Inverse Trigonometric Functions

SOLUTIONS

EXERCISE - O

1. **Ans. (B)**

$\cos^{-1}\{\sqrt{x}\} > 0$ and $-1 \leq \{\sqrt{x}\} \leq 1$ and $x \geq 0$

It is true It is true

$x \in [0, \infty)$

2. **Ans. (A)**

Range of $ax^2 + bx + c = 0$ when $\alpha < 0$ is $\left(-\infty, -\frac{-D}{4a}\right]$

$-\infty, < -x^2 + 2x + 3 \leq 4$

log function is defined only when

$0 < -x^2 + 2x + 3 \leq 4$

$-\infty < \log_2(-x^2 + 2x + 3) \leq 2$

$\because$ $\sin^{-1} M$ is defined only when $-1 \leq M \leq 1$

So, $-1 \leq \log_2(-x^2 + 2x + 3) \leq 1$

$-\frac{\pi}{2} \leq \sin^{-1}\left(\log_2\left(-x^2 + 2x + 3\right)\right) \leq \frac{\pi}{2}$

Range of function $f(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

3. **Ans. (C)**

$\because$ $0 \leq x^2 < \infty$

$\Rightarrow$ $-\infty < 1 - x^2 \leq 1$

$\Rightarrow$ $0 < 1 - x^2 \leq 1$ (to define $\log_e\left(1 - x^2\right)$)

$\Rightarrow$ $-\infty < \log_e(1 - x^2) \leq 0$

$\frac{\pi}{2} \leq \cot^{-1}\left(\log_e\left(1 - x2\right)\right) < \pi$

So Range of function $f(x)$ is $\left[\frac{\pi}{2}, \pi\right)$

4. **Ans. (C)**

$f(x) = \cot^{-1}\sqrt{(x+3)x} + \cos^{-1}\sqrt{x^2 + 3x + 1}$

to define of $\cot^{-1}\sqrt{(x+3)x}$

$\Rightarrow (x+3)(x) \geq 0$ ...(1)

\+ – +

−3 0

$x \in (-0, -3] \cup [0, \infty)$

to define of $\cos^{-1}\sqrt{x^2 + 3x + 1}$

$0 \leq x^2 + 3x + 1 \leq 1$

$-1 \le x^2 + 3x \le 0$

$-1 \le (x)(x+3) \le 0$ ...(2)

from (1) and (2)

$x^2 + 3x = 0$

$x = 0, -3$

domain of $f(x) = \{0, -3\}$

5. **Ans. (B)**

$$\cot^{-1}\left\{\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right\} \qquad \frac{\pi}{2}<x<\pi$$

We know that $\sin x > \cos y\ \forall\ x \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$

$\cos x > \sin y\ \forall\ x \in \left(0,\frac{\pi}{4}\right)$

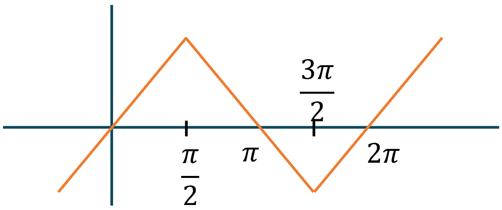
$$= \cot^{-1}\left\{\frac{\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|+\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|}{\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|-\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|}\right\} \qquad \frac{x}{2} \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$$

$$= \cot^{-1}\left\{\frac{\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}-\cos\frac{x}{2}-\sin\frac{x}{2}-\cos\frac{x}{2}}\right\} = \cot^{-1}\left\{\frac{2\sin\frac{x}{2}}{-2\cos\frac{x}{2}}\right\}$$

$$= \cot^{-1}\left(-\frac{\tan x}{2}\right) = \frac{\pi}{2} - \tan^{-1}\left(-\tan\frac{x}{2}\right)$$

$$= \frac{\pi}{2}+\frac{x}{2}$$

6. **Ans. (A)**



$\sin^{-1}(\sin 5) = 5 - 2\pi$

$\sin^{-1}(\sin 5) > x^2 - 4x$

$\Rightarrow x^2 - 4x < 5 - 2\pi$

$\Rightarrow x^2 - 4x + 2\pi - 5 < 0$

$$\Rightarrow \left(x - 2 - \sqrt{9-2\pi}\,\right)\left(x - 2 + \sqrt{9-2\pi}\,\right) < 0 \qquad \left\{\begin{array}{l}\text{Roots of Quardatic} \\ x^2 - 4x + 2\pi - 5 = 0 \\ x = 2 \pm \sqrt{9-2\pi}\end{array}\right\}$$

$$\begin{array}{ccc} +\text{ve} & -\text{ve} & +\text{ve} \\ 2-\sqrt{9-2\pi} & & 2+\sqrt{9-2\pi} \end{array}$$

$\Rightarrow x \in \left(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi}\right)$

7. **Ans. (B)**

$(x)(x-2)(3x-7) = 2$

$(x^2-2x)(3x-7) = 2$

$3x^3 - 6x^2 - 7x^2 + 14x = 2$

$3x^2 - 13x^2 + 14x - 2 = 0$ r s t

$r+s+t = \frac{13}{3}$

$rs+st+tr = \frac{14}{3}$

$rst = \frac{2}{3}$

$\Rightarrow \tan^{-1}(r) + \tan^{-1}(s) + \tan^{-1}(t)$

$\Rightarrow \tan^{-1}\left(\frac{r+s}{1-rs}\right) + \tan^{-1}(t)$

$\Rightarrow \tan^{-1}\left(\frac{\frac{r+s}{1-rs}+t}{1-\left(\frac{r+s}{1-rs}\right)t}\right)$

$\Rightarrow \tan^{-1}\left(\frac{r+s+t-rst}{1-rs-rt-st}\right)$

$\Rightarrow \tan^{-1}\left(\frac{\frac{13}{3}-\frac{2}{3}}{1-\frac{14}{3}}\right)$

$\Rightarrow \tan^{-1}\left(\frac{\frac{11}{3}}{\frac{-11}{3}}\right)$

$\Rightarrow \tan^{-1}(-1) = \frac{3\pi}{4}$

8. **Ans. (A)**

$[\cot^{-1} x]^2 - 6[\cot^{-1} x] + 9 \le 0$

Let $[\cot^{-1} x] = t$

$t^2 - 6t + 9 \le 0$

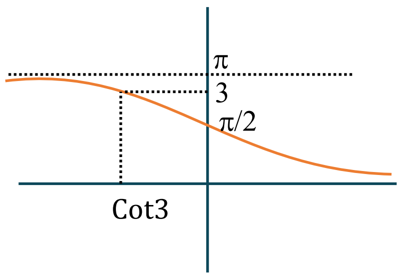
$(t-3)^2 \le 0$

$t = 3$

$[\cot^{-1} x] = 3$

$[\cot^{-1} x] = 3$

$\boxed{x \in (-\infty, \cot 3]}$



9. **Ans. (A)**

Put $x = \cos\theta$ i.e. $\theta = \cos^{-1} x$

$\therefore \cos^{-1}(2\cos^2\theta -) = 2\pi - 2\theta$

$\cos^{-1}(\cos 2\theta) = 2\pi - 2\theta$

possible when

$0 \le 2\pi - 2\theta \le \pi$

$\Rightarrow 0 - 2\pi \le -2\theta \le \pi - 2\pi$

$\Rightarrow -2\pi \le -2\theta \le -\pi$

$\Rightarrow \pi \le 2\theta \le 2\pi$

$\Rightarrow \frac{\pi}{2} \le \theta \le \pi$

$\Rightarrow \frac{\pi}{2} \le \cos^{-1} x \le \pi$

$\Rightarrow -1 \le x \le 0$

10. **Ans. (C)**

$\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$

$\Rightarrow \cos^{-1}\left(\frac{xy}{2} + \sqrt{\left(1-x^2\right)\left(1-\frac{y^2}{4}\right)}\right) = \alpha$

$\Rightarrow \cos^{-1}\left(\frac{xy + \sqrt{4 - y^2 - 4x^2 + x^2y^2}}{2}\right) = \alpha$

$\Rightarrow xy + \sqrt{4 - y^2 - 4x^2 + x^2y^2} = 2\cos\alpha$

$\Rightarrow \sqrt{4 - y^2 - 4x^2 + x^2y^2} = 2\cos\alpha - xy$

Square on both side

$\Rightarrow 4 - y^2 - 4x^2 + x^2y^2 = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$

$\Rightarrow 4x^2 + y^2 - 4xy\cos\alpha = 4\sin^2\alpha$

11. **Ans. (C)**

$\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1} x\right)$

Let $\frac{1}{2}\cos^{-1} x = \theta$

$\Rightarrow \cos^{-1} x = 2\theta \;\Rightarrow\; x = \cos 2\theta$

$\Rightarrow \tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right)$

$= \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta} + \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}$ $\left\{\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right\}$

$= \frac{1+\tan\theta}{1-\tan\theta} + \frac{1-\tan\theta}{1+\tan\theta}$ $\left\{\because \tan\frac{\pi}{4} = 1\right\}$

$$= \frac{(1+\tan\theta)^2 + (1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)}$$

$$= \frac{1+\tan^2\theta+2\tan\theta+1+\tan^2\theta-2\tan\theta}{1-\tan^2\theta}$$

$$= \frac{2+2\tan^2\theta}{1-\tan^2\theta}$$

$$= 2\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right) = \frac{2}{\cos 2\theta} = \frac{2}{x} \qquad \left\{\because \ \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right\}$$

12. Ans. (A)

$$\alpha = 2 \text{ arc } \tan\left(\frac{1+x}{1-x}\right)$$

$$= 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$$

Put $x = \tan\theta > 1 \Rightarrow \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$\Rightarrow 2\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= 2\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$$

$$= 2\left[\frac{\pi}{4}+\theta-\pi\right] \qquad \left\{as \ \frac{\pi}{4}+\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)\right\}$$

$$\Rightarrow 2\theta - \frac{3\pi}{2} \Rightarrow 2\tan^{-1} x - \frac{3\pi}{2}$$

$$\Rightarrow \beta = \text{ arc } \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \frac{\pi}{2} - \left(2\tan^{-1} x\right) \qquad \left\{as \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1} x \text{ for } x > 0\right\}$$

$$\alpha + \beta = \left(2\tan^{-1} - \frac{3\pi}{2}\right) + \left(\frac{\pi}{2} - 2\tan^{-1} x\right) = -\pi$$

13. Ans. (A)

Let $\theta = \tan^{-1}\frac{1}{5}$

$$\tan\theta = \frac{1}{5}$$

$$\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$$

$$\Rightarrow \tan\left(2\theta-\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\tan 2\theta-\tan\frac{\pi}{4}}{1+\tan 2\theta\cdot\tan\frac{\pi}{4}}=\frac{\tan 2\theta-1}{1+\tan 2\theta}$$

Now $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

$$=\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}=\frac{\frac{2}{5}}{\frac{24}{25}}\Rightarrow \frac{10}{24}=\frac{5}{12}$$

$$\Rightarrow \frac{\tan 2\theta-1}{1+\tan 2\theta}=\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\Rightarrow \frac{-7}{17}$$

14. **Ans. (B)**

$$\tan^{-1}\left(\frac{r((r+1)!)}{(r+1)+((r+1)!)^2}\right)$$

$$=\tan^{-1}\left(\frac{r\frac{((r+1)!)}{r+1}}{1+\frac{((r+1)!)^2}{r+1}}\right)=\tan^{-1}\left(\frac{r(r!)}{1+\left(\frac{(r+1)!}{r+1}\right)((r+1)!)}\right)$$

$$=\tan^{-1}\left(\frac{(r+1-1)r!}{1+(r!)(r+1)!}\right)$$

$$=\tan^{-1}\left[\frac{(r+1)!-r!}{1+(r!)(r+1)!}\right]$$

$$=\tan^{-1}[(r+1)!]-\tan^{-1}(r!) \qquad \left[\because \tan^{-1}\left(\frac{x-y}{1+xy}\right)=\tan^{-1}x-\tan^{-1}y\right]$$

$$\sum_{r=0}^{\infty}\left[\tan^{-1}((r+1)!)-\tan^{-1}(r!)\right]$$

$= \tan^{-1}(1!) - \tan^{-1}(0!)$
$+\tan^{-1}(2!) - (\tan^{-1}(1!)$
$+\tan^{-1}(3!) - \tan^{-1}(2!)$
$+ \quad \vdots \qquad\qquad \vdots$
$\quad \vdots \qquad\qquad \vdots$
$\quad \vdots \qquad\qquad \vdots$

$\tan^{-1}(n+1)! - \tan^{-}n!$

$$\Rightarrow \lim_{n\to\infty}\tan^{-1}((n+1)!)-\tan^{-1}(0!)$$

$$\Rightarrow \frac{\pi}{2}-\tan^{-1}1$$

$$\Rightarrow \frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

15. Ans. (A)

$$\tan^{-1}\left(\frac{2r+1}{r^4+2r^3+r^2+1}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{2r+1}{1+r^2\left(r^2+2r+1\right)}\right)$$

$$\Rightarrow \tan^{-1}\left[\frac{(r^2+2r+1)-r^2}{1+r^2(r^2+2r+1)}\right]$$

$$\Rightarrow \tan^{-1}(r^2+2r+1)-\tan^{-1}r^2 \qquad \left[\because \tan^{-1}\left(\frac{x-y}{1+xy}\right)=\tan^{-1}x-\tan^{-1}y\right]$$

$$\sum_{r=1}^{n}\left[\tan^{-1}(r+1)^2-\tan^{-1}r^2\right]$$

$\Rightarrow \tan^{-1}2^2-\tan^{-1}1^2$

$+\tan^{-1}3^2-\tan^{-1}2^2$

$+\tan^{-1}4^2-\tan^{-1}3^2$

$+ \quad \vdots \qquad \vdots$

$\quad \vdots \qquad \vdots$

$\quad \vdots \qquad \vdots$

$+\tan^{-1}(n+1)^2-\tan^{-1}n^2$

$\Rightarrow \tan^{-1}(n+1)^2-\tan^{-1}1$

$$\lim_{n\to\infty}\left(\tan^{-1}(n+1)^2-\tan^{-1}1\right)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

16. Ans. (B,C)

$f(x)=e^x,\ g(x)=\sin^{-1}x$

$gof(x)=sin^{-1}(e^x)$

Domain of $gof(x)$

$-1\le e^x\le 1 \Rightarrow -\infty<x\le 0$

& domain of $f(x)$ is R

Domain of $gof(x)$ is $x\in(-\infty,0]$

Range of $gof(x)$

$\because e^x\in(0,1]\forall x\in(-\infty,0]$

$\Rightarrow \sin^{-1}\left(e^x\right)\in\left(0,\frac{\pi}{2}\right]$

Option (B) & (C) are correct

17. Ans. (A,B,C,D)

$f(x)=\sin^{-1}(\tan x)+\cos^{-1}(\cot x)$

$f(x)$ is defined then

$-1\le tanx\le 1$ ...(i)

and $-1\le cotx\le 1$

$tanx\in(-\infty,-1]\cup[1,\infty)$ ...(ii)

From (i) & (ii)

$tanx = 1, -1 \qquad \therefore f(x) = \frac{\pi}{2}$

$\Rightarrow x = n\pi \pm \frac{\pi}{4}, \qquad n \in I$

For $n \in I$, $f(x)$ is periodic with period $\frac{\pi}{2}$

Also for diff. value of n, we have diff. values of x for which $f(x) = \frac{\pi}{2}$

Hence it is many one function.

