Differentiability

SOLUTIONS

EXERCISE - O

1. **Ans. (A)**

$$\lim_{h\to 0}\frac{f(h)-f(-2h)}{h}=?$$

$$=\lim_{h\to 0}\frac{f(h)-f(0)-f(-2h)+f(0)}{h}$$

$$=\lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}-\frac{f(-2h)-f(0)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}\right)+2\lim_{h\to 0}\left(\frac{f(-2h)-f(0)}{-2h}\right)$$

$= f'(0) + 2f'(0) = 1 + 2(1) = 3.$

2. **Ans. (A)**

$f(x) = (x^5 + 1)|(x + 1)(x - 5)| + \sin|x| + \cos(x - 1)$

So $f(x)$ is not differentiable

at $x = 5$ & $x = 0$.

3. **Ans. (D)**

$f(x)$ is continuous at $x = 1$

$\therefore f(1^+) = f(1^-) = f(1)$

$\Rightarrow -(1)^3 + 2(1)^2 + a(1) = 1^3 + 2(1)^2$

$\Rightarrow -1 + 2 + a = 3$

$\Rightarrow a = 2$

Also $f(x)$ is differentiable at $x = 1$

$$\therefore\ f'(x)=\begin{cases}3x^2+4x & ; \quad x\in Q\\ -3x^2+4x+a & ; \quad x\notin Q\end{cases}$$

$f'(1) = f'(1^-) = f'(1^+)$

$\Rightarrow 3(1)^2 + 4(1) = -3(1)^2 + 4(1) + a$

$\Rightarrow 3 + 4 = -3 + 4 + a$

$\Rightarrow a = 6$

$\therefore$ Here is no unique value of 'a'

so answer not possible.

4. **Ans. (D)**

Replace x by $x + h$ & y by x where $h \to 0$

$\lim_{h\to 0}|f(x + h) - f(x)| \le |h|^3$

$$\lim_{x\to 0}\left|\frac{f(x+h)-f(x)}{h}\right|\le\left|h^2\right|$$

$\Rightarrow |f'(x)|\le 0$

$\Rightarrow f(x) = 100 \qquad (\because f(10) = 100)$

So $f(20) = 100$.

5. Ans. (D)

$f(x) = (x^2 - 1)|(x - 1)(x - 2)| + \cos x$

clearly not differentiable at $x = 2$

6. Ans. (A)

$f(x)$ is differentiable at $x = 0$

RHD = LHD

$f'(0^+) = f'(0^-)$

If $f(x)$ is differentiable then it is continuous also

$$f'(x)=\begin{cases}2xe^{x^2}+3x^2 & x>0\\ 2ax+b & x\le 0\end{cases}$$

$$\left.\begin{matrix}f'(0^+)=0\\ f'(0^-)=b\end{matrix}\right]\text{LHD = RHD}$$

$b = 0, a \in R.$

7. Ans. (B)

In $[0, 2\pi)$, $f(x)$ is defined as :

$$f(x)=\begin{cases}\sqrt{\sin x} & ,0\le x<\frac{\pi}{2}\\ 1 & ,x=\frac{\pi}{2}\\ \sqrt{\sin x} & ,\frac{\pi}{2}<x\le\pi\\ -1+\sqrt{\sin x+1} & ,\pi<x<2\pi\end{cases}$$

$$\& \quad f'(x)=\begin{cases}\dfrac{\cos x}{2\sqrt{\sin x}} & ,0<x<\frac{\pi}{2}\\ \dfrac{\cos x}{2\sqrt{\sin x}} & ,\frac{\pi}{2}<x<\pi\\ \dfrac{\cos x}{2\sqrt{\sin x+1}} & ,\pi<x<2\pi\end{cases}$$

$f(x)$ is continuous at $x=\frac{\pi}{2}$ & π

Also $f(x)$ is differentiable at $x=\frac{\pi}{2}$ but not differentiable at $x = \pi$.

8. Ans. (D)

$$f(x)=\begin{cases}\sin^{-1}x+a\cos\pi x & -1<x<0\\ b\cos^{-1}x+\sin\pi x & 0\le x<1\end{cases}$$

$f(x)$ is differentiable in $(-1, 1)$ so $f(x)$ is continuous in $(-1, 1)$

$f(0) = f(0^+) = f(0^-)$

$a=\frac{b\pi}{2}$

$$f'(x)=\begin{cases}\dfrac{1}{\sqrt{1-x^2}}+(-a\pi\sin\pi x) & -1<x<0\\ \dfrac{-b}{\sqrt{1-x^2}}+\pi\cos\pi x & 0\le x<1\end{cases}$$

because $f(x)$ is differentiable in $(-1, 1)$

$f'(0^+) = f'(0^-)$

$1 = -b + \pi$

$b = \pi - 1$

$\Rightarrow 2a + b$

$\Rightarrow \frac{2(\pi)(\pi-1)}{2} + \pi - 1$

$= \pi^2 - 1$ **Ans. (D).**

9. **Ans. (D)**

$$\text{Sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

$$f(x) = \begin{cases} |x+1|, & x < 0 \\ |x|, & x = 0 \\ |x-1|, & x > 0 \end{cases}$$

$$f(x) = \begin{cases} -(x+1), & x \le -1 \\ x+1 & -1 < x < 0 \\ 0, & x = 0 \\ -(x-1), & 0 < x < 1 \\ (x-1), & x \ge 1 \end{cases}$$

Function is non-differentiable at every breaking point

i.e. $x = -1, 0, 1$

Number of points of non-differentiability are 3

10. **Ans. (A)**

$f(x)$ is differentiable at $x = 0$

So, LHD,

$$\Rightarrow f'(0^-) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$\Rightarrow f'(0^-) = \lim_{h\to 0} \frac{\frac{\ln\cos h}{-ah} - 0}{-h}$$

$$\Rightarrow \frac{1}{4} = \lim_{h\to 0} \frac{\ln\cos h}{ah^2}$$

apply L'Hospital rule.

$$\Rightarrow \frac{1}{4} = \lim_{h\to 0} \frac{1(-\sin h)}{a\cos h \times 2h}$$

$$\Rightarrow \frac{1}{4} = -\frac{1}{2a} \lim_{h\to 0} \frac{\sin h}{h}$$

$\Rightarrow 2a = -4 \Rightarrow a = -2$

RHD,

$$\Rightarrow f'(0^+) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$\Rightarrow f'(0^+) = \lim_{h\to 0} \frac{\frac{e^{h^2}-1}{bh} - 0}{h}$$

$$\Rightarrow \frac{1}{4} = \lim_{h\to 0} \frac{e^{h^2}-1}{bh^2}$$

$$\Rightarrow \frac{1}{4} = \frac{1}{b}\lim_{h\to 0} \frac{e^{h^2}-1}{h^2}$$

$$\Rightarrow \frac{1}{b} = \frac{1}{4} \quad \Rightarrow b = 4$$

$a + b = -2 + 4 = 2$

11. Ans. (B,C)

(a) At $x = 1, f(1) = 5$

R.H.L. at $x = 1 \Rightarrow 5$

$\Rightarrow y = f(x)$ is continuous at $x = 1$

$$\Rightarrow f'(x) = \begin{cases} 4x & , x \le 1 \\ 3 & , x > 1 \end{cases}$$

R.H.D. at $x = 1 \Rightarrow 3$

L.H.D. at $x = 1 \Rightarrow 4$

(b) $$f(x) = \begin{cases} x^3 & x \ge 0 \\ -x^3 & x < 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 3x^2 & , x > 0 \\ -3x^2 & , x < 0 \end{cases} ; f'(0) = 0$$

$$\Rightarrow f''(x) = \begin{cases} 6x & x > 0 \\ -6x & x < 0 \end{cases} ; f''(0) = 0$$

$\Rightarrow f'(0^+) = f'(0^-) = 0$

$\Rightarrow y = f(x)$ is derivable at $x = 0$

$\Rightarrow f''(0^+) = f''(0^-) = 0$

$\Rightarrow f(x)$ is twice differentiable at $x = 0$

(c) $f(5^+) = f(5^-) = f(5) = 2$

$$\Rightarrow \lim_{x\to 2} f(4x^2 - 11) = 2$$

(d) $$\lim_{x\to a}\left(f(x) + g(x)\right) = 2$$

$$\Rightarrow \lim_{x\to a}\left(f(x) - g(x)\right) = 1$$

$$\Rightarrow \lim_{x\to a}\left(f(x) + g(x)\right) + \left(f(x) - g(x)\right)$$

$$\Rightarrow \lim_{x\to a}\left(2f(x)\right) = \lim_{x\to a}\left(f(x) + g(x)\right) + \lim_{x\to a}\left(f(x) - g(x)\right)$$

$$\Rightarrow 2\left(\lim_{x\to a} f(x)\right) = 3$$

$$\Rightarrow \lim_{x\to a} f(x) = \frac{3}{2}$$

$$\Rightarrow \lim_{x\to a}\left(f(x) + g(x)\right) - \left(f(x) - g(x)\right) = \lim_{x\to a} 2g(x)$$

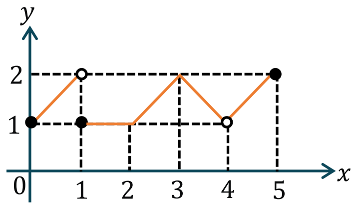
$\Rightarrow 2-1=2\lim_{x\to a} g(x)$

$\Rightarrow \lim_{x\to a} g(x)=\frac{1}{2}$

$\Rightarrow \lim_{x\to a} f(x).g(x)=\lim_{x\to a} f(x).\lim_{x\to a} g(x)$

$\Rightarrow \frac{3}{2}\times\frac{1}{2}=\frac{3}{4}$

12. Ans. (B,C)



for removable discontinuity at $x = a$,

LHL. = R.H.L. $\neq f(a)$

$\Rightarrow f(x)$ is discontinuous at $x = 1, 4$

$\Rightarrow f(x)$ has non-removable discontinuity at $x = 1$

$\Rightarrow f(x)$ is non-differentiable at $x = 1, 4, 3$

$\Rightarrow \lim_{x\to 1} f(f(x))$

R.H.L. $\lim_{x\to 1^+} f(f(x))=\lim_{x\to 1^+} f(1)=1$

L.H.L. $\lim_{x\to 1^-} f(f(x))=\lim_{x\to 1^-} f(2^-)=1$

$\Rightarrow \lim_{x\to 1} f(f(x))=1$

13. Ans. (A,B,C,D)

$f(x)=\sqrt[5]{x^2|x|^3}-\sqrt[3]{x^2|x|}-1$

For $x > 0$, $f(x)=\sqrt[5]{x^5}-\sqrt[3]{x^3}-1$

$= x - x - 1$

$= -1$

for $x < 0$, $f(x)=\sqrt[5]{-x^5}-\sqrt[3]{-x^3}-1$

$= (-x) - (-x) - 1$

$= -1$

for $x = 0, f(x) = -1$

$\Rightarrow y = f(x)$ is differentiable $\forall\, x \in R$

$\Rightarrow S = \phi$

14. Ans. (A,B,C,D)

(a) Let $f(x)=\begin{bmatrix} x+1, & x\in Q, & x\in[0,1] \\ -x-1, & x\notin Q, & x\in[0,1] \end{bmatrix}$

$|f(x)| = x + 1 \;\forall\, x \in R$

$y = f(x)$ is discontinuous at all $x \in [0,1]$

$y = |f(x)|$ is continuous at all $x \in [0,1]$

(b) $\lim_{h\to0}\frac{f(a+h)\cdot g(a+h)-f(a)\cdot g(a)}{h}$

$=\lim_{h\to0}\frac{f(a+h)\cdot g(a+h)}{h}$

$=\lim_{h\to0}g(a+h)\cdot\frac{(f(a+h)-f(a))}{h}$

$= g(a)\cdot f'(a)$ exists

$\Rightarrow y = f(x)\cdot g(x)$ is differentiable at $x = a$

(c) $f'(a^+) = 2$

$\Rightarrow y = f(x)$ is right continuous at $x = a$

$f'(a^-) = 3$

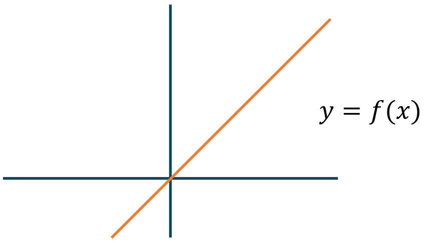
$\Rightarrow y = f(x)$ is left continuous at $x = a$

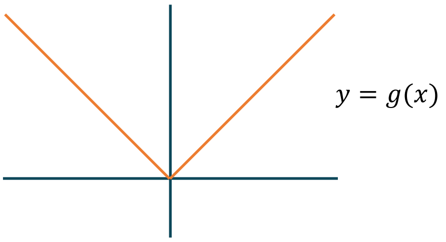
$\Rightarrow y = f(x)$ is continuous at $x = a$

(d) By intermediate value theorem, (D) is true.

