03 Inverse Trigonometric Functions

1. Introduction:

The inverse trigonometric functions, denoted by $\sin^{-1}x$ or $(\arcsin x)$, $\cos^{-1}x$ etc., denote the angles whose sine, cosine etc, is equal to x. The angles are usually the numerically smallest angles, except in the case of $\cot^{-1}x$ and if positive & negative angles have same numerical value, the positive angle has been chosen.

It is worthwhile noting that the functions $\sin x, \cos x$ etc. are in general not invertible. Their inverse is defined by choosing an appropriate domain & co-domain so that they become invertible. For this reason, the chosen value is usually the simplest and easy to remember.

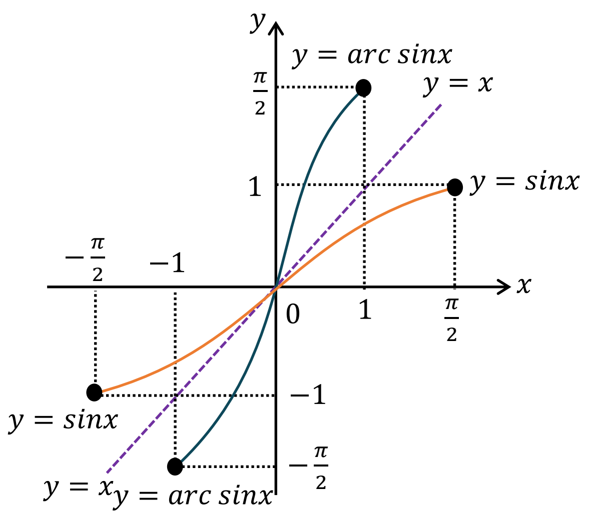
2. Graph of Inverse Trigonometric Functions:

(a) $f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1,1]$

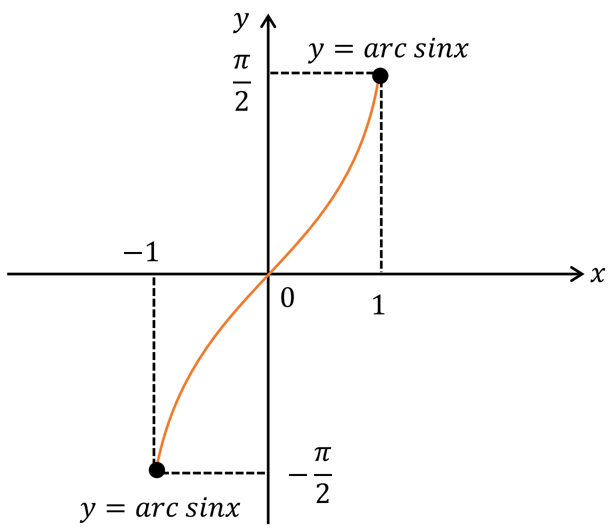
$f(x) = \sin x$

$f^{-1} : [-1, 1] \rightarrow [-\pi/2, \pi/2]$

$f^{-1}(x) = \sin^{-1}(x)$



(Taking image of $\sin x$ about $y = x$ to get $\sin^{-1}x$)



($y = \sin^{-1}x$)

Illustration 1:

The value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{5\pi}{12}$ (C) $\frac{3\pi}{4}$ (D) $\frac{13\pi}{12}$

Ans.(C)

Solution:

$$\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

Illustration 2:

The value of $\sin^{-1}(-\sqrt{3}/2)$ is -

(A) $-\pi/3$ (B) $-2\pi/3$ (C) $4\pi/3$ (D) $5\pi/3$

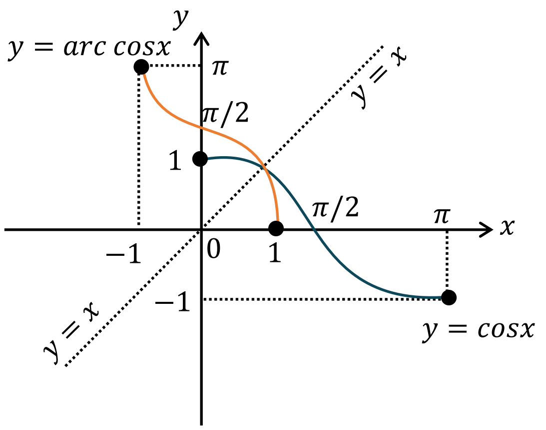
Ans. (A)

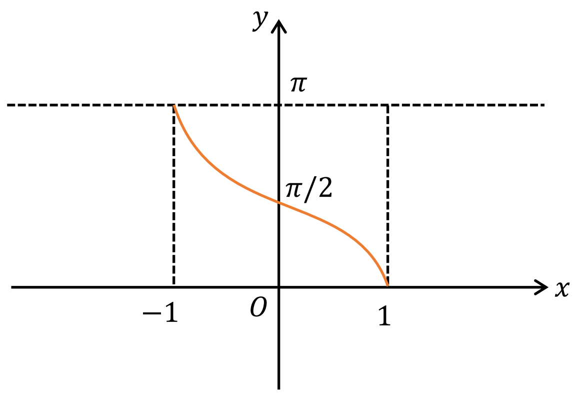
Solution:

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

(b) $f:[0,\pi] \to [-1,1]$ $f^{-1}:[-1,1] \to [0,\pi]$

$f(x) = \cos x$ $f^{-1}(x) = \cos^{-1} x$

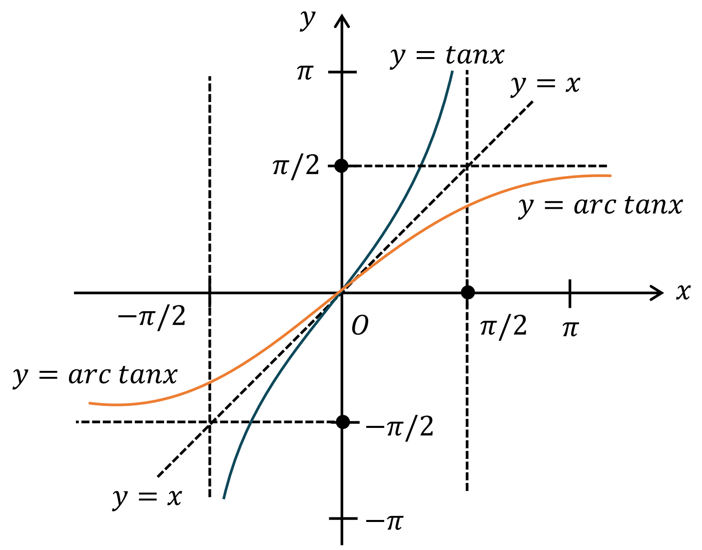


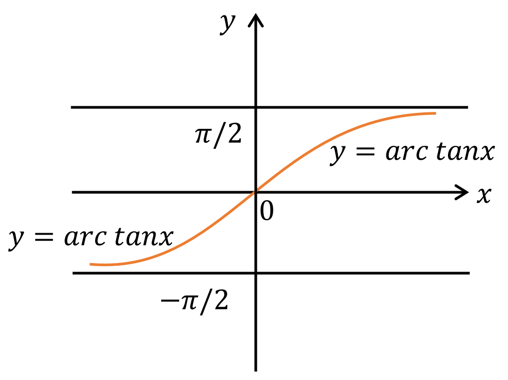


(Taking image of $\cos x$ about $y = x$) ($y = \cos^{-1}x$)

(c) $f:(-\pi/2,\pi/2) \to \mathbb{R}$ $f^{-1}:\mathbb{R} \to (-\pi/2,\pi/2)$

$f(x) = \tan x$ $f^{-1}(x) = \tan^{-1} x$



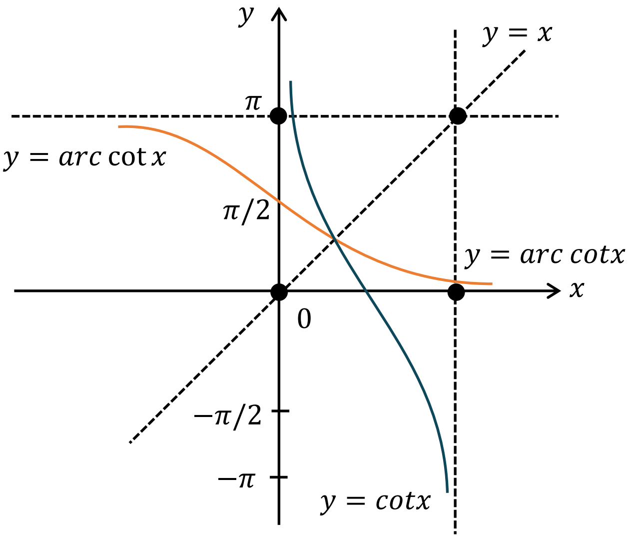


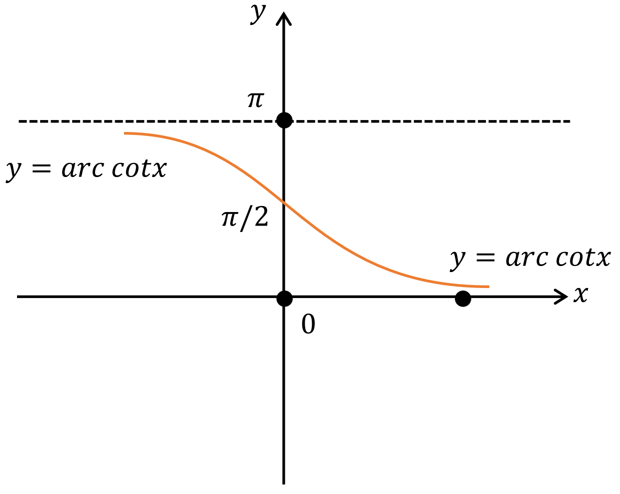
($y = \tan^{-1}x$)

(Taking image of $\tan x$ about $y = x$)

(d) $f:(0,\pi)\to\mathbb{R}$ $\quad$ $f^{-1}:\mathbb{R}\to(0,\pi)$

$f(x)=\cot x$ $\quad$ $f^{-1}(x)=\cot^{-1}x$





($y=\cot^{-1}x$)

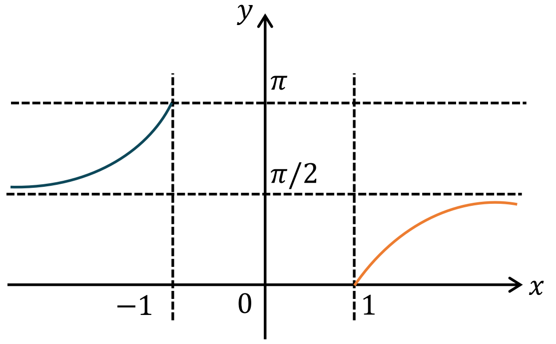
(Taking image of $\cot x$ about $y=x$)

