02 Trigonometric Equations

Definition:

Trigonometric Equation:

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometric equation.

e.g. (i) $\sin x = \frac{1}{2}$. (ii) $\cos^2 x - 4\sin x = 1$

- There are millions and billions of solutions which satisfy say $\tan x = -1$ but our main object is to write down those infinite solutions in one line.
- Since all trigonometric function are periodic and therefore solution of all trigonometric equation can be generalised with the help of periodicity of trigonometric function.

Solution of Trigonometric Equation:

A value of the unknown angle which satisfies the given equation is called a solution of the trigonometric equation.

(a) **Principal solution:** The solution of the trigonometric equation lying in the interval $[0, 2\pi)$.

(b) **Particular solution:** The solution of the trigonometric equation lying in the given interval.

(c) **General solution:** Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solution of trigonometric equation.

Three Types of Solution

- Principal solution $0 \leq \theta < 2\pi$
- Particular solution (Solution lying in given interval)
- General solution

Illustration 1:

Find the principal solution of $\sin x = \frac{1}{2}$.

Solution:

$$\sin x = \frac{1}{2}$$

$$\sin x = \sin\frac{\pi}{6} \text{ or } \sin x = \sin\frac{5\pi}{6}$$

Hence $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

Illustration 2:

Find the principal solution of $\sin x = \frac{1}{\sqrt{2}}$.

Solution:

$\sin x = \frac{1}{\sqrt{2}}$

$\sin x = \sin\frac{\pi}{4}$ or $\sin x = \sin\frac{3\pi}{4}$

Hence $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$.

General Solution of Standard Trigonometric Equations:

General Solutions of Some Trigonometric Equations (to be remembered):

(a) If $\sin\theta = 0$, then $\theta = n\pi, \forall n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}, \forall n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi, \forall n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha, \forall n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha, \forall n \in I$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha, \forall n \in I$

(g) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$, then $\theta = n\pi \pm \alpha, \forall n \in I$

Special Cases:

- **General solution of extreme values of sin x and cos x :**

(i) If $\sin\theta = 1$, then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}, \forall n \in I$

(ii) If $\sin\theta = -1$, then $\theta = 2n\pi - \frac{\pi}{2} = (4n-1)\frac{\pi}{2}, \forall n \in I$

(iii) If $\cos\theta = 1$ then $\theta = 2n\pi, \forall n \in I$

(iv) If $\cos\theta = -1$ then $\theta = (2n+1)\pi, \forall n \in I$

- **Frequently used derived results:**

(i) $\sin(n\pi + \theta) = (-1)^n \sin\theta, \forall n \in I$,
Hence $\sin n\pi = 0 \ \forall\ n \in I$.

(ii) $\cos(n\pi + \theta) = (-1)^n \cos\theta, \forall n \in I$,
Hence $\cos n\pi = (-1)^n, \forall n \in I$.

(iii) If n is an odd integer, then

(a) $\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}}\cos\theta$

Hence, $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$

(b) $\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}}\sin\theta$

Hence, $\cos\frac{n\pi}{2} = (-1)^{\frac{n+1}{2}}$

Illustration 3:

Solve $\sin\theta = \frac{1}{2}$.

Solution:

$\because \sin\theta = \frac{1}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{6}$

$\therefore \theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I$

Illustration 4:

Solve: $\sec 2\theta = -\frac{2}{\sqrt{3}}$

Solution:

$\because \sec 2\theta = -\frac{2}{\sqrt{3}}$

$\Rightarrow \cos 2\theta = -\frac{\sqrt{3}}{2} \Rightarrow \cos 2\theta = \cos\frac{5\pi}{6}$

$\Rightarrow 2\theta = 2n\pi \pm \frac{5\pi}{6}, n \in I \Rightarrow \theta = n\pi \pm \frac{5\pi}{12}, n \in I$

Illustration 5:

Solve $\tan\theta = 1$

Solution:

$\because \tan\theta = 1$ ...(i)

$\tan\theta = \tan\frac{\pi}{4} \Rightarrow \theta = n\pi + \frac{\pi}{4}, n \in I$

Types of Trigonometric Equations:

Type -1(a) Solving Trigonometric Equations by Factorisation.

e.g. $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\therefore$ $(2\sin x - \cos x)(1 + \cos x) - (1 - \cos^2 x) = 0$

$\therefore$ $(1 + \cos x)(2\sin x - \cos x - 1 + \cos x) = 0$

$\therefore$ $(1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow$ $\cos x = -1$ or $\sin x = \frac{1}{2}$

$\Rightarrow$ $\cos x = -1 = \cos\pi \Rightarrow x = 2n\pi \pm \pi = (2n+1)\pi, n \in I$

or $\sin x = \frac{1}{2} = \sin\frac{\pi}{6} \Rightarrow x = k\pi + (-1)^k \frac{\pi}{6}, k \in I$

Type -1 (b) Solving of Trigonometric Equation by Reducing it to a Quadratic Equation.

e.g. $6 - 10\cos x = 3\sin^2 x$

$\therefore$ $6 - 10\cos x = 3 - 3\cos^2 x \Rightarrow 3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow$ $(3\cos x - 1)(\cos x - 3) = 0 \Rightarrow \cos x = \frac{1}{3}$ or $\cos x = 3$

Since $\cos x = 3$ is not possible as $-1 \le \cos x \le 1$

$\therefore$ $\cos x = \frac{1}{3} = \cos\left(\cos^{-1}\frac{1}{3}\right) \Rightarrow x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right), n \in I$

Illustration 6:

If $\frac{1}{6}$ $\sin\theta, \cos\theta$ and $\tan\theta$ are in G.P. then the general solution for θ is -

(A) $2n\pi \pm \frac{\pi}{3}$ (B) $2n\pi \pm \frac{\pi}{6}$ (C) $n\pi \pm \frac{\pi}{3}$ (D) none of these

Ans. (A)

Solution:

Since, $\frac{1}{6}\sin\theta, \cos\theta, \tan\theta$ are in G.P.

$\Rightarrow \cos^2\theta = \frac{1}{6}\sin\theta \,.\tan\theta \Rightarrow 6\cos^3\theta + \cos^2\theta - 1 = 0$

$\therefore (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ (other values of $\cos\theta$ are imaginary)

$\Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in I.$

Illustration 7:

Solve $\sin^2\theta - \cos\theta = \frac{1}{4}$ for θ and write the values of θ in the interval $0 \le \theta \le 2\pi$.

Solution:

The given equation can be written as

$1 - \cos^2\theta - \cos\theta = \frac{1}{4} \Rightarrow \cos^2\theta + \cos\theta - \frac{3}{4} = 0$

$\Rightarrow 4\cos^2\theta + 4\cos\theta - 3 = 0 \Rightarrow (2\cos\theta - 1)(2\cos\theta + 3) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}, -\frac{3}{2}$

$\cos\theta = -3/2$ is not possible as $-1 \le \cos\theta \le 1$

$\therefore \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

For the given interval, $n = 0$ and $n = 1$.

$\Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Illustration 8:

Find the number of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$.

Solution:

Here, $\tan x + \sec x = 2\cos x \Rightarrow \sin x + 1 = 2\cos^2 x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2}, -1$

But $\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$ for which $\tan x + \sec x = 2\cos x$ is not defined.

Thus $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow$ number of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$ is 2.

