01 Determinants

Introduction:

If the equations $a_1x + b_1 = 0, a_2x + b_2 = 0$ are satisfied by the same value of x, then $a_1b_2 - a_2b_1 = 0$.
The expression $a_1b_2 - a_2b_1$ is called a determinant of the second order and is denoted by:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

A determinant of second order consists of two rows and two columns.
Next consider the system of equations $a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0, a_3x + b_3y + c_3 = 0$
If these equations are satisfied by the same values of x and y, then on eliminating x and y we get.

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1a_2b_3 - a_3b_2) = 0$$

The expression on the left is called a determinant of the third order, and is denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

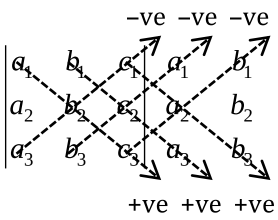
A determinant of third order consists of three rows and three columns.

Value of a Determinant:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Note: Sarrus diagram to get the value of determinant of order three:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$



$$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)$$

Note that the product of the terms in first bracket (i.e. $a_1a_2a_3b_1b_2b_3c_1c_2c_3$) is same as the product of the terms in second bracket.

Illustration 1:

The value of $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ is

(A) 213 (B) –231 (C) 231 (D) 39

Ans. (C)

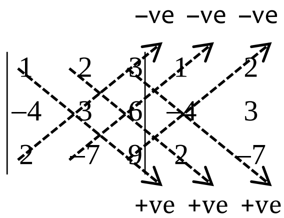
Solution:

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = 1\begin{vmatrix} 3 & 6 \\ -7 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 3 \\ 2 & -7 \end{vmatrix}$$

$= (27 + 42) - 2(-36 - 12) + 3(28 - 6) = 231$

Alternative: By Sarrus diagram

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} =$$



$= (27 + 24 + 84) - (18 - 42 - 72) = 135 - (18 - 114) = 231$

Illustration 2:

If $\begin{vmatrix} x+3 & 1 & -2 \\ 3 & -2 & 1 \\ -x & -3 & 3 \end{vmatrix} = 0$, find x.

Solution:

$$\begin{vmatrix} x+3 & 1 & -2 \\ 3 & -2 & 1 \\ -x & -3 & 3 \end{vmatrix} = 0$$

$\Rightarrow (x+3)[-6-(-3)] - 1[9 + x] - 2[-9 - 2x]$

$\Rightarrow (x+3)(-6+3) - (9+x) - 2(-9-2x) = 0$

$\Rightarrow -3(x+3) - 9 - x + 18 + 4x = 0$

$\Rightarrow -3x - 9 - 9 - x + 18 + 4x = 0$

$\Rightarrow -4x + 4x - 18 + 18 = 0$

$\therefore\ x \in R.$

Cofactor and Minors of an Element:

Minors:

The minor of a given element of determinant is the determinant obtained by deleting the row and the column in which the given element stands.

e.g., The minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ and the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order three will have "9 minors".

Cofactors:

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element is given by : $\boldsymbol{C_{ij} = (-1)^{i+j} . M_{ij}}$

Illustration 3:

Find the minors and cofactors of elements $'-3', '5', '-1'$ and $'7'$ in the determinant $\begin{vmatrix} 2 & -3 & 1 \\ 4 & 0 & 5 \\ -1 & 6 & 7 \end{vmatrix}$.

Solution:

Minor of –3 = $\begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix}$ = 33; Cofactor of – 3 = – 33

Minor of 5= $\begin{vmatrix} 2 & -3 \\ -1 & 6 \end{vmatrix}$ = 9 ; Cofactor of 5 = – 9

Minor of –1 = $\begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix}$ = –15 ; Cofactor of –1= –15

Minor of 7 = $\begin{vmatrix} 2 & -3 \\ 4 & 0 \end{vmatrix}$ = 12 ; Cofactor of 7=12

Expansion of a Determinant in Terms of the Elements of any Row or Column:

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(i) The sum of the product of elements of any row (column) with their corresponding cofactors is always equal to the value of the determinant.

D can be expressed in any of the six forms:

$a_1A_1 + b_1B_1 + c_1C_1, a_1A_1 + a_2A_2 + a_3A_3,$

$a_2A_2 + b_2B_2 + c_2C_2, b_1B_1 + b_2B_2 + b_3B_3,$

$a_3A_3 + b_3B_3 + c_3C_3, c_1C_1 + c_2C_2 + c_3C_3,$

where A_i, B_i and $C_i (i = 1, 2, 3)$ denote cofactors of a_i, b_i and c_i respectively.

(ii) The sum of the product of elements of any row (column) with the cofactors of other row (column) is always equal to zero.