(e) $f:\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]\to(-\infty,-1]\cup[1,\infty)$

$f(x)=\sec x$

$f^{-1}:(-\infty,-1]\cup[1,\infty)\to[0,\pi/2)\cup(\pi/2,\pi]$

$f^{-1}(x)=\sec^{-1}x$

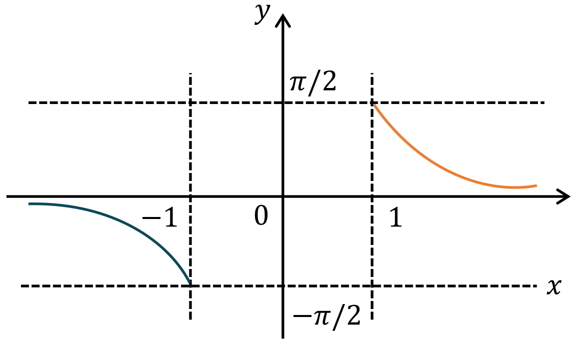


(f) $f:\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]\to(-\infty,-1]\cup[1,\infty)$

$f(x)=\operatorname{cosec}x$

$f^{-1}:(-\infty,-1]\cup[1,\infty)\to[-\pi/2,0)\cup(0,\pi/2]$

$f^{-1}(x)=\operatorname{cosec}^{-1}x$



S.No	$f(x)$	Domain	Range
(1)	$\sin^{-1}x$	$\lvert x\rvert \le 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(2)	$\cos^{-1}x$	$\lvert x\rvert \le 1$	$[0, \pi]$
(3)	$\tan^{-1}x$	$x \in \mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(4)	$\sec^{-1}x$	$\lvert x\rvert \ge 1$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$ or $\left[0, \frac{\pi}{2}\right)\cup\left(\frac{\pi}{2}, \pi\right]$
(5)	$\operatorname{cosec}^{-1}x$	$\lvert x\rvert \ge 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$
(6)	$\cot^{-1}x$	$x \in \mathbb{R}$	$(0, \pi)$

From the above discussions following IMPORTANT points can be concluded:

(i) All the inverse trigonometric functions represent an angle.

(ii) If $x > 0$, then all six inverse trigonometric functions viz $\sin^{-1}x, \cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$, $\cot^{-1}x$ represent an acute angle.

(iii) If $x < 0$, then $\sin^{-1}x, \tan^{-1}x$ & $\operatorname{cosec}^{-1}x$ represent an angle from $-\pi/2$ to 0 (IV$^{\text{th}}$ quadrant)

(iv) If $x < 0$, then $\cos^{-1}x, \cot^{-1}x$ & $\sec^{-1}x$ represent an obtuse angle. (II$^{\text{nd}}$ quadrant)

(v) III$^{\text{rd}}$ quadrant is never used in range of inverse trigonometric function.

Illustration 3:

Domain of the function $y = \sqrt{\sin^{-1}(\log_2 x)}$

(A) $[1, 2]$ (B) $[-1, 2]$ (C) $[-2, 2]$ (D) $[-3, 3]$

Ans. (A)

Solution:

$\sin^{-1}(\log_2 x) \ge 0$ & $-1 \le \log_2 x \le 1$ & $x > 0$

$0 \le \sin^{-1}(\log_2 x) \le \frac{\pi}{2}$

$0 \le (\log_2 x) \le 1$

$1 \le x \le 2$

$D = [1, 2]$

Illustration 4:

Domain of the function $f(x) = \sin^{-1}\left(\frac{2-3[x]}{4}\right)$ (Where [.] is G.I.F.)

(A) $[-3, 3]$ (B) $[0, 3]$ (C) $[0, 3)$ (D) $(-\infty, 3]$

Ans. (C)

Solution:

$$-1 \le \frac{2-3[x]}{4} \le 1 \Rightarrow -4 \le 2-3[x] \le 4 \Rightarrow -6 \le -3[x] \le 2 \Rightarrow -\frac{2}{3} \le [x] \le 2$$

$[x] = 0, 1, 2 \quad \therefore x \in [0, 3)$

Illustration 5:

Domain of the function $y = \sin^{-1}\left(\frac{x-3}{2}\right) + \log_{10}(4-x)$ is

(A) [1, 4] (B) [−4, 4] (C) [1, 4) (D) (0, 4)

Ans. (C)

Solution:

$y = \sin^{-1}\left(\frac{x-3}{2}\right) + \log_{10}(4-x)$

$-1 \le \frac{x-3}{2} \le 1 \qquad 4 - x > 0$

$-2 \le x - 3 \le 2 \qquad x < 4$

$1 \le x \le 5$

$D_1 = [1, 5] \qquad D_2 = (-\infty, 4) \qquad \therefore\ D = D_1 \cap D_2$

$D = [1, 4)$

Illustration 6:

Domain of the function $f(x) = \sin^{-1}(|x-1| - 2)$ is

(A) $[-2, 0] \cup [2, 4]$ (B) $[-2, -1] \cup [1, 4]$ (C) $(-2, 4)$ (D) $[-2, 4]$

Ans. (A)

Solution:

$-1 \le |x-1| - 2 \le 1$

$1 \le |x-1| \le 3$

$x - 1 \in [-3, -1] \cup [1, 3]$

$x \in [-2, 0] \cup [2, 4]$

Illustration 7:

Domain of the function $f(x) = \cos^{-1}\left(\frac{6-3x}{4}\right) + \operatorname{cosec}^{-1}\left(\frac{x-1}{2}\right)$ is

(A) $[3, \infty)$ (B) $\left[3, \frac{10}{3}\right]$ (C) $\left[-3, \frac{10}{3}\right]$ (D) $\left[0, \frac{10}{3}\right]$

Ans. (B)

Solution:

$-1 \le \frac{6-3x}{4} \le 1$

$-4 \le 6 - 3x \le 4$

$-10 \le -3x \le -2$

$2 \le 3x \le 10$

$\frac{2}{3} \le x \le \frac{10}{3}$

$D_1 = \left[\frac{2}{3}, \frac{10}{3}\right]$ and $\left|\frac{x-1}{2}\right| \ge 1$

$\frac{x-1}{2} \le -1$ or $\frac{x-1}{2} \ge 1$

$x - 1 \le -2$ or $x - 1 \ge 2$

$x \le -1$ or $x \ge 3$

$D_2 = (-\infty, -1] \cup [3, \infty)$

$\therefore D = D_1 \cap D_2$

$= \left[3, \frac{10}{3}\right]$

Illustration 8:

Range of $\sin^{-1}x + \operatorname{cosec}^{-1}x$ is -

(A) $\{-\pi, \pi\}$ (B) $(-\pi, \pi)$ (C) $\left\{0, \frac{\pi}{2}\right\}$ (D) none of these

Ans. (A)

Solution:

Domain of $\sin^{-1}x$ is $[-1, 1]$

domain of $\operatorname{cosec}^{-1}x$ is $(-\infty, -1] \cup [1, \infty)$

common domain is $\{-1, 1\}$ only

range = $\sin^{-1}(-1) + \operatorname{cosec}^{-1}(-1) = -\pi$

or $\sin^{-1}(1) + \operatorname{cosec}^{-1}(1) = \pi$

Hence, range = $\{-\pi, \pi\}$

Illustration 9:

Domain of $f(x) = \sin^{-1}(2x-2) + \sqrt{\log_{\{x\}} x}$ is, (where {.} represents fractional part function):-

(A) $\left[\frac{1}{2}, \frac{3}{2}\right]$ (B) $\left[\frac{1}{2}, 1\right)$ (C) $\left[\frac{1}{2}, 1\right]$ (D) $\left(\frac{1}{2}, 1\right]$

Ans. (B)

Solution:

$-1 \le 2x - 2 \le 1$ & $0 < x < 1$

$\frac{1}{2} \le x \le \frac{3}{2}$ & $0 < x < 1$

Intersection $\frac{1}{2} \le x < 1$

Illustration 10:

Find domain & range of $y = \tan^{-1}(\sqrt{x})$

Solution:

$D_f : [0, \infty)$

$$R_f : \underbrace{\tan^{-1}\underbrace{\left(\sqrt{x}\right)}_{[0,\infty)}}_{\left[0,\frac{\pi}{2}\right)}$$

$R_f : \left[0, \frac{\pi}{2}\right)$

Illustration 11:

Find domain & range of $y = \sin^{-1}(x) + \cos^{-1}(x) + \tan^{-1}(x)$

Solution:

$D_f : x \in [-1,1]$ & $x \in [-1,1]$ & $x \in R$

$\Rightarrow x \in [-1,1]$

$\therefore D_f : [-1,1]$

$\frac{\pi}{2} + \tan^{-1}(x)$

$(-1,1)$

$R_f :$ $\quad -\frac{\pi}{4}, \frac{\pi}{4}$

$\frac{\pi}{4}, \frac{3\pi}{4}$

$\therefore R_f : \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

Illustration 12:

The range of the function $f(x) = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + cos^{-1}\left[x^2 - \frac{1}{2}\right]$, where [.] is the greatest integer function, is:

Ans. $\{\pi\}$

Solution:

$$f(x) = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$$

Domain : $-1 \le \left[x^2 - \frac{1}{2}\right] \le 1 \Rightarrow x \in \left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right)$

And $-1 \le \left[x^2 + \frac{1}{2}\right] \le 1 \Rightarrow x \in \left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$

$\Rightarrow$ domain is $x \in \left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$ or $x^2 \in \left[0, \frac{3}{2}\right)$

If (i) $x^2 \in \left[0, \frac{1}{2}\right)$, then $f(x) = \pi$

If (ii) $x^2 \in \left[\frac{1}{2}, 1\right)$, then $f(x) = \pi$

If (iii) $x^2 \in \left[1, \frac{3}{2}\right)$, then $f(x) = \pi \Rightarrow$ range = $\{\pi\}$

Illustration 13:

The domain of the function $f(x) = \sqrt{\dfrac{1}{(|x|-1)\cos^{-1}(2x+1).\tan 3x}}$ is:

(A) $(-1, 0)$ (B) $(-1, 0) - \left\{-\dfrac{\pi}{6}\right\}$

(C) $(-1, 0] - \left\{-\dfrac{\pi}{6}, -\dfrac{\pi}{2}\right\}$ (D) $\left(-\dfrac{\pi}{6}, 0\right)$

Ans. (D)

Solution:

$f(x) = \sqrt{\dfrac{1}{(|x|-1)\cos^{-1}(2x+1).\tan 3x}}$

Here $-1 \le 2x+1 < 1 \Rightarrow -2 \le 2x < 0 \Rightarrow -1 \le x < 0$

$\Rightarrow x \in [-1, 0)$

But $x \neq -1$ as $|x| - 1 \neq 0$

$\therefore\ x \in (-1, 0)$

for $x \in (-1, 0)$, $(|x| - 1|$ is –ve

$\therefore$ $\tan 3x < 0$

$0 > 3x > -\dfrac{\pi}{2}$ or $x \in \left(-\dfrac{\pi}{6}, 0\right)$

Domain: $\left(-\dfrac{\pi}{6}, 0\right) \cap (-1, 0) \equiv \left(-\dfrac{\pi}{6}, 0\right)$

Illustration 14:

Let $D \equiv [-1, 1]$ is the domain of the following functions, state which of them are injective.

