left[\frac{\pi}{6}, \frac{\pi}{2}\right)$

Illustration 81:

Let $f(x) = \log_2 \log_3 \log_4 \log_5 (\sin x + a^2)$. Find the set of values of a for which domain of $f(x)$ is R.

Ans. $a \in (-\infty, -\sqrt{626}) \cup (\sqrt{626}, \infty)$

Solution:

Given $f(x) = \log_2 \log_3 \log_4 \log_5 (\sin x + a^2)$

$f(x)$ is defined only if $\log_3 \log_4 \log_5 (\sin x + a^2) > 0, \forall x \in R$

$\Rightarrow \log_4 \log_5 (\sin x + a^2) > 1, \forall x \in R \Rightarrow \log_5 (\sin x + a^2) > 4, \forall x \in R$

$\Rightarrow (\sin x + a^2) > 5^4, \forall x \in R \Rightarrow a^2 > 625 - \sin x, \forall x \in R$

$\Rightarrow a^2$ must be greater than maximum value of $625 - \sin x$ which is 626 (when $\sin x =-1$)

$\Rightarrow a^2 > 626 \Rightarrow a \in (-\infty, -\sqrt{626}) \cup (\sqrt{626}, \infty)$

Illustration 82:

Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(fogogof)(x) = (gogof)(x)$, where $(fog)(x) = f(g(x))$, is

(A) $\pm\sqrt{n\pi}, n \in \{0, 1, 2, \ldots\}$ (B) $\pm\sqrt{n\pi}, n \in \{1, 2, \ldots\}$

(C) $\frac{\pi}{2} + 2n\pi, n \in \{\ldots -2, -1, 0, 1, 2, \ldots\}$ (D) $2n\pi, n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Ans. (A)

Solution:

$f(x) = x^2; g(x) = \sin x \Rightarrow gof(x) = \sin x^2 \Rightarrow gogof(x) = \sin(\sin x^2)$

$\Rightarrow (fogogof)(x) = (\sin(\sin x^2))^2 = \sin^2(\sin x^2)$

Now $\sin^2(\sin x^2) = \sin(\sin x^2) \Rightarrow \sin(\sin x^2) = 0, 1$

$\Rightarrow \sin x^2 = n\pi, (4n+1)\frac{\pi}{2};\ \eta \in I \Rightarrow \sin x^2 = 0$

$\Rightarrow x^2 = n\pi \Rightarrow x = \pm\sqrt{n\pi}\ ;\ n \in W$

Illustration 83:

If $f(x) = \frac{ax+b}{cx+d}$, then $(fof)(x) = x$, provided that

(A) $d + a = 0$ (B) $d - a = 0$ (C) $a = b = c = d = 1$ (D) $a = b = 1$

Ans. (A)

Solution:

$$f(x) = \frac{ax+b}{cx+d} \Rightarrow fof(x) = \frac{a\left(\frac{ax+b}{cx+d}\right)+b}{c\left(\frac{ax+b}{cx+d}\right)+d} \Rightarrow fof(x) = \frac{a^2x+ab+bcx+bd}{acx+bc+cdx+d^2}$$

$$fof(x) = \frac{(a^2+bc)x+(ab+bd)}{(ac+cd)x+(bc+d^2)} = x$$

on comparing coefficient of both side $(a^2+bc)x + (ab+bd) = (ac+cd)x^2 + (bc+d^2)x$

$a^2 + bc = bc + d^2 \Rightarrow a = d$ or $a = -d$

and $ab + bd = 0 \Rightarrow b = 0$ or $a = -d$

and $ac + cd = 0 \Rightarrow c = 0$ or $a = -d$

which can be simultaneously true for $a = -d$

Illustration 84:

If $f(x) = \begin{cases} x^2 & x \le 1 \\ 1-x & x > 1 \end{cases}$ & composite function $h(x) = |f(x)| + f(x+2)$, then

(A) $h(x) = 2x^2 + 4x + 4\ \forall\ x \le -1$

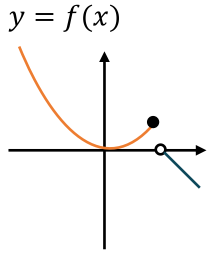
(B) $h(x) = x^2 + x + 1\ \forall\ -1 < x \le 1$

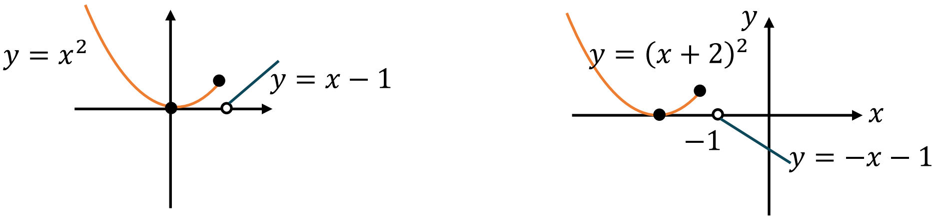
(C) $h(x) = x^2 - x - 1\ \forall\ -1 < x \le 1$