18. **Ans. (B,C)**

$$\sin\left(\underbrace{2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)}_{\alpha}\right) + \cos\left(\underbrace{2\tan^{-1}\left(\frac{1}{3}\right)}_{\beta}\right) = \frac{p}{q}$$

$$\Rightarrow \alpha = 2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \quad : \quad \beta = 2\tan^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \cos\frac{\alpha}{2} = \frac{1}{\sqrt{5}}, \sin\frac{\alpha}{2} = \frac{2}{\sqrt{5}} \quad : \quad \tan\frac{\beta}{2} = \frac{1}{3}$$

$$\Rightarrow \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \quad : \quad \cos\beta = \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}$$

$$\Rightarrow \sin\alpha = 2\times\frac{2}{\sqrt{5}}\times\frac{1}{\sqrt{5}} = \frac{4}{5} \quad : \quad \cos\beta = \frac{1-\frac{1}{9}}{1+\frac{1}{9}} = \frac{8}{10} = \frac{4}{5}$$

$$\Rightarrow \sin\alpha + \cos\beta = \frac{8}{5}$$

$\Rightarrow p = 8, q = 5$

$\Rightarrow (p-q) = (3)^{2k+1}, \quad k \in n$

$\Rightarrow 2k + 1 \in$ odd no.

Unit digit place of 3^{2k+1} are 3,7

19. **Ans. (A,B)**

$$3\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow 3\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow 3\left[\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right)\right]$$

$$\Rightarrow 3\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\times\frac{1}{2}}\right) = 3\tan^{-1}(1) = \frac{3\pi}{4}$$

Option A & B are correct

20. Ans. (A,C)

$$\alpha = 2\tan^{-1}\sqrt{3-2\sqrt{2}} + \sin^{-1}\left(\frac{1}{\sqrt{6}-\sqrt{2}}\right)$$

$$\Rightarrow 2\tan^{-1}\sqrt{\left(\sqrt{2}-1\right)^{2}} + \sin^{-1}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$$\Rightarrow\ 2\tan^{-1}\left(\sqrt{2}-1\right)+\sin^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

$$\Rightarrow\ 2\times\frac{\pi}{8}+\frac{5\pi}{12}=\frac{2\pi}{3}$$

$$\Rightarrow \beta = \cot^{-1}\left(\sqrt{3}-2\right)+\frac{1}{8}\left(\pi-\sec^{-1}(2)\right)$$

$$\Rightarrow \beta = \cot^{-1}\left(-\left(2-\sqrt{3}\right)\right)+\frac{1}{8}\left(\pi-\frac{\pi}{3}\right)$$

$$\Rightarrow \beta = \pi-\frac{5\pi}{12}+\frac{\pi}{12}=\frac{7\pi}{12}+\frac{\pi}{2}=\frac{2\pi}{3}$$

$$\Rightarrow \gamma = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad \left\{\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)=\tan^{-1}\left(\sqrt{2}\right)\right\}$$

$$\Rightarrow \gamma = \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}+\sqrt{2}}{1-\frac{1}{\sqrt{2}}\times\sqrt{2}}\right)=\frac{\pi}{2}$$

Option (A) & (C) are correct.

21. Ans. (A,C)

$$\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{4-x}}{2}\right)+\tan^{-1}y=\frac{2\pi}{3}$$

Domain $x \geq 0$ & $4 - x \geq 0$

$x \geq 0$ & $x \leq 4$

$x \in [0,4], y \in R$

$$\Rightarrow\ \sin^{-1}\left(\frac{\sqrt{x}}{2}\right)=\tan^{-1}\left(\frac{\sqrt{x}}{\sqrt{4-x}}\right)$$

$$\Rightarrow\ \sin^{-1}\left(\frac{\sqrt{4-x}}{2}\right)=\tan^{-1}\left(\frac{\sqrt{4-x}}{\sqrt{x}}\right)$$

Now,

$$\tan^{-1}\left(\frac{\sqrt{x}}{\sqrt{4-x}}\right)+\tan^{-1}\left(\frac{\sqrt{4-x}}{\sqrt{x}}\right)+\tan^{-1}y=\frac{2\pi}{3}$$

$$\Rightarrow\ \tan^{-1}\left[\frac{\frac{\sqrt{x}}{4-x}+\frac{\sqrt{4-x}}{\sqrt{x}}}{1-\frac{\sqrt{x}}{\sqrt{4-x}}\times\frac{\sqrt{4-x}}{\sqrt{x}}}\right]+\tan^{-1}y=\frac{2\pi}{3}$$

$$\Rightarrow \frac{\pi}{2}+\tan^{-1} y=\frac{2\pi}{3}$$

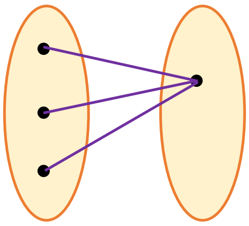
$$\Rightarrow y=\frac{1}{\sqrt{3}}$$

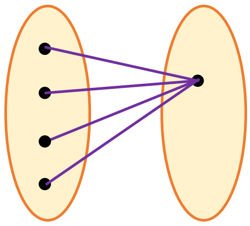
$$\left(x^2+y^2\right)_{\min}=0+\frac{1}{3}=\frac{1}{3}$$

$$\left(x^2+y^2\right)_{\max}=16+\frac{1}{3}=\frac{49}{3}$$

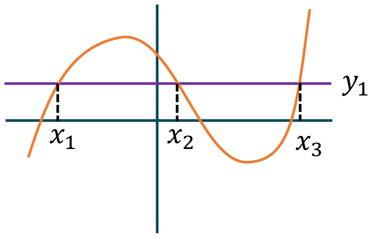
22. **Ans. (C)**

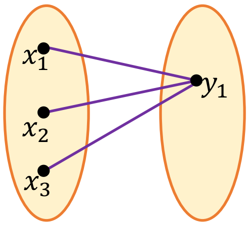
Three one function mean of type



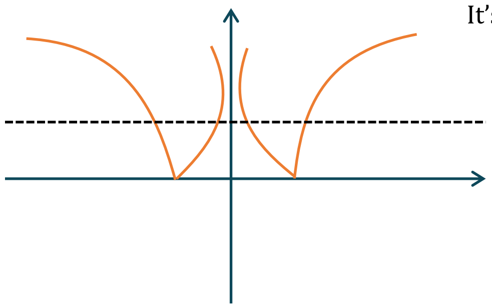


Graphically,



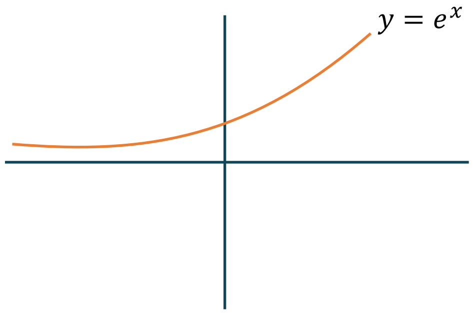


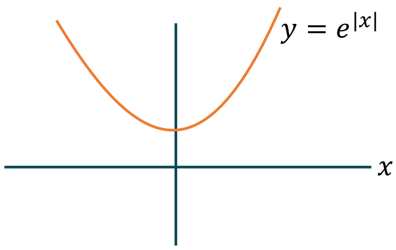
(A) $y=|\ell n|x||$



It's four one

(B) $y=e^{|x|}$

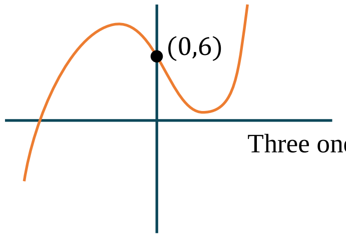




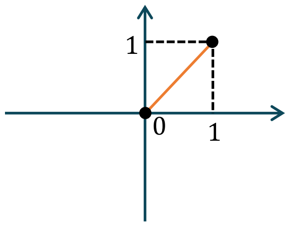
It is two one

(C) $y = x^3 + 3x^2 - 7x + 6$

$\frac{dy}{dx} = 3x^2 + 6x - 7$ has $D > 0$ so two zeroes of $\frac{dy}{dx}$

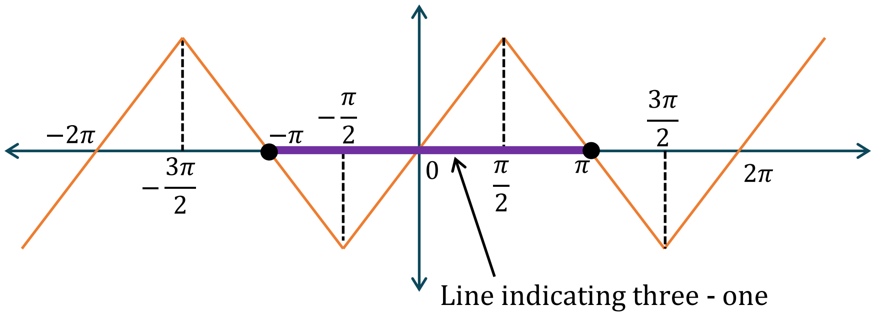


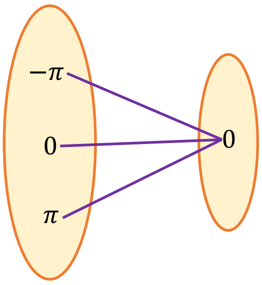
(D) $y = \cos(\cos^{-1} x)$



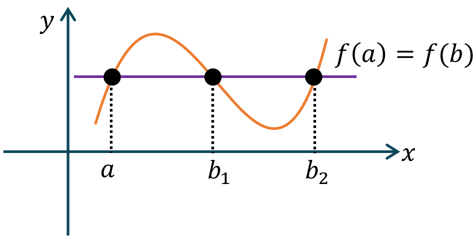
23. **Ans. (A)**

$y = \sin^{-1}(\sin x)$





24. **Ans. (B)**



b_1 & b_2 are two other position of b.

25. Ans. (A,C,D)

$$\alpha = 2\tan^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5} \quad \left\{\because \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}\right\}$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{3}{4}\right)$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{\frac{1}{2}+\frac{3}{4}}{1-\frac{3}{8}}\right)$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(2) = \cot^{-1}(2) + \tan^{-1}(2)$$

$$\Rightarrow \alpha = \frac{\pi}{2}$$

$$\Rightarrow \beta = \sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \cot^{-1}\left(\frac{16}{63}\right)$$

$$\Rightarrow \beta = \tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{63}{16}\right)$$

$$\Rightarrow \beta = \pi + \tan^{-1}\left[\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}+\frac{3}{4}}\right] + \tan^{-1}\left(\frac{63}{16}\right)$$

$$\Rightarrow \beta = \pi + \tan^{-1}\left(\frac{-63}{16}\right) + \tan^{-1}\left(\frac{63}{16}\right)$$

$$\Rightarrow \beta = \pi - \tan^{-1}\left(\frac{63}{16}\right) + \tan^{-1}\left(\frac{63}{16}\right)$$

$$\Rightarrow \beta = \pi$$

$x^2 - ax + b = 0$ — roots $2sin\alpha$, $cos\beta$

$2\sin\alpha\cos\beta = b$

$2\sin\alpha + \cos\beta = a$

$b = -2$

$a = 1$

26. Ans. (A,B,D)

$$\cot^{-1}\left(x^2-2x\right)$$

$$\Rightarrow \cot^{-1}\left(x^2-2x+1-1\right) = \cot^{-1}\left((x-1)^2-1\right)$$

$$\Rightarrow (x-1)^2 - 1 \in [-1,\infty)$$

$$\Rightarrow \cot^{-1}\left((x-1)^2-1\right) \in \left(0,\frac{3\pi}{4}\right]$$

$$\Rightarrow c = 0,\ d = \frac{3\pi}{4}$$

27. Ans. (C)

(R) $\tan^{-1}\left(\sin^2\theta-2\sin\theta+3\right)+\cot^{-1}\left(5\sec^2 y+1\right)=\frac{\pi}{2}$

$\Rightarrow \tan^{-1}\left(\left(\sin\theta-1\right)^2+2\right)+\cot^{-1}\left(5\sec^2 y+1\right)=\frac{\pi}{2}$

Recall that $\tan^{-1}\alpha+\cot^{-1}\alpha=\frac{\pi}{2}$

So, $\alpha=\left(\sin\theta-1\right)^2+2=5\sec^2 y+1$

$\Rightarrow \underbrace{\left(\sin\theta-1\right)^2+1}_{[1,5]} \quad = \underbrace{5\sec^2 y}_{[5,\infty]}$

Think,

$\Rightarrow \sec^2 y\geq 1\Rightarrow 5\sec^2 y\geq 5$

& $-1\leq\sin\theta\leq 1\Rightarrow -2\leq\sin\theta-1\leq 0$

$\Rightarrow 0\leq\left(\sin\theta-1\right)^2\leq 4$

$\Rightarrow 1\leq\left(\sin\theta-1\right)^2+1\leq 5$

So, LHS = RHS = 5 (only)

$\therefore \sec^2 y=1 \,\&\, \sin\theta=-1$

$\Rightarrow \cos^2 y-\sin\theta=1-(-1)=2$

Match (R) $\rightarrow$ (2)

28. Ans. (A)

(P) $\sin^{-1}x^2+\tan^{-1}x^2=2k+1; k\in\mathbb{Z}$

As $0\leq x^2\leq 1$

So, $\sin^{-1}x^2\in\left[0,\frac{\pi}{2}\right]$

And $\tan^{-1}x^2\in\left[0,\frac{\pi}{4}\right]$

Both $\sin^{-1}x^2$ & $\tan^{-1}x^2$ are increasing in $0\leq x^2\leq 1$

So, $\sin^{-1}x^2+\tan^{-1}x^2\Big|_{\min}=0+0=0$

& $\sin^{-1}x^2+\tan^{-1}x^2\Big|_{\max}=\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}$

So, $0\leq 2k+1\leq\frac{3\pi}{4}\simeq 2.3$

So, integral value of k satisfying is/are $k\in\{0\}$

Number of values of k = 1

Match of (P) $\rightarrow$ (1)

(Q) $\sum_{k=1}^{\infty} \cot^{-1}\left(\frac{k^2}{8}\right)$

k^{th} term, $T_k = \cot^{-1}\left(\frac{k^2}{8}\right) = \tan^{-1}\frac{8}{k^2}$

$= \tan^{-1}\frac{2}{\left(\frac{k}{2}\right)^2}$

$= \tan^{-1}\frac{2}{1+\left(\frac{k^2}{2^2}-1\right)}$

$= \tan^{-1}\frac{2}{1+\left(\frac{k}{2}-1\right)\left(\frac{k}{2}+1\right)}$

$\Rightarrow \left\{\text{Think of } \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y; x, y > 0\right\}$

So, $T_k = \tan^{-1}\left(\frac{\left(\frac{k}{2}+1\right)-\left(\frac{k}{2}-1\right)}{1+\left(\frac{k}{2}-1\right)\left(\frac{k}{2}+1\right)}\right)$

$\Rightarrow T_k = \tan^{-1}\left(\frac{k}{2}+1\right) - \tan^{-1}\left(\frac{k}{2}-1\right)$

So, $T_k = \tan^{-1}\left(\frac{k+2}{2}\right) - \tan^{-1}\left(\frac{k-2}{2}\right)$

$\Rightarrow T_1 = \tan^{-1}\left(\frac{3}{2}\right) - \tan^{-1}\left(\frac{-1}{2}\right)$

$\Rightarrow T_2 = \tan^{-1}\left(\frac{4}{2}\right) - \tan^{-1}\left(\frac{0}{2}\right)$

$\Rightarrow T_3 = \tan^{-1}\left(\frac{5}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow T_4 = \tan^{-1}\left(\frac{6}{2}\right) - \tan^{-1}\left(\frac{2}{2}\right)$