(A) $f(x) = \begin{cases} \tan^{-1}\dfrac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ (B) $g(x) = x^3$

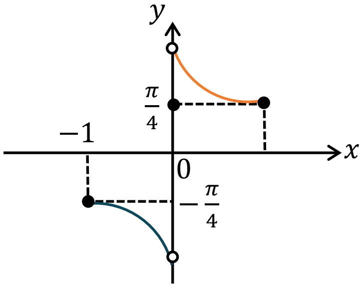
(C) $h(x) = \sin 2x$ (D) $k(x) = \sin(\pi x/2)$

Ans. (B,D)

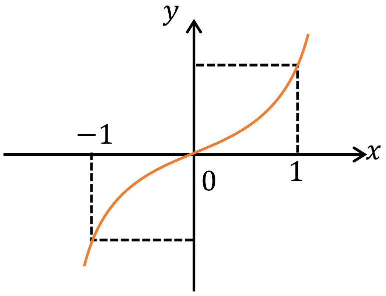
Solution:

Domain $D \in [-1, 1]$

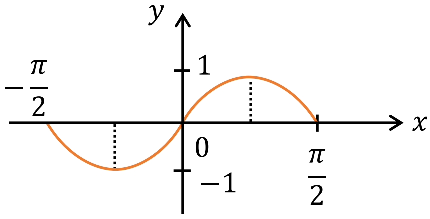
(A) $f(x) = \begin{cases} \tan^{-1}\dfrac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ many -one



(B) $g(x) = x^3$ one - one



(C) $h(x) = \sin 2x$ many - one



(D) $k(x) = \sin\left(\frac{\pi x}{2}\right)$ one-one function

Illustration 15:

Consider $f(x) = \sin^{-1}[2x] + \cos^{-1}([x] - 1)$ (where [.] denotes greatest integer function.) If domain of $f(x)$ is $[a, b)$ & the range of $f(x)$ is $\{c, d\}$ then $a + b + \frac{2d}{c}$ is equal to (where $c < d$)

Ans. (4)

Solution:

$f(x) = \sin^{-1}[2x] + \cos^{-1}([x] - 1)$

$-1 \leq [2x] \leq 1 \& -1 \leq [x] - 1 \leq 1$

$\Rightarrow -1 \leq 2x < 2 \& 0 \leq [x] \leq 2$

$\Rightarrow -\frac{1}{2} \leq x < 1 \ \& \ 0 \leq x < 3$

$\Rightarrow 0 \leq x \leq 1$ Domain

$\Rightarrow [x] = 0 \Rightarrow 0 \leq 2x < 2$

$\Rightarrow [2x] = 0$ or 1

Now $f(x) = \sin^{-1}[2x] + \cos^{-1}(-1)$

$= \left(0 \text{ or } \frac{\pi}{2}\right) + \pi = \pi \text{ or } \frac{3\pi}{2}$

$\Rightarrow a + b + \frac{2d}{c} = 1 + 3 = 4$

Illustration 16:

The value of $\cos\left(\sin^{-1}\left(\sqrt{\frac{3-\sqrt{5}}{8}}\right) + 2\cos^{-1}\left(\sqrt{\frac{3+\sqrt{5}}{8}}\right)\right)$ is equal to -

(A) 1 (B) −1 (C) $\frac{1}{2}$ (D) 0

Ans.(D)

Solution:

$$\sqrt{\frac{3-\sqrt{5}}{8}}=\sqrt{\frac{6-2\sqrt{5}}{16}}=\frac{(\sqrt{5}-1)}{4}$$

$$\sin^{-1}\left(\frac{\sqrt{5}-1}{4}\right)=\frac{\pi}{10}$$

Similarly, $\cos^{-1}\sqrt{\frac{3+\sqrt{5}}{8}}=\frac{\pi}{5}$

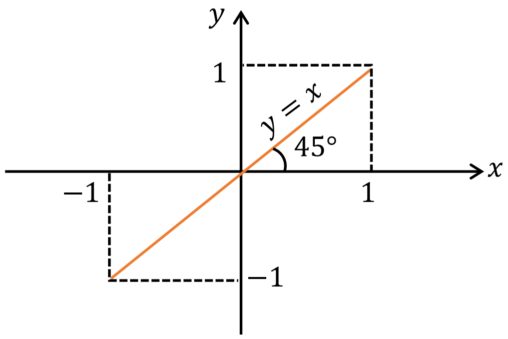
$$\cos\left(\frac{\pi}{10}+\frac{2\pi}{5}\right)=\cos\left(\frac{5\pi}{10}\right)=0$$

3. Properties of Inverse Circular Functions:

3.1 Property –1

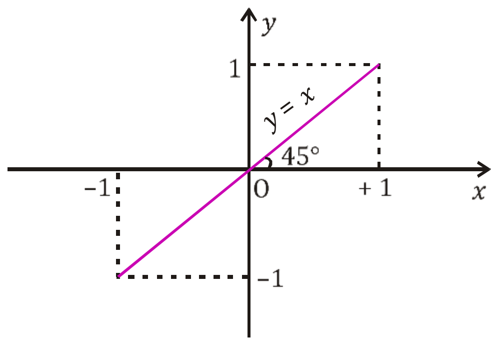
(i) $y = \sin(\sin^{-1}x) = x$

$x \in [-1,1], y \in [-1,1]$



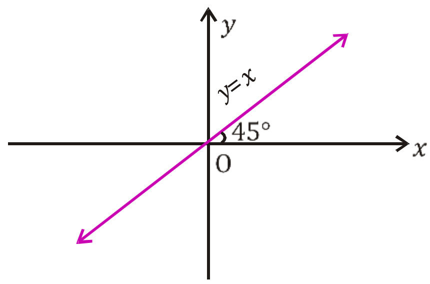
(ii) $y = \cos(\cos^{-1} x) = x$

$x \in [-1,1], y \in [-1,1]$

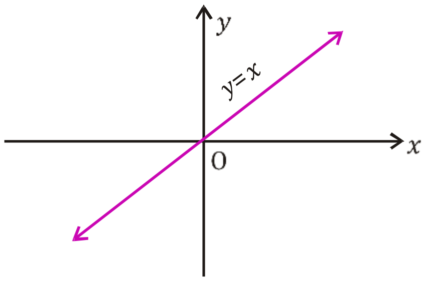


(iii) $y = \tan(\tan^{-1} x) = x$

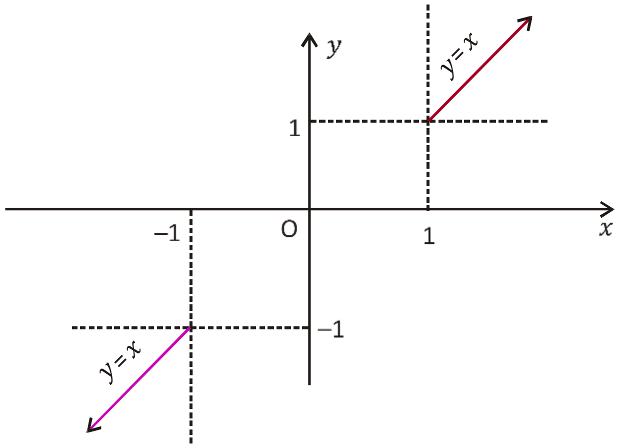
$x \in \mathbb{R}, y \in \mathbb{R}$



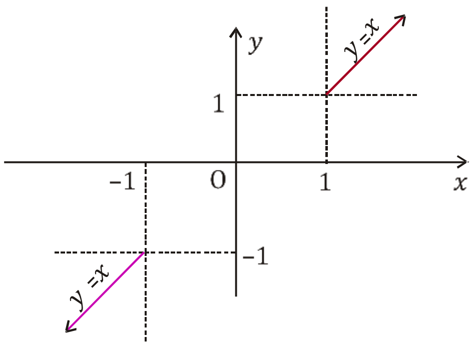
(iv) $y = \cot(\cot^{-1} x) = x$,
$x \in \mathbb{R}; y \in \mathbb{R}$



(v) $y = \text{cosec}(\text{cosec}^{-1} x) = x$
$|x| \geq 1, |y| \geq 1$



(vi) $y = \sec(\sec^{-1} x) = x$
$|x| \geq 1 ; |y| \geq 1$



Note: All the above functions are aperiodic.

