06 Differentiability

Introduction of Derivability:

Two-fold meaning of derivative

Geometrical meaning of derivative
Slope of the tangent drawn to the curve
At any point (if exists)

Physical meaning of derivative
Instantaneous rate of change of function

1. Meaning of Derivative :

The instantaneous rate of change of a function with respect to the dependent variable is called derivative. Let 'f' be a given function of one variable and let Δx denote a number (positive or negative) to be added to the number x. Let Δf denote the corresponding change of 'f' then $\Delta f = f(x + \Delta x) - f(x)$

$$\Rightarrow \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

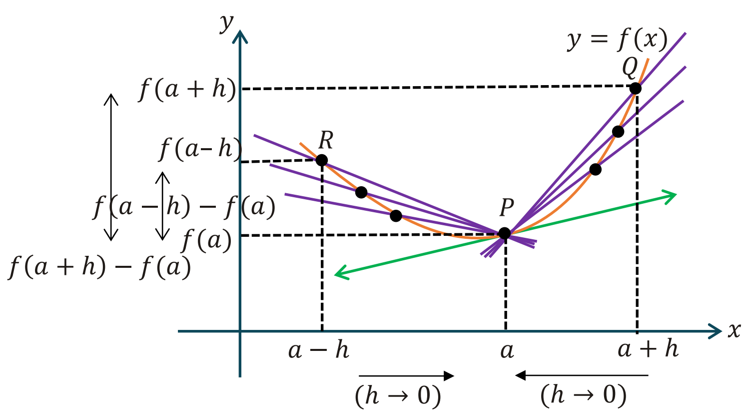
If $\Delta f / \Delta x$ approaches a limit as Δx approaches zero, this limit is the derivative of 'f' at the point x. The derivative of a function 'f' is a function; this function is denoted by symbols such as

$$f'(x), \frac{df}{dx}, \frac{d}{dx} f(x) \text{ or } \frac{df(x)}{dx}$$

$$\Rightarrow \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative evaluated at a point a, is written, $f'(a), \left.\frac{df(x)}{dx}\right|_{x=a}, f'(x)_{x=a}$, etc.

2. Existence of Derivative at $x = a$:



(a) Right hand derivative:

The right hand derivative of $f(x)$ at $x = a$ denoted by $Rf'(a)$ is defined as:

$Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided the limit exists & is finite. $(h > 0)$

(b) Left hand derivative:

The left hand derivative of $f(x)$ at $x = a$ denoted by $Lf'(a)$ is defined as:

$\mathrm{Lf}'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$, provided the limit exists & is finite. $(h > 0)$

Hence $f(x)$ is said to be **derivable or differentiable at $x = a$**.

If $\mathrm{R}f'(a) = \mathrm{Lf}'(a) =$ finite quantity

and it is denoted by $f'(a)$; where $f'(a) = \mathrm{Lf}'(a) = Rf'(a)$ & it is called derivative or differential coefficient of $f(x)$ at $x = a$.

3. Differentiability & Continuity:

Theorem: If a function $f(x)$ is derivable at $x = a$, then $f(x)$ is continuous at $x = a$.

Proof: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

Also $f(a+h) - f(a) = \frac{f(a+h) - f(a)}{h} . h \quad [h \neq 0]$

$\therefore \quad \lim_{h \to 0} [f(a+h) - f(a)] = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} . h = f'(a).0 = 0$

$\Rightarrow \quad \lim_{h \to 0} [f(a+h) - f(a)] = 0 \Rightarrow \lim_{h \to 0} f(a+h) = f(a) \Rightarrow f(x)$ is continuous at $x = a$.

Note :

(i)

Differentiable $\Rightarrow$ Continuous ; Continuous $\not\Rightarrow$ Differentiable ; Not Differentiable $\not\Rightarrow$ Not Continuous But Not Continuous $\Rightarrow$ Not Differentiable

(ii) All polynomial, rational, trigonometric, logarithmic and exponential function are continuous and differentiable in their domains.

(iii) If $f(x)$ & $g(x)$ are differentiable at $x = a$, then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).\, g(x)$ will also be differentiable at $x = a$ & if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be differentiable at $x = a$.