Hence, $a_2A_1 + b_2B_1 + c_2C_1 = 0,$

$b_1A_1 + b_2A_2 + b_3A_3 = 0$ and so on.

where A_i, B_i and $C_i (i = 1, 2, 3)$ denote cofactors of a_i, b_i and c_i respectively.

Illustration 4:

If in the determinant $\begin{vmatrix} x & 3 & 3 \\ 3 & 3 & x \\ 2 & 3 & 3 \end{vmatrix}$, $C_{11} = C_{22}$, where C_{ij} is cofactor of element a_{ij} then $x =$

(A) 2 (B) –2 (C) $\frac{5}{2}$ (D) $-\frac{9}{8}$

Ans. (C)

Solution:

$C_{11} = C_{22} \Rightarrow (-1)^{1+1} \begin{vmatrix} 3 & x \\ 3 & 3 \end{vmatrix} = (-1)^{2+2} \begin{vmatrix} x & 3 \\ 2 & 3 \end{vmatrix}$

$\Rightarrow 9 - 3x = 3x - 6 \Rightarrow -6x = -15 \Rightarrow x = \frac{5}{2}$

Properties of Determinants (Part-1):

(a) The value of a determinant remains unaltered, if the rows and columns are inter-changed,

e.g. if $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.

e.g. Let $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_1=\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then $D_1 = -D$.

(c) If all the elements of a row (or column) are zero, then the value of the determinant is zero.

(d) If all the elements of any row (or column) are multiplied by the same number, then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D_1 = KD$

(e) If all the elements of a row (or column) are proportional (or identical) to the element of any other row, then the determinant vanishes, i.e. its value is zero.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D = 0$; If $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D_1 = 0$

Illustration 5:

Prove that $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$

Solution:

$D = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$ (By interchanging rows and columns)

$= -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$ $(C_1 \leftrightarrow C_2)$

$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$ $(R_1 \leftrightarrow R_2)$

Illustration 6:

Find the value of the determinant $\begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix}$

Solution:

$$D = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = a\begin{vmatrix} a & b & c \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = abc\begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0$$

Since all rows are same, hence value of the determinant is zero.

(f) If each element of any row (or column) is expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

e.g. $$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Note: If $D_r = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

where $r \in N$ and a, b, c, a_1, b_1, c_1 are constants, then

$$\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

Illustration 7:

If $D_r = \begin{vmatrix} r & r^3 & 2 \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix}$, find $\sum_{r=0}^{n} D_r$.

Solution:

$$\sum_{r=0}^{n} D_r = \begin{vmatrix} \sum_{r=0}^{n} r & \sum_{r=0}^{n} r^3 & \sum_{r=0}^{n} 2 \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \\ n & n^3 & 2n \\ \frac{n(n+1)}{2} & \left(\frac{n(n+1)}{2}\right)^2 & 2(n+1) \end{vmatrix} = 0$$

Properties of Determinants (Part-2):

(g) **Row - column operation:** The value of a determinant remains unaltered under a column (C_i) operation of the form $C_i \to C_i + \alpha C_j + \beta C_k\ (j, k \neq i)$ or row (R_i) operation of the form $R_i \to R_i + \alpha R_j + \beta R_k\ (j, k \neq i)$. In other words, the value of a determinant is not altered by adding the elements of any row (or column) to the same multiples of the corresponding elements of any other row
(or column)

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$D = \begin{vmatrix} a_1+\alpha a_2 & b_1+\alpha b_2 & c_1+\alpha c_2 \\ a_2 & b_2 & c_2 \\ a_3+\beta a_2 & b_3+\beta b_2 & c_3+\beta c_2 \end{vmatrix} (R_1 \to R_1 + \alpha R_2;\ R_3 \to R_3 + \beta R_2)$$

Note:

(i) By using the operation $R_i \to xR_i + yR_j + zR_k\ (j, k \neq i)$, the value of the determinant becomes x times the original one.

(ii) While applying this property **ATLEAST ONE ROW (OR COLUMN)** must remain unchanged.

