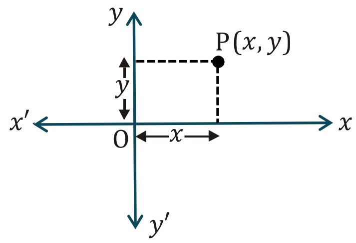
01 Point and Straight Lines

Introduction of Coordinate Geometry:

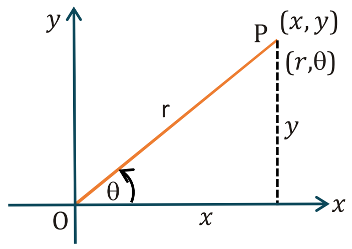
Coordinate geometry is the combination of algebra and geometry. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes. The resulting combination of analysis and geometry is referred as ***analytical geometry***.

Cartesian Co-ordinates System:



In two-dimensional coordinate system, two lines are used; the lines are at right angles, forming a rectangular coordinate system. The horizontal axis is the x-axis and the vertical axis is y −axis. The point of intersection O is the origin of the coordinate system. Distances along the x −axis to the right of the origin are taken as positive, distances to the left as negative. Distances along the y −axis above the origin are positive; distances below are negative. The position of a point anywhere in the plane can be specified by two numbers, the coordinates of the point, written as (x, y). The x −coordinate (or abscissa) is the distance of the point from the y −axis in a direction parallel to the x −axis (i.e. horizontally). The y −coordinate (or ordinate) is the distance from the x −axis in a direction parallel to the y −axis (vertically). The origin O is the point $(0, 0)$.

Polar Co-ordinates System:



A coordinate system in which the position of a point is determined by the length of a line segment from a fixed origin together with the angle that the line segment makes with a fixed line. The origin is called the pole and the line segment is the radius vector (r). The angle θ between the polar axis and the radius vector is called the vectoral angle. By convention, positive values of θ are measured in an anticlockwise sense, negative values in clockwise sense. The coordinates of the point are then specified as (r, θ). If (x, y) are cartesian co-ordinates of a point P, then:

$$x = r\cos\theta, y = r\sin\theta \text{ and } r = \sqrt{x^2+y^2}, \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Distance Formula and its Applications:

If $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ are two points, then $AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Note:

(i) Three given points A, B and C are collinear, when sum of any two distances out of AB, BC, CA is equal to the remaining third otherwise the points will be the vertices of a triangle.

(ii) Let A, B, C & D be the four given points in a plane. Then the quadrilateral will be:

(a) Square if $AB = BC = CD = DA$ & $AC = BD;\ AC \perp BD$

(b) Rhombus if $AB = BC = CD = DA$ and $AC \neq BD;\ AC \perp BD$

(c) Parallelogram if $AB = DC, BC = AD;\ AC \neq BD;\ AC \not\perp BD$

(d) Rectangle if $AB = CD, BC = DA, AC = BD;\ AC \not\perp BD$

Illustration 1:

The number of points on x –axis which are at a distance $c (c < 3)$ from the point $(2, 3)$ is

(A) 2 (B) 1 (C) infinite (D) no point

Ans. (D)

Solution:

Let a point on x –axis is $(x_1, 0)$ then its distance from the point $(2, 3)$

$$= \sqrt{(x_1 - 2)^2 + 9} = c \text{ or } (x_1 - 2)^2 = c^2 - 9$$

$$\therefore\ x_1 - 2 = \pm\sqrt{c^2 - 9} \text{ since } c < 3 \Rightarrow c^2 - 9 < 0$$

$\therefore\ x_1$ will be imaginary

Illustration 2:

The distance between the point $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\beta, a\sin\beta)$ is -

(A) $\left|4a\sin\frac{\alpha-\beta}{2}\right|$ (B) $\left|2a\sin\frac{\alpha+\beta}{2}\right|$ (C) $\left|2a\sin\frac{\alpha-\beta}{2}\right|$ (D) $\left|2a\cos\frac{\alpha-\beta}{2}\right|$

Ans. (C)

Solution:

$$d^2 = (a\cos\alpha - a\cos\beta)^2 + (a\sin\alpha - a\sin\beta)^2 = a^2(\cos\alpha - \cos\beta)^2 + a^2(\sin\alpha - \sin\beta)^2$$

$$= a^2\left\{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}\right\}^2 + a^2\left\{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right\}^2$$

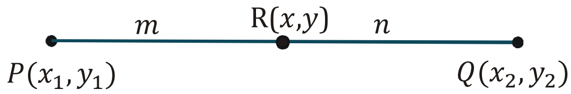
$$= 4a^2\sin^2\frac{\alpha-\beta}{2}\left\{\sin^2\frac{\alpha+\beta}{2} + \cos^2\frac{\alpha+\beta}{2}\right\} = 4a^2\sin^2\frac{\alpha-\beta}{2} \Rightarrow d = \left|2a\sin\frac{\alpha-\beta}{2}\right|$$

Section Formula:

The co-ordinates of a point dividing a line joining the points $P(x_1, y_1)$ and $P(x_2, y_2)$ in the ratio m:n is given by :

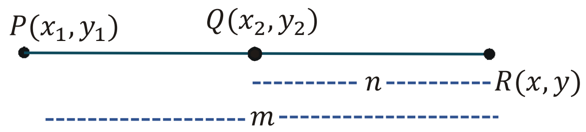
(a) For internal division: $P - R - Q \Rightarrow R$ divides line segment PQ, internally.

$$(x, y) \equiv \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$



(b) **For external division:** $R-P-Q$ or $P-Q-R \Rightarrow R$ divides line segment PQ, externally.

$$(x,y) \equiv \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$



$\frac{(PR)}{(QR)} < 1 \Rightarrow R$ lies on the left of P & $\frac{(PR)}{(QR)} > 1 \Rightarrow R$ lies on the right of Q

(c) **Harmonic conjugate :** If P divides AB internally in the ratio $m:n$ & Q divides AB externally in the ratio $m:n$ then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically; $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$ i.e. AP, AB & AQ are in H.P.

Illustration 3:

Determine the ratio in which $y-x+2=0$ divides the line joining $(3,-1)$ and $(8,9)$.

Solution:

Suppose the line $y-x+2=0$ divides the line segment joining $A(3,-1)$ and $B(8,9)$ in the ratio $\lambda:1$ at a point P, then the co-ordinates of the point P are $\left(\frac{8\lambda+3}{\lambda+1}, \frac{9\lambda-1}{\lambda+1}\right)$

But P lies on $y-x+2=0$ therefore $\left(\frac{9\lambda-1}{\lambda+1}\right)-\left(\frac{8\lambda+3}{\lambda+1}\right)+2=0$

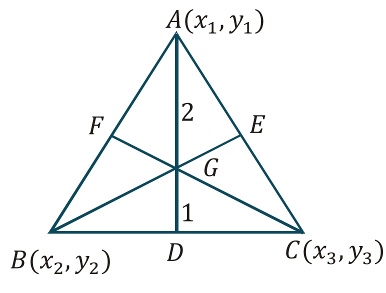
$\Rightarrow 9\lambda-1-8\lambda-3+2\lambda+2=0$

$\Rightarrow 3\lambda-2=0$ or $\lambda=\frac{2}{3}$

So, the required ratio is $\frac{2}{3}:1$, i.e., $2:3$ (internally) since here λ is positive.

Co-ordinates of Some Particular Points:

Let $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of any triangle ABC, then



Centroid:

The centroid is the point of intersection of the medians (line joining the mid-point of sides and opposite vertices). Centroid divides each median in the ratio of $2:1$.

Co-ordinates of centroid $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Illustration 4:

Two vertices of a triangle are $(1, 4)$ and $(3, 1)$. If the centroid of the triangle is the origin, find the third vertex.

Solution:

Let the coordinates of the third vertex are (h, k).

Therefore, the coordinates of the centroid of the triangle $\left(\frac{1+3+h}{3}, \frac{4+1+k}{3}\right)$

According to the problem we know that the centroid of the given triangle is (0, 0)

Therefore,

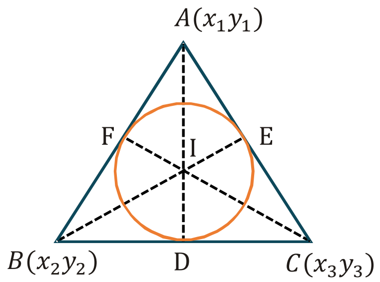
$\frac{1+3+h}{3} = 0$ and $\frac{4+1+k}{3} = 0$

$\Rightarrow h = -4$ and $k = -5$

Therefore, the third vertex of the given triangle are $(-4, -5)$.

Incenter:

The incenter is the point of intersection of internal bisectors of the angles of a triangle. Also, it is a centre of the circle touching all the sides of a triangle.



Co-ordinates of incenter $I\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

where a, b, c are the sides of triangle ABC.

Note:

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g. $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

(ii) Incenter divides the angle bisectors in the ratio $(b + c) : a$, $(c + a) : b$, $(a + b) : c$.

Illustration 5:

Find the co-ordinates of (i) centroid (ii) in-Centre of the triangle whose vertices are $(0, 0)$, $(6, 0)$ and $(0, 8)$.

Solution:

(i) We know that the co-ordinates of the centroid of a triangle whose angular points are (x_1, y_1), (x_2, y_2) (x_3, y_3) are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

So, the co-ordinates of the centroid of a triangle whose vertices are (0, 0), (6, 0) and (0, 8) are $\left(\frac{0+0+6}{3}, \frac{0+0+8}{3}\right)$ or $\left(2, \frac{8}{3}\right)$.

(ii) Let $A\ (0,0), B\ (6,0)$ and $C(0,8)$ be the vertices of triangle ABC.

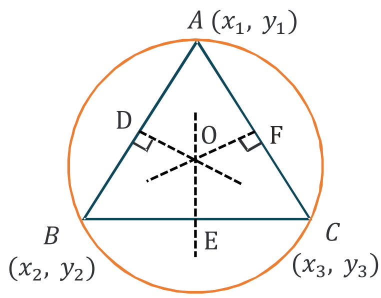
Then $a = BC = \sqrt{(0-8)^2+(6-0)^2} = 10, b = CA = 8$ and $c = AB = 6$

The co-ordinates of the in-Centre are $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

Or $\left(\frac{10\times0+6\times8+0\times6}{10+8+6}, \frac{10\times0+6\times0+8\times6}{10+8+6}\right)$ or $(2,2)$

Circumcenter:

It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcenter of any triangle ABC, then $OA^2 = OB^2 = OC^2$. Also, it is a centre of a circle touching all the vertices of a triangle.



Note:

(i) If the triangle is right angled, then its circumcenter is the mid-point of hypotenuse.

(ii) Co-ordinates of circumcenter $\left(\frac{x_1\sin2A+x_2\sin2B+x_3\sin2C}{\sin2A+\sin2B+\sin2C}, \frac{y_1\sin2A+y_2\sin2B+y_3\sin2C}{\sin2A+\sin2B+\sin2C}\right)$

Illustration 6:

Using the circumcenter formula, find the circumcenter of ΔABC whose vertices $A(0,2), B(0,0)$ and $C(2,0)$ and respective measures of angles A, B and C are $45°, 90°$ and $45°$.