15. **Ans. (B,C,D)**

Let





For options B, C and D consider above graphs.

16. **Ans. (A,B,D)**

$\phi(x) = [|x| - \sin|x|]$ is even function

$\phi(0) = 0$

$\phi(0^+) = 0$

$\phi(0^-) = 0$

$\Rightarrow y = \phi(x)$ is continuous at $x = 0$

In neighbourhood of $x = 0$

$\phi(x) = 0$

$\phi(x)$ acts as a constant function.

$\Rightarrow \phi'(x) = 0$ at $x = 0$.

17. **Ans. (A,C,D)**

$f(x+y) + f(x-y) = 2f(x)$

$\because\ (f(x) + f(y) = f(x+y))$

$\Rightarrow f(x) = kx$

$\Rightarrow f'(x) = k$

$\Rightarrow f'(1) = f'(2) = f'(3)$

it is odd function $(f(-x) = -f(x))$

18. Ans. (A,B)

LHL $\lim_{x\to1^-} f(0^+)=1-x\left(f(x)\to0^+\right)=1$

RHL $\lim_{x\to1^+} f(3^+)=4-x=1$

$f(1) = f(f(1)) = f(0) = 1$

LHD $f'f(1^-)\times f'(1^-) = f'(0^+)\times(-1)$

$= (-1)\times(-1) = 1$

RHD $f'(f(1^+))\times f'(1^+) = f'(3^+)\times(1)$

$= (-1)\times(1) = -1$

LHD $\neq$ RHD it is not differentiable at $x = 1$

Now at $x = 2$

$\text{LHL} = \lim_{x\to2^-} f(f(2^-)) = f(3^+) = 4-3 = 1$

$\text{RHL} = \lim_{x\to2^+} f(f(2^+)) = f(1^+) = 1+2 = 3$

LHL $\neq$ RHL

discontinuous at $x = 2$

Now at $x = 3$

$\text{LHL} = \lim_{x\to3^-} f(f(3^-)) = f(1^+) = 3$

$\text{RHL} = \lim_{x\to3^+} f(f(3^+)) = f(1^-) = 0$

discontinuous at $x = 3$

option (A, B) are correct.

19. Ans. (A,C)

$f(x)=e^{-|x|}$

$\Rightarrow x < 0$

$\Rightarrow f(x) = e^x (\because |x| = -x)$

$\Rightarrow (f'(0) = e^0 = 1)$

$$\Rightarrow \lim_{x\to0}\frac{1-\cos\left(2\sin^2\frac{x}{4}\right)}{2^m x^n}=1$$

$$\Rightarrow \lim_{x\to0}\left(\frac{1-\cos\left(2\sin^2\frac{x}{4}\right)}{\left(2\sin^2\frac{x}{4}\right)^2}\right)\times\frac{4\sin^4\frac{x}{4}}{\left(\frac{x}{4}\right)^4\times2^m x^n}\times\left(\frac{x}{4}\right)^4=1$$

$$\Rightarrow \frac{1}{2}\times\frac{x^{4-n}}{4^3\times2^m}=1 \Rightarrow n = 4$$

and $m + 7 = 0 \Rightarrow\ m = -7$

$|4 - 7(2)| = 10$, option (A,C) are correct.

20. **Ans. (C)**

$f(x) = tgn\ (x) \sin(x)\,|x|$

$\Rightarrow 0 \le x < 1, [x] = 0, f(x) = \sin x \times x$

$\Rightarrow 1 \le x < 2, [x] = 1, f(x) = -x \sin x$

$\Rightarrow 2 \le x < 3, [x] = 2, f(x) = x \sin x$

$\Rightarrow 3 \le x < 4, [x] = 3, f(x) = -x \sin x$

$\Rightarrow 4 \le x < 5, [x] = 4, f(x) = x \sin x$

$\Rightarrow 5 \le x < 6, [x] = 5, f(x) = x \sin x$

$\Rightarrow 6 \le x < 7, [x] = 6, f(x) = x \sin x$

$\Rightarrow 7 \le x < 8, [x] = 7, f(x) = -x \sin x$

$\Rightarrow 8 \le x < 9, [x] = 8, f(x) = x \sin x$

$\Rightarrow 9 \le x < 10, [x] = 9, f(x) = -x \sin x$

$\Rightarrow x = 10, [x] = 10, f(x) = x \sin x$

$f(x)$ is discontinuous at

$x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ (LHL $\neq$ RHL)

21. **Ans. (B)**

$f(x) = \text{tgn}\ (x) \sin x\,|x|$

$x = -10 f(x) = -x \sin x$

$-10 \le x < -9, f(x) = -x \sin x$

$-9 \le x < -8, f(x) = x \sin x$

$-8 \le x < -7, f(x) = -x \sin x$

$-7 \le x < -6, f(x) = x \sin x$

$-6 \le x < -5, f(x) = -x \sin x$

$-5 \le x < -4, f(x) = x \sin x$

$-4 \le x < -3, f(x) = -x \sin x$

$-3 \le x < -2, f(x) = x \sin x$

$-2 \le x < -1, f(x) = -x \sin x$

$-1 \le x < 0, f(x) = x \sin x$

$f(x)$ is discontinuous at $x = -9, -8, -7, -6, -5, -4, -3, -2, -1,$

$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

in $[-10, 10]$ (LHL $\neq$ RHL)

Here it is not diff.

22. **Ans. (A,C)**

$f(x)$ is continuous in $[-2, 10]$

$f(x)$ is continuous at $x = 4$

$\Rightarrow \text{LHL} = \lim\limits_{x\to 4^-}\left(x^2 - 5x + 6\right) = 2$

$\Rightarrow \text{RHL} = \lim\limits_{x\to 4^+}\left(k - \tan\left(\frac{\pi x}{4}\right)\right) = k$

$\Rightarrow f(4) = 2$

LHL = RHL = $f(4)$

$\Rightarrow \boxed{k = 2}$

$f(x)$ is continuous at $x = 5$

$\Rightarrow$ LHL $= \lim_{x \to 5^-} k - \tan\left(\frac{5\pi}{4}\right) = k - 1 = 1$

$\Rightarrow$ RHL $= \lim_{x \to 5^+} \log_{10}(\alpha x) = \log_{10}(5\alpha)$

$\Rightarrow f(5) = k - 1 = 1$

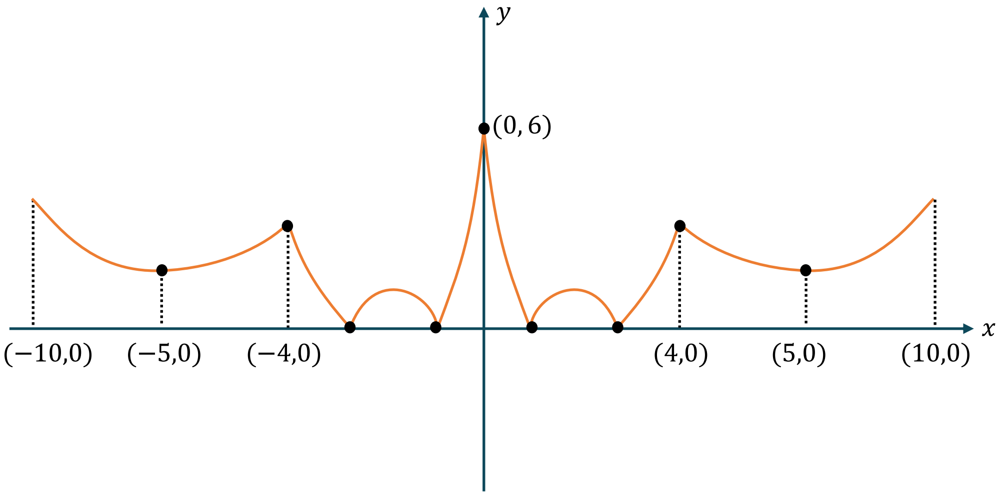
$\Rightarrow 1 = \log_{10}(5\alpha)$

$\Rightarrow 10 = 5\alpha$

$\Rightarrow \alpha = 2$

23. **Ans. (A)**

$f_1(x) = |f(|x|)|$



No. of points where $f_1(x)$ is not diff. = 9

24. **Ans. (C)**

(P) $\lim_{h \to 0} \frac{f(3+h^2) - f(3-h^2)}{(3+h^2)-(3-h^2)} = f'(3) = 2$

(Q) $f(x) = f(-x)$

diff. wrt (x)

$\Rightarrow f'(x) = -f'(-x)$

Put $x = 0$

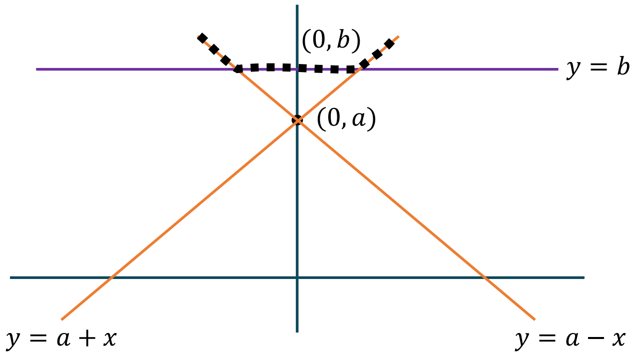
$\Rightarrow f'(0) = -f'(0)$

$\Rightarrow f'(0) = 0$

(R) $f'(0^-) = \lim_{h \to 0^+} \frac{f(0-h) - f(0)}{-h}$

$$\Rightarrow f'(0^-) = \lim_{h \to 0^+} \left(\frac{-h}{1+e^{-\frac{1}{h}}}\right)\left(-\frac{1}{h}\right) = \lim_{h \to 0^+} \frac{e^{\frac{1}{h}}}{e^{\frac{1}{h}}+1} = 1$$

(S)



clearly $f(x)$ is N.D. at two points.

EXERCISE - S

1. **Ans. (1)**

For continuity at $x = 0$

$$f(0) = LHL = \lim_{x \to 0^-} \frac{-x^2}{2} = 0$$

$$\text{RHL} = \lim_{x \to 0^+} x^n \sin\left(\frac{1}{x}\right) = 0 \Rightarrow n > 0 \qquad \text{...(i)}$$

For differentiability at $x = 0$

$$f'(0^-) = 0$$

$$f'(0^+) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} h^{n-1} \sin\left(\frac{1}{h}\right) \neq 0 \qquad [\because f(x) \text{ is non differentiable at } x = 0]$$

$$\Rightarrow n - 1 \leq 0 \Rightarrow n \leq 1 \qquad \text{...(ii)}$$

from (i) and (ii)

$n \in (0, 1]$.