(D) $h(x) = -2\ \forall\ x > 1$

Ans. (A,C,D)

Solution:





$y = |f(x)| + f(x+2) = \begin{cases} x^2 + (x+2)^2 & x \le -1 \\ x^2 + (-x-1) & -1 < x \le 1 \\ x - 1 - x - 1 & x > 1 \end{cases}$; $y = |f(x)| + f(x+2)$

$= \begin{cases} 2x^2 + 4x + 4 & x \le -1 \\ x^2 - x - 1 & -1 < x \le 1 \\ -2 & x > 1 \end{cases}$

16. Inverse of a Function:

Let $f: A \to B$ be a one-one & onto function, then there exists a unique function $g: B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall x \in A\ \&\ y \in B$. Then g is said to be inverse of f.

Thus $g = f^{-1}: B \to A = \{(f(x), x)) | (x, f(x)) \in f\}$.

Properties of inverse function:

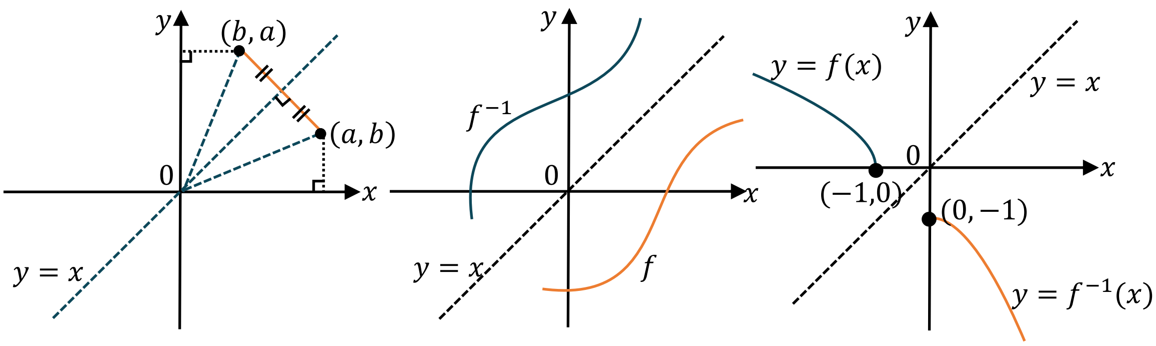
(A) The inverse of a bijection is unique.

(B) If $f: A \to B$ is a bijection & $g: B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively. If $fof = I$, then f is inverse of itself.

(C) The inverse of a bijection is also a bijection.

(D) If f & g are two bijections $f: A \to B, g: B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.

(E) Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of 'f' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$.



The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Drawing the graph of $y = f^{-1}(x)$ from the known graph of $y = f(x)$

For drawing the graph of $y = f^{-1}(x)$ take the reflection of $y = f(x)$ about the line $y = x$. The reflected part would give us the graph of $y = f^{-1}(x)$.

e.g. let us draw the graph of $y = \sin^{-1} x$. We know that $y = f(x) = \sin x$ is invertible if $f:\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

$\Rightarrow$ the inverse mapping would be $f^{-1}:[-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

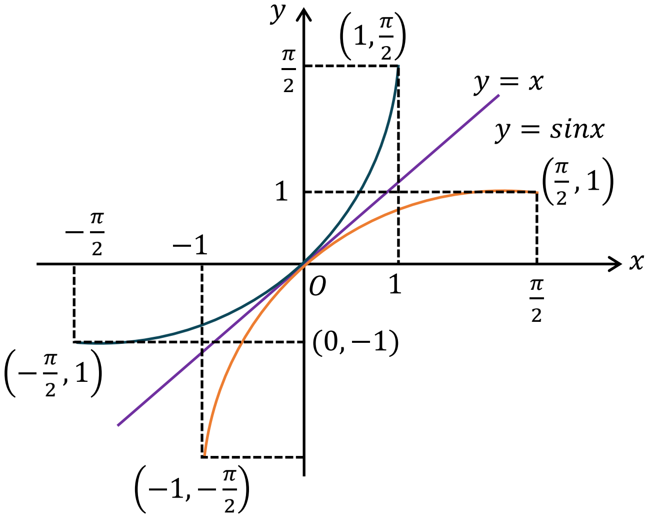


Illustration 85:

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (e^x - e^{-x})/2$. Is $f(x)$ invertible? If so, find its inverse.

Solution:

Let us check for invertibility of $f(x)$:

(A) One-One :

$$f(x) == \frac{1}{2}\left(e^x - e^{-x}\right) \Rightarrow f'(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$\Rightarrow$ $f'(x) > 0, f(x)$ is increasing function

$\therefore$ $f(x)$ is one-one function.

(B) Onto:

As x tends to larger and larger values so does $f(x)$ and when $x \to \infty, f(x) \to \infty$.

Similarly, as $x \to -\infty, f(x) \to -\infty$ i.e. $-\infty < f(x) < \infty$ so long as $x \in (-\infty, \infty)$

Hence the range of f is same as the set R. Therefore $f(x)$ is onto.