Solution:

We know, if $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$ are the vertices of the ΔABC an A, B, C are their respective angles. Then,

Circumcenter = $O(x,y)=\left(\frac{x_1\sin2A+x_2\sin2B+x_3\sin2C}{\sin2A+\sin2B+\sin2C}, \frac{y_1\sin2A+y_2\sin2B+y_3\sin2C}{\sin2A+\sin2B+\sin2C}\right)$

putting the corresponding values of coordinates of vertices and angle measures of the ΔABC in the above formula. We get

$$O(x,y)=\left(\frac{0\sin2(45)+0\sin2(90)+2\sin2(45)}{\sin2(45)+\sin2(90)+\sin2(45)}, \frac{2\sin2(45)+2\sin2(90)+0\sin2(45)}{\sin2(45)+\sin2(90)+\sin2(45)}\right)$$

$$O(x,y)=\left(\frac{2(\sin90)}{\sin(90)+\sin(180)+\sin(90)}, \frac{2(\sin90)}{\sin(90)+\sin(180)+\sin(90)}\right)$$

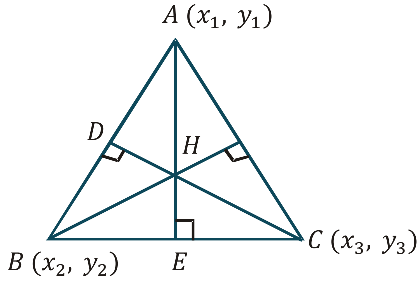
$$\Rightarrow O(x,y) = \left(\frac{2}{1+0+1}, \frac{2}{1+0+1}\right)$$

$$\Rightarrow O(x,y) = \left(\frac{2}{2}, \frac{2}{2}\right)$$

$$\Rightarrow O(x,y) = (1,1)$$

Orthocenter:

It is the point of intersection of perpendiculars drawn from vertices on the opposite sides of a triangle and it can be obtained by solving the equation of any two altitudes.



Note :

(i) If a triangle is right angled, then orthocenter is the point where right angle is formed.

(ii) Co-ordinates of orthocenter $\left(\dfrac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C},\dfrac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right)$

Remarks :

(i) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter coincide.

(ii) Orthocenter, centroid and circumcenter are always collinear and centroid divides the line joining orthocenter and circumcenter in the ratio 2 : 1

(iii) In an isosceles triangle centroid, orthocenter, incentre & circumcenter lie on the same line.

Illustration 7:

Find the orthocentre of a triangle having vertices (1,1) , (3,1) & (1,3)

Solution:

Here $AB = \sqrt{(1-1)^2+(3-1)^2} = 2$

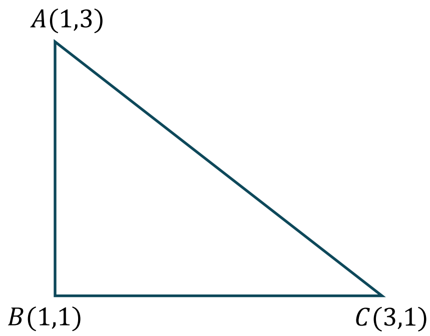
$BC = \sqrt{(1-3)^2+(1-1)^2} = 2$

$CA = \sqrt{(1-3)^2+(3-1)^2} = 2\sqrt{2}$

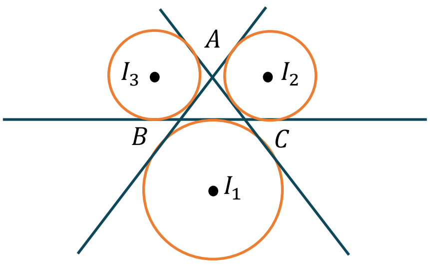
$AB^2 + BC^2 = CA^2$

$\Rightarrow$ It's right – angled triangle at B

Hence, orthocenter will be at (1,1)



Ex-Centers:



The center of a circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are

$$I_1\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c},\frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

Similarly, ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively , and

$$I_2\left(\frac{ax_1-bx_2+cx_3}{a-b+c},\frac{ay_1-by_2+cy_3}{a-b+c}\right),\ I_3\left(\frac{ax_1+bx_2-cx_3}{a+b-c},\frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

Illustration 8:

If $\left(\frac{5}{3}, 3\right)$ is the centroid of a triangle and its two vertices are (0, 1) and (2, 3), then find its third vertex, circumcenter, circumradius & orthocenter.

Solution:

Let the third vertex of triangle be (x, y), then

$\frac{5}{3}=\frac{x+0+2}{3} \Rightarrow x=3$ and $3=\frac{y+1+3}{3} \Rightarrow y=5$. So third vertex is (3, 5).

Now three vertices are $A(0,1), B(2,3)$ and $C(3,5)$

Let circumcenter be $P(h,k)$,

then $AP = BP = CP = R$ (circumradius) $\Rightarrow AP^2 = BP^2 = CP^2 = R^2$

$h^2 + (k-1)^2 = (h-2)^2 + (k-3)^2 = (h-3)^2 + (k-5)^2 = R^2$ ...(i)

from the first two equations, we have

$h + k = 3$ ...(ii)

from the first and third equation, we obtain

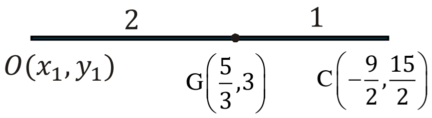
$6h + 8k = 33$ ...(iii)

On solving, (ii) & (iii), we get

$h = -\frac{9}{2},\ k = \frac{15}{2}$

Substituting these values in (i), we have

$R = \frac{5}{2}\sqrt{10}$



Let $O(x_1, y_1)$ be the orthocenter, then $\frac{x_1+2\left(-\frac{9}{2}\right)}{3}=\frac{5}{3} \Rightarrow x_1=14$, $\frac{y_1+2\left(\frac{15}{2}\right)}{3}=3$

$\Rightarrow y_1 = -6$. Hence orthocentre of the triangle is $(14, -6)$.

Illustration 9:

The vertices of a triangle are $A(0,-6), B(-6,0)$ and $C(1,1)$ respectively, then coordinates of the ex-centre opposite to vertex A is :

(A) $\left(\frac{-3}{2}, \frac{-3}{2}\right)$ (B) $\left(-4, \frac{3}{2}\right)$ (C) $\left(\frac{-3}{2}, \frac{3}{2}\right)$ (D) $(-4, 6)$

Ans. (D)

Solution:

$a = BC = \sqrt{(-6-1)^2 + (0-1)^2} = \sqrt{50} = 5\sqrt{2}$

$b = CA = \sqrt{(1-0)^2 + (1+6)^2} = \sqrt{50} = 5\sqrt{2}$

$c = AB = \sqrt{(0+6)^2 + (-6-0)^2} = \sqrt{72} = 6\sqrt{2}$

coordinates of ex-centre opposite to vertex A will be :

$$x=\frac{-ax_1+bx_2+cx_3}{-a+b+c}=\frac{-5\sqrt{2}.0+5\sqrt{2}(-6)+6\sqrt{2}(1)}{-5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}=\frac{-24\sqrt{2}}{6\sqrt{2}}=-4$$

$$y=\frac{-ay_1+by_2+cy_3}{-a+b+c}=\frac{-5\sqrt{2}(-6)+5\sqrt{2}.0+6\sqrt{2}(1)}{-5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}=\frac{36\sqrt{2}}{6\sqrt{2}}=6$$

Hence coordinates of ex-centre is (– 4, 6)

Area of Triangle:

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of a triangle, then

Area of $\Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\,|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]|$

To remember the above formula, take the help of the following method:

$$=\frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{bmatrix}=\frac{1}{2}\,|\,[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3\,y_1 - x_1y_3)]|$$

Remarks :

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) **Area of Equilateral triangle :** If altitude of any equilateral triangle is P, then its area = $\frac{P^2}{\sqrt{3}}$.

If 'a' be the side of equilateral triangle, then its area = $\left(\frac{a^2\sqrt{3}}{4}\right)$.

(iii) Area of quadrilateral with given vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), D(x_4, y_4)$

Area of quad. $ABCD = \frac{1}{2}\left|\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{bmatrix}\right|$

Note: Area of a polygon can be obtained by dividing the polygon into disjoined triangles and then adding their areas.

Illustration 10:

The point A divides the join of the points (– 5, 1) and (3, 5) in the ratio k : 1 and coordinates of points B and C are (1,5) and (7, –2) respectively. If the area of ΔABC be 2 units, then k equals -

(A) 7, 9 (B) 6, 7 (C) 7, $\frac{31}{9}$ (D) 9, $\frac{31}{9}$

Ans. (C)

Solution:

$$A\equiv\left(\frac{3k-5}{k+1},\frac{5k+1}{k+1}\right)$$

Area of ΔABC = 2 units $\Rightarrow \frac{1}{2}\left[\frac{3k-5}{k+1}(5+2)+1\left(-2-\frac{5k+1}{k+1}\right)+7\left(\frac{5k+1}{k+1}-5\right)\right]=\pm 2$

$$\Rightarrow 14k-66=\pm 4(k+1)\ \Rightarrow k=7 \text{ or } \frac{31}{9}$$

Illustration 11:

Prove that the co-ordinates of the vertices of an equilateral triangle cannot all be rational.

Solution:

Let $A(x_1\ y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC. If possible let $x_1, y_1, x_2, y_2, x_3, y_3$ be all rational.

Now area of $\Delta\ ABC = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$ = Rational ...(i)

Since ΔABC is equilateral

$\therefore$ Area of $\Delta ABC = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}(AB)^2 = \frac{\sqrt{3}}{4}\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$ = Irrational ...(ii)

From (i) and (ii),

Rational = Irrational

which is contradiction.

Hence $x_1, y_1, x_2, y_2, x_3, y_3$ cannot all be rational.

Conditions for Collinearity of Three Given Points:

Three given points $A\ (x_1, y_1), B\ (x_2, y_2), C\ (x_3, y_3)$ are collinear if any one of the following conditions are satisfied.

(a) Area of triangle ABC is zero i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(b) Slope of AB =slope of BC =slope of AC. i.e. $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$

(c) Find the equation of line passing through 2 given points, if the third point satisfies the given equation of the line, then three points are collinear.

Illustration 12:

Find the values of K for which the points $(K + 1, 2 - K)$; $(1 - K, -K)$ and $(2 + K, 3 - K)$ are collinear.

Ans. $\boldsymbol{K = 1}$

Solution:

On equating the slopes,

$\frac{(2-K)-(-K)}{(K+1)-(1-K)} = \frac{-K-(3-K)}{1-K-(2+K)} \Rightarrow -2K - 1 = -3K \Rightarrow K = 1$

Illustration 13:

Show that $(b, c + a), (c, a + b)$ and $(a, b + c)$ are collinear.

Solution:

$\frac{(a+b)-(c+a)}{c-b} = \frac{(b+c)-(a+b)}{a-c} = \frac{(c+a)-(b+c)}{b-a} = -1$ (equating the slopes)

Locus:

The locus of a moving point is the path traced out by that point under one or more geometrical conditions.

(a) **Equation of Locus:**

The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

(b) Procedure for finding the equation of the locus of a point:

(i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given condition as equations in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.

(iii) Eliminate the parameters, so that the eliminant contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the eliminant. The resulting equation would be the equation of the locus of P.

