CHAPTER 7

Gravitation

Chapter Highlights

The universal law of gravitation. Acceleration due to gravity and its variation with altitude and depth. Kepler's laws of planetary motion. Gravitational potential energy; gravitational potential. Escape velocity. Orbital velocity of a satellite. Geo-stationary satellites.

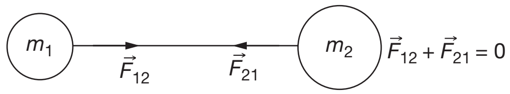
NEWTON'S LAW OF GRAVITATION

Newton's law of gravitation states that every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the particles.

Therefore, from Newton's law of gravitation

$$\vec{F} = \frac{Gm_1 m_2}{r^2}\hat{r} \qquad (7.1)$$

where G is called the gravitational constant and $\hat{r}$ is the unit vector along the line joining the two mass particles.



The gravitational force between two particles forms an action-reaction pair.

SOLVED EXAMPLES

1. A mass M is split into two parts m and $(M - m)$, which are then separated by a certain distance. What ratio $\left(\frac{m}{M}\right)$ maximizes the gravitational force between the parts?

Solution:

If r is the distance between m and $(M - m)$, the gravitational force will be

$$F = G\frac{(M-m)m}{r^2} = \frac{G}{r^2}[Mm - m^2]$$

For F to be maximum, $\frac{dF}{dm} = 0$,

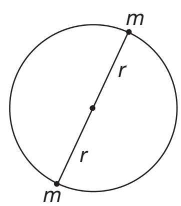
i.e., $$\frac{d}{dm}\left[\frac{G}{r^2}(Mm - m^2)\right] = 0$$

or, $$M - 2m = 0 \qquad \left[\because \frac{G}{r^2} \neq 0\right]$$

or, $$\frac{m}{M} = \frac{1}{2},$$

i.e., the force will be maximum when two parts are equal.

2. Two particles of equal mass m are moving in a circle of radius r under the action of their mutual gravitational attraction. Find the speed of each particle.



Solution:

The particles will always remain diametrically opposite so that the force on each particle will be directed along the radius.

Considering the circular motion of one particle, we have,

$$\frac{mv^2}{r} = \frac{Gm.m}{(2r)^2}$$

$$\therefore \qquad v = \sqrt{\frac{Gm}{4r}}.$$

3. Three equal particles each of mass m are placed at the three corners of an equilateral triangle of side a. Find the force exerted by this system on another particle of mass m placed at (a) the mid-point of a side (b) at the centre of the triangle.

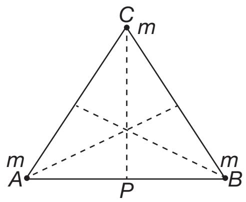


Fig. 7.1

Solution:

As gravitational force is a two body interaction, the principle of superposition is valid, i.e., resultant force on the particle of mass m at P is

$$\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C.$$

(A) As shown in the above Fig. 7.1, when P is at the mid-point of a side, $\vec{F}_A$ and $\vec{F}_B$ will be equal in magnitude but opposite in direction. So they will cancel each other. So the point mass m at P will experience a force due to C only, i.e.,

$$F = F_C = \frac{Gmm}{(CP)^2} = \frac{Gm^2}{(a\sin 60°)^2}$$

$$= \frac{4Gm^2}{3a^2} \text{ along } PC.$$

(B) From symmetry, the net force on the particle at the centre of triangle = 0.

4. Find the gravitational force of attraction on the point mass m placed at O by a thin rod of mass M and length L as shown in Fig. 7.2.

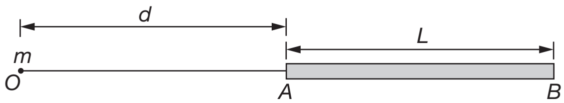


Fig. 7.2

Solution:

First we need to find the force due to an element of length dx. The mass of the element is $dm = \left(\frac{M}{L}\right)dx$. so,

$$dF = G\frac{Mm}{L}\frac{dx}{x^2}$$

$\therefore$ The net gravitational force is

$$F = \frac{GMm}{L}\int_d^{d+L}\frac{dx}{x^2} = \frac{GMm}{L}\left[\frac{1}{d} - \frac{1}{L+d}\right]$$

$$= \frac{Mm}{d(L+d)}.$$

Notice that when $d >> L$, we find $F = \frac{GMm}{d^2}$, the result for two point masses.

5. Find the gravitational force of attraction between a uniform sphere of mass M and a uniform rod of length ℓ and mass M oriented as shown in the Fig. 7.3.

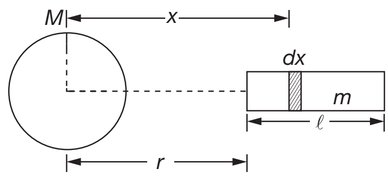


Fig. 7.3

Solution:

Since the sphere is uniform its entire mass may be considered to be concentrated at its centre. The force on the elementary mass dm is

$$dF = \frac{GMdm}{x^2}.$$

But, $$dm = \frac{m}{\ell}dx,$$

$$F = \int_r^{r+\ell}\frac{GMm}{\ell x^2}dx = -\frac{GMm}{\ell}\left[\frac{1}{x}\right]_r^{r+\ell}$$

$$= -\frac{GMm}{\ell}\left[\frac{1}{r+\ell} - \frac{1}{r}\right]$$

$$F = \frac{GmM}{\ell}\frac{\ell}{r(r+\ell)} = \frac{GMm}{r(r+\ell)}.$$

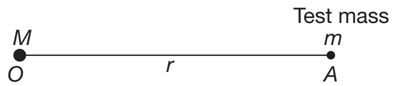
GRAVITATIONAL FIELD

Gravitational field due to a mass is defined as the region of space in which it interacts with other masses. In order to get the extent of interaction between two masses we defined another quantity called gravitational field intensity. Gravitational field intensity due to a mass m at a distance r is defined as the force acting on unit mass kept at a distance r. The gravitational field intensity is a vector quantity and its direction is the direction along which the unit mass has a tendency to move. The unit of gravitational field intensity is N/kg and its dimensions are $[LT^{-2}]$.

Calculation of Gravitational Field

Gravitational Field Intensity due to a Point Mass

Consider a point mass M at O and let us calculate gravitational intensity at A due to this point mass.



Suppose a test mass is placed at A.
By Newton's law of gravitation, force on test mass

$$F = \frac{GMm}{r^2} \text{ along } \overrightarrow{AO}$$

$$E = \frac{F}{m} = -\frac{GM}{r^2}\hat{e}_r \quad (7.2)$$

Gravitational field intensity due to a uniform circular ring at a point on its axis.

Figure 7.4 shows a ring of mass m and radius R. Let P is the point at a distance r from the centre of the ring. By symmetry the field must be towards the centre that is along $\overrightarrow{PO}$.

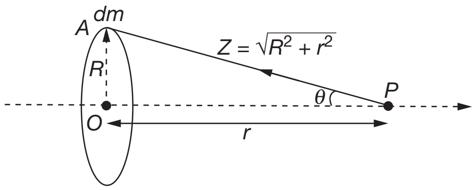


Fig. 7.4

Let us assume that a particle of mass dm on the ring say, at point A. Now the distance AP is $\sqrt{R^2 + r^2}$. Again the gravitational field at P due to dm is along $\overrightarrow{PA}$ and its magnitude is

$$dE = \frac{Gdm}{Z^2}$$

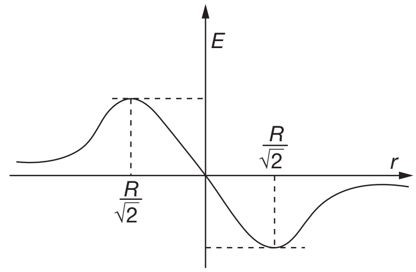
$$\therefore \quad dE \cos\theta = \frac{Gdm}{Z^2}\cos\theta$$

Net gravitational field $E = \dfrac{G\cos\theta}{Z^2}\int dm$

$$= \frac{GM}{Z^2}\frac{r}{Z}$$

$$= \frac{GMr}{(r^2 + R^2)^{3/2}} \text{ along } \overrightarrow{PO} \quad (7.3)$$

Variation of gravitational field due to a ring as a function of its axial distance.



IMPORTANT POINTS

- If $r >> R$, $r^2 + R^2 \simeq r^2$

  $\therefore \quad E = -\dfrac{GMr}{r^3} = -\dfrac{GM}{r^2}$ [where negative sign is because of attraction]

  Thus, for a distant point, a ring behaves as a point mass placed at the centre of the ring.
- If $r << R$, $r^2 + R^2 \simeq R^2$

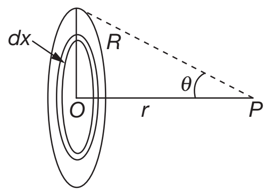
  $\therefore \quad E = -\dfrac{GM\,r}{R^3}$

  i.e., $E \propto r$.

Gravitational Field Intensity due to a Uniform Disc at a Point an its Axis

Let the mass of disc be M and its radius is R and P is the point on its axis where gravitational field is to be calculated.

Let us draw a circle of radius x and centre at O. We draw another concentric circle of radius $x + dx$. The part of disc enclosed between two circle can be treated as a uniform ring of radius x. The area of this ring is $2\pi x dx$.



Therefore mass dm of the ring $= \dfrac{M}{\pi R^2}2\pi x dx$

$$= \frac{2Mxdx}{R^2}.$$

Gravitational field at P due to the ring is,

$$dE = \frac{G\left(\dfrac{2Mx\,dx}{R^2}\right)r}{(r^2 + x^2)^{3/2}}$$

$$\int dE = \frac{2GMr}{R^2}\int_O^a \frac{xdx}{(r^2 + x^2)^{3/2}}$$

$$= \frac{2GMr}{R^2}\left[-\frac{1}{\sqrt{r^2 + x^2}}\right]_0^R$$

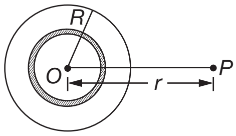
$$= \frac{2GMr}{R^2}\left[\frac{1}{r} - \frac{1}{\sqrt{r^2+R^2}}\right]$$

in terms of θ

$$E = \frac{2GM}{R^2}(1-\cos\theta) \tag{7.4}$$

Gravitational Field due to a Uniform Solid Sphere

Case I: Field at an external point.
Let the mass of sphere is M and its radius is R we have to calculate the gravitational field at P.

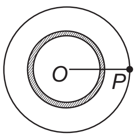


$$\int dE = \int \frac{Gdm}{r^2}$$

$$= \frac{G}{r^2}\int dm = \frac{GM}{r^2} \tag{7.5}$$

Thus, a uniform sphere may be treated as a single particle of equal mass placed at its centre for calculating the gravitational field at an external point.

Case II: Field at an internal point.
Suppose the point P is inside the solid sphere, in this case $r < R$ the sphere may be divided into thin spherical shells all centered at O. Suppose the mass of such a shell is dm. Then



$$dE = \frac{Gdm}{r^2} \text{ along } PO$$

$$= \frac{G}{r^2}\int dm$$

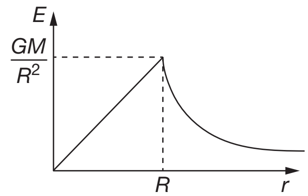
where

$$\int dm = \frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = \frac{Mr^3}{R^3}$$

$$\therefore \quad E = \frac{GM}{R^3}r \tag{7.6}$$

Therefore gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere. At the centre $r = 0$ the field is zero. At the surface of the sphere $r = R$

$$E = \frac{GM}{R^2} \tag{7.7}$$

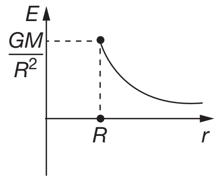


Gravitational field due to solid sphere is continuous but it is not differentiable function.

Field due to Uniform Thin Spherical Shell

Case I: When point lies inside the spherical shell

$$\int dE = \frac{G}{r^2}\int m_{\text{enclosed}} = 0 \tag{7.8}$$



Case II: When point P lies outside the spherical shell

$$\int dE = \frac{G}{r^2}\int dm = \frac{GM}{r^2} \tag{7.9}$$

Gravitational field due to thin spherical shell is both discontinuous and non-differentiable function.

SOLVED EXAMPLES

6. Two concentric shells of masses M_1 and M_2 are situated as shown in Fig. 7.5. Find the force on a particle of mass m when the particle is located at

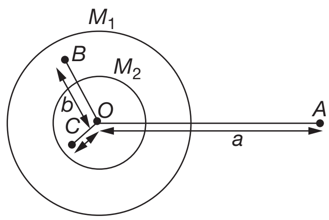


Fig. 7.5

(A) $r = a$ (B) $r = b$ (C) $r = c$.
The distance r is measured from the centre of the shell.