Illustration 8:

If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, then the value of k is

(A) 2 (B) 1 (C) –1 (D) 0

Ans. (B)

Solution:

Applying $(C_3 \to C_3 - C_1)$

$$D = \begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 9 & 11 & 1 \end{vmatrix} = 0\ (R_3 \to R_3 - R_2;\ R_2 \to R_2 - R_1)$$

$\Rightarrow\ k - 1 = 0 \Rightarrow k = 1$

Illustration 9:

Prove that $\begin{vmatrix} 23 & 66 & 11 \\ 36 & 55 & 26 \\ 63 & 143 & 37 \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} 23 & 66 & 11 \\ 36 & 55 & 26 \\ 63 & 143 & 37 \end{vmatrix}$$

Taking 11 common from C_2

$$= 11\begin{vmatrix} 23 & 6 & 11 \\ 36 & 5 & 26 \\ 63 & 13 & 37 \end{vmatrix}$$

$C_1 \to C_1 - (2C_2 + C_3)$

$$= 11\begin{vmatrix} 0 & 6 & 11 \\ 0 & 5 & 26 \\ 0 & 13 & 37 \end{vmatrix} = 0$$

Special Determinants:

(a) Cyclic Determinant:

The elements of the rows (or columns) are in cyclic arrangement.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc) = -(a+b+c)\,(a^2+b^2+c^2-ab-bc-ac)$$

$$= -\frac{1}{2}(a+b+c) \times \{(a-b)^2 + (b-c)^2 + (c-a)^2\}$$

$= -(a+b+c)\,(a+b\omega+c\omega^2)\,(a+b\omega^2+c\omega)$, where ω, ω^2 are cube roots of unity

(b) Other Important Determinants:

(i) $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$

Illustration 10:

Prove that = $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix} = -(1-\alpha^3)^2$.

Solution:

This is a cyclic determinant.

$\Rightarrow \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix} = -(1+\alpha+\alpha^2)(1+\alpha^2+\alpha^4-\alpha-\alpha^2-\alpha^3)$

$= -(1+\alpha+\alpha^2)(-\alpha+1-\alpha^3+\alpha^4) = -(1+\alpha+\alpha^2)(1-\alpha)^2(1+\alpha+\alpha^2)$

$= -(1-\alpha)^2(1+\alpha+\alpha^2)^2 = -(1-\alpha^3)^2$

Illustration 11:

If α, β and γ are the roots of the equation $x^3+px+q=0$ then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is-

(A) p (B) q (C) p^2-2q (D) 0

Ans. (D)

Solution:

α, β , γ are roots of $x^3+px+q=0$

$\therefore \alpha+\beta+\gamma=0$

$\Rightarrow$ on $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$, Applying $C_1 \to C_1+C_2+C_3$

$\Rightarrow (\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta & \gamma \\ 1 & \gamma & \alpha \\ 1 & \alpha & \beta \end{vmatrix} = 0$

Factor Theorem:

If the elements of a determinant D are rational integral functions of x and two rows (or columns) become identical when $x=a$ then $(x-a)$ is a factor of D.

Note: If r rows become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of D.

Illustration 12:

Prove that $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = m(x-a)(x-b)$

Solution:

Using factor theorem,

Put $x=a$

$D = \begin{vmatrix} a & a & a \\ m & m & m \\ b & a & b \end{vmatrix} = 0$

Since R_1 and R_2 are proportional which makes $D=0$, therefore $(x-a)$ is a factor of D.

Similarly, by putting $x = b, D$ becomes zero, therefore $(x - b)$ is a factor of D.

$$\Rightarrow D = \begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = \lambda(x-a)(x-b) \qquad \text{...(i)}$$

To get the value of λ, put $x = 0$ in equation (i)

$$\Rightarrow \begin{vmatrix} a & a & 0 \\ m & m & m \\ b & 0 & b \end{vmatrix} = \lambda ab$$

$\Rightarrow amb = \lambda ab \Rightarrow \lambda = m$

$\therefore D = m(x-a)(x-b)$

Illustration 13:

Using factor property of determinants prove that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Solution:

On putting $x = y, y = z, z = x$ we are getting value of determinant is 0 so by factor theorems $(x - y), (y - z)$ and $(z - x)$ are factors. So by factor theorem

$$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \lambda(x-y)(y-z)(z-x)$$

For λ put $x = 0, y = 1, z = 2$

We get $\boxed{\lambda = 1}$

So R.H.S. = $(x - y)(y - z)\,(z - x)$

Multiplication of Two Determinants (Part-1):

If A and B are two square matrices of same order, then $|AB| = |A|\,|B|$.

Multiplication of Two Determinants of Order 2×2

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 & a_1m_1 + b_1m_2 \\ a_2\ell_1 + b_2\ell_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

Multiplication of Two Determinants of Order 3×3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 + c_1\ell_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2\ell_1 + b_2\ell_2 + c_2\ell_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3\ell_1 + b_3\ell_2 + c_3\ell_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$$

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D_1 is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D_1 = D^{n-1}$

Illustration 14:

Find the value of $\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \times \begin{vmatrix} 3 & 0 \\ -1 & 4 \end{vmatrix}$ and prove that it is equal to $\begin{vmatrix} 1 & 8 \\ -6 & 12 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \times \begin{vmatrix} 3 & 0 \\ -1 & 4 \end{vmatrix} = \begin{vmatrix} 1\times3-2\times1 & 1\times0+2\times4 \\ -1\times3+3\times(-1) & -1\times0+3\times4 \end{vmatrix} = \begin{vmatrix} 1 & 8 \\ -6 & 12 \end{vmatrix} = 60$$

Illustration 15:

Show that $\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ac+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} = \lambda^3(\lambda^2+a^2+b^2+c^2)^3$

Solution:

We observe that the elements in the first determinant are the cofactors of corresponding elements of second determinant.