Illustration 14:

The ends of the rod of length ℓ moves on two mutually perpendicular lines, find the locus of the point on the rod which divides it in the ratio $m_1 : m_2$

(A) $m_1^2x^2 + m_2^2y^2 = \dfrac{\ell^2}{(m_1+m_2)^2}$ (B) $(m_2x)^2 + (m_1y)^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$

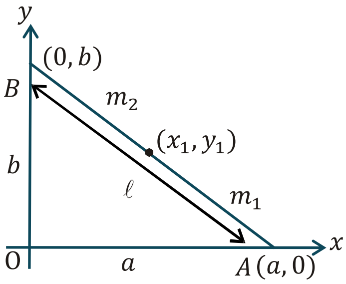
(C) $(m_1x)^2 + (m_2y)^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$ (D) none of these

Ans. (C)

Solution:

Let (x_1, y_1) be the point that divide the rod $AB = \ell$, in the ratio $m_1 : m_2$, and $OA = a$, $OB = b$ say

$\therefore \quad a^2 + b^2 = \ell^2$ ...(i)



Now $x_1 = \left(\dfrac{m_2a}{m_1+m_2}\right) \Rightarrow a = \left(\dfrac{m_1+m_2}{m_2}\right)x_1$

$y_1 = \left(\dfrac{m_1b}{m_1+m_2}\right) \Rightarrow b = \left(\dfrac{m_1+m_2}{m_1}\right)y_1$

putting these values in (i) $\dfrac{(m_1+m_2)^2}{m_2^2}x_1^2 + \dfrac{(m_1+m_2)^2}{m_1^2}y_1^2 = \ell^2$

Locus of (x_1, y_1) is $m_1^2x^2 + m_2^2y^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$

Illustration 15:

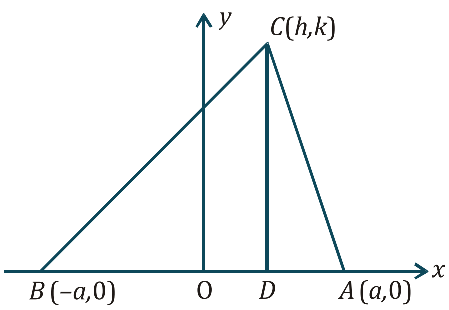
$A(a, 0)$ and $B(-a, 0)$ are two fixed points of ΔABC. If its vertex C moves in such a way that $\cot A + \cot B = \lambda$, where λ is a constant, then the locus of the point C is -

(A) $y\lambda = 2a$ (B) $y = \lambda a$ (C) $ya = 2\lambda$ (D) none of these

Ans. (A)

Solution:

Given that coordinates of two fixed points A and B are $(a, 0)$ and $(-a, 0)$ respectively. Let variable point C is (h, k). From the adjoining figure



$\cot A = \dfrac{DA}{CD} = \dfrac{a-h}{k}$

$\cot B = \dfrac{BD}{CD} = \dfrac{a+h}{k}$

But $\cot A + \cot B = \lambda$,

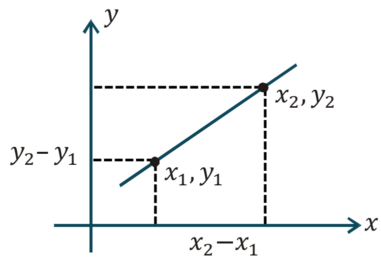
so, we have $\dfrac{a-h}{k} + \dfrac{a+h}{k} = \lambda \Rightarrow \dfrac{2a}{k} = \lambda$ sd

Hence locus of C is $y\lambda = 2a$

Slope of Line:

If a given line makes an angle θ $(0° \leq \theta < 180°, \theta \neq 90°)$ with the positive direction of x-axis, then slope of this line will be $\tan\theta$ and is usually denoted by the letter m

i.e. $m = \tan\theta$. If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line $AB = \frac{y_2 - y_1}{x_2 - x_1}$



Remark:

(i) If $\theta = 90°$, m does not exist and line is parallel to y-axis.

(ii) If $\theta = 0°$, $m = 0$ and the line is parallel to x-axis.

(iii) Let m_1 and m_2 be slopes of two given lines (none of them is parallel to y-axis)

(a) If lines are parallel, $m_1 = m_2$ and vice-versa.

(b) If lines are perpendicular, $m_1 m_2 = -1$ and vice-versa

Illustration 16:

Find the slope

(i) $\theta = 45°$ (ii) $\theta = 120°$

Solution:

(i) $m = \tan 45° = 1$(ii) $m = \tan 120° = -\sqrt{3}$

Illustration 17:

Slope of the line passing through $(1, 2)$ and $(0, -1)$.

Solution:

$$m = \frac{2-(-1)}{1-0} = 3$$

Straight Line:

Introduction: A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here, remember that every one-degree equation in variable x and y always represents a straight line i.e. $ax + by + c = 0$; a & $b \neq 0$ simultaneously.

(a) Equation of a line parallel to x-axis at a distance $'a'$ is $y = a$ or $y = -a$

(b) Equation of x −axis is $y = 0$

(c) Equation of a line parallel to y − axis at a distance $'b'$ is $x = b$ or $x = -b$

(d) Equation of y − axis is $x = 0$

Illustration 18:

Prove that every first-degree equation in x, y represents a straight line.

Solution:

Let $ax + by + c = 0$ be a first − degree equation in x, y

where a, b, c are constants.

Let $P(x_1, y_1)$ & $Q(x_2, y_2)$ be any two points on the curve represented by $ax + by + c = 0$. Then $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$

Let R be any point on the line segment joining P & Q

Suppose R divides PQ in the ratio $\lambda: 1$. Then, the coordinates of R are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$

We have $a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c = 0$

$\therefore R\left(\frac{\lambda x_2+x_1}{\lambda+1},\frac{\lambda y_2+y_1}{\lambda+1}\right)$ lies on the curve represented by $ax+by+c=0$. Thus every point on the line segment joining P & Q lies on $ax+by+c=0$.

Hence $ax+by+c=0$ represents a straight line.

Standard Forms of Equations of A Straight Line:

(a) Slope Intercept form : Let m be the slope of a line and c its intercept on y –axis. Then the equation of this straight line is written as : $y = mx + c$. If the line passes through origin, its equation is written as $y = mx$

(b) Point Slope form : If m be the slope of a line and it passes through a point (x_1, y_1), then its equation is written as: $y - y_1 = m(x - x_1)$

(c) Two point form : Equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is written as :

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

(d) Intercept form : If a and b are the intercepts made by a line on the axes of x and y, its equation is written as :

$$\frac{x}{a}+\frac{y}{b}=1$$

(i) Length of intercept of line between the coordinate axes $=\sqrt{a^2+b^2}$

(ii) Area of triangle AOB $=\frac{1}{2}OA.OB=\left|\frac{1}{2}ab\right|$

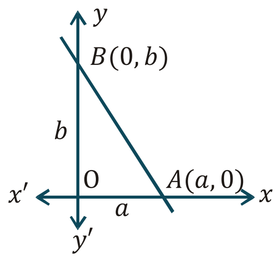


Illustration 19:

The equation of the lines which passes through the point (3, 4) and the sum of its intercepts on the axes is 14 is -

(A) $4x - 3y = 24, x - y = 7$
(B) $4x + 3y = 24, x + y = 7$
(C) $4x + 3y + 24 = 0, x + y + 7 = 0$
(D) $4x - 3y + 24 = 0, x - y + 7 = 0$

Ans. (B)

Solution:

Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ ...(i)

This passes through (3, 4), therefore $\frac{3}{a}+\frac{4}{b}=1$ ...(ii)

It is given that $a + b = 14 \Rightarrow b = 14 - a$. Putting $b = 14 - a$ in (ii), we get

$\frac{3}{a}+\frac{4}{14-a}=1 \Rightarrow a^2 - 13a + 42 = 0 \Rightarrow (a-7)(a-6) = 0 \Rightarrow a = 7, 6$

For $a = 7, b = 14 - 7 = 7$ and for $a = 6, b = 14 - 6 = 8$

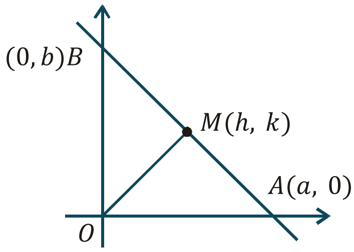
Putting the values of a and b in (i), we get the equations of the lines

$\frac{x}{7}+\frac{y}{7}=1$ and $\frac{x}{6}+\frac{y}{8}=1$ or $x + y = 7$ and $4x + 3y = 24$

Illustration 20:

Two points A and B move on the positive direction of $x-$ axis and y-axis respectively, such that $OA + OB = K$. Show that the locus of the foot of the perpendicular from the origin O on the line AB is $(x+y)(x^2+y^2) = Kxy$.

Solution:



Let the equation of AB be $\frac{x}{a}+\frac{y}{b}=1$ ...(i)

given, $a+b=K$ ...(ii)

now, $m_{AB} \times m_{OM} = -1 \Rightarrow ah = bk$ ...(iii)

from (ii) and (iii),

$a=\frac{kK}{h+k}$ and $b=\frac{hK}{h+k}$

$\therefore$ from (i) $\frac{x(h+k)}{k.K}+\frac{y(h+k)}{h.K}=1$

as it passes through (h,k)

$\frac{h(h+k)}{k.K}+\frac{k(h+k)}{h.K}=1 \Rightarrow (h+k)(h^2+k^2) = Khk$

$\therefore$ locus of (h,k) is $(x+y)(x^2+y^2) = Kxy$.

(e) General form : We know that a first degree equation in x and y, $ax+by+c=0$ always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $=\frac{-a}{b}=-\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on $x-$ axis $=-\frac{c}{a}$ and intercept by this line on y-axis $=-\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2+b^2}$.

Equation of Lines Parallel and Perpendicular to A Given Line:

(a) Equation of line parallel to line $ax+by+c=0$

$ax+by+\lambda=0$

(b) Equation of line perpendicular to line $ax+by+c=0$

$bx-ay+k=0$

Here λ, k are parameters and their values are obtained with the help of additional information given in the problem.

Illustration 21:

If $x+4y-5=0$ and $4x+ky+7=0$ are two perpendicular lines then k is -

(A) 3 (B) 4 (C) -1 (D) -4

Ans. (C)

Solution:

$m_1 = -\frac{1}{4}\, m_2 = -\frac{4}{k}$

Two lines are perpendicular if $m_1\, m_2 = -1$

$\Rightarrow \left(-\frac{1}{4}\right)\times\left(-\frac{4}{k}\right) = -1 \quad k = -1$

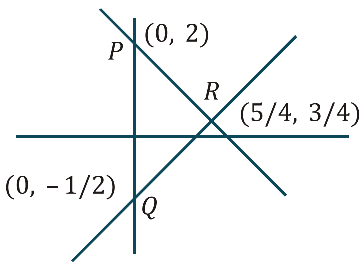
Illustration 22:

A line L passes through the points (1, 1) and (0, 2) and another line M which is perpendicular to L passes through the point $\left(0, -\frac{1}{2}\right)$. The area of the triangle formed by these lines with y-axis is -

(A) 25/8 (B) 25/16 (C) 25/4 (D) 25/32

Ans. (B)

Solution:



Equation of the line L is $y - 1 = \frac{-1}{1}\,(x - 1) \Rightarrow y = -x + 2$

Equation of the line M is $y = x - 1/2$.