Solution:
We know that attraction at an external point due to spherical shell of mass M is $\frac{GM}{r^2}$ while at an internal point it is zero. So,

(A) For $r = a$, the point is external for both the shell; so

$$E_A = \frac{G(M_1+M_2)}{a^2}$$

$$\therefore \quad F_A = mE_A = \frac{GM[M_1+M_2]}{a^2}$$

(B) For $r = b$, the point is external to the shell of mass M_2 and internal to the shell of mass M_1; so,

$$E_B = \frac{GM_2}{b^2} + O$$

$\therefore$ $$F_B = mE_B = \frac{GMm}{b^2}$$

(C) For $r = c$, the point is internal to both the shells, so,

$$E_C = 0 + 0 = 0$$

$\therefore$ $$F_C = m\,E_C = 0.$$

7. A uniform solid sphere of mass M and radius a is surrounded symmetrically by a thin spherical shell of equal mass and radius $2a$. Find the gravitational field at a distance.

(A) $\frac{3}{2}a$ from centre (B) $\frac{5}{2}a$ from centre.

Solution:

(A) The situation is indicated in Fig. 7.6 in the two cases.

The point P_1 is at a distance $\frac{3}{2}a$ from centre and P_2 is at a distance $\frac{5}{2}a$ from centre. As P_1 is inside the cavity of the thin spherical shell the field here due to the shell is zero. The field due to the solid sphere is

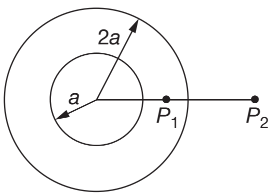


Fig. 7.6

$$E = \frac{GM}{\left(\frac{3}{2}a\right)^2} = \frac{4\,GM}{9a^2}.$$

This also represents the resultant field at P_1.

Resultant field $= \frac{4GM}{9a^2}$. The direction is towards centre.

(B) In this case P_2 is outside the sphere as well as the shell. Both may be replaced by single particles of same mass at the centre.

The field due to each of them at

$$P_2 = \frac{GM}{\left(\frac{5}{2}a\right)^2} = \frac{4\,GM}{25a^2}.$$

$$\text{Resultant field} = \frac{4\,GM}{25a^2} + \frac{4\,GM}{25a^2} = \frac{8\,GM}{25a^2}.$$

This is also acting towards the common centre.

Variation in Acceleration due to Gravity

With Altitude

At the surface of the earth,

$$g = \frac{GM}{R^2}.$$

For a height h above the surface of the earth,

$$g' = \frac{GM}{(R+h)^2}$$

$$\frac{g'}{g} = \frac{R^2}{(h+R)^2} = \frac{1}{\left(1+\frac{h}{R}\right)^2}$$

or, $$g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}$$

So, with increase in height, g decreases. If $h << R$.

$$g' = g\left[1+\frac{h}{R}\right]^{-2} = g\left[1-\frac{2h}{R}\right]$$

or, $$g' = g\left[1-\frac{2h}{R}\right] \qquad (7.10)$$

With Depth

At the surface of the earth,

$$g = \frac{GM}{R^2}$$

For a point at a depth d below the surface,

$$g' = \frac{GM}{R^3}[R-d]$$

$\therefore$ $$\frac{g'}{g} = \left(\frac{R-d}{R}\right)$$

i.e., $$g' = g\left[1-\frac{d}{R}\right] \qquad (7.11)$$

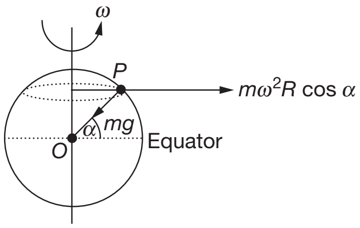
So with increase in depth below the surface of the earth, g keeps decreasing and at the centre of the earth it becomes zero.

It is to be noted that value of g decreases if we move above the surface or below the surface of the earth.

Due to Rotation of Earth

The earth is rotating about its axis from west to east. So, the earth is a non-inertial frame of reference. Every body on its surface experiences a centrifugal force $m\omega^2 R \cos \alpha$, where α is latitude of the place.

The net force on a particle on the surface of the earth



$$F = \sqrt{m^2g^2 + m^2\omega^4R^2\cos^2\alpha + 2(mg)(m\omega^2R\cos\alpha)[\cos(180-\alpha)]}$$

$$= mg' \qquad (7.12)$$

Therefore,

1. g is maximum ($= g$) when $\cos\alpha$ = minimum = 0, i.e., $\alpha = 90°$, i.e., at the pole.
2. g is minimum ($= g - \omega^2R$) when $\cos\alpha$ = maximum = 1, i.e., $\alpha = 0°$, i.e., at the equator.

SOLVED EXAMPLES

8. Calculate the acceleration due to gravity at the surface of Mars if its diameter is 6760 km and mass one tenth that of the earth. The diameter of earth is 12742 km and acceleration due to gravity on the earth is 9.8 m/s^2.

Solution:

We know that

$$g = \frac{GM}{R^2}$$

$$\therefore \quad \frac{g_M}{g_E} = \left(\frac{M_M}{M_E}\right)\left(\frac{R_E}{R_M}\right)^2 = \left[\frac{1}{10}\right]\left[\frac{12742}{6760}\right]^2$$

$$\text{or,} \quad \frac{g_M}{g_E} = 0.35$$

$$\therefore \quad g_M = 0.35 \times g_E = 9.8 \times 0.35 = 3.48 \text{ m/s}^2.$$

9. Compute the mass and density of the moon if acceleration due to gravity on its surface is 1.62 m/s^2 and its radius is 1.74×10^6 m. ($G = 6.67 \times 10^{-11}$ MKS units)

Solution:

We know that

$$g = \frac{GM}{R^2}$$

$$\therefore \quad M = \frac{gR^2}{G} = \frac{1.62 \times (1.74 \times 10^6)^2}{6.67 \times 10^{-11}}$$

$$\text{or,} \quad M = 7.35 \times 10^{22} \text{ kg}$$

$$\text{Now,} \quad \rho = \frac{M}{V} = \frac{gR^2}{G\left(\frac{4}{3}\pi R^3\right)} \quad \frac{3g}{4\pi GR}$$

$$\therefore \quad \rho = \frac{3 \times 1.62}{4 \times 3.14 \times 6.67 \times 10^{-11} \times 1.74 \times 10^6}$$

$$= 3.3 \times 10^3 \text{ kg/m}^3.$$

10. Two bodies of masses M_1 and M_2 are placed at a distance d apart. What is the potential at the position where the gravitational field due to them is zero?

Solution:

Let the field be zero at a point distant x from M_1.

$$\therefore \quad \frac{GM_1}{x^2} = \frac{GM_2}{(d-x)^2}$$

$$\therefore \quad \frac{x}{d-x} = \sqrt{\frac{M_1}{M_2}}$$

$$x\sqrt{M_2} = \sqrt{M_1} \cdot d - x\sqrt{M_1}$$

$$x\left[\sqrt{M_1} + \sqrt{M_2}\right] = \sqrt{M_1} \cdot d$$

$$x = \frac{d\sqrt{M_1}}{\sqrt{M_1} + \sqrt{M_2}}$$

$$d - x = \frac{d\sqrt{M_2}}{\sqrt{M_1} + \sqrt{M_2}}$$

Potential at this point due to both the masses will be

$$= -\frac{GM_1}{x} - \frac{GM_2}{(d-x)}$$

$$= -G\left[\frac{M_1\left(\sqrt{M_1} + \sqrt{M_2}\right)}{d\sqrt{M_1}} + \frac{M_2\left(\sqrt{M_1} + \sqrt{M_2}\right)}{d\sqrt{M_2}}\right]$$

$$= -\frac{G}{d}\left(\sqrt{M_1} + \sqrt{M_2}\right)^2 = -\frac{G}{d}\left(M_1 + M_2 + 2\sqrt{M_1M_2}\right).$$

11. The distance between earth and moon is 3.8×10^5 km and the mass of earth is 81 times the mass of moon. Deduce the position of a point on the line joining the centres of earth and moon, where the gravitational field is zero. What would be the value of gravitational field at this point, due to earth and moon individually?

Solution:

Let x be the distance of the point of no net field from earth. The distance of this point from moon is $(r - x)$, where $r = 3.8 \times 10^5$ km.

The gravitational field due to earth $= \dfrac{GM_e}{x^2}$ and that due to moon $= \dfrac{GM_m}{(r-x)^2}$. For the net field to be zero, these are equal and opposite.

$$\frac{GM_e}{x^2} = \frac{GM_m}{(r-x)^2}$$

$$\frac{M_e}{M_m} = \frac{x^2}{(r-x)^2}.$$

But $$\frac{M_e}{M_m} = 81$$

$\therefore$ $$81 = \frac{x^2}{(r-x)^2}$$

$$\frac{x}{r-x} = 9$$

$$9r - 9x = x$$

$$10x = 9r$$

$$x = \frac{9}{10}r = \frac{9}{10}\times 3.8\times 10^5 = 3.42 \times 10^5 \text{ km}$$

The intensity of the field

$$= \frac{GM_e}{x^2} = \frac{R_e^2\, g}{x^2} = \frac{\left(6.4\times 10^6\right)^2 \times 9.8}{\left(3.42\times 10^8\right)^2}$$

$$= 3.43 \times 10^{-3} \text{ N/kg.}$$

GRAVITATIONAL POTENTIAL

At a point in a gravitational field, potential (V) is defined as the work done by the external agent against the gravitational field in bringing unit mass from infinity to that point.

Mathematically,

$$V = \frac{W}{m}.$$

It is a scalar having dimensions $[L^2T^{-2}]$ and SI unit J/kg.

$\Rightarrow$ By the definition of potential energy, $U = W$

So, $$V = \frac{U}{m}$$

i.e., $$U = mV.$$

Thus, gravitational potential at a point represents potential energy of a unit point mass at that point.

$\Rightarrow$ by definition of work $W = \int \vec{F}_{\text{ext}} \cdot \vec{dr}$

$\therefore$ But, $$\vec{F}_{\text{ext}} = -\vec{F}_{\text{gravitation}}$$

$\therefore$ $$W = -\int \vec{F}_{\text{gravitational}} \cdot \vec{dr}$$

So, $$V = -\frac{\int \vec{F}_{\text{gravitational}} \cdot \vec{dr}}{m} = -\int \vec{E} \cdot \vec{dr}$$

$$\left[\because \frac{\vec{F}_{\text{gravitational}}}{m} = \vec{E}\right]$$

i.e., $$dV = -Edr$$

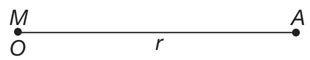
or $$E = -\frac{dV}{dr} \tag{7.13}$$

So the potential can also be defined as a scalar function of position whose negative gradient. i.e., space derivative gives field intensity.

$\Rightarrow$ Negative of the slope of V *vs* r graph gives intensity.

Calculation of Gravitational Potential

Gravitational Potential at a Point due to a Point Mass



We have, gravitational field due to a point mass

$$E = -\frac{GM}{r^2}$$

[The negative sign used as gravitational force is attractive]

$\therefore$ $$V = -\int Edr = -\frac{GM}{r} + C$$

when, $r = \infty$, $V = 0$;

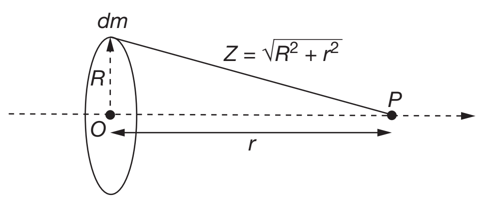
so $C = 0$

$\therefore$ $$V = -\frac{GM}{r}. \tag{7.14}$$

Gravitational Potential at a Point due to a Ring

Let M be the mass and R be the radius of a thin ring.

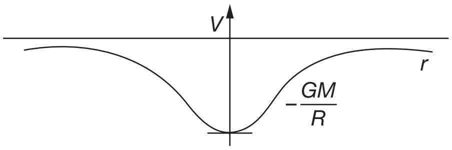
Considering a small element of the ring and treating it as a point mass, the potential at the point P is



$$dV = -\frac{G\,dm}{Z} = -\frac{G\,dm}{\sqrt{R^2 + r^2}}.$$

Hence, the total potential at the point P is given by

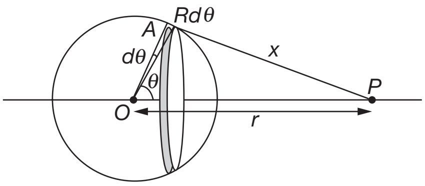
$$V = -\int \frac{G\,dm}{\sqrt{R^2+r^2}} = -\frac{GM}{\sqrt{R^2+r^2}} \quad (7.15)$$



at $r = 0 \quad \frac{dV}{dr} = 0$

∴ gravitational field is zero at the centre.