Illustration 9:

Solve the equation $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4$

Solution:

To solve this equation, we use the trigonometric identity ,

$\sin^2 x + \cos^2 x = 1$

Re-writing the given equation as,

$5\sin^2 x - 7\sin x \cdot \cos x + 16\cos^2 x = 4(\sin^2 x + \cos^2 x)$

$\Rightarrow \sin^2 x - 7\sin x\cos x + 12\cos^2 x = 0$

dividing by $\cos^2 x$ on both sides we get,

$\tan^2 x - 7\tan x + 12 = 0$

Now it can be factorized as:

$(\tan x - 3)(\tan x - 4) = 0$

$\Rightarrow \tan x = 3, 4$

i.e., $\tan x = \tan(\tan^{-1} 3)$ or $\tan x = \tan(\tan^{-1} 4)$

$\Rightarrow x = n\pi + \tan^{-1} 3$ or $x = n\pi + \tan^{-1} 4, n \in I$.

Type -2 Solving Trigonometric Equations by Introducing an Auxiliary Argument.

Consider, $a\sin\theta + b\cos\theta = c$ ...(i)

$$\therefore \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has a solution only if $|c| \le \sqrt{a^2+b^2}$

Let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$ & $\phi = \tan^{-1}\frac{b}{a}$

by introducing this auxiliary argument ϕ, equation (i) reduces to

$$\sin(\theta + \phi) = \frac{c}{\sqrt{a^2+b^2}}$$

Now this equation can be solved easily.

Illustration 10:

Find the number of distinct solutions of $\sec x + \tan x = \sqrt{3}$, where $0 \le x \le 3\pi$.

Solution:

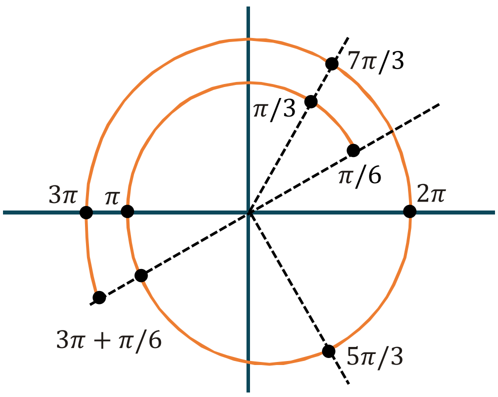
Here, $\sec x + \tan x = \sqrt{3} \Rightarrow 1 + \sin x = \sqrt{3}\cos x$

Or $\sqrt{3}\cos x - \sin x = 1$

dividing both sides by $\sqrt{a^2+b^2}$ i.e. $\sqrt{4} = 2$, we get

$$\Rightarrow \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x = \frac{1}{2} \Rightarrow \cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$



As $0 \le x \le 3\pi$

$$\frac{\pi}{6} \le x + \frac{\pi}{6} \le 3\pi + \frac{\pi}{6}$$

$$\Rightarrow x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \Rightarrow x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}$$

But at $x = \frac{3\pi}{2}$, $\tan x$ and $\sec x$ is not defined.

$\therefore$ Total number of solutions are 2.

Type -3 Solving Trigonometric Equations by Transforming Sum of Trigonometric Functions into Product.

e.g. $\cos 3x + \sin 2x - \sin 4x = 0$

$\cos 3x - 2\sin x \cos 3x = 0$

$\Rightarrow$ $(\cos 3x)(1 - 2\sin x) = 0$

$\Rightarrow$ $\cos 3x = 0$ or $\sin x = \frac{1}{2}$

$\Rightarrow$ $\cos 3x = 0 = \cos\frac{\pi}{2}$ or $\sin x = \frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow$ $3x = 2n\pi \pm \frac{\pi}{2}$ or $x = m\pi + (-1)^m \frac{\pi}{6}$

$\Rightarrow$ $x = \frac{2n\pi}{3} \pm \frac{\pi}{6}$ or $x = m\pi + (-1)^m \frac{\pi}{6}$; $(n, m \in I)$

Illustration 11:

Solve : $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$

Solution:

We have $\cos\theta + \cos 7\theta + \cos 3\theta + \cos 5\theta = 0$

$\Rightarrow 2\cos 4\theta \cos 3\theta + 2\cos 4\theta \cos\theta = 0 \Rightarrow \cos 4\theta(\cos 3\theta + \cos\theta) = 0$

$\Rightarrow \cos 4\theta(2\cos 2\theta \cos\theta) = 0$

$\Rightarrow$ Either $\cos\theta = 0 \Rightarrow \theta = (2n_1 + 1)\pi/2, n_1 \in I$

Or $\cos 2\theta = 0 \Rightarrow \theta = (2n_2 + 1)\frac{\pi}{4}, n_2 \in I$

or $\cos 4\theta = 0 \Rightarrow \theta = (2n_3 + 1)\frac{\pi}{8}, n_3 \in I$

Type -4 Solving Trigonometric Equations by Transforming a Product into Sum.

e.g. $\sin 5x . \cos 3x = \sin 6x . \cos 2x$

$\sin 8x + \sin 2x = \sin 8x + \sin 4x$

$\therefore$ $2\sin 2x . \cos 2x - \sin 2x = 0$

$\Rightarrow$ $\sin 2x(2\cos 2x - 1) = 0$

$\Rightarrow$ $\sin 2x = 0$ or $\cos 2x = \frac{1}{2}$

$\Rightarrow$ $\sin 2x = 0 = \sin 0$ or $\cos 2x = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow$ $2x = n\pi + (-1)^n \times 0, n \in I$ or $2x = 2m\pi \pm \frac{\pi}{3}, m \in I$

$\Rightarrow$ $x = \frac{n\pi}{2}, n \in I$ or $x = m\pi \pm \frac{\pi}{6}, m \in I$

Illustration 12:

Solve : $\cos\theta \cos2\theta \cos3\theta = \frac{1}{4}$; where $0 \le \theta \le \pi$.

Solution:

$\frac{1}{2}(2\cos\theta \cos3\theta)\cos2\theta = \frac{1}{4} \Rightarrow (\cos2\theta + \cos4\theta)\cos2\theta = \frac{1}{2}$

$\Rightarrow \frac{1}{2}[2\cos^2 2\theta + 2\cos4\theta \cos2\theta] = \frac{1}{2} \Rightarrow 1 + \cos4\theta + 2\cos4\theta \cos2\theta = 1$

$\therefore \cos4\theta(1 + 2\cos2\theta) = 0$

$\cos4\theta = 0$ or $(1 + 2\cos2\theta) = 0$

Now from the first equation: $2\cos4\theta = 0 = \cos(\pi/2)$

$\therefore 4\theta = \left(n+\frac{1}{2}\right)\pi \Rightarrow \theta = (2n+1)\frac{\pi}{8}, n \in I$

For $n = 0, \theta = \frac{\pi}{8}$; $n = 1, \theta = \frac{3\pi}{8}$; $n = 2, \theta = \frac{5\pi}{8}$; $n = 3, \theta = \frac{7\pi}{8}$ $(\because 0 \le \theta \le \pi)$

and from the second equation :

$\cos2\theta = -\frac{1}{2} = -\cos(\pi/3) = \cos(\pi - \pi/3) = \cos(2\pi/3)$

$\therefore 2\theta = 2k\pi \pm 2\pi/3 \quad \therefore \theta = k\pi \pm \frac{\pi}{3}, k \in I$

again for $k = 0, \theta = \frac{\pi}{3}$; $k = 1, \theta = \frac{2\pi}{3}$ $\qquad (\because 0 \le \theta \le \pi)$

$\therefore \theta = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$

Type -5 Solving Equations by a Change of Variable:

Equations of the form $P(\sin x \pm \cos x, \sin x.\cos x) = 0$, where $P(y, z)$ is a polynomial, can be solved by the substitution:

$\cos x \pm \sin x = t \Rightarrow 1 \pm 2\sin x.\cos x = t^2$.

e.g. $\sin x + \cos x = 1 + \sin x.\cos x$.