2. **Ans. (2)**

$f(x) = |x - 1|([x] - [-x])$

RHD at $x = 1$

$$f'(1^+) = \lim_{h \to 0^+} \frac{|1+h-1|\left([1+h]-[-1-h]\right)-0}{h}$$

$$= \lim_{h \to 0^+} \frac{|h|\left(1-(-2)\right)}{h}$$

$$\lim_{h \to 0^+} \frac{(h)(3)}{(h)} = 3$$

LHD at $x = 1$

$$f'(1^-) = \lim_{h \to 0^+} \frac{|(1-h-1)|\left([1-h]-[-1+h]\right)}{-h}$$

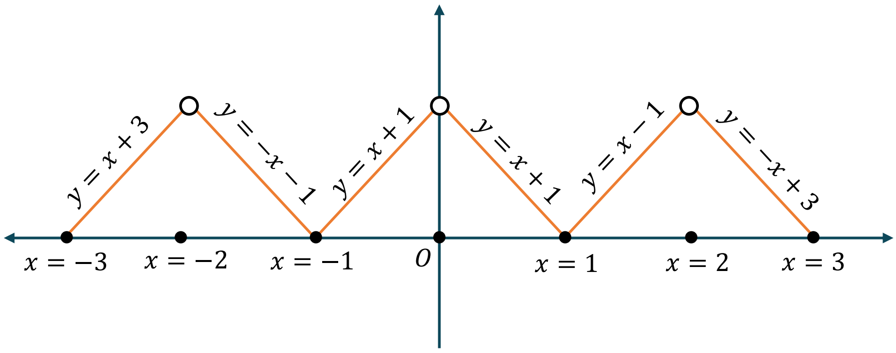
$$= \lim_{h \to 0^+} \frac{|-h|\left(0-(-1)\right)}{-h}$$

= – 1

$Rf'(1) + Lf'(1) = 3 - 1 = 2.$

3. Ans. (8)

$$f(x)=\begin{cases} x+3 & x\in[-3,-2) \\ -x-1 & x\in[-2,-1) \\ x+1 & x\in[-1,0) \\ -x+1 & x\in[0,1) \\ x-1 & x\in[1,2) \\ -x+3 & x\in[2,3) \\ x-3 & x=3 \end{cases}$$



$f(x)$ is discontinuous at $x = -2, 0$ & 2

So, $L = 3$

$f(x)$ is non differentiable at $x = -2, -1, 0, 1$ & 2

So, $M = 5 \Rightarrow L + M = 8$.

4. Ans. (5150)

$$f(x)-f(y)\geq \ell n\left(\frac{x}{y}\right)+x-y \qquad x, y \in \mathbb{R}^+$$

$$f(x) - f(y) \geq \ell nx - lny + x - y$$

$$\frac{f(x)-f(y)}{x-y}\geq \frac{\ell nx-\ell ny}{x-y}+1$$

$$\lim_{x\to y}\frac{f(x)-f(y)}{x-y}\geq \lim_{x\to y}\frac{\ell nx-\ell ny}{x-y}+1$$

$$f'(y)\geq \frac{1}{y}+1$$

$$f'(y)\geq \frac{y+1}{y} \qquad ...(1)$$

Now calculate for $y > x$ in similar way

$$f'(y)\leq \frac{y+1}{y} \qquad ...(2)$$

From 1 of (2) both equation

$$g(x)=1+\frac{1}{x} \qquad f'(y)=\frac{y+1}{y}$$

$$g\left(\frac{1}{n}\right)=1+n$$

$$\sum_{n=1}^{100} g\left(\frac{1}{n}\right) = 100 + \frac{100\times101}{2}$$

= 5150.

5. **Ans. (1)**

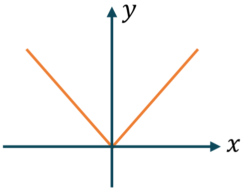
$f(x) = [[x] + \{x^2\}] + \{[x^2] + \{x\}\}$

$\Rightarrow f(x) = [x] + [\{x^2\}] + \{\{x\}\}$

$\Rightarrow f(x) = [x] + 0 + \{x\}$

$\Rightarrow f(x) = x$

graph of $|f(x)| = |x|$



at $x = 0$, $|f(x)|$ is non differentiable.

6. **Ans. (2)**

Put $y = 2x$ then

$$f(x) = \frac{f(x) + f(2x) + f(0)}{3}$$

$\Rightarrow 2f(x) = f(2x) + f(0)$

$\Rightarrow f(x) - f(0) = f(2x) - f(x)$

$\Rightarrow \lim_{x\to0} \frac{f(x) - f(0)}{x} = \lim_{x\to0} \frac{f(2x) - f(x)}{x}$

$\Rightarrow f'(0) = f'(x) = 2$

$\Rightarrow f(x) = 2x + C$ where C is constant.

7. **Ans. (4)**

Since, its $\frac{0}{0}$ forms, so, use L'Hospitals Rule,

$\lim_{x\to0} f'(3-\sin x)(-\cos x) - f'(3+x) = 8$

$\Rightarrow -f'(3) - f'(3) = 8$

$\Rightarrow |f'(3)| = 4.$

8. **Ans. (120)**

$2f(x + y) + f(x - y) = 3f(x) + 3f(y) + 2xy$

diff. w.r.t. 'y' we get.

$2f'(x + y) - f'(x - y) = 3f'(y) + 2x$

Also, $f'(0) = 0$ So, put $y = 0$

$2f'(x) - f'(x) = 3f'(0) + 2x$

$\Rightarrow f'(x) = 2x \Rightarrow f(x) = x^2 + \lambda$

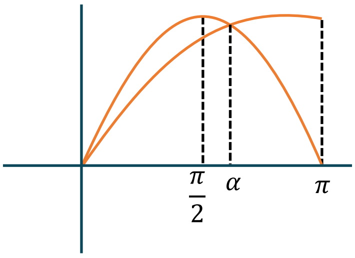
$\Rightarrow f(0) = 0 \Rightarrow \lambda = 0$

$\Rightarrow f(x) = x^2$

$\therefore f(10) = 100 f'(10) = 20$

$\therefore f(10) + f'(10) = 120.$

9. Ans. (3)



max(2 sin x , (− cos x)

$f(x)$ is non-differentiable at $x = \alpha$

where $2 \sin \alpha = 1 - \cos \alpha$

squaring $4 \sin^2 \alpha = 1 + \cos^2 \alpha - 2\cos \alpha$

$\Rightarrow 5 \cos^2 \alpha - 2 \cos \alpha - 3 = 0$

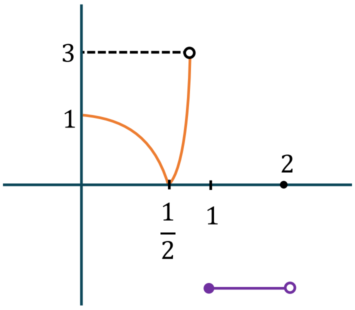
$\Rightarrow (5 \cos \alpha + 3)(\cos \alpha - 1) = 0$

$\Rightarrow \cos\alpha = -\frac{3}{5}, \cos \alpha = 1$

$\Rightarrow \alpha = \cos^{-1}\left(-\frac{3}{5}\right)$ $\qquad \alpha \neq 0$

$\Rightarrow \pi - \cos^{-1}\left(\frac{3}{5}\right)$

10. Ans. (2)



Non differentiable at $x = \frac{1}{2}, 1$

$\Rightarrow 1 - 4x^2$ $\qquad \Rightarrow 0 \le x \le \frac{1}{2}$

$\Rightarrow 4x^2 - 1$ $\qquad \Rightarrow \frac{1}{2} < x < 1$

$\Rightarrow$ -1 $\qquad \Rightarrow -1 < x < 2$

$\Rightarrow 0$ $\qquad \Rightarrow x = 2$

EXERCISE - JEE (Main) PYQ

1. Ans. (4)

$$\lim_{x\to 0}\left(\frac{1+f(3+x)-f(3)}{1+f(2-x)-f(2)}\right)^{\frac{1}{x}} \quad (1^{\infty}\text{ form})$$

$$\Rightarrow e^{\lim\limits_{x\to 0}\frac{f(3+x)-f(2-x)-f(3)+f(2)}{x(1+f(2-x)-f(2))}}$$

using L'Hopital

$$\Rightarrow e^{\lim\limits_{x\to 0}\frac{f'(3+x)+f'(2-x)}{-xf'(2-x)+(1+f(2-x)-f(2))}}$$

$$\Rightarrow e^{\frac{f'(3)+f'(2)}{1}} = 1$$

2. Ans. (4)

$|f(x) - f(y)|, \le 2|x-y|^{\frac{3}{2}}$

divide both sides by $|x-y|$

$$\Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right| \le 2.|x-y|^{1/2}$$

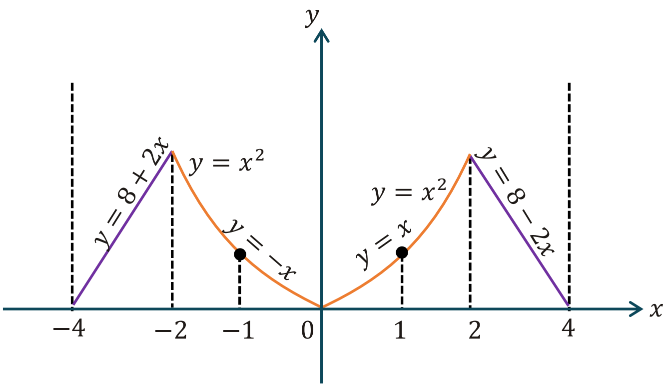
apply limit $x \to y$

$\Rightarrow |f'(y)| \le 0 \Longrightarrow f'(y) = 0$

$\Rightarrow f(y) = c \Longrightarrow f(x) = 1$

$$\Rightarrow \int_0^1 1.dx = 1$$

3. Ans. (3)



$$\Rightarrow f(x) = \begin{cases} 8+2x, & -4 \le x < -2 \\ x^2, & -2 \le x \le -1 \\ |x|, & -1 < x < 1 \\ x^2, & 1 \le x \le 2 \\ 8-2x, & 2 < x \le 4 \end{cases}$$

$f(x)$ is not differentiable at $x = \{-2,-1,0,1,2\}$

$\Rightarrow S = \{-2, -1, 0, 1, 2\}$

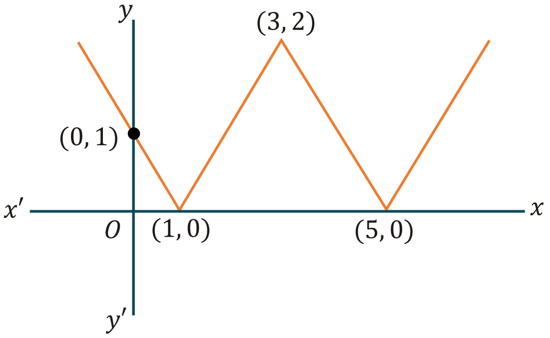
4. Ans. (4)

$$|f(x)| = \begin{cases} 1 & , \ -2 \le x < 0 \\ 1-x^2 & , \ 0 \le x < 1 \\ x^2-1 & , \ 1 \le x \le 2 \end{cases} \text{ and } f(|x|) = x^2 - 1, x \in [-2,2]$$

$$\text{Hence } g(x) = \begin{cases} x^2 & , \ x \in [-2,0) \\ 0 & , \ x \in [0,1) \\ 2(x^2-1) & , \ x \in [1,2] \end{cases}$$

It is not differentiable at $x = 1$

5. Ans. (3)



$f(x) = |2 - |x - 3||$

f is not differentiable at

$x = 1, 3, 5$

$\Rightarrow \sum_{x \in s} f(f(x)) = f(f(1)) + f(f(3)) + f(f(5))$

$= f(0) + f(2) + f(0)$

$= 1 + 1 + 1 = 3$

6. Ans. (10)

Since, $\lim_{x \to 0} \frac{f(x)}{x}$ exist $\Rightarrow f(0) = 0$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{f(h) + xh^2 + x^2h}{h}$ (take $y = h$) $= \lim_{h \to 0} \frac{f(h)}{h} + \lim_{h \to 0}(xh) + x^2$