Illustration 1:

Determine the values of x for which the following functions fails to be continuous or differentiable

$$f(x) = \begin{cases} (1-x), & x < 1 \\ (1-x)(2-x), & 1 \le x \le 2 \\ (3-x), & x > 2 \end{cases}$$. Justify your answer.

Solution:

By the given definition it is clear that the function f is continuous and differentiable at all points except possibility at $x = 1$ and $x = 2$.

Check the differentiability at $x = 1$

$\Rightarrow q = \text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{1 - (1-h) - 0}{-h} = -1$

$\Rightarrow p$ = RHD = $\lim_{h\to 0}\dfrac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\dfrac{\{1-(1+h)\}\{2-(1+h)-0\}}{h}=-1$

$\because q = p$ $\therefore$ Differentiable at $x = 1 \Rightarrow$ Continuous at $x = 1$.

Check the differentiability at $x = 2$

$\Rightarrow q$ = LHD = $\lim_{h\to 0}\dfrac{f(2-h)-f(2)}{-h}=\lim_{h\to 0}\dfrac{(1-2+h)(2-2+h)-0}{-h}=1$ = finite

$\Rightarrow p$ = RHD = $\lim_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\dfrac{(3-2-h)-0}{h}\to\infty$ (not finite)

$\because q \neq p$ $\therefore$ not differentiable at $x = 2$.

Now we have to check the continuity at $x = 2$

$\Rightarrow$ LHL = $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(1-x)(2-x)=\lim_{h\to 0}(1-(2-h))(2-(2-h))=0$

$\Rightarrow$ RHL = $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(3-x)=\lim_{h\to 0}(3-(2+h))=1$

$\therefore$ LHL $\neq$ RHL

$\Rightarrow$ not continuous at $x = 2$.

Illustration 2:

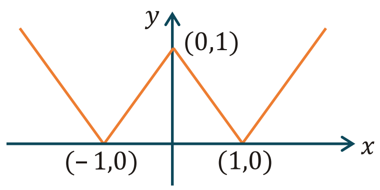
$f(x)=\left||x|-1\right|$ is not differentiable at

(A) 0 (B) ±1,0 (C) 1 (D) ±1

Ans. (B)

Solution:

$$=\begin{cases}|x|-1, & |x|-1\geq 0\\ -|x|+1, & |x|-1<0\end{cases}$$



$$=\begin{cases}|x|-1, & x\leq -1 \text{ or } x\geq 1\\ -|x|+1, & -1<x<1\end{cases}$$

$$=\begin{cases}-x-1, & x\leq -1\\ x+1, & -1<x<0\\ -x+1, & 0\leq x<1\\ x-1, & x\geq 1\end{cases}$$

From the graph. It is clear that $f(x)$ is not differentiable at $x = -1,0$ and 1.

Illustration 3:

For what triplets of real numbers (a, b, c) with $a \neq 0$ the function

$f(x)=\begin{cases} x \quad , & x \leq 1\\ ax^2+bx+c \quad , & \text{otherwise}\end{cases}$ is differentiable for all real x?

(A) $\{(a, 1-2a, a) | a \in R, a \neq 0\}$

(B) $\{(a, 1-2a, c) | a, c \in R, a \neq 0\}$

(C) $\{(a, b, c) | a, b, c \in R, a+b+c = 1\}$

(D) $\{(a, 1-2a, 0) | a \in R, a \neq 0\}$

Ans. (A)

Solution:

$$f(x)=\begin{cases} x & , \quad x\le 1 \\ ax^2+bx+c & , \quad \text{otherwise}\end{cases}$$

$\Rightarrow f(x)$ should be continuous at $x = 1$

it gives $a + b + c = 1$

$\Rightarrow f(x)$ should be differentiable at $x = 1$

it gives $2a + b = 1 \Rightarrow b = 1 - 2a$

$\Rightarrow c = 1 - a - b = a$

Important Note:

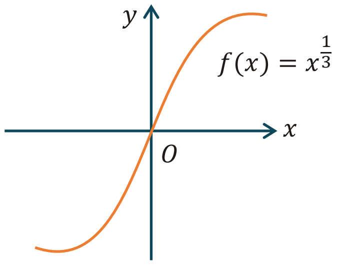
(a) Let $Rf'(a) = p$ & $Lf'(a) = q$ where p & q are finite then :

(i) $p = q \Rightarrow f$ is differentiable at $x = a \Rightarrow f$ is continuous at $x = a$

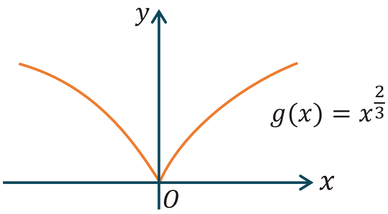
(ii) $p \neq q \Rightarrow f$ is not differentiable at $x = a$, but f is continuous at $x = a$.