2^{nd} term of T_1 to T_4 are left

$\Rightarrow T_5 = \tan^{-1}\left(\frac{7}{2}\right) - \tan^{-1}\left(\frac{3}{2}\right)$

$\Rightarrow T_6 = \tan^{-1}\left(\frac{8}{2}\right) - \tan^{-1}\left(\frac{4}{2}\right)$

$\vdots \qquad \vdots \qquad \vdots$

$$\Rightarrow T_{n-5}=\tan^{-1}\left(\frac{n-3}{2}\right)-\tan^{-1}\left(\frac{n-7}{2}\right)$$

$$\Rightarrow T_{n-4}=\tan^{-1}\left(\frac{n-2}{2}\right)-\tan^{-1}\left(\frac{n-6}{2}\right)$$

$$\Rightarrow T_{n-2}=\tan^{-1}\left(\frac{n}{2}\right)-\tan^{-1}\left(\frac{n-4}{2}\right)$$

$$\Rightarrow T_{n-1}=\tan^{-1}\left(\frac{n+1}{2}\right)-\tan^{-1}\left(\frac{n-3}{2}\right)$$

1st term of T_{n-3} to T_n are left

$$\text{So, } S_n=\tan^{-1}\left(\frac{n+2}{2}\right)+\tan^{-1}\left(\frac{n+1}{2}\right)+\tan^{-1}\left(\frac{n}{2}\right)+\tan^{-1}\left(\frac{n-1}{2}\right)+\tan^{-1}\frac{1}{2}-0-\tan^{-1}\frac{1}{2}-\tan^{-1}1$$

$$\text{So, } S_\infty=\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}+\tan^{-1}\frac{1}{2}-0-\tan^{-1}\frac{1}{2}-\tan^{-1}1$$

$$=2\pi-\frac{\pi}{4}=\frac{7\pi}{4}=\frac{p\pi}{q}$$

Hence, $|p-q|=|7-4|=3$

(Q) $\rightarrow$ (3)

(S) $f(x)=\sqrt{\sec^{-1}\left(\frac{1-|x|}{2}\right)}$

$$\Rightarrow \frac{1-|x|}{2}\geq 1 \text{ or } \frac{1-|x|}{2}\leq -1$$

$\Rightarrow 1-|x|\geq 2$ or $1-|x|\leq -2$

$\Rightarrow -1\geq |x|$ or $3\leq |x|$

$\Rightarrow |x|\geq 3$

$\Rightarrow x\in(-\infty,-3]\cup[3,\infty)$

$x_{\text{least positive}}=3$

Match (S) $\rightarrow$ (3)

29. **Ans. (A $\rightarrow$ p ; B $\rightarrow$ p ; C $\rightarrow$ p ; D $\rightarrow$ s)**

(A) $a,b,c\in\mathbb{R}^+$ $\quad x+y+z=xyz$

$$\theta=\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

Think, Here if

$$\Rightarrow x=\sqrt{\frac{a(a+b+c)}{bc}}=a\sqrt{\frac{a+b+c}{abc}}$$

$$\Rightarrow y=\sqrt{\frac{b(a+b+c)}{ca}}=b\sqrt{\frac{a+b+c}{abc}}$$

$$\Rightarrow z=\sqrt{\frac{c(a+b+c)}{ab}}=c\sqrt{\frac{a+b+c}{abc}}$$

& observe, $x+y+z=\dfrac{(a+b+c)^{\frac{3}{2}}}{(abc)^{\frac{1}{2}}}$

& $xyz=(abc)\dfrac{(a+b+c)^{\frac{3}{2}}}{(abc)^{\frac{3}{2}}}$

$=\dfrac{(a+b+c)^{\frac{3}{2}}}{(abc)^{\frac{1}{2}}}$

So, $x+y+z=xyz$

$\Rightarrow \tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$

(A) $\rightarrow$ (P)

(B) $E_1=\tan^{-1}\left(\dfrac{1}{2}\tan 2A\right)=\tan^{-1}\left(\dfrac{1}{2}.\dfrac{2\tan A}{1-\tan^2 A}\right)$

$=\tan^{-1}\left(\dfrac{\tan A}{1-\tan^2 A}\right)$ $\qquad 0<A<\dfrac{\pi}{4}$

$E_2=\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=\pi+\tan^{-1}\left(\dfrac{\cot A+\cot^3 A}{1-\cot A.\cot^3 A}\right)$

$=\pi+\tan^{-1}\left(\dfrac{\cot A(1+\cot^2 A)}{1-\cot^4 A}\right)$

$=\pi+\tan^{-1}\left(\dfrac{\cot A}{1-\cot^2 A}\right)$

$=\pi+\tan^{-1}\left(\dfrac{\tan A}{\tan^2 A-1}\right)$

$=\pi-\tan^{-1}\left(\dfrac{\tan A}{1-\tan^2 A}\right)$

Think, If $x>0; y>0$ & $xy>1$

Then $\tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

Here, $x=\cot A$ & $y=\cot^3 A$ & $0<A<\dfrac{\pi}{4}$

So, $\cot A>1$ & $\cot^3 A>1$

i.e. $xy>1$

So, required expression = $E_1+E_2=\pi$

(B) $\rightarrow$ (P)

(C) If $x<0$, then

$E=\dfrac{1}{2}\left\{\cos^{-1}(2x^2-1)+2\cos^{-1}x\right\}$

Let $\cos^{-1}x=\theta \Rightarrow \cos\theta=x$

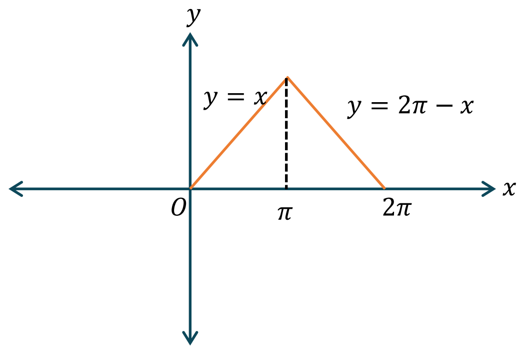
As $-1 \le x < 0$ so, $\frac{\pi}{2} < \theta \le \pi$

Now, $\cos^{-1}(2x^2 - 1) = \cos^{-1}(2\cos^2\theta - 1)$

$= \cos^{-1}(\cos 2\theta) = 2\pi - 2\theta$

$= 2\pi - 2\cos^{-1} x$

$\left\{as\ \frac{\pi}{2} < \theta \le \pi \Rightarrow \pi < 2\theta \le 2\pi\right\}$

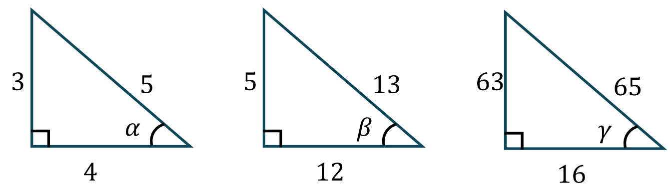


So, $E = \frac{1}{2}\{2\pi - 2\cos^{-1} x + 2\cos^{-1} x\} = \pi$

(C) $\rightarrow$ (P)

(D) $\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{16}{65}\right)$

$= \tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{63}{16}\right)$



$= \tan^{-1}\left(\frac{\frac{3}{4} - \frac{5}{12}}{1 + \frac{3}{4} \times \frac{5}{12}}\right) + \tan^{-1}\left(\frac{63}{16}\right)$ $\left[\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) x, y > 0\right]$

$= \tan^{-1}\left(\frac{36-20}{48+15}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$= \tan^{-1}\left(\frac{16}{63}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$= \tan^{-1}\left(\frac{16}{63}\right) + \cot^{-1}\left(\frac{16}{63}\right)$

$= \frac{\pi}{2}$

(D) $\rightarrow$ (S)

30. Ans. (P → 2; Q → 1; R → 4; S → 3)

(P) $\sin^{-1} x + \sin^{-1}(2x) = \frac{\pi}{3}$

To satisfy above equation, clearly 'x' must be positive.

$\Rightarrow \sin^{-1} x + \sin^{-1}(2x) = \frac{\pi}{3}$

$\Rightarrow \sin^{-1}(2x) = \frac{\pi}{3} - \sin^{-1} x$

Take sin both sides

$\sin\left(\sin^{-1}(2x)\right) = \sin\left(\frac{\pi}{3} - \sin^{-1} x\right)$

$2x = \frac{\sqrt{3}}{2}\left(\sqrt{1-x^2}\right) - \frac{1}{2} \cdot x$

$\frac{5x}{2} = \frac{\sqrt{3}}{2}\sqrt{1-x^2}$

$\Rightarrow$ squaring both sides

$25x^2 = 3\left(1-x^2\right)$

$\Rightarrow 28x^2 = 3$

$\Rightarrow x^2 = \frac{3}{28}$

$\Rightarrow x = \frac{\sqrt{3}}{2\sqrt{7}}$

(Q) $\tan^{-1}\left(\frac{1}{1+2x}\right) + \tan^{-1}\left(\frac{1}{1+4x}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$

Let $x > 0$

$$\tan^{-1}\left(\frac{\frac{1}{1+2x} + \frac{1}{1+4x}}{1 - \left(\frac{1}{1+2x}\right)\left(\frac{1}{1+4x}\right)}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$\Rightarrow \frac{2+6x}{8x^2+6x} = \frac{2}{x^2}$

$\Rightarrow \frac{3x+1}{4x+3} = \frac{2}{x}$

$\Rightarrow 3x^2 + x = 8x + 6$

$\Rightarrow 3x^2 - 7x - 6 = 0$

$\Rightarrow 3x^2 - 9x + 2x - 6 = 0$

$\Rightarrow (3x+2)(x-3) = 0$

$\Rightarrow x = 3$

(R) $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}(3x)$

$\tan^{-1}(x-1)+\tan^{-1}(x+1)=\tan^{-1}(3x)-\tan^{-1}(x)$

$\tan^{-1}\left(\frac{(x-1)+(x+1)}{1-(x^2-1)}\right)=\tan^{-1}\left(\frac{3x-x}{1+3x^2}\right)$

$\tan^{-1}\left(\frac{2x}{2-x^2}\right)=\tan^{-1}\left(\frac{2x}{3x^2+1}\right)$

$\frac{2x}{2-x^2}=\frac{2x}{3x^2+1}\Rightarrow x=0$

$2-x^2=3x^2+1$

$4x^2=1$

$x^2=\frac{1}{4}\Rightarrow x=\pm\frac{1}{2}$

(S) $3\cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}.(4x^2-1)\right)$

x must be ≥ 0

If $x<0$, then LHS lies in $\left(\frac{3\pi}{2},3\pi\right]$

And RHS lies in $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

Let $x=\cos\theta$

$\theta=\cos^{-1}x \qquad \theta\in\left[0,\frac{\pi}{2}\right]$

LHS $3\cos^{-1}(\cos\theta)=3\theta$

RHS $\sin^{-1}\left(\sin\theta.\left(4\cos^2\theta-1\right)\right)$

$3\theta=\sin^{-1}\left(\sin\theta.\left(3-4\sin^2\theta\right)\right)$

$3\theta=\sin^{-1}\left(3\sin\theta-4\sin^3\theta\right)$

$3\theta=\sin^{-1}\left(\sin(3\theta)\right)$ ...(1)

(1) Is true only if $0\leq 3\theta\leq\frac{\pi}{2}$

$0\leq\theta\leq\frac{\pi}{6}$

$0\leq\cos^{-1}x\leq\frac{\pi}{6}\Rightarrow x\in\left[\frac{\sqrt{3}}{2},1\right]$

EXERCISE - S

1. **Ans. (2)**

$$f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$f(1)=\sin^{-1}\left(\frac{2\times1}{1+1^2}\right)+\cos^{-1}\left(\frac{1-1^2}{1+1^2}\right)$$

$\Rightarrow \sin^{-1}1+\cos^{-1}0$

$$\Rightarrow \frac{\pi}{2}+\frac{\pi}{2}=\pi$$

$$f(2)=\sin^{-1}\left(\frac{2\times2}{1+2^2}\right)+\cos^{-1}\left(\frac{1-2^2}{1+2^2}\right)$$

$$\Rightarrow \sin^{-1}\frac{4}{5}+\cos^{-1}\left(\frac{-3}{5}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{4}{3}\right)+\pi-\cos^{-1}\left(\frac{3}{5}\right) \qquad \{\because \cos^{-1}(-x)=\pi-\cos^{-1}x\}$$

$$\Rightarrow \tan^{-1}\left(\frac{4}{3}\right)+\pi-\tan^{-1}\left(\frac{4}{3}\right) \qquad \left\{\because \sin^{-1}\left(\frac{x}{y}\right)=\tan^{-1}\frac{x}{\sqrt{y^2-x^2}}\right\}$$

$$=\pi \qquad \left\{\because \cos^{-1}\left(\frac{x}{y}\right)=\tan^{-1}\frac{\sqrt{y^2-x^2}}{x}\right\}$$

$f(1)+f(2)=2\pi=k\pi$

$\Rightarrow k=2$

2. **Ans. (20)**

$$\cos^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}-\cos^{-1}\left(\frac{16}{x}\right)$$

taking cos both sides

$$\Rightarrow \cos\ \cos^{-1}\left(\frac{12}{x}\right)=\cos\left(\frac{\pi}{2}-\cos^{-1}\frac{16}{x}\right)$$

$$\Rightarrow \frac{12}{x}=\sin\left(\cos^{-1}\frac{16}{x}\right) \qquad \{\because \cos\ \cos^{-1}x=x\}$$

$$\Rightarrow \frac{12}{x}=\sqrt{1-\left(\frac{16}{x}\right)^2}$$

on squaring

$$\Rightarrow \left(\frac{12}{x}\right)^2=1-\left(\frac{16}{x}\right)^2$$

$$\Rightarrow \frac{144}{x^2}=1-\frac{256}{x^2}$$

$$\Rightarrow \frac{144+256}{x^2}=1$$

$\Rightarrow x^2=400$

$\Rightarrow x=\pm20$

but $x>0$ so $x=20$

3. **Ans. (0)**

$$\sin^{-1}\left(1+b+b^2+\ldots\infty\right)+\cos^{-1}\left(a-\frac{a^2}{3}+\frac{a^3}{9}\ldots\infty\right)=\frac{\pi}{2}$$

for above expression to be defined

$-1 \leq 1 + b + b^2 + \ldots.\infty \leq 1$

which is only possible for $b = 0$

If $b = 0$

$$\Rightarrow a-\frac{a^2}{3}+\frac{a^3}{9}\ldots\infty=1 \qquad \left\{\sin^{-1}1+\cos^{-1}1=\frac{\pi}{2}\right\}$$

$$\Rightarrow \frac{a}{1+\frac{a}{3}}=1\Rightarrow 3a = 3 + a$$

$\Rightarrow 2a = 3$

$$\Rightarrow a=\frac{3}{2} \text{ (Not integer)}$$

So no integral ordered pairs (a, b) are possible.

4. **Ans. (0)**

$$\cos^{-1}\sqrt{x}-\sin^{-1}\sqrt{x-1}+\cos^{-1}\sqrt{1-x}-\sin^{-1}\frac{1}{\sqrt{x}}=\frac{\pi}{2}$$

First we find domain of the expression

$$\left.\begin{array}{l} x\geq 0 \\ x-1\geq 0 \\ 1-x\geq 0 \end{array}\right\}\Rightarrow \text{Domain } x = 1$$

Now check for $x = 1$

$$\cos^{-1}\sqrt{1}-\sin^{-1}\sqrt{1-1}+\cos^{-1}\sqrt{1-1}-\sin^{-1}\frac{1}{\sqrt{1}}=\frac{\pi}{2}$$

$$\Rightarrow 0-0+\frac{\pi}{2}-\frac{\pi}{2}=\frac{\pi}{2} \text{ Not satisfying}$$

So, no solution.