$\Rightarrow f'(x) = 1 + 0 + x^2 \Rightarrow f'(3) = 10$

7. Ans. (1)

$$f(x) = \begin{cases} \frac{\pi}{4} + \tan^{-1} x & , \quad x \in (-\infty, -1] \cup [1, \infty) \\ -\frac{(x+1)}{2} & , \quad x \in (-1, 0) \\ \frac{x-1}{2} & , \quad x \in (0, 1) \end{cases}$$

for continuity at $x = -1$

$\text{L.H.L.} = \frac{\pi}{4} - \frac{\pi}{4} = 0$

R.H.L. = 0

so, continuous at $x = -1$

for continuity at $x = 1$

L.H.L. = 0

$\text{R.H.L.} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

so, not continuous at $x = 1$

For differentiability at $x = -1$

$\text{L.H.D.} = \frac{1}{1+1} = \frac{1}{2}$

$\text{R.H.D.} = -\frac{1}{2}$

so, non-differentiable at $x = -1$

8. Ans. (1)

$f(x)$ is continuous and differentiable

$\Rightarrow f(\pi^-) = f(\pi) = f(\pi^+)$

$\Rightarrow -1 = -k_2$

$\Rightarrow \boxed{k_2 = 1}$

$\Rightarrow f'(x) = \begin{cases} 2k_1(x-\pi); & x \le \pi \\ -k_2 \sin x; & x > \pi \end{cases}$

$\Rightarrow f'(\pi^-) = f'(\pi^+)$

$\Rightarrow 0 = 0$

so, differentiable at $x = 0$

$\Rightarrow f''(x) = \begin{cases} 2k_1 & ; x \le \pi \\ -k_2 \cos x & ; x > \pi \end{cases}$

$\Rightarrow f''(\pi^-) = f''(\pi^+)$

$\Rightarrow 2k_1 = k_2$

$\Rightarrow \boxed{k_1 = \frac{1}{2}}$

$\Rightarrow (k_1, k_2) = \left(\frac{1}{2}, 1\right)$

9. Ans. (5)

$f(x) = x^5 . \sin\frac{1}{x} + 5x^2$ if $x < 0$

$\Rightarrow f(x) = 0 \Rightarrow x = 0$

$\Rightarrow f(x) = x^5 . \cos\frac{1}{x} + \lambda x^2 \Rightarrow x > 0$

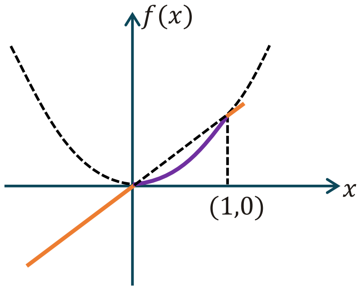
LHD of $f'(x)$ at $x = 0$ is 10

RHD of $f'(x)$ at $x = 0$ is 2λ

if $f''(0)$ exists then

$2\lambda = 10 \Rightarrow \lambda = 5$

10. Ans. (1)



$f(x) = \max(x, x^2)$

Non-differentiable at $x = 0, 1$

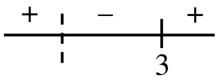
$\Rightarrow S = \{0, 1\}$

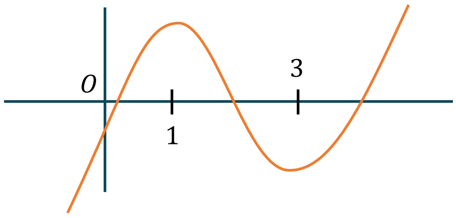
11. Ans. (1)

$f(x) = x^3 - 6x^2 + 9x - 3$

$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

$f(1) = 1 \Rightarrow f(3) = -3$





$$g(x)=\begin{cases} f(x) & 0\le x\le 1 \\ 1 & 1\le x\le 3 \\ 4-x & 3<x\le 4\end{cases}$$

$g(x)$ is continuous

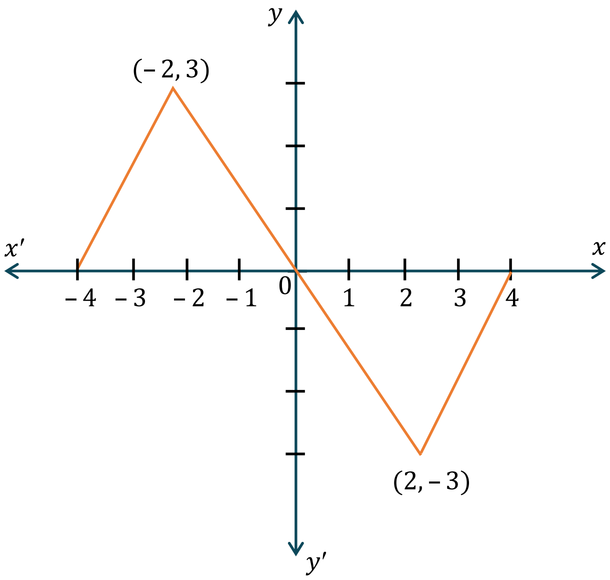
$$g'(x)=\begin{cases} 3(x-1)(x-3) & 0\le x\le 1 \\ 0 & 1\le x\le 3 \\ -1 & 3<x\le 4\end{cases}$$

$g(x)$ is non-differentiable at $x = 3$

12. **Ans. (4)**

$$f(x+2)=\begin{cases} \frac{3x}{2}+6 & -4\le x\le -2 \\ -\frac{3x}{2} & -2<x\le 0 \\ 0 & x\in(-\infty,-4)\cup(0,+\infty)\end{cases}$$

$$f(x-2)=\begin{cases} \frac{3x}{2} & 0\le x\le 2 \\ -\frac{3x}{2}+6 & 2\le x\le 4 \\ 0 & x\in(-\infty,0)\cup(4,+\infty)\end{cases}$$



$$g(x)=f(x+2)-f(x-2)\begin{cases} \frac{3x}{2}+6 & -4\le x\le -2 \\ -\frac{3x}{2} & -2<x<2 \\ \frac{3x}{2}-6 & 2\le x\le 4 \\ 0 & x\in(-\infty,-4)\cup(4,\infty)\end{cases}$$

$n = 0$

$m = 4 \Rightarrow (n + m = 4)$

13. Ans. (2)

$f(x)=|2x+1|-3|x+2|+|x^2+x-2|$

$\Rightarrow |2x+1|-3|x+2|+|x+2||x-1|$

$\Rightarrow |2x+1|+|x+2|(|x-1|-3)$

Critical points are $x=\frac{-1}{2},-2,1$

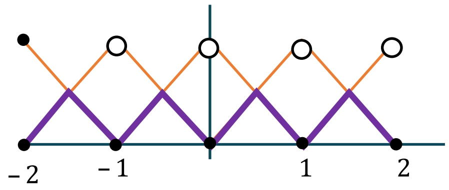
but $x=-2$ is making a zero.

twice in product so, points of non-different ability are $\boxed{x=\frac{-1}{2} \text{ and } x=1}$

14. Ans. (1)

$\min\{x-[x],1-x+[x]\}$

$h(x)=\min\{x-[x],1-[x-[x]]\}$



$\Rightarrow$ always continuous in [–2, 2]

but non-differentiable at 7 Points

15. Ans. (3)

$f(x)=|(x-3)(x+1)|\,.\,e^{(3x-2)^2}$

$$f(x)=\begin{cases}(x-3)(x+1).e^{(3x-2)^2} & ;\quad x\in(3,\infty)\\ -(x-3)(x+1).e^{(3x-2)^2} & ;\quad x\in[-1,3]\\ (x-3).(x+1).e^{(3x-2)^2} & ;\quad x\in(-\infty,-1)\end{cases}$$

Clearly, non-differentiable at $x=-1$ & $x=3$.

16. Ans. (79)

$f(x)=4|2x+3|+9\left[x+\frac{1}{2}\right]-12[x+20]$

$x\in(-20,20)$

$f(x)$ is not Diff. at $x=I\in\{-19,-18,....0,...19\}=39$

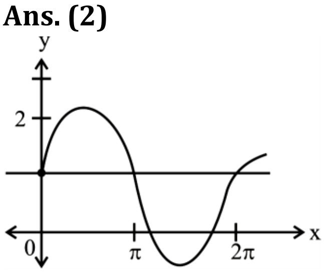
at $x=\text{I}+\frac{1}{2}$, $f(x)$ Non diff. at 39 points

Check at $x=\frac{-3}{2}$ Discount at $x=\frac{-3}{2}$ $\therefore$ N. R(1)

No. of point of non-differentiability

= 39 + 39 + 1 = 79

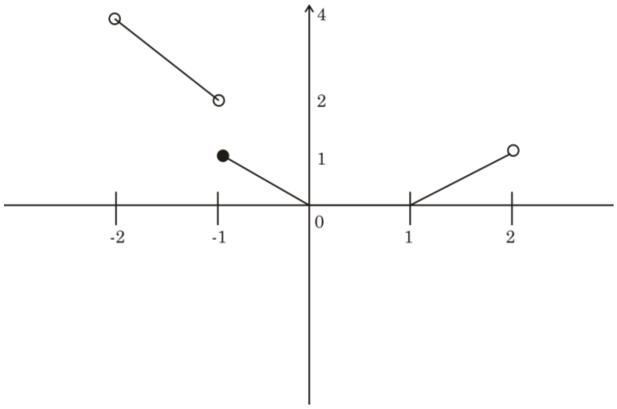
17. Ans. (2)



No. of non-differentiable points = 1 (m)
No. of not continuous points = 0 (n)
(m, n) = (1, 0)

18. Ans. (4)

$$f(x)=\begin{cases} x[x] & ,-2<x<0 \\ (x-1)[x] & ,0\le x<2 \end{cases}$$



$|f(x)|$ = Remain same

$m = 1, n = 3$

$m + n = 4$

19. Ans. (4)

$$Let\ g(x)=1+x+[x]=\begin{cases} 1+x; & x\in[0,1) \\ 2+x; & x\in[1,2) \\ 5; & x=2 \end{cases}$$

$$\lambda(x)=x+2[x]=\begin{cases} x; & x\in[0,1) \\ x+2; & x\in[1,2) \\ 6; & x=2 \end{cases}$$

$r(x) = 2 + x$

$$f(x)=\begin{cases} 2+x; & x\in[0,2) \\ 6; & x=2 \end{cases}$$

$f(x)$ is discontinuous only at $x = 2 \Rightarrow m = 1$
$f(x)$ is differentiable in $(0,2) \Rightarrow n = 0$
$(m + n)^2 + 2 = 3$

20. Ans. (2)

Continuity of $f(x)$ **:** $f(0^+) = h^2.\sin\frac{1}{h}=0$

$$f(0^-)=(-h)^2.\sin\left(\frac{-1}{h}\right)=0$$

$f(0) = 0$
$f(x)$ is continuous

$$f'(0^+)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\frac{h^2.\sin\left(\frac{1}{h}\right)-0}{h}=0$$

$$f'(0^-)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\frac{h^2.\sin\left(\frac{1}{-h}\right)-0}{-h}=0$$

$f(x)$ is differentiable.

$$f'(x)=2x.\sin\left(\frac{1}{x}\right)+x^2.\cos\left(\frac{1}{x}\right).\frac{-1}{x^2}$$

$$f'(x)=\begin{cases}2x.\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$$

$\Rightarrow f'(x)$ is not continuous (as $\cos\left(\frac{1}{x}\right)$ is highly oscillating at $x = 0$)

EXERCISE - JEE (Advanced) PYQ

1. **Ans. (B)**

$$f(x)=\begin{cases}x^2\left|\cos\frac{\pi}{x}\right| & , \quad x\neq 0\\ 0 & , \quad x=0\end{cases},$$

For $x = 0$

R.H.D. $f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$$\Rightarrow \lim_{h\to 0}\frac{h^2\left|\cos\left(\frac{\pi}{h}\right)\right|}{h}$$

$$\Rightarrow \lim_{h\to 0}h\left|\cos\left(\frac{\pi}{h}\right)\right|=0$$

L.H.D. $f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$$= \lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{h}\right|}{-h}=0$$