If these lines meet y-axis at $P(0, -1/2)$ and $Q\ (0, 2)$ then $PQ = 5/2$.

Also, x-coordinate of their point of intersection $R = 5/4$

$\therefore$ area of the $\Delta PQR = \frac{1}{2}\left(\frac{5}{2}\times\frac{5}{4}\right) = 25/16.$

Illustration 23:

If the straight line $3x + 4y + 5 - k(x + y + 3) = 0$ is parallel to y –axis, then the value of k is -

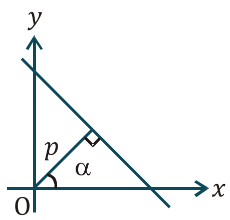
(A) 1 (B) 2 (C) 3 (D) 4

Ans. (D)

Solution:

A straight line is parallel to y –axis, if its y –coefficient is zero, i.e. $4 - k = 0$ i.e. $k = 4$

Normal Form:



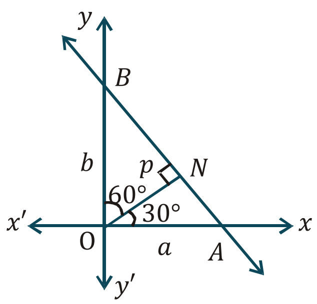
If p is the length of perpendicular on a line from the origin, and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as :

$x\cos\alpha + y\sin\alpha = p$ (p is always positive) where $0 \le \alpha < 2\pi$.

Illustration 24:

Find the equation of the straight line on which the perpendicular from origin makes an angle 30° with positive x-axis and which forms a triangle of area $\left(\frac{50}{\sqrt{3}}\right)$ sq. units with the co-ordinate's axes.

Solution:



$\angle NOA = 30°$

Let $ON = p > 0, OA = a, OB = b$

In ΔONA, $\cos 30° = \frac{ON}{OA} = \frac{p}{a} \Rightarrow \frac{\sqrt{3}}{2} = \frac{p}{a}$ or $a = \frac{2p}{\sqrt{3}}$

and in ΔONB, $\cos 60° = \frac{ON}{OB} = \frac{p}{b} \Rightarrow \frac{1}{2} = \frac{p}{b}$ or $b = 2p$

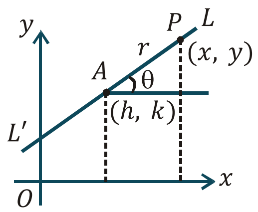
$\because$ Area of $\Delta OAB = \frac{1}{2}ab = \frac{1}{2}\left(\frac{2p}{\sqrt{3}}\right)(2p) = \frac{2p^2}{\sqrt{3}}$

$\therefore \frac{2p^2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \Rightarrow p^2 = 25$ or $p = 5$

$\therefore$ Using $x\cos\alpha + y\sin\alpha = p$, the equation of the line AB is $x\cos 30° + y\sin 30° = 5$ or $x\sqrt{3} + y = 10$

Parametric Form:

To find the equation of a straight line which passes through a given point $A(h, k)$ and makes a given angle θ with the positive direction of the x –axis. $P(x, y)$ is any point on the line LAL'.



Let $AP = r$, then $x - h = r\cos\theta$, $y - k = r\sin\theta$ & $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$ is the equation of the straight line LAL'. Any point P on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).

Illustration 25:

Equation of a line which passes through point $A(2, 3)$ and makes an angle of 45° with x axis. If this line meet the line $x + y + 1 = 0$ at point P then distance AP is -

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $5\sqrt{2}$ (D) $2\sqrt{5}$

Ans. (B)

Solution:

Here $x_1 = 2$, $y_1 = 3$ and $\theta = 45°$ hence $\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°} = r$

from first two parts $\Rightarrow x - 2 = y - 3 \Rightarrow x - y + 1 = 0$

Co-ordinate of point P on this line is $\left(2+\frac{r}{\sqrt{2}}, 3+\frac{r}{\sqrt{2}}\right)$.

If this point is on line $x + y + 1 = 0$ then $\left(2+\frac{r}{\sqrt{2}}\right)+\left(3+\frac{r}{\sqrt{2}}\right) + 1 = 0 \Rightarrow r = -3\sqrt{2}$; $|r| = 3\sqrt{2}$

Illustration 26:

A straight line through $P(-2, -3)$ cuts the pair of straight lines $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ in Q and R. Find the equation of the line if $PQ.PR = 20$.

Solution:

Let line be $\frac{x+2}{\cos\theta} = \frac{y+3}{\sin\theta} = r$

$\Rightarrow x = r\cos\theta - 2, y = r\sin\theta - 3$ ...(i)

Now, $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ ...(ii)

Taking intersection of (i) with (ii) and considering terms of r^2 and constant

(as we need $PQ.PR = r_1.r_2$ = product of the roots)

$r^2(\cos^2\theta + 3\sin^2\theta + 4\sin\theta\cos\theta)$ + (some terms)$r + 80 = 0$

$\therefore r_1.r_2 = PQ.PR = \frac{80}{\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta}$

$\therefore \cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta = 4$ $\qquad (\because PQ\,.\,PR = 20)$

$\therefore \sin^2\theta - 4\sin\theta\cos\theta + 3\cos^2\theta = 0$

$\Rightarrow (\sin\theta - \cos\theta)(\sin\theta - 3\cos\theta) = 0$

$\therefore \tan\theta = 1, \tan\theta = 3$

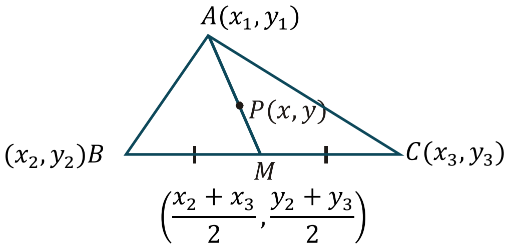
hence equation of the line is $y + 3 = 1(x + 2) \Rightarrow x - y = 1$ and $y + 3 = 3(x + 2) \Rightarrow 3x - y + 3 = 0.$

Lines in Determinant Form:

Line passing through two points (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Illustration 27:

Find the equation of the median through $A(x_1, y_1)$ (in the given figure)



Solution:

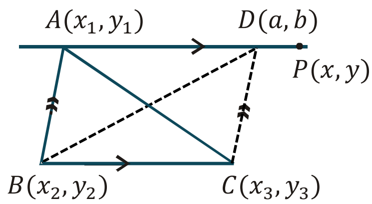
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \dfrac{x_2+x_3}{2} & \dfrac{y_2+y_3}{2} & 1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Illustration 28:

Find the equation of the line through A and parallel to the base BC, where (a, b) are assumed to be co-ordinates of D. (in the given figure)

Solution:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ a & b & 1 \end{vmatrix} = 0$$



Now, equating the middle point of BD and AC

$a + x_2 = x_1 + x_3 \Rightarrow a = x_1 - x_2 + x_3$

$b + y_2 = y_1 + y_3 \Rightarrow b = y_1 - y_2 + y_3$

Hence the equation of the line is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_1 + x_3 - x_2 & y_1 + y_3 - y_2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 - x_2 & y_3 - y_2 & 1 - 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Position of Two Points with Respect to a given Line:

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1), Q(x_2, y_2)$ be two points. If the expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

Illustration 29:

If the point (1,2) and (3,4) are to be on the opposite side of the line $3x - 5y + a = 0$, then-

(A) $7 < a < 11$ (B) $b = 7$ (C) $a = 11$ (D) $a < 7$ or $a > 11$

Ans. (A)

Solution:

(1, 2) & (3, 4) lies opposite side of line $3x - 5y + 4 = 0$

So $(3 - 10 + a)(9 - 20 + a) < 0$ & $(a - 7)(a - 11) < 0$

$a > 7$ & $a < 11$

So $7 < a < 11$

Illustration 30:

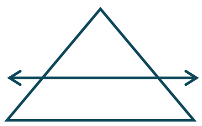
Prove that line $x + y - 1 = 0$ passes through inside the Δ formed by points (0, 0), (3, 0) and (2, 1).

Solution:

$L(0, 0) \to$ –ve

$L(3, 0) \to$ +ve

$L(2, 1) \to$ +ve



Hence two points on one side and one point on opposite side.

Length of Perpendicular From A Point on A Line:

Length of perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

In particular, the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \frac{|c|}{\sqrt{a^2 + b^2}}$

Illustration 31:

If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Solution:

Let n given points be (x_i, y_i) where $i = 1, 2 \dots n$ and the variable straight line is $ax + by + c = 0$.

Given that $\sum_{i=1}^{n}\left(\frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}}\right) = 0 \Rightarrow a\sum x_i + b\sum y_i + cn = 0 \Rightarrow a\frac{\Sigma x_i}{n} + b\frac{\Sigma y_i}{n} + c = 0$.

Hence the variable straight line always passes through the fixed point $\left(\frac{\Sigma x_i}{n}, \frac{\Sigma y_i}{n}\right)$.

Illustration 32:

Prove that no line can be drawn through the point $(4, -5)$ so that its distance from $(-2, 3)$ will be equal to 12.

Solution:

Suppose, if possible.

Equation of line through $(4, -5)$ with slope of m is $y + 5 = m(x - 4)$

$\Rightarrow mx - y - 4m - 5 = 0$

Then $\dfrac{|m(-2)-3-4m-5|}{\sqrt{m^2+1}}=12$

$\Rightarrow |-6m-8| = 12\sqrt{(m^2+1)}$

On squaring, $(6m+8)^2 = 144(m^2+1)$

$\Rightarrow 4(3m+4)^2 = 144(m^2+1)$

$\Rightarrow (3m+4)^2 = 36(m^2+1)$

$\Rightarrow 27m^2 - 24m + 20 = 0$ ...(i)

Since the discriminant of (i) is $(-24)^2 - 4.27.20 = -1584$ which is negative, there is no real value of m. Hence no such line is possible.

Distance Between Two Parallel Lines:

(a) The distance between two parallel lines $ax+by+c_1=0$ and $ax+by+c_2=0$ is $=\dfrac{|c_1-c_2|}{\sqrt{a^2+b^2}}$

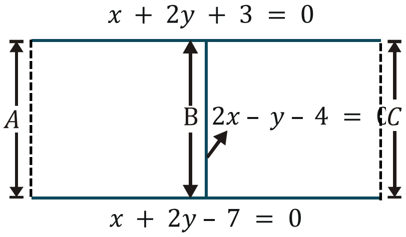
(**Note :** The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram = $\dfrac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides . Note that area of the parallelogram bounded by the lines $y=m_1x+c_1, y=m_1x+c_2$ and $y=m_2x+d_1, y=m_2x+d_2$ is given by $\left|\dfrac{(c_1-c_2)(d_1-d_2)}{m_1-m_2}\right|$.

Illustration 33:

Three lines $x+2y+3=0, x+2y-7=0$ and $2x-y-4=0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.

Solution:



Distance between the two parallel lines is $\dfrac{|7+3|}{\sqrt{5}}=2\sqrt{5}$.