Gravitational Potential at a Point due to a Spherical Shell (Hollow Sphere)



Consider a spherical shell of mass M and radius R. P is a point at a distance r from the centre O of the shell.

Consider a ring at right angles to OP. Let θ be the angular position of the ring from the line OP.

The radius of the ring = $R\sin\theta$

The width of the ring = $R\,d\theta$

The surface area of the ring = $(2\pi R\sin\theta)\cdot R\,d\theta$

$$= 2\pi R^2 \sin\theta\,d\theta$$

The mass of the ring = $(2\pi R^2 \sin\theta\,d\theta)\times\frac{M}{4\pi R^2}$

$$= \frac{M\sin\theta\,d\theta}{2}$$

If x is the distance of the point P from a point of the ring, then the potential at P due to the ring.

$$dV = -\frac{GM\sin\theta\,d\theta}{2x} \quad (7.16)$$

From the 'cosine-property' of the triangle OAP,

$$x^2 = R^2 + r^2 - 2Rr\cos\theta$$

Differentiating,

$$2x\,dx = 2\,Rr\sin\theta\,d\theta$$

$$\therefore \quad \sin\theta\,d\theta = \frac{x\,dx}{Rr}$$

Substituting the above value of $\sin\theta\,d\theta$ in equation (7.16), we get

$$dV = -\frac{GM}{2x}\times\frac{x\,dx}{Rr}$$

$$= -\frac{GM}{2Rr}dx.$$

Case I: When the point P lies outside the shell.

$$V = -\frac{GM}{2Rr}\int_{r-R}^{R+r} dx = -\frac{GM}{2Rr}[x]_{r-R}^{r+R}$$

$$V = -\frac{GM}{2Rr}\left[(R+r)-(r-R)\right]$$

$$V = -\frac{GM}{r} \quad (7.17)$$

This is the potential at P due to a point mass M at O.

For external point, a spherical shell behaves as a point mass supposed to be placed at it centre.

Case II: When the point P lies inside the spherical shell.

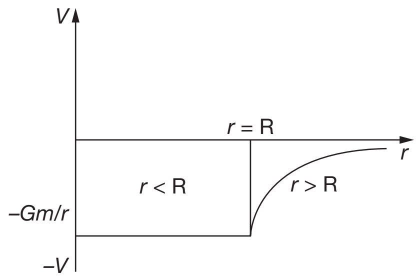
$$V = -\frac{GM}{2Rr}\int_{R-r}^{R+r} dx = -\frac{GM}{2Rr}[x]_{R-r}^{R+r}$$

or $$V = -\frac{GM}{R} \quad (7.18)$$

This expression is independent of r. Thus, the potential at every point inside the spherical shell is the same and it is equal to the potential of the surface of the shell.

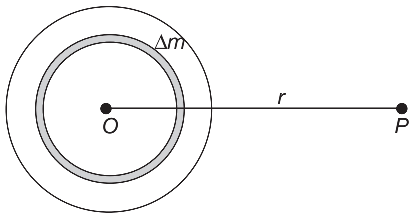
Thus, the gravitational field inside a spherical shell is zero everywhere.

Graphical representation of the variation of V with r in case of a hollow spherical shell.



Gravitational Potential due to a Homogeneous Solid Sphere

Case I: When the point P lies outside the sphere



Consider a homogeneous solid sphere of mass M and radius R. P is a point at a distance r from the centre of the sphere.

The solid sphere may be supposed to be made up of large number of thin concentric spherical shells. Consider one such shell of mass Δm.

The potential at P due to the shell $= -\dfrac{G\,\Delta m}{r}$

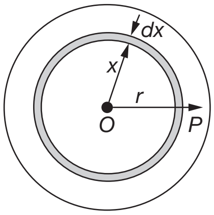
So, the potential at P due to the entire sphere.

$$V = -\sum \frac{G\,\Delta m}{r} = -\frac{G}{r}\sum \Delta m$$

$$V = -\frac{GM}{r} \qquad [\because M = \Sigma\Delta m] \qquad (7.19)$$

Hence, for an external point, a solid sphere behaves as if the whole of its mass is concentrated at the centre.

Case II: When the point P lies inside the sphere



Let us consider a concentric spherical surface thorough the point P. The potential at P arises out of the inner sphere and the outer thick spherical shell.

$\therefore\ V = V_1 + V_2$, where V_1 = potential due to the inner sphere and V_2 = potential due to the outer thick shell.

The mass of the inner sphere $= \dfrac{4\pi r^3}{3}\rho$,

where ρ = density of the sphere $= \dfrac{M}{\frac{4}{3}\pi R^3} = \dfrac{3M}{4\pi R^3}$

The potential at P due to this sphere

$$V_1 = -\frac{G\left[\frac{4\pi r^3}{3}\right]\rho}{r} = -\frac{4\pi G\rho}{3}r^2 \qquad (7.20)$$

To find V_2, consider a thin concentric shell of radius x and thickness dx.

The volume of the shell $= 4\pi x^2\, d\,x$.

The mass of the shell $= 4\pi x^2\, d\,x\,\rho$.

The potential at P due to this shell

$$V_2 = -\int_r^R 4\pi G\rho x\, dx$$

$$= -4\pi G\rho\left[\frac{x^2}{2}\right]_r^R = -4\pi G\rho\left[\frac{R^2}{2} - \frac{r^2}{2}\right]$$

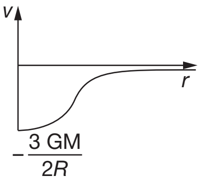
$$-2\pi G\rho\left[R^2 - r^2\right].$$

$$\therefore\ V_1 + V_2 = -\frac{4\pi G\rho r^2}{3} - 2\pi G\rho\left[R^2 - r^2\right]$$

$$= -\frac{4\pi G\rho}{3}\left[r^2 + \frac{3R^2}{2} - \frac{3r^2}{2}\right]$$

$$= -\frac{4\pi G\rho}{3}\left[\frac{3R^2}{2} - \frac{r^2}{2}\right]$$

$$= -\frac{4\pi G}{3}\cdot\frac{3M}{4\pi R^3}\left[\frac{3R^2 - r^2}{2}\right]$$



or, $$V = -\frac{GM}{2R^3}\left[3R^2 - r^2\right] \qquad (7.21)$$

at $r = 0$ $\dfrac{dV}{dr} = 0.$

Hence gravitational field is 0 at the centre of a solid sphere.

GRAVITATIONAL POTENTIAL ENERGY

The potential energy of a system corresponding to a conservative force is defined as

$$\int_{U_i}^{U_f} dU = -\int \vec{F}\cdot\vec{dr}$$

i.e., The change in potential energy is equal to negative of work done.

Let a particle of mass m_1 be kept fixed at point A and another particle of mass m_2 be taken from a point B to C. Initially, the distance between the particle is r_1 and finally it becomes $AC = r_2$. We have to calculate the change in gravitation in potential energy of the system of two particles.

Consider a small displacement when the distance between the particles changes from r to $r + dr$. In the Fig. 7.7 this corresponds to the second particle going from D to E.

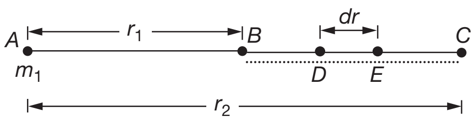


Fig. 7.7

Force acting on second particle is

$$\vec{F} = \frac{Gm_1m_2}{r_2} \text{ along } \overrightarrow{DA}$$

$$\therefore \quad dW = \vec{F}\cdot\overrightarrow{dr}$$

$$= -\frac{Gm_1m_2}{r^2}dr$$

$$dU = -dW = \int \frac{Gm_1m_2\ dr}{r^2}$$

$$\int_{r_1}^{r_2} dU = Gm_1m_2 \int_{r_1}^{r_2} \frac{dr}{r^2} = Gm_1m_2\left(-\frac{1}{r}\right)_{r_1}^{r_2}$$

$$U(r_2) - U(r_1) = Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad (7.22)$$

We choose the potential energy of the two particles system to be zero when the distance between them is infinity. This means $U(\infty) = 0$.

Hence, $U(r) = U(r) - U(\infty)$

Now in equation (18)

Take $r_1 = r$ and $r_2 = \infty$

$$U(\infty) - U(r) = Gm_1m_2\left(\frac{1}{r} - \frac{1}{\infty}\right)$$

$$\therefore \quad U(r) = -\frac{Gm_1m_2}{r_1} \qquad (7.23)$$

Hence, when two masses m_1 and m_2 separated by a distance their gravitational potential energy is $U(r) = \frac{-Gm_1m_2}{r}$.

SOLVED EXAMPLE

12. Three particles each of mass m are placed at the corners of an equilateral triangle of side d. Calculate

(A) the potential energy of the system.

(B) work done on the system if the side of the triangle is changed from d to $2d$.

Solution:

(A) $U = -\frac{3Gmm}{d}$

$$= -\frac{3Gm^2}{d}$$

(B) $U_i = -\frac{3Gm^2}{d}$

$$U_f = -\frac{3Gm^2}{2d}$$

$\therefore$ Work done $= w = \Delta U = U_f - U_i$

$$= \frac{3Gm^2}{2d}.$$

SOLVED EXAMPLE

13. A rocket starts vertically upwards with speed v_0. Show that its speed v at a height is given by $v_0^2 - v^2 = \frac{2gh}{1+\frac{h}{R}}$, where R is the radius of earth. Hence deduce the maximum height reached by the rocket fired with a speed of 90% to escape velocity.

Solution:

Kinetic energy on the surface of earth $= \frac{1}{2}mv_0^2$

Potential energy on the surface of earth $= \frac{-GMm}{R}$

Total energy $= \frac{1}{2}mv_0^2 - \frac{GMm}{R}$

Kinetic energy at a height $h = \frac{1}{2}mv^2$

Potential energy at such height $= \frac{-GMm}{(R+h)}$

Total energy $= \frac{1}{2}mv^2 - \frac{GMm}{R+h}$

By the principle of conservation of energy,

$$\frac{1}{2}mv^2 - \frac{GMm}{R+h} = \frac{1}{2}mv_0^2 - \frac{GMm}{R}$$

$$\frac{1}{2}\left(v_0^2 - v^2\right) = \frac{GM}{R} - \frac{GM}{(R+h)}.$$

But $\qquad GM = gR^2$

$$\frac{1}{2}\left(v_0^2 - v^2\right) = \frac{gR^2h}{R(R+h)}$$

$$v_0^2 - v^2 = \frac{2gRh}{R+h} = \frac{2gh}{1+\frac{h}{R}}$$

For maximum height $v = 0$,

$v_0 = 90\%$ of escape velocity

$$= 0.9\sqrt{2gR}$$

$$\therefore \quad \left(0.9\sqrt{2gR}\right)^2 - 0 = \frac{2gh_{max}}{1+\frac{h_{max}}{R}}$$

$$0.81R = \frac{h_{max}}{1+\frac{h_{max}}{R}}$$

$$0.81R + 0.81\, h_{max} = h_{max}$$

$$0.19\, h_{max} = 0.81R$$

$$h_{max} = \frac{0.81R}{0.19} = 4.26\text{ R}.$$

ESCAPE SPEED

It is the minimum speed with which a body must be projected from the surface of a planet (usually the earth) so that it permanently overcomes and escapes from the gravitational field of the planet (the earth). We can also say that a body projected with escape speed will be able to go to a point which is at infinite distance from the earth.

If a body of mass m is projected with speed v from the surface of a planet of mass M and radius R, then

$$\text{KE} = \frac{1}{2}mv^2; \ \text{GPE} = -\frac{GMm}{R}$$

Total mechanical energy (TME) of the body $= \frac{1}{2}mv^2 - \frac{GMm}{R}$

If the v' is the speed of the body at infinity, then

$$\text{TME at infinity} = 0 + \frac{1}{2}mv'^2 = \frac{1}{2}mv'^2$$

Applying the principle of conservation of mechanical energy, we have

$$\frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{1}{2}mv'^2$$

or, $$v^2 = \frac{2GM}{R} + v'^2$$

v will be minimum when $v' \to$ o, i.e.,

$$v_e = v_{min} = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} \qquad (7.24)$$

$$\left[\because g = \frac{GM}{R^2}\right]$$

IMPORTANT POINTS

- Escape speed is independent of the mass and direction of projection of the body.
- For earth as $g = 9.8$ m/s^2 and $R = 6400$ km

  $v_e = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} = 11.2$ km/s.

