

05

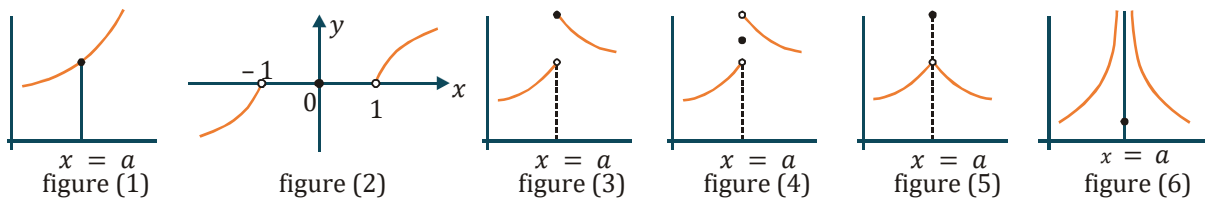
Continuity

1. Continuous Functions:

A function for which a small change in the independent variable causes only a small change and not a sudden jump in the dependent variable are called continuous functions. Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function around that point without lifting the pen from the plane of the paper.

A function $f(x)$ is said to be continuous at $x = a$, if $\lim_{x \rightarrow a} f(x)$ exists and is equal to $f(a)$. Symbolically $f(x)$ is continuous at $x = a$ if $\lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} f(a+h) = f(a) = \text{finite quantity}$.

i.e. $\text{LHL}|_{x=a} = \text{RHL}|_{x=a} = \text{value of } f(x)|_{x=a} = \text{finite quantity. } (h > 0)$



In figure (1) and (2) $f(x)$ is continuous at $x = a$ and $x = 0$ respectively and in figure (3) to (6) $f(x)$ is discontinuous at $x = a$.

Note 1 : Continuity of a function must be discussed only at points which are in the domain of the function.

Note 2 : If $x = a$ is an isolated point of domain then $f(x)$ is always considered to be continuous at $x = a$.

Illustration 1:

If $f(x) = \begin{cases} \sin \frac{\pi x}{2}, & x < 1 \\ [x] & x \geq 1 \end{cases}$ then find whether $f(x)$ is continuous or not at $x = 1$, where $[.]$ denotes greatest integer function.

integer function.

Solution:

$$f(x) = \begin{cases} \sin \frac{\pi x}{2}, & x < 1 \\ [x], & x \geq 1 \end{cases}$$

For continuity at $x = 1$, we determine, $f(1)$, $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$.

$$\text{Now, } f(1) = [1] = 1$$

$$f(x) \lim_{x \rightarrow 1^-} = \lim_{x \rightarrow 1^-} \sin \frac{\pi x}{2} = \sin \frac{\pi}{2} = 1 \text{ and } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} [x] = 1$$

$$\text{so } f(1) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$$

$\therefore f(x)$ is continuous at $x = 1$

Illustration 2:

$$\text{Let } f(x) = \begin{cases} \frac{a(1-x \sin x) + b \cos x + 5}{x^2} & x < 0 \\ 3 & x = 0 \\ \left(1 + \left(\frac{cx + dx^3}{x^2}\right)\right)^{\frac{1}{x}} & x > 0 \end{cases}$$

If f is continuous at $x = 0$, then find out the values of a, b, c and d .

Solution:

Since $f(x)$ is continuous at $x = 0$, so at $x = 0$, both left and right limits must exist and both must be equal to 3.

$$\text{Now } \lim_{x \rightarrow 0^-} \frac{a(1-x \sin x) + b \cos x + 5}{x^2} = \lim_{x \rightarrow 0^-} \frac{(a+b+5) + \left(-a - \frac{b}{2}\right)x^2 + \dots}{x^2} = 3 \text{ (By the expansions of } \sin x \text{ and } \cos x)$$

$$\text{If } \lim_{x \rightarrow 0^-} f(x) \text{ exists then } a + b + 5 = 0 \text{ and } -a - \frac{b}{2} = 3 \Rightarrow a = -1 \text{ and } b = -4$$

$$\text{since } \lim_{x \rightarrow 0^+} \left(1 + \left(\frac{cx + dx^3}{x^2}\right)\right)^{\frac{1}{x}} \text{ exists } \Rightarrow \lim_{x \rightarrow 0^+} \frac{cx + dx^3}{x^2} = 0 \Rightarrow c = 0$$

$$\text{Now } \lim_{x \rightarrow 0^+} (1 + dx)^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \left[(1 + dx)^{\frac{1}{dx}} \right]^d = e^d$$

$$\text{So } e^d = 3 \Rightarrow d = \ln 3,$$

Hence $a = -1, b = -4, c = 0$ and $d = \ln 3$.