$\Rightarrow f(x)$ is differentiable at $x = 0$

$$f(2)=2^2\cdot\left|\cos\left(\frac{\pi}{2}\right)\right|=0$$

At $x = 2$

R.H.D = $f'(2^+) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$

$$\Rightarrow \lim_{h\to 0}\frac{(2+h)^2\cdot\left|\cos\left(\frac{\pi}{2+h}\right)\right|}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\left(4+h^2+2h\right)\cdot\left(\cos\frac{\pi}{(2+h)}\right)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\left(4+h^2+2h\right)\cdot\sin\left(\frac{\pi}{2}-\frac{\pi}{2+h}\right)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\left(4+h^2+2h\right)\cdot\sin\left(\frac{\pi h}{2(2+h)}\right)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\left(4+h^2+2h\right)\cdot\sin\left(\frac{\pi h}{2(2+h)}\right)}{\left(\frac{\pi h}{2(2+h)}\right)}\times\frac{\pi}{2(2+h)}$$

$$\Rightarrow \lim_{h\to 0}\frac{\pi(2+h)}{2}=\pi$$

L.H.D $f'(2^-)=\lim_{h\to 0}\dfrac{(2-h)^2\cdot\left|\cos\left(\frac{\pi}{(2-h)}\right)\right|-0}{(-h)}$

$$\Rightarrow \lim_{h\to 0}\frac{(2-h)^2\cdot\left(-\cos\left(\frac{\pi}{2-h}\right)\right)}{(-h)}$$

$$\Rightarrow \lim_{h\to 0}\frac{(2-h)^2\cdot\cos\left(\frac{\pi}{2-h}\right)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{(2-h)^2\cdot\sin\left(\frac{\pi}{2}-\frac{\pi}{2-h}\right)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{(2-h)^2\cdot\sin\left(\frac{-h\pi}{2(2-h)}\right)}{h\left(-\frac{\pi}{2(2-h)}\right)}\times\frac{-\pi}{2(2-h)}$$

$$\Rightarrow \lim_{h\to 0}(2-h)\left(\frac{-\pi}{2}\right)=-\pi$$

L.H.D. $\neq$ R.H.D.

$f(x)$ is not differentiable at $x = 2$.

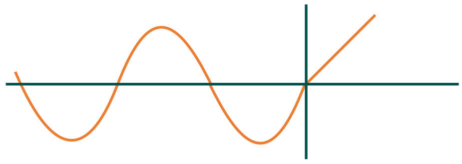
2. **Ans. (D)**

$$f_3(x) = \begin{cases}\sin x & \text{if } x<0,\\ x & \text{if } x\geq 0\end{cases}$$

at $x = 0$

$f_3(0^+)=0,\ \ f_3(0)=0,\ \ f_3(0^-)=0$

$\Rightarrow f_3(x)$ is continuous at $x=0$



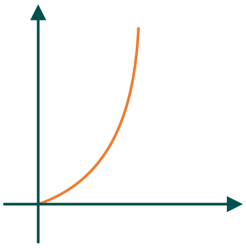
R.H.D. = $f'_3(0^+)=\lim\limits_{h\to0}\dfrac{f_3(0+h)-f_3(0)}{h}$

= $\lim\limits_{h\to0}\dfrac{h-0}{h}=1$

L.H.D. = $f_3(0^-)=\lim\limits_{h\to0}\dfrac{f_3(0-h)-f_3(0)}{(-h)}$

$\lim\limits_{h\to0}\dfrac{\sin(-h)}{(-h)}=1$

$\Rightarrow f_3(x)$ is differentiable at $x = 0$



$\Rightarrow f_3(x)$ is not one-one

$\Rightarrow$ Q→3

(S) $f_2:[0,\infty)\to R$

$f_2(x)=x^2$

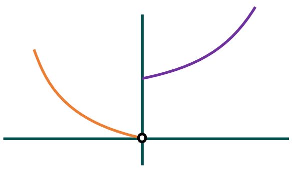
$\Rightarrow f_2$ is continuous and one-one

$\Rightarrow$ S → 4

$$\Rightarrow f_1(x)=\begin{cases}-x & x<0\\ e^x & x\ge 0\end{cases}$$

$\Rightarrow f_2(f_1(x))=(f_1(x))^2$

$$\Rightarrow g(x)=f_2(f_1(x))\begin{cases}x^2 & x<0\\ e^{2x} & x\ge 0\end{cases}$$



$\Rightarrow g(0^+)=g(0)=1$

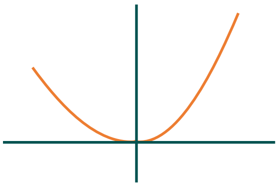
$\Rightarrow g(0^-)=0$

$\Rightarrow g(x)$ is discontinuous at $x=0$

$\Rightarrow g(x)$ is not one-one

R → 2

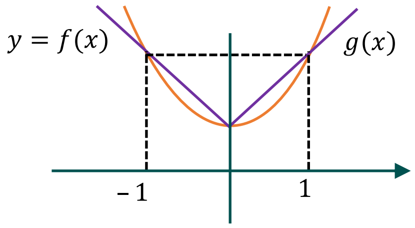
$$\text{(P)}\ f_4(x)\begin{cases}x^2 & x<0\\ e^{2x}-1 & x\ge 0\end{cases}$$



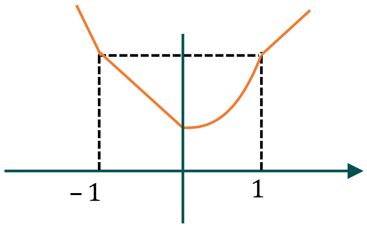
Onto but not one-one

P → 1

3. **Ans. (3)**



$$h(x)=\begin{cases} x^2+1 & x\le -1 \\ |x|+1 & -1<x<0 \\ x^2+1 & 0\le x<1 \\ |x|+1 & x\ge 1 \end{cases}$$



$y = h(x)$ is not differentiable at $x = 0, 1, -1$

4. **Ans. (A,B)**

If $x^3 - x \ge 0 \Rightarrow \cos|x^3 - x| = \cos(x^3 - x)$

$x^3 - x < 0 \Rightarrow \cos|x^3 - x| = \cos(x^3 - x)$

Similarly, $b|x|\sin|x^3 + x| = bx\sin(x^3 + x)$ for all $x \in R$

$\therefore f(x) = a\cos(x^3 - x) + bx\sin(x^3 + x)$

which is composition and sum of differentiable functions

therefore, always continuous and differentiable.

5. **Ans. (B,C)**

$f(x) = [x^2] - 3$

$g(x) = (|x| + |4x - 7|)([x^2] - 3)$

$\because f$ is discontinuous at $x = 1, \sqrt{2}, \sqrt{3}, 2$ in $\left[-\frac{1}{2}, 2\right]$

and $|x| + |4x - 7| \neq 0$ at $x = 1, \sqrt{2}, \sqrt{3}, 2$

$\Rightarrow g(x)$ is discontinuous at $x = 1, \sqrt{2}, \sqrt{3}$ in $\left(-\frac{1}{2}, 2\right)$

In $(0 - \delta, 0 + \delta)$

$g(x) = (|x| + |4x - 7|).(-3)$

$\Rightarrow$ 'g' is non derivable at $x = 0$.

In $\left(\frac{7}{4} - \delta, \frac{7}{4} + \delta\right)$

$g(x) = 0$ as $f(x) = 0$

$\Rightarrow$ Derivable at $x = \frac{7}{4}$

$\therefore$ 'g' is non-derivable at $0, 1, \sqrt{2}, \sqrt{3}$

6. Ans. (2)

$P(x,y): f(x+y) = f(x)f'(y) + f'(x)f(y)\ \forall\, x,y \in R$

$P(0,0): f(0) = f(0)f'(0) + f'(0)f(0)$

$\Rightarrow 1 = 2f'(0) \Rightarrow f'(0) = \frac{1}{2}$

$P(x,0): f(x) = f(x).f'(0) + f'(x).f(0)$

$\Rightarrow f(x) = \frac{1}{2}f(x) + f'(x) \Rightarrow f'(x) = \frac{1}{2}f(x)$

$\Rightarrow f(x) = e^{\frac{1}{2}x} \Rightarrow ln(f(4)) = 2$

7. Ans. (D)

(i) $f(x) = \sin\sqrt{1-e^{-x^2}}$

$f_1'(x) = \cos\sqrt{1-e^{-x^2}}.\frac{1}{2\sqrt{1-e^{-x^2}}}\left(0 - e^{-x^2}.(-2x)\right)$

at $x = 0$ $f_1'(x)$ does not exist

So, $P \to 2$

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1}x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

$\lim\limits_{x\to 0^+} \frac{\sin x}{x}\frac{x}{\tan^{-1}x} = 1$

$\Rightarrow f_2(x)$ does not continuous at $x = 0$

So $Q \to 1$

(iii) $f_3(x) = [\sin \ell n(x+2)] = 0$

$1 < x+2 < e^{\pi/2}$

$\Rightarrow 0 < \ell n(x+2) < \frac{\pi}{2} \Rightarrow 0 < \sin(\ell n(x+2) < 1 \Rightarrow f_3(x) = 0$

So $R \to 4$

(iv) $f_4(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

So $S \to 3$

8. Ans. (A,C)

$f: \mathbb{R} \to \mathbb{R}\ f(x) = (x^2 + \sin x)(x-1)\ \ f(1^+) = f(1^-) = f(1) = 0$

$fg(x): f(x).g(x) \qquad fg: \mathbb{R} \to \mathbb{R}$

let $fg(x) = h(x) = f(x).g(x) \qquad h: \mathbb{R} \to \mathbb{R}$

option (C)

$h'(x) = f'(x)g(x) + f(x)\,g'(x)$

$h'(1) = f'(1)\,g(1) + 0,$

{as $f(1) = 0, g'(x)$ exists}

$\Rightarrow$ if $g(x)$ is differentiable then $h(x)$ is also differentiable (true)

option (A)

If $g(x)$ is continuous at $x = 1$ then $g(1^+) = g(1^-) = g(1)$

$h'(1^+) = \lim\limits_{h\to 0^+} \frac{h(1+h)-h(1)}{h}$

$$h'(1^+) = \lim_{h\to0^+}\frac{f(1+h)g(1+h)-0}{h} = f'(1)g(1)$$
$$h'(1^-) = \lim_{h\to0^+}\frac{f(1-h)g(1-h)-0}{-h} = f'(1)g(1)$$

So $h(x) = f(x).g(x)$ is differentiable

at $x = 1$ (True)

option (B), (D)

$$h'(1^+) = \lim_{h\to0^+}\frac{h(1+h)-h(1)}{-h}$$
$$h'(1^+) = \lim_{h\to0^+}\frac{f(1+h)g(1+h)}{h} = f'(1)g(1^+)$$
$$h'(1^-) = \lim_{h\to0^+}\frac{f(1-h)g(1-h)}{-h} = f'(1).g(1^-)$$

$\Rightarrow g(1^+) = g(1^-)$

So, we cannot comment on the continuity and differentiability of the function.