put $\sin x + \cos x = t \Rightarrow \sin^2 x + \cos^2 x + 2\sin x.\cos x = t^2$

$\Rightarrow 2\sin x \cos x = t^2 - 1 (\because \sin^2 x + \cos^2 x = 1)$

$\Rightarrow \sin x.\cos x = \left(\frac{t^2-1}{2}\right)$

Substituting above result in given equation, we get :

$t = 1 + \frac{t^2-1}{2} \Rightarrow 2t = t^2 + 1 \Rightarrow t^2 - 2t + 1 = 0$

$\Rightarrow (t-1)^2 = 0 \Rightarrow t = 1$

$\Rightarrow \sin x + \cos x = 1$

Dividing both sides by $\sqrt{1^2+1^2}$ i.e. $\sqrt{2}$, we get

$\Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}} \Rightarrow \cos x \cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \Rightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow x = 2n\pi$ or $x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}, n \in I$

Type -6 Solving Trigonometric Equations With the Use of the Boundness of the Functions Involved.

e.g. $\sin x\left(\cos\frac{x}{4}-2\sin x\right)+\left(1+\sin\frac{x}{4}-2\cos x\right).\cos x=0$

$\therefore\quad \sin x\cos\frac{x}{4}+\cos x\sin\frac{x}{4}+\cos x=2$

$\therefore\quad \sin\left(\frac{5x}{4}\right)+\cos x=2$

$\Rightarrow\quad \sin\left(\frac{5x}{4}\right)=1\ \&\ \cos x=1\ \ (\text{as } \sin\theta\le 1\ \&\ \cos\theta\le 1)$

Now consider

$\cos x=1\Rightarrow\ x=2\pi,4\pi,6\pi,8\pi\ \ldots\ldots$

and $\sin\frac{5x}{4}=1\Rightarrow x=\frac{2\pi}{5},\frac{10\pi}{5},\frac{18\pi}{5}\ \ldots\ldots$

Common solution to above APs will be the AP having

First term $=2\pi$

Common difference = LCM of 2π and $\frac{8\pi}{5}=\frac{40\pi}{5}=8\pi$

$\therefore$ General solution will be general term of this AP i.e. $2\pi+(8\pi)n, n\in I$

$\Rightarrow x=2(4n+1)\pi, n\in I$

Illustration 13:

Solve the equation $(\sin x+\cos x)^{1+\sin 2x}=2$, when $0\le x\le\pi$.

Solution:

We know, $-\sqrt{a^2+b^2}\le a\sin\theta+b\cos\theta\le\sqrt{a^2+b^2}$ and $-1\le\sin\theta\le 1$.

$\therefore$ $(\sin x+\cos x)$ admits the maximum value as $\sqrt{2}$

and $(1+\sin 2x)$ admits the maximum value as 2.

Also $(\sqrt{2})^2=2$.

$\therefore$ the equation could hold only when, $\sin x+\cos x=\sqrt{2}$ and $1+\sin 2x=2$

Now, $\sin x+\cos x=\sqrt{2}\Rightarrow\cos\left(x-\frac{\pi}{4}\right)=1$

$\Rightarrow x=2n\pi+\frac{\pi}{4}, n\in I$ ...(i)

And $1+\sin 2x=2\Rightarrow\sin 2x=1=\sin\frac{\pi}{2}$

$\Rightarrow 2x=m\pi+(-1)^m\frac{\pi}{2}, m\in I\Rightarrow x=\frac{m\pi}{2}+(-1)^m\frac{\pi}{4}$ ...(ii)

The value of x in $[0,\pi]$ satisfying equations (i) and (ii) is $x=\frac{\pi}{4}$ (when $n=0\ \&\ m=0$)

Note: $\sin x+\cos x=-\sqrt{2}$ and $1+\sin 2x=2$ also satisfies but as $x\ge 0$, this solution is not in domain.

Illustration 14:

Solve for x and y : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+1/2}\le 1$

Solution:

$2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+1/2}\le 1$ ...(i)

$2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\le 1$

Minimum value of $2^{\frac{1}{\cos^2 x}} = 2$

Minimum value of $\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{2}$

$\Rightarrow$ Minimum value of $2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+\frac{1}{2}}$ is 1

$\Rightarrow$ (i) is possible when $2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=1$

$\Rightarrow \cos^2 x = 1$ and $y = 1/2 \Rightarrow \cos x = \pm 1 \Rightarrow x = n\pi$, where $n \in I$.

Hence $x = n\pi, n \in I$ and $y = 1/2$.

Type -7 Solution of Trigonometric Equation of the Form of $f(x) = \sqrt{g(x)}$

Illustration 15:

The total number of solutions of $\cos x=\sqrt{1-\sin 2x}$ in $[0, 2\pi]$ is equal to

(A) 2 (B) 3 (C) 5 (D) none of these

Ans. (A)

Solution:

$\cos x=\sqrt{1-\sin 2x} = |\sin x - \cos x|$

(a) $\sin x < \cos x \Rightarrow x\in\left[0,\frac{\pi}{4}\right)\cup\left(\frac{5\pi}{4},2\pi\right]$ ...(i)

Then the given equation is $\cos x = \cos x - \sin x \Rightarrow \sin x = 0 \Rightarrow x = \pi, 2\pi$

$\Rightarrow x = 2\pi$ [From Eq. (i)]

(b) $\sin x \ge \cos x \Rightarrow x\in\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$

$\Rightarrow \cos x = \sin x - \cos x$

$\Rightarrow \tan x = 2$

$\Rightarrow x = \tan^{-1}(2)$ ...(ii)

Hence, there are two solutions.

Illustration 16:

The set of all x in $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ satisfying $|4\sin x - 1| < \sqrt{5}$ is given by

(A) $\left[\frac{\pi}{10},\frac{3\pi}{10}\right]$ (B) $\left(\frac{\pi}{10},\frac{3\pi}{10}\right)$ (C) $\left(-\frac{\pi}{10},\frac{3\pi}{10}\right)$ (D) none of these

Ans. (C)

Solution:

We have $|4\sin x - 1| < \sqrt{5}$

$\Rightarrow -\sqrt{5} < 4\sin x - 1 < \sqrt{5}$

$$\Rightarrow -\left(\frac{\sqrt{5}-1}{4}\right) < \sin x < \left(\frac{\sqrt{5}+1}{4}\right)$$

$$\Rightarrow -\sin\frac{\pi}{10} < \sin x < \cos\frac{\pi}{5}$$

$$\Rightarrow \sin\left(-\frac{\pi}{10}\right) < \sin x < \sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right)$$

$$\Rightarrow \sin\left(-\frac{\pi}{10}\right) < \sin x < \sin\left(\frac{3\pi}{10}\right)$$

$$\Rightarrow x \in \left(-\frac{\pi}{10},\frac{3\pi}{10}\right)$$

Illustration 17:

If $\sin x + \cos x = \sqrt{y+\frac{1}{y}}$ for $x \in [0,\pi]$, then

(A) $x = \pi/4$ (B) $y = 0$ (C) $y = 1$ (D) $x = 3\pi/4$

Ans. (A, C)

Solution:

$$y+\frac{1}{y} \geq 2 \Rightarrow \sqrt{y+\frac{1}{y}} \geq \sqrt{2}$$

But $\sin x + \cos x \leq \sqrt{2}$

$$\Rightarrow y+\frac{1}{y} = 2 \text{ and } \sin x + \cos x = \sqrt{2}$$

$$\Rightarrow y = 1 \text{ and } \sin\left(x+\frac{\pi}{4}\right) = 1 \text{ or } y = 1 \text{ and } x = \frac{\pi}{4}$$

Illustration 18:

$$\sqrt{(\sin x)} + 2^{1/4}\cos x = 0$$

Solution:

$$\sqrt{(\sin x)} + 2^{1/4}\cos x = 0$$

$\therefore \sqrt{(\sin x)} > 0$ and so $\cos x < 0$,

Also $\sin x > 0 \Rightarrow x$ lies in 2^{nd} quadrant

Equation (i) can be rewritten as $2^{1/4}\cos x = -\sqrt{(\sin x)}$

On squaring, we get $\sqrt{2}\cos^2 x = \sin x$

$\Rightarrow \sqrt{2}\sin^2 x + \sin x - \sqrt{2} = 0 \Rightarrow (\sqrt{2}\sin x - 1)(\sin x + \sqrt{2}) = 0$

$\sin x \neq -\sqrt{2}$, therefore $\sin x = 1/\sqrt{2}$

$\Rightarrow x = 3\pi/4$ and so, the general value of x is given by $x = 2n\pi + 3\pi/4, n \in \mathbb{Z}$

Illustration 19:

Equation $\sqrt{1-\sin 2x} = \sin x$ has how many total numbers of solutions for $x \in [0, \pi/4]$.