So $\begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}^3 = [\lambda\,(\lambda^2+a^2+b^2+c^2)]^3$

$= \lambda^3(\lambda^2+a^2+b^2+c^2)^3$

Multiplication of Two Determinants (Part-2):

Illustration 16:

Prove that $\begin{vmatrix} a_1x_1+b_1y_1 & a_1x_2+b_1y_2 & a_1x_3+b_1y_3 \\ a_2x_1+b_2y_1 & a_2x_2+b_2y_2 & a_2x_3+b_2y_3 \\ a_3x_1+b_3y_1 & a_3x_2+b_3y_2 & a_3x_3+b_3y_3 \end{vmatrix} = 0$

Solution:

Given determinant can be splatted into product of two determinants

i.e. $\begin{vmatrix} a_1x_1+b_1y_1 & a_1x_2+b_1y_2 & a_1x_3+b_1y_3 \\ a_2x_1+b_2y_1 & a_2x_2+b_2y_2 & a_2x_3+b_2y_3 \\ a_3x_1+b_3y_1 & a_3x_2+b_3y_2 & a_3x_3+b_3y_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$

Illustration 17:

Let α & β be the roots of equation $ax^2+bx+c=0$ and $S_n = \alpha^n + \beta^n$ for $n \geq 1$. Evaluate the value of the determinant $\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$.

Solution:

$$D = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix} = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}^2 = [(1-\alpha)(1-\beta)(\alpha-\beta)]^2$$

$\Rightarrow D = (\alpha-\beta)^2\,(\alpha+\beta-\alpha\beta-1)^2$

$\because$ α & β are roots of the equation $ax^2+bx+c=0$

$$\Rightarrow \alpha+\beta = \frac{-b}{a} \;\&\; \alpha\,\beta = \frac{c}{a} \Rightarrow |\alpha-\beta| = \frac{\sqrt{b^2-4ac}}{|a|}$$

$$\Rightarrow D = \frac{(b^2-4ac)}{a^2}\left(\frac{a+b+c}{a}\right)^2 = \frac{(b^2-4ac)(a+b+c)^2}{a^4}$$

Illustration 18:

Prove that $\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\,\beta+\gamma\,\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\,\beta(\gamma+\delta)+\gamma\,\delta(\alpha+\beta) \\ \alpha\,\beta+\gamma\,\delta & \alpha\,\beta(\gamma+\delta)+\gamma\,\delta(\alpha+\beta) & 2\alpha\,\beta\,\gamma\,\delta \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\,\beta & \gamma\,\delta & 0 \end{vmatrix}\begin{vmatrix} 1 & \gamma+\delta & \gamma\,\delta \\ 1 & \alpha+\beta & \alpha\,\beta \\ 0 & 0 & 0 \end{vmatrix}$$

$= 0.0$

$= 0$

Illustration 19:

If α, β & γ are real numbers, then prove that $D = \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0$

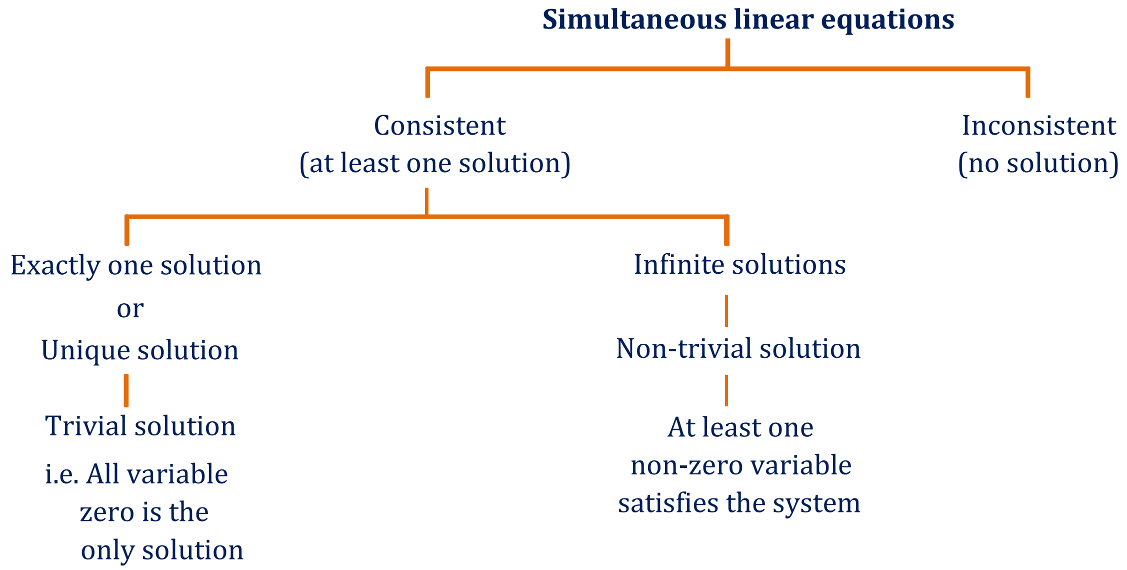
Solution:

$$\begin{vmatrix} 1 & \cos\beta\cos\alpha+\sin\alpha\sin\beta & \cos\gamma\cos\alpha+\sin\gamma\sin\alpha \\ \cos\alpha\cos\beta+\sin\alpha\sin\beta & 1 & \cos\gamma\cos\beta+\sin\gamma\sin\beta \\ \cos\alpha\cos\gamma+\sin\alpha\sin\gamma & \cos\beta\cos\gamma+\sin\beta\sin\gamma & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2\alpha+\sin^2\alpha & \cos\beta\cos\alpha+\sin\alpha\sin\beta & \cos\gamma\cos\alpha+\sin\gamma\sin\alpha \\ \cos\alpha\cos\beta+\sin\alpha\sin\beta & \cos^2\beta+\sin^2\beta & \cos\gamma\cos\beta+\sin\gamma\sin\beta \\ \cos\alpha\cos\gamma+\sin\alpha\sin\gamma & \cos\beta\cos\gamma+\sin\beta\sin\gamma & \cos^2\gamma+\sin^2\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \sin\alpha & \sin\beta & \sin\gamma \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Cramer's Rule (System of Linear Equation) (Part-1):



(a) Equations Involving Two Variables:

(i) Consistent Equations : Definite & unique solution (Intersecting lines)
(ii) Inconsistent Equations : No solution (Parallel lines)
(iii) Dependent Equations : Infinite solutions (Identical lines)

Let, $a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$ then:

(1) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow$ Given equations are consistent with unique solution

(2) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent

(3) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are consistent with infinite solutions

(b) Equations Involving Three Variables:

Let $a_1x + b_1y + c_1z = d_1$ ...(i)
$a_2x + b_2y + c_2z = d_2$ ...(ii)
$a_3x + b_3y + c_3z = d_3$ ...(iii)

Then, $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Note: (i) If $D \neq 0$ and atleast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations is consistent and has unique non-trivial solution.

(ii) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent and has trivial solution only.

(iii) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

(iv) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations may have infinite or no solution.

Note: In case $\left.\begin{array}{l}a_1x+b_1y+c_1z=d_1\\a_1x+b_1y+c_1z=d_2\\a_1x+b_1y+c_1z=d_3\end{array}\right\}$ (Atleast two of d_1, d_2 & d_3 are not equal)

$D = D_1 = D_2 = D_3 = 0$. But these three equations represent three parallel planes. Hence the system is inconsistent.

Illustration 20:

If the system of equations $x + \lambda y + 1 = 0, \lambda x + y + 1 = 0$ & $x + y + \lambda = 0$ is consistent, then find the value of λ.

Solution:

For consistency of the given system of equations

$$D=\begin{vmatrix}1 & \lambda & 1\\ \lambda & 1 & 1\\ 1 & 1 & \lambda\end{vmatrix}=0$$

$\Rightarrow 3\lambda = 1 + 1 + \lambda^3$ or $\lambda^3 - 3\lambda + 2 = 0$

$\Rightarrow (\lambda - 1)^2 (\lambda + 2) = 0$

$\Rightarrow \lambda = 1$ or $\lambda = -2$

Illustration 21:

Find the nature of solution for the given system of equations:

$x + 2y + 3z = 1;\ 2x + 3y + 4z = 3;\ 3x + 4y + 5z = 0$

Solution:

$$D = \begin{vmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 3 & 4 & 5\end{vmatrix} = 0$$

Now, $D_1 = \begin{vmatrix}1 & 2 & 3\\ 3 & 3 & 4\\ 0 & 4 & 5\end{vmatrix} = 5$

$\because\ D = 0$ but $D_1 \neq 0$

Hence no solution.