SOLVED EXAMPLES

14. The masses and radii of the earth and the moon are M_1, R_1 and M_2, R_2, respectively. Their centres are at a distance d apart. What is the minimum speed with which a particle of mass m should be projected from a point midway between the two centres so as to escape to infinity?

Solution:

Potential energy of m which is midway between M_1 and M_2 is

$$U = m\,(V_1 + V_2) = m\left(-\frac{GM_1}{d/2} + \frac{-GM_2}{d/2}\right)$$

$$= -\frac{2Gm}{d}\,(M_1 + M_2)$$

Let v be the required speed, then

$$(\text{TME})_{\text{initial}} = -\frac{2Gm}{d}(M_1 + M_2) + \frac{1}{2}mv^2$$

As the particle reaches infinity,

$$(\text{TME})_{\text{final}} = 0$$

From the principle of conservation of mechanical energy, we have

$$-\frac{2Gm}{d}(M_1 + M_2) + \frac{1}{2}mv^2 = 0$$

$$v = 2\sqrt{\frac{G(M_1 + M_2)}{d}}.$$

15. What will be the acceleration due to gravity on the surface of the moon if its radius were $\frac{1}{4}$ th the radius of the earth and its mass $\left(\frac{1}{80}\right)$th the mass of the earth. What will be the escape speed on the surface of the moon if it is 11.2 km/s on the surface of the earth. Given $g = 9.8$ m/s^2.

Solution:

As on the surface of planet,

$$g = \frac{GM}{R^2},$$

we have

$$\frac{g_M}{g_E} = \frac{M_M}{M_E} \times \left(\frac{R_E}{R_M}\right)^2 = \frac{1}{80} \times (4)^2 = \frac{1}{5}$$

$$\therefore \quad g_M = \frac{g}{5} = \frac{9.8}{5} = 1.96 \text{ m/s}^2.$$

Further, as escape speed $= v_e = \sqrt{\frac{2GM}{R}}$,

So,

$$\frac{v_M}{v_E} = \sqrt{\frac{M_M}{M_E} \times \frac{R_E}{R_M}} = \sqrt{\frac{1}{80} \times 4} = \frac{1}{\sqrt{20}}$$

$$\therefore \quad v_M = \frac{v_E}{\sqrt{20}} = \frac{11.2}{4.47} = 2.5 \text{ km/s}.$$

MOTION OF A SATELLITE

Consider a satellite of mass m revolving in a circle around the earth. If the satellite is at a height h above the earth's surface, the radius of its orbit is $r = R + h$, where R is the radius of the earth. The gravitational force between m and M provides the necessary centripetal force for circular motion.

Orbital Velocity (v_0)

The velocity of a satellite in its orbit is called orbital velocity. Let v_0 be the orbital velocity of the satellite, then

$$\frac{GMm}{r^2} = \frac{mv_0^{\,2}}{r}$$

$$\Rightarrow \qquad v_0 = \sqrt{\frac{GM}{r}} \qquad (7.25)$$

or,

$$v = \sqrt{\frac{GM}{R+h}} \qquad (\because r = R + h)$$

Important Points

Orbital velocity is independent of the mass of the orbiting body and is always along the tangent to the orbital.

Close to the surface of the earth, $r = R$ as $h = 0$.

$$\therefore \quad v_0 = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \sqrt{10 \times 6.4 \times 10^6} \approx 8 \text{ km/s}$$

Close to the surface of the planet,

$$v_0 = \sqrt{\frac{GM}{R}} = \frac{v_e}{\sqrt{2}}$$

i.e.,

$$v_e = \sqrt{2} v_0$$

Time Period of a Satellite

The time taken by a satellite to complete one revolution is called the time period (T) of the satellite.

It is given by

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM}}$$

or,

$$T = \frac{2\pi r\sqrt{r}}{\sqrt{GM}}$$

or,

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \qquad (7.26)$$

$$\Rightarrow \qquad T^2 \propto r^3$$

Angular Momentum of a Satellite (L)

In case of satellite motion, angular momentum will be given by

$$L = mvr = mr\sqrt{\frac{GM}{r}}$$

$$L = \left(m^2 GMr\right)^{1/2} \qquad (7.27)$$

In case of satellite motion, the net force on the satellite is centripetal force. The torque of this force about the centre of the orbit is zero. Hence, angular momentum of the satellite is conserved, i.e., L = constant.

Energy of a Satellite

The PE of a satellite is

$$U = mv = -\frac{GMm}{r} \qquad \left[\because v = -\frac{GM}{r}\right]$$

The kinetic energy of the satellite is

$$K = \frac{1}{2}mv_0^2 = \frac{GMm}{2r} \qquad \left[\because v_0 = \sqrt{\frac{GM}{r}}\right]$$

Total mechanical energy of the satellite

$$= -\frac{GMm}{r} + \frac{GMm}{2r} = -\frac{GMm}{2r} \qquad (7.28)$$

Important Points

We have,

$$\frac{K}{E} = -1$$

i.e.,

$$K = -E$$

Also,

$$\frac{U}{E} = 2$$

$$\Rightarrow \qquad U = 2E$$

Total energy of a satellite in its orbit is negative. Negative energy means that the satellite is bound to the central body by an attractive force and energy must be supplied to remove it from the orbit to infinity.

Binding Energy of the Satellite

The energy required to remove the satellite from its orbit to infinity is called binding energy of the satellite, i.e.,

$$\text{Binding energy} = -E = \frac{GMm}{2r}.$$

GEO-STATIONARY SATELLITE

If there is a satellite rotating in the direction of earth's rotation, i.e., from west to east, then for an observer on the earth, the angular velocity of the satellite will be ($\omega_S - \omega_E$).

However, if $\omega_S - \omega_E = 0$, satellite will appear stationary relative to the earth. Such a satellite is called 'geo-stationary satellite' and is used for communication purposes.

The orbit of a geostationary satellite is called ‘Parking orbit’.

We know that,

$$T^2 = \frac{4\pi^2}{GM} r^3 \qquad (7.29)$$

For geostationary satellite, $T = 24$ hours.

Putting this value of T in the above equation, we get

$$r \approx 42000 \text{ km}$$

or, $$h \approx 36000 \text{ km}$$

where h is the height of the satellite from the surface of the earth.

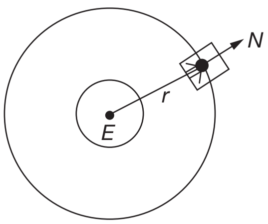
WEIGHTLESSNESS IN A SATELLITE

The radial acceleration of the satellite is given by

$$a_r = \frac{F_r}{m} = \frac{GMm}{r^2 \times m} = \frac{GM}{r^2}.$$

For a astronaut inside the satellite, we have

$$\frac{GMm_a}{r^2} - N - m_a a_r = 0$$



where m_a is mass of astronaut, a_r is radial acceleration of satellite and N is normal reaction on the astronaut

or, $$\frac{GMm_a}{r^2} - N - \frac{GMm_a}{r^2} = 0$$

$\Rightarrow$ $$N = 0$$

Hence, the astronaut feels weightlessness.

SOLVED EXAMPLES

16. Calculate the orbital velocity of a satellite revolving at a height h above the earth’s surface if $h = R$. Also calculate the time period of this satellite. ($g = 9$ m/s^2, $R = 6400$ km)

Solution:

For the orbital velocity in a circular orbit, we have

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R+h}} \qquad (\because r = R + h)$$

$$\Rightarrow \quad v = \sqrt{\frac{gR^2}{2R}} = \sqrt{\frac{gR}{2}} \qquad (\because GM = gR^2 \text{ and } h = R)$$

$$\Rightarrow \quad v = \sqrt{\frac{9.8 \times 6400 \times 10^3}{2}} = 5.6 \text{ km/s}$$

$$\text{Time period} = T = \frac{2\pi r}{v} = \frac{2\pi(2R)}{4\sqrt{2} \times 10^3}$$

$$\Rightarrow \quad T = \frac{4\pi \times 6400 \times 10^3}{4\sqrt{2} \times 10^3} = 3.95 \text{ hrs.}$$

17. Two planets have masses in the ratio 1 : 10 and radii in the ratio 2 : 5. Compare

(A) their densities.
(B) the acceleration due to gravity on their surface.
(C) the escape velocities from their surfaces.
(D) the periods of revolutions of satellites near to their surfaces.

Solution:

Let M_1, M_2 by the masses and R_1, R_2 be the radii of the planets.

$$\Rightarrow \quad \frac{M_1}{M_2} = \frac{1}{10} \text{ and } \frac{R_1}{R_2} = \frac{2}{5}$$

(A) Ratio of densities = $\frac{d_1}{d_2}$

or, $$\frac{d_1}{d_2} = \left[\frac{M_1}{\frac{4}{3}\pi R_1^3}\right]\left[\frac{\frac{4}{3}\pi R_2^3}{M_2}\right]$$

or, $$\frac{d_1}{d_2} = \frac{M_1}{M_2}\left[\frac{R_2}{R_1}\right]^3$$

$$\Rightarrow \quad \frac{d_1}{d_2} = \left[\frac{1}{10}\right]\left[\frac{5}{2}\right]^3 = \frac{25}{16}$$

(B) Acceleration due to gravity at the surface = $g = \frac{GM}{R^2}$

$$\therefore \quad \frac{g_1}{g_2} = \frac{M_1}{M_2}\left[\frac{R_2}{R_1}\right]^2$$

$$= \frac{1}{10}\left[\frac{5}{2}\right]^2 = \frac{5}{8}$$

(C) Escape velocity = $\sqrt{\frac{2GM}{R}}$

$$\Rightarrow \quad \frac{v_1}{v_2} = \sqrt{\frac{M_1}{M_2}}\sqrt{\frac{R_2}{R_1}} = \sqrt{\frac{1}{10} \times \frac{5}{2}} = \frac{1}{2}$$

(D) Time period of a satellite near the surface (orbit radius = R) = $\dfrac{2\pi}{\sqrt{GM}} R\sqrt{R}$

$$\Rightarrow \quad \frac{T_1}{T_2} = \sqrt{\frac{M_2}{M_1}}\left[\frac{R_1}{R_2}\right]\left[\sqrt{\frac{R_1}{R_2}}\right]$$

$$= \sqrt{\frac{10}{1}}\left[\frac{2}{5}\right]\left[\sqrt{\frac{2}{5}}\right] = \frac{4}{5}.$$

18. The mean distance of mars from the sun is 1.524 times that of the earth from the sun. Find the number of years required for mars to make one revolution about the sun.

Solution:

For planets revolving around the sun, $T^2 \propto r^3$

$$\Rightarrow \quad \frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

[T_1 : time period of mars and T_2 : time period of earth]

$$\Rightarrow \quad \frac{T_1}{T_2} = \left[\frac{r_1}{r_2}\right]^{3/2}$$

$$\Rightarrow \quad T_1 = T_2\left[\frac{r_1}{r_2}\right]^{3/2} = (1 \text{ year})\,[1.524]^{3/2} = 1.88 \text{ years.}$$

19. An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from earth. Determine

(A) the height of satellite above earth's surface.

(B) if the satellite is suddenly stopped, find the speed with which the satellite will hit the earth's surface after falling down.

Solution:

Escape velocity $= \sqrt{2gR}$, where g is acceleration due to gravity on surface of earth and R the radius of earth.

$$\text{Orbital velocity } = \frac{1}{2}v_e = \frac{1}{2}\sqrt{2gR}$$

$$= \sqrt{\frac{gR}{2}}$$

(A) If h is the height of satellite above earth,

$$\frac{mv_0^2}{R+h} = \frac{GMm}{(R+h)^2}$$

$$v_0^2 = \frac{GM}{R+h} = \frac{gR^2}{(R+h)}$$

$$\therefore \quad \left(\frac{1}{2}v_e\right)^2 = \frac{gR^2}{R+h}$$

$$\frac{gR}{2} = \frac{gR^2}{R+h}$$

from equation (1)

$$R + h = 2R$$

$$h = R.$$

(B) If the satellite is stopped in orbit, the kinetic energy is zero and its potential energy is $\dfrac{-GMm}{2R}$,

Total energy = $\dfrac{-GMm}{2R}$

When it reaches the earth let v be its velocity.

Hence, the kinetic energy $= \dfrac{1}{2}mv^2$

Potential energy $= -\dfrac{GMm}{R}$

$$\therefore \quad \frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{GMm}{2R}$$

$$v^2 = 2GM\left(\frac{1}{R} - \frac{1}{2R}\right) = \frac{2gR^2}{2R}$$

$$= gR$$

$$v = \sqrt{gR}.$$

20. Two satellites S_1 and S_2 revolve round a planet in coplanar circular orbits in the same sense. Their periods of revolutions are 1 hour and 8 hours respectively. The radius of orbit of S_1 is 10^4 km. When S_2 is closest to S_1, find

(A) the speed of S_2 relative to S_1.