The equations of sides A and C are of the form $2x-y+k=0$. Since distance between sides A and B = distance between sides

B and C $\dfrac{|k-(-4)|}{\sqrt{5}}=2\sqrt{5} \Rightarrow \dfrac{k+4}{\sqrt{5}}=\pm 2\sqrt{5} \Rightarrow k=6,-14.$

Hence the fourth sides of the two squares are (i) $2x-y+6=0$ (ii) $2x-y-14=0$.

Foot of Perpendicular of a Point about a Line:

Foot of the perpendicular from a point (x_1, y_1) on the line $ax+by+c=0$ is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=-\left(\frac{ax_1+by_1+c}{a^2+b^2}\right)$$

Illustration 34:

Find the foot of perpendicular of the line drawn from $P(-3,5)$ on the line $x-y+2=0$.

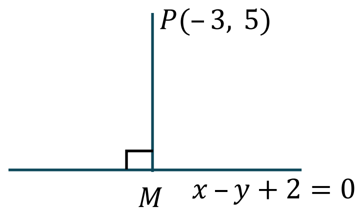
Solution:

Slope of $PM=-1$

$\therefore$ Equation of PM is

$x+y-2=0$ ...(i)

solving equation (i) with $x-y+2=0$, we get co-ordinates of M $(0,2)$



Aliter Here, $\frac{x+3}{1}=\frac{y-5}{-1}=-\frac{(1\times(-3)+(-1)\times 5+2)}{(1)^2+(-1)^2}$

$\Rightarrow \frac{x+3}{1}=\frac{y-5}{-1}=3 \Rightarrow x+3=3 \Rightarrow x=0$ and $y-5=-3$

$\Rightarrow y=2$

$\therefore$ M *is* $(0,2)$

Reflection of A Point:

Let $P(x,y)$ be any point, then its image with respect to

(a) $x-$ axis is $Q(x,-y)$

(b) $y-$ axis is $R(-x,y)$

(c) origin is $S(-x,-y)$

(d) line $y=x$ is $T(y,x)$

(e) Reflection of a point about any arbitrary line : The image (h,k) of a point $P(x_1,y_1)$ about the line $ax+by+c=0$ is given by following formula.

$$\frac{h-x_1}{a}=\frac{k-y_1}{b}=-2\frac{(ax_1+by_1+c)}{a^2+b^2}$$

and the foot of perpendicular (α,β) from a point (x_1,y_1) on the line $ax+by+c=0$ is given by following formula.

$$\frac{\alpha-x_1}{a}=\frac{\beta-y_1}{b}=-\frac{ax_1+by_1+c}{a^2+b^2}$$

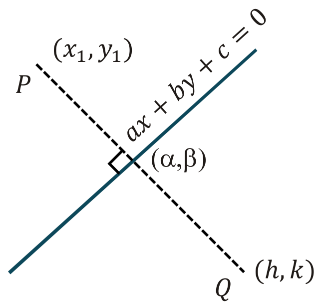


Illustration 35:

Find the image of the point $P(-1,2)$ in the line mirror $2x-3y+4=0$.

Solution:

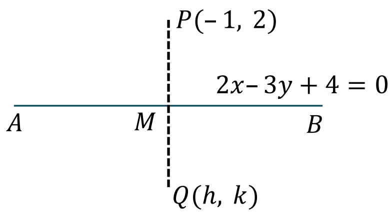
Let image of P is Q.

$\therefore PM = MQ$ & $PQ \perp AB$

Let Q is (h,k) $\therefore$ M is $\left(\frac{h-1}{2},\frac{k+2}{2}\right)$

It lies on $2x-3y+4=0$.

$\therefore 2\left(\frac{h-1}{2}\right)-3\left(\frac{k+2}{2}\right)+4=0$. Or $2h-3k=0$ ...(i)



slope of $PQ=\frac{k-2}{h+1}$

$PQ \perp AB$

$\therefore \frac{k-2}{h+1} \times \frac{2}{3} = -1. \Rightarrow 3h + 2k - 1 = 0.$ ...(ii)

solving (i) & (ii), we get $h = \frac{3}{13}, k = \frac{2}{13}$

$\therefore$ Image of $P(-1, 2)$ is $Q\left(\frac{3}{13}, \frac{2}{13}\right)$

Aliter: The image of $P(-1,2)$ about the line $2x - 3y + 4 = 0$ is $\frac{x+1}{2} = \frac{y-2}{-3} = -2\frac{[2(-1)-3(2)+4]}{2^2+(-3)^2}$

$$\frac{x+1}{2} = \frac{y-2}{-3} = \frac{8}{13} \Rightarrow 13x + 13 = 16 \Rightarrow x = \frac{3}{13} \ \& \ 13y - 26 = -24$$

$\Rightarrow y = \frac{2}{13}$ $\therefore$ image is $\left(\frac{3}{13}, \frac{2}{13}\right)$

Internal Angle of A Triangle:

If slopes of sides are $m_1 > m_2 > m_3$, then tangents of internal angles of triangle are given by $\frac{m_1 - m_2}{1 + m_1 m_2}, \frac{m_2 - m_3}{1 + m_2 m_3}$ and $\frac{m_3 - m_1}{1 + m_1 m_3}$

Illustration 36:

Find the nature of the triangle having side along $x - y = 0, 7x - y - 6 = 0$ and $2x - y - 6 = 0$

Solution:

Let's take the slopes in decreasing order for these three given sides of the triangle and A, B, C as their internal angles.

$m_1 = 7, m_2 = 2, m_3 = 1$

$$tanA = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{7-2}{1+14} = \frac{1}{3}$$

$$tanB = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{2-1}{1+2} = \frac{1}{3}$$

$$tanC = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{1-7}{1+7} = -\frac{3}{4}$$

Since $tan = C$ is negative it is obtuse angled triangle.

Linear Inequalities:

Linear inequalities $ax + by + c > 0$ or $ax + by + c < 0$ represents region above or below the line $ax + by + c = 0$, which can be divided by taking one particular point in the region.

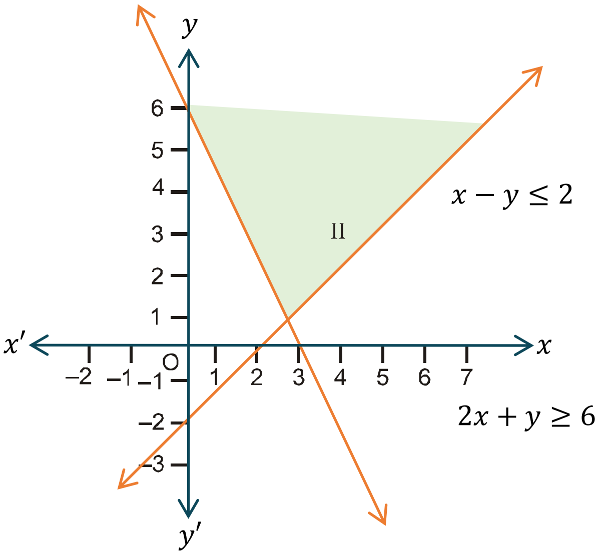
Illustration 37:

Solve the following system of linear inequalities graphically.

$2x + y \geq 6$ ...(1)

$x - y \leq 2$ ...(2)

Solution:



The graph of linear equation $2x + y = 6$ is drawn in fig. We note that solution of inequality (1) is represented by the shaded region above the line $2x + y = 6$, including the point on the line on the same set of axes, we draw graph of the equation $x - y = 2$ as shown in fig. Then we note that inequality (2) represents the shaded region above the line $x - y = 2$ including the points on the line. Clearly, the double shaded region, common to the above two shaded regions is the required solution region of the given system of inequalities.

Illustration 38:

Solve the following system of inequalities graphically

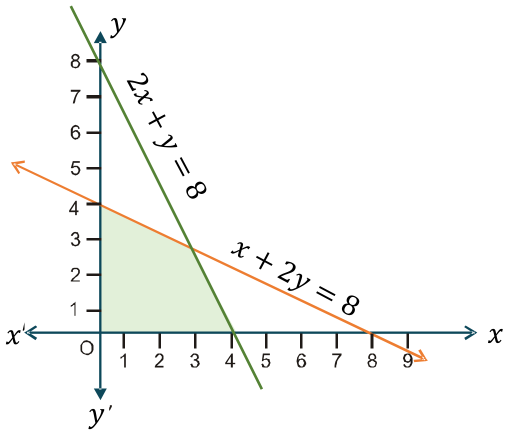
$x + 2y \le 8$ ...(1)

$2x + y \le 8$ ...(2)

$x \ge 0$ ...(3)

$y \ge 0$ ...(4)

Solution:



We draw the graphs of the lines $x + 2y = 8$ and $2x + y = 8$. The inequality (1) and (2) represent the region below the two lines, including the point on the respective lines.

Since $x \ge 0$, $y \ge 0$, every point in the shaded region in the first quadrant represent a solution of the given system of inequalities.

Straight Line Making A Given Angle With A Line:

Equations of lines passing through a point (x_1, y_1) and making an angle α, with the line $y = mx + c$ is written as:

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Illustration 39:

Find the equation of the straight line which passes through the origin and making angle 60° with the line $x - \sqrt{3}y = 0$.

Solution:

Given line is $x - \sqrt{3}\,y = 0$.

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30°$$

so, α can be 90° or 30° − 60° = −30°

so line can be $x = 0$ or $y = -\frac{1}{\sqrt{3}}x$

Concurrency of Straight Lines

The condition for 3 lines $a_1x + b_1y + c_1 = 0$ $a_2x + b_2y + c_2 = 0, a_3x + b_3y + c_3 = 0$ to be concurrent is -

Point of intersection L_1 & L_2 is $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1}\right)$ which lies on L_3

$$\Rightarrow a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

(i) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(ii) If determinant = 0 then it is not always necessary that lines are concurrent but may be parallel.

(iii) There exist 3 constants l, m, n (not all zero at the same time) such that $lL_1 + mL_2 + nL_3 = 0$, where $L_1 = 0, L_2 = 0$ and $L_3 = 0$ are the three given straight lines.

(iv) The three lines are concurrent if any one of the lines passes through the point of intersection of the other two lines.

Illustration 40:

Prove that the straight lines $4x + 7y = 9, 5x - 8y + 15 = 0$ and $9x - y + 6 = 0$ are concurrent.

Solution:

Given lines are

$4x + 7y - 9 = 0$ ...(1)

$5x - 8y + 15 = 0$ ...(2) and

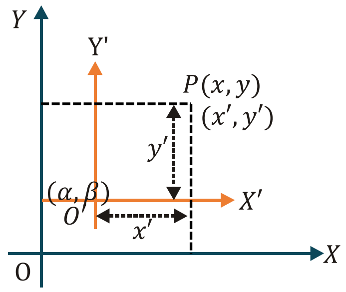
$9x - y + 6 = 0$ ...(3)

$$\Delta = \begin{vmatrix} 4 & 7 & -9 \\ 5 & -8 & 15 \\ 9 & -1 & 6 \end{vmatrix} = 4(-48 + 15) - 7(30 - 135) - 9(-5 + 72) = -132 + 735 - 603 = 0$$

Hence lines (1), (2) and (3) are concurrent.