5. **Ans. (1)**

$2\sin^{-1}x + \cos^{-1}(1 - x) = 0$

$\sin^{-1}x$ should be less than or equal to 0

$-1\leq x\leq 0,$

and $-1\leq 1-x\leq 1$

$\Rightarrow -1\leq x-1\leq 1$

$\Rightarrow \ 0\leq x\leq 2$

Domain $x = 0$

Put $x = 0$

$\sin^{-1}0 + \cos^{-1}(1 - 0) = -\sin^{-1}(0)$

$\Rightarrow x = 0$

So, $x = 0$ is the only solution.

6. Ans. (1)

$$\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}.......\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}.......\right)=\frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{1+\frac{x}{2}}\right)+\cos^{-1}\left(\frac{x^2}{1+\frac{x^2}{2}}\right)=\frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\frac{2x}{2+x}\right)+\cos^{-1}\left(\frac{2x^2}{2+x^2}\right)=\frac{\pi}{2}$$

Now for above expression to be designed

$\Rightarrow \frac{2x}{2+x}=\frac{2x^2}{2+x^2}$

$\Rightarrow x(2+x^2)=x^2(2+x)$

$\Rightarrow 2x+x^3=2x^2+x^3$

$\Rightarrow 2x^2-2x=0$

$\Rightarrow x(x-1)=0$

$\Rightarrow x=0,1$

for $0<|x|<\sqrt{2}$

$x\neq 0$

So $x=1$ is the only solution.

7. Ans. (53)

$y=\sin^{-1}(\sin 8)-\tan^{-1}(\tan 10)+\cos^{-1}(\cos 12)-\sec^{-1}(\sec 9)+\cot^{-1}(\cot 6)-\mathrm{cosec}^{-1}(\mathrm{cosec} 7)$

$\sin^{-1}(\sin 8)=3\pi-8$

$\tan^{-1}(\tan 10)=10-3\pi$

$\cos^{-1}(\cos 12)=4\pi-12$

$\sec^{-1}(\sec 9)=9-2\pi$

$\cot^{-1}(\cot 6)=6-\pi$

$\mathrm{cosec}^{-1}(\mathrm{cosec}\ 7)=7-2\pi$

$y=(3\pi-8)-(10-3\pi)+(4\pi-12)-(9-2\pi)+6-\pi-(7-2\pi)$

$\Rightarrow 3\pi+3\pi+4\pi+2\pi-\pi+2\pi-8-10-12-9+6-7$

$\Rightarrow 13\pi-40=a\pi+b$

$a=13, b=-40$

$a-b=13-(-40)=53$

8. Ans. (1)

$$\sin^{-1}\sin\left(\frac{10\pi}{3}\right)+\cos^{-1}\cos\left(\frac{22\pi}{3}\right)+\tan^{-1}\tan 10=a\pi+b$$

$$\Rightarrow \sin^{-1}\sin\left(3\pi+\frac{\pi}{3}\right)+\cos^{-1}\cos\left(7\pi+\frac{\pi}{3}\right)+\tan^{-1}\tan 10$$

$$\Rightarrow \sin^{-1}\left(-\sin\frac{\pi}{3}\right)+\cos^{-1}\left(-\cos\frac{\pi}{3}\right)+10-3\pi$$

$$\Rightarrow \sin^{-1}\sin\left(-\frac{\pi}{3}\right)+\cos^{-1}\cos\left(\frac{2\pi}{3}\right)+10-3\pi$$

$$\Rightarrow -\frac{\pi}{3}+\frac{2\pi}{3}+10-3\pi$$

$$\Rightarrow \frac{\pi}{3}+10-3\pi$$

$$\Rightarrow \frac{-8\pi}{3}+10$$

$$\Rightarrow a=\frac{-8}{3}, b = 10$$

$$\Rightarrow \left|\frac{3ab}{80}\right| \Rightarrow \left|\frac{3\times\left(\frac{-8}{3}\right)\times 10}{80}\right|=1$$

9. **Ans. (1)**

$$f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$$

Domain

(i) $-1 \le 2x \le 1$

$$\Rightarrow -\frac{1}{2}\le x\le\frac{1}{2} \qquad \text{...(i)}$$

(ii) $\sin^{-1}(2x)+\frac{\pi}{6}\ge 0$ & $-\frac{\pi}{2}\le \sin^{-1}(2x)\le\frac{\pi}{2}$

$$\Rightarrow \sin^{-1}(2x)\ge-\frac{\pi}{6} \ \& \ -1\le 2x\le 1$$

$$\Rightarrow 2x\ge-\frac{1}{2} \ \& \ -\frac{1}{2}\le x\le\frac{1}{2}$$

$$\Rightarrow x\ge-\frac{1}{4} \ \& \ -\frac{1}{2}\le x\le\frac{1}{2}$$

$$\Rightarrow -\frac{1}{4}\le x\le\frac{1}{2} \qquad \text{...(ii)}$$

(i) ∩ (ii)

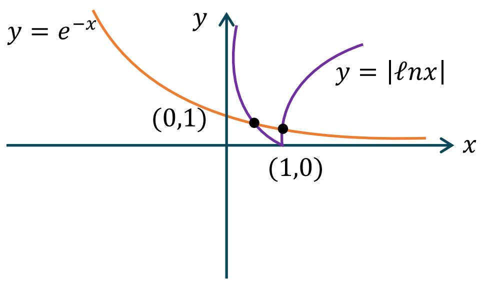
$$\Rightarrow x\in\left[-\frac{1}{4},\frac{1}{2}\right]$$

$$\Rightarrow \alpha=-\frac{1}{4}, \beta=\frac{1}{2}$$

$$\Rightarrow 4(\alpha+\beta)=1$$

10. Ans. (2)

$$\tan^{-1}\left(e^{-x}\right)+\cot^{-1}\left(\left|\ell nx\right|\right)=\frac{\pi}{2}$$



Total no. of point of intersections of e^{-x} & $|\ell nx|$ are 2

No. of solution = 2

11. Ans. (4)

$$\tan^{-1}x+\cot^{-1}\left(\frac{1}{y}\right)+2\tan^{-1}z=\pi$$

$$\Rightarrow \tan\left(\tan^{-1}(x)+\tan^{-1}y\right)=\tan\left(\pi-2\tan^{-1}z\right)$$

$$\Rightarrow \frac{x+y}{1-xy}=-\frac{2z}{1-z^2}$$

$\Rightarrow x - xz^2 + y - yz^2 = -2z + 2xyz$

$\Rightarrow xz^2 + yz^2 + 2xyz = x + y + 2z$

Comparing with $xz^2 + yz^2 + \lambda xyz = x + y + \mu z$

$\lambda = 2, \mu = 2$

$\boxed{\lambda + \mu = 4}$

12. Ans. (23)

$$\cot\left(\operatorname{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right)$$

$$\Rightarrow \cot\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right) \qquad \left(\because \operatorname{cosec}^{-1}\frac{5}{3}=\tan^{-1}\frac{3}{4}\right)$$

$$\Rightarrow \cot\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right) \qquad \left(\because \tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)\right)$$

$$\Rightarrow \cot\left[\tan^{-1}\left(\frac{9+8}{12}\times\frac{12}{6}\right)\right]=\cot\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$$

$$\Rightarrow \cot\left(\cot^{-1}\left(\frac{6}{17}\right)\right)=\frac{6}{17}$$

EXERCISE - JEE (Main) PYQ

1. Ans. (1)

$$\cos^{-1}\left(\frac{2}{3x}\right)+\cos^{-1}\left(\frac{3}{4x}\right)=\frac{\pi}{2},\ x>\frac{3}{4}$$

$$\Rightarrow \cos^{-1}\left(\frac{3}{4x}\right)=\frac{\pi}{2}-\cos^{-1}\left(\frac{2}{3x}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{3}{4x}\right)=\sin^{-1}\left(\frac{2}{3x}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{3}{4x}\right)=\cos^{-1}\left(\sqrt{1-\frac{4}{9x^2}}\right) \qquad (\because \sin^{-1}x=\cos^{-1}\sqrt{1-x^2}\,)$$

$$\Rightarrow \frac{3}{4x}=\sqrt{1-\frac{4}{9x^2}}$$

Square both side

$$\Rightarrow \frac{9}{16x^2}=1-\frac{4}{9x^2}$$

$$\Rightarrow \frac{9}{16x^2}+\frac{4}{9x^2}=1$$

$$\Rightarrow \frac{9}{16}+\frac{4}{9}=x^2$$

$$\Rightarrow x^2=\frac{81+64}{16\times 9}$$

$$\Rightarrow x=\frac{\sqrt{145}}{12} \qquad x=\frac{\sqrt{145}}{12}$$

2. Ans. (1)

$x=\sin^{-1}(\sin 10)$

$=\sin^{-1}[\sin(3\pi+\theta)]$

$=\sin^{-1}(-\sin\theta)$

$x=-\theta$

$y=\cos^{-1}(\cos 10)$

$=\cos^{-1}[(\cos(3\pi+\theta)]$

$=\cos^{-1}(-\cos\theta)$

$y=\pi-\theta$

$\therefore\ y-x=\pi-\theta-(-\theta)$

$=\pi$

3. Ans. (3)

$$\cot\sum_{n=1}^{19}\left[\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right]$$

$$=\cot\sum_{n=1}^{19}\left[\cot^{-1}\left(1+n\left(n+1\right)\right)\right]$$

$$=\cot\sum_{n=1}^{19}\tan^{-1}\left(\frac{1}{n^2+n+1}\right)$$

$$=\cot\sum_{n=1}^{19}\tan^{-1}\left[\frac{1}{1+n(n+1)}\right]$$

$$=\cot\sum_{n=1}^{19}\tan^{-1}\left[\frac{(n+1)-n}{1+n(n+1)}\right]$$

$$=\cot\sum_{n=1}^{19}\left[\tan^{-1}(n+1)-\tan^{-1}n\right]$$

$$=\cot\left[(\tan^{-1}2-\tan^{-1}1)+(\tan^{-1}3-\tan^{-1}2)+.....+\left(\tan^{-1}20-\tan 19\right)\right]$$

$$=\cot\left[\tan^{-1}20-\tan^{-1}1\right]$$

$$=\cot\left[\tan^{-1}\left(\frac{20-1}{1+20}\right)\right]$$

$$=\cot\tan^{-1}\left(\frac{19}{21}\right)$$

$$=\cot\cot^{-1}\left(\frac{21}{19}\right)$$

$$=\frac{21}{19}$$

4. **Ans. (4)**

$$\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x+3x}{1-2x\times 3x}\right]=\frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{1-6x^2}=\tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{5x}{1-6x^2}=1$$

$\Rightarrow 5x = 1 - 6x^2$

$\Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow (6x - 1)(x + 1) = 0$

$$\Rightarrow x=\frac{1}{6}, x = -1$$

$$\Rightarrow x=\frac{1}{6} \qquad \because x \geq 0$$

5. **Ans. (3)**

$$\sin^{-1}\left(\frac{12}{13}\right)-\sin^{-1}\left(\frac{3}{5}\right)$$

Let $\sin^{-1}\left(\frac{12}{13}\right)=A\Rightarrow \sin A=\frac{12}{13}, \cos A=\frac{5}{13}$

$\sin^{-1}\left(\frac{3}{5}\right)=B\Rightarrow \sin B=\frac{3}{5}, \cos B=\frac{4}{5}$

Now $\sin(A - B) = \sin A\cos B - \cos A\sin B$

$$=\frac{12}{13}\times\frac{4}{5}-\frac{5}{13}\times\frac{3}{5}$$
$$=\frac{48-15}{65}$$
$$\Rightarrow \sin(A-B)=\frac{33}{65}\Rightarrow\cos(A-B)=\frac{56}{65}$$
$$\Rightarrow A-B=\cos^{-1}\left(\frac{56}{65}\right)$$
$$\Rightarrow A-B=\cos^{-1}\left(\frac{56}{65}\right)$$

6. **Ans. (4)**

$$S=\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+......$$
$$=\tan^{-1}\left(\frac{2-1}{1+1.2}\right)+\tan^{-1}\left(\frac{3-2}{1+2.3}\right)+....+\tan^{-1}\left(\frac{11-10}{1+10.11}\right)$$
$$=\left(\tan^{-1}2-\tan^{-1}1\right)+\left(\tan^{-1}3-\tan 2\right)+....+\left(\tan^{-1}11-\tan 10\right)$$
$$=\tan^{-1}11-\tan^{-1}1$$
$$S=\tan^{-1}\left(\frac{11-1}{1+11}\right)$$
$$\tan S=\frac{10}{12}\Rightarrow \tan S=\frac{5}{6}$$

7. **Ans. (1)**

$$f(x)=\sin\left(\frac{|x|+5}{x^2+1}\right)$$

For domain :

$$-1\le\frac{|x|+5}{x^2+1}\le 1$$

Since $|x|+5$ & x^2+1 is always positive

So $\frac{|x|+5}{x^2+1}\ge 0\ \forall x\in R$

So for domain :

$$\frac{|x|+5}{x^2+1}\le 1$$
$$\Rightarrow |x|+5\le x^2+1$$
$$\Rightarrow 0\le x^2-|x|-4$$
$$\Rightarrow 0\le\left(|x|-\frac{1+\sqrt{17}}{2}\right)\left(|x|-\frac{1-\sqrt{17}}{2}\right)$$
$$\Rightarrow |x|\ge\frac{1+\sqrt{17}}{2}\ or\ |x|\le\frac{1-\sqrt{17}}{2}\quad \text{(Rejected)}$$
$$\Rightarrow x\in\left(-\infty,-\frac{1+\sqrt{17}}{2}\right]\cup\left[\frac{1+\sqrt{17}}{2},\infty\right)$$

So, $a=\frac{1+\sqrt{17}}{2}$

8. Ans. (3)

$$2\pi-\left(\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)\right)$$

$$=2\pi-\left(\tan^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{16}{63}\right)\right)$$

$$=2\pi-\left(\tan^{-1}\left(\frac{63}{16}\right)+\tan^{-1}\left(\frac{16}{63}\right)\right)$$

$$=2\pi-\frac{\pi}{2}=\frac{3\pi}{2}$$

9. Ans. (4)

$$\because\ 1\le \sin x+\cos x\le\sqrt{2},\ x\in\left[0,\frac{\pi}{2}\right]$$

$$\tan^{-1}(1)\le\tan^{-1}(\sin x+\cos x)\le\tan^{-1}\sqrt{2}$$

$$\therefore\ M=\tan^{-1}\sqrt{2}\ \&\ m=\tan^{-1}(1)$$

$$\Rightarrow M-m=\tan^{-1}\sqrt{2}-\tan^{-1}(1)$$

$$\Rightarrow\ M-m=\tan^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$\Rightarrow\ \tan(M-m)=\frac{\sqrt{2}-1}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}$$

$$=\left(\sqrt{2}-1\right)^2$$

$$=3-2\sqrt{2}$$

10. Ans. (2)

$$\sum_{r=1}^{50}\tan^{-1}\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{50}\tan^{-1}\left(\frac{2}{4r^2}\right)$$

$$=\sum_{r=1}^{50}\tan^{-1}\left\{\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}\right\}$$

$$=\sum_{r=1}^{50}\left[\tan^{-1}(2r+1)-\tan^{-1}(2r-1)\right]$$

$$=\left(\tan^{-1}3+\tan^{-1}5+\tan^{-1}7+\ldots\ldots+\tan^{-1}101\right)-\left(\tan^{-1}1+\tan^{-1}3+\ldots\ldots+\tan^{-1}99\right)$$

$$=\tan^{-1}101-\tan^{-1}1$$

$$=\tan^{-1}\left[\frac{101-1}{1+101}\right]$$

$$=\tan^{-1}\left[\frac{100}{102}\right]$$

$$=\tan^{-1}\left(\frac{50}{51}\right)$$

11. Ans. (1)

$\because \tan^{-1} a + \tan^{-1} b = \frac{\pi}{4}$, $0 < a, b < 1$

$\Rightarrow \tan^{-1}\left[\frac{a+b}{1-ab}\right] = \frac{\pi}{4}$

$\Rightarrow \frac{a+b}{1-ab} = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \frac{a+b}{1-ab} = 1$

$\Rightarrow a + b = 1 - ab$

$\Rightarrow (a+1)(b+1) = 2$ ...(1)

Now

$\Rightarrow (a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right)$.........