Since $f(x)$ is both one-one and onto, $f(x)$ is invertible.

(C) To find $f^{-1}(x)$: Interchange x & y

$$\frac{e^y - e^{-y}}{2} = x \Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \Rightarrow e^y = x \pm \sqrt{1 + x^2}$$

Since $e^y > 0$, hence negative sign is ruled out and

Hence $e^y = x + \sqrt{1 + x^2}$

Taking logarithm, we have $y = \ell n(x + \sqrt{1 + x^2})$ or $f^{-1}(x) = \ell n(x + \sqrt{1 + x^2})$

Illustration 86:

Find the inverse of the function $f(x)=\log_a\left(x+\sqrt{(x^2+1)}\right);a>1$ and assuming it to be an onto function.

Solution:

Given $f(x)=\log_a\left(x+\sqrt{x^2+1}\right)$

$\therefore\ f'(x)=\dfrac{\log_a e}{\sqrt{(1+x^2)}}>0$

which is a strictly increasing functions.
Thus, $f(x)$ is injective, given that $f(x)$ is onto. Hence the given function $f(x)$ is invertible.
Interchanging x & y

$\Rightarrow\ \log_a\left(y+\sqrt{(y)^2+1}\right)=x$

$\Rightarrow\ y+\sqrt{(y)^2+1}=a^x$ ...(i)

and $\sqrt{(y)^2+1}-y=a^{-x}$ ...(ii)

From (i) and (ii), we get $y=\frac{1}{2}(a^x-a^{-x})$ or $f^{-1}(x)=\frac{1}{2}(a^x-a^{-x})$

Illustration 87:

Find the inverse of the function $f(x)=\ell n(x^2+3x+1);\ x\in\ [1,3]$ and assuming it to be an onto function.

Solution:

Given $f(x)=\ell n(x^2+3x+1)$

$\therefore\ f'(x)=\dfrac{2x+3}{(x^2+3x+1)}>0\ \forall\ x\in[1,3]$

Which is a strictly increasing function. Thus $f(x)$ is injective, given that $f(x)$ is onto. Hence the given function $f(x)$ is invertible.
Interchanging x & y

$\Rightarrow\ (y)^2+3(y)+1-e^x=0$

$\therefore\ y=\dfrac{-3\pm\sqrt{9-4.(1-e^x)}}{2}=\dfrac{-3\pm\sqrt{(5+4e^x)}}{2}$

$\Rightarrow\ \ y=\dfrac{-3+\sqrt{(5+4e^x)}}{2}$ (as $y\in[1,3]$)

Hence $f^{-1}(x)=\dfrac{-3+\sqrt{(5+4e^x)}}{2}$.

Illustration 88:

Let $f:\left[-\frac{3\pi}{4},\frac{\pi}{4}\right]\to Y$, $f(x)=\sin x+\cos x+2\sqrt{2}$ be invertible function then Y is :

Solution:

$f(x)=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)+2\sqrt{2}$

for $x\in\left[-\frac{3\pi}{4},\frac{\pi}{4}\right]\Rightarrow Y\in\left[\sqrt{2},3\sqrt{2}\right]$

Illustration 89:

Let $f(x)=\dfrac{x^3}{3}+\dfrac{x^2}{2}+ax+b \quad \therefore \forall x \in R$ it is invertible if

(A) $a \in \left[\dfrac{1}{4},\infty\right), b \in R$

(B) $a \in \left[\dfrac{1}{8},\infty\right), b \in R$

(C) $a \in \left(-\infty,\dfrac{1}{4}\right], b \in R$

(D) $a \in \left(-\infty,\dfrac{1}{4}\right), b \in R$

Ans. (A)

Solution:

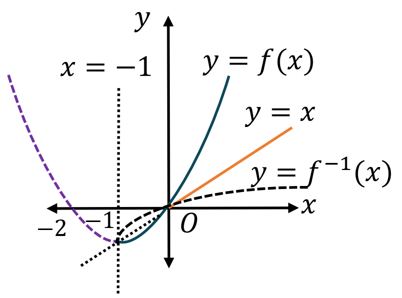
For $f(x)$ to be invertible $f(x)$ should be one-one that is $f'(x) \geq 0$,

$\Rightarrow a \in \left[\dfrac{1}{4},\infty\right)$

Illustration 90:

Let $f(x) = x^2 + 2x;\ x \geq -1$. Draw graph of $f^{-1}(x)$ also find the number of solutions of the equation, $f(x) = f^{-1}(x)$

Solution:



$f(x) = f^{-1}(x)$ is equivalent to $f(x) = x$

$\Rightarrow x^2 + 2x = x \Rightarrow x(x+1) = 0 \Rightarrow x = 0, -1$

Hence two solutions for $f(x) = f^{-1}(x)$

Illustration 91:

If $y = f(x) = x^2 - 3x + 1, x \geq 2$. Find the value of $g'(1)$ where g is inverse of f

Solution:

$y = 1 \Rightarrow x^2 - 3x + 1 = 1$

$\Rightarrow x(x-3) = 0 \Rightarrow x = 0, 3$

But $x \geq 2 \therefore x = 3$

Now $g(f(x)) = x$

Differentiating both sides $w.r.t.x$

$\Rightarrow g'(f(x)).f'(x) = 1$

$\Rightarrow g'(f(x)) = \dfrac{1}{f'(x)} \Rightarrow g'(f(3)) = \dfrac{1}{f'(3)}$

$\Rightarrow g'(1) = \dfrac{1}{6-3}$

$= \dfrac{1}{3}$ = (As $f'(x) = 2x - 3$)

Alternate Method

$y = x^2 - 3x + 1$

$x^2 - 3x + 1 - y = 0$

$$x = \frac{3 \pm \sqrt{9 - 4(1-y)}}{2} = \frac{3 \pm \sqrt{5+4y}}{2}$$

$x \geq 2$

$$x = \frac{3 + \sqrt{5+4y}}{2}$$

$$g(x) = \frac{3 + \sqrt{5+4x}}{2}$$

$$g'(x) = 0 + \frac{1}{4\sqrt{5+4x}}4$$

$$g'(1) = \frac{1}{\sqrt{5+4}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

Illustration 92:

If $f(x) = \begin{bmatrix} x & , & x < 1 \\ x^2 & , & 1 \leq x \leq 4 \\ 8\sqrt{x} & , & x > 4 \end{bmatrix}$, then find $f^{-1}(x)$.

Ans. $f^{-1}(x) = \begin{cases} x & , & x < 1 \\ \sqrt{x} & , & 1 \leq x \leq 16 \\ \dfrac{x^2}{64} & , & x > 16 \end{cases}$

Solution:

Case I $y = x \; x < 1$

$x = y y < 1$

$f^{-1}(x) = x, x < 1$

Case II $y = x^2 \; 1 \leq x \leq 4$

$x^2 = y \; 1 \leq y \leq 16$

$x = \sqrt{y} \; 1 \leq y \leq 16$

$f^{-1}(x) = \sqrt{x} \; 1 \leq x \leq 16$

Case III $y = 8\sqrt{x}x > 4$

$x = \dfrac{y^2}{64} y > 16$

$f^{-1}(x) = \dfrac{x^2}{64} x > 16$

Illustration 93:

Let f be a one–one function with domain $\{21, 22, 23\}$ and range $\{x, y, z\}$. It is given that exactly one of the following statements is true and the remaining two are false. $f(21) = x; f(22) \neq x; f(23) \neq y$. Then $f^{-1}(x)$ is :

Ans. (22)

Solution:

21 x

22 y

23 z

$f(21) = x f(22) \neq x f(23) \neq y$

case-I T F F

case-II F T F

case-III F F T

case-I $f(22) = x, f(23) = y$

then $f(21) = x$ is not true

case-II $f(23) = y, f(22) = z, f(21) = x$

not possible

case-III $f(22) = x, f(23) = z, f(21) = y$

$\therefore f^{-1}(x) = 22$

JEE ADVANCED PART–04:

Illustration 94:

Let $f : [-\sqrt{2} + 1, \sqrt{2} + 1] \to \left[\frac{-\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2}\right]$ be a function defined by $f(x) = \frac{1-x}{1+x^2}$.

If $f^{-1}(x) = \begin{cases} \frac{-1+\lambda\left(\sqrt{4x-4x^2+1}\right)}{2x}, & x \neq 0 \\ \mu, & x = 0 \end{cases}$, then $\lambda + \mu$ is.

Ans. (2)

Solution:

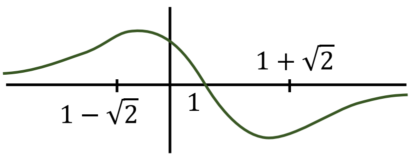
$f(x) = \frac{1-x}{1+x^2} \Rightarrow f'(x) = 0$ at $x = 1 \pm \sqrt{2}$

for $x \in \left[-\sqrt{2}+1, 1+\sqrt{2}\right]$ f is bijective function hence f is invertible.

$\frac{1-x}{1+x^2} = y$

Or $x^2 y + x + (y - 1) = 0$

Or $x = \frac{-1 \pm \sqrt{1-4y(y-1)}}{2y} = \frac{-1 \pm \sqrt{4y-4y^2+1}}{2y}$



$f^{-1}(x) = \begin{cases} \frac{-1+\sqrt{4x-4x^2+1}}{2x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ as $f(1) = 0$

$\Rightarrow \lambda = 1, \mu = 1 \Rightarrow \lambda + \mu = 2$

Illustration 95:

The number of real solutions of the equation $x^3 + 1 = 2\sqrt[3]{2x-1}$, is:

Ans. (3)

Solution:

$$\frac{x^3+1}{2} = \sqrt[3]{2x-1}$$

Let $f(x) = \frac{x^3+1}{2} \Rightarrow f^{-1}(x) = \sqrt[3]{2x-1}$

Equation becomes $f(x) = f^{-1}(x)$

$\Rightarrow f(x) = x \Rightarrow \frac{x^3+1}{2} = x \Rightarrow x^3 - 2x + 1 = 0 \Rightarrow (x-1)(x^2+x-1) = 0 \Rightarrow x = 1, \frac{-1\pm\sqrt{5}}{2}$

17. Odd and Even Functions:

Consider a function $f(x)$ such that both x and $-x$ are in its domain then

If $\begin{cases} f(-x) = f(x) \text{ then } f \text{ is said to be an even function} \\ f(-x) = -f(x) \text{ then } f \text{ is said to be an odd function} \end{cases}$

Note:

(i) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

(ii) A function may neither be odd nor even.