9. **Ans. (A,B,D)**

Since $f(x) = xg(x)$

$\lim_{x\to0} f(x) = \lim_{x\to0} xg(x)$

$\lim_{x\to0} f(x) = \left(\lim_{x\to0} x\right).\left(\lim_{x\to0} g(x)\right)$

$\lim_{x\to0} f(x) = 0\times1 = 0$ ...(1)

$f(x+y) = f(x)+f(x)+f(x)f(y)$

Now we check continuity of $f(x)$

at $x = a$

$\lim_{h\to0} f(a+h) = f(a)+f(h)+f(a)\cdot f(h)$

$\lim_{h\to0}\left(f(a)+f(h)\,(1+f(a))\right)$

$\lim_{h\to0} f(a+h) = f(a)$ $\therefore$ $f(x)$ is continuous $\forall\, x\in R$

$\lim_{x\to0} f(x) = f(0) = 0\left(\lim_{x\to0} f(x) = 0\right)$ $\therefore f(0) = 0$

and $\lim_{x\to0}\frac{f'(x)}{1} = 1$ $\therefore f'(0) = 1$

Now

$f(x+y) = f(x)+f(y)+f(x)\,f(y)$

using partial derivative (w.r.t. y)

$f'(x+y)+f'(y)+f(x)+f'(y)$

put $y = 0$

$f'(x) = f'(0)+f(x)f'(0)$

$f'(x) = 1+f(x)$

$$\int\frac{f'(x)}{1+f(x)}dx = \int 1dx$$

$\ell n|(1+f(x))| = x+C$

$f(0) = 0\,;\ c = 0 \therefore |1+f(x)| = e^x$

$1+f(x) = \pm e^x$ or $f(x) = \pm\, e^x - 1$

Now $f(0) = 0 \therefore f(x) = e^x - 1$ $\therefore f(x) = e^x - 1$

option (A) is correct and $f'(x) = e^x$

$f'(0) = 1$ option(D) is correct

$$g(x) = \frac{f(x)}{x} = \begin{cases} \dfrac{e^x - 1}{x} & ; \quad x \neq 0 \\ 1 & ; \quad x = 0 \end{cases}$$

$$g'(0 + h) = \lim_{h \to 0} \frac{g(0 + h) - g(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{e^h - 1}{h} - 1}{h} = \frac{1}{2}$$

option (B) is correct.

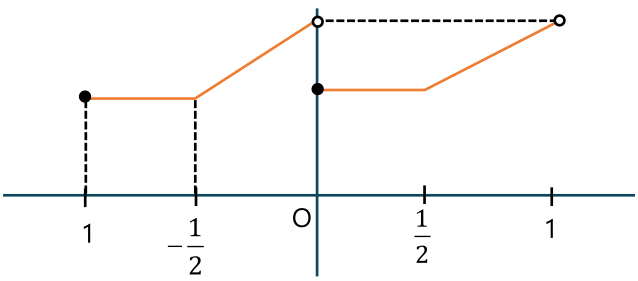
10. Ans. (4)

$$f(x) = \begin{cases} 4x & x \geq \frac{1}{2} \\ 2 & -\frac{1}{2} < x < \frac{1}{2} \\ -4x & x \leq \frac{-1}{2} \end{cases}$$

$g(x) = \{x\}$

$$\Rightarrow f(g(x)) \begin{cases} 4\{x\} & \{x\} \geq \frac{1}{2} \\ 2 & 0 \leq \{x\} < \frac{1}{2} \end{cases}$$

$\Rightarrow f(g(x))$ is periodic with period 1



$\Rightarrow fog$ is discontinuous at $x = 0 \Rightarrow c = 1$

$\Rightarrow fog$ is non-differentiable and continuous at $x = \pm\frac{1}{2}$

$\Rightarrow d = 3$

$c + d = 4$

11. Ans. (A,B)

$$f(x)=\begin{cases} 0 & ; \ 0<x<\frac{1}{4} \\ \left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) & ; \ \frac{1}{4}\le x<\frac{1}{2} \\ 2\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) & ; \ \frac{1}{2}\le x<\frac{3}{4} \\ 3\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) & ; \ \frac{3}{4}\le x<1 \end{cases}$$

$f(x)$ is discontinuous at $x=\frac{3}{4}$ only

$$f'(x)=\begin{cases} 0 & ; \ 0<x<\frac{1}{4} \\ 2\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+\left(x-\frac{1}{4}\right)^2 & ; \ \frac{1}{4}<x<\frac{1}{2} \\ 4\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+2\left(x-\frac{1}{4}\right)^2 & ; \ \frac{1}{2}<x<\frac{3}{4} \\ 6\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right)+3\left(x-\frac{1}{4}\right)^2 & ; \ \frac{3}{4}<x<1 \end{cases}$$

$f(x)$ is non-differentiable at $x=\frac{1}{2}$ and $\frac{3}{4}$

minimum values of $f(x)$ occur at $x=\frac{5}{12}$ whose value is $-\frac{1}{432}$

12. Ans. (A,C,D)

$S = (0, 1) \cup (1, 2) \cup (3, 4)$

$T = \{0, 1, 2, 3\}$

Number of functions :

Each element of S have 4 choice

Let n be the number of element in set S.

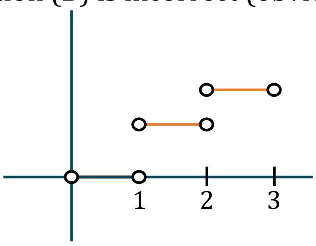
Number of function = 4^n

Here $n \to \infty$

$\Rightarrow$ Option (A) is correct.

Option (B) is incorrect (obvious)

(C)



For continuous function

Each interval will have 4 choices.

$\Rightarrow$ Number of continuous functions

$= 4 \times 4 \times 4 = 64$

$\Rightarrow$ Option (C) is correct.

(D) Every continuous function is piecewise constant functions

$\Rightarrow$ Differentiable.

Option (D) is correct.

JEE (Main) Practice Paper

SECTION–A

1. **Ans. (2)**

$f(x) = x\left(\sqrt{x}-\sqrt{x+1}\right)$ is continuous at $x = 0$

$f'(x) = x\left(\sqrt{x}-\sqrt{x+1}\right) + x\left(\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x+1}}\right)$

$f'(x) = \sqrt{x}-\sqrt{x+1}+\frac{\sqrt{2}}{2}-\frac{x}{2\sqrt{x+1}}$

$f'(0^+)\ f'(0^-) = -1$

Hence it is differentiable at $x = 0$

2. **Ans. (2)**

$f(x) = \frac{x}{\sqrt{x+1}-\sqrt{x}}$ $\qquad f(0) = 0$ $\qquad$ Domain $x \geq 0$

$\because \lim\limits_{x \to 0^+} f(x) = 0$ $\qquad \therefore f(x)$ is continous

$\because$ RHD $(x = 0) = \lim\limits_{h \to 0} \frac{\frac{h}{\sqrt{h+1}-\sqrt{h}}-0}{h} = 1$

$\therefore$ $f(x)$ is differntiable at $x = 0$

3. **Ans. (2)**

$$f(x) = \sin^{-1}(\cos x) = \frac{\pi}{2}-\cos^{-1}\cos x = \begin{cases} \frac{\pi}{2}+x, & x\in[-\pi,\ 0] \\ \frac{\pi}{2}-x, & x\in[0,\ \pi] \end{cases}$$

continuous but not differentiable at $x = 0$

4. **Ans. (4)**

$f(0) = \lim\limits_{x\to 0} f(x) = 0-1+0.\sin(-1) = -1$

$f(0^+) = \lim\limits_{x\to 0} f(x) = 0+0+0.\sin 0 = 0 = f(0)$

$f(x)$ is not continuous at $x = 0$ at $x = 2$,

$f(2^+) = 2 + 2 + 2 \sin 2 = 4 + 2 \sin 2$

$f(2^-) = 2 + 1 + 2 \sin 1 = 3 + 2 \sin 1$

$f(x)$ is not continuous at $x = 2$

5. **Ans. (1)**

$$f(x) = \begin{cases} \frac{x}{1-x}, & x<0 \\ \frac{x}{1+x}, & x\geq 0 \end{cases}$$

$f(x)$ is continous and differntiable for $x \in \mathrm{R}$ $f(x)$

$$f(x) = \begin{cases} \frac{x}{1-x}, & x<0 \\ \frac{x}{1+x}, & x\geq 0 \end{cases}$$

$f(x)$ is discontinous at $|x| = 1$

6. **Ans. (2)**

If 'f' is differentiable

then $|f|$ is differentiable at each point x, where $f(x) \neq 0$

if $f(\alpha) = 0$ and $f'(\alpha) = 0$, then $|f|$ is differentiable at $x = \alpha$

if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $|f|$ is not differentiable at $x = \alpha$

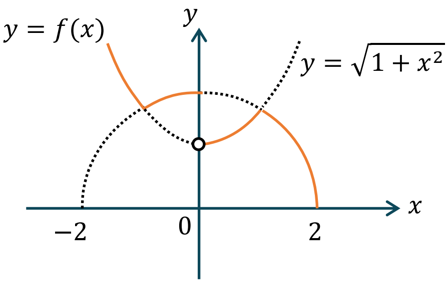
$\Rightarrow$ If f is differentiable then $|f|$ may or may not be differentiable, [option A, C, D not necessarily true]

Now $|f|^2 = f^2$

$(f^2)' = 2.f.f'$ since f is differentiable

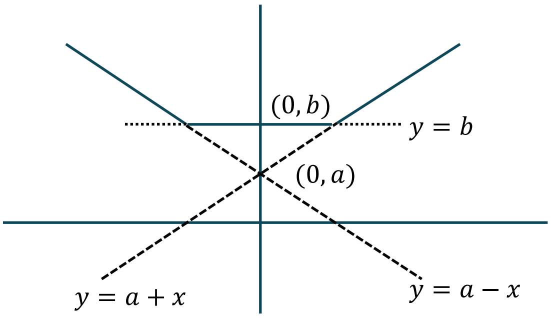
$\therefore$ f^2 is also differentiable

7. **Ans. (4)**



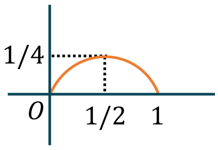
8. **Ans. (2)**

$y = f(x) = \max\{a-x, a+x, b\}$, $0 < a < b$

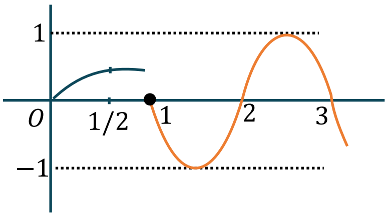


$f(x)$ is non-differentiable at 2 points

9. **Ans. (3)**

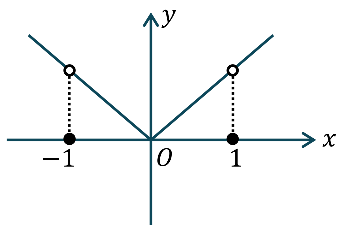


$$y = g(x) = \begin{cases} x - x^2 & 0 \le x \le 1/2 \\ 1/4 & 1/2 < x \le 1 \\ \sin \pi x & x > 1 \end{cases}$$



10. Ans. (1)

$S_1 : f(x) = |x\ \text{sgn}\ (1 - x^2)| = \begin{cases} -x & x \in (-\infty, -1) \cup (-1, \ \infty) \\ 0 & x = -1, \ 0, \ 1 \\ x & x \in (0, \ 1) \cup (1, \ \infty) \end{cases}$



function is discontinuous at $x = -1, 1$

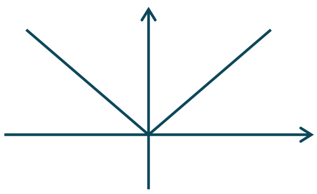
and non-differentiable at $x = -1, 0, 1$

$S_2 : f(x) = \text{a} \sin \frac{\pi}{2} \ (x + 1), x \ \ 0 \qquad = \frac{\tan x - \sin x}{x^3}, \ x > 0$

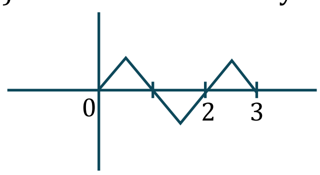
$a = \lim\limits_{x \to 0^+} \frac{\tan x - \sin x}{x^3} = \lim\limits_{x \to 0^+} \frac{\tan x(1 - \cos x)}{x^3} = \frac{1}{2}$

$\therefore \ a = \frac{1}{2}$

$S_3 : f(x) = (x^2\ |x|)^{1/3} = \left[\begin{array}{lll} (x^3)^{\frac{1}{3}} = x & , & x \geq 0 \\ (-x^5)^{\frac{1}{3}} = (-1)^{\frac{1}{3}} x = -x & , & x < 0 \end{array} \right. = -x$



$f(x)$ is differentiable every where except at $x = 0$



S_4 **:**

$f(x)$ will be non differentiable if $\sin^{-1}(\sin x) = 0$ or graph of $f(x)$ has a sharp point. Hence number of points of non differentiable will be 5.