$= \left(a - \frac{a^2}{2} + \frac{a^3}{3} + \text{........}\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} + \text{..........}\right)$

$= \log(1+a) + \log(1+b))$

$= \log|(1+a)(1+b)|$

$= \log(2)$ using (1)

12. Ans. (3)

$f(x) = \sin^{-1}\left(\frac{3x^2+x-1}{(x-1)^2}\right) + \cos^{-1}\left(\frac{x-1}{x+1}\right)$

$-1 \le \frac{x-1}{x+1} \le 1 \Rightarrow 0 \le x < \infty$ ...(1)

$-1 \le \frac{3x^2+x-1}{(x-1)^2} \le 1 \Rightarrow x \in \left[\frac{-1}{4}, \frac{1}{2}\right] \cup \{0\}$ ...(2)

(1) & (2)

$\Rightarrow$ Domain = $\left[\frac{1}{4}, \frac{1}{2}\right] \cup \{0\}$

13. Ans. (2)

$\operatorname{cosec}\left[2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{3}{4}\right)\right]$

$\operatorname{cosec}\left[\tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{3}{4}\right)\right]$

$= \operatorname{cosec}\left[\tan^{-1}\left(\frac{56}{33}\right)\right] = \frac{65}{56}$ option (2)

14. Ans. (4)

$$\because\ -1 \le \frac{2\sin^{-1}\left(\dfrac{1}{4x^2-1}\right)}{\pi} \le 1$$

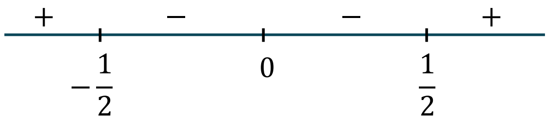
$$\Rightarrow -\frac{\pi}{2} \le \sin^{-1}\left(\frac{1}{4x^2-1}\right) \le \frac{\pi}{2}$$

$$\Rightarrow -1 \le \frac{1}{4x^2-1} \le 1 \qquad \ldots(1)$$

Case-I

$$\Rightarrow \frac{1}{4x^2-1} \ge -1$$

$$\Rightarrow \frac{1}{4x^2-1} + 1 \ge 0$$



$$\Rightarrow \frac{1+4x^2-1}{4x^2-1} \ge 0$$

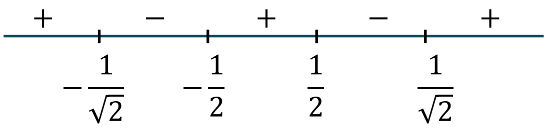
$$\Rightarrow \frac{4x^2}{4x^2-1} \ge 0$$

$$\Rightarrow x \in \left(-\infty, \frac{-1}{2}\right) \cup \left(\frac{1}{2}, \infty\right) \cup \{0\}$$

Case-II

$$\Rightarrow \frac{1}{4x^2-1} \le 1$$

$$\Rightarrow \frac{1}{4x^2-1} - 1 \le 0$$



$$\Rightarrow \frac{1-4x^2+1}{4x^2-1} \le 0$$

$$\Rightarrow \frac{2-4x^2}{4x^2-1} \le 0$$

$$\Rightarrow \frac{4x^2-2}{4x^2-1} \ge 0$$

$$\Rightarrow x \in \left(-\infty, \frac{-1}{\sqrt{2}}\right) \cup \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$$

15. Ans. (1)

$(\tan^{-1} x)^3 + (\cot^{-1} x)^3$

$$\left\{(a+b)^3 = a^3 + b^3 + 3ab(a+b)\right\}$$

$$= (\tan^{-1} x + \cot^{-1} x)^3 - 3\tan^{-1} x \cot^{-1} x\,(\tan^{-1} x + \cot^{-1} x)$$

$$= \left(\frac{\pi}{2}\right)^3 - 3\tan^{-1}\cdot\left(\frac{\pi}{2} - \tan^{-1} x\right)\cdot\frac{\pi}{2}$$

$$= \left(\frac{\pi}{2}\right)^3 - \frac{3\pi}{2}\tan^{-1} x\left(\frac{\pi}{2} - \tan^{-1} x\right)$$

$$=\frac{3\pi}{2}\left[\left(\tan^{-1}x\right)^2-\frac{\pi}{2}\tan^{-1}x\right]+\left(\frac{\pi}{2}\right)^3$$

$$=\frac{3\pi}{2}\left(\tan^{-1}x-\frac{\pi}{4}\right)^2-\frac{3\pi^3}{32}+\frac{\pi^3}{8}$$

$$=\frac{3\pi}{2}\left(\tan^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^3}{32}$$

$$\Rightarrow \frac{\pi^3}{32}\le S\le\frac{7\pi^3}{8}$$

$$\Rightarrow \frac{\pi^3}{32}\le k\pi^3<\frac{7}{8}\pi^3$$

$$\Rightarrow \frac{1}{32}\le k<\frac{7}{8}$$

16. Ans. (3)

$$\cos^{-1}\left[\frac{3}{10}\cos\left(\tan^{-1}\frac{4}{3}\right)+\frac{2}{5}\sin\left(\tan^{-1}\frac{4}{3}\right)\right]$$

$$=\cos^{-1}\left[\frac{3}{10}\cos\left(\cos^{-1}\frac{3}{5}\right)+\frac{2}{5}\sin\left(\sin^{-1}\frac{4}{5}\right)\right]$$

$$=\cos^{-1}\left[\frac{3}{10}\times\frac{3}{5}+\frac{2}{5}\times\frac{4}{5}\right]$$

$$=\cos^{-1}\left[\frac{9}{50}+\frac{8}{25}\right]$$

$$=\cos^{-1}\left[\frac{9+16}{50}\right]$$

$$=\cos^{-1}\left(\frac{25}{50}\right)$$

$$=\frac{\pi}{3}$$

17. Ans. (4)

$$-1\le\frac{x^2-5x+6}{x^2-9}\le 1$$

$$\Rightarrow \frac{x^2-5x+6}{x^2-9}-1\le 0$$

$$\Rightarrow \frac{1}{x+3}\ge 0$$

$$\Rightarrow x\in(-3,\infty) \qquad \ldots(1)$$

$$\Rightarrow \frac{x^2-5x+6}{x^2-9}+1\ge 0$$

$$\Rightarrow \frac{2x+1}{x+3}\ge 0$$

$$\Rightarrow x\in(-\infty,-3)\cup\left[-\frac{1}{2},\infty\right) \qquad \ldots(2)$$

after taking intersection

$$\Rightarrow x \in \left[-\frac{1}{2}, \infty\right)$$

$\Rightarrow x^2 - 3x + 2 > 0$

$\Rightarrow x \in (-\infty, 1) \cup (2, \infty)$

$\Rightarrow x^2 - 3x + 2 \neq 1$

$$\Rightarrow x \neq \frac{3 \pm \sqrt{5}}{2}$$

after taking intersection of each solution

$$\Rightarrow \left[-\frac{1}{2}, 1\right) \cup (2, \infty) - \left\{\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right\}$$

18. Ans. (2)

$$f(x) = \frac{-2}{\sqrt{1-x^2}} - \frac{4}{1+x^2} - 6x - 2$$

$$= -2\left[\frac{1}{\sqrt{1-x^2}} + \frac{2}{1+x^2} + 3x + 1\right]$$

$f'(x) < 0 \Rightarrow f(x)$ is a dec. function

$f(1) = \pi + 5$

$f(-1) = 5\pi + 5$

Range : $[a, b] \equiv [\pi + 5, 5\pi + 5]$

$a = \pi + 5$, $b = 5\pi + 5 \Rightarrow 4a - b = 11 - \pi$.

19. Ans. (29)

$$50\tan\left(3\tan^{-1}\frac{1}{2} + 2\cos^{-1}\frac{1}{\sqrt{5}}\right) + 4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}2\sqrt{2}\right)$$

$$= 50\tan\left(\tan^{-1}\frac{1}{2} + 2(\tan^{-1}\frac{1}{2} + \tan^{-1}2)\right) + 4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}2\sqrt{2}\right)$$

$$= 50\tan\left(\tan^{-1}\frac{1}{2} + 2.\frac{\pi}{2})\right) + 4\sqrt{2} \times \frac{1}{\sqrt{2}}$$

$$= 50\left(\tan\tan^{-1}\frac{1}{2}\right) + 4$$

$= 25 + 4 = 29$

20. Ans. (2)

$$f(x) = 4\sin^{-1}\left(\frac{x^2}{x^2+1}\right)$$

$$\frac{x^2+1-1}{x^2+1} = 1 - \frac{1}{x^2+1} \Rightarrow [0,1)$$

Range of $f(x) = [0, 2\pi)$

21. Ans. (3)

$$\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right)+\sec^{-1}\left(\sqrt{\frac{8+4\sqrt{3}}{6+3\sqrt{3}}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)+\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)=\frac{\pi}{3}$$

22. Ans. (2)

Case I : $x > 0$

$$\tan^{-1}\frac{2x}{1-x^2}+\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$$

$$x=2-\sqrt{3}$$

Case II : $x < 0$

$$\tan^{-1}\frac{2x}{1-x^2}+\tan^{-1}\frac{2x}{1-x^2}+\pi=\frac{\pi}{3}$$

$$x=\frac{-1}{\sqrt{3}} \Rightarrow \alpha=2$$

23. Ans. (1,3)

$$a_2-a_1=a_3-a_2=.....=a_{2022}-a_{2021}=1.$$

$$\therefore \tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right)+.....+\tan^{-1}\left(\frac{a_{2022}-a_{2021}}{1+a_{2021}a_{2022}}\right)$$

$$=\left[\left(\tan^{-1}a_2\right)-\tan^{-1}a_1\right]+\left[\tan^{-1}a_3-\tan^{-1}a_2\right]+\ .....\ +\left[\tan^{-1}a_{2022}-\tan^{-1}a_{2021}\right]$$

$$=\tan^{-1}a_{2022}-\tan^{-1}a_1$$

$$=\tan^{-1}(2022)-\tan^{-1}1=\tan^{-1}2022-\frac{\pi}{4}$$ **(option 3)**

$$=\left(\frac{\pi}{2}-\cot^{-1}(2022)\right)-\frac{\pi}{4}$$

$$=\frac{\pi}{4}-\cot^{-1}(2022)$$ **(option 1)**

24. Ans. (4)

$$\sin^{-1}\sin\theta-\left(\frac{\pi}{2}-\sin^{-1}\sin\theta\right)>0$$

$$\Rightarrow \sin^{-1}\sin\theta>\frac{\pi}{4}$$

$$\Rightarrow \sin\theta>\frac{1}{\sqrt{2}}$$

So, $\theta\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$

$$\theta\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right)=(a,b)$$

$$b-a=\frac{\pi}{2}=\alpha-\beta$$

$$\Rightarrow \beta=\alpha-\frac{\pi}{2}$$

$$\Rightarrow \alpha x^2+\beta x+\sin^{-1}\left[(x-3)^2+1\right]+\cos^{-1}\left[(x-3)^2+1\right]=0$$

$$x=3,\ 9\alpha+3\beta+\frac{\pi}{2}+0=0$$

$$\Rightarrow 9\alpha+3\left(\alpha-\frac{\pi}{2}\right)+\frac{\pi}{2}=0$$

$$\Rightarrow 12\alpha-\pi=0$$

$$\alpha=\frac{\pi}{12}$$

25. **Ans. (1)**

$$\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{\alpha}{17}=\tan^{-1}\frac{77}{36}-\tan^{-1}\frac{3}{4}=\tan^{-1}\left(\frac{\frac{77}{36}-\frac{3}{4}}{1+\frac{77}{36}.\frac{3}{4}}\right)$$

$$\sin^{-1}\frac{\alpha}{17}=\tan^{-1}\frac{8}{15}=\sin^{-1}\frac{8}{17}$$

$$\Rightarrow \frac{\alpha}{17}=\frac{8}{17}\Rightarrow \alpha=8$$

$$\therefore \sin^{-1}(\sin 8)+\cos^{-1}(\cos 8)$$

$$=3\pi-8+8-2\pi$$

$$=\pi$$

EXERCISE - JEE (Advanced) PYQ

1. **Ans. (B)**

$$cot\left(\sum_{n=1}^{23} cot^{-1}\left(1+\sum_{k=1}^{n} 2k\right)\right)$$

$$= cot\left(\sum_{n=1}^{23} cot^{-1}\left(1+\frac{2n(n+1)}{2}\right)\right)$$

$$= cot\left(\sum_{n=1}^{23} tan^{-1}\frac{1}{1+n(n+1)}\right)$$

$$= cot\left(\sum_{n=1}^{23} tan^{-1}\frac{(n+1)-n}{1+n(n+1)}\right)$$

$$= cot\left(\sum_{n=1}^{23}(tan^{-1}(n+1)-tan^{-1}n)\right)$$

$$= cot(tan^{-1}24-tan^{-1}1)$$

$$= cot\left(tan^{-1}\frac{23}{25}\right)=cot\left(cot^{-1}\left(\frac{25}{23}\right)\right)=\frac{25}{23}$$

2. Ans. (B)

(P) $\left(\frac{1}{y^2}\left(\frac{\frac{1}{\sqrt{1+y^2}}+\frac{y^2}{\sqrt{1+y^2}}}{\frac{\sqrt{1-y^2}}{y}+\frac{y}{\sqrt{1-y^2}}}\right)^2+y^4\right)^{\frac{1}{2}}$

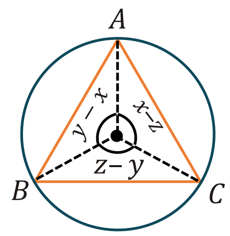
$\Rightarrow \left(\left(\frac{(1+y^2)y^2(1-y^2)}{y^2}\right)+y^4\right)^{\frac{1}{2}}$

$\Rightarrow ((1-y^4)+y^4)^{\frac{1}{2}}=1$

(Q) $A(\cos x, \sin x), B(\cos y, \sin y)$ & $C(\cos z, \sin z)$ lie on circle $x^2+y^2=1$

$\because$ (0,0) is circumcentre as well as centroid of ΔABC

$\Rightarrow \Delta ABC$ is an equilateral triangle



$y-x=\frac{2\pi}{3}$

$\cos\frac{y-x}{2}=\cos\frac{\pi}{3}=\frac{1}{2}$

(R) $\cos 2x\left(\cos\left(\frac{\pi}{4}-x\right)-\cos\left(\frac{\pi}{4}-x\right)\right)$

$=\sin 2x(1-\tan x)$

$\sqrt{2}\sin x\cos 2x=\sin 2x(1-\tan x)$

$\sin x\left(\sqrt{2}\cos 2x-2(\sin x-\cos x)\right)=0$

$\Rightarrow \sin x=0$ or $\sin x=\cos x$ or $\sin x+\cos x=\sqrt{2}$

$\Rightarrow \sec x = \pm 1$ or $\sec x=\pm\sqrt{2}$

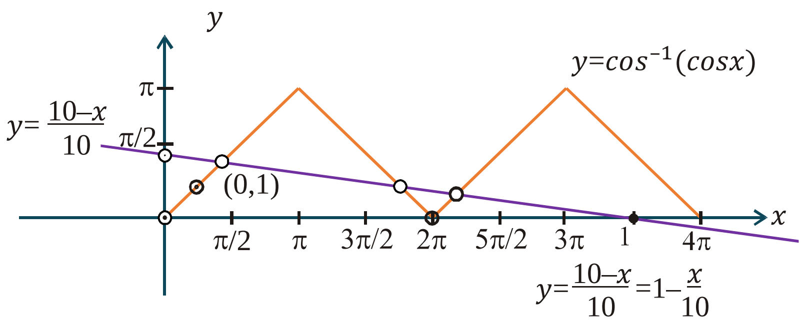
(S) $\cot\left(\sin^{-1}\sqrt{1-x^2}\right)=\sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$

$\frac{|x|}{\sqrt{1-x^2}}=\frac{\sqrt{6}x}{\sqrt{1+6x^2}}\ (x>0)$

$\Rightarrow 6-6x^2=1+6x^2$

$\Rightarrow x=\sqrt{\frac{5}{12}}=\frac{\sqrt{5}}{2\sqrt{3}}$

3. Ans. (3)



from above figure it is clear that $y=\frac{10-x}{10}$ and $y=\cos^{-1}(\cos x)$ intersect at 3 distinct points, so number of solutions = 3.