(b) **Vertical tangent:**

(i) If $y = f(x)$ is continuous at $x = a$ and $\lim_{x\to a}|f'(x)|$ approaches to ∞, then $y = f(x)$ has a vertical tangent at $x = a$. If a function has vertical tangent at $x = a$ then it is non differentiable at $x = a$.



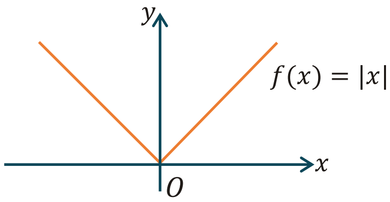
e.g. (1) $f(x) = x^{\frac{1}{3}}$ has vertical tangent at $x = 0$ since $f_+'(0) \to \infty$ and $f_-'(0) \to \infty$ hence $f(x)$ is not differentiable at $x = 0$



(2) $g(x) = x^{\frac{2}{3}}$ have vertical tangent at $x = 0$ since $g_+'(0) \to \infty$ and $g_-'(0) \to -\infty$ hence $g(x)$ is not differentiable at $x = 0$.

(c) **Geometrical interpretation of differentiability:**

(i) If the function $y = f(x)$ is differentiable at $x = a$, then a unique non vertical tangent can be drawn to the curve $y = f(x)$ at the point $P(a, f(a))$ & $f'(a)$ represent the slope of the tangent at point P.



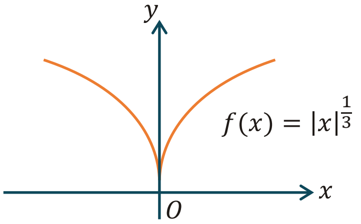
(ii) If a function $f(x)$ does not have a unique tangent (p & q are finite but unequal), then f is continuous at $x = a$, it geometrically implies a corner at $x = a$.

e.g. $f(x) = |x|$ is continuous but not differentiable at $x = 0$ & there is corner at $x = 0$.

(does not have unique tangent, corner at $x=0$) $\begin{cases} p = 1 \\ q = -1 \end{cases}$

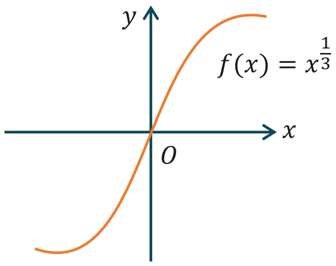
(iii) If one of p & q tends to ∞ and other tends to $-\infty$, then their will be a cusp at $x = a$. Where $p = Rf'(a)$ and $q = Lf'(a)$

e.g. (1) $f(x) = |x|^{\frac{1}{3}}$ is continuous but not differentiable at $x = 0$ & there is cusp at $x = 0$.



(has a vertical tangent, cusp at $x = 0$) $\begin{cases} p \to -\infty \\ q \to +\infty \end{cases}$

(2) $f(x) = x^{\frac{1}{3}}$ is continuous but not differentiable at $x = 0$ because $Rf'(0) \to \infty$ and $Lf'(0) \to \infty$.



(have a unique vertical tangent but does not have corner) $\begin{cases} p \to +\infty \\ q \to +\infty \end{cases}$

Note: corner/vertical tangent $\Rightarrow$ non differentiable

non differentiable $\not\Rightarrow$ corner/vertical tangent

Illustration 4:

$f(x) = \begin{cases} [\cos \pi x] & x \le 1 \\ 2\{x\} - 1 & x > 1 \end{cases}$ comment on the derivability at $x = 1$, where [.] denotes greatest integer function & {.} denotes fractional part function.