Solution:

$\sqrt{1 - \sin 2x} = \sin x$

$\Rightarrow \sqrt{(\sin x - \cos x)^2} = \sin x$

$\Rightarrow |\sin x - \cos x| = \sin x$

$\Rightarrow \cos x - \sin x = \sin x$

$\Rightarrow 2\sin x = \cos x$

$\Rightarrow \tan x = \frac{1}{2}$ which has only one solution for $x \in [0, \pi/4]$ for these values of x.

Some More Solved Examples:

Illustration 20:

The most general solution of $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$ is:

(A) $n\pi + \frac{7\pi}{4}, n \in I$

(B) $n\pi + (-1)^n \frac{7\pi}{4}, n \in I$

(C) $2n\pi + \frac{7\pi}{4}, n \in I$

(D) $n\pi + (-1)^n \frac{\pi}{4}, n \in I$

Ans. (C)

Solution:

$\tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ in $[0, 2\pi]$

$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}, \frac{7\pi}{4}$ in $[0, 2\pi]$

common value is $x = \frac{7\pi}{4}$

general solution is $2n\pi + \frac{7\pi}{4}, n \in I$.

Illustration 21:

The most general value of θ which satisfies both the equations $\tan\theta = \sqrt{3}$ and $\text{cosec}\,\theta = -\frac{2}{\sqrt{3}}$ is-

(A) $n\pi + \frac{4\pi}{3}; n \in I$ (B) $n\pi + \frac{2\pi}{3}; n \in I$ (C) $2n\pi + \frac{4\pi}{3}; n \in I$ (D) $2n\pi + \frac{2\pi}{3}; n \in I$

Ans. (C)

Solution:

$\tan\theta = \sqrt{3}$ and $\sin\theta = -\dfrac{\sqrt{3}}{2}$ are both satisfied for only one value of $\theta = \dfrac{4\pi}{3}, \theta \in [0, 2\pi]$

$\therefore$ general solution is $2n\pi + \dfrac{4\pi}{3}, n \in I$

Illustration 22:

If $0 \le x \le 2\pi, 0 \le y \le 2\pi$ and $\sin x + \sin y = 2$ then the value of $x + y$ is-

(A) π (B) $\dfrac{\pi}{2}$ (C) 3π (D) 0

Ans. (A)

Solution:

As $\sin x + \sin y = 2, \sin x = \sin y = 1$

$\therefore x = \dfrac{\pi}{2}, y = \dfrac{\pi}{2}$

$\therefore x + y = \pi$

Illustration 23:

The solution set of equation $\cos^5 x = 1 + \sin^4 x$ is

(A) $n\pi, n \in I$ (B) $2n\pi, n \in I$ (C) $4n\pi, n \in I$ (D) $\dfrac{n\pi}{2}, n \in I$

Ans. (B)

Solution:

$\cos^5 x = 1 + \sin^4 x$

As $\cos^5 x \le 1$

$\therefore \cos x = 1, \sin x = 0$

$\Rightarrow x = 2n\pi, n \in I$

Illustration 24:

The general solution of the equation $2\cos 2x = 3.2\cos^2 x - 4$ is

(A) $x = 2n\pi, n \in I$ (B) $x = n\pi, n \in I$

(C) $x = \dfrac{n\pi}{4}, n \in I$ (D) $x = \dfrac{n\pi}{2}, n \in I$

Ans. (B)

Solution:

$2\cos 2x = 6\cos^2 x - 4$

$\Rightarrow 2(2\cos^2 x - 1) = 6\cos^2 x - 4$

$\Rightarrow 2\cos^2 x = 2 \Rightarrow \cos^2 x = 1 \Rightarrow x = n\pi.$

Illustration 25:

The general solution of equation $\sin x + \sin 5x = \sin 2x + \sin 4x$ is :

(A) $\dfrac{n\pi}{2}; n \in I$ (B) $\dfrac{n\pi}{5}; n \in I$ (C) $\dfrac{n\pi}{3}; n \in I$ (D) $\dfrac{2\ n\pi}{3}; n \in I$

Ans. (C)

Solution:

$\sin x + \sin 5x = \sin 2x + \sin 4x \Rightarrow 2\sin 3x \cos 2x = 2\sin 3x \cos x$

$\Rightarrow \sin 3x = 0$ or $\cos 2x = \cos x \Rightarrow 3x = n\pi$ or $2x = 2n\pi \pm x$

$\Rightarrow x = \frac{n\pi}{3}, 2n\pi, \frac{2n\pi}{3}$

$\Rightarrow x = \frac{n\pi}{3}$ (It includes all three possible solutions)

Illustration 26:

$\cos 4x \cos 8x - \cos 5x \cos 9x = 0$ if

(A) $\cos 12x = \cos 14x$ (B) $\sin 13x = 0$

(C) $\sin x = 0$ (D) all of these

Ans. (D)

Solution:

$\cos 4x \cos 8x - \cos 5x \cos 9x = 0$

$\Rightarrow 2\cos 4x \cos 8x = 2\cos 5x \cos 9x \Rightarrow \cos 12x + \cos 4x = \cos 14x + \cos 4x$

$\Rightarrow 14x = 2n\pi \pm (12x) \Rightarrow 2x = 2n\pi$ or $26x = 2n\pi$

$\Rightarrow x = n\pi$ or $\frac{n\pi}{13}$ $\therefore \sin x = 0$ or $\sin 13x = 0$

Illustration 27:

If $x \in \left[0, \frac{\pi}{2}\right]$, the number of solutions of the equation $\sin 7x + \sin 4x + \sin x = 0$ is:

(A) 3 (B) 5 (C) 6 (D) 2

Ans. (B)

Solution:

$\sin 7x + \sin 4x + \sin x = 0$

$\Rightarrow 2\sin 4x \cos 3x + \sin 4x = 0$

$\Rightarrow \sin 4x = 0$ or $\cos 3x = -\frac{1}{2} \Rightarrow 4x = n\pi$ or $3x = 2n\pi \pm \frac{2\pi}{3}$

$\Rightarrow x = \frac{n\pi}{4}, \frac{2n\pi}{3} \pm \frac{2\pi}{9} = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{9}, \frac{4\pi}{9}.$

Illustration 28:

General solution of equation $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \ \tan 2x} = 1$ is

(A) $n\pi + \frac{\pi}{4}, n \in I$ (B) $n\pi - \frac{\pi}{4}, n \in I$ (C) $n\pi, n \in I$ (D) ϕ

Ans. (D)

Solution:

$\frac{\tan 3x - \tan 2x}{1 + \tan 3x \ \tan 2x} = 1 \Rightarrow \tan(3x - 2x) = 1$

$\Rightarrow x = n\pi + \frac{\pi}{4}, n \in I$ But then $2x = 2n\pi + \frac{\pi}{2}$

at which $\tan 2x$ is undefined, so no solution is possible.