11. Ans. (3)

$S_1 : f(x) = \frac{\sin\ (\pi\ [x - \pi]\ )}{1 + [\ x\ ]^2}$

$[x - \pi]$ is an integer for $x \in R$

$\therefore \ f(x) = 0 \ \forall \ x \in R.$

Hence $f(x)$ is always continuous. **(False)**

$S_2 : f(x) = p[x + 1] + q\ [x - 1]$

$= (p + q)\ [x] + p - q$

$f(1) = 2p$

$f(1^+) = 2p$

$f(1^-) = p - q$

But $f(x)$ is continous at $x = 1$

$2p = p - q \quad p + q = 0$ [True]

S_3 : $f(x) = |[x]\, x| = \begin{cases} -x & , \; -1 \le x < 0 \\ 0 & , \; 0 \le x < 1 \\ x & , \; 1 \le x < 2 \\ 4 & , \; x = 2 \end{cases}$

$\because$ function is not continuous at $x = 2$

$\therefore$ non-differentiable also **(True)**

S_4 : $f(0)$ = constant

$f'(0) = 0 \;\forall\; x \in R$

$f'(10) = 0$ [False]

12. Ans. (1)

$f(x) = \begin{cases} x & , \quad x \le 1 \\ ax^2 + bx + c & , \quad \text{otherwise} \end{cases}$

$f(x)$ should be continous at $x = 1$

it gives $a+b+c = 1$

$f(x)$ should be differentiable at $x = 1$

it gives $2a+b=1 \Rightarrow b = 1-2a \qquad c = 1-a-b = a$

13. Ans. (4)

$$\lim_{h\to 0} \frac{6}{4} = \lim_{h\to 0} \frac{f(2h+2+h^2)-f(2)}{(2h+2+h^2)-(2)} \; \frac{(h-h^2+1)-(1)}{f(h-h^2+1)-f(1)} \frac{(2h+2+h^2)-(2)}{(h-h^2+1)-(1)} = \lim_{h\to 0} \frac{f'(2)}{f'(1)} . \frac{h(2+h)}{h(1-h)} = 3$$

14. Ans. (2)

$f(x + y) = f(x) \,.\, f(y)$ and $f(1) = 2$

$$\sum_{n=1}^{10} f(n) = f(1) + f(2) + \dots\dots\dots + f(10) = 2^1 + 2^2 + 2^3 + \dots\dots + 2^{10} = 2\left(\frac{2^{10}-1}{2-1}\right) = 2046$$

15. Ans. (3)

$f(1) = 1 = 2 - 1$

$f(n + 1) = 2f(n) + 1$

$\therefore \quad f(2) = 2f(1) + 1 = 2.\,1 + 1 = 3 = 2^2 - 1 \Rightarrow f(3) = 7 = 2^3 - 1$

$f(4) = 15 = 2^4 - 1$

Similarly $f(n) = 2^n - 1$

16. Ans. (4)

$$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$$

Replace $x + \dfrac{1}{x} = t$, where $|t| \ge 2$

$\therefore f(t) = t^2 - 2, \; |t| \ge 2$

17. **Ans. (2)**

Method 1 : (usual but lengthy)

$x^2 f(x) + f(1-x) = 2x - x^4$ ...(1)

replace x by $(1-x)$ in equation (1)

$(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4$ ...(2)

eliminate $f(1-x)$ by equation (1) and (2)

we get, $f(x) = 1 - x^2$

Method 2 :

Since R.H.S. is polynomial of 4th degree and also by options consider $f(x) = ax^2 + bx + c$

$x^2 f(x) + f(1-x) = 2x - x^4$

$\Rightarrow x^2(ax^2 + bx + c) + a(1-x)^2 + b(1-x) + c = 2x - x^4$

by comparing coefficients

$a = -1$

$b = 0$

$c = 1$

$\therefore f(x) = -x^2 + 1$

18. **Ans. (4)**

$f(x + 2y) = f(x) + f(2y) + 4xy \quad \forall \quad x, y \in R$

Replace $2y$ with y we have

$f(x + y) = f(x) + f(y) + 2xy \quad \forall \quad x, y \in R$

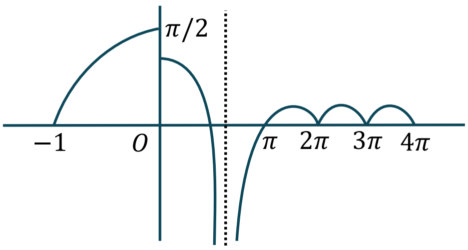
diff. w.r.t. x

$f'(x+y) = f'(x) + 2y$

Put $x = 1 \quad y = -1 \qquad f'(0) = f'(1) - 2$

19. **Ans. (2)**

Graph of $f(x)$



Now solve

20. **Ans. (4)**

$$f'(1^+) = \lim_{h\to 0} \frac{h\sqrt{|\ell n(1+h)|} - 0}{h} = 0$$

$$f'(1^-) = \lim_{h\to 0} \frac{-h\sqrt{|\ell n(1-h)|} - 0}{-h} = 0$$

SECTION–B

1. Ans. (11)

$f(10-x) = f(x) = f(4-x) \Rightarrow f(10-x) = f(4-x)$

Let $4-x = t \Rightarrow f(6+t) = t$

$\Rightarrow f(x)$ is periodic with period 6. $\Rightarrow f(x) = 101$ at $x = 0,6,12, 18, 24, 30$

Since $f(2+x) = f(2-x) \Rightarrow f(x)$ is symmetric about $x = 2$

$\Rightarrow f(0) = f(4) \Rightarrow$ using periodic nature

$f(x) = 101$ at $x = 4, 10, 16, 22, 28 \Rightarrow f(5+x) = f(5-x)$

x is symmetric about $x = 5$

$f(0) = f(10) \Rightarrow x = 4, 10, 16, 22$

$f(6) = f(4) \quad \Rightarrow \quad x = 0, 6, 12, 18,$

Total different values of x are 0,4, 6, 10, 12, 16, 18, 22, 24, 28, 30

2. Ans. (2)

$f(x)$ is continous at $x = 1 \Rightarrow a-b = -1$

$$\text{L.H.D.} = \lim_{h\to 0} \frac{a(1-h)^2 - b + 1}{-h} = 2a$$

$$\text{R.H.D.} = \lim_{h\to 0} \frac{\frac{-1}{1+h}+1}{h} = 1$$

it gives $\Rightarrow a = \frac{1}{2}$, $b = \frac{3}{2}$

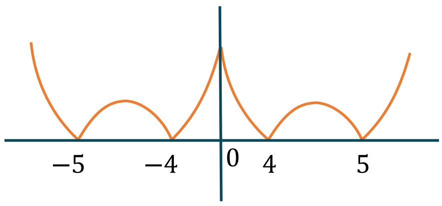
3. Ans. (9)

$f(x) \,.\, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \quad \Rightarrow \quad f(x) = 1 \pm x^n$

$f(3) = -26 \quad \Rightarrow \quad f(x) = 1 - x^3$

$f'(x) = -3x^2 \quad$ or $\quad f'(1) = -3$

4. Ans. (5)



Not differentiable at 5 sharp corner points x = 0, ±4, ±5

5. Ans. (6)

$f(x)$ must be continuous and differentiable at $x = 0$

So, $a = 5\sqrt{2}$, $7 + b = 4\sqrt{3}$.

$$a-7 = \sqrt{50}-\sqrt{49} = \frac{1}{\sqrt{50}+\sqrt{49}} < \frac{1}{\sqrt{49}+\sqrt{48}} = \sqrt{49}-\sqrt{48} = -b \Rightarrow a+b<7$$

6. Ans. (1)

$x = -1, y = 0 : f(0) + f(-1) + f(0) = -1$

$x = 0, y = -1 : f(0) + f(1) + f(0) = 1$

$x = -1, y = -1 : f(1) + f(0) + f(-1) = 0$

So, $f(0) = 0$

$x = 0, y = t : f(-t) + f(t+1) = 1$

$x = t, y = -1 : f(-t) + f(t + 1) + f(t) = t+1$

So, $f(t) = t$

$P(x) = \frac{2016^t}{2016^t + \sqrt{2016}}$ and $P(1 - x) = \frac{2016^{1-t}}{2016^{1-t} + \sqrt{2016}}$

$P(x) + P(1 - x) = 1$

Put $x = \frac{1001}{2016}$

7. **Ans. (5)**

$$\prod_{k=1}^{\infty}\left(\frac{1+2\cos\frac{2x}{3^k}}{3}\right)$$

$$\prod_{k=1}^{\infty}\frac{1}{3}\left[1+2-4\sin^2\frac{x}{3^k}\right]=\prod_{k=1}^{\infty}\frac{1}{3\sin\frac{x}{3^k}}\left\{3\sin\frac{x}{3^k-4}\sin^3\frac{x}{3^k}\right\}$$

$$\prod_{k=1}^{\infty}\left[\frac{\sin\frac{x}{3^{k-1}}}{3\sin\frac{x}{3^k}}\right]$$

$$\lim_{k\to\infty}\frac{1}{3^k}\left\{\frac{\sin x}{\sin\frac{x}{3}}\times\frac{\sin\frac{x}{3}}{\sin\frac{x}{9}}.....\frac{\sin\frac{x}{3^{k-1}}}{\sin\frac{x}{3^k}}\right\}$$

$$\lim_{k\to\infty}\frac{\sin x}{\frac{\sin\left(\frac{x}{3^k}\right)}{\left(\frac{x}{3^k}\right)}}\Rightarrow f(x)=\frac{\sin x}{x}$$

$xf(x) = \sin x$

$[\sin x] + |\sin x| + (x - 1)\,|(x - 1)(x - 2)|$

N.D. at 5 points

8. **Ans. (1)**

For 11 points of discontinuity

$11 < 3n\sin x \le 12 \Rightarrow n \in \left(\frac{11}{3}, 4\right]$

9. **Ans. (2)**

Both roots of equation $x^2 - x + k - 2 = 0$ must be distinct and positive

then, $k \in \left(2, \frac{9}{4}\right)$

$\therefore\ 8(b - a) = 2$

10. Ans. (2)

If is possible when all equation have one common root.

$$\left.\begin{aligned} x^2+\lambda x+12=0 \\ x^2+\mu x+15=0 \\ x^2+(\lambda+\mu)x+36=0 \end{aligned}\right\} \text{common root} \quad \begin{aligned} \lambda=7,\ \mu=8 \\ \text{or} \\ \lambda=-7, \mu=-8 \end{aligned}$$

$x^2 + (\lambda + \mu)x + 36 = 0$ has equal roots and

$r(x), f(x)$ have a common factor.

We get $\lambda, \mu \in \phi$

$x^2 + (\lambda + \mu)x + 36 = 0$

have equal roots and $f(x)$, g(x) have a common factor

we get $\lambda, \mu \in \phi$

Exactly two possibility that

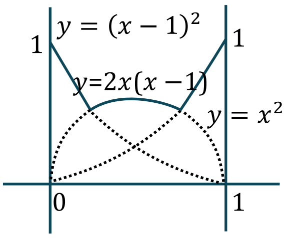
$\lambda = 7, \mu = 8$ or $\lambda = -7, \mu = -8$

JEE (Advanced) Practice Paper

1. Ans. (C)

$$y = f(x) = max\{x^2, (x-1)^2, 2x(1-x)\}$$

$y = f(x)$

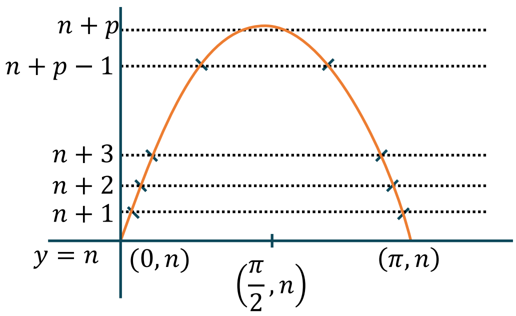


Non-differentiable at two points.