(iii) The only function which is defined on the entire number line & is even and odd at the same time is $f(x) = 0$.

(iv) Every constant function is even function.

(v) Inverse of an even function is not defined.

(vi) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

Special Note:

If a function $f(x)$ is defined as $f(a+x) = f(a-x)$ then this function is symmetric about line $x = a$

(vii) Every function which has '$-x$' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function.

i.e. $$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\textbf{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\textbf{ODD}}$$

(viii) If $f(x)$ is odd and defined at $x = 0$, then $f(0) = 0$.

f(x)	g(x)	f(x) + g(x)	f(x) – g(x)	f(x) . g(x)	f(x)/g(x)	(go$f$)(x)	($f$og)(x)
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Illustration 96:

Which of the following functions is (are) even, odd or neither:

(i) $f(x) = x^2 \sin x$ (ii) $f(x) = \sin x - \cos x$ (iii) $f(x) = \frac{e^x + e^{-x}}{2}$

Solution:

(i) $f(-x) = (-x)^2\sin(-x) = -x^2\sin x = -f(x)$. Hence $f(x)$ is odd.

(ii) $f(-x) = \sin(-x) - \cos(-x) = -\sin x - \cos x$.

Hence $f(x)$ is neither even nor odd.

(iii) $f(-x) = \dfrac{e^{-x}+e^{-(-x)}}{2} = \dfrac{e^{-x}+e^{x}}{2} = f(x)$. Hence $f(x)$ is even

Illustration 97:

Identify the given functions as odd, even or neither:

$f(x) = \dfrac{x}{e^x-1} + \dfrac{x}{2} + 1$

Solution:

$f(x) = \dfrac{x}{e^x-1} + \dfrac{x}{2} + 1$

Clearly domain of $f(x)$ is $\mathbb{R} \sim \{0\}$. We have,

$$f(-x) = \frac{-x}{e^{-x}-1} - \frac{x}{2} + 1 = \frac{-e^x.x}{1-e^x} - \frac{x}{2} + 1 = \frac{(e^x-1+1)x}{(e^x-1)} - \frac{x}{2} + 1$$

$$= x + \frac{x}{e^x-1} - \frac{x}{2} + 1 = \frac{x}{e^x-1} + \frac{x}{2} + 1 = f(x)$$

Hence $f(x)$ is an even function.

Illustration 98:

Identify the given functions as odd, even or neither:

$f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$

Solution:

$f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$

Replacing x, y by zero, we get $f(0) = 2f(0) \Rightarrow f(0) = 0$

Replacing y by$-x$, we get $f(x) + f(-x) = f(0) = 0 \Rightarrow f(x = -f(-x)$

Hence $f(x)$ is an odd function.

Illustration 99:

Show that $\log\left(x+\sqrt{x^2+1}\right)$ is an odd function.

Solution:

Let $f(x) = \log\left(x+\sqrt{x^2+1}\right)$.

Then $f(x) = \log\left(-x+\sqrt{(-x)^2+1}\right)$

$$= \log\left(\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}\right) = \log\frac{1}{\sqrt{x^2+1}+x} = -\log\left(x+\sqrt{x^2+1}\right) = -f(x)$$

or $f(x) + f(-x) = 0$

Hence $f(x)$ is an odd function.

Illustration 100:

Show that $a^x + a^{-x}$ is an even function.

Solution:

Let $f(x) = a^x + a^{-x}$

Then $f(-x) = a^{-x} + a^{-(-x)} = a^{-x} + a^x = f(x)$.Hence $f(x)$ is an even function

Illustration 101:

If $f: R \to R, f(x) = x^5 + \sin^3 x + \left[\frac{5\cos^2 x + 7}{p}\right]$ is an odd function, then the value of p (where [.] denotes the greatest integer function) is:

(A) $[7, 12]$ (B) $(12, \infty)$ (C) $(0, 12]$ (D) $(-\infty, \infty)$

Ans. (B)

Solution:

For odd function $\left[\frac{5\cos^2 x + 7}{p}\right] = 0$

$\Rightarrow p > 5\cos^2 x + 7, \forall x \in R \Rightarrow p > 12$

Illustration 102:

$$f(x) = \begin{cases} x^2 \sin\frac{\pi x}{2} ; & |x| < 1 \\ x|x| ; & |x| \geq 1 \end{cases}$$, then $f(x)$ is:

(A) An even function (B) An odd function

(C) Neither even nor odd (D) Both even & odd

Ans. (B)

Solution:

$$f(-x) = \begin{cases} -x^2 \sin\left(\frac{\pi x}{2}\right); & |-x| < 1 \\ -x|x| ; & |x| \geq 1 \end{cases} = -f(x) \Rightarrow \text{odd function}$$

Illustration 103:

If $f(x) = 2017^x + 2017^{-x}, g(x) = x^2 - 1$ then $g(f(x))$ is

(A) An even function (B) An odd function

(C) Neither even nor odd (D) a periodic function

Ans. (A)

Solution:

$g(f(x)) = (f(x))^2 - 1$

$= (2017)^{2x} + (2017)^{-2x} + 2 - 1$

$= (2017)^{2x} + (2017)^{-2x} + 1 \geq 3$

$\Rightarrow g[f(x)]$ is an even function

Illustration 104:

Let $f(x) = ([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)x - (\tan x)\, sgn(x)$ be an even function $\forall x \in R$, then the sum of all possible values of $'3a'$ is

(where $[\cdot]$ denotes G.I.F. and $\{\cdot\}$ fractional part functional part function)

Ans. (35)

Solution:

$f(x) = ([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)x - \tan x(sgnx)$

for $x > 0$

$f(x) = ([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)x - \tan x$

for $x < 0$

$f(x) = ([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)\,x + \tan x$

Given that function is even function $\forall\, x \in R$

So $f(x) - f(-x) = 0\ \forall\, n \in R$

$2x^3([a]^2 - 5[a] + 4) - 2x(6\{a\}^2 - 5\{a\} + 1) = 0$

So this equation should be independent from x $\therefore$ coff. of x^3 & x will be zero.

$[a]^2 - 5[a] + 4 = 0; 6\{a\}^2 - 5\{a\} + 1 = 0$

$[a] = 1, 4; \{a\} = 1/2, 1/3$

$a = 1 + 1/2, 4 + 1/2, 1 + 1/3, 4 + 1/3 = 3/2, 9/2, 4/3, 13/3$

Sum $= 3/2 + 9/2 + 4/3 + 13/3 = 6 + 17/3 = 35/3$

Illustration 105:

If $f: [-2, 2] \to R$ where $f(x) = x^3 + \tan x + \left[\frac{x^2+1}{P}\right]$ is a odd function, then the value of parametric P, where [.] denotes the greatest integer function, can be

(A) $5 < P < 10$ (B) $P < 5$ (C) $P > 5$ (D) $P = 15$

Ans. (A,C,D)

Solution:

$g(x) = x^3 + \tan x + \left[\frac{x^2+1}{P}\right] \Rightarrow g(-x) = (-x)^3 + \tan(-x) + \left[\frac{(-x)^2+1}{P}\right]$

$\Rightarrow g(-x) = -x^3 - \tan x + \left[\frac{x^2+1}{P}\right] \Rightarrow g(x) + g(-x) = 0$

because $g(x)$ is a odd function

$\therefore \left(-x^3 - \tan x + \left[\frac{x^2+1}{P}\right]\right) + \left(-x^3 - \tan x + \left[\frac{x^2+1}{P}\right]\right) = 0$

$\Rightarrow 2\left[\frac{(x^2+1)}{P}\right] = 0 \Rightarrow 0 \le \frac{x^2+1}{P} < 1$

Now $x \in [-2, 2] \Rightarrow 0 < \frac{5}{P} < 1 \Rightarrow P > 5$

18. Periodic Functions:

A function $f(x)$ is called periodic if there exists a positive number T such that $f(x + T) = f(x) = f(x - T)$, for all values of x within the domain of f. Smallest positive T (if exists) is called fundamental period of function $f(x)$.

Note:

(i) Odd powers of $\sin x, \cos x, \sec x, \operatorname{cosec} x$ are periodic with period 2π.

(ii) Non-zero integral powers of $\tan x, \cot x$ are periodic with period π.

(iii) Non-zero even powers or modulus of $\sin x, \cos x, \sec x, \operatorname{cosec} x$ are periodic with period π.

(iv) $f(T) = f(0) = f(-T)$, where 'T' is the period.

(v) if $f(x)$ has a period T then $f(ax + b)$ has a period $T/|a| (a \neq 0)$.

Proof:

Let $f(x+T)=f(x)$ and $f[a(x+T')+b]=f(ax+b)$

$f(ax+b+aT')=f(ax+b)$

$f(y+aT')=f(y)=f(y+T)\Rightarrow T=aT'\Rightarrow T'=\frac{T}{a}$

(vi) If $f(x)$ & $g(x)$ are periodic with period T_1 & T_2 respectively, then a period (need not be fundamental) of $f(x)\pm g(x)$ is L. C. M. of (T_1, T_2).

(A) LCM of T_1&T_2 is defined when T_1/T_2 is rational.