Solution:

$\Rightarrow Lf'(1) = \lim\limits_{h \to 0} \dfrac{f(1-h) - f(1)}{-h} = \lim\limits_{h \to 0} \dfrac{[\cos(\pi - \pi h)] + 1}{-h} = \lim\limits_{h \to 0} \dfrac{-1+1}{-h} = 0$

$\Rightarrow Rf'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0} \dfrac{2\{1+h\} - 1 + 1}{h} = \lim\limits_{h \to 0} \dfrac{2h}{h} = 2$

Hence $f(x)$ is not differentiable at $x = 1$.

Illustration 5:

If $f(x) = \begin{cases} x - 3 & x < 0 \\ x^2 - 3x + 2 & x \ge 0 \end{cases}$. Draw the graph of the function & discuss the continuity and differentiability of $f(|x|)$ and $|f(x)|$.

Solution:

$f(|x|) = \begin{cases} |x| - 3; & |x| < 0 \to \text{not possible} \\ |x|^2 - 3|x| + 2; & |x| \ge 0 \end{cases}$

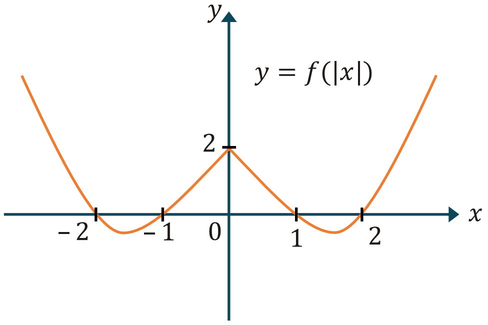
$\Rightarrow f(|x|) = \begin{cases} x^2 + 3x + 2, & x < 0 \\ x^2 - 3x + 2, & x \ge 0 \end{cases}$

at $x = 0$

$\Rightarrow q$=LHD=$\lim\limits_{h\to 0}\dfrac{f(0-h)-f(0)}{-h} = \lim\limits_{h\to 0}\dfrac{h^2-3h+2-2}{-h} = 3$

$\Rightarrow p$=RHD=$\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0}\dfrac{h^2-3h+2-2}{h} = -3$

$\Rightarrow q \neq p$



$\therefore$ not differentiable at $x = 0$. but p & q are both are finite

$\Rightarrow$ continuous at $x = 0$

Now, $|f(x)| = \begin{cases} 3-x, & x<0 \\ (x^2-3x+2), & 0 \le x < 1 \\ -(x^2-3x+2), & 1 \le x \le 2 \\ (x^2-3x+2), & x > 2 \end{cases}$

To check differentiability at $x = 0$

$\left.\begin{array}{l} q = \text{LHD} = \lim\limits_{h\to 0}\dfrac{f(0-h)-f(0)}{-h} = \lim\limits_{h\to 0}\dfrac{3+h-2}{-h} = \lim\limits_{h\to 0}\dfrac{(1+h)}{-h} \to -\infty \\ p = \text{RHD} = \lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0}\dfrac{h^2-3h+2-2}{h} = -3 \end{array}\right\} \Rightarrow$ not differentiable at $x = 0$

Now to check continuity at $x = 0$

$\left.\begin{array}{l} \text{LHL} = \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 3-x = 3 \\ \text{RHL} = \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x^2-3x+2 = 2 \end{array}\right\} \Rightarrow$ not continuous at $x = 0$

To check differentiability at $x = 1$

$\Rightarrow q = \text{LHD} = \lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{-h}$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{(1-h)^2-3(1-h)+2-0}{-h} = \lim\limits_{h\to 0}\dfrac{h^2+h}{-h} = -1$

$\Rightarrow p$ = RHD = $\lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0}\dfrac{-(h^2+2h+1-3+3h+2)-0}{h} = \lim\limits_{h\to 0}\dfrac{-(h^2-h)}{h} = 1$

$\Rightarrow$ not differentiable at $x = 1$.

but $|f(x)|$ is continuous at $x = 1$, because $p \neq q$ and both are finite.

To check differentiability at $x = 2$

$\Rightarrow q = \text{LHD} = \lim\limits_{h\to 0}\dfrac{f(2-h)-f(2)}{-h}$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{-(4+h^2-4h-6+3h+2)-0}{-h} = \lim\limits_{h\to 0}\dfrac{h^2-h}{h} = -1$

$\Rightarrow p = \text{RHD} = \lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0}\dfrac{(h^2+4h+4-6-3h+2)-0}{h} = \lim\limits_{h\to 0}\dfrac{(h^2+h)}{h} = 1$

$\Rightarrow$ not differentiable at $x = 2$.

but $|f(x)|$ is continuous at $x = 2$, because $p \neq q$ and both are finite.