(B) the angular speed of S_2 actually observed by an astronaut in S_1.

Solution:

If r_1 and r_2 are radii of orbits of S_1 and S_2, T_1 and T_2 their respective periods, we have by Kepler's third law

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

$$r_2^3 = r_1^3\left(\frac{T_2}{T_1}\right)^2 = (10^4)^3\left(\frac{8}{1}\right)^2 = (10^4 \times 4)^3$$

$$\therefore \quad = 4 \times 10^4 \text{ km.}$$

(A) If the orbital speeds of satellites S_1 and S_2 are v_1 and v_2.

$$v_1 = \frac{2\pi r_1}{T_1}$$

$$v_2 = \frac{2\pi r_2}{T_2}$$

Speed of S_2 relative to $S_1 = |v_2 - v_1|$

$$= \frac{2\pi r_2}{T_2} - \frac{2\pi r_1}{T_1} = 2\pi\left(\frac{r_2}{T_2} - \frac{r_1}{T_1}\right)$$

$$= 2\pi\left(\frac{4\times10^4}{8} - \frac{10^4}{1}\right)$$

$$= -2\pi\times10^4\times\frac{1}{2} = -3.14\times10^4 \text{ km/hr.}$$

(B) Angular speed of S_2 relative to

$$S_1 = \frac{v_2 - v_1}{r_2 - r_1} = -\frac{3.14\times10^4}{4\times10^4 - 10^4}$$

$$= -1.046 \text{ rad/h} = -\frac{1.046}{3600} \text{ rad/s}$$

$$= -2.906\times10^{-4} \text{ rad/s.}$$

21. Two satellites of same masses are launched in the same orbit round the earth so as to rotate opposite to each other. They collide inelastically and stick together as wreckage. Obtain the total energy of the system before and after collision. Describe the subsequent motion of wreckage.

Solution:

Potential energy of satellite in orbit $= -\frac{GMm}{r}$

If v is the velocity in orbit, we have

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v^2 = \frac{GM}{r}$$

$\therefore$ Kinetic energy $= \frac{1}{2}mv^2 = \frac{GMm}{2r}$

$\therefore$ Total energy $= \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$

For the two satellites, the total energy before collision

$$= 2\left[-\frac{GMm}{2r}\right] = -\frac{GMm}{r}.$$

After collision, let v' be the velocity of the wreckage. By law of conservation of momentum, $m\vec{v}' - m\vec{v}' = 2mv'$. (since they are rotating opposite to each other)

$\therefore$ $$v' = 0$$

The wreckage has no kinetic energy after collision but has potential energy

$$= \frac{-GM(2m)}{r}$$

$\therefore$ Total energy after collision $= \frac{-2GMm}{r}$

After collision the centripetal force disappears and the wreckage falls down under the action of gravity.

22. An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth.

(A) Determine the height of the satellite above the earth's surface.

(B) If the satellite is stopped suddenly in its orbit and allowed to fall freely onto the earth, find the speed with which it hits the surface of the earth. [$g = 9.8$ m/s^2 and $Re = 6400$ km]

Solution:

(A) We know that for satellite motion.

$$v_0 = \sqrt{\frac{GM}{r}} = R\sqrt{\frac{g}{(R+h)}}$$

$$\left[\text{as } g = \frac{GM}{R^2} \text{ and } r = R+h\right]$$

In this problem,

$$v_0 = \frac{1}{2}v_e = \frac{1}{2}\sqrt{2gR}.$$

So, $$\frac{R^2 g}{R+h} = \frac{1}{2}gR$$

i.e., $$h = R = 6400 \text{ km.}$$

(B) By conservation of ME

$$0 + \left(-\frac{GMm}{r}\right) = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right)$$

or, $$v^2 = 2GM\left[\frac{1}{R} - \frac{1}{2R}\right]$$

[as $r = R + h = R + R = 2R$]

or, $$v = \sqrt{\frac{GM}{R}} = \sqrt{gR} = 8 \text{ km/s.}$$

23. A planet of mass m moves along an ellipse around the sun so that its maximum and minimum distances from the sun are r_1 and r_2. Find the angular momentum of the planet relative to centre of sun. (Mass of sun = M)

Solution:

The angular momentum of planet is constant

i.e., $$mv_1 r_1 = mv_2 r_2$$

or $$v_1 r_1 = v_2 r_2.$$

Total energy of planet is constant

i.e., $$\frac{-GMm}{r_1}+\frac{1}{2}mv_1^2=\frac{-GMm}{r_2}+\frac{1}{2}mv_2^2$$

i.e., $$GM\left\{\frac{1}{r_2}-\frac{1}{r_1}\right\}=\frac{v_2^2-v_1^2}{2}=\frac{v_2^2}{2}-\frac{v_1^2}{2}$$

or, $$GM\left\{\frac{r_1-r_2}{r_1r_2}\right\}=\frac{\left(\frac{v_1r_1}{r_2}\right)^2}{2}-\frac{v_1^2}{2}=\frac{v_1^2}{2}\left\{\frac{r_1^2}{r_2^2}-1\right\}$$

i.e., $$GM\left\{\frac{r_1-r_2}{r_1r_2}\right\}=\frac{v_1^2}{2}\frac{\left(r_1^2-r_2^2\right)}{r_2^2}$$

or, $$v_1^2=\frac{2GM(r_1-r_2)r_2^2}{(r_1^2-r_2^2)r_1r_2}=\frac{2GM\times r_2}{r_1(r_1+r_2)}$$

$$v_1=\sqrt{\frac{2GMr_2}{r_1(r_1+r_2)}}.$$

Angular momentum of the planet

$$=mv_1r_1=m\sqrt{\frac{2GMr_1r_2}{r_1+r_2}}.$$

KEPLER'S LAWS

Kepler discovered three empirical laws that accurately described the motions of the planets. The three laws may be stated as,

1. Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse. This law is also known as the law of elliptical orbits and obviously gives the shape of the orbits of the planets round the sun.
2. The radius vector, drawn from the sun to a planet, sweeps out equal areas in equal time, i.e., its areal velocity (or the area swept out by it per unit time) is constant. This is referred to as the law of areas and gives the relationship between the orbital speed of the planet and its distance from the sun.
3. The square of the planet's time period is proportional to the cube of the semi-major axis of its orbit. This is known as the harmonic law and gives the relationship between the size of the orbit of a planet and its time of revolution.

Kepler did not know why the planets move in this way. Three generations later when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's s can be derived. They are consequences of Newton's law of motion and the law of gravitation.

Let us first consider the elliptical orbits described in Kepler's first law. Figure 7.8 shows the geometry of the ellipse. The longest dimension is the major axis with half length a. This half length is called the semi-major axis.

$$SP+S'P=\text{constant}$$

Here, S and S' are the foci and P any point on the ellipse.

The sun is at S and planet at P.

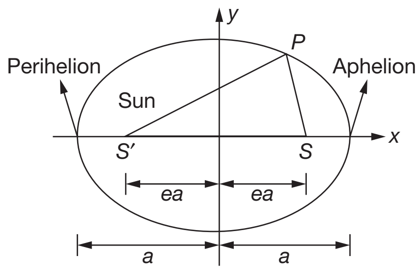


Fig. 7.8

The distance of each focus from the centre of ellipse is ea, where e is the dimensionless number between 0 to 1 called the eccentricity. If $e = 0$, the ellipse is a circle. The actual orbits of the planets are nearly circular, their eccentricities range from 0.007 for Venus to 0.248 for Pluto. For earth $e = 0.017$. The point in the planet's orbit closest to the sun is the perihelion and the point most distant from the sun is aphelion.

BRAIN MAP

GRAVITATION

1. Newton's law of gravitation $\vec{F} = \dfrac{Gm_1m_2}{r^2}\hat{r}$.

2. Gravitational force when a particle of mass m is placedin gravitational field $\vec{F} = m\vec{E}$.

3. Field due to a point mass $= \dfrac{Gm}{r2}\hat{r}$.

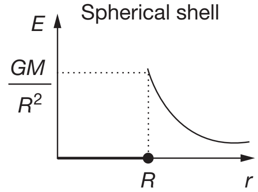
4. Field due to ring at an axial distance $= \dfrac{Gmr}{(R^2+r^2)^{3/2}}$.

5. Field due to disc $= \dfrac{2Gmr}{R^2}\left(\dfrac{1}{r} - \dfrac{1}{\sqrt{r^2+R^2}}\right)$.

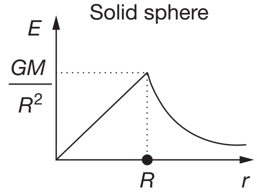
6. Field due to thin spherical shell: Inside = 0 Outside $= \dfrac{GM}{r^2}$.

7. Field due to solid sphere: Inside $= \dfrac{GMr}{R^3}$ Outside $= \dfrac{GM}{r^2}$.

Graph
Spherical shell



Graph
Solid sphere



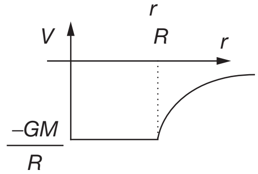
8. Gravitational potential due to a point mass $= -\dfrac{Gm}{r}$.

9. Gravitational potential due to a ring of radius R $= \dfrac{-GM}{\sqrt{R^2+r^2}}$.

10. Gravitational potential due to uniform spherical shell

Inside $= \dfrac{-GM}{R}$; constant.

Outside $= \dfrac{-GM}{r}$.



11. Escape velocity $= \sqrt{\dfrac{2GM}{R}}$.

12. Orbital speed $= \sqrt{\dfrac{GM}{R}}$.

13. Time period (T) $= \dfrac{2\pi r\sqrt{r}}{\sqrt{GM}}$.

14. Gravitational potential energy when two masses m_1 and m_2 are at a distance r apart $= \dfrac{-Gm_1m_2}{r}$.

EXERCISES

Single Option Correct Type

1. The Fig. 7.9 shows a planet in elliptical orbit around the sun S. Where is the kinetic energy of the planet maximum?

(A) P_1 (B) P_2 (C) P_3 (D) P_4

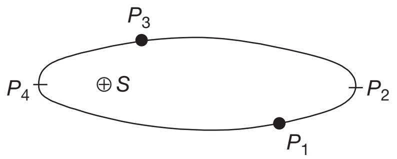


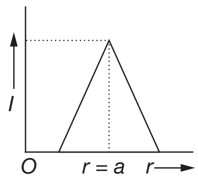
Fig. 7.9

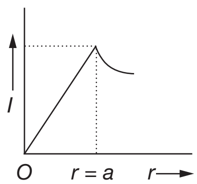
2. The ratio of the radii of the planets P_1 and P_2 is k_1. The ratio of the acceleration due to the gravity on them is k_2. The ratio of the escape velocities from them will be

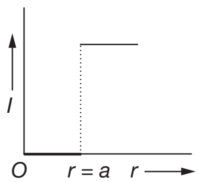
(A) $k_1 k_2$ (B) $\sqrt{k_1 k_2}$

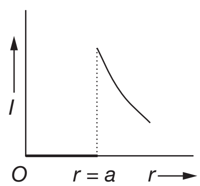
(C) $\sqrt{(k_1/k_2)}$ (D) $\sqrt{(k_2/k_1)}$

3. Which of the following graphs represent correctly the variation of intensity of gravitational field I with the distance r from the centre of a spherical shell of mass M and radius a?

(A)


(B)


(C)


(D)


4. The orbital speed of Jupiter is

(A) greater than the orbital speed of earth.
(B) lesser than the orbital speed of earth.
(C) equal to the orbital speed of earth.
(D) Zero.

5. The period of a satellite in a circular orbit of radius R is T. The period of another satellite in a circular orbit of radius 4 R is

(A) 4 T (B) $T/4$ (C) 8 T (D) $T/8$

6. The period of a satellite in a circular orbit around a planet is independent of,

(A) the mass of the planet.
(B) the radius of the planet.
(C) the mass of the satellite.
(D) all of three parameters a, b and c.