2. Continuity of The Function in an Interval:

- (a) A function is said to be continuous in (a, b) if f is continuous at each & every point belonging to (a, b) .
- (b) A function is said to be continuous in a closed interval $[a, b]$ if :
 - (i) f is continuous in the open interval (a, b)
 - (ii) f is right continuous at 'a' i.e. $\lim_{x \rightarrow a^+} f(x) = f(a) =$ a finite quantity
 - (iii) f is left continuous at 'b' i.e. $\lim_{x \rightarrow b^-} f(x) = f(b) =$ a finite quantity

Note :

- (i) All polynomials, trigonometrical functions, exponential & logarithmic functions are continuous in their domains.
- (ii) If $f(x)$ & $g(x)$ are two functions that are continuous at $x = c$ then the function defined by :
 $F_1(x) = f(x) \pm g(x); F_2(x) = K f(x)$, where K is any real number $F_3(x) = f(x).g(x)$; are also continuous at $x = c$.

Further, if $g(c)$ is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at $x = c$.

Illustration 3:

Discuss the continuity of $f(x) = \begin{cases} |x+1| & , x < -2 \\ 2x+3 & , -2 \leq x < 0 \\ x^2+3 & , 0 \leq x < 3 \\ x^3-15 & , x \geq 3 \end{cases}$

Solution:

We write $f(x)$ as $f(x) = \begin{cases} -x-1 & , x < -2 \\ 2x+3 & , -2 \leq x < 0 \\ x^2+3 & , 0 \leq x < 3 \\ x^3-15 & , x \geq 3 \end{cases}$

As we can see, $f(x)$ is defined as a polynomial function in each of intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 3)$ and $(3, \infty)$. Therefore, it is continuous in each of these four open intervals. Thus, we check the continuity at $x = -2, 0, 3$.

At the point $x = -2$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (-x - 1) = +2 - 1 = 1$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} (2x + 3) = 2 \cdot (-2) + 3 = -1$$

Therefore, $\lim_{x \rightarrow -2} f(x)$ does not exist and hence $f(x)$ is discontinuous at $x = -2$.

At the point $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (2x + 3) = 3$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x^2 + 3) = 3$$

$$f(0) = 0^2 + 3 = 3$$

Therefore $f(x)$ is continuous at $x = 0$.

At the point $x = 3$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x^2 + 3) = 3^2 + 3 = 12$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x^3 - 15) = 3^3 - 15 = 12$$

$$f(3) = 3^3 - 15 = 12$$

Therefore, $f(x)$ is continuous at $x = 3$.

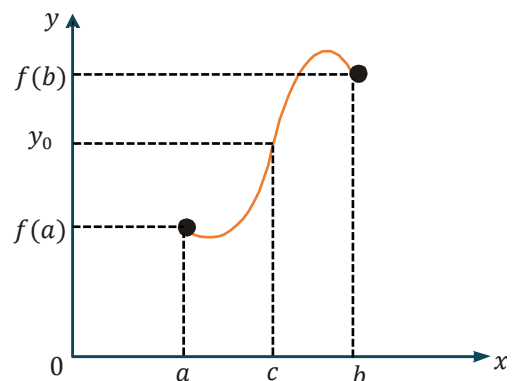
We find that $f(x)$ is continuous at all points in \mathbb{R} except at $x = -2$

3. The Intermediate Value Theorem:

Suppose $f(x)$ is continuous on an interval I , and a and b are any two points of I . Then if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c) = y_0$

Note that a function f which is continuous in $[a, b]$ possesses the following properties :

- (i) If $f(a)$ & $f(b)$ posses opposite signs, then there exists atleast one root of the equation $f(x) = 0$ in the open interval (a, b) .