Derivability Over an Interval:

$f(x)$ is said to be differentiable over an open interval (a, b) if it is differentiable at each & every point of the open interval (a, b)

$f(x)$ is said to be differentiable over the closed interval $[a, b]$ if

(i) $f(x)$ is differentiable in (a, b) &

(ii) for the points a and b, $f'_+(a)$ & $f'_-(a)$ exist.

Note:

(i) If $f(x)$ is differentiable at $x = a$ & $g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x).g(x)$ can still be differentiable at $x = a$.

e.g. Consider $f(x) = x$ & $g(x) = |x|$. f is differentiable at $x = 0$ & g is non-differentiable at $x = 0$, but $f(x).g(x)$ is still differentiable at $x = 0$.

(ii) If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function; $F(x) = f(x).g(x)$ can still be differentiable at $x = a$.

e.g. Consider $f(x) = |x|$ & $g(x) = -|x|$. f & g are both non-differentiable at $x = 0$, but $f(x).g(x)$ still differentiable at $x = 0$.

(iii) If $f(x)$ & $g(x)$ both are non-differentiable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function.

e.g. $f(x) = |x|$ & $g(x) =-|x|$. f & g are both non-differentiable at $x = 0$, but $(f + g)(x)$ still differentiable at $x = 0$.

(iv) If $f(x)$ is differentiable at $x = a \nRightarrow f'(x)$ is continuous at $x = a$.

e.g. $f(x)=\begin{bmatrix} x^2\sin\dfrac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{bmatrix}$

Illustration 6:

If $f(x) = \begin{cases} e^{-|x|}, & -5 < x < 0 \\ -e^{-|x-1|} + e^{-1} + 1, & 0 \le x < 2. \\ e^{-|x-2|}, & 2 \le x < 4 \end{cases}$ Discuss the continuity and differentiability of $f(x)$ in the interval $(-5, 4)$.

Solution:

$$f(x) = \begin{cases} e^{+x} & -5 < x < 0 \\ -e^{x-1} + e^{-1} + 1 & 0 \le x \le 1 \\ -e^{-x+1} + e^{-1} + 1 & 1 < x < 2 \\ e^{-x+2} & 2 \le x < 4 \end{cases}$$

Check the differentiability at $x = 0$

$$\Rightarrow \text{LHD} = \lim_{h\to 0}\frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0}\frac{e^{-h} - 1}{-h} = 1$$

$$\Rightarrow \text{RHD} = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{-e^{h-1} + e^{-1} + 1 - 1}{h} = -e^{-1}$$

$\Rightarrow$ LHD $\neq$ RHD

$\therefore$ Not differentiable at $x = 0$, but continuous at $x = 0$ since LHD and RHD both are finite.

Check the differentiability at $x = 1$

$$\Rightarrow \text{LHD} = \lim_{h\to 0}\frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0}\frac{-e^{1-h-1} + e^{-1} + 1 - e^{-1}}{-h} = -1$$

$\Rightarrow$ RHD = $\lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h}=\lim\limits_{h\to 0}\dfrac{-e^{1-h-1}+e^{-1}+1-e^{-1}}{h}=1$

$\Rightarrow$ LHD $\neq$ RHD

$\therefore$ Not differentiable at $x=1$, but continuous at $x=1$ since LHD and RHD both are finite.

Check the differentiability at $x=2$

$\Rightarrow$ LHD=$\lim\limits_{h\to 0}\dfrac{f(2-h)-f(2)}{-h}=\lim\limits_{h\to 0}\dfrac{-e^{-2+h+1}+e^{-1}+1-1}{-h}=\lim\limits_{h\to 0}\dfrac{-e^{-1}(e^{h}-1)}{-h}=e^{-1}$

$\Rightarrow$ RHD=$\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=\lim\limits_{h\to 0}\dfrac{e^{-h}-1}{h}=-1$

$\Rightarrow$ LHD $\neq$ RHD

$\therefore$ Not differentiable at $x=2$, but continuous at $x=2$ since LHD & RHD both are finite.