Illustration 22:

Let p, q, r be real numbers such that $p + q + r \neq 0$. The system of linear equations

$x + 2y - 3z = p$

$2x + 6y - 11z = q$

$x - 2y + 7z = r$

has at least one solution if :

(A) $5p + 2q - r = 0$

(B) $5p - 2q - r = 0$

(C) $5p + 2q + r = 0$

(D) $5p - 2q + r = 0$

Ans. (B)

Solution:

$x + 2y - 3z = p$

$2x + 6y - 11z = q$

$x - 2y + 7z = r$

$$\Rightarrow D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{vmatrix}$$

$= 1(42 - 22) - 2(14 + 11) - 3\,(-4 - 6)$

$= 20 - 50 + 30 = 0$

$$\Rightarrow D_1 = \begin{vmatrix} p & 2 & -3 \\ q & 6 & -11 \\ r & -2 & 7 \end{vmatrix} = p(20) - 2(7q + 11r) - 3(-2q - 6r)$$

$= 20p - 14q - 22r + 6q + 18r$

$= 20p - 8q - 4r = 4(5p - 2q - r)$

If $D_1 = 0$, then there are infinite solutions which confirm at least one solution.

$\therefore 5p - 2q - r = 0$

Cramer's Rule (System of Linear Equation) (Part-2):

Homogeneous system of linear equation:

If x, y, z are not all zero, the condition for

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

to be consistent in x, y, z is that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

Remember that if a given system of linear equations have **Only Zero** Solution for all its variables then the given equations are said to have **TRIVIAL SOLUTION**.

Application of Determinants in Geometry:

(a) The lines: $a_1x + b_1y + c_1 = 0$ ...(i)

$a_2x + b_2y + c_2 = 0$ ...(ii)

$a_3x + b_3y + c_3 = 0$ ...(iii)

are concurrent or all three parallel if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

This is the necessary condition for consistency of three simultaneous linear equations in 2 variables but may not be sufficient.

(b) Equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(c) Area of a triangle whose vertices are (x_r, y_r); $r = 1, 2, 3$ is $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

If $D = 0$, then the three points are collinear.

(d) Equation of a straight line passing through points (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Illustration 23:

Find the value of λ, if the following equations are consistent:

$x + y - 3 = 0; (1+\lambda)x + (2+\lambda)y - 8 = 0;\ x - (1+\lambda)y + (2+\lambda) = 0$

Solution:

The given equations in two unknowns are consistent, then $\Delta = 0$

i.e. $\begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$

$\therefore \begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & 3\lambda-5 \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$

$\Rightarrow (5+\lambda) - (3\lambda-5)(-2-\lambda) = 0 \Rightarrow 3\lambda^2 + 2\lambda - 5 = 0$

$\therefore \lambda = 1, -5/3$

Illustration 24:

If the trivial solution is the only solution of the system of equations

$x - ky + z = 0$

$kx + 3y - kz = 0$

$3x + y - z = 0$

then the set of all values of k is :

(A) $R - \{2, -3\}$ (B) $R - \{2\}$ (C) $R - \{-3\}$ (D) $\{2, -3\}$

Ans. (A)

Solution:

$x - ky + z = 0$

$kx + 3y - kz = 0$

$3x + y - z = 0$

this system will have non trivial solution if (non-trival solution)

$\Rightarrow \begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow 1(-3+k) + k(-k+3k) + 1(k-9) = 0$

$\Rightarrow k - 3 + 2k^2 + k - 9 = 0$

$\Rightarrow 2k^2 + 2k - 12 = 0$

$\Rightarrow k^2 + k - 6 = 0$

$\Rightarrow k = -3, k = 2$

So the system of equations will have only trivial solution when $k \in R - \{2, -3\}$.

Illustration 25:

If the system of linear equations, $x + 2ay + az = 0, x + 3by + bz = 0$ and $x + 4cy + cz = 0$ has a non-zero solution, then a, b, c satisfy

(A) $2ac = ab + bc$ (B) $2ab = ac + bc$ (C) $2b = a + c$ (D) $b^2 = ac$

Ans. (A)

Solution:

$$\Delta = 0 \Rightarrow \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0 \Rightarrow 2ac = ab + bc$$

JEE Advance (Part-1):

Comprehension (For problem 26 to 28)

Suppose $f(x)$ is a function satisfying the following conditions:

(i) $f(0) = 2, f(1) = 1,$

(ii) f has a minimum value at $x = 5/2$

(iii) For all x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Illustration 26:

The value of $f(2)$ is

(A) 1/4 (B) -2 (C) -1 (D) 3

Illustration 27:

$f(x) = 0$ has

(A) Both roots positive (B) Both roots negative

(C) Roots of opposite sign (D) imaginary roots

Illustration 28:

Range of $f(x)$ is

(A) $[7/16, \infty)$ (B) $(-\infty, 15/16]$ (C) $[3/4, \infty)$ (D) none of these