Illustration 17:

Evaluate the following:

(i) $\sin(\cos^{-1} 13/5)$ (ii) $\cos(\tan^{-1} 13/4)$

Solution:

(i) Let $\cos^{-1} 3/5 = \theta$. Then,
$\cos\theta = 3/5 \Rightarrow \sin\theta = 4/5$ $\quad \therefore \sin(\cos^{-1} 13/5) = \sin\theta = 4/5$

(ii) Let $\tan^{-1} 3/4 = \theta$. Then,
$\tan\theta = 3/4$

$\Rightarrow \cos\theta = \dfrac{4}{5}$ $\quad \left\{\because \text{ as } \cos^2\theta = \dfrac{1}{1+\tan^2\theta}\right\}$ $\quad \therefore \cos(\tan^{-1} 3/4) = \cos\theta = 4/5$

Illustration 18:

(1) $\sin\left(\frac{\pi}{2}-\sin^{-1}\left(-\frac{1}{2}\right)\right)$ (2) $\tan\left\{2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right\}$

Solution:

(1) $\sin\left(\frac{\pi}{2}-\sin^{-1}\left(\frac{-1}{2}\right)\right)=\sin\left(\frac{\pi}{2}-\left(-\frac{\pi}{6}\right)\right)=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$

(2) Let $\tan^{-1}\left(\frac{1}{5}\right)=\theta\Rightarrow\tan\theta=\frac{1}{5}$

$$\tan\left(2\theta-\frac{\pi}{4}\right)=\frac{\tan(2\theta)-1}{1+\tan 2\theta} \text{ and } \tan 2\theta=\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}=\frac{5}{12} \quad (\because \tan\theta=\frac{1}{5})$$

$$\Rightarrow\tan\left(2\theta-\frac{\pi}{4}\right)=\frac{\frac{5}{12}-1}{1+\frac{5}{12}}=-\frac{7}{17}$$

Illustration 19:

Prove that $\sec^2(\tan^{-1}2)+\text{cosec}^2(\cot^{-1}3)=15$

Solution:

We have,

$\sec^2(\tan^{-1}2)+\text{cosec}^2(\cot^{-1}3)$

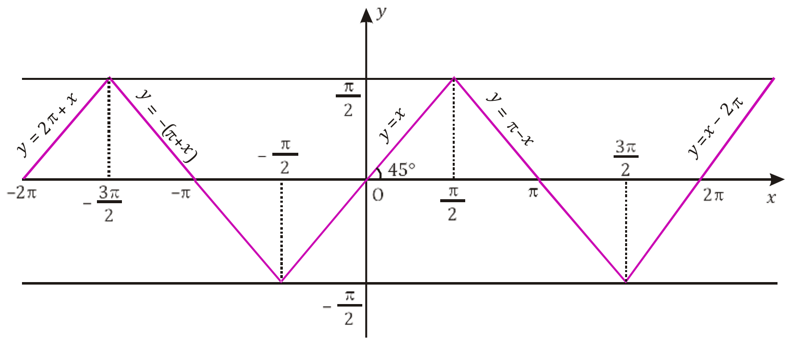
$$=\left\{\sec\left(\tan^{-1}2\right)\right\}^2+\left\{\text{cosec}\left(\cot^{-1}3\right)\right\}^2=\left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2+\left\{\text{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$$

$$=\left\{\sec\left(\sec^{-1}\sqrt{5}\right)\right\}^2+\left\{\text{cosec}\left(\text{cosec}^{-1}\sqrt{10}\right)\right\}^2=\left(\sqrt{5}\right)^2+\left(\sqrt{10}\right)^2=15$$

3.2 Property –2

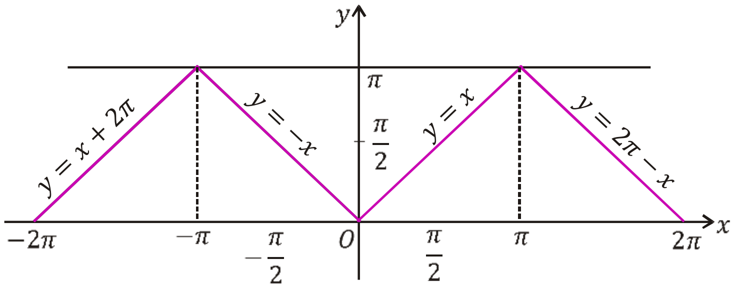
(i) $y=\sin^{-1}(\sin x), x\in\mathbb{R},\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ periodic with period 2π and it is an odd function.

$$\sin^{-1}(\sin x)=\begin{cases}-\pi-x, & -\pi\le x\le-\frac{\pi}{2}\\ x, & -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \pi-x, & \frac{\pi}{2}\le x\le\pi\end{cases}$$



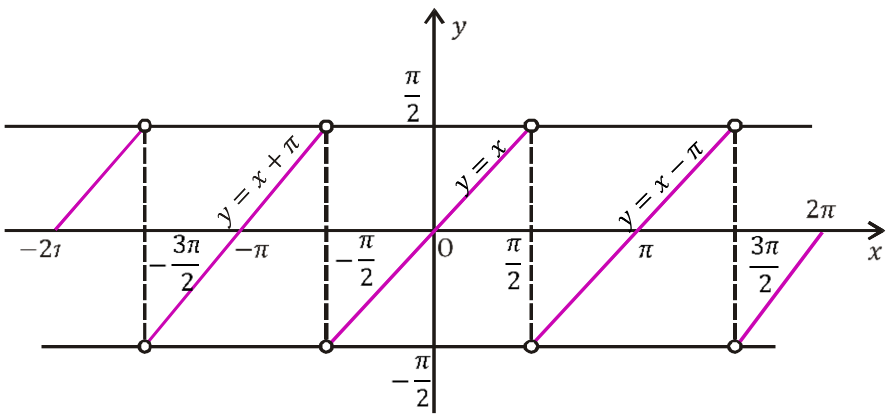
(ii) y = $\cos^{-1}(\cos x)$, $x \in \mathbb{R}$, $y \in [0,\pi]$, periodic with period 2π and it is an even function.

$$\cos^{-1}(\cos x)=\begin{cases}-x & , \quad -\pi \le x \le 0 \\ x & , \quad 0 < x \le \pi\end{cases}$$



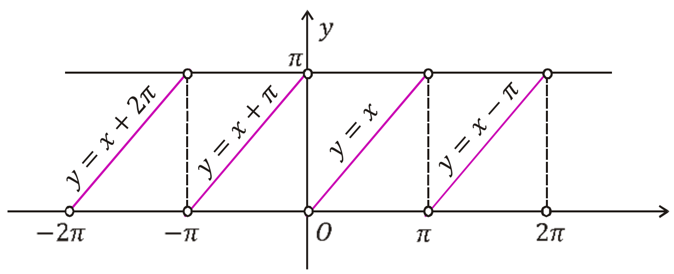
(iii) y = $\tan^{-1}(\tan x)$ $x \in \mathbb{R} - \left\{(2n-1)\frac{\pi}{2}, n \in \mathrm{I}\right\}$; $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, periodic with period π and it is an odd function.

$$\tan^{-1}(\tan x)=\begin{cases}x+\pi & , \quad -\frac{3\pi}{2} < x < -\frac{\pi}{2} \\ x & , \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x-\pi & , \quad \frac{\pi}{2} < x < \frac{3\pi}{2}\end{cases}$$

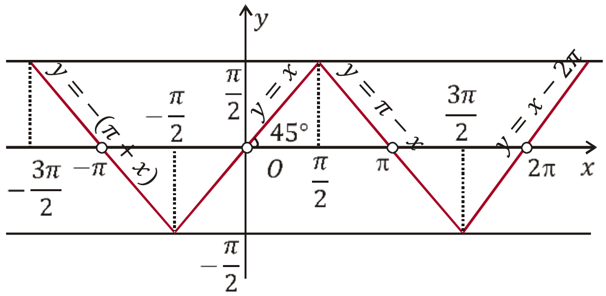


(iv) $y = \cot^{-1}(\cot x)$, $x \in \mathbb{R} - \{n\pi, n \in I\}$, $y \in (0, \pi)$, periodic with period π and neither even nor odd function.

$$\cot^{-1}(\cot x)=\begin{cases}x+\pi & , \quad -\pi < x < 0 \\ x & , \quad 0 < x < \pi \\ x-\pi & , \quad \pi < x < 2\pi\end{cases}$$



(v) $y = \text{cosec}^{-1}(\text{cosec}\, x), x \in \mathbb{R} - \{n\pi, n \in I\}$ $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, is periodic with period 2π and it is an odd function.



(vi) $y = \sec^{-1}(\sec x)$, y is periodic with period 2π and it is an even function.

$x \in \mathbb{R} - \left\{(2n-1)\frac{\pi}{2} n \in \text{I}\right\},\ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

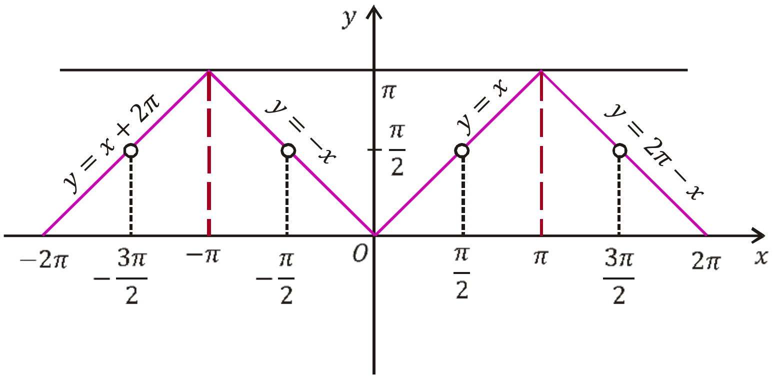


Illustration 20:

The value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6))$ is -

(A) $5\pi/6$ (B) $\pi/2$ (C) $3\pi/2$ (D) none of these

Ans.(B)

Solution:

$\sin^{-1}\left(-\sqrt{3}/2\right) = -\sin^{-1}\left(\sqrt{3}/2\right) = -\pi/3$

and $\cos^{-1}(\cos(7\pi/6)) = \cos^{-1}\cos(2\pi - 5\pi/6) = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

Hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos 7\pi/6) = -\frac{\pi}{3} + \frac{5\pi}{6} = \frac{\pi}{2}$

Illustration 21:

Evaluate the following:

(i) $\sin^{-1}(\sin 10)$ (ii) $\tan^{-1}(\tan(-6))$

Solution:

(i) We know that $\sin^{-1}(\sin\theta) = \theta$, if $-\pi/2 \le \theta \le \pi/2$

Here, $\theta = 10$ radians which does not lie between $-\pi/2$ and $\pi/2$

But, $3\pi - \theta$ i.e., $3\pi - 10$ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Also, $\sin(3\pi - 10) = \sin 10$

$\therefore \sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = (3\pi - 10)$

(ii) We know that,

$\tan^{-1}(\tan\theta) = \theta$, if $-\pi/2 < \theta < \pi/2$. Here, $\theta = -6$, radians which does not lie between $-\pi/2$ and $\pi/2$. We find that $2\pi - 6$ lies between $-\pi/2$ and $\pi/2$ such that;

$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$

$\therefore \tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6)) = (2\pi - 6)$

Illustration 22:

Evaluate the following:

$\cot^{-1}(\cot 4)$

Solution:

$\cot^{-1}(\cot 4) = \cot^{-1}(\cot(\pi + (4 - \pi))) = \cot^{-1}(\cot(4 - \pi)) = (4 - \pi)$

Illustration 23:

The principle value of $\cos^{-1}\left(-\sin\frac{7\pi}{6}\right)$ is -

(A) $\frac{5\pi}{3}$ (B) $\frac{7\pi}{6}$ (C) $\frac{\pi}{3}$ (D) none of these

Solution:

Ans. (C)

$$\frac{\pi}{2} - \sin^{-1}\left(-\sin\left(\frac{7\pi}{6}\right)\right) = \frac{\pi}{2} + \sin^{-1}\sin\left(\frac{7\pi}{6}\right) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

Illustration 24:

$\sin^{-1}\left(\sin\frac{4\pi}{7}\right) + \cos^{-1}\left(\cos\frac{4\pi}{7}\right) + \tan^{-1}\left(\tan\frac{4\pi}{7}\right)$ is equal to -

(A) π (B) $\frac{8\pi}{7}$ (C) $\frac{4\pi}{7}$ (D) $\frac{12\pi}{7}$

Ans.(C)

Solution:

$$\because \frac{\pi}{2} < \frac{4\pi}{7} < \pi$$

$$\therefore \sin^{-1}\left(\sin\frac{4\pi}{7}\right) = \pi - \frac{4\pi}{7}, \tan^{-1}\left(\tan\frac{4\pi}{7}\right) = \frac{4\pi}{7} - \pi$$

$$\cos^{-1}\left(\cos\frac{4\pi}{7}\right) = \frac{4\pi}{7}$$

Illustration 25:

Identify the correct statement(s) -

(A) $\tan^{-1}\tan\left(\frac{47}{7}\pi\right)$ is negative

(B) $\cos^{-1}(\cos(1 + \sin x)) = 1 + \sin x$ for all $x \in R$

(C) $\sin^{-1}\left(\sin\frac{23\pi}{5}\right) + \cos^{-1}\left(\cos\frac{23\pi}{5}\right) = \pi$

(D) $\tan^{-1} 2 > \cot^{-1} 2$

Ans. (A,B,C,D)

Solution:

(A) $\tan^{-1}\left(\tan\left(\frac{47\pi}{7}-7\pi\right)\right)=-\frac{2\pi}{7}$

(B) $1+\sin x \in [0,2], \forall x \in R$

$\Rightarrow \cos^{-1}(\cos(1+\sin x)) = 1+\sin x$

(C) $\sin^{-1}\left(\sin\frac{23\pi}{5}\right)=\sin^{-1}\left(\sin\left(5\pi-\frac{23\pi}{5}\right)\right)=\frac{2\pi}{5}$

$\cos^{-1}\left(\cos\frac{23\pi}{5}\right)=\cos^{-1}\left(\cos\left(4\pi+\frac{3\pi}{5}\right)\right)=\frac{3\pi}{5}$

$\frac{2\pi}{5}+\frac{3\pi}{5}=\pi$

(D) $\tan^{-1}2 > \frac{\pi}{2}-\tan^{-1}2 \Leftrightarrow \tan^{-1}2>\frac{\pi}{4} \Leftrightarrow \tan^{-1}2 > \tan^{-1}1$ which is true

3.3 Property –3

(i)	$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$;	$\lvert x\rvert \le 1$
(ii)	$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$;	$x \in \mathbb{R}$
(iii)	$\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$;	$\lvert x\rvert \ge 1$

3.4 Property –4

(i)	$\sin^{-1}(-x) = -\sin^{-1}x$;	$\lvert x\rvert \le 1$
(ii)	$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$;	$\lvert x\rvert \ge 1$
(iii)	$\tan^{-1}(-x) = -\tan^{-1}x$;	$x \in \mathbb{R}$
(iv)	$\cot^{-1}(-x) = \pi - \cot^{-1}x$;	$x \in \mathbb{R}$
(v)	$\cos^{-1}(-x) = \pi - \cos^{-1}x$;	$\lvert x\rvert \le 1$
(vi)	$\sec^{-1}(-x) = \pi - \sec^{-1}x$;	$\lvert x\rvert \ge 1$

Illustration 26:

Let $y = \cos^{-1}x + \cot^{-1}x$, then y can be equal to -

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{2}$ (C) $\frac{3\pi}{2}$ (D) $\frac{11\pi}{6}$

Ans. (B,C)

Solution:

$y = \cos^{-1}x + \cot^{-1}x$

Domain of y is $x \in [-1,1]$

$\because$ both $\cos^{-1}x$ & $\cot^{-1}x$ are decreasing

$\Rightarrow y_{max} = \cos^{-1}(-1)+\cos^{-1}(-1)$

$=\pi+\frac{3\pi}{4}=\frac{7\pi}{4}$

$y_{min} = \cos^{-1}(1)+\cot^{-1}(1) = 0+\frac{\pi}{4}$

$\Rightarrow y\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right]$

$\Rightarrow$ (B),(C) are correct.

3.5 Property –5

(i) $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$; $|x| \geq 1$

(ii) $\sec^{-1} x = \cos^{-1}\frac{1}{x}$; $|x| \geq 1$

(iii) $\cot^{-1} x = \begin{cases} \tan^{-1}\frac{1}{x} & ; \quad x > 0 \\ \pi + \tan^{-1}\frac{1}{x} & ; \quad x < 0 \end{cases}$

3.6 Property –6

(i) (a) $\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\frac{x+y}{1-xy} & \text{where} \quad x>0, y>0 \,\&\, xy<1 \\ \pi + \tan^{-1}\frac{x+y}{1-xy} & \text{where} \quad x>0, y>0 \,\&\, xy>1 \\ \frac{\pi}{2}, & \text{where} \quad x>0, y>0 \,\&\, xy=1 \end{cases}$

(b) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$ where $x > 0, y > 0$

(c) $\tan^{-1}x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

if $x > 0, y > 0, z > 0 \,\&\, xy + yz + zx < 1$

(ii) (a) $\sin^{-1} x + \sin^{-1} y = \begin{cases} \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}] & \text{where } x>0, y>0 \,\&\, (x^2+y^2) \leq 1 \\ \pi - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}] & \text{where } x>0, y>0 \,\&\, x^2+y^2 > 1 \end{cases}$

(b) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ where $x > 0, y > 0$

(iii) (a) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$ where $x > 0, y > 0$

(b) $\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ; \quad x<y, \quad x, y>0 \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right); & \quad x>y, x, y>0 \end{cases}$

Note: In the above results x & y are taken positive. In case if these are given as negative, we first apply P–4 and then use above results.

Illustration 27:

Prove that $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}$

Solution:

L. H. S. $= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13}$

$= \tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right\}$ $\quad \left\{\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); \text{if } xy<1\right\}$

$\tan^{-1}\left(\frac{20}{90}\right) = \tan^{-1}\left(\frac{2}{9}\right) = R.H.S.$

Illustration 28:

Prove that $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Solution:

$$\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$$

$$=\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)=\tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)$$

$$=\tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)=\tan^{-1}\left(\frac{325}{325}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

Illustration 29:

Prove that $\sin^{-1}\frac{12}{13}+\cot^{-1}\frac{4}{3}+\tan^{-1}\frac{63}{16}=\pi$

Solution:

We have,

$$\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}$$

$$=\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{63}{16} \qquad \left[\because\ \sin^{-1}\frac{12}{13}=\tan^{-1}\frac{12}{5}\text{ and }\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}\right]$$

$$=\pi+\tan^{-1}\left\{\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right\}+\tan^{-1}\frac{63}{16} \qquad \left[\because\ \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right), if\ \ xy>1\right]$$

$$=\pi+\tan^{-1}\left(\frac{63}{-16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16} \qquad \left[\because \tan^{-1}(-x)=-\tan^{-1}x\right]=\pi$$

Illustration 30:

Prove that: $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Solution:

We have, L.H.S. = $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{3}{4}\right)$

$$\left[\because \cos^{-1}\left(\frac{12}{13}\right)=\tan^{-1}\left(\frac{5}{12}\right)\ \&\ \sin^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{3}{4}\right)\right]$$

L.H.S. $=\tan^{-1}\left(\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}\times\frac{3}{4}}\right)=\tan^{-1}\left(\frac{56}{33}\right)$

R.H.S. = $\sin^{-1}\left(\frac{56}{65}\right)=\tan^{-1}\left(\frac{56}{33}\right)$

L.H.S. = R.H.S. Hence Proved

Illustration 31:

Which of the following is true -

(A) $\cos^{-1}\sqrt{\frac{1+x}{2}}=\sin^{-1}\sqrt{\frac{1-x}{2}}\ \forall x\in[-1,1]$ (B) $\cos^{-1}\sqrt{\frac{1+x}{2}}=2\cos^{-1}x\ \forall x\in[-1,1]$

(C) $\sin^{-1}\sqrt{\frac{1-x}{2}}=\frac{1}{2}\cos^{-1}x\ \forall x\in[-1,1]$ (D) $\sin^{-1}\sqrt{\frac{1-x}{2}}=\frac{1}{2}\sin^{-1}x\ \forall x\in[-1,1]$

Ans. (A, C)

Solution:

Let $\cos^{-1}x=\theta\in[0,\pi]$

$$\Rightarrow\quad \cos^{-1}\sqrt{\frac{1+x}{2}}=\cos^{-1}\left(\cos\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\cos^{-1}x$$

also $\sin^{-1}\sqrt{\frac{1-x}{2}}=\sin^{-1}\left(\sin\frac{\theta}{2}\right)=\frac{1}{2}\cos^{-1}x$

$\therefore$ A, C are correct.

Illustration 32:

If $\theta=\tan^{-1}(2\tan^2\theta)-\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$then find the sum of all possible values of $\tan\theta$.