Illustration 7:

Number of points of non-differentiability of the function

$g(x)=[x^2]\{\cos^2 4x\}+\{x^2\}[\cos^2 4x]+x^2\sin^2 4x+[x^2][\cos^2 4x]+\{x^2\}\{\cos^2 4x\}$

In $(-50, 50)$ where $[x]$ and $\{x\}$ denotes the greatest integer function and fractional part function of x respectively, is equal to :-

(A) 98 (B) 99 (C) 100 (D) 0

Ans. (D)

Solution:

$g(x)=\{\cos^2 4x\}([x^2]+\{x^2\})+[\cos^2 4x]([x^2]+\{x^2\})+x^2\sin^2 4x$

(Using $x=[x]+\{x\}$)

$\Rightarrow g(x)=x^2\{\cos^2 4x\}+x^2[\cos^2 4x]+x^2\sin^2 4x$

$\Rightarrow g(x)=x^2(\{\cos^2 4x\}+[\cos^2 4x])+x^2\sin^2 4x$

$\Rightarrow g(x)=x^2(\cos^2 4x+\sin^2 4x)$

$\Rightarrow g(x)=x^2$

$g(x)$ is always differentiable.

Illustration 8:

The function $f(x)=(x^2-1)|x^2-3x+2|+\cos(|x|)$ is NOT differentiable at :

(A) –1 (B) 0 (C) 1 (D) 2

Ans. (D)

Solution:

$f(x)=(x^2-1)|(x-1)(x-2)|$

Clearly not differentiable at $x=2$

Illustration 9:

Let S denotes the set of all points where $\sqrt[5]{x^2|x|^3}-\sqrt[3]{x^2|x|}-1$ is not differentiable then S is a subset of –

(A) {0, 1} (B) {0, 1, –1} (C) {0, 1} (D) {0}

Ans. (A,B,C,D)

Solution:

$f(x) = \sqrt[5]{|x|^5} - \sqrt[3]{|x|^3} - 1$

$= |x| - |x| - 1$

$f(x)$ is always differentiable $\forall\, x \in \mathbb{R}$

Important Concepts:

Let $f(x)$ be a function whose derivative is $f'(x)$ then $\lim\limits_{y \to x} \frac{f(x)-f(y)}{x-y} = f'(x)$

Determination of Function Satisfying the Given Functional Equation:

Write down the expression for $f'(x)$ as $f'(x) = \frac{f(x+h)-f(x)}{h}$

Manipulate $f(x+h) - f(x)$ in such a way that the given functional rule is applicable. Now apply the functional rule and simplify the R.H.S to get $f'(x)$ as a function of x along with constant if any Integrate $f'(x)$ to get $f(x)$ as a function of x and a constant of integration. In some cases a Differential Equation is formed which can be solved to get $f(x)$ Apply the boundary value conditions to determine the value of the constant

Illustration 10:

A derivable function $f: \mathbb{R}^+ \to \mathbb{R}$ satisfies the condition $f(x) - f(y) \ge \ell n\left(\frac{x}{y}\right) + x - y$ for every $x, y \in \mathbb{R}^+$.

If g denotes the derivative of f then compute the value of the sum $\sum\limits_{n=1}^{100} g\left(\frac{1}{n}\right)$.

Solution:

Let $x = y + h$, then

$f(y+h) - f(y) \ge \ell n(y+h) - \ell n y + h$

$\Rightarrow \lim\limits_{h\to 0} \frac{f(y+h)-f(y)}{h} \ge \lim\limits_{h\to 0} \frac{\ell n(y+h) - \ell n y}{h} + 1$

$\Rightarrow f'(y) \ge \frac{d}{dx}(\ell n y) + 1$ ...(i)

Put $y = x + h$ then

$\Rightarrow f(x) - f(x+h) \ge \ell n(x) - \ell n(x+h) - h$

$\Rightarrow f(x+h) - f(x) \le \ell n(x+h) - \ell n x + h$

$\Rightarrow \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} \le \lim\limits_{h\to 0} \frac{\ell n(x+h) - \ell n x}{h} + 1$