Ans. 26(B), 27(D), 28(A)

Solution: (26 to 28)

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$

$$\Rightarrow f('x) = \begin{vmatrix} -(b+1) & -(b+2) & 2ax+b+1 \\ (b+1) & (b+2) & -1 \\ b & b+1 & 2ax+b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - R_2$, we get

$$\Rightarrow f'(x) = \begin{vmatrix} 0 & 0 & 2ax+b \\ b+1 & b+2 & -1 \\ -1 & -1 & 2ax+b+1 \end{vmatrix}$$

$= (2ax + b)[-b - 1 + b + 2]$

$\therefore\ f'(x) = 2ax + b$

$\therefore\ f(x) = ax^2 + bx + c$

$\Rightarrow f(0) = 2$

$\Rightarrow c = 2$

$\Rightarrow f(1) = 1 \Rightarrow a + b + 2 = 1$

$\Rightarrow a + b = -1$

$\Rightarrow f'\left(\frac{5}{2}\right) = 0$

$\Rightarrow 5a + b = 0$

$\Rightarrow a = \frac{1}{4}, b = \frac{-5}{4}$

Hence, $f(x) = \frac{1}{4}x^2 - \frac{5}{2}x + 2$

Clearly, discriminant (D) of the equation $f(x) = 0$ is less than 0.

Hence, $f(x) = 0$ has imaginary roots. Also, $f(2) = -2$.

and minimum value of $f(x)$ is $= -\frac{D}{4a} = -\left(\frac{\frac{25}{16} - 4\frac{1}{4}(2)}{4\frac{1}{4}}\right) = \frac{7}{16}$

Hence, range of the $f(x)$ is $\left[\frac{7}{16}, \infty\right)$

Illustration 29:

Column I	Column II
A. The value of determinant $\begin{vmatrix} x+2 & x+3 & x+5 \\ x+4 & x+6 & x+9 \\ x+8 & x+11 & x+15 \end{vmatrix}$	p. 1
B. If one of the roots of the equation $\begin{vmatrix} 7 & 6 & x^2-13 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix} = 0$ is $x - 2$, then sum of all other five roots is	q. -6
C. The value of = $\begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6} \\ \sqrt{12} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{18} & \sqrt{2}+\sqrt{12}\ i & \sqrt{27}+2i \end{vmatrix}$	r. 2
D. If $f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$ = then $f(\pi/3)$	s. -2

Ans. $(\boldsymbol{a \rightarrow s;\ b \rightarrow r; c \rightarrow q, r;\ d \rightarrow p})$

Solution:

A. The given determinant is $\Delta = \begin{vmatrix} x+2 & x+3 & x+5 \\ x+4 & x+6 & x+9 \\ x+8 & x+11 & x+15 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we have

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+5 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$$

$$=2\begin{vmatrix} x & x & x+1 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 - R_2 \text{ and } R_3 \to R_3 - R_2]$$

$$=2\begin{vmatrix} x & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_2]$$

$= -2$ [Expanding along R_3]

B. $\begin{vmatrix} 7 & 6 & x^2-13 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix}$

Let $x^2 - 13 = t$. Then

$\Rightarrow t^3 - 67t + 126 = 0$

$\Rightarrow t = -9, 2, 7 \;\Rightarrow x = \pm 2, \pm\sqrt{20}, \pm\sqrt{15}$

Hence sum of other five roots is 2.

C. $\Delta = \begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6} \\ \sqrt{12} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{18} & \sqrt{2}+\sqrt{12}\,i & \sqrt{27}+2i \end{vmatrix}$

Taking $\sqrt{6}$ common from C_1, we get

$$\Rightarrow \Delta = \sqrt{6}\begin{vmatrix} 1 & 2i & 3+\sqrt{6} \\ \sqrt{2} & \sqrt{3}+2\sqrt{2}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{3} & \sqrt{2}+2\sqrt{3}i & 3\sqrt{3}+2i \end{vmatrix}$$

Applying $R_2 \to R_2 - \sqrt{2}R_1$ and $R_3 \to R_3 - \sqrt{3}R_1$, we get

$$\Rightarrow \Delta = \sqrt{6}\begin{vmatrix} 1 & 2i & 3+\sqrt{6} \\ 0 & \sqrt{3} & \sqrt{6}i-2\sqrt{3} \\ 0 & \sqrt{2} & 2i-3\sqrt{2} \end{vmatrix} = \sqrt{6}\begin{vmatrix} \sqrt{3} & \sqrt{6}i-2\sqrt{3} \\ \sqrt{2} & 2i-3\sqrt{2} \end{vmatrix}$$

$$= \sqrt{6}\begin{vmatrix} \sqrt{3} & -2\sqrt{3} \\ \sqrt{2} & -3\sqrt{2} \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - \sqrt{2}iC_1]$$