Transformation of Axes:

(a) Shifting of origin without rotation of axes:

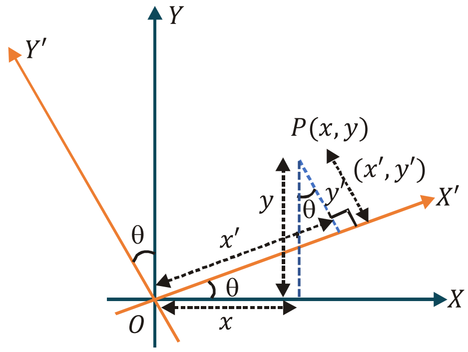


Let $P(x, y)$ with respect to axes OX and OY. Let O' (α, β) is new origin with respect to axes OX and OY and let $P\ (x', y')$ with respect to axes $O'X'$ and $O'Y'$, where OX and $O'X'$ are parallel and OY and $O'Y'$ are parallel.

Then $x = x' + \alpha, y = y' + \beta$ or $x' = x - \alpha, y' = y - \beta$

Thus, if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

(b) Rotation of axes without shifting the origin:



Let O be the origin. Let $P(x, y)$ with respect to axes OX and OY and let $P(x', y')$ with respect to axes OX' and OY' where $\angle X'OX = \angle YOY' = \theta$, where θ is measured in anticlockwise direction.

then $x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$ and

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

New \ Old	$x\downarrow$	$y\downarrow$
$x'\rightarrow$	$\cos\theta$	$\sin\theta$
$y'\rightarrow$	$-\sin\theta$	$\cos\theta$

Illustration 41:

Through what angle should the axes be rotated so that the equation $9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$?

Solution:

Let angle be θ then replacing (x, y) by $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

Then $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ becomes

$9(x\cos\theta - y\sin\theta)^2 - 2\sqrt{3}(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + 7(x\sin\theta + y\cos\theta)^2 = 10$

$\Rightarrow x^2(9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta) + 2xy(-9\sin\theta\cos\theta - \sqrt{3}\cos 2\theta + 7\sin\theta\cos\theta)$

$+ y^2(9\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta) = 10$

On comparing with $3x^2 + 5y^2 = 5$(coefficient of $xy = 0$) We get $-9\sin\theta\cos\theta - \sqrt{3}\cos 2\theta + 7\sin\theta\cos\theta = 0$

Or $\sin 2\theta = -\sqrt{3}\cos 2\theta$ or $\tan 2\theta = -\sqrt{3} = \tan(180° - 60°)$

Or $2\theta = 120°$

$\therefore \theta = 60°$

Family of Lines:

If equation of two lines be $P \equiv a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is:

$P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$.

The value of λ is obtained with the help of the additional informations given in the problem.

Illustration 42:

Prove that each member of the family of straight lines

$(3\sin\theta + 4\cos\theta)x + (2\sin\theta - 7\cos\theta)y + (\sin\theta + 2\cos\theta) = 0$ (θ is a parameter) passes through a fixed point.

Solution:

The given family of straight lines can be rewritten as $(3x + 2y + 1)\sin\theta + (4x - 7y + 2)\cos\theta = 0$

or, $(4x - 7y + 2) + \tan\theta(3x + 2y + 1) = 0$ which is of the form $L_1 + \lambda L_2 = 0$

Hence each member of it will pass through a fixed point which is the intersection of $4x - 7y + 2 = 0$ and $3x + 2y + 1 = 0$ i.e. $\left(\frac{-11}{29}, \frac{2}{29}\right)$.

Illustration 43:

Find the straight line passing through the point of intersection of the lines $3x - 2y - 12 = 0$, $x + 5y + 13 = 0$ and perpendicular to the first line.

Solution:

The gradient of the first line is $\frac{3}{2}$; hence, the gradient of the required line is $-\frac{2}{3}$.

Method - I

The solution of the given equations given $P(2, -3)$ as the point of intersection. The required line is then

$y + 3 = -\frac{2}{3}(x - 2)$ or $2x + 3y + 5 = 0$. ...(i)

Method - II

Any line through the point of intersection of the given lines is

$3x - 2y - 12 + \lambda(x + 5y + 13) = 0$

Or $(3 + \lambda)x - (2 - 5\lambda)y + (13\lambda - 12) = 0$ ...(ii)

The gradient of (ii) is $\frac{3+\lambda}{2-5\lambda}$ and since the gradient of the required line is $-\frac{2}{3}$, we have $\frac{3+\lambda}{2-5\lambda} = -\frac{2}{3}$

From which $\lambda = \frac{13}{7}$. Substitute this value of λ in (ii), then the required line is

$7(3x - 2y - 12) + 13(x + 5y + 13) = 0$

or $2x + 3y + 5 = 0$, as before.

Illustration 44:

If $3a + 2b + 5c = 0$ and the set of lines $ax + by + c = 0$ passes through a fixed point. Find co-ordinates of that point.

Solution:

$3a + 2b + 5c = 0$ ...(i)

$ax + by + c = 0$ ...(ii)

Eliminating c, we get.

$$ax + by - \frac{1}{5}(3a + 2b) = 0 \Rightarrow a\left(x - \frac{3}{5}\right) + b\left(y - \frac{2}{5}\right) = 0 \Rightarrow \left(x - \frac{3}{5}\right) + \frac{b}{a}\left(y - \frac{2}{5}\right) = 0$$

It is of the form $L_1 + \lambda L_2 = 0$

Which passes through the point of intersection $\left(\frac{3}{5}, \frac{2}{5}\right)$ of $L_1 = 0$ & $L_2 = 0$ for all real values of a & b

Aliter :

$3a + 2b + 5c = 0 \Rightarrow \frac{3}{5}a + \frac{2}{5}b + c = 0$

$\left(\frac{3}{5}, \frac{2}{5}\right)$ lies on the line $ax + by + c = 0$ Hence, fixed point $\left(\frac{3}{5}, \frac{2}{5}\right)$

Equation of Bisectors of Angles Between Two Lines:

If equation of two intersecting lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad \text{...(i)}$$

Equation of bisector of angle containing origin:

If the equation of the lines are written with constant terms $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (i)

Illustration 45:

For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$ find the equation of the bisector of the angle which contains the origin.

Solution:

To find the bisector of the angle between the lines which contains the origin, We first down the equations of the given lines in such a form that the constant terms in the equations of the lines are positive. The equations of the given lines are

$4x + 3y - 6 = 0 \Rightarrow -4x - 3y + 6 = 0$ ...(i)

$5x + 12y + 9 = 0$ ...(ii)

$\Rightarrow -52x - 39y + 78 = 25x + 60y + 45$

$\Rightarrow 7x + 9y - 3 = 0$

From (i) and (ii), we have $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$.

Therefore, the origin is situated in an acute angle region and the bisector of this angle is $7x + 9y - 3 = 0$

If equation of two intersecting lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad \text{...(i)}$$

Equation of Bisector of Acute/Obtuse Angles:

To find the equation of the bisector of the acute or obtuse angle:

(i) let ϕ be the angle between one of the two bisectors and one of two given lines.

Then if tanϕ < 1 i.e. ϕ < 45° i.e. 2 ϕ < 90°, the angle bisector will be bisector of acute angle.

(ii) See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by –1 to make the constant terms positive.

Determine the sign of $a_1a_2 + b_1b_2$

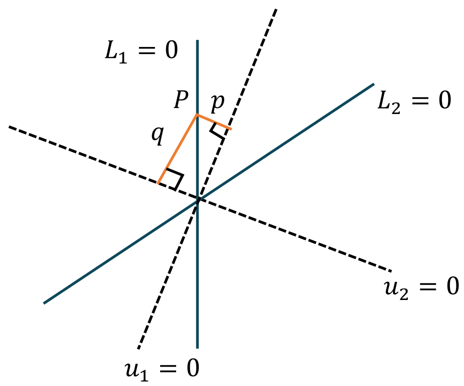
If sign of $a_1a_2+b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	Use + sign in eq. (1)	use – sign in eq. (1)
–	use – sign in eq. (1)	Use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

(iii) Another way of identifying an acute and obtuse angle bisector is as follows :

Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown. If,



$|p| < |q| \Rightarrow u_1$ is the acute angle bisector .

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector .

$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular.

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

Illustration 46:

For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) bisector of the obtuse angle between them.

(ii) bisector of the acute angle between them.

(iii) bisector of the angle which contains origin.

(iv) bisector of the angle which contains (1, 2).

Solution:

Equations of bisectors of the angles between the given lines are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}}=\pm\frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 9x-7y-41=0 \text{ and } 7x+9y-3=0$$

If θ is the acute angle between the line $4x+3y-6=0$ and the bisector

$9x-7y-41=0$, then $\tan\theta=\left|\dfrac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right|=\dfrac{11}{3}>1$

Hence

(i) bisector of the obtuse angle is $9x-7y-41=0$

(ii) bisector of the acute angle is $7x+9y-3=0$

(iii) bisector of the angle which contains origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}}=\frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$$

(iv) $L_1(1,2)=4\times1+3\times2-6=4>0$

$L_2(1,2)=5\times1+12\times2+9=38>0$

$+ve$ sign will give the required bisector, $\dfrac{4x+3y-6}{5}=+\dfrac{5x+12y+9}{13}$

$\Rightarrow 9x-7y-41=0.$

Alternative :

Making c_1 and c_2 positive in the given equation, we get $-4x-3y+6=0$ and $5x+12y+9=0$

Since $a_1a_2+b_1b_2=-20-36=-56<0$, so the origin will lie in the acute angle.

Hence bisector of the acute angle is given by

$$\frac{-4x-3y+6}{\sqrt{4^2+3^2}}=\frac{5x+12y+9}{\sqrt{5^2+12^2}} \quad\Rightarrow\quad 7x+9y-3=0$$

Similarly, bisector of obtuse angle is $9x-7y-41=0.$

Pair of Straight Lines:

(a) Homogeneous equation of second degree:

(i) Let us consider the homogeneous equation of 2nd degree as

$ax^2+2hxy+by^2=0$ ...(i)

which represents pair of straight lines passing through the origin.

Now, we divide by x^2, we get

$$a+2h\left(\frac{y}{x}\right)+b\left(\frac{y}{x}\right)^2=0$$

$\dfrac{y}{x}=m$ (say)

then $a+2hm+bm^2=0$ ...(ii)

if m_1 & m_2 are the roots of equation (ii), then $m_1+m_2=-\dfrac{2h}{b}, m_1m_2=\dfrac{a}{b}$

and also,

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right| = \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2}\right| = \left|\frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1+\frac{a}{b}}\right| = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$

These lines will be:

(1) Real and different, if $h^2 - ab > 0$

(2) Real and coincident, if $h^2 - ab = 0$

(3) Imaginary, if $h^2 - ab < 0$

(ii) The condition that these lines are:

(1) At right angles to each other is $a + b = 0$. i.e. coefficient of x^2 + coefficient of $y^2 = 0$.

(2) Coincident is $h^2 = ab$.

(3) Equally inclined to the axes of x is $h = 0$. i.e. coefficient of $xy = 0$.

(iii) Homogeneous equation of 2nd degree $ax^2 + 2hxy + by^2 = 0$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x \equiv y = m_1x \ \& \ y = m_2x \text{ and } m_1 + m_2 = -\frac{2h}{b}\,;\ m_1m_2 = \frac{a}{b}$$

These straight lines passes through the origin.

(iv) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

(v) The product of the perpendiculars drawn from the point (x_1, y_1) on the lines

$ax^2 + 2hxy + by^2 = 0$ is $\left|\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$

Note: A homogeneous equation of degree n represents n straight lines passing through **origin**.