Illustration 29:

General solution of equation $5\sin\theta + 2\cos\theta = 5$ is

(A) $2n\pi - \frac{\pi}{2}, n \in I$

(B) $2n\pi - 2\alpha$ where $\alpha = \tan^{-1}\frac{2}{7}, n \in I$

(C) $2n\pi + \frac{\pi}{2}, n \in I$

(D) $2n\pi + 2\alpha$ where $\alpha = \tan^{-1}\frac{2}{7}, n \in I$

Ans. (C)

Solution:

$5\sin\theta + 2\cos\theta = 5$

$\Rightarrow \frac{5}{\sqrt{29}}\sin\theta + \frac{2}{\sqrt{29}}\cos\theta = \frac{5}{\sqrt{29}} \Rightarrow \sin\phi\sin\theta + \cos\phi\cos\theta = \frac{5}{\sqrt{29}}$

$\Rightarrow \cos(\theta - \phi) = \sin\phi = \cos\left(\frac{\pi}{2} - \phi\right) \Rightarrow \theta - \phi = 2n\pi \pm \left(\frac{\pi}{2} - \phi\right)$

$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{2} \mp \phi + \phi$

$\theta = 2n\pi + \frac{\pi}{2}, 2n\pi - \frac{\pi}{2} + 2\phi \Rightarrow \theta = 2n\pi + \frac{\pi}{2}, 2n\pi - \frac{\pi}{2} + 2\phi$

For $\theta = 2n\pi - \frac{\pi}{2} + 2\phi$,

We have $\theta = 2n\pi + 2\left(\phi - \frac{\pi}{4}\right) = 2n\pi + 2\left(\tan^{-1}\frac{5}{2} - \tan^{-1}1\right)$

$= 2n\pi + 2\tan^{-1}\left(\frac{\frac{5}{2} - 1}{1 + \frac{5}{2}}\right) = 2n\pi + 2\tan^{-1}\left(\frac{3}{7}\right)$

$\therefore\ \theta = 2n\pi + \frac{\pi}{2}$ or $2n\pi + 2\alpha$ where $\tan^{-1}\frac{3}{7} = \alpha$

Illustration 30:

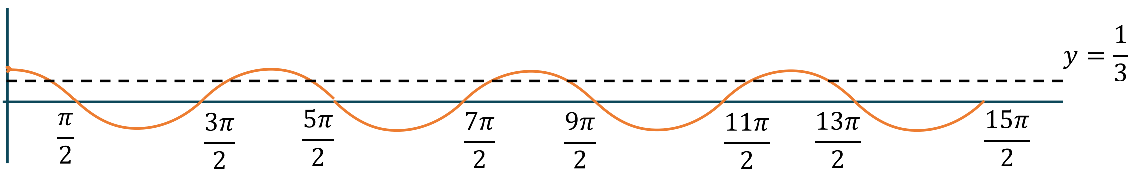
If $2\tan^2 x - 5\sec x - 1 = 0$ has 7 different roots in $\left[0, \frac{n\pi}{2}\right]$, $n \in N$, then greatest value of n is

(A) 8 (B) 10 (C) 13 (D) 15

Ans. (D)

Solution:

$2\tan^2 x - 5\sec x - 1 = 0$



$\Rightarrow 2(\sec^2 x - 1) - 5\sec x - 1 = 0$

$\Rightarrow 2\sec^2 x - 5\sec x - 3 = 0$

$\Rightarrow \sec x = \frac{6}{2}, \frac{-1}{2} = 3, \frac{-1}{2}$

$\Rightarrow \sec x = 3 \qquad \left(\sec x \neq \frac{-1}{2}\right)$

$\Rightarrow \cos x = \frac{1}{3} \Rightarrow 7$ solutions in $\left[0, \frac{15\pi}{2}\right]$

$\therefore n = 15$

Illustration 31:

$\frac{\cos 3\theta}{2\cos 2\theta - 1} = \frac{1}{2}$ if

(A) $\theta = n\pi + \frac{\pi}{3}, n \in I$ (B) $\theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

(C) $\theta = 2n\pi \pm \frac{\pi}{6}, n \in I$ (D) $\theta = n\pi + \frac{\pi}{6}, n \in I$

Ans. (B)

Solution:

$\frac{\cos 3\theta}{2\cos 2\theta - 1} = \frac{1}{2}$

$\Rightarrow 2(4\cos^3\theta - 3\cos\theta) = 2(2\cos^2\theta - 1) - 1$

$\Rightarrow 8\cos^3\theta - 4\cos^2\theta - 6\cos\theta + 3 = 0$

$\Rightarrow (4\cos^2\theta - 3)(2\cos\theta - 1) = 0 \Rightarrow \cos\theta = \frac{1}{2}, \pm\frac{\sqrt{3}}{2}$

But when $\cos\theta = \pm\frac{\sqrt{3}}{2}$

then $2\cos 2\theta - 1 = 0$

$\therefore$ rejecting this value,

$\cos\theta = \frac{1}{2}$ is valid only $\Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$

Illustration 32:

The equation $\sin 3\theta = 4\sin\theta . \sin 2\theta . \sin 4\theta$ in $0 \leq \theta \leq \pi$ has:

(A) 2 real solutions (B) 4 real solutions (C) 6 real solutions (D) 8 real solutions.

Ans. (D)

Solution:

$\sin 3\theta = 4\sin\theta\sin 2\theta\sin 4\theta$

$\Rightarrow \sin 3\theta = (2\sin\theta)(2\sin 2\theta\sin 4\theta) \Rightarrow 3\sin\theta - 4\sin^3\theta = 2\sin\theta(\cos 2\theta - \cos 6\theta)$

$\Rightarrow 3 - 4\sin^2\theta = 2(\cos 2\theta - \cos 6\theta)$ or $\sin\theta = 0$

$\Rightarrow 3 - 2(1 - \cos 2\theta) = 2\cos 2\theta - 2\cos 6\theta$ or $\sin\theta = 0$

$\Rightarrow 1 = -2\cos 6\theta \Rightarrow \cos 6\theta = \frac{-1}{2}$ or $\sin\theta = 0$

$\therefore \sin\theta = 0$ or $\cos 6\theta = \frac{-1}{2}$

$\Rightarrow \theta = n\pi$ or $\theta = \frac{2n\pi \pm \left(\frac{2\pi}{3}\right)}{6} = \frac{n\pi}{3} \mp \frac{\pi}{9} \Rightarrow \theta = 0, \pi, \frac{\pi}{9}, \frac{\pi}{3} \mp \frac{\pi}{9}, \frac{2\pi}{3} \mp \frac{\pi}{9}, \pi - \frac{\pi}{9}$

So, eight solutions.