$= \sqrt{6}(-3\sqrt{6} + 2\sqrt{6})$

$= -6$, which is an integer

D. $f(\theta)=\begin{vmatrix}\cos^2\theta & \cos\theta\sin\theta & -\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta\\ \sin\theta & -\cos\theta & 0\end{vmatrix}$

Applying $R_1 \to R_1 + (\sin\theta)R_3$ and $R_2 \to R_2 - (\cos\theta)R_3$, we get

$\Rightarrow f(\theta)=\begin{vmatrix}1 & 0 & -\sin\theta\\ 0 & 1 & \cos\theta\\ \sin\theta & -\cos\theta & 0\end{vmatrix}$

$= \sin^2\theta + \cos^2\theta = 1$

$\Rightarrow f\left(\frac{\pi}{3}\right) = 1$

JEE Advance (Part-2):

Comprehension (For problem 30 to 32)

If $x > m, y > n, z > r (x, y, z > 0)$ such that $\begin{vmatrix}x & n & r\\ m & y & r\\ m & n & z\end{vmatrix}=0$

Illustration 30:

The value of $\frac{x}{x-m}+\frac{y}{y-n}+\frac{z}{z-r}$ is

(A) 1 (B) –1 (C) 2 (D) –2

Illustration 31:

The value of $\frac{x}{x-m}-1+\frac{y}{y-n}-1+\frac{z}{z-r}-1$ is

(A) –2 (B) –4 (C) 0 (D) –1

Illustration 32:

The greatest value of $\frac{xyz}{(x-m)(y-n)(z-r)}$ is

(A) 27 (B) $\frac{-8}{27}$ (C) $\frac{64}{27}$ (D) none of these

Ans. 30(C), 31(D), 32(B)

Solution: (30 to 32)

$\begin{vmatrix}x & n & r\\ m & y & r\\ m & n & z\end{vmatrix}=0$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$\Rightarrow (x-m)(x-y)z + (n-y)(r-z)m - n(r-z)(x-m) = 0$

Dividing by $(x-m)(y-n)(z-r)$, we have

$\Rightarrow \frac{z}{z-r}+\frac{m}{x-m}+\frac{n}{y-n}=0$

$\Rightarrow \frac{z}{z-r}+\frac{m}{x-m}+\frac{n}{y-n}=0$

$\Rightarrow \frac{z}{z-r}+\frac{m}{x-m}+1+\frac{n}{y-n}+1=2$

$$\Rightarrow \frac{z}{z-r}+\frac{x}{x-m}+\frac{y}{y-n}=2 \Rightarrow \frac{z}{z-r}-1+\frac{m}{x-m}-1+\frac{n}{y-n}-1=-1$$

$$\Rightarrow \frac{m}{x-m}+\frac{n}{y-n}+\frac{r}{z-r}=-1$$

Now, $A.M. \geq G.M.$

$$\frac{\frac{z}{z-r}+\frac{x}{x-m}+\frac{y}{y-n}}{3} \geq \left(\frac{z}{(z-r)}\times\frac{x}{(x-m)}\times\frac{y}{(y-n)}\right)^{1/3}$$

$$\frac{xyz}{(x-m)(y-n)(z-r)} \leq \frac{-8}{27}$$

Illustration 33:

Column I	Column II
A. Coefficient of x in $f(x)=\begin{vmatrix} x & (1+\sin x)^3 & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & 1+x^2 & 0 \end{vmatrix}$	p. 10
B. Maximum Value of $\begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$ is	q. 0
C. If a, b, c are in $A.P.$ and $f(x)$ = $\begin{vmatrix} x+a & x^2+1 & 1 \\ x+b & 2x^2-1 & 1 \\ x+c & 3x^2-2 & 1 \end{vmatrix}$ = ?	r. −12
D. If $\begin{vmatrix} x & 2 & x \\ 1 & x & 6 \\ x & x & x+1 \end{vmatrix}$ = $a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$, then a_0 is	s. −2

Ans. ($a \to s$; $b \to p$; $c \to s$; $d \to s$)

Solution:

A. Coefficient of x in $f(x)$ is coefficient of x in $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 2 \\ x^2 & 1 & 0 \end{vmatrix}$ (by using expansions).

Therefore, coefficient of x is-2

B. Let D = $\begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$

$= (3\cos\theta - \sin\theta)^2$

$\Delta_{\max} = 10$

C. $f(x) = 0$

D. $a_0 = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 0 & 6 \\ 0 & 0 & 1 \end{vmatrix} = -2(1) = -2.$