4. Ans. (B,C,D)

$\alpha = 3\sin^{-1}\dfrac{6}{11}$ & $\beta = 3\cos^{-1}\dfrac{4}{9}$

$\because \dfrac{6}{11} > \dfrac{1}{2} \Rightarrow \sin^{-1}\dfrac{6}{11} > \sin^{-1}\dfrac{1}{2} \Rightarrow 3\sin^{-1}\dfrac{6}{11} > 3\sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{2}$

$\therefore \alpha > \dfrac{\pi}{2}$ $\quad\quad\quad\quad \therefore \cos\alpha < 0$

Now, $\beta = 3\cos^{-1}\dfrac{4}{9}$

$\because \dfrac{4}{9} < \dfrac{1}{2} \Rightarrow 3\cos^{-1}\dfrac{4}{9} > 3\cos^{-1}\dfrac{1}{2}$

$\therefore \beta > \pi$

$\therefore \cos\beta < 0$ & $\sin\beta < 0$

Now, α is slightly greater than $\dfrac{\pi}{2}$ & β is slightly greater than π

$\therefore \cos(\alpha+\beta) > 0$

5. Ans. (2)

$$\sum_{i=1}^{\infty} x^{i+1} = \frac{x^2}{1-x}$$

$$\sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i = \frac{x}{2-x}$$

$$\sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i = \frac{-x}{2+x}$$

$$\sum_{i=1}^{\infty} (-x)^i = \frac{-x}{1+x}$$

To have real solutions

$$\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i = \sum_{i=1}^{\infty}\left(\frac{-x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i$$

$$\frac{x^2}{1-x} - \frac{x^2}{2-x} = \frac{-x}{2+x} + \frac{x}{1+x}$$

$x(x^3 + 2x^2 + 5x - 2) = 0$

$\therefore\ x = 0$ and let $f(x) = x^3 + 2x^2 + 5x - 2$

$f\left(\dfrac{1}{2}\right).f\left(-\dfrac{1}{2}\right) < 0$

Hence two solutions exist.

6. Ans. (A)

$E_1 : \dfrac{x}{x-1} > 0$

$\Rightarrow E_1 : x \in (-\infty, 0) \cup (1, \infty)$

$E_2 : -1 \le \ell n\left(\dfrac{x}{x+1}\right) \le 1$

Now $\dfrac{x}{x-1} - \dfrac{1}{e} \ge 0$

$$\Rightarrow \frac{(e-1)x+1}{e(x-1)} \geq 0$$

+   −   +

$-1/(e-1)$ 1

$$\Rightarrow x \in \left(-\infty, \frac{1}{1-e}\right] \cup (1, \infty)$$

Also $\dfrac{x}{x-1} - e \leq 0$

$$\frac{(e-1)x-e}{x-1} \geq 0$$

+   −   +

1 $e/(e-1)$

$$\Rightarrow x \in (-\infty, 1) \cup \left[\frac{e}{e-1}, \infty\right]$$

So $E_2 : \left(-\infty, \dfrac{1}{1-e}\right) \cup \left[\dfrac{e}{e-1}, \infty\right]$

as Range of $\dfrac{x}{x-1}$ is $R^+ - \{1\}$

$\Rightarrow$ Range of f is $R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$

Range of g is $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] \backslash\{0\}$ or $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Now P $\to$ 4, Q $\to$ 2, R $\to$ 1, S $\to$ 1

Hence A is correct.

7. **Ans. (0.00)**

$$sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{1}{cos\left(\frac{7\pi}{12}+\frac{k\pi}{12}\right)cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$= sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{sin\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}-\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)}{cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right).cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$= sec^{-1}\left(\frac{1}{4}\left(\sum_{k=0}^{10} tan\left(\frac{7\pi}{12}+(k+1)\frac{\pi}{2}\right) - tan\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)\right)$$

$$= sec^{-1}\left(\frac{1}{4}\left(tan\left(\frac{11\pi}{2}+\frac{7\pi}{12}\right) - tan\left(\frac{7\pi}{12}\right)\right)\right)$$

$$= sec^{-1}\left(\frac{1}{4}\left(-cot\frac{7\pi}{12} - tan\frac{7\pi}{12}\right)\right)$$

$$= sec^{-1}\left(\frac{1}{4}\left(-\frac{1}{sin\frac{7\pi}{12}cos\frac{7\pi}{12}}\right)\right)$$

$$= sec^{-1}\left(-\frac{1}{2}\times\frac{1}{sin\frac{7\pi}{6}}\right) = sec^{-1}(1) = 0.00$$

8. Ans. (A,B)

$$S_n(x)=\sum_{k=1}^{n}\tan^{-1}\left(\frac{x}{1+kx(kx+x)}\right)$$

$$=\sum_{k=1}^{n}\tan^{-1}\left(\frac{(kx+x)-(kx)}{1+(kx+x)(kx)}\right)$$

$$S_n(x) = \tan^{-1}(nx + x) - \tan^{-1}x = \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$$

(A) $S_{10}(x) = \tan^{-1}\dfrac{10x}{1+11x^2} = \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{1+11x^2}{10x}\right)$ $(x > 0)$

(B) $\lim\limits_{n\to\infty} \cot(S_n(x)) = \lim\limits_{n\to\infty}\dfrac{\frac{1}{n}+\left(1+\frac{1}{n}\right)x^2}{x} = x$ $(x > 0)$

(C) $S_3(x) = \tan^{-1}\dfrac{3x}{1+4x^2} = \dfrac{\pi}{4} \Rightarrow 4x^2 - 3x + 1 = 0 \Rightarrow x \notin \mathbb{R}$

(D) $\tan(S_n(x)) = \dfrac{nx}{1+(n+1)x^2}$; $\forall\, n \geq 1$; $x > 0$

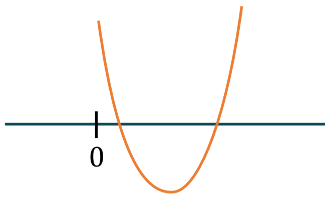
We need to check the validity of $\dfrac{nx}{1+(n+1)x^2} \leq \dfrac{1}{2}$ $\forall\, n \geq 1$; $x > 0$; $n \in \mathbb{N}$

$\Rightarrow 2nx \leq (n + 1)x^2 + 1$

$\Rightarrow (n + 1)x^2 - 2nx + 1 \geq 0$ $\forall\, n \geq 1$; $x > 0$; $n \in \mathbb{N}$

Discriminant of $y = (n + 1)x^2 - 2nx + 1$ is

$D = 4n^2 - 4(n + 1)$ and $n \in \mathbb{N}$



$D < 0$ for $n = 1$; true for $x > 0$

$D > 0$ for $n \geq 2 \Rightarrow \exists$ some $x > 0$

for which $y < 0$ as both roots of

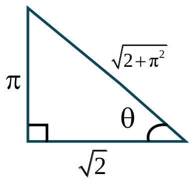
$y = 0$ will be positive.

$y = (n + 1)x^2 - 2nx + 1, n \geq 2$

So, $y \geq 0$ $\forall\, n \geq 1$; $\forall\, x > 0$; $n \in N$ is false.

9. Ans. (2.35 or 2.36)

$$\cos^{-1}\sqrt{\frac{2}{2+\pi^2}}=\tan^{-1}\frac{\pi}{\sqrt{2}}$$



$$\sin^{-1}\left(\frac{2\sqrt{2}\pi}{2+\pi^2}\right)=\sin^{-1}\left(\frac{2\times\frac{\pi}{\sqrt{2}}}{1+\left(\frac{\pi}{\sqrt{2}}\right)^2}\right)$$

$=\pi-2\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$

$\left(As,\ \sin^{-1}\left(\frac{2x}{1+x^2}\right)=\pi-2\tan^{-1}x, x\geq 1\right)$

and $\tan^{-1}\frac{\sqrt{2}}{\pi}=\cot^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$

$\therefore$ Expression $=\frac{3}{2}\left(\tan^{-1}\frac{\pi}{\sqrt{2}}\right)+\frac{1}{4}\left(\pi-2\tan^{-1}\frac{\pi}{\sqrt{2}}\right)+\cot^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$

$=\left(\frac{3}{2}-\frac{2}{4}\right)\tan^{-1}\frac{\pi}{\sqrt{2}}+\frac{\pi}{4}+\cot^{-1}\frac{\pi}{\sqrt{2}}$

$=\left(\tan^{-1}\frac{\pi}{\sqrt{2}}+\cot^{-1}\frac{\pi}{\sqrt{2}}\right)+\frac{\pi}{4}$

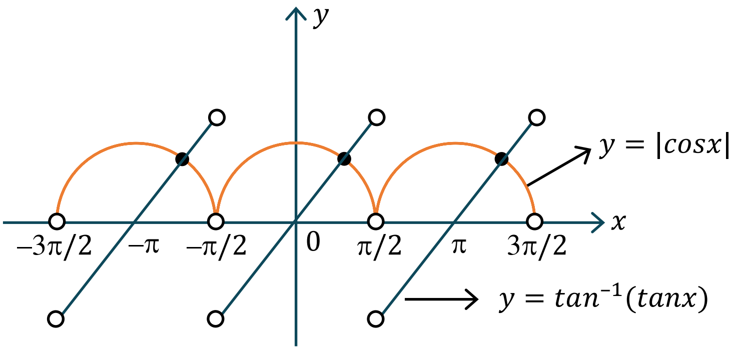
$=\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}$

= 2.35 or 2.36

10. Ans. (3)

$\sqrt{2}|\cos x|=\sqrt{2}.\tan^{-1}(\tan x)$

$|\cos x|=\tan^{-1}\tan x$



No. of solutions = 3

11. Ans. (C)

Case-I : $y \in (-3,0)$

$\tan^{-1}\left(\frac{6y}{9-y^2}\right)+\pi+\tan^{-1}\left(\frac{6y}{9-y^2}\right)=\frac{2\pi}{3}$

$2\tan^{-1}\left(\frac{6y}{9-y^2}\right)=-\frac{\pi}{3}$

$y^2-6\sqrt{3}y-9=0 \Rightarrow y=3\sqrt{3}-6\ (\because y\in(-3,0))$

Case-I : $y \in (0,3)$

$2\tan^{-1}\left(\frac{6y}{9-y^2}\right)=\frac{2\pi}{3}\Rightarrow\sqrt{3}y^2+6y-9\sqrt{3}=0$

$y=\sqrt{3}$ or $y=-3\sqrt{3}$ (rejected)

sum $=\sqrt{3}+3\sqrt{3}-6=4\sqrt{3}-6$

JEE (Main) Practice Paper

SECTION–A

1. **Ans. (1)**

$f(x)=\sin^{-1}(|x-1|-2)$. For domain $-1 \le |x-1|-2 \le 1$

$\Rightarrow 1 \le |x-1| \le 3 \Rightarrow x-1 \in [-3,-1] \cup [1,3] \Rightarrow x \in [-2,0] \cup [2,4]$

2. **Ans. (4)**

$\therefore\ -1 \le x \le 1$ ...(1)

$x \in R$ ...(2)

$x \le -1$ or $x \ge 1$ ...(3)

By $(1) \cap (2) \cap (3)$

$\Rightarrow x \in \{-1, 1\}$

3. **Ans. (4)**

$\text{cosec}^{-1}(\cos x)$ is defined if

$\cos x \ge 1$ or $\cos x \le -1 \Rightarrow \cos x = \pm 1 \Rightarrow x = n\pi$

4. **Ans. (1)**

$$f(x) = \sin^{-1}\left[x^2+\frac{1}{2}\right] + \cos^{-1}\left[x^2-\frac{1}{2}\right]$$

$$\text{Domain :} -1 \le \left[x^2-\frac{1}{2}\right] \le 1 \Rightarrow x \in \left(-\sqrt{\frac{5}{2}},\sqrt{\frac{5}{2}}\right)$$

$$\text{and } -1 \le \left[x^2+\frac{1}{2}\right] \le 1 \Rightarrow x \in \left(-\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}\right)$$

$$\Rightarrow \text{ domain is } x \in \left(-\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}\right) \text{ or } x^2 \in \left[0,\frac{3}{2}\right)$$

if (i) $x^2 \in \left[0,\frac{1}{2}\right)$, then $f(x) = \pi$

if (ii) $x^2 \in \left[\frac{1}{2},1\right)$, then $f(x) = \pi$

if (iii) $x^2 \in \left[1,\frac{3}{2}\right)$, then $f(x) = \pi$ $\Rightarrow$ range = $\{\pi\}$

5. **Ans. (4)**

$\pi \le x \le 2\pi$

$\cos^{-1}\cos x = 2\pi - x$

6. **Ans. (2)**

$$\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2} - \cos^{-1}x - \cos^{-1}y + \frac{\pi}{2} = \pi - (\cos^{-1}x + \cos^{-1}y) = \frac{2\pi}{3}$$

$$\Rightarrow \cos^{-1}x + \cos^{-1}y = \frac{\pi}{3}$$

7. **Ans. (4)**

$\theta = \sin^{-1} x + \cos^{-1} x - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1} x$

Domain $x \in [-1, 1]$ But given $x \ge 0$

$\Rightarrow x \in [0, 1] \Rightarrow \theta = \frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

for $x \in [0, 1] \Rightarrow \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

8. **Ans. (2)**

$= \cot\left(\frac{\pi}{2} + \sin^{-1}\frac{3}{5}\right) = -\tan\sin^{-1}\frac{3}{5} = \frac{-3}{4}$

9. **Ans. (3)**

$\tan^2(\sec^{-1} 2) + \cot^2(\operatorname{cosec}^{-1} 3) \Rightarrow \tan^2(\tan^{-1}\sqrt{3}) + \cot^2(\cot^{-1}\sqrt{8}) \Rightarrow 3 + 8 = 11$