The function f , being continuous on $[a, b]$ takes on every value between $f(a)$ and $f(b)$

(ii) If K is any real number between $f(a)$ & $f(b)$, then there exists atleast one root of the equation $f(x) = K$ in the open interval (a, b) .

Note: In above cases the number of roots is always odd.

Illustration 4:

Show that the function, $f(x) = (x - a)^2(x - b)^2 + x$, takes the value $\frac{a+b}{2}$ for some $x_0 \in (a, b)$

Solution:

$$f(x) = (x - a)^2(x - b)^2 + x$$

$$f(a) = a$$

$$f(b) = b$$

$$\& \frac{a+b}{2} \in (f(a), f(b))$$

\therefore By intermediate value theorem, there is atleast one $x_0 \in (a, b)$ such that $f(x_0) = \frac{a+b}{2}$.

Illustration 5:

Let $f: [0, 1] \xrightarrow{\text{onto}} [0, 1]$ be a continuous function, then prove that $f(x) = x$ for atleast one $x \in [0, 1]$

Solution:

Consider $g(x) = f(x) - x$

$$g(0) = f(0) - 0 = f(0) \geq 0 \quad \{\because 0 \leq f(x) \leq 1\}$$

$$g(1) = f(1) - 1 \leq 0$$

$$\Rightarrow g(0) \cdot g(1) \leq 0$$

$\Rightarrow g(x) = 0$ has atleast one root in $[0, 1]$

$\Rightarrow f(x) = x$ for atleast one $x \in [0, 1]$

4. Some Important Points:

(a) If $f(x)$ is continuous & $g(x)$ is discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ will **not necessarily be discontinuous at $x = a$** , e.g.

$$f(x) = x \ \& \ g(x) = \begin{cases} \sin \frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f(x)$ is continuous at $x = 0$ & $g(x)$ is discontinuous at $x = 0$, but $f(x) \cdot g(x)$ is continuous at $x = 0$.

(b) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ **is not necessarily be discontinuous at $x = a$** , e.g.

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

$f(x)$ & $g(x)$ both are discontinuous at $x = 0$ but the product function $f \cdot g(x)$ is still continuous at $x = 0$

(c) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$ then $f(x) \pm g(x)$ is not necessarily be discontinuous at $x = a$

(d) A continuous function whose domain is closed must have a range also in closed interval.

(e) If f is continuous at $x = a$ & g is continuous at $x = f(a)$ then the composite $g[f(x)]$ is continuous

at $x = a$. eg. $f(x) = \frac{x \sin x}{x^2 + 2}$ & $g(x) = |x|$ are continuous at $x = 0$, hence the composite $(g \circ f)(x) = \left| \frac{x \sin x}{x^2 + 2} \right|$

will also be continuous at $x = 0$

Continuity

Illustration 6:

If $f(x) = \frac{x+1}{x-1}$ and $g(x) = \frac{1}{x-2}$, then discuss the continuity of $f(x)$, $g(x)$ and $f \circ g(x)$ in its domain.

Solution:

$$f(x) = \frac{x+1}{x-1}$$

$f(x)$ is a rational function it must be continuous in its domain and f is not defined at $x = 1$.

$$g(x) = \frac{1}{x-2}$$

$g(x)$ is also a rational function. It must be continuous in its domain and g is not defined at $x = 2$.

Consider $g(x) = 1$

$$\frac{1}{x-2} = 1 \Rightarrow x = 3$$

$\therefore f \circ g(x)$ is continuous in its domain : $\mathbb{R} - \{2, 3\}$

5. Continuity Over Countable Set:

There are functions which are continuous over a countable set and else where discontinuous.

Illustration 7:

If $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{cases}$, find the points where $f(x)$ is continuous

Solution:

Let $x = a$ be the point at which $f(x)$ is continuous.

$$\Rightarrow \lim_{\substack{x \rightarrow a \\ \text{through rational}}} f(x) = \lim_{\substack{x \rightarrow a \\ \text{through irrational}}} f(x)$$

$$\Rightarrow a = -a$$

$$\Rightarrow a = 0 \Rightarrow \text{function is continuous at } x = 0.$$