Illustration 33:

Total number of solution of $16^{\cos^2 x}+16^{\sin^2 x}=10$ in $x \in [0,3\pi]$ is equal to-

(A) 4 (B) 8 (C) 12 (D) 16

Ans. (C)

Solution:

$16^{\cos^2 x}+16^{1-\cos^2 x}=10$

Let $16^{\cos^2 x}=y,\ \ y+\dfrac{16}{y}=10, y^2-10y+16=0$

$\Rightarrow y=2,8 \Rightarrow 16^{\cos^2 x}=2,8 \Rightarrow \cos^2 x=\dfrac{1}{4},\dfrac{3}{4}$

$$\left.\begin{array}{l}\cos^2 x=\dfrac{1}{4}\Rightarrow x=\dfrac{\pi}{3},\dfrac{2\pi}{3},\dfrac{4\pi}{3},\dfrac{5\pi}{3},\dfrac{7\pi}{3},\dfrac{8\pi}{3}\\ \\ \cos^2 x=\dfrac{3}{4}\Rightarrow x=\dfrac{\pi}{6},\dfrac{5\pi}{6},\dfrac{7\pi}{6},\dfrac{11\pi}{6},\dfrac{13\pi}{6},\dfrac{17\pi}{6}\end{array}\right\}\text{12 solutions}$$

Illustration 34:

If $\tan\theta+\tan 4\theta+\tan 7\theta=\tan\theta\tan 4\theta\tan 7\theta$, then θ =

(A) $\dfrac{n\pi}{4}, n\in I$ (B) $\dfrac{n\pi}{7}, n\in I$

(C) $\dfrac{n\pi}{12}; n\neq 6(2k+1), (n,k\in I)$ (D) $n\pi, n\in I$

Ans. (C)

Solution:

$$\tan(7\theta+4\theta+\theta)=\frac{\tan 7\theta+\tan 4\theta+\tan\theta-\tan 7\theta\tan 4\theta\tan\theta}{1-\tan 7\theta\tan 4\theta-\tan 4\theta\tan\theta-\tan\theta\tan 7\theta}$$

$\Rightarrow \tan 12\theta=0$

$\therefore 12\theta=n\pi, \theta=\dfrac{n\pi}{12}$

Clearly $\dfrac{n\pi}{12}\neq(2k+1)\dfrac{\pi}{2}\Rightarrow n\neq 6(2k+1)$

$n.k\in I$

Illustration 35:

$\sin^2 x-\cos 2x=2-\sin 2x$ if

(A) $x=\dfrac{n\pi}{2}, n\in I$ (B) $\tan x=\dfrac{1}{2}$

(C) $x=(2n+1)\dfrac{\pi}{2}, n\in I$ (D) $x=n\pi+(-1)n\sin^{-1}\dfrac{2}{3}, n\in I$

Ans. (C)

Solution:

$\sin^2 x - \cos 2x = 2 - \sin 2x$

$\Rightarrow \sin^2 x - (1 - 2\sin^2 x) = 2 - 2\sin x \cos x$

$\Rightarrow 3\sin^2 x + 2\sin x \cos x = 3$

case-I: $\cos x \neq 0$ $\quad \therefore 3\tan^2 x + 2\tan x = 3(1 + \tan^2 x) \Rightarrow \tan x = \frac{3}{2}$

case-II: $\cos x = 0$ $\quad \therefore 3(1) + 2(\pm 1)(0) = 3$ which is true

$\therefore x = (2n+1)\frac{\pi}{2}$

Illustration 36:

Solve : $\cos x + \sqrt{3}\sin x = \sqrt{3}$.

Solution:

$\because \cos x + \sqrt{3}\sin x = \sqrt{3}$ by changing $\sin x$ and $\cos x$ to there half angles ...(i)

$\because \cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ & $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\therefore$ equation (i) becomes

$$\Rightarrow \left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + \sqrt{3}\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = \sqrt{3} \quad \text{...(ii)}$$

Let $\tan\frac{x}{2} = t$

$\therefore$ Equation (ii) becomes $\left(\dfrac{1-t^2}{1+t^2}\right) + \sqrt{3}\left(\dfrac{2t}{1+t^2}\right) = \sqrt{3}$

$\Rightarrow (\sqrt{3}+1)t^2 - 2\sqrt{3}\,t + \sqrt{3} - 1 = 0 \Rightarrow (\sqrt{3}+1)t^2 - (\sqrt{3}+1)t + (1-\sqrt{3})t + \sqrt{3} - 1 = 0$

$\Rightarrow [(\sqrt{3}+1)t + (1-\sqrt{3})](t-1) = 0 \Rightarrow \tan\frac{x}{2} = 1$ or $\tan\left(\frac{x}{2}\right) = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

So $x = 2n\pi + \frac{\pi}{2}$ or $x = 2n\pi + \frac{\pi}{6}$

Illustration 37:

Solve : $5(\sin x + \cos x) = 5 + 2\sin x.\cos x$

Solution:

$\because 5(\sin x + \cos x) = 5 + 2\sin x.\cos x$ ...(i)

Let $\sin x + \cos x = t$

$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x.\cos x = t^2$

$\Rightarrow \sin x.\cos x = \dfrac{t^2-1}{2}$

Now put $\sin x + \cos x = t$ and $\sin x.\cos x = \frac{t^2-1}{2}$ in (i)

$\Rightarrow t^2 - 5t + 4 = 0$

$\Rightarrow t = 1$ or $t = 4$

$\Rightarrow t \neq 4$ because $-\sqrt{2} \leq \sin x + \cos x \leq \sqrt{2}$

So $t = 1$

$\Rightarrow \sin x + \cos x = 1$ ...(ii)

divide both sides of equation (ii) by $\sqrt{2}$, we get

$\Rightarrow \sin x.\ \frac{1}{\sqrt{2}} + \cos x.\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\sin\left(x+\frac{\pi}{4}\right)=\sin\frac{\pi}{4}$

$x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}$

Illustration 38:

Solve : $\cos^{50}x - \sin^{50}x = 1$

Solution:

$\cos^{50}x - \sin^{50}x = 1$

$\Rightarrow \cos^{50}x = \sin^{50}x + 1$

$\Rightarrow$ L.H.S. ≤ 1 & R.H.S. ≥ 1

so, both sides should be equal to 1

$\cos^{50}x = 1$ & $\sin^{50}x = 0$

Or $\sin x = 0$ or $x = n\pi$ where $n \in I$

Illustration 39:

Find number of values of x in the interval $[0, 10\pi]$ satisfying the equation $1 + 2\text{cosec}\, x = -\frac{\sec^2\frac{x}{2}}{2}$.

Solution:

$1 + 2\ \text{cosec}\ x = \frac{-\sec^2 x/2}{2}$

$\sin x + 2 = -\tan\frac{x}{2}$

$\frac{2\tan x/2}{1+\tan^2 x/2} + 2 = -\ \tan\frac{x}{2}$

$2\tan\frac{x}{2} + 2 + 2\tan^2\frac{x}{2} = -\tan\frac{x}{2} - \tan^3\frac{x}{2}$

$\tan^3\frac{x}{2} + 2\tan^2\frac{x}{2} + 3\tan\frac{x}{2} + 2 = 0$

$\left(\tan\frac{x}{2}+1\right)\left(\tan^2\frac{x}{2}+\tan\frac{x}{2}+2\right)=0$

$\tan\frac{x}{2}=-1$

$\frac{x}{2}= n\pi - \frac{\pi}{4}$

$\boxed{x=2n\pi-\frac{\pi}{2}}$

Now put $n = 1, 2, 3, 4, 5$

$x=\frac{3\pi}{2},\frac{7\pi}{2},\frac{11\pi}{2},\frac{15\pi}{2},\frac{19\pi}{2}$

Illustration 40:

Find number of values of x in the interval $[0, 5\pi]$ satisfying the equation $\frac{\sqrt{3}}{2}\sin x - \cos x = \cos^2 x$.

Solution:

$\frac{\sqrt{3}}{2}\sin x-\cos x=\cos^2 x$

$\Rightarrow \frac{3}{4}(1-\cos^2 x)= (\cos^2 x + \cos x)^2 \Rightarrow 3(1 - \cos^2 x) - 4\cos^2 x\,(1 + \cos x)^2 = 0$

$\Rightarrow (1 + \cos x)\,[\,3 - 3\cos x - 4\cos^2 x - 4\cos^3 x] = 0 \Rightarrow (1 + \cos x)[-4\cos^3 x - 4\cos^2 x - 3\cos x + 3] = 0$

$\Rightarrow (1 + \cos x)(\cos x - \frac{1}{2})(4\cos^2 x + 6\cos x + 6) = 0$

$\Rightarrow \cos x = -1, \cos x = \frac{1}{2}, 4\cos^2 x + 6\cos x + 6 = 0$

$x = \pi, \frac{\pi}{3}; x = \frac{5\pi}{3}; D < 0$

But $\frac{5\pi}{3}$ does not satisfy the equation

$\therefore x = 2n\pi + \frac{\pi}{3}$

Or $x = 2n\pi \pm \pi, n \in I$

Now vary n.