$\Rightarrow f'(x) \le \frac{d}{dx}(\ell n x) + 1 = \frac{1}{x} + 1$ ...(ii)

From (i) & (ii)

$\Rightarrow f'(x) = \frac{1}{x} + 1 = g(x) \Rightarrow g\left(\frac{1}{n}\right) = n + 1$

$\Rightarrow \sum\limits_{n=1}^{100} g\left(\frac{1}{n}\right) = \sum\limits_{n=1}^{100} (n+1)$

$\Rightarrow \frac{100 \times 101}{2} + 100$

$\Rightarrow 5050 + 100$

$\Rightarrow 5150$

Illustration 11:

If $\lim\limits_{x\to 0}\frac{f(3-\sin x)-f(3+x)}{x}=8$, then $|f'(3)|$ is

Solution:

$$\lim_{x\to 0}\frac{f(3-\sin x)-f(3)-(f(3+x)-f(3))}{x}$$

$\Rightarrow -f'(3)-f'(3)=8 \Rightarrow f'(3)=-4$

$\therefore |f'(3)|=4$

Illustration 12:

Let f be differentiable at $x=0$ and $f'(0)=1$. Then $\lim\limits_{h\to 0}\frac{f(h)-f(-2h)}{h}=$

(A) 3 (B) 2 (C) 1 (D) –1

Ans. (A)

Solution:

$$\lim_{h\to 0}\frac{f(h)-f(-2h)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}+\frac{2(f(0-2h)-f(0))}{(-2h)}$$

$\Rightarrow f'(0)+2f'(0)=3f'(0) = 3$

Illustration 13:

If $2x+3|y|=4y$, then y as a function of x i.e. $y=f(x)$, is -

(A) discontinuous at one point

(B) non-differentiable at one point

(C) discontinuous & non-differentiable at same point

(D) continuous & differentiable everywhere

Ans. (B)

Solution:

(I) Let $y\geq 0$

$2x+3y=4y$

$y=2x \Rightarrow x\geq 0$

(II) Let $y<0$

$2x-3y=4y$

$7y=2x$

$y=\frac{2x}{7}\Rightarrow x<0$

$$\Rightarrow y=\begin{cases}2x; & x\geq 0\\ \frac{2x}{7}; & x<0\end{cases}$$

$\Rightarrow$ Non-differentiable at $x=0$

Illustration 14:

Let $\mathbb{R}$ be the set of real numbers and $f: \mathbb{R} \to \mathbb{R}$, be a differentiable function such that $|f(x) - f(y)| \leq |x - y|^3 \ \forall \ x, y \in \mathbb{R}$. If $f(10) = 100$, then the value of $f(20)$ is equal to -

(A) 0 (B) 10 (C) 20 (D) 100

Ans. (D)

Solution:

Replace x by $x + h$ & y by x where $h \to 0$

$\Rightarrow \lim_{h\to 0} |f(x+h) - f(x)| \leq |h|^3$

$\Rightarrow \lim_{h\to 0} \left|\frac{f(x+h) - f(x)}{h}\right| \leq |h^2|$

$\Rightarrow |f'(x)| \leq 0 \Rightarrow f'(x) = 0; f(x) = \text{constant}$

$\Rightarrow f(x) = 100$

So $f(20) = 100$

Illustration 15:

Let $f(x)$ be a differentiable function such that $2f(x + y) + f(x - y) = 3f(x) + 3f(y) + 2xy \ \forall \ x, y \in \mathbb{R}$ & $f'(0) = 0$, then $f(10) + f'(10)$ is equal to

Solution:

$2f(x + y) + f(x - y) = 3f(x) + 3f(y) + 2xy$

Diff. w.r.t 'y' we get

$\Rightarrow 2f'(x + y) - f'(x - y) = 3f'(y) + 2x$

Also $f(0) = 0$ So, put $y = 0$.

$\Rightarrow 2f'(x) - f'(x) = 3f'(0) + 2x$

$\Rightarrow f'(x) = 2x \Rightarrow f(x) = x^2 + \lambda$

$\Rightarrow f(0) = 0 \Rightarrow \lambda = 0$

$\Rightarrow f(x) = x^2$

$\Rightarrow f(10) = 100 \quad f'(10) = 20$

$\Rightarrow f(10) + f'(10) = 120$