Solution:

$$\theta=\tan^{-1}(2\tan^2\theta)-\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)\Rightarrow\theta=\tan^{-1}(2\tan^2\theta)-\frac{1}{2}\sin^{-1}\left(\frac{6\tan\theta}{9+\tan^2\theta}\right)$$

$$\Rightarrow\quad\theta=\tan^{-1}(2\tan^2\theta)-\frac{1}{2}\sin^{-1}\left[\frac{2\left(\frac{1}{3}\tan\theta\right)}{1+\left(\frac{1}{3}\tan\theta\right)^2}\right]\Rightarrow\theta=\tan^{-1}(2\tan^2\theta)-\frac{2}{2}\tan^{-1}\left(\frac{1}{3}\tan\theta\right)$$

$$\Rightarrow\theta=\tan^{-1}(2\tan^2\theta)-\tan^{-1}\left(\frac{1}{3}\tan\theta\right)\qquad\text{...(i)}$$

taking tangent on both sides

$$\Rightarrow\ \tan\theta=\frac{6\tan^2\theta-\tan\theta}{3+2\tan^3\theta}\Rightarrow 2\tan^4\theta-6\tan^2\theta+4\tan\theta=0$$

$\Rightarrow 2\tan\theta\,(\tan^3\theta-3\tan\theta+2)=0\Rightarrow 2\tan\theta\,(\tan\theta-1)^2\,(\tan\theta+2)=0$

$\Rightarrow \tan\theta=0,1,-2$ which satisfy equation (i)

$\therefore$ sum $=0+1-2=-1$

Illustration 33:

If $\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)+\sin^{-1}\left(\sqrt{1-\frac{x}{4}}\right)+\tan^{-1}(y)=\frac{2\pi}{3}$, then

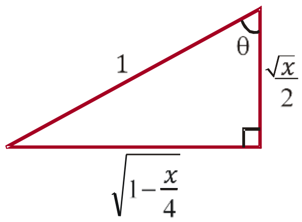
(A) Maximum value of x^2+y^2 is $\frac{49}{3}$ (B) Maximum value of x^2+y^2 is 4

(C) Minimum value of x^2+y^2 is $\frac{1}{3}$ (D) Minimum value of x^2+y^2 is 3

Ans. (A,C)

Solution:

$\sin^{-1}\left(\sqrt{1-\frac{x}{4}}\right)=\theta \Rightarrow \sin\theta=\sqrt{1-\frac{x}{4}}$



$\frac{\pi}{2}+\tan^{-1}y=\frac{2\pi}{3} \Rightarrow y=\frac{1}{\sqrt{3}} \Rightarrow 0 \le x \le 4$

Max. value of x^2+y^2 is $\frac{49}{3}$

Min. value of $x^2 + y^2$ is $\frac{1}{3}$

Illustration 34:

Let x_1, x_2, x_3 be the solution of $\tan^{-1}\left(\frac{2x+1}{x+1}\right)+\tan^{-1}\left(\frac{2x-1}{x-1}\right)=2\tan^{-1}(x+1)$ where $x_1 < x_2 < x_3$ then $2x_1+x_2+x_3^2$ is equal to

Ans. (1)

Solution:

Let $\alpha+\beta=2\gamma$ ...(i)

where $\tan\alpha=\frac{2x+1}{x+1}, \tan\beta=\frac{2x-1}{x-1}$ & $\tan\gamma=x+1$

Taking tan in (i) we get

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{2\tan\gamma}{1-\tan^2\gamma}$$

$$\Rightarrow \frac{\frac{2x+1}{x+1}+\frac{2x-1}{x-1}}{1-\frac{4x^2-1}{x^2-1}}=\frac{2(x+1)}{1-(x+1)^2} \Rightarrow \frac{2x^2-x-1+2x^2+x-1}{x^2-1-4x^2+1}=\frac{2(x+1)}{-x^2-2x}$$

$$\Rightarrow \frac{2(2x^2-1)}{-3x^2}=\frac{2(x+1)}{-(x^2+2x)}$$

$x=0$

Or $2x^3+4x^2-x-2=3x^2+3x$

$\Rightarrow 2x^3+x^2-4x-2=0 \Rightarrow x^2(2x+1)-2(2x+1)=0$

$\Rightarrow (x^2-2)(2x+1)=0 \Rightarrow x=\sqrt{2}, -\sqrt{2}$ or $x=-\frac{1}{2}$

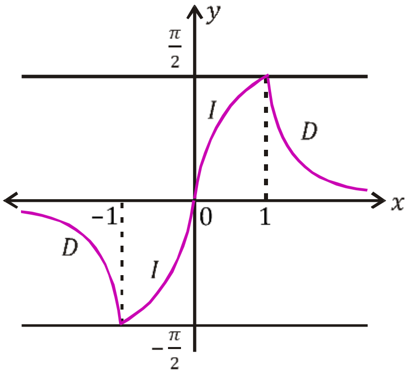
$x=-\sqrt{2}$ is rejected

$\because$ it does not satisfy (i)

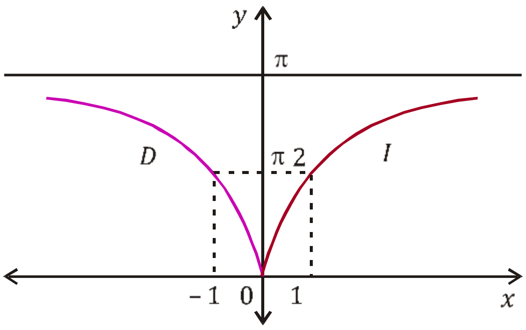
$\Rightarrow x_1=-\frac{1}{2}, x_2=0$ & $x_3=\sqrt{2} \Rightarrow 2x_1+x_2+x_3^2=1$

4. Simplified Inverse Trigonometric Functions:

(a) $y = f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$



(b) $y = f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$



(c) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$

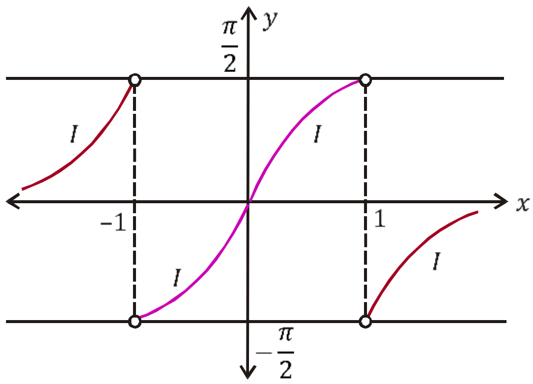


Illustration 35:

Prove that: $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Solution:

We have, $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$

$$=\tan^{-1}\left\{\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right\}+\tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{if } -1<x<1\right]$$

$$=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left\{\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right\}=\tan^{-1}\frac{31}{17}$$

Illustration 36:

Prove that $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in[0,1]$

Solution:

We have, $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left\{\frac{1-\left(\sqrt{x}\right)^2}{1+\left(\sqrt{x}\right)^2}\right\}=\frac{1}{2}\times 2\tan^{-1}\sqrt{x}=\tan^{-1}\sqrt{x}.$

Alter: Putting $\sqrt{x}$ = tan θ, we have ⇒ $\theta\in\left[0,\frac{\pi}{4}\right]$

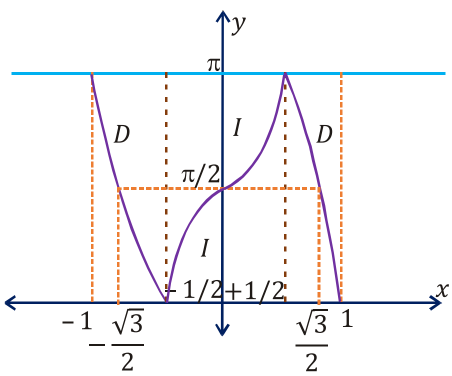
RHS = $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\frac{1}{2}\cos^{-1}(\cos 2\theta)=\theta \qquad \because\left(2\theta\in\left[0,\frac{\pi}{2}\right]\right)$

$=\tan^{-1}\sqrt{x}$ = LHS

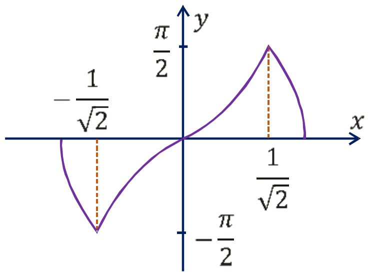
(d) $y=f(x)=\sin^{-1}(3x-4x^3)=\begin{cases}-(\pi+3\sin^{-1}x) & \text{if } -1\le x\le -\frac{1}{2}\\ 3\sin^{-1}x & \text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ \pi-3\sin^{-1}x & \text{if } \frac{1}{2}\le x\le 1\end{cases}$

(e) $y=f(x)=\cos^{-1}(4x^3-3x)$

$=\begin{cases}3\cos^{-1}x-2\pi & \text{if } -1\le x\le -\frac{1}{2}\\ 2\pi-3\cos^{-1}x & \text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ 3\cos^{-1}x & \text{if } \frac{1}{2}\le x\le 1\end{cases}$



(f) $$\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\begin{cases}-\left(\pi+2\sin^{-1}x\right) & -1\le x\le -\dfrac{1}{\sqrt{2}}\\ 2\sin^{-1}x & -\dfrac{1}{\sqrt{2}}\le x\le \dfrac{1}{\sqrt{2}}\\ \pi-2\sin^{-1}x & \dfrac{1}{\sqrt{2}}\le x\le 1\end{cases}$$



(g) $$\cos^{-1}\left(2x^2-1\right)=\begin{cases}2\cos^{-1}x & 0\le x\le 1\\ 2\pi-2\cos^{-1}x & -1\le x\le 0\end{cases}$$

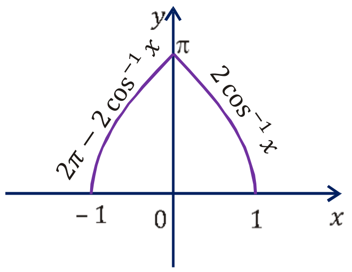


Illustration 37:

Prove that: $4\tan^{-1}\dfrac{1}{5}-\tan^{-1}\dfrac{1}{70}+\tan^{-1}\dfrac{1}{99}=\dfrac{\pi}{4}$

Solution:

$$4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=2\left\{2\tan^{-1}\frac{1}{5}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$$

$$=2\left\{\tan^{-1}\frac{2\times 1/5}{1-(1/5)^2}\right\}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}\quad\left[\begin{array}{c}\because 2\tan^{-1}x\\ =\tan^{-1}\dfrac{2x}{1-x^2},\text{if }|x|<1\end{array}\right]$$

$$=2\tan^{-1}\frac{5}{12}-\left\{\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}\right\}=\tan^{-1}\left\{\frac{2\times 5/12}{1-(5/12)^2}\right\}-\tan^{-1}\left\{\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right\}$$

$$=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{29}{6931}=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{1}{239}=\tan^{-1}\left\{\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right\}=\tan^{-1}1=\frac{\pi}{4}$$

Illustration 38:

Prove that: $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Solution:

$$2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=2\left\{\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right\}+\sec^{-1}\frac{5\sqrt{2}}{7}$$

$$=2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\}+\tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2-1} \qquad \left[\because \sec^{-1}x=\tan^{-1}\sqrt{x^2-1}\right]$$

$$=2\tan^{-1}\frac{13}{39}+\tan^{-1}\frac{1}{7}=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left\{\frac{2\times 1/3}{1-(1/3)^2}\right\}+\tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2},\ \text{if}\ |x|<1\right]$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\}=\tan^{-1}1=\frac{\pi}{4}$$

5. Equation and Identities Involving ITF:

Illustration 39:

Solve $\sin^{-1}(x^2-2x+1)+\cos^{-1}(x^2-x)=\frac{\pi}{2}$

Solution:

$\sin^{-1}(f(x))+\cos^{-1}(g(x))=\frac{\pi}{2}\Leftrightarrow f(x)=g(x)$ and $-1\le f(x), g(x)\le 1$

$x^2-2x+1=x^2-x\Leftrightarrow x=1$, accepted as a solution

Illustration 40:

Find the value of $\cos(2\cos^{-1}x+\sin^{-1}x)$ when $x=\frac{1}{5}$

Solution:

$$\cos\left(2\cos^{-1}\frac{1}{5}+\sin^{-1}\frac{1}{5}\right)=\cos\left(\cos^{-1}\frac{1}{5}+\sin^{-1}\frac{1}{5}+\cos^{-1}\frac{1}{5}\right)$$

$$=\cos\left(\frac{\pi}{2}+\cos^{-1}\frac{1}{5}\right)=-\sin\left(\cos^{-1}\left(\frac{1}{5}\right)\right) \qquad ...(i)$$

$$=-\sqrt{1-\left(\frac{1}{5}\right)^2}=-\frac{2\sqrt{6}}{5}.$$

***<u>Aliter</u>*:** Let $\cos^{-1}\frac{1}{5}=\theta\Rightarrow\cos\theta=\frac{1}{5}$ and $\theta\in\left(0,\frac{\pi}{2}\right)$

$\therefore \sin\theta=\frac{\sqrt{24}}{5}$

$\therefore \sin^{-1}(\sin\theta)=\sin^{-1}\left(\frac{\sqrt{24}}{5}\right)$ ...(ii)

$\because\ \theta \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \sin^{-1}(\sin\theta) = \theta$

$\therefore$ equation (ii) can be written as

$\theta = \sin^{-1}\left(\frac{\sqrt{24}}{5}\right)$

$\because\ \theta = \cos^{-1}\left(\frac{1}{5}\right)$

$\Rightarrow \cos^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{\sqrt{24}}{5}\right)$

Now equation (i) can be written as $y = -\sin\left\{\sin^{-1}\left(\frac{\sqrt{24}}{5}\right)\right\}$ ...(iii)

$\because\ \frac{\sqrt{24}}{5} \in [-1, 1]$ $\qquad \therefore\ \sin\left\{\sin^{-1}\left(\frac{\sqrt{24}}{5}\right)\right\} = \frac{\sqrt{24}}{5}$

$\therefore$ from equation (iii), we get $y = -\frac{\sqrt{24}}{5}$

Illustration 41:

Find value of x if $\cos^{-1}(-x) + \tan^{-1}(-x) - 2\sin^{-1}(x) + \sec^{-1}\left(-\frac{1}{x}\right) = \frac{\pi}{4}$ for $|x| \le 1$.

Solution:

$\pi - \cos^{-1}(x) - \tan^{-1}(x) - 2\sin^{-1}(x) + \cos^{-1}(-x) = \frac{\pi}{4}$

$\pi - \cos^{-1}(x) - \tan^{-1}(x) - 2\sin^{-1}(x) + \pi - \cos^{-1}(x) = \frac{\pi}{4}$

$2\pi - 2(\sin^{-1}x + \cos^{-1}x) - \frac{\pi}{4} = \tan^{-1}x$

$2\pi - \pi - \frac{\pi}{4} = \tan^{-1}x \Rightarrow \tan^{-1}x = \frac{3\pi}{4}$. Hence no solution

Illustration 42:

The value of 'a' for which $ax + \sin^{-1}(4x - x^2 - 5) + \cos^{-1}(4x - x^2 - 5) = 0$ has a real solution, is -

(A) $-\frac{\pi}{2}$ (B) $-\frac{\pi}{4}$ (C) $\frac{\pi}{4}$ (D) infinitely many solutions

Ans. (B)

Solution:

Given is

$ax + \sin^{-1}\{-1-(x-2)^2\} + \cos^{-1}\{-1-(x-2)^2\} = 0$

$\because\ -1 \le -1 - (x-2)^2 \le 1 \Rightarrow x = 2$

so, $2a + \frac{\pi}{2} = 0 \Rightarrow a = -\frac{\pi}{4}$

Illustration 43:

The value of x that satisfies $\tan^{-1}(\tan 3) = \tan^2 x$ is -

(A) $3 - \pi$ (B) $\sqrt{\pi - 3}$ (C) $\sqrt{2\pi - 3}$ (D) not possible

Ans. (D)

Solution:

$\tan^2 x = \tan^{-1}(\tan 3)$

$\tan^{-1}(\tan 3) = 3 - \pi$

$\tan^2 x = 3 - \pi$

but $3 - \pi$ is (–ve)

hence not possible.

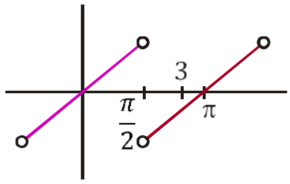


Illustration 44:

Number of values of x satisfying $\tan^{-1} x + \tan^{-1}(1 - x) = \frac{3\pi}{16}\tan\left(2\tan^{-1}\frac{1}{2}\right)$ is-

(A) 0 (B) 1 (C) 2 (D) Infinite

Ans.(C)

Solution:

$\tan^{-1} x + \tan^{-1}(1 - x) = \frac{3\pi}{16}\tan 2\theta$, where $\tan\theta = \frac{1}{2}$

$$\Rightarrow \tan^{-1} x + \tan^{-1}(1 - x) = \frac{3\pi}{16}\left(\frac{2.\frac{1}{2}}{1 - \frac{1}{4}}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{x + 1 - x}{1 - x + x^2} = 1 \Rightarrow x = 0,1$$

Illustration 45:

If $\tan^{-1}y = 4\tan^{-1}x$, $\left(|x| < \tan\frac{\pi}{8}\right)$, find y as an algebraic function of x and hence prove that $\tan\frac{\pi}{8}$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

Solution:

We have $\tan^{-1}y = 4\tan^{-1}x$

$$\Rightarrow \tan^{-1}y = 2\tan^{-1}\frac{2x}{1 - x^2} \text{ (as } |x| < 1)$$

$$= \tan^{-1}\frac{\frac{4x}{1 - x^2}}{1 - \frac{4x^2}{(1 - x^2)^2}} = \tan^{-1}\frac{4x(1 - x^2)}{x^4 - 6x^2 + 1} \qquad \left(\text{as } \left|\frac{2x}{1 - x^2}\right| < 1\right)$$

$$\Rightarrow y = \frac{4x(1 - x^2)}{x^4 - 6x^2 + 1}$$

If $x = \tan\frac{\pi}{8}$

$$\Rightarrow \tan^{-1} y = 4\tan^{-1} x = \frac{\pi}{2} \Rightarrow y \text{ is not defined} \Rightarrow x^4 - 6x^2 + 1 = 0$$

Illustration 46:

Let ${}^{50}C_{15} + 5\,{}^{50}C_{14} + 10\,{}^{50}C_{13} + 10\,{}^{50}C_{12} + 5\,{}^{50}C_{11} + {}^{50}C_{10} = {}^{55}C_p$

and ${}^{5}C_5 + {}^{6}C_5 + {}^{7}C_5 + \ldots\ldots\ldots + {}^{54}C_5 = {}^{55}C_q$, $p, q < 30$.

On the basis of above information, answer the following questions:

(i) $\sin^{-1}\sin\left((p+q)\frac{\pi}{4}\right)+\cos^{-1}\cos(p)+\tan^{-1}\tan(q)$ equals

(A) $\frac{13\pi}{4}-9$ (B) $21-\frac{25\pi}{4}$ (C) $\frac{11\pi}{4}-9$ (D) $\frac{27\pi}{4}-21$

(ii) If $\sin^{-1}(x-x^2+x^3-x^4+\ldots\ldots\infty)+\cos^{-1}(x^2+x^3+x^4+\ldots\ldots\infty)=(p-2q-1)\frac{\pi}{4}, (|x|<1)$ then number of possible real value(s) of x is

(A) 1 (B) 2 (C) 3 (D) more than 3

(iii) If $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{q\pi}{2(p-3)}$, then sum of all possible real value(s) of x is

(A) – 1 (B) – $\frac{5}{6}$ (C) 0 (D) $\frac{1}{6}$

Solution:

${}^{5}C_0\,{}^{50}C_{15} + {}^{5}C_1\,{}^{50}C_{14} + {}^{5}C_2\,{}^{50}C_{13} + {}^{5}C_3\,{}^{50}C_{12} + {}^{5}C_4 + {}^{50}C_{11} + {}^{5}C_5\,{}^{50}C_{10} = {}^{55}C_{15}$

so $p = 15$

${}^{5}C_5 + {}^{6}C_5 + {}^{7}C_5 + \ldots\ldots\ldots\ldots + {}^{54}C_5 = {}^{55}C_6$

$q = 6$

(i) Ans.(B)

$$\sin^{-1}\sin(15+6)\frac{\pi}{4}=\sin^{-1}\sin\frac{21\pi}{4}=5\pi-\frac{21\pi}{4}=-\frac{\pi}{4}$$

$$\cos^{-1}\cos(15) = 15-4\pi$$

$$\tan^{-1}\tan(6)=6-2\pi$$

(ii) Ans.(A)

$$\sin^{-1}(x-x^2+x^3-x^4+\ldots\ldots\infty)+\cos^{-1}(x^2+x^3+x^4+\ldots\ldots\infty)=\frac{\pi}{2}$$

$$\sin^{-1}\left(\frac{x}{1+x}\right)+\cos^{-1}\left(\frac{x^2}{1-x}\right)=\frac{\pi}{2} \Rightarrow \frac{x}{1+x}=\frac{x^2}{1-x}$$

$$x = 0 \text{ or } \frac{1}{1+x}=\frac{x}{1-x}$$

$\Rightarrow 1-x = x^2 + x$

$\Rightarrow x^2 + 2x-1 = 0$

$$\Rightarrow x=\frac{-2\pm\sqrt{8}}{2}$$

$\Rightarrow x = -1 \pm \sqrt{2}$

$\Rightarrow x = -1+\sqrt{2}, x = -1-\sqrt{2}$ (rejected)

so, one value of x.

(iii) Ans.(D)

$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

$\Rightarrow \frac{2x+3x}{1-6x^2}=1$

$\Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow x=\frac{1}{6}$, $x = -1$ (rejected)

so, one value of x.