$x=\frac{\pi}{3},2\pi+\frac{\pi}{3},4\pi+\frac{\pi}{3},\pi,3\pi,5\pi$

Illustration 41:

Find number of values of x in the interval $[0, 2\pi]$ satisfying the equation $\cot x - 2\sin 2x = 1$.

Solution:

$\cot x - 2\sin 2x = 1$

$\Rightarrow \frac{\cos x-\sin x}{\sin x} = 2\sin 2x \Rightarrow \cos x - \sin x = 2\sin 2x \sin x$

$\Rightarrow \cos x - \sin x = -\cos 3x + \cos(x)$

$\Rightarrow \sin x = \cos 3x \Rightarrow \cos\left(\frac{\pi}{4}-x\right)\sin\left(2x-\frac{\pi}{4}\right) = 0$

$\therefore \boxed{x=\frac{\pi}{8}+\frac{n\pi}{2} \text{ or } x=\frac{3\pi}{4}+n\pi, n\in I}$

Now vary n.

$x=\frac{\pi}{8},\frac{5\pi}{8},\frac{9\pi}{8},\frac{13\pi}{8},\frac{3\pi}{4},\frac{7\pi}{4}$

Illustration 42:

Find number of values of x in the interval $(-2\pi, 2\pi)$ satisfying the equation $\sqrt{13-18\tan x} = 6\tan x - 3$.

Solution:

$\sqrt{13-18\tan x} = 6\tan x - 3$

$13 - 18\tan x = 36\tan^2 x + 9 - 36\tan x$

$36\tan^2 x - 18\tan x - 4 = 0$

$18\tan^2 x - 9\tan x - 2 = 0$

$\tan x = \frac{2}{3}, \frac{-1}{6}$

But $\tan x = \frac{-1}{6}$ does not satisfy the original equation

Number of solutions in $(-2\pi, 2\pi) = 4$.

Illustration 43:

Find number of values of θ in the interval $[0, \pi]$ satisfying the equation $\sin\theta + \sin 5\theta = \sin 3\theta$.

Solution:

$\sin\theta + \sin 5\theta = \sin 3\theta$

$2\sin 3\theta \times \cos 2\theta - \sin 3\theta = 0$

$\sin 3\theta[2\cos 2\theta - 1] = 0$

$\sin 3\theta = 0$ or $\cos 2\theta = \frac{1}{2}$

$3\theta = k\pi$ or $2\theta = 2n\pi + \frac{\pi}{3}$

$\therefore$ Solution in $[0, \pi]$ are, $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

Solution of Trigonometric Inequalities and System of Inequalities:

There is no general rule to solve trigonometric inequations and the same rules of algebra are valid provided the domain and range of trigonometric functions should be kept in mind.

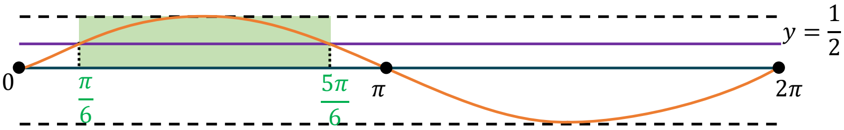
Illustration 44:

Find the solution set of inequality $\sin x > 1/2$.

Solution:

When $\sin x = \frac{1}{2}$, the two values of x between 0 and 2π are $\pi/6$ and $5\pi/6$.

From the graph of $y = \sin x$, it is obvious that between 0 and 2π,



$\sin x > \frac{1}{2}$ for $\pi/6 < x < 5\pi/6$

Hence, $\sin x > 1/2$

$\Rightarrow 2n\pi + \pi/6 < x < 2n\pi + 5\pi/6, n \in I$

Thus, the required solution set is $\bigcup_{n\in I}\left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$

Illustration 45:

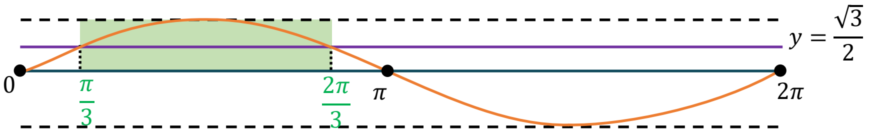
Set of values of x satisfying inequality $2\sin x - \sqrt{3} \geq 0$ is

(A) $\bigcup\limits_{n\in I}\left[2n\pi+\frac{2\pi}{3},2n\pi+\frac{4\pi}{3}\right]$ (B) $\bigcup\limits_{n\in I}\left[2n\pi+\frac{\pi}{3},2n\pi+\frac{2\pi}{3}\right]$

(C) $\bigcup\limits_{n\in I}\left[2n\pi-\frac{\pi}{3},2n\pi+\frac{\pi}{3}\right]$ (D) $\bigcup\limits_{n\in I}\left[2n\pi,2n\pi+\frac{\pi}{3}\right]$

Ans. (B)

Solution:



$2\sin x - \sqrt{3} \geq 0$

$\sin x \geq \frac{\sqrt{3}}{2} \Rightarrow x \in \left[\frac{\pi}{3},\frac{2\pi}{3}\right]$ in $[0,2\pi)$;

The general solution is

$x \in \bigcup\limits_{n\in I}\left[2n\pi+\frac{\pi}{3},2n\pi+\frac{2\pi}{3}\right]$

Illustration 46:

The set of values of x for which $\sin x.\cos^3 x > \cos x.\sin^3 x, 0 \leq x \leq 2\pi$, is-

(A) $(0,\pi)$ (B) $\left(\pi,\frac{3\pi}{2}\right)$ (C) $\left(\frac{\pi}{4},\pi\right)$ (D) $\left(0,\frac{\pi}{4}\right)$

Ans. (D)

Solution:

$\sin x \cos x\,(\cos^2 x - \sin^2 x) > 0$

$\therefore \frac{1}{2}\sin 2x \cos 2x > 0$

$\Rightarrow \frac{1}{4}\sin 4x > 0$

This is satisfied when

$4x \in (0,\pi)\cup(2\pi,3\pi)\cup(4\pi,5\pi)\cup(6\pi,7\pi)$

$\Rightarrow x \in \left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)\cup\left(\pi,\frac{5\pi}{4}\right)\cup\left(\frac{3\pi}{2},\frac{7\pi}{4}\right)$

Illustration 47:

Find the value of x in the interval $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$ for which $\sqrt{2}\sin 2x + 1 \leq 2\sin x + \sqrt{2}\cos x$

Solution:

We have, $\sqrt{2}\sin 2x+1\leq 2\sin x+\sqrt{2}\cos x \Rightarrow 2\sqrt{2}\sin x\cos x-2\sin x-\sqrt{2}\cos x+1\leq 0$

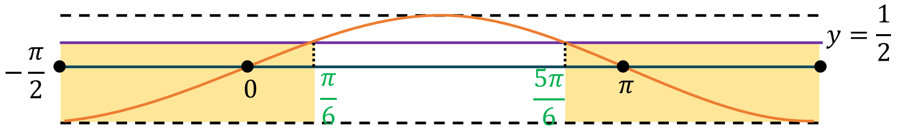
$\Rightarrow 2\sin x(\sqrt{2}\cos x-1)-1(\sqrt{2}\cos x-1)\leq 0 \Rightarrow (2\sin x-1)(\sqrt{2}\cos x-1)\leq 0$