7. The acceleration due to gravity on the surface of the moon is $\frac{1}{6}$ that of the surface of earth and the diameter of the moon is $\frac{1}{4}$ that of earth. The ratio of escape velocities on earth and moon will be

(A) $\frac{\sqrt{6}}{2}$ (B) $\sqrt{24}$ (C) 3 (D) $\frac{\sqrt{3}}{2}$

8. At a height above the surface of the earth equal to the radius of the earth the value of g (acceleration due to gravity on the surface of the earth) will be nearly

(A) Zero (B) $\sqrt{g}$ (C) $\frac{g}{4}$ (D) $\frac{g}{2}$

9. Two satellites S_1 and S_2 describe circular orbits of radii r and $2r$ respectively around a planet. If the orbital angular velocity of S_1 is ω, the orbital angular velocity of S_2 is

(A) $\frac{\omega}{2\sqrt{2}}$ (B) $\frac{\omega\sqrt{2}}{3}$ (C) $\frac{\omega}{\sqrt{2}}$ (D) $\omega\sqrt{2}$

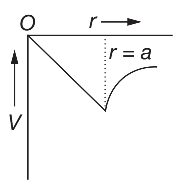
10. A person brings a mass of 1 kg from infinity to a point A. Initially the mass was at rest but it moves at a speed of 2 m/s as it reaches A. The work done by the person on the mass is –3 J. The potential at A is

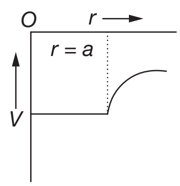
(A) –3 J/kg (B) –2 J/kg
(C) –5 J/kg (D) None of these

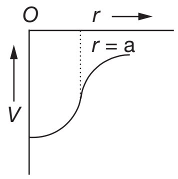
11. An artificial satellite moving in a circular orbit around the earth has a total energy (KE + PE) is E_0. Its potential energy is

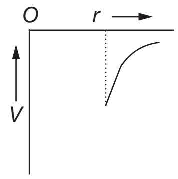
(A) $-E_0$ (B) 1.5 E_0
(C) 2 E_0 (D) E_0

12. P is a point at a distance r from the centre of a solid sphere of radius a. The gravitational potential at P is V. If V is plotted as a function of r, which is the correct curve?

(A)


(B)


(C) 

(D) 

13. Four particles of equal mass M move along a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is

(A) $\frac{GM}{R}$ (B) $\sqrt{\left(\frac{GM}{R}\right)}$

(C) $\sqrt{\left[\frac{GM}{R}\left(\frac{2\sqrt{2}+1}{4}\right)\right]}$ (D) $\sqrt{\left[\frac{GM}{R}(\sqrt{2}+1)\right]}$

14. A simple pendulum has a time period T_1 when on the earth's surface, and T_2 when taken to a height R above the earth's surface, where R is radius of earth. The value of T_2/T_1 is

(A) 1 (B) $\sqrt{2}$ (C) 4 (D) 2

15. A uniform spherical shell gradually shrinks maintaining its shape. The gravitational potential at the centre

(A) Increases (B) Decreases
(C) Remains constant (D) Oscillates

16. A body of mass m rises to a height $h = R/5$ from the earth's surface where R is radius of the earth. If g is acceleration due to gravity at the earth surface, the increase in potential energy is

(A) mgR (B) $(4/5)\ mgR$
(C) $(1/6)\ mgR$ (D) $(6/7)\ mgR$

17. If the distance between the earth and the sun were half its present value, the number of days in a year would have been

(A) 64.5 (B) 129
(C) 182.5 (D) 730

18. Imagine a light planet revolving around a very massive star in a circular orbit of radius R with a period of revolution T. If the gravitational force of attraction between the planet and the star is proportional to $R^{-5/2}$, T^2 is proportional to

(A) R^3 (B) $R^{7/2}$ (C) $R^{3/2}$ (D) $R^{3.75}$

19. A body is suspended from a spring balance kept in a satellite. The reading of the balance is W_1 when the satellite goes in an orbit of radius R and is W_2 when it goes in an orbit of radius 2 R.

(A) $W_1 = W_2$ (B) $W_1 < W_2$
(C) $W_1 > W_2$ (D) $W_1 \neq W_2$

20. A planet is revolving around the sun in elliptical orbit. Its closest distance from the sun is r and the farthest distance is R. If the orbital velocity of the planet closest to the sun be v, then what is the velocity at the farthest point?

(A) vr/R (B) vR/r

(C) $v\left(\frac{r}{R}\right)^{1/2}$ (D) $v\left(\frac{R}{r}\right)^{1/2}$

21. The orbital velocity of an artificial satellite in a circular orbit just above earth's surface is v_0. For a satellite orbiting in a circular orbit at an altitude of half of earth's radius is

(A) $\sqrt{\frac{3}{2}}\,v_0$ (B) $\sqrt{\frac{2}{3}}\,v_0$ (C) $\frac{3}{2}v_0$ (D) $\frac{2}{3}v_0$

22. A particle is placed in a field characterized by a value of gravitational potential given by $V = -kxy$, where k is a constant. If $\vec{E}_g$ is the gravitational field then,

(A) $\vec{E}_g = k(x\hat{i} + y\hat{j})$ and is conservative in nature.
(B) $\vec{E}_g = k(y\hat{i} + x\hat{j})$ and is conservative in nature.
(C) $\vec{E}_g = k(x\hat{i} + y\hat{j})$ and is non-conservative in nature
(D) $\vec{E}_g = k(y\hat{i} + x\hat{j})$ and is non-conservative in nature.

23. Three equal masses m kg are placed at the vertices of an equilateral triangle of side a metre. The gravitational potential energy equals to

(A) $-\frac{3Gm^2}{a}$ (B) $-\frac{3Gm}{a^2}$ (C) $-\frac{3Gm}{a}$ (D) $\frac{3Gm^2}{a}$

24. If three uniform spheres, each having mass M and radius R, are kept in such a way that each touches the other two, the magnitude of the gravitational force on any sphere due to the other two is

(A) $\frac{GM^2}{4R^2}$ (B) $\frac{2GM^2}{R^2}$

(C) $\frac{2GM^2}{4R^2}$ (D) $\frac{\sqrt{3}GM^2}{4R^2}$

25. The period of revolution of planet A around the sun is 8 times that of B. The distance of A from the sun is how many times greater than that of B from the sun?

(A) 2 (B) 3 (C) 4 (D) 5

26. If the length of a simple pendulum is equal to the radius R of the earth, its time period will be

(A) $2\pi\sqrt{R/g}$ (B) $2\pi\sqrt{R/2g}$

(C) $2\pi\sqrt{2R/g}$ (D) $\pi\sqrt{R/2g}$

27. Given that mass of the earth is M and its radius is R. A body is dropped from a height equal to the radius of the earth above the surface of earth. When it reaches the ground its velocity will be

(A) $\frac{GM}{R}$ (B) $\left[\frac{GM}{R}\right]^{1/2}$

(C) $\left[\frac{2GM}{R}\right]^{1/2}$ (D) $\left[\frac{2GM}{R}\right]$

28. The earth is an approximate sphere. If the interior contained matter which is not of the same density everywhere, then on the surface of the earth, the acceleration due to gravity.

(A) Will be directed towards the centre but not the same everywhere.
(B) Will have the same value everywhere but not directed towards the centre.
(C) Will be same everywhere in magnitude directed towards the centre.
(D) Cannot be zero at any point.

29. As observed from the earth, the sum appears to move in an approximate circular orbit. For the motion of another planet like mercury as observed from the earth, this would

(A) be similarly true.
(B) not be true because the force between the earth and mercury is not inverse square law.
(C) not be true because the major gravitational force on mercury is due to the sun.
(D) not be true because mercury is influenced by forces other than gravitational forces.

30. Different points in the earth are at slightly different distances from the sun and hence experience different forces due to gravitation. For a rigid body, we know that if various forces act at various points in it, the resultant motion is as if a net force acts on the CM (centre of mass) causing translation and a net torque at the CM causing rotation around an axis through the CM. For the earth-sun system (approximating the earth as a uniform density sphere).

(A) The torque is zero.
(B) The torque cause the earth to spin.
(C) The rigid body result is not applicable since the earth is not even approximately a rigid body.
(D) The torque causes the earth to move around the sun.

31. Satellites orbiting the earth have finite life and sometimes debris of satellites fall to the earth, This is because,

(A) the solar cells and batteries in satellites run out.
(B) the laws of gravitation predict a trajectory spiralling inwards.
(C) of viscous forces causing the speed of satellite and hence height to gradually decrease.
(D) of collisions with other satellites.

32. Both the earth and the moon are subject to the gravitational force of the sun. As observed from the sun, the orbit of the moon

(A) will be elliptical.
(B) will not be strictly elliptical because the total gravitational force on it is not central.
(C) is not elliptical but will necessarily be a closed curve.
(D) deviates considerably from being elliptical due to influence of planets other than the earth.

33. In the solar system, the inter-planetary region has chunks of matter (much smaller in size compared to planets) called asteroids. They

(A) will not move around the sun, since they have very small masses compared to the sun.
(B) will move in an irregular way because of their small masses and will drift away into outer space.
(C) will move around the sun in closed orbits but not obey Kepler's laws.
(D) will move in orbits like planets and obey Kepler's laws.

34. Choose the wrong option.

(A) Inertial mass is a measure of difficulty of accelerating a body by an external force whereas the gravitational mass is relevant in determining the gravitational force on it by an external mass.
(B) That the gravitational mass and inertial mass are equal is an experimental result.
(C) That the acceleration due to gravity on the earth is the same for all bodies is due to the equality of gravitational mass and inertial mass.
(D) Gravitational mass of a particle like proton can depend on the presence of neighbouring heavy objects but the inertial mass cannot.

35. Particles of mass $2M$, m and M are, respectively, at point A, B and C with $AB = \frac{1}{2}(BC)$. m is very much smaller than M and at time $t = 0$, they are all at rest as given in Fig. 7.10.

At subsequent times before any collision takes place.

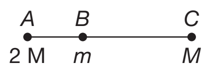


Fig. 7.10

(A) m will remain at rest.
(B) m will move towards M.
(C) m will move towards 2 M.
(D) m will have oscillatory motion.

More than One Option Correct Type

36. A particle initially at rest is displaced by applying a non-conservative force F in a uniform gravitational field. In the process, following physical quantities associated with the particle are measured.

ΔU = change in gravitational potential energy
ΔK = change in kinetic energy
ΔW_1 = work done by the force F
ΔW_2 = work done by the gravitational force

(A) $\Delta W_2 = -\Delta U$ (B) $\Delta K = \Delta W_1 + \Delta W_2$
(C) $\Delta K + \Delta U = \Delta W_1 + \Delta W_2$ (D) $\Delta W_1 = \Delta W_2$

37. Three planets of same density have radii R_1, R_2 and R_3 such that $R_1 = 2R_2 = 3R_3$. The gravitational field at their respective surfaces are g_1, g_2 and g_3 and escape velocities from their surfaces are v_1, v_2 and v_3 respectively, then

(A) $g_1/g_2 = 2$ (B) $g_1/g_3 = 3$
(C) $v_1/v_2 = 1/4$ (D) $v_1/v_3 = 3$

38. Two spherical planets have the same mass but densities in the ratio 1 : 8. For these planets, the

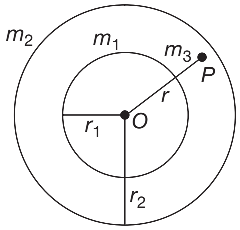
(A) acceleration due to gravity will be in the ratio 4 : 1.
(B) acceleration due to gravity will be in the ratio 1 : 4.
(C) escape velocities from their surfaces will be in the ratio $\sqrt{2} : 1$.
(D) escape velocities from their surfaces will be in the ratio $1 : \sqrt{2}$.

39. Consider an attractive force which is central but is inversely proportional to the first power of distance. If such a particle is in circular orbit under such a force, which of the following statements are correct.

(A) $v \propto r$ (B) $v \propto r^{o}$ (C) $T \propto r$ (D) $T \propto r^{o}$

40. Two concentric spherical shells masses m_1 and m_2 and radii r_1 and r_2. Then

(A) outer shell will have no contribution in gravitational field at point P.
(B) force on P is directed towards O.
(C) force on P is $\dfrac{Gm_1m_2}{r^2}$.
(D) force on P is $\dfrac{Gm_1m_3}{r^2}$.



41. Two concentric shells of uniform density of mass M_1 and M_2 are situated as shown in Fig 7.11. The forces experienced by a particle of mass m when placed at position A, B and C, respectively, are ($OA = r_1$, $OB = r_2$ and $OC = r_3$)

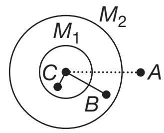


Fig. 7.11

(A) $F_A = \dfrac{G(M_1 + M_2)m}{r_1^2}$ (B) $F_B = 0$

(C) $F_B = \dfrac{G(M_1)m}{r_2^2}$ (D) $F_C = 0$

42. A solid sphere of uniform density and radius 4 units is located with its centre at the origin O of co-ordinates. Two spheres of equal radii 1 units, with their centres at $A(-2, 0, 0)$ and B (2, 0, 0) respectively, are taken out of the solid leaving behind spherical cavities as shown in Fig. 7.12. Then

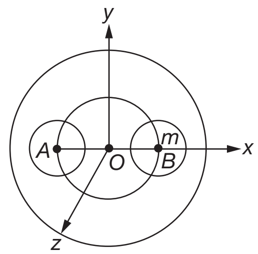


Fig. 7.12

(A) the gravitational field due to this object at the origin is zero.
(B) the gravitational field at the point $B(2, 0, 0)$ is zero.
(C) the gravitational potential is same at all points on the circle $y^2 + z^2 = 36$.
(D) the gravitational potential is same at all points on the circle $y^2 + z^2 = 4$.