(B) LCM of $\left\{\frac{a}{b},\frac{p}{q}\right\}=\frac{LCM \text{ of (a, p)}}{\text{HCF of (b, q)}}$ In case if .there exists a positive K such that $K<LCM$ of T_1 and T_2 and overall function repeats itself after every K, then fundamental period of the function will be K.

(vii) Every constant function is always periodic, whose fundamental period is undefined.

(viii) Periodic functions are non invertible.

Illustration 106:

Find the fundamental periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x)=e^{\ell n(\sin x)}+\tan^3 x-\operatorname{cosec}(3x-5)$

(ii) $f(x)=x-[x-b], b\in\mathbb{R}$

(iii) $f(x)=\frac{|\sin x+\cos x|}{|\sin x|+|\cos x|}$

(iv) $f(x)=\tan\frac{\pi}{2}[x]$

(v) $f(x)=\cos(\sin x)+\cos(\cos x)$

(vi) $f(x)=\frac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)}$

(vii) $f(x)=e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+........+|\cos n\pi|}$

Solution:

(i) $f(x)=e^{\ell n(\sin x)}+\tan^3 x-\operatorname{cosec}(3x-5)$

Period of $e^{\ell n\sin x}=2\pi, \tan^3 x=\pi$

$\operatorname{cosec}(3x-5)=\frac{2\pi}{3}$

$\therefore$ Period $=2\pi$

(ii) $f(x)=x-[x-b]=b+\{x-b\}$

$\therefore$ Period $=1$

(iii) $f(x)=\frac{|\sin x+\cos x|}{|\sin x|+|\cos x|}$

Since period of $|\sin x+\cos x|=\pi$ and period of $|\sin x|+|\cos x|$ is $\frac{\pi}{2}$. Hence $f(x)$ is periodic with π as its period

(iv) $f(x)=\tan\frac{\pi}{2}[x]$

$\tan\frac{\pi}{2}[x+T]=\tan\frac{\pi}{2}[x]\Rightarrow\frac{\pi}{2}[x+T]=n\pi+\frac{\pi}{2}[x]$

$\therefore T=2$

$\therefore$ Period $=2$

(v) Let $f(x)$ is periodic then $f(x+T) = f(x)$

$\Rightarrow \cos(\sin(x+T)) + \cos(\cos(x+T)) = \cos(\sin x) + \cos(\cos x)$

If $x = 0$ then $\cos(\sin T) + \cos(\cos T) = \cos(0) + \cos(1) = \cos\left(\cos\frac{\pi}{2}\right) + \cos\left(\sin\frac{\pi}{2}\right)$

On comparing $T = \frac{\pi}{2}$

(vi) $f(x) = \frac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)} = \frac{(1+\sin x)(1+\cos x)\sin x}{\cos x(1+\sin x)(1+\cos x)} \Rightarrow f(x) = \tan x$

Hence $f(x)$ has period π.

(vii) $f(x) = e^{x-[x]+|\cos\pi x|+|\cos 2\pi x|+........+|\cos n\pi x|}$

Period of $x - [x] = 1$

Period of $|\cos\pi x| = 1$

Period of $|\cos 2\pi x| = \frac{1}{2}$

....

Period of $|\cos n\pi x| = \frac{1}{n}$

So period of $f(x)$ will be $L.C.M.$ of all period = 1

Illustration 107:

Find the fundamental periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{x-[x]} + \sin x$

(ii) $f(x) = \sin\frac{\pi x}{\sqrt{2}} + \cos\frac{\pi x}{\sqrt{3}}$

(iii) $f(x) = \sin\frac{\pi x}{\sqrt{3}} + \cos\frac{\pi x}{2\sqrt{3}}$

Solution:

(i) Period of $e^{x-[x]} = 1$

period of $\sin x = 2\pi$

$\because$ $L.C.M.$ of rational and an irrational number does not exist.

$\therefore$ not periodic.

(ii) Period of $\sin\frac{\pi x}{\sqrt{2}} = \frac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}$

Period of $\cos\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

$\because$ $L.C.M.$ of two different kinds of irrational number does not exist.

$\therefore$ not periodic.

(iii) Period of $\sin\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

Period of $\cos\frac{\pi x}{2\sqrt{3}} = \frac{2\pi}{\pi/2\sqrt{3}} = 4\sqrt{3}$

$\because$ $L.C.M.$ of two similar irrational number exist.

$\therefore$ Periodic with period = $4\sqrt{3}$

Illustration 108:

Find period of the following functions

(i) $f(x) = \sin\frac{x}{2} + \cos\frac{x}{3}$

(ii) $f(x) = \{x\} + \sin x$, where {.}denotes fractional part function

Solution:

(i) Period of $\sin\frac{x}{2}$ is 4π while period of $\cos\frac{x}{3}$ is 6π.