2. Ans. (D)

$f(x) = [n + p\sin x], x \in (0, \pi)$

graph of $y = n + p \sin x$



obviously

$f(x) = [n + p\sin x]$ is discontinous at points mark in above curve

$\Rightarrow$ number of such points $\Rightarrow (p-1) + 1 + p - 1 = 2p - 1$

3. Ans. (C)

$f(x)=x^3-x^2+x+1$

$f'(x)=3x^2-2x+1>1, \quad \forall \quad x\in R$

$f(x)$ is strictly increasing

So $g(x) = \begin{cases} \max\ \{f(t)\ ;\ 0\le t\le x\}, & 0\le x\le 1 \\ 3-x+x^2, & 1<x\le 2 \end{cases} = \begin{cases} f(x) & 0\le x\le 1 \\ 3-x+x^2 & 1<x\le 2 \end{cases}$

$g(1) = f(1) = 1 - 1 + 1 + 1 = 2 = \lim\limits_{x\to 1^-} g(x)$

$\lim\limits_{x\to 1^+} g(x) = 3 - 1 + 1 = 3$

$g(x)$ is neither continous nor differentiable at $x = 1$

4. Ans. (D)

$f\left(\frac{x+y}{3}\right)=\frac{4-2f(x)-2f(y)}{3} \qquad \forall, x, y\in R \qquad$ ......(1)

differentiate w.r.t. y

$\frac{1}{3}f'\left(\frac{x+y}{3}\right)=-\frac{2}{3}f'(y)$

replace x with $3x$ and y with 0

$f'(x) = -2f'(0)$

put $x = 0$ we get $f'(0) = 0 \Rightarrow f'(x) = 0 \Rightarrow f(x)$ = constant

$\Rightarrow f(x) = f(0)$

in equation (1) put $x = 0 = y$ it gives $f(0) = \frac{4}{7} \Rightarrow f(x) = \frac{4}{7}$

5. Ans. (B)

$f(x) = \begin{cases} \frac{x(3e^{1/x}+4)}{2-e^{1/x}}, & x\neq 0 \\ 0 & x=0 \end{cases}$

$f'(0^-) = \lim\limits_{h\to 0}\frac{\frac{-h(3e^{-1/h}+4)}{2-e^{-1/h}}-0}{-h}=2$

$f'(0+) = \lim\limits_{x\to 0} f(x)=0=\lim\limits_{h\to 0^+} f(x)=f(0)$ not differentiable

continuous

6. Ans. (A,B,D)

$f(x)=\begin{cases} \pi\log_\pi(-x) & x<-\pi \\ -x & -\pi\le x<0 \\ x & 0\le x\le\pi \\ \pi\log_\pi x & x>\pi \end{cases}$

$\because f(x)$ is continuous everywhere.

$\Rightarrow$ check differentiability at $x = \pi$

$\Rightarrow \begin{cases} f'(\pi^+) = \pi.\dfrac{1}{\pi\ell n\pi} \\ f'(\pi^-) = 1 \end{cases}$ Not differentiable

at $x = 0$

$\begin{cases} f'(0^+) = 1 \\ f'(0^-) = -1 \end{cases}$ Not differentiable at $x = 0$

at $x = -\pi$

$\begin{cases} f'(-\pi^+) = -1 \\ f'(-\pi^-) = -\dfrac{1}{\ell n\pi} \end{cases}$ Not differentiable at $x = 0$

7. Ans. (A, B, D)

$f(x) = \left|x - \frac{1}{2}\right| + |x - 1| + \tan x$

$\left|x - \frac{1}{2}\right|$ is non-differentiable at $x = \frac{1}{2}$

$|x - 1|$ is non-differentiable at $x = 1$

$\tan x$ is non-differentiable at $x = \frac{\pi}{2}$

8. Ans. (B, D)

$y = f\ (x) = \begin{cases} (\sin^{-1} x)^2 \ \ \cos 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f(x)$ can be discontinuous only at $x = 0$ in $[-1,1]$

So we check only at $x = 0$

LHD $(x = 0) = \lim\limits_{h\to 0} h \ \dfrac{(\sin^{-1}(-h))^2 \cos\left(-\frac{1}{h}\right) - 0}{-h}$

$\lim\limits_{h\to 0}\left(\dfrac{\sin^{-1}(-h)}{h}\right)^2 . \ h\cos\left(\dfrac{1}{h}\right)$ = –1. 0. [finite quantity between [–1,1]] = 0

RHD $(x = 0)$ is $\lim\limits_{h\to 0} h \ \dfrac{(\sin^{-1}(-h))^2}{h^2} . \ \cos\left(\dfrac{1}{h}\right) = 0$

Hence $f(x)$ is differentiable as well as continuous in $[-1,1]$

9. Ans. (A, C)

$f(x) = \sum\limits_{k=0}^{n} a_k |x|^k = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3 + \ldots\ldots\ldots\ldots + a_n|x|^n$

$f(0) = a_0$ we know that $\lim\limits_{x\to 0}|x| = 0$

$\lim\limits_{x\to 0} f(x) = a_0$

$f(x)$ is continous for $x = 0$

$|x|^n$ is differentiable if $n \neq 1, n \in N$

$f(x)$ is not differentiable at $x = 0$, due to presence of $|x|$

If all $a_{2k+1} = 0$, $f(x)$ does not contains $|x| \Rightarrow f(x)$ is differentiable at $x = 0$

10. Ans. (C)

$g(t) = \lim\limits_{x \to 0} (1 + a \tan x)^{t/x}$

$g(t) = e^{\lim\limits_{x \to 0} \frac{t}{x} a \tan x} = e^{\lim\limits_{x \to 0} ta \frac{\tan x}{x}}$

$g(t) = e^{ta} = e^{ta}$

$g(x) = e^{ax}$

$\because a = 2,\ g(x) = e^{2x}$

$g(2) = e^4$

11. Ans. (C)

$f(x) = \begin{cases} x \quad e^{ax} & , \quad x \leq 0 \\ x + ax^2 - x^3 & , \quad x > 0 \end{cases}$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x + ax^2 - x^3 = 0$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x\, e^{ax} = 0$

$f(0) = 0$

$a \in (0, \infty)$

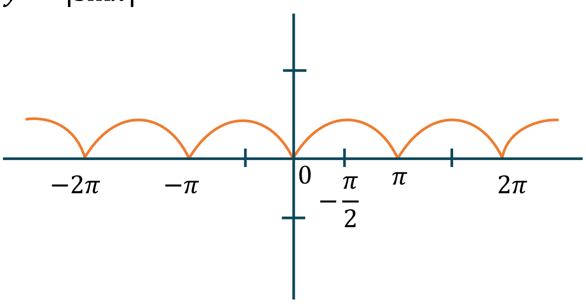
12. Ans. (C)

$f'(0^+) = \lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{h + ah^2 - h^3}{h} = 1$

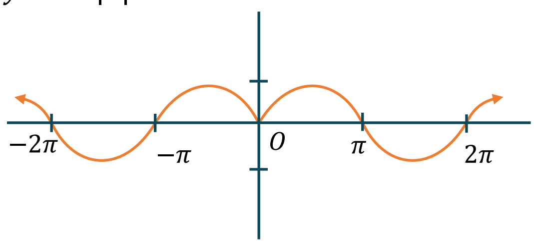
$f'(0^-) = \lim\limits_{h \to 0^-} \dfrac{f(o-h) - f(o)}{-\ h} = \lim\limits_{h \to 0^-} \dfrac{he^{-ah}}{h} = 1$

13. Ans. (7)

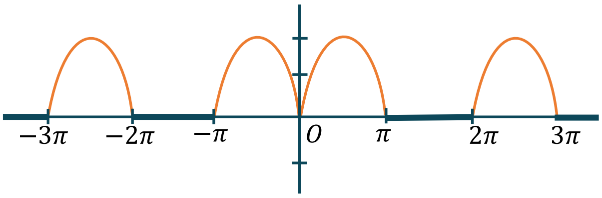
$y = |\sin x|$



$y = \sin|x|$



$y = f(x) = |\sin x| + \sin|x|$



$f(x)$ is continous every where

$f(x)$ is not differentiable at $x = n\pi$

$f(x)$ is not periodic

14. Ans. (4)

Differentiability at $x = 1$

$$f'(1^-) = \lim_{h\to 0} \frac{\frac{\sin[(1-h)^2]\quad \pi}{(1-h)^2-3\quad(1-h)+8}+a\quad(1-h)^3+b-(a+b)}{-h}$$

$$= \lim_{h\to 0} \frac{a\quad(1-h)^3-a}{-h}\left(\frac{0}{0}\text{ form}\right) = \lim_{h\to 0}\frac{3a\quad(1-h)^2}{1}$$

$$f'(1^-) = 3a$$

$$f'(1^+) = \lim_{h\to 0}\frac{2\cos(1+h)\quad\pi+\tan^{-1}(1+h)-a-b}{h} = \lim_{h\to 0}\frac{(-2\cos\quad\pi h+\tan^{-1}(1+h)-a-b)}{h}$$

Function is differentiable

$\therefore\quad -2+\frac{\pi}{4} = a+b \qquad .....(1)$

$$= \lim_{h\to 0}\frac{-2\cos\pi h\quad+\tan^{-1}(1+h)-2+\pi/2}{h} = \lim_{h\to 0} 2\pi\sin\pi h + \frac{1}{1+(1+h)^2} = \frac{1}{2}$$

Now $f'(1^-) = f'(1^+)$

$3a = \frac{1}{2}$

$a = \frac{1}{6} \qquad ....(2)$

by (1) and (2) $b = \frac{\pi}{4} - \frac{13}{6}$

15. Ans. (7)

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}\quad x^2e^{2(x-1)} = 1$

$f(1) = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} a\text{ sgn }(x+1)\cos 2(x-1) + bx^2 = a.1.1 + b$

for continuity $a + b = 1$

$$\text{LHD } (x = 1) \text{ is } \lim_{h\to 0}\frac{(1-h)^2e^{-2h}-1}{h} = \lim_{h\to 0} 2e^{-2h}+he^{-2h}+\left(\frac{e^{-2h}-1}{h}\right)$$

$$= 2 + 0 + 2 = 4$$

$$\text{RHD } (x = 1) \text{ is } \lim_{h\to 0}\frac{a\,\text{sgn}(2+h)\cos 2h+b(1+h)^2-1}{h} = \lim_{h\to 0}\frac{a\cos 2h+b+bh^2+2bh-(a+b)}{h}$$

$= \lim_{h\to 0} a\left(\frac{\cos 2h - 1}{h}\right) + bh + 2b = 2b$

$f(x)$ is differentiable at $x = 1$ if $2b = 4$ $b = 2$ $a = -1$

16. **Ans. (0)**

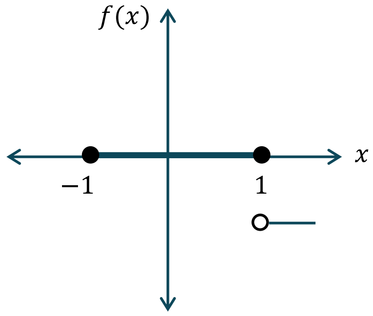
As $0 < \{e^x\} < 1$

$\therefore \lim_{n\to\infty} \frac{\{e^x\}^n - 1}{\{e^x\}^n + 1} = -1 \Rightarrow f(x) = -1 \ \forall\ x \in R$

17. **Ans. (3)**

$f(x) = [x \sin \pi x]$

graph of $f(x)$ is as shown in the figure



18. **Ans. (2)**

$g(x) = (|x - 1| + |4x - 11|)\ [(x - 1)^2 - 3] = (|x - 1| + |4x - 11|)\ ([(x - 1)^2] - 3)$

Now $|x - 1| + |4x - 11|$ is continuous every where

& $[(x - 1)^2 - 3]$ is discontinuous at $x = 1, 2\ ;\ \sqrt{2} + 1$

At $x = 1$, $g(x)$ is continuous

At $x = 2$, $g(x)$ is discontinuous

At $x = (\sqrt{2} + 1)$, $g(x)$ is discontinuous