43. Two objects of masses m and $4m$ are at rest at an infinite separation. They move towards each other under mutual gravitational force of attraction. If G is the universal gravitational constant. Then at separation r

(A) the total energy of the two objects is zero.
(B) their relative velocity of approach is $\left(\dfrac{10Gm}{r}\right)^{\frac{1}{2}}$ in magnitude.
(C) the total kinetic energy of the objects is $\dfrac{4Gm^2}{r}$.
(D) net angular momentum of both the particles is zero about any point.

Passage Based Questions

Passage 1

A sphere of density ρ and radius a has a concentric cavity of radius b as shown in the Fig. 7.13.

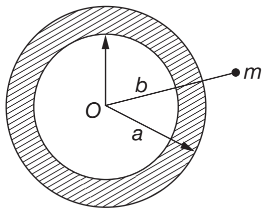


Fig. 7.13

44. Gravitation force F exerted by the sphere on the particle of mass m, located at a distance r from the centre of sphere as a function r when $b < r < a$.

(A) $F_r = 0$

(B) $F_r = \frac{4}{3}\pi G\rho m\left(\frac{a^3 - b^3}{r^2}\right)$

(C) $F_r = \frac{4}{3}\pi G\rho m\left(r - \frac{b^3}{r^2}\right)$

(D) None of these

45. Gravitational potential energy as a function of r, where r is the distance from the centre of the sphere. When $0 < r < b$,

(A) $U(r) = -2\pi G\rho m(a^2 - b^2)$.

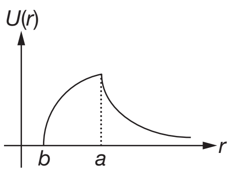
(B) $U(r) = \frac{-4\pi G\rho m}{3r^2}(a^3 - b^3)$.

(C) $U(r) = -\frac{2\pi G\rho m}{3r}(3ra^2 - 2b^2 - r^3)$.

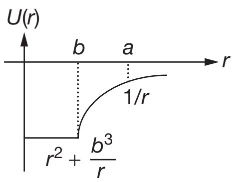
(D) None of these.

46. Which one is correct sketch for potential as a function of r.

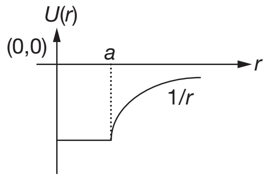
(A)



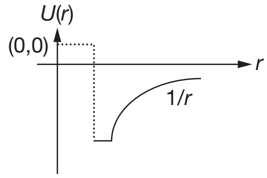
(B)



(C)



(D)



Passage 2

In 1783, John Mitchell noted that if a body having same density as that of the sun but radius 500 times that of the sun, magnitude of its escape velocity will be greater than c, the speed of light. All the light emitted by such a body will return to it. He, thus, suggested the existence of a black hole. $v = c = \sqrt{\frac{2GM}{R}}$ suggests that a body of mass M will act as a black hole if its radius R is less than or equal to a certain critical radius. Karl Schwarzchild, in 1926, derived the expression for the critical radius R_S called Schwarzchild radius. The surface of the sphere with radius R_S surrounding a black hole is called event horizon.

47. Schwarzchild radius R_S is

(A) $> \frac{2GM}{c^2}$ (B) $\frac{GM}{c^2}$

(C) $< \frac{2GM}{c^2}$ (D) $\frac{2GM}{c^2}$

48. Density of the sun is

(A) 14.1 kg m^{-3} (B) 141.1 kg m^{-3}

(C) 1410 kg m^{-3} (D) None of these

49. To make black hole with density of the sun, the ratio of radius of the object to that of sun should be

(A) 5 (B) 50 (C) 500 (D) 2.5

Passage 3

Suppose you are, as the header of a group of scientists working in NASA, sent to a planet named NSP2009 to study that planet you have following data with you.

1. The planet NSP2009 is spherical in shape and of radius R.
2. The mass of the planet NSP2009 is M.

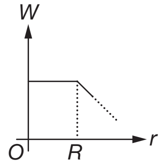
50. If your team finds that the weight of any body inside the planet remains same as on its surface, what could be the possible value of mass density of the planet as a function of radial distance (where ρ_0 is a content) ($R \geq >0$)

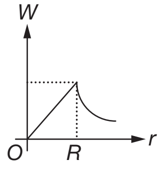
(A) $\rho = \rho_0 r^0$ (B) $\rho = \rho_0 r$
(C) $\rho = \rho_0 r^{-1}$ (D) $\rho = \rho_0 r^{-1/2}$

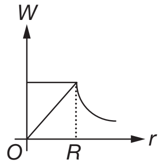
51. Being a good physicist and mathematician also we expect that you have chosen the right choice in constant question. Now tell the value of content ρ_0.

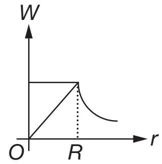
(A) $\rho_0 = \dfrac{M}{2\pi R^3}$ (B) $\rho_0 = \dfrac{M}{2\pi R^2}$
(C) $\rho_0 = \dfrac{3M}{4\pi R^3}$ (D) $\rho_0 = \dfrac{M}{2\pi R^4}$

52. Suppose weight of a body at the surface of the planet is w_0. Suggest the possible variation of weight of the body with radial distance.

(A)


(B)


(C)


(D)


Match the Column Type

53. From two particles describing circular path under mutual gravitational force of attraction.

Column-I	Column-II
(A) If speed of the particles decreases such that velocity of centre of mass remains zero then	(1) Particle separation will increase
(B) If one of the particles stops then	(2) Particle separation will decrease
(C) If both particle stops for a moment then	(3) Common centre will lie on centre of mass
(D) If speed of both particles increase such that velocity of centre of mass remains zero.	(4) Particle collide at centre of mass

54. Gravitational potential on the surface of an isolated uniform solid sphere of mass M and radius R is found to be V_0. A spherical cavity having radius $R/2$ is created inside the sphere which is touching the surface of original sphere. The cavity is then filled with material having density 16 times that of original sphere. A, B, C and D are consecutive points as shown in Fig. 7.14 each $R/2$ apart. V_A, V_B, V_C and V_D are gravitational potentials found at points A, B, C and D, respectively.

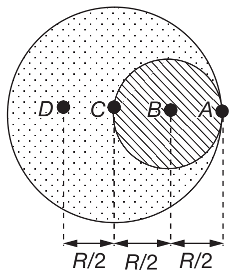
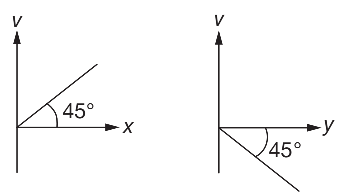


Fig. 7.14

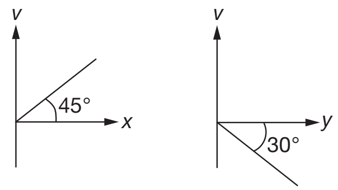
Column-I	Column-II
(A) V_A	(1) $7V_0$
(B) V_B	(2) $13/4\ V_0$
(C) V_C	(3) $19/4\ V_0$
(D) V_D	(4) $21/4\ V_0$
	(5) $25/4\ V_0$

55. Column-I shows the parabola angle of the eclectic field vector makes with positive x-direction. Column-II shows graph of gravitational potential V versus x and y.

Column-I	Column-II
(A) $0 \leq \theta \leq 35°$	(1) 

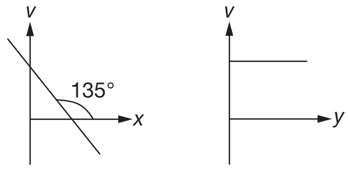
(B) $0 \le \theta \le 45°$

(2)



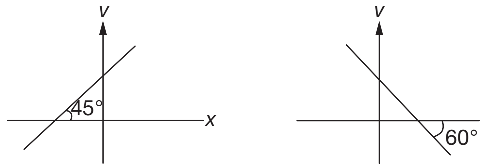
(C) $45° \le \theta \le 90°$

(3)

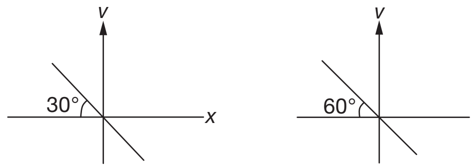


(D) $90° \le \theta \le 180°$

(4)



(5)



Assertion-Reason Type

56. Assertion: If the radius of earth is decreased keeping its mass constant, effective value of g increases at pole while increasing or decreasing at equator.

Reason: Value of g on the surface of earth is given by $g = \dfrac{GM}{R^2}$.

(A) A (B) B (C) C (D) D

57. Assertion: A particle of mass m is projected towards the heavy planet as shown in Fig. 7.15. The angular momentum of the particle is conserved about point O.

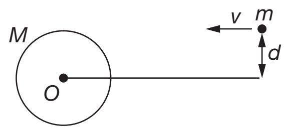


Fig. 7.15

Reason: Torque of gravitational force about point O is always zero.

(A) A (B) B (C) C (D) D

Integer Type

58. A particle is projected vertically upwards from the surface of the earth with a kinetic energy equal to $\dfrac{1}{3}$ times the minimum kinetic energy needed to escape. If radius of the earth is 6400 km, the maximum height attained by the particle (in km) from the surface of the earth is $\dfrac{nR}{2}$ then the value of n is.

59. Assuming that the law of gravitation is of the form $F = \dfrac{GMm}{r^3}$. Find on what power of r, will the square of time period depend.

60. A particle of mass m is dropped from the earth surface into a tunnel dug through a diameter of the earth. The velocity with which it cross the centre of the earth is

$\sqrt{ngR}$, then the value of n is? Assume the earth to be of uniform density. Express your answer in terms of radius R of the earth and the acceleration due to gravity g at the surface of the earth.

61. Calculate the ratio m_0/m for a rocket if it is to escape from the earth. Given escape velocity = 11.2 km/s and exhaust speed of gases is 2 km/s.

62. An artificial satellite is moving in a circular around the earth with a speed equal to half the magnitude of escape velocity from earth. The height of satellite above earth's surface is nR Then the value of n is.

Previous Years' Questions

63. The kinetic energy needed to project a body of mass m from the earth's surface (radius R) to infinity is **[2002]**

(A) $\frac{mgR}{2}$ (B) $2mgR$

(C) mgR (D) $\frac{mgR}{4}$

64. The escape velocity of a body depends upon mass as **[2002]**

(A) m^0 (B) m^1 (C) m^2 (D) m^3

65. Two spherical bodies of mass M and $5M$ and radii R and $2R$ respectively, are released in free space with initial separation between their centres equal to 12 R. If they attract each other due to gravitational force only, then the distance covered by the smaller body just before collision is **[2003]**

(A) 2.5 R (B) 4.5 R (C) 7.5 R (D) 1.5 R

66. The escape velocity for a body projected vertically upwards from the surface of earth is 11 km/s. If the body is projected at an angle of $45°$ with the vertical, the escape velocity will be **[2003]**

(A) $11\sqrt{2}$ km/s (B) 22 km/s

(C) 11 km/s (D) $\frac{11}{\sqrt{2}}$ m/s

67. A satellite of mass m revolves around the earth of radius R at a height x from its surface. If g is the acceleration due to gravity on the surface of the earth, the orbital speed of the satellite is **[2004]**

(A) gx (B) $\frac{gR}{R-x}$

(C) $\frac{gR^2}{R+x}$ (D) $\left(\frac{gR^2}{R+x}\right)^{1/2}$

68. The time period of an earth satellite in circular orbit is independent of **[2004]**

(A) the mass of the satellite.
(B) radius of its orbit.
(C) both the mass and radius of the orbit.
(D) neither the mass of the satellite nor the radius of its orbit.