10. **Ans. (3)**

Let $f(x) = x^3 + 3x - \tan 2$

$f'(x) = 3x^2 + 3 > 0 \ \forall\ x \in R \Rightarrow f(x)$ is increasing and has exactly one real root

$f(0) = -\tan 2 =$ positive

$f(-1) = -1 - 3 - \tan 2 = -4 - \tan 2 =$ negative

$\therefore \alpha$ lies between $(-1,0)$

$\therefore \alpha, (-1,0)$

Now $\cot^{-1}\alpha + \cot^{-1}\frac{1}{\alpha} - \frac{\pi}{2} = \cot^{-1}\alpha + \pi + \tan^{-1}\alpha - \frac{\pi}{2} = \pi$

11. **Ans. (1)**

Here, $x \in [0,4]$

Now, we have

$\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \cos^{-1}\left(\frac{\sqrt{x}}{2}\right) + \tan^{-1} y = \frac{2\pi}{3} \Rightarrow y = \frac{1}{\sqrt{3}}$

$\therefore$ Maximum value of $(x^2 + y^2) = 16 + \frac{1}{3} = \frac{49}{3}$

and minimum value of $(x^2 + y^2) = (0)^2 + \frac{1}{3} = \frac{1}{3}$

12. **Ans. (2)**

By property if $x < 0$ $\tan^{-1}\frac{1}{x} = \cot^{-1} x - \pi$

$\therefore \tan^{-1} x + \tan^{-1}\frac{1}{x} = \tan^{-1} x + \cot^{-1} x - \pi = \frac{\pi}{2} - \pi \Rightarrow \tan^{-1} x + \tan^{-1}\frac{1}{x} = -\frac{\pi}{2}$

13. **Ans. (2)**

$\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} \Rightarrow \sin^{-1} x + \cos^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{2} \Rightarrow x = \frac{1}{\sqrt{5}}$

14. Ans. (2)

$$f(x) = \tan^{-1}\frac{\frac{x}{3}-\frac{x}{\sqrt{3}}}{1+\frac{x^2}{3\sqrt{3}}} = \tan^{-1}\frac{x}{3} - \tan^{-1}\frac{x}{\sqrt{3}} \quad ; 0 \le x \le 3.$$

Hence, $f(x) = \tan^{-1}\left(\frac{x}{3}\right) \Rightarrow$ range of $f(x)$ is $\left[0, \frac{\pi}{4}\right]$

15. Ans. (1)

Given that $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$ ...(i)

We know that, $\cot^{-1}(\sqrt{\cos\alpha}) + \tan^{-1}(\sqrt{\cos\alpha}) = \frac{\pi}{2}$ ...(ii)

On adding equations (i) and (ii), we get

$$2\cot^{-1}(\sqrt{\cos\alpha}) = \frac{\pi}{2} + x \Rightarrow \sqrt{\cos\alpha} = \cot\left(\frac{\pi}{4}+\frac{x}{2}\right) \Rightarrow \sqrt{\cos\alpha} = \frac{\cot\frac{x}{2}-1}{1+\cot\frac{x}{2}}$$

$$\Rightarrow \sqrt{\cos\alpha} = \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}} \Rightarrow \cos\alpha = \frac{1-\sin x}{1+\sin x} \Rightarrow \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-\sin x}{1+\sin x}$$

Applying componendo and dividendo rule, we get $\sin x = \tan^2\left(\frac{\alpha}{2}\right)$

16. Ans. (3)

Given that, $\sin^{-1} x = 2\sin^{-1}\alpha$

Since, $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le 2\sin^{-1}\alpha \le \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4} \le \sin^{-1}\alpha \le \frac{\pi}{4} \Rightarrow \sin\left(-\frac{\pi}{4}\right) \le \alpha \le \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow -\frac{1}{\sqrt{2}} \le \alpha \le \frac{1}{\sqrt{2}} \Rightarrow |\alpha| \le \frac{1}{\sqrt{2}}$

17. Ans. (3)

$f(x) = \cot^{-1}x \qquad R^+ \to \left(0, \frac{\pi}{2}\right); g(x) = 2x - x^2 \qquad R \to R$

$f(g(x)) = \cot^{-1}(2x - x^2)$, where $x \in (0, 2)$ & $2x - x^2 \in (0, 1]$

hence $f(g(x)) \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$

18. Ans. (1)

$$f(x) = e^{\cos^{-1}\left(\sin\left(x+\frac{\pi}{3}\right)\right)}$$

Domain $-1 \le \sin\left(x+\frac{\pi}{3}\right) \le 1 \;+\; \frac{\pi}{2} \le x + \frac{\pi}{3} \le \frac{3\pi}{2} \Rightarrow \frac{\pi}{6} \le x \le \frac{7\pi}{6}$

$g(x) = \text{cosec}^{-1}\left(\frac{4-2\cos x}{3}\right) \Rightarrow \frac{4-2\cos x}{3} \geq 1$

Or $\cos x \leq \frac{1}{2} \Rightarrow x \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$

Domain of $h(x) : x \in \left[\frac{\pi}{3}, \frac{7\pi}{6}\right] \Rightarrow h(x) = e^{\cos^{-1}\left(\sin\left(x+\frac{\pi}{3}\right)\right)} = e^{\cos^{-1}\left[-1, \frac{\sqrt{3}}{2}\right]}$ range of $h(x)$: $[e^{\pi/6}, e^{\pi}]$

19. Ans. (2)

$\tan\left(\sin^{-1} x\right) > 1$ or $\tan\left(\sin^{-1} x\right) < -1$

$\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\quad \frac{-\pi}{2} \leq \sin^{-1} x < \frac{-\pi}{4}$

$\frac{1}{\sqrt{2}} < x \leq 1$ ...(1)

$\frac{\pi}{4} < \sin^{-1} x \leq \frac{\pi}{2}$ $\quad -1 \leq x < -\frac{1}{\sqrt{2}}$ ...(2)

From (1) & (2)

$x \in \left[-1, \frac{-1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right]$

20. Ans. (1)

$2\tan^{-1} x = \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) - \cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)$ $a > 0, b > 0$

Let, $a = \tan\alpha$, $b = \tan\beta$, $b = \tan\beta$, $\quad 0 < \alpha, \beta < \frac{\pi}{2}$

$\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right) = \cos^{-1}\cos 2\alpha = 2\alpha$ $\quad \{2\alpha \in (0, \pi)\}$

$\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\beta}{1+\tan^2\beta}\right) = \cos^{-1}\cos 2\beta = 2\beta$ $\quad \{2\beta \in (0, \pi)\}$

$\Rightarrow 2\tan^{-1} x = 2\alpha - 2\beta$

$\Rightarrow x = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{a-b}{1+ab}$

SECTION–B

1. Ans. (1)

$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$-1 \leq x, y, z \leq 1$

let $x = \cos A$, $y = \cos B$, $z = \cos C$

where $0 \leq A, B, C \leq \pi$

$$A + B + C = \pi$$

$x^2 + y^2 + z^2 + 2xyz = \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C$

$= \sum\left(\frac{1+\cos 2A}{2}\right) + 2\cos A\cos B\cos C$

$= \frac{3}{2} + \frac{1}{2}(-1 - 4\cos A\cos B\cos C) + 2\cos A\cos B\cos C = 1$

2. **Ans. (30)**

Case-I : $x \geq 0$

Let $\cot^{-1} x = \theta$

$\therefore\ \theta \in \left(0, \frac{\pi}{2}\right] \Rightarrow x = \cot\theta$

$\therefore\ \sin\theta = \frac{1}{\sqrt{1+x^2}} \Rightarrow \sin^{-1}\sin\theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}} \Rightarrow \theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

Case-II : $x < 0$

Let $\cot^{-1} x = \theta$

$\therefore\ \theta \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \cot\theta = x$

$\therefore\ \sin\theta = \frac{1}{\sqrt{1+x^2}} \Rightarrow \sin^{-1}\sin\theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}} \Rightarrow \pi - \theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \theta = \pi - \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

Therefore,

$$\text{LHS} = \begin{cases} \cos\tan^{-1}\sin\sin^{-1}\frac{1}{\sqrt{1+x^2}} & , \text{if } x \geq 0 \\ \cos\tan^{-1}\sin\left(\pi - \sin^{-1}\frac{1}{\sqrt{1+x^2}}\right) & , \text{if } x < 0 \end{cases} = \cos\tan^{-1}\sin\sin^{-1}\frac{1}{\sqrt{1+x^2}}\ ;$$

$x \in R = \cos\tan^{-1}\frac{1}{\sqrt{1+x^2}}$

Let $\phi = \tan^{-1}\frac{1}{\sqrt{1+x^2}}$

As $\frac{1}{\sqrt{1+x^2}} \in (0, 1]$ $\quad\therefore\ \phi \in \left(0, \frac{\pi}{4}\right]$ $\quad\therefore\ \tan\phi = \frac{1}{\sqrt{1+x^2}}$ $\quad\therefore\ \cos\phi = \sqrt{\frac{1+x^2}{2+x^2}}$

3. **Ans. (1)**

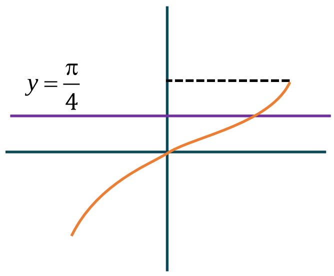
$\sin^{-1} x > \cos^{-1} x$

$2\sin^{-1} x > \sin^{-1} x + \cos^{-1} x$

$2\sin^{-1} x > \frac{\pi}{2}$

$\sin^{-1} x > \frac{\pi}{4}$

$\frac{1}{\sqrt{2}} < x \leq 1$



4. **Ans. (3)**

$\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) = \frac{\pi}{2}$

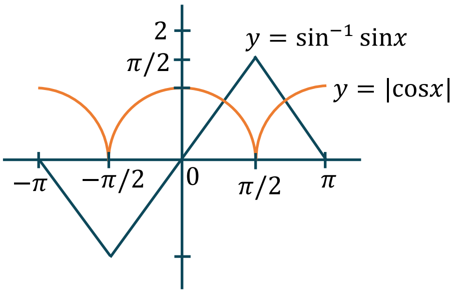
Taking sin on both side

$\frac{3\sin 2\theta}{5+4\cos 2\theta} = 1 \Rightarrow 3\sin 2\theta = 5 + 4\cos 2\theta \quad \Rightarrow \frac{6\tan\theta}{1+\tan^2\theta} = 5 + 4\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\tan^2\theta - 6\tan\theta + 9 = 0 \quad \Rightarrow \tan\theta = 3$

5. **Ans. (20)**

Given equation is $|\cos x| = \sin^{-1}(\sin x) - \pi \le x \le \pi$



Number of solution = 2 in $-\pi \le x \le \pi$

So number of solutions = 20 in $-10\pi \le x \le 10\pi$

6. **Ans. (6)**

$$f(x) = \sqrt{\frac{1}{(|x|-1)\cos^{-1}(2x+1)\ \tan 3x}}$$

here $-1 \le 2x + 1 < 1 \Rightarrow -2 \le 2x < 0 \Rightarrow -1 \le x < 0 \Rightarrow x \in [-1, 0)$

But $x \ne -1$ as $|x| - 1 \ne 0$

$\therefore x \in (-1, 0)$

for $x \in (-1, 0)$, $(|x| - 1)$ is $-$ve

$\therefore \tan 3x < 0$

$0 > 3x > -\frac{\pi}{2}$ or $x \in \left(-\frac{\pi}{6}, 0\right)$

Domain : $\left(-\frac{\pi}{6}, 0\right) \cap (-1, 0) \equiv \left(-\frac{\pi}{6}, 0\right)$

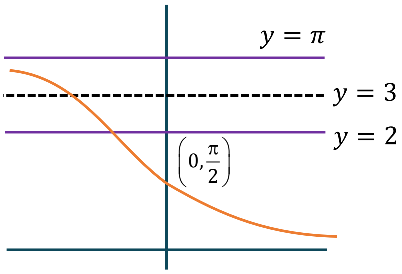
7. **Ans. (5)**

$$\left(\cot^{-1} x\right)^2 - 5\left(\cot^{-1} x\right) + 6 > 0$$

$$\left(\cot^{-1} x - 2\right)\left(\cot^{-1} x - 3\right) > 0$$

$\cot^{-1} x < 2$ or $\cot^{-1} x > 3$

$x \in (\cot 2, \infty)$ $\quad$ $x \in (-\infty, \cot 3)$



8. **Ans. (5)**

$\cot^{-1} \frac{n}{\pi} > \frac{\pi}{6}$ $\quad \Rightarrow -\infty < \frac{n}{\pi} < \cot \frac{\pi}{6}$

$n < \pi\sqrt{3}$ $\quad n \in N$, $n_{max} = 5$

9. **Ans. (19)**

$\alpha = \cos^{-1}\left(\frac{3}{5}\right)$

$\Rightarrow \begin{matrix} \cos\alpha = \frac{3}{5} \\ \tan\alpha = \frac{4}{3} \end{matrix}$

$\Rightarrow \beta = \tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \tan\beta = \frac{1}{3}$

$\Rightarrow \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

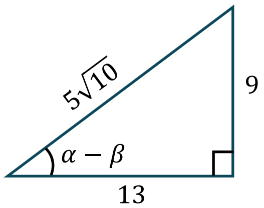
$= \frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{3}\times\frac{1}{3}}$

$= \frac{1}{\frac{9+4}{9}}$

$\Rightarrow \tan(\alpha-\beta) = \frac{9}{13}$

$\Rightarrow \sin(\alpha-\beta) = \frac{9}{5\sqrt{10}}$

$\Rightarrow \alpha-\beta = \sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$



10. **Ans. (2)**

$\sum_{n=1}^{\infty} \tan^{-1}\frac{4n}{n^4-2n^2+2} = \lim_{k\to\infty}\sum_{n=1}^{k}\left\{\tan^{-1}(n+1)^2 - \tan^{-1}(n-1)^2\right\}$

$= \lim_{k\to\infty}\left\{\tan^{-1}(k+1)^2 + \tan^{-1}k^2 - \tan^{-1}1 - \tan^{-1}0\right\} = \frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{4}-0 = \frac{3\pi}{4}$

Also $\tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}\left(\frac{3+2}{1-3.2}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$, ($\because xy = 6 > 1$)

JEE (Advanced) Practice Paper

1. **Ans. (B)**

$\cot^{-1}\left\{\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right\} \qquad \frac{\pi}{2} < x < \pi$

Rationalize the term in the bracket

$= \cot^{-1}\left(\frac{2+2\sqrt{1-\sin^2 x}}{-2\sin x}\right) = \cot^{-1}\left(\frac{1-\cos x}{-\sin x}\right)$

$= \cot^{-1}\left(-\tan\frac{x}{2}\right) = \frac{\pi}{2} - \tan^{-1}\left(-\tan\frac{x}{2}\right) = \frac{\pi}{2}+\tan^{-1}\tan\frac{x}{2}$

since $\frac{x}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \frac{\pi}{2}+\frac{x}{2}$

2. Ans. (D)

$$f(x) = \sin^{-1}\left(\frac{1+x^3}{2x^{3/2}}\right) + \sqrt{\sin(\sin x)} + \log_{(3\{x\}+1)}\left(x^2+1\right)$$

Domain : $3\{x\} + 1 \neq 1$ or $0 \Rightarrow x \notin I$ and $-1 \leq \frac{1+x^3}{2x^{3/2}} \leq 1$