(b) The combined equation of angle bisectors:

The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by $\dfrac{x^2 - y^2}{a-b} = \dfrac{xy}{h}$, $a \neq b, h \neq 0$.

Note :

(i) If $a = b$, the bisectors are $x^2 - y^2 = 0$ i.e. $x - y = 0, x + y = 0$

(ii) If $h = 0$, the bisectors are $xy = 0$ i.e. $x = 0, y = 0$.

(iii) The two bisectors are always at right angles, since we have coefficient of x^2 + coefficient of $y^2 = 0$

General Equation and Homogeneous Equation of Second Degree:

(i) The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) If θ be the angle between the lines, then $\tan\theta = \left|\dfrac{2\sqrt{h^2-ab}}{a+b}\right|$

Obviously, these lines are

(1) Parallel, if $\Delta = 0, h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(2) Perpendicular, if $a + b = 0$ i. e. coeff. of x^2 + coeff. of $y^2 = 0$.

(iii) The product of the perpendiculars drawn from the origin to the lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left|\dfrac{c}{\sqrt{(a-b)^2+4h^2}}\right|$

Illustration 47:

If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then λ is equal to -

(A) 4 (B) 3 (C) 2 (D) 1

Ans. (C)

Solution:

Here $a = \lambda, b = 12, c = -3, f = -8, g = 5/2, h = -5$

Using condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, we have

$\lambda(12)(-3) + 2(-8)(5/2)(-5) - \lambda(64) - 12(25/4) + 3(25) = 0$

$\Rightarrow -36\lambda + 200 - 64\lambda - 75 + 75 = 0 \Rightarrow 100\lambda = 200$

$\therefore \lambda = 2$

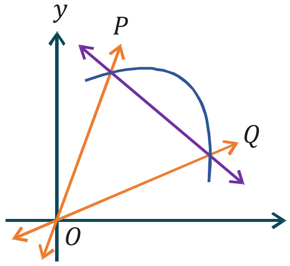
Equations of Lines Joining the Points of Intersection of *a* Line and a Curve to the Origin:

(a) Let the equation of curve be :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

and straight line be

$\ell x + my + n = 0$ ...(ii)



Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation (i) homogenous with the help of equation of the line. Thus, required equation is given by

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0$$

$\Rightarrow (an^2 + 2g\ell n + c\ell^2)x^2 + 2(hn^2 + gmn + f\ell n + c\ell m)xy + (bn^2 + 2fmn + cm^2)y^2 = 0$ ....(iii)

All points which satisfy (i) and (ii) simultaneously, will satisfy (iii)

(b) Any second-degree curve through the four points of intersection of $f(x, y) = 0$ & $xy = 0$ is given by $f(x, y) + \lambda xy = 0$ where $f(x, y) = 0$ is also a second-degree curve.

Illustration 48:

The chord $\sqrt{6}y = \sqrt{8}px + \sqrt{2}$ of the curve $py^2 + 1 = 4x$ subtends a right angle at origin then find the value of p.

Solution:

$\sqrt{3}y - 2px = 1$ is the given chord. Homogenizing the equation of the curve, we get,

$py^2 - 4x(\sqrt{3}\,y - 2px) + \left(\sqrt{3}\,y - 2px\right)^2 = 0$

$\Rightarrow (4p^2 + 8p)x^2 + (p+3)y^2 - 4xy - 4pxy = 0$

Now, angle at origin is 90°

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore 4p^2 + 8p + p + 3 = 0 \Rightarrow 4p^2 + 9p + 3 = 0$

$\therefore p = \dfrac{-9 \pm \sqrt{81-48}}{8} = \dfrac{-9 \pm \sqrt{33}}{8}$.

Problems on Locii:

Illustration 49:

ABC is a variable triangle with the fixed vertex $C(1,2)$ and A,B having the coordinates $(\cos t,\ \sin t), (\sin t,\ -\ \cos t)$ respectively where t is a parameter. Find the locus of the centroid of the ΔABC.

Solution:

Let $G(\alpha, \beta)$ be the centroid in any position. Then $G(\alpha, \beta) = \left(\dfrac{1+\cos t+\sin t}{3}, \dfrac{2+\sin t-\cos t}{3}\right)$

$\therefore \alpha = \dfrac{1+\cos t+\sin t}{3}$, $\beta = \dfrac{2+\sin t-\cos t}{3}$

or $3\alpha - 1 = \cos t + \sin t$, ...(i)

$3\beta - 2 = \sin t - \cos t$ ...(ii)

Squaring and adding, (i) and (ii) equation

$(3\alpha-1)^2 + (3\beta-2)^2 = (\cos t + \sin t)^2 + (\sin t - \cos t)^2$

$\Rightarrow 2(\cos^2 t + \sin^2 t) = 2$

$\therefore$ the equation of the locus of the centroid is

$(3x-1)^2 + (3y-2)^2 = 2$

or $9(x^2+y^2) - 6x - 12y + 3 = 0$

$\therefore 3(x^2+y^2) - 2x - 4y + 1 = 0$.

Illustration 50:

A variable straight line drawn through the point of inter-section of lines $\dfrac{x}{a}+\dfrac{y}{b}=1$ & $\dfrac{x}{b}+\dfrac{y}{a}=1$, meets the co-ordinate axes in A and B. Show that the locus of the mid-point of AB is the curve $2xy(a+b) = ab(x+y)$.

Solution:

The equation of the variable line through the point of inter-section of the given lines is of the type :

$\left(\dfrac{x}{a}+\dfrac{y}{b}-1\right) + k\left(\dfrac{x}{b}+\dfrac{y}{a}-1\right) = 0$

or $\left(\dfrac{1}{a}+\dfrac{k}{b}\right)x + \left(\dfrac{1}{b}+\dfrac{k}{a}\right)y = (1+k)$or $(ak+b)\,x + (bk+a)\,y = ab\,(1+k)$

$\therefore$ Points A and B are

$\left(\dfrac{ab(1+k)}{ak+b}, 0\right)$ and $\left(0, \dfrac{ab(1+k)}{bk+a}\right)$.

Let $P(x_1, y_1)$ be the mid $-$ point of AB.

Then $x_1 = \dfrac{ab(1+k)}{2(ak+b)}$ and $y_1 = \dfrac{ab(1+k)}{2(bk+a)}$ ...(i)

(Observe that the required equation of the locus of the same as $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{2(a+b)}{ab}$), therefore, we write (i) in the form.

$$\frac{1}{x_1}=\frac{2(ak+b)}{ab(1+k)},\ \frac{1}{y_1}=\frac{2(bk+a)}{ab(1+k)}$$

$$\frac{1}{x_1}=\frac{1}{y_1}=\frac{2(a+b)(1+k)}{ab(1+k)}=\frac{x_1+y_1}{x_1y_1}=\frac{2(a+b)}{ab}$$

(This eliminates parameter k).

Hence the locus of $P(x_1, y_1)$ is

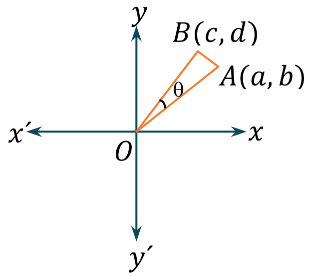
$2xy\ (a+b) = ab(x+y)$.

Illustration 51:

If the segments joining the points $A(a, b)$ and $B(c, d)$ subtend an angle θ at the origin, prove that

$$\cos\theta=\frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

Solution:



Let O be the origin. Then $OA^2 = a^2 + b^2$,

$OB^2 = c^2 + d^2$ and $AB^2 = (c-a)^2 + (d-b)^2$

Using cosine formula in ΔOAB, we have

$AB^2 = OA^2 + OB^2 - 2\,.\,OA\,.\,OB\cos\theta$

$\Rightarrow (c-a)^2 + (d-b)^2 = a^2 + b^2 + c^2 + d^2 - 2\ \sqrt{a^2+b^2}\ \sqrt{c^2+d^2}\cos\theta$

$\Rightarrow c^2 + a^2 - 2ac + d^2 + b^2 - 2bd$

$= a^2 + b^2 + c^2 + d^2 - 2\sqrt{a^2+b^2}\ \sqrt{c^2+d^2}\cos\theta$

$\Rightarrow 2(ac+bd) = 2\sqrt{a^2+b^2}\ \sqrt{c^2+d^2}\cos\theta$

$$\Rightarrow\quad \cos\theta=\frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

Illustration 52:

Change to Cartesian coordinates the equations

(1) $r = a\sin\theta$, and (2) $r^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{\theta}{2}$.

Solution:

(1) Multiplying the equation No. 1 by r, it becomes $r^2 = ar\sin\theta$,

i.e. by using the relations b/w cartesian and polar coordinate the equation becomes, $x^2 + y^2 = ay$.

(2) Squaring the equation (2), it becomes $r = a\cos^2\frac{\theta}{2} = \frac{a}{2}(1+\cos\theta)$, now by using the relation between cartesian and polar coordinate. The equation becomes

i.e. $2r^2 = ar + ar\cos\theta$,

i.e. $2(x^2+y^2) = a\sqrt{x^2+y^2} + ax$

i.e. $(2x^2 + 2y^2 - ax)^2 = a^2(x^2+y^2)$.

Illustration 53:

If t_1, t_2 and t_3 are distinct, the points $(t_1, 2at_1 + at_1^3), (t_2, 2at_2 + at_2^3), (t_3, 2at_3 + at_3^3)$ are collinear if

(A) $t_1\,t_2\,t_3 = 1$ (B) $t_1 + t_2 + t_3 = t_1\,t_2\,t_3$

(C) $t_1 + t_2 + t_3 = 0$ (D) $t_1 + t_2 + t_3 = -1$

Ans. (C)

Solution:

The given points are collinear if $\Delta = 0$

$$\begin{vmatrix} t_1 & 2at_1 + at_1^3 & 1 \\ t_2 & 2at_2 + at_2^3 & 1 \\ t_3 & 2at_3 + at_3^3 & 1 \end{vmatrix} = 0 \Rightarrow a \begin{vmatrix} t_1 & 2t_1 + t_1^3 & 1 \\ t_2 & 2t_2 + t_2^3 & 1 \\ t_3 & 2t_3 + t_3^3 & 1 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} t_1 & 2t1 + t_1^3 & 1 \\ t_2 - t_1 & 2(t_2 - t_1) + (t_2^3 - t_1^3) & 0 \\ t_3 - t_1 & 2(t_3 - t_1)(t_3^3 - t_1^3) & 0 \end{vmatrix} = 0$$

$$\Rightarrow (t_2 - t_1)(t_3 - t_1) \begin{vmatrix} t_1 & 2t_1 + t_1^3 & 1 \\ 1 & 2 + t_2^2 + t_1^2 + t_2 t_2 & 0 \\ 1 & 2 + t_3^2 + t_1^2 + t_3 t_1 & 0 \end{vmatrix} = 0$$

$\Rightarrow (t_2 - t_1)\,(t_3 - t_1)\,(t_3 - t_2)(t_3 + t_2 + t_1) = 0$

$\Rightarrow t_1 + t_2 + t_3 = 0 \quad [\because t_1 \neq t_2 \neq t_3]$

Illustration 54:

The points $(k, 2 - 2k), (-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ are collinear for

(A) all values of k (B) $k = -1$ (C) $k = 1/2$ (D) no value of k.