6. Simultaneous Inequations and Inequations Involving I.T.F.

For $|x| \le 1$, $|y| \le 1$

$$x > y \Leftrightarrow \begin{cases} \sin^{-1} x > \sin^{-1} y \\ \cos^{-1} x < \cos^{-1} y \end{cases}$$

and for x, y $\in R$

$x > y \Leftrightarrow \tan^{-1} x > \tan^{-1} y$

Illustration 47:

The solution of the inequality $\pi - \sin^{-1}\sin\left(\frac{2x^2+3}{x^2+1}\right) \le 3$ -

(A) [2, 3] (B) (2, 3] (C) R^+ (D) R

Ans. (D)

Solution:

$\frac{2x^2+3}{x^2+1} = \frac{2(x^2+1)+1}{x^2+1} = 2 + \frac{1}{x^2+1}$

$0 < \frac{1}{x^2+1} \le 1 \Rightarrow 2 < 2 + \frac{1}{x^2+1} \le 3$

$\sin^{-1}\sin\left(\frac{2x^2+3}{x^2+1}\right) = \pi - \frac{2x^2+3}{x^2+1}$

Given inequality becomes, $\pi - \left(\pi - \frac{2x^2+3}{x^2+1}\right) \le 3$

$2x^2 + 3 \le 3x^2 + 3$

$x^2 \ge 0 \Rightarrow x \in R$.

Illustration 48:

Find the complete solution set of $\sin^{-1}(\sin 5) > x^2 - 4x$.

Solution:

$\sin^{-1}(\sin 5) > x^2 - 4x \Rightarrow \sin^{-1}[\sin(5 - 2\pi)] > x^2 - 4x$

$\Rightarrow x^2 - 4x < 5 - 2\pi \Rightarrow x^2 - 4x + (2\pi - 5) < 0$

$\Rightarrow 2-\sqrt{9-2\pi} < x < 2+\sqrt{9-2\pi} \Rightarrow x \in (2-\sqrt{9-2\pi},\ 2+\sqrt{9-2\pi})$

Illustration 49:

Find the complete solution set of $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$, where [.] denotes the greatest integer function.

Solution:

$[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$

$\Rightarrow ([\cot^{-1}x] - 3)^2 \le 0 \Rightarrow [\cot^{-1}x] = 3 \Rightarrow 3 \le \cot^{-1}x < 4 \Rightarrow x \in (-\infty, \cot 3]$

Illustration 50:

If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in N$,then the maximum value of n is -

(A) 1 (B) 5 (C) 9 (D) none of these

Ans. (B)

Solution:

$\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$

$\Rightarrow \cot\left(\cot^{-1}\left(\frac{n}{\pi}\right)\right) < \cot\left(\frac{\pi}{6}\right) \Rightarrow \frac{n}{\pi} < \sqrt{3}$

$\Rightarrow n < \pi\sqrt{3} \Rightarrow n < 5.5$ (approx.)

$\Rightarrow \; n = 5 \qquad \because (n \in N)$

7. Summation of Series Involving $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y$

Illustration 51:

If $A = 2\tan^{-1}(2\sqrt{2} - 1)$ and $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$, then show $A > B$.

Solution:

We have, $A = 2\tan^{-1}(2\sqrt{2} - 1) = 2\tan^{-1}(1.828)$

$\Rightarrow A > 2\tan^{-1}(\sqrt{3}) \Rightarrow A > \frac{2\pi}{3}$ ...(i)

also, we have, $\sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{1}{2}\right) \Rightarrow \sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{6}$

$\Rightarrow 3\sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{2}$

also, $3\sin^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left(3.\frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right) = \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852)$

$\Rightarrow 3\sin^{-1}(1/3) < \sin^{-1}(\sqrt{3}/2) \Rightarrow 3\sin^{-1}(1/3) < \pi/3$

also, $\sin^{-1}(3/5) = \sin^{-1}(0.6) < \sin^{-1}(\sqrt{3}/2) \Rightarrow \sin^{-1}(3/5) < \pi/3$

Hence, $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5) < \frac{2\pi}{3}$ ...(ii)

From (i) and (ii), we have $A > B$.

Illustration 52:

The value of $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2-n+1}\right)$ is $\frac{\pi}{a}$, then 'a' is equal to

Ans. (2)

Solution:

$T_n = \tan^{-1}\left(\frac{1}{1+n(n-1)}\right) = \tan^{-1}\left(\frac{n-(n-1)}{1+n(n-1)}\right)$

$T_n = \tan^{-1}(n) - \tan^{-1}(n-1)$

$T_1 = \tan^{-1}(1) - \tan^{-1}(0)$

$T_2 = \tan^{-1}(2) - \tan^{-1}(1)$

$T_n = \tan^{-1}(n) - \tan^{-1}(n-1)$

$S_n = \tan^{-1}(n)$

$S_\infty = \tan^{-1}(\infty) = \frac{\pi}{2}$

Illustration 53:

Prove that: $\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+...+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)=\tan^{-1}\left(\frac{x}{y}\right)$

Solution:

L.H.S.

$\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+...+\tan^{-1}\left(\frac{c_n-c_{n-1}}{1+c_nc_{n-1}}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$

$=\tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}.\frac{1}{c_1}}\right)+\left(\tan^{-1}c_2-\tan^{-1}c_1\right)+\left(\tan^{-1}c_3-\tan^{-1}c_2\right)+....+\left(\tan^{-1}c_n-\tan^{-1}c_{n-1}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)$

$=\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{1}{c_1}\right)-\tan^{-1}c_1+\tan^{-1}c_n+\tan^{-1}\left(\frac{1}{c_n}\right)$

$=\tan^{-1}\left(\frac{x}{y}\right)-\left(\cot^{-1}c_1+\tan^{-1}c_1\right)+\left(\tan^{-1}c_n+\cot^{-1}c_n\right)$

$=\tan^{-1}\left(\frac{x}{y}\right)-\frac{\pi}{2}+\frac{\pi}{2}=\tan^{-1}\left(\frac{x}{y}\right)$ = R.H.S.

Illustration 54:

If $\sum_{n=1}^{x}\tan^{-1}\left(\frac{4}{n^2+3}\right)=\pi-\tan^{-1}\frac{26}{7}$ is satisfied by $x \in N$, then the value of $\left[\sin^{-1}(\cos x)\right]$ (where [.] denotes greatest integer function), is -

(A) – 2 (B) – 1 (C) 0 (D) 1

Ans.(B)

Solution:

$$\pi-\tan^{-1}\frac{26}{7}=\sum_{n=1}^{x}\tan^{-1}\left(\frac{n+1}{2}\right)-\tan^{-1}\left(\frac{n-1}{2}\right)$$

$$=\tan^{-1}(1)-\tan^{-1}(0)+\tan^{-1}\left(\frac{3}{2}\right)-\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}(2)-\tan^{-1}(1)+....+\tan^{-1}\left(\frac{x+1}{2}\right)-\tan^{-1}\left(\frac{x-1}{2}\right)$$

$$=\tan^{-1}\left(\frac{x}{2}\right)-\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{x+1}{2}\right)=\pi+\tan^{-1}\left\{\frac{\frac{x+1}{2}+\frac{x}{2}}{1-\frac{(x+1)x}{4}}\right\}-\tan^{-1}\left(\frac{1}{2}\right)$$

$$=\pi+\tan^{-1}\left(\frac{4x+2}{4-x(x+1)}\right)-\tan^{-1}\frac{1}{2}\Rightarrow\frac{x^2+9x}{2(x^2-x-5)}=\frac{26}{7}\Rightarrow x=4$$

again, $\left[\sin^{-1}(\cos 4)\right]=\left[\frac{\pi}{2}-\cos^{-1}(\cos 4)\right]=\left[\frac{\pi}{2}-2\pi+4\right]=-1$

Illustration 55:

Sum of series $\sum_{n=0}^{\infty}\cot^{-1}(2^{n+1}+2^{-n})$ is equal to-

(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}+\tan^{-1}2$ (D) $\frac{\pi}{4}+\cot^{-1}2$

Ans. (B)

Solution:

$$\sum_{n=0}^{\infty}\cot^{-1}(2^{n+1}+2^{-n})=\sum_{n=0}^{\infty}\tan^{-1}\left(\frac{1}{2^{n+1}+2^{-n}}\right)=\sum\tan^{-1}\left(\frac{2^{n+1}-2^n}{1+2^n.2^{n+1}}\right)=\sum_{n=0}^{\infty}\tan^{-1}(2^{n+1})-\tan^{-1}(2^n)$$

$$(\cancel{\tan^{-1}2}-\tan^{-1}1)+(\cancel{\tan^{-1}4}-\cancel{\tan^{-1}2})+.......\infty$$

$$=\frac{\pi}{2}-\tan^{-1}(1)=\frac{\pi}{4}$$

Illustration 56:

If $\tan^{-1}\frac{2}{1^2}+\tan^{-1}\frac{2}{2^2}+\tan^{-1}\frac{2}{3^2}+\tan^{-1}\frac{2}{4^2}+.........\tan^{-1}\frac{2}{17^2}=\pi-\tan^{-1}\frac{p}{q}$, where p & q are co-prime, then $(p-q)$ is

Ans. (7)

Solution:

$$T_n=\tan^{-1}\frac{2}{n^2}=\tan^{-1}\left[\frac{(n+1)-(n-1)}{1+(n-1)(n+1)}\right]$$

$$\sum_{n=1}^{n}\tan^{-1}\frac{2}{n^2}=\tan^{-1}18+\tan^{-1}17-\tan^{-1}1=\pi-\tan^{-1}\frac{34}{27}$$

$p-q=7$

Illustration 57:

If $\sum\limits_{r=1}^{\infty}\cot^{-1}\left(\frac{(r+1)^2}{2}\right)=\tan^{-1}x$, then x is equal to -

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) 3 (D) $\frac{3}{2}$

Ans. (C)

Solution:

$\sum\limits_{r=1}^{n}\tan^{-1}\left(\frac{2}{(r+1)^2}\right)=\tan^{-1}\left(\frac{(r+2)-r}{1+r(r+2)}\right)$

$=(\tan^{-1}3-\tan^{-1}1)+(\tan^{-1}4-\tan^{-1}2)+(\tan^{-1}5-\tan^{-1}3)+..$

$..+(\tan^{-1}(n+1)-\tan^{-1}(n-1))+(\tan^{-1}(n+2)-\tan^{-1}n)$

$=\tan^{-1}(n+1)+\tan^{-1}(n+2)-\tan^{-1}1-\tan^{-1}2$

$\sum\limits_{r=1}^{\infty}\tan^{-1}\left(\frac{2}{(r+1)^2}\right)=\frac{\pi}{2}+\frac{\pi}{2}-\tan^{-1}1-\tan^{-1}2=\tan^{-1}3$