$\Rightarrow -2x^{3/2} \leq 1 + x^3 \leq 2x^{3/2} \Rightarrow 1 + x^3 + 2x^{3/2} \geq 0 \Rightarrow \left(1 + x^{3/2}\right)^2 \geq 0$

$\Rightarrow x \in R \Rightarrow 1 + x^3 - 2x^{3/2} \leq 0$

or $\left(1-x^{3/2}\right)^2 \leq 0$ or $1-x^{3/2}=0$ or $x = 1$

Hence domain $x \in \phi$

3. Ans. (A)

$\left(\left[\cot^{-1}x\right]-3\right)^2 \leq 0 \Rightarrow [\cot^{-1}x]=3$

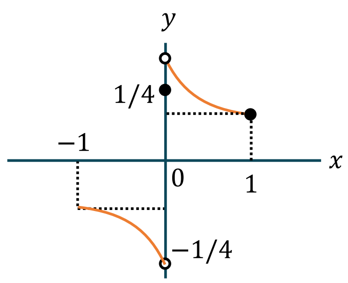
$\Rightarrow 3 \leq \cot^{-1}x < 4 \Rightarrow 3 \leq \cot^{-1}x < \pi < 4$

$\Rightarrow -\infty < x \leq \cot 3$

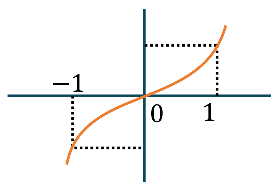
4. Ans. (B)

Domain $D \in [-1, 1]$

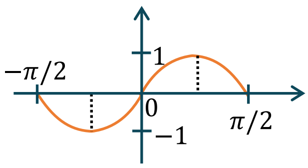
(A) $f(x) = \begin{cases} \tan^{-1}\frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ many -one



(B) $g(x)=x^3$ one - one



(C) $h(x)=\sin 2x$ many - one



(D) $k(x) = \cos\left(\frac{\pi x}{2}\right)$ many-one function

5. Ans. (B,C,D)

(A) $f(x) = e^{\ell n(\sec^{-1}x)} = \sec^{-1}x, x \in (-\infty, -1] \cup (1, \infty)$

$g(x) = \sec^{-1}x, \quad x \in (-\infty, -1] \cup [1, \infty)$

non-identical functions

(B) $f(x) = \tan(\tan^{-1}x) = x, x \in R$

$g(x) = \cot(\cot^{-1}x) = x, x \in R$

identical functions

(C) $f(x) = sgn\,(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases} \quad \Rightarrow \quad g(x) = sgn(sgn\,x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$

Identical functions

(D) $f(x) = \cot^2 x \,.\, \cos^2 x, x \in R - \{n\,\pi\}, n \in I$

$g(x) = \cot^2 x - \cos^2 x$

$= \cot^2 x(1 - \sin^2 x) = \cot^2 x.\, \cos^2 x \qquad x \in R - \{n\,\pi\}, n \in I$

Identical functions

6. Ans. (A,B)

$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \Rightarrow x = y = z = 1 \Rightarrow x^{100} + y^{100} + z^{100} - \dfrac{9}{x^{101} + y^{101} + z^{101}} = 0.$

7. Ans. (A,B)

$X = \text{cosec } \tan^{-1} \cos \cot^{-1} \sec \sin^{-1} a = \text{cosec } \tan^{-1} \cos \cot^{-1}\left(\dfrac{1}{\sqrt{1-a^2}}\right)$

$= \text{cosec}\left\{\tan^{-1}\left(\dfrac{1}{\sqrt{2-a^2}}\right)\right\} = \sqrt{3-a^2}$

again $Y = \sec \cot^{-1} \sin \tan^{-1} \text{cosec} \cos^{-1} a \Rightarrow Y = \sec \cot^{-1} \sin \tan^{-1}\left(\dfrac{1}{\sqrt{1-a^2}}\right)$

$= \sec \cot^{-1}\left(\dfrac{1}{\sqrt{2-a^2}}\right) = \sqrt{(3-a^2)} \Rightarrow X = Y = \sqrt{3-a^2}$

8. Ans. (C,D)

$x^2 - x - 2 > 0 \qquad \alpha$ satisfies it

$\Rightarrow \alpha^2 - \alpha - 2 > 0 \Rightarrow (\alpha - 2)(\alpha + 1) > 0 \Rightarrow \alpha < -1$ or $\alpha > 2$

so C & D are correct.

9. Ans. (B,C)

$f(x) = \ell n(\sin^{-1}(\log_2 x))$

Domain $0 < \log_2 x \le 1, x \in (1, 2]$

Range $\left(-\infty, \ell n\dfrac{\pi}{2}\right]$

10. Ans. (B,C,D)

$f:[-1,1]\to[-1,1]$

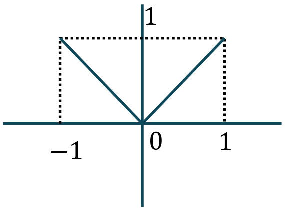
(A) $f(x) = \sin(\sin^{-1}x) = x,\ x \in [-1, 1]$ Bijective function $y \in [-1, 1]$

(B) $f(x) = \frac{2}{\pi}\sin^{-1}(\sin x) = \frac{2}{\pi},\quad x \in [-1, 1]$

Not bijective $y \in \left[\frac{-2}{\pi}, \frac{2}{\pi}\right]$

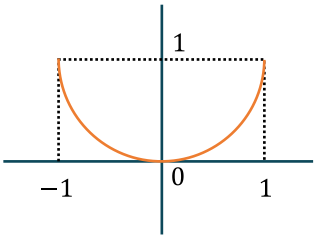
(C) $f(x) = (\text{sgn } x)(\ell n\ e^{x})$

$$= \begin{cases} x & , \quad x \in (0,1] \\ 0 & , \quad x = 0 \\ -x & , \quad x \in [-1,0) \end{cases}$$



Not bijective

(D)



$f(x) = x^3 sgn\, x$

$$= \begin{cases} x^3 & , \quad x > 0 \\ 0 & , \quad x = 0 \\ -x^3 & , \quad x < 0 \end{cases}$$

Not bijective

11. Ans. (A,D)

$\sin \cot^{-1}\cos\tan^{-1}t = \sin\cot^{-1}\frac{1}{\sqrt{1+t^2}} = \frac{\sqrt{1+t^2}}{\sqrt{2+t^2}}$

similarly $\cos\tan^{-1}\sin\cot^{-1}\sqrt{2}\ t = \frac{\sqrt{1+2t^2}}{\sqrt{2+2t^2}}$

so $\frac{1}{\sqrt{2}}\left\{\frac{\sin\cot^{-1}\cos\tan^{-1}t}{\cos\tan^{-1}\sin\cot^{-1}\sqrt{2}\,t}\right\}\cdot\left\{\sqrt{\frac{1+2t^2}{2+t^2}}\right\} = \dfrac{\frac{1}{\sqrt{2}}\frac{\sqrt{1+t^2}}{\sqrt{2+t^2}}\times\frac{\sqrt{1+2t^2}}{\sqrt{2+t^2}}}{\frac{\sqrt{1+2t^2}}{\sqrt{2}\sqrt{1+t^2}}} = \frac{1+t^2}{2+t^2} = 1-\frac{1}{2+t^2}$

$0 < \frac{1}{t^2+2} \le \frac{1}{2}$; $\frac{1}{2} \le 1-\frac{1}{t^2+2} < 1$

12. Ans. (A,B)

$\tan^{-1}\dfrac{\sqrt{1-x^2}}{1+x}$ since $0 < x < 1 = \tan^{-1}\left(\dfrac{\sin\theta}{1+\cos\theta}\right)$ (let $\cos^{-1}x = \theta$, $0 < \theta < \dfrac{\pi}{2}$)

$= \tan^{-1}\tan\dfrac{\theta}{2}$ $\quad\because\ \dfrac{\theta}{2} \in \left(0, \dfrac{\pi}{4}\right)$

$\Rightarrow \dfrac{\theta}{2} = \dfrac{1}{2}\cos^{-1}x$ ...(1)

also $\cos\theta = 2\cos^2\dfrac{\theta}{2} - 1$

$\cos\dfrac{\theta}{2} = \sqrt{\dfrac{1+x}{2}}$ (taking $\cos^{-1}$ on both side)

$\cos^{-1}\cos\dfrac{\theta}{2} = \cos^{-1}\left(\sqrt{\dfrac{1+x}{2}}\right)$ since $\dfrac{\theta}{2} \in \left(0, \dfrac{\pi}{4}\right)$

$\Rightarrow \dfrac{\theta}{2} = \cos^{-1}\left(\sqrt{\dfrac{1+x}{2}}\right)$ ...(2)

similarly $\sin\dfrac{\theta}{2} = \sqrt{\dfrac{1-x}{2}}$

$\sin^{-1}\sin\dfrac{\theta}{2} = \dfrac{\theta}{2} = \sin^{-1}\sqrt{\dfrac{1-x}{2}}$ ...(3)

also $\dfrac{\theta}{2} = \tan^{-1}\sqrt{\dfrac{1+x}{1-x}}$ ...(4)

13. Ans. (A,D)

Let $\theta = \cos^{-1}x$

$$f(x) = \theta + \cos^{-1}\left(\cos\left(\theta - \frac{\pi}{3}\right)\right) = \begin{cases} \theta + \theta - \dfrac{\pi}{3} & \dfrac{\pi}{3} < \theta \le \pi \\ \theta - \theta + \dfrac{\pi}{3} & 0 \le \theta \le \dfrac{\pi}{3} \end{cases} = \begin{cases} 2\cos^{-1}x - \dfrac{\pi}{3} & -1 \le x < \dfrac{1}{2} \\ \dfrac{\pi}{3} & \dfrac{1}{2} \le x \le 1 \end{cases}$$

(i) $f(2/3) = \dfrac{\pi}{3}$ $\qquad$ (ii) $f(1/3) = 2\cos^{-1}\dfrac{1}{3} - \dfrac{\pi}{3}$

14. Ans. (A,B,C,D)

$\tan x = \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}$

If $\tan\alpha = \dfrac{1}{\sqrt{2}} \Rightarrow x = n\pi \pm \alpha \quad n \in z$

15. Ans. (C, D)

If $-1 \le x < 0$, then $-\dfrac{\pi}{2} \le \sin^{-1}x < 0$. Also $0 < 2\cot^{-1}(y^2 - 2y) < 2\pi$

$\therefore\ -\dfrac{\pi}{2} < \sin^{-1}x + 2\cot^{-1}(y^2 - 2y) < 2\pi$

$\therefore$ there is no solution in this case.

thus x can not be negative ....(i)

Now if $x \ge 0$, then $0 \le \sin^{-1}x \le \dfrac{\pi}{2} \Rightarrow \dfrac{3\pi}{4} \le \cot^{-1}(y^2 - 2y) < \pi$

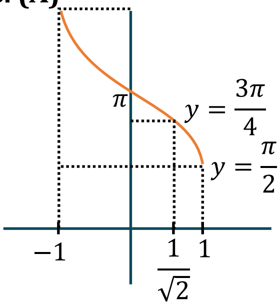
$\Rightarrow y^2 - 2y \le -1 \Rightarrow y = 1$

since for $y = 1$, we have $2\cot^{-1}\left(y^2 - 2y\right) = 2\cot^{-1}(-1) = \frac{3\pi}{2}$

$\therefore \sin^{-1} x = \frac{\pi}{2}$ i.e. $x = 1$

$\therefore$ the solution is $x = 1, y = 1$

16. **Ans. (A)**

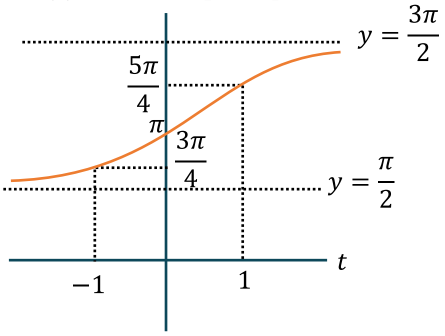


17. **Ans. (C)**

Clearly $\pi - x$.

18. **Ans. (B)**

$= \tan^{-1}(t)$, $t = -x \in [-1, 1]$



19. **Ans. ($A \to q$; $B \to s$; $C \to p$; $D \to r$)**

(A) $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2} \Rightarrow x \in [0, 1]$

(B) $\sin^{-1}\sqrt{x} + \cos^{-1}\left(\sqrt{1-x^2}\right) = 0 \Rightarrow \cos^{-1}\sqrt{1-x^2} = -\sin^{-1}(x) \Rightarrow x \in [-1, 0]$

(C) $g\left(\frac{1-x^2}{1+x^2}\right) = 2h(x) \Rightarrow \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1} x \Rightarrow x \in [0, \infty)$

(D) $h(x) + h(1) = h\left(\frac{1+x}{1-x}\right) \Rightarrow \tan^{-1} x + \tan^{-1} 1 = \tan^{-1}\left(\frac{1+x}{1-x}\right) \Rightarrow x \in (-\infty, 1)$

20. **Ans. ($P \to 3$; $Q \to 4$; $R \to 1$; $S \to 2$)**

(P) $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

$\tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$

$\Rightarrow \tan\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right) \Rightarrow \tan\left(\tan^{-1}\left(\frac{9+8}{12}\times\frac{12}{6}\right)\right)$

$$\Rightarrow \tan\left(\tan^{-1}\left(\frac{17}{6}\right)\right)=\frac{17}{6}$$

(Q) $$\cos\left(\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{2}{3}\right)$$

$$\cos\left(\tan^{-1}\frac{4}{3}+\tan^{-1}\left(\frac{\sqrt{5}}{2}\right)\right)$$

$$\Rightarrow \cos\left(\tan^{-1}\left[\frac{\frac{4}{3}+\frac{\sqrt{5}}{2}}{1-\frac{4\sqrt{5}}{6}}\right]\right)$$

$$\Rightarrow \cos\left(\tan^{-1}\left(\frac{8+3\sqrt{5}}{6-4\sqrt{5}}\right)\right)$$

$$\Rightarrow \cos\left(\cos^{-1}\left(\frac{6-4\sqrt{5}}{15}\right)\right)=\frac{6-4\sqrt{5}}{15}$$

(R) $$\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)=\sin\alpha$$

$$\Rightarrow \sin^{-1}\frac{\sqrt{63}}{8}=4\alpha$$

$$\Rightarrow \sin 4\alpha=\frac{\sqrt{63}}{8}\Rightarrow \cos 4\alpha=\frac{1}{8}$$

$$\Rightarrow 2\cos^2 2\alpha-1=\frac{1}{8}$$

$$\Rightarrow 2\cos^2 2\alpha=\frac{9}{8}\Rightarrow \cos^2 2\alpha=\frac{9}{16}$$

$$\Rightarrow \cos 2\alpha=\frac{3}{4}$$

$$\Rightarrow 2\cos^2\alpha-1=\frac{3}{4}\Rightarrow 2\cos^2\alpha=\frac{7}{4}$$

$$\cos^2\alpha=\frac{7}{8},\sin^2\alpha=\frac{1}{8}$$

$$\sin\alpha=\frac{1}{2\sqrt{2}}$$

(S) $$\cot\left[\underbrace{\cos^{-1}\left(\frac{3}{4}\right)+\sin^{-1}\left(\frac{3}{4}\right)}_{\frac{\pi}{2}}-\sec^{-1}3\right]$$

$$\Rightarrow \cot\left[\frac{\pi}{2}-\sec^{-1}3\right]\Rightarrow \tan\left(\sec^{-1}3\right)$$

$$\Rightarrow \tan\left(\tan^{-1}\sqrt{8}\right)=\sqrt{8}$$