Ans.(B)

Solution:

The given points are collinear if

$$\Rightarrow \begin{vmatrix} k & 2-2k & 1 \\ -k+1 & 2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} k & 2-2k & 1 \\ -2k+1 & 4k-2 & 0 \\ -4-2k & 4 & 0 \end{vmatrix} = 0$$

$[R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$

$\Rightarrow 4(-2k + 1) - (-4 - 2k)\,(4k - 2) = 0$

$\Rightarrow (1 - 2k)\,(4 - 8 - 4k) = 0$

$\Rightarrow (1 - 2k)\,(k + 1) = 0 \Rightarrow k = -1$

Note: for $k = 1/2$ first two points are identical.

Illustration 55:

If the vertices of a triangle have integral coordinates, prove that the triangle cannot be equilateral.

Solution:

Let $A \equiv (x_1, y_1),\ B \equiv (x_2, y_2), C \equiv (x_3, x_3)$ be a triangle and $x_1, x_2, x_3, y_1, y_2, y_3$ be integers.

$\therefore BC^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2$ a positive integer.

If the triangle is equilateral,

then $AB = BC = CA = a$ (say) and

$\angle A = \angle B = \angle C = 60°$.

$\therefore$ Area of the triangle$=\left(\frac{1}{2}\right) bc \sin A$

$= \left(\frac{1}{2}\right) a^2 \sin 60° = (a^2/2).(\sqrt{3}/2) = \left(\frac{\sqrt{3}}{4}\right) a^2$

which is irrational,

$\because a^2$ is a positive integer.

Now, the area of the triangle in terms of the coordinates

$= (1/2)\,[x_1\,(y_2 - y_3) + x_2\,(y_3 - y_1) + x_3\,(y_1 - y_2)]$

which is a rational number.

This contradicts that the area is an irrational number, if the triangle is equilateral.

Illustration 56:

The line $x + y = 1$ meets x–axis at A and y –axis at B. P is the mid-point of AB(Fig.) P_1 is the foot of the perpendicular from P to OA ; M_1 is that of P_1 from OP ; P_2 is that of M_1 from OA; M_2 is that of P_2 from OP; P_3 is that of M_2 from OA and so on. If P_n denotes the n^{th} foot of the perpendicular on OA from M_{n-1}, then OP_n =

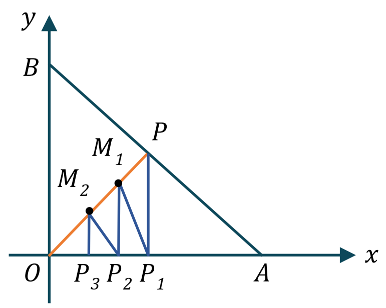
Solution:

Let $x + y = 1$ meets x-axis at $A(1, 0)$ and y-axis at $B(0, 1)$

The coordinates of P are $(1/2, 1/2)$ and PP_1 is perpendicular to OA.

$\Rightarrow OP_1 = P_1P = 1/2$

Equation of line OP is $y = x$.



We have

$OM_{n-1}^2 = OP_n^2 + P_nM_{n-1}^2 = 2OP_n^2 = 2p^2$ (say)

Also, $OP_{n-1}^2 = OM_{n-1}^2 + P_{n-1} + p_{n-1}M_{n-1}^2$

$= 2p_n^2 + \frac{1}{2} p_{n-1}^2$

$\Rightarrow p_n^2 = \frac{1}{4} p_{n-1}^2 \quad \Rightarrow p_n = \frac{1}{2} p_{n-1}$

$\therefore OP_n = P_n = \frac{1}{2} p_{n-1} = \frac{1}{2^2} p_{n-2} = \ldots\ldots\ldots = \frac{1}{2^{n-1}} p_1 = \frac{1}{2^n}$

Illustration 57:

Show that the area of the triangle with vertices $[(a + 1)\,(a + 2), (a + 2)], [(a + 2)\,(a + 3), (a + 3)]$ and $[(a + 3)\,(a + 4), (a + 4)]$ is independent of a.

Solution:

The area of the given triangle is

$$\Delta = \frac{1}{2}\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_2$, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ 2(a+2) & 1 & 0 \\ 2(a+3) & 1 & 0 \end{vmatrix} = |-1| = 1$$

Which is independent of a.

Illustration 58:

The point A divides the join of $P \equiv (-5,1)$ and $Q = (3,5)$ in the ratio $k:1$. Find the two values of k for which the area of ΔABC where $B \equiv (1,5), C \equiv (7,2)$ is equal to 2 square units.

Solution:

Co-ordinates of A, dividing the join of $P \equiv (-5,1)$ and $Q \equiv (3,5)$ in the ratio $k : 1$ are given by

$$\left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$$

Also, area of the ΔABC is given by

$$\Delta = \left|\frac{1}{2}\sum x_1(y_2-y_3)\right|$$

$$= \frac{1}{2}\left|\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]\right|$$

$$= \left|\frac{1}{2}\left\{\frac{3k-5}{k+1}(7)+\left(-2-\frac{5k+1}{k+1}\right)+7\left(\frac{5k+1}{k+1}-5\right)\right\}\right| = 2.$$

$$\left(\frac{1}{2}\right)\left\{\frac{3k-5}{k+1}(7)+\left(-2-\frac{5k+1}{k+1}\right)+7\left(\frac{5k+1}{k+1}-5\right)\right\} = \pm 2 \Rightarrow 14k-66 = 4k+4, 10k = 70, k = 7$$

or $14k - 66 = -4k - 4, 18k = 62$,

$k = \left(\frac{31}{9}\right)$.

Therefore, value of the $k = 7, \frac{31}{9}$

Illustration 59:

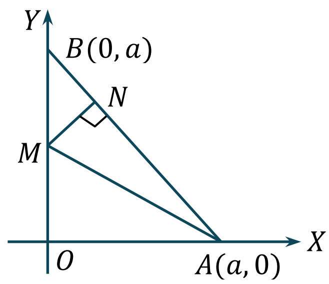
The line $x + y = a$, meets the axis of x and y at A and B respectively. A triangle AMN is inscribed in the triangle OAB, O being the origin, with right angle at $N. M$ and N lie respectively on OB and AB. If the area of the triangle AMN is 3/8 of the area of the triangle OAB, then AN/BN =

Solution:

Let $\frac{AN}{BN} = \lambda$.

Then the coordinates of N are $\left(\frac{a}{1+\lambda}, \frac{\lambda a}{1+\lambda}\right)$

where $(a, 0)$ and $(0, a)$ are the coordinates of A and B respectively.



Now equation of MN perpendicular to AB

is $y - \frac{\lambda a}{1+\lambda} = x - \frac{a}{1+\lambda}$ or $x - y = \frac{1-\lambda}{1+\lambda}a$.

So, the coordinates of M are $\left(0, \frac{\lambda-1}{\lambda+1}a\right)$

Therefore, area of the triangle AMN is

$$= \frac{1}{2}\left|\left[a\left(\frac{-a}{\lambda+1}\right)+\frac{1-\lambda}{(1+\lambda)^2}a^2\right]\right| = \frac{\lambda a^2}{(1+\lambda)^2}$$

Also area of the triangle $OAB = a^2/2$.
So that according to the given condition.

$$\frac{\lambda a^2}{(1+\lambda)^2} = \frac{3}{8}.\frac{1}{2}a^2$$

$\Rightarrow 3\lambda^2 - 10\lambda + 3 = 0$

$\Rightarrow \lambda = 3$ or $\lambda = 1/3$.

for $\lambda = 1/3$, M lies outside the segment OB and hence the required value of λ is 3.

Illustration 60:

Prove that the sum of the reciprocals of the intercepts made on the coordinate axes by any line not passing through the origin and through the point of intersection of the lines $2x + 3y = 6$ and $3x + 2y = 6$ is constant.

Solution:

Equation of any line through the points of intersection of the given lines is

$2x + 3y - 6 + k(3x + 2y - 6) = 0$

$\Rightarrow (2 + 3k)x + (3 + 2k)y - 6(k + 1) = 0$

$$\Rightarrow \frac{x}{\frac{6(k+1)}{2+3k}} + \frac{y}{\frac{6(k+1)}{3+2k}} = 1, \text{ when } k \neq -1$$

and in this case sum of the reciprocals of the intercepts made by this line on the coordinate axis is equal to

$$\frac{2+3k+3+2k}{6(k+1)} = \frac{5(k+1)}{6(k+1)} = \frac{5}{6}.$$

However, for $k = -1$, the line becomes $x = y$ which passes through the origin.

Illustration 61:

The line $x\cos\theta + y\sin\theta = p$ meets the axes of co-ordinates at A and B respectively. Through A and B lines are drawn parallel to axes so as to meet the perpendicular drawn from origin to given line in P and Q respectively; then show that $|PQ| = \frac{4p|\cos 2\theta|}{\sin^2 2\theta}$.

Solution:

$$A\left(\frac{p}{\cos\theta}, 0\right), \quad B\left(0, \frac{p}{\sin\theta}\right)$$

Lines through A, B parallel to axes are $x = \frac{p}{\cos\theta}, y = \frac{p}{\sin\theta}$

These meet the line through origin and $\perp$ to give line i.e., $x\sin\theta - y\cos\theta = p$ in P and Q.

$$\therefore P\left(\frac{p}{\cos\theta}, \frac{p\sin\theta}{\cos^2\theta}\right), Q\left(\frac{p\cos\theta}{\sin^2\theta}, \frac{p}{\sin\theta}\right)$$

$\therefore PQ^2$ (by distance formula)

$$= p^2\cos^2 2\theta\left[\frac{1}{\sin^4\theta\cos^2\theta} + \frac{1}{\cos^4\theta\sin^2\theta}\right]$$

$$= 16p^2\cos^2 2\theta \frac{1}{(2\sin\theta\cos\theta)^4}$$

$$\therefore PQ = \frac{4p|\cos 2\theta|}{\sin^2 2\theta}$$

Illustration 62:

Find the incentre of the triangle whose sides are $x + y = 1, x - y + 1 = 0, 7x - y = 6$.

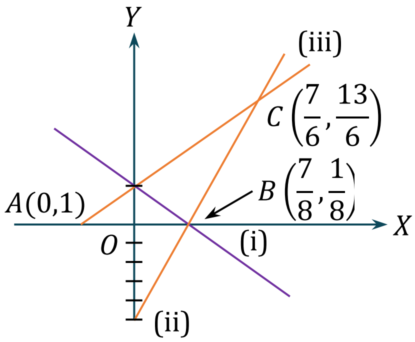
Solution:

Given lines of ΔABC as

$AB: x + y - 1 = 0$ ...(i)

$BC: 7x - y - 6 = 0$ ...(ii)

$CA: x - y + 1 = 0$ ...(iii)



Solving (iii) and (i), co-ordinates of $A \equiv (0, 1)$

Solving (i) and (ii) $\Rightarrow B \equiv \left(\frac{7}{8}, \frac{1}{8}\right)$

Solving (ii) and (iii) $\Rightarrow C \equiv \left(\frac{7}{6}, \frac{13}{6}\right)$

To obtain the coordinates of incentre use

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

we get incentre as $\left(\frac{7}{12}, 1\right)$.