$\Rightarrow \left(\sin x-\frac{1}{2}\right)\left(\cos x-\frac{1}{\sqrt{2}}\right)\leq 0$

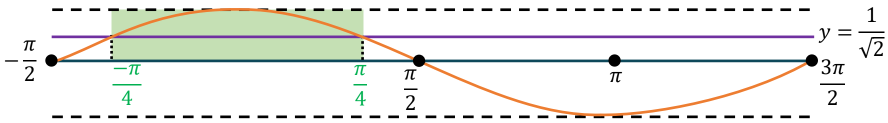
Above inequality holds when :

Case-I: $\sin x - \frac{1}{2} \le 0$ and $\cos x - \frac{1}{\sqrt{2}} \ge 0 \Rightarrow \sin x \le \frac{1}{2}$ and $\cos x \ge \frac{1}{\sqrt{2}}$

Now considering the given interval of x :



for $\sin x \le \frac{1}{2}$: $x \in \left[-\frac{\pi}{2}, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, \frac{3\pi}{2}\right]$ and

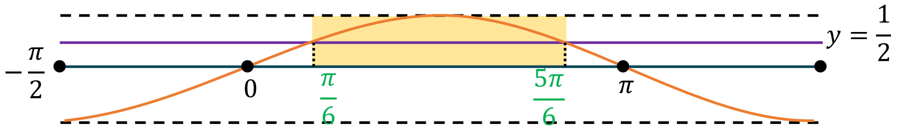


for $\cos x \ge \frac{1}{\sqrt{2}}$: $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

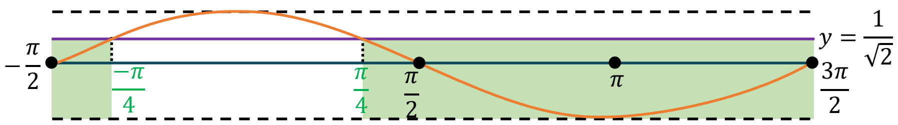
For both results to simultaneously hold true : $x \in \left[-\frac{\pi}{4}, \frac{\pi}{6}\right]$

Case-II: $\sin x - \frac{1}{2} \ge 0$ and $\cos x \le \frac{1}{\sqrt{2}}$

Again, for the given interval of x :



for $\sin x \ge \frac{1}{2}$: $x \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ and



for $\cos x \le \frac{1}{\sqrt{2}}$: $x \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{3\pi}{2}\right]$

For both results to simultaneously hold true : $x \in \left[\frac{\pi}{4}, \frac{5\pi}{6}\right]$

$\therefore$ Given inequality holds for $x \in \left[-\frac{\pi}{4}, \frac{\pi}{6}\right] \cup \left[\frac{\pi}{4}, \frac{5\pi}{6}\right]$

Illustration 48:

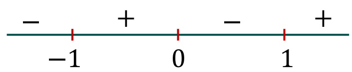
Find the values of α lying between 0 and π for which the inequality: $\tan\alpha > \tan^3\alpha$ is valid.

Solution:

We have: $\tan\alpha - \tan^3\alpha > 0 \Rightarrow \tan\alpha\,(1 - \tan^2\alpha) > 0$

$\Rightarrow (\tan\alpha)(\tan\alpha + 1)(\tan\alpha - 1) < 0$

So $\tan\alpha < -1, 0 < \tan\alpha < 1$



$\therefore$ Given inequality holds for $\alpha \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

Illustration 49:

Solve the inequality : $2\sin^2 x - 7\sin x + 3 > 0$

Solution:

$2\sin^2 x - 7\sin x + 3 > 0$

$\Rightarrow 2\sin^2 x - 6\sin x - \sin x + 3 > 0 \Rightarrow 2\sin x(\sin x - 3) - 1(\sin x - 3) > 0$

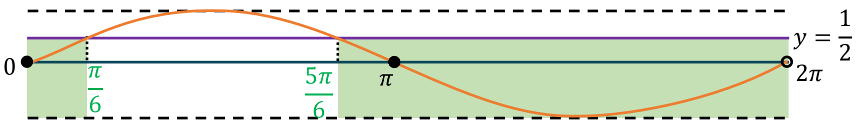
$\Rightarrow (2\sin x - 1)(\sin x - 3) > 0$

$\because \sin x \in [-1, 1]\ \forall\ x \in R$

$\therefore \sin x - 3 < 0\ \forall\ x \in R$

$\therefore$ the above inequality $\Rightarrow\ 2\sin x - 1 < 0$

$\Rightarrow\ \sin x < \dfrac{1}{2}$



From the graph, it is clear that, the solution set in $[0, 2\pi)$ is $\left[0, \dfrac{\pi}{6}\right) \cup \left(\dfrac{5\pi}{6}, 2\pi\right)$

Hence, the general solution is

$$x \in \bigcup_{n\in I}\left[2n\pi, 2n\pi + \frac{\pi}{6}\right) U \left(2n\pi + \frac{5\pi}{6}, 2n\pi + 2\pi\right)$$

Illustration 50:

Solve the inequality : $\cos x + \cos 2x + \cos 3x > 0$

Solution:

Transforming the sum of the end terms into a product, we get an inequality

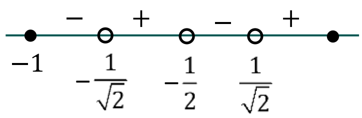
$\cos 2x + 2\cos 2x \cos x > 0$

Substituting $\cos 2x$ before the brackets, we obtain

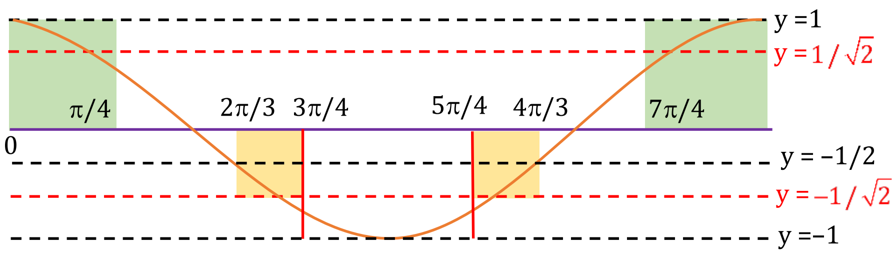
$\cos 2x(2\cos x + 1) > 0$

$\Rightarrow (2\cos^2 x - 1)(2\cos x + 1) > 0$

$$\Rightarrow 4\left(\cos x - \frac{1}{\sqrt{2}}\right)\left(\cos x + \frac{1}{\sqrt{2}}\right)\left(\cos x + \frac{1}{2}\right) > 0$$



Hence, $\cos x \in \left(\dfrac{-1}{\sqrt{2}}, \dfrac{-1}{2}\right) \cup \left(\dfrac{1}{\sqrt{2}}, 1\right)$



From the graph it is clear that in $[0, 2\pi)$, the solution set is

$$\left[0, \frac{\pi}{4}\right) \cup \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{4\pi}{3}\right) \cup \left[\frac{7\pi}{4}, 2\pi\right)$$

$\therefore$ The general solution set is

$$\bigcup_{n\in I}\left[2n\pi, 2n\pi + \frac{\pi}{4}\right) \cup \left(2n\pi + \frac{2\pi}{3}, 2n\pi + \frac{3\pi}{4}\right) \cup \left(2n\pi + \frac{5\pi}{4}, 2n\pi + \frac{4\pi}{3}\right) \cup \left(2n\pi + \frac{7\pi}{4}, 2n\pi + 2\pi\right)$$