Hence period of $\sin\frac{x}{2} + \cos\frac{x}{3}$ is 12π\{L. C. M. of 4 and 6 is 12\}

(ii) Period of $\sin x = 2\pi$

Period of $\{x\} = 1$

but L.C.M. of 2π and 1 is not possible as their ratio is irrational number

$\therefore$ it is aperiodic

Illustration 109:

Find period of the following functions

(i) $f(x) = \cos x . \cos 3x$

(ii) $f(x) = \sin\frac{3x}{2} - \cos\frac{x}{3} - \tan\frac{2x}{3}$

Solution:

(i) $f(x) = \cos x . \cos 3x = \frac{1}{2}(\cos 4x + \cos 2x)$

period of $f(x)$ is L.C.M. of $\frac{2\pi}{4}, \frac{2\pi}{2} = \pi$

(ii) Period of $f(x)$ is L.C.M. of $\frac{2\pi}{3/2}, \frac{2\pi}{1/3}, \frac{\pi}{2/3}$ = L.C.M. of $\frac{4\pi}{3}, 6\pi, \frac{3\pi}{2} = 12\pi$

Illustration 110:

If fundamental period of $f(x) = |\sin(ax)| + |\cos(ax)|$ is $\frac{\pi}{12}$ then a is :

(A) 3 (B) 6 (C) 12 (D) 1

Ans. (B)

Solution:

Period of $g(x) = |\sin(x)| + |\cos(x)|$ is $\frac{\pi}{2}$

Hence period of $f(x) = |\sin(ax)| + |\cos(ax)|$ will be $\frac{\pi}{2a}$

$\frac{\pi/2}{a} = \frac{\pi}{12} \Rightarrow a = 6$

Illustration 111:

Given a function $f_1(x) = e^{\tan\left\{\frac{x}{4}\right\}} + \cos\pi\left(\frac{(1-2[x])}{2}\right) + \sin\left(\frac{\pi[x]}{2}\right)$ whose fundamental period is p, (where {.} and [.] represent fractional part and greatest integral part functions respectively)

and $y = \sqrt{2p + \frac{p}{2}[x] - [x]^2}$ the domain of y is $[q, r)$

then on the basis of above information answer the following:

(i) The period p of $f_1(x)$ is:

(A) An irrational number (B) Prime number
(C) A composite number (D) Neither prime nor a composite number

Ans. (C)

Solution:

Period of $e^{\tan\left\{\frac{x}{4}\right\}}$ is 4

$\cos\pi\left(\frac{(1-2[x])}{2}\right) = 0 \quad \forall\, x \in R$

Period of $\sin\left(\frac{\pi[x]}{2}\right)$ is 4

$\therefore$ Period of $f_1(x)$ is 4, which is a composite number.

(ii) Value of $r - q - 1$ is equal to

(A) 6 (B) 7 (C) 8 (D) 9

Ans. (A)

Solution:

$\therefore p = 4$ then $y = \sqrt{8+2[x]-[x]^2}$

$\therefore -[x]^2 + 2[x] + 8 \geq 0$

$\therefore [x]^2 - 2[x] - 8 \leq 0$

i.e., $([x] - 4)([x] + 2) \leq 0$

$\therefore -2 \leq [x] \leq 4$

$\therefore -2 \leq x < 5$

$\therefore q = -2, r = 5$

$\therefore r - q - 1 = 5 + 2 - 1 = 6$

Illustration 112:

If the range of the function $f(x) = \left\{\frac{x}{4}\right\} + \cos\pi\left(\frac{(1-2[x])}{2}\right) + \sin\left(\frac{\pi[x]}{2}\right)$ is $\left[\frac{\alpha}{4},\frac{\beta}{4}\right) \cup \left[\frac{\gamma}{4},\frac{\delta}{4}\right) \cup \left[\frac{2\gamma+1}{4},\frac{\delta}{2}\right)$, (where {.} and [.] represent fractional part and greatest integer part functions respectively), then $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is

Ans. (15)

Solution:

Period of $\left\{\frac{x}{4}\right\}$ is 4

$\cos\pi\left(\frac{(1-2[x])}{2}\right) = 0 \quad \forall\, x \in R$

Period of $\sin\left(\frac{\pi[x]}{2}\right)$ is 4

$\therefore$ Period of $f(x)$ is 4. For periodic function $f(x)$ range can be calculated for $x \in [0, 4]$

If $x \in [0, 1)$; $f(x) = \frac{x}{4}, f(x) \in \left[0,\frac{1}{4}\right)$; If $x \in [1, 2)$; $f(x) = \frac{x}{4} + 1, f(x) \in \left[\frac{5}{4},\frac{3}{2}\right)$

If $x \in [2, 3)$; $f(x) = \frac{x}{4}, f(x) \in \left[\frac{2}{4},\frac{3}{4}\right)$; If $x \in [3, 4)$; $f(x) = \frac{x}{4} - 1, f(x) \in \left[-\frac{1}{4},0\right)$

$\therefore$ Range $\in \left[-\frac{1}{4},\frac{1}{4}\right) \cup \left[\frac{2}{4},\frac{3}{4}\right) \cup \left[\frac{5}{4},\frac{3}{2}\right)$

$\Rightarrow \alpha = -1,\ \beta = 1,\ \gamma = 2,\ \delta = 3$

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 15$