69. If g is the acceleration due to gravity on the earth's surface, the gain in the potential energy of an object of mass m raised from the surface of the earth to a height equal to the radius R of the earth, is **[2004]**

(A) 2 mgR (B) $\frac{1}{2}mgR$

(C) $\frac{1}{4}mgR$ (D) mgR

70. Suppose the gravitational force varies inversely as the n^{th} power of distance. Then the time period of a planet in circular orbit of radius R around the sun will be proportional to **[2004]**

(A) $R^{\left(\frac{n+1}{2}\right)}$ (B) $R^{\left(\frac{n-1}{2}\right)}$ (C) R^n (D) $R^{\left(\frac{n-2}{2}\right)}$

71. The change in the value of g at a height h above the surface of the earth is the same as at a depth d below the surface of earth. When both d and h are much smaller than the radius of earth, then which one of the following is correct? **[2005]**

(A) $d = \frac{h}{2}$ (B) $d = \frac{3h}{2}$

(C) $d = 2h$ (D) $d = h$

72. Average density of the earth **[2005]**

(A) does not depend on g.
(B) is a complex function of g.
(C) is directly proportional to g.
(D) is inversely proportional to g.

73. The change in the value of g at a height h above the surface of the earth is the same as that of a depth d below the surface of earth. When both d and h are much smaller than the radius of earth, then which one of the following is correct? **[2005]**

(A) $d = \frac{h}{2}$ (B) $d = \frac{3h}{2}$
(C) $d = 2h$ (D) $d = h$

74. A particle of mass 10 g is kept on the surface of a uniform sphere of mass 100 kg and radius 10 cm. Find the work to the done against be gravitational force between them, to take the particle far away from the sphere. (you may take $G = 6.67\times10^{-11}$ Nm^2/Kg^{-2}) **[2005]**

(A) 13.34×10^{-10} J (B) 3.33×10^{-10} J
(C) 6.67×10^{-9} J (D) 6.67×10^{-10} J

75. A planet in a distance solar system is 10 times more massive than the earth and its radius is 10 times smaller. Given that the escape velocity from the earth is 11 kms^{-1}, the escape velocity from the surface of the planet would be. **[2008]**

(A) 1.1 kms^{-1} (B) 11 kms^{-1}
(C) 110 kms^{-1} (D) 0.11 kms^{-1}

76. This question contains Statement 1 and Statement 2 of the four choices given after the statements, choose the one that best describes the two statements. **[2008]**

Statement 1: For a mass *M* kept at the centre of a cube of side *a* the flux of gravitational field passing through its sides $4\pi GM$.
Statement 2: If the direction of a field due to a point source is radial and its dependence on the distance *r* from the source is given as $\frac{1}{r^2}$, its flux through a closed surface depends only on the strength of the source enclosed by the surface and not on the size or shape of the surface.
(A) Statement 1 is false, Statement 2 is true.
(B) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1.
(C) Statement 1 is false, Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.
(D) Statement 1 is true, Statement 2 is false.

77. The height at which the acceleration due to gravity becomes $\frac{g}{9}$ (where g = the acceleration due to gravity on the surface of the earth) in terms of *R*, the radius of the earth is **[2009]**

(A) $\frac{R}{\sqrt{2}}$ (B) $R/2$ (C) $\sqrt{2}R$ (D) $2R$

78. The mass of a spaceship is 1000 kg. It is to be launched from the earth's surface out into free space. The value of *g* and *r* (radius of earth) are 10 m/s^2 and 6400 km respectively. The required energy for this work will be **[2012]**

(A) 6.4×10^{11} J (B) 6.4×10^{8} J
(C) 6.4×10^{9} J (D) 6.4×10^{10} J

79. This question has Statement 1 and Statement 2. Of the four choices given after the statements, choose the one that best describes the two statements. **[2013]**

Statement 1: Higher the range, greater is the resistance of ammeter.
Statement 2: To increase the range of ammeter, additional shunt needs to be used across it.
(A) Statement 1 is true, Statement 2 is true, Statement 2 is **not** the correct explanation of Statement 1.
(B) Statement 1 is true, Statement 2 is false.
(C) Statement 1 is false, Statement 2 is true.
(D) Statement 1 is true, Statement 2 is true, Statement 2 is **correct** explanation of Statement 1.

80. Four particles, each of mass *M* and equidistant from each other, move along a circle of radius *R* under the action of their mutual gravitational attraction. The speed of each particle is **[2014]**

(A) $\sqrt{\frac{GM}{R}}$ (B) $\sqrt{2\sqrt{2}\frac{GM}{R}}$
(C) $\sqrt{\frac{GM}{R}\left(1+2\sqrt{2}\right)}$ (D) $\frac{1}{2}\sqrt{\frac{GM}{R}\left(1+2\sqrt{2}\right)}$

81. From a solid sphere of mass *M* and radius *R*, a spherical portion of radius $\frac{R}{2}$ is removed, as shown in Fig. 7.16. Taking gravitational potential $V = 0$ at $r = \infty$, the potential at the centre of the cavity thus formed is (G = gravitational constant) **[2015]**

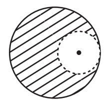


Fig. 7.16

(A) $\frac{-GM}{R}$ (B) $\frac{-2GM}{3R}$
(C) $\frac{-2GM}{R}$ (D) $\frac{-GM}{2R}$

82. The height at which the acceleration due to gravity becomes $\frac{g}{9}$ (where g = the acceleration due to gravity on the surface of the earth) in terms of *R*, the radius of the earth, is **[2010]**

(A) $2R$ (B) $\frac{R}{\sqrt{3}}$ (C) $\frac{R}{2}$ (D) $\sqrt{2}R$

83. Two particle of equal mass *m* go around a circle of radius *R* under the action of their mutual gravitational attraction. The speed of each particle with respect to their centre of mass is **[2011]**

(A) $\sqrt{\frac{Gm}{R}}$ (B) $\sqrt{\frac{Gm}{4R}}$

(C) $\sqrt{\frac{GM}{3R}}$ (D) $\sqrt{\frac{Gm}{2R}}$

84. Two bodies of masses m and $4m$ are placed at a distance r. The gravitational potential at a point on the line joining them where the gravitational field is zero is **[2011]**

(A) $-\frac{4Gm}{r}$ (B) $-\frac{6Gm}{r}$

(C) $-\frac{9Gm}{r}$ (D) Zero

85. The mass of a spaceship is 1000 kg. It is to be launched from the earth's surface out into free space. The value of g and r (radius of earth) are 10 m/s^2 and 6400 km respectively. The required energy for this work will be: **[2012]**

(A) 6.4×10^{11} J (B) 6.4×10^{8} J

(C) 6.4×10^{9} J (D) 6.4×10^{10} J

86. This question has Statement 1 and Statement 2. Of the four choices given after the statements, choose the one that best describes the two statements. **[2013]**

Statement 1: Higher the range, greater is the resistance of ammeter.

Statement 2: To increase the range of ammeter, additional shunt needs to be used across it.

(A) Statement 1 is true, Statement 2 is true, Statement 2 is **not** the correct explanation of Statement 1.
(B) Statement 1 is true, Statement 2 is false.
(C) Statement 1 is false, Statement 2 is true.
(D) Statement 1 is true, Statement 2 is true, Statement 2 is **correct** explanation of Statement 1.

87. Four particles, each of mass M and equidistant from each other, move along a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is **[2014]**

(A) $\sqrt{\frac{GM}{R}}$ (B) $\sqrt{2\sqrt{2}\frac{GM}{R}}$

(C) $\sqrt{\frac{GM}{R}\left(1+2\sqrt{2}\right)}$ (D) $\frac{1}{2}\sqrt{\frac{GM}{R}\left(1+2\sqrt{2}\right)}$

88. From a solid sphere of mass M and radius R, a spherical portion of radius $\frac{R}{2}$ is removed, as shown in Fig. 7.17. Taking gravitational potential $V=0$ at $r=\infty$, the potential at the centre of the cavity thus formed is (G = gravitational constant) **[2015]**

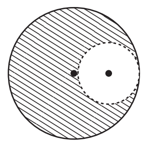


Fig. 7.17

(A) $\frac{-GM}{R}$ (B) $\frac{-2GM}{3R}$

(C) $\frac{-2GM}{R}$ (D) $\frac{-GM}{2R}$

89. A satellite is revolving in a circular orbit at a height h from the earth's surface (radius of earth R. $h << R$). The minimum increase in its orbital velocity required, so that the satellite could escape from the earth's gravitational field, is close to: (Neglect the effect of atmosphere) **[2016]**

(A) $\sqrt{gR}$ (B) $\sqrt{gR/2}$

(C) $\sqrt{gR}(\sqrt{2}-1)$ (D) $\sqrt{2gR}$

ANSWER KEYS

Single Option Correct Type

1. (D)	**2.** (B)	**3.** (D)	**4.** (B)	**5.** (C)	**6.** (C)	**7.** (B)	**8.** (C)	**9.** (A)	**10.** (C)
11. (C)	**12.** (C)	**13.** (C)	**14.** (D)	**15.** (B)	**16.** (C)	**17.** (B)	**18.** (B)	**19.** (A)	**20.** (A)
21. (B)	**22.** (B)	**23.** (A)	**24.** (D)	**25.** (C)	**26.** (B)	**27.** (B)	**28.** (D)	**29.** (C)	**30.** (A)
31. (C)	**32.** (B)	**33.** (D)	**34.** (D)	**35.** (C)					

More than One Option Correct Type

36. (A) and (B) **37.** (A), (B) and (D) **38.** (B) and (D) **39**. (B) and (C) **40.** (A), (B) and (D)
41. (A), (C) and (D) **42.** (A), (C) and (D) **43.** (A), (C) and (D)

Passage Based Questions

Passage 1

44. (C) **45.** (A) **46.** (B)

Passage 2

47. (D) **48.** (C) **49.** (C)

Passage 3

50. (A) **51.** (B) **52**. (C)

Match the Column Type

53. (A) → 1, 3; (B) → 1; (C) → 2, 4; (D) → 2, 3
54. (A) → 3; (B) → 1; (C) → 4; (D) → 2
55. (A) → (1, 3, 4, 5); (B) → 3; (C) → 3; (D) → (1, 2, 4)

Assertion-Reason Type

56. (B) **57.** (A)

Integer Type

58. 1 **59.** 4 **60.** 1 **61.** 270 **62.** 1

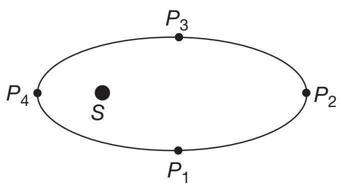
Previous Years' Questions

63. (C) **64.** (A) **65.** (C) **66.** (C) **67.** (D) **68.** (A) **69.** (B) **70.** (A) **71.** (C) **72.** (B)
73. (A) **74.** (D) **75.** (C) **76.** (B) **77.** (D) **78.** (D) **79.** (C) **80.** (D) **81.** (A) **82.** (A)
83. (B) **84.** (C) **85.** (D) **86.** (C) **87.** (D) **88.** (A) **89.** (C)

HINTS AND SOLUTIONS

Single Option Correct Type

1.



Angular momentum of the planet about S is conserved. So, mvr = constant.

v is maximum when r is minimum. So, v is maximum at point P_4.

The correct option is (D)

2. $v_e = \sqrt{2gR}$

$$\frac{v_{e_1}}{v_{e_2}} = \sqrt{\frac{g_1}{g_2}\frac{R_1}{R_2}} = \sqrt{k_1 k_2}$$

The correct option is (B)

3. Inside the shell gravitation field $\vec{I} = 0$

Outside the shell $\vec{I} = \frac{GM}{r^2}$

The correct option is (D)

4. $v=\sqrt{\frac{GM}{R}}$

Distance of Jupiter is more than earth from sun.

The correct option is (B)

5. $T^2 \propto R^3$

$\Rightarrow \quad \frac{T^2}{T_1^2}=\frac{R^3}{64R^3} \Rightarrow T_1 = 8T$

The correct option is (C)

6. $T=2\pi\sqrt{\frac{R^3}{GM_P}}$ $\quad$ (M_p = mass of planet)

The correct option is (C)

7. $v_e=\sqrt{2g_eR_e}$

$\Rightarrow \quad v_m=\sqrt{2g_mR_m}$

$\frac{v_e}{v_m}=\sqrt{\frac{g_e}{g_m}\frac{R_e}{R_m}}=\sqrt{24}$

The correct option is (B)

8. $g'=\frac{GM_e}{(R_e+R_e)^2}=\frac{g}{4} \quad \left(\because \quad g=\frac{GM_e}{R_e^2}\right)$

The correct option is (C)

9. $\frac{T_1^2}{T_2^2}=\frac{R_1^3}{R_2^3}=\frac{1}{8}$

$\Rightarrow \quad \frac{\omega_1}{\omega_2}=\frac{T_2}{T_1}=\sqrt{8}$

$\Rightarrow \quad \omega_2=\frac{\omega}{\sqrt{8}}=\frac{\omega}{2\sqrt{2}}$

The correct option is (A)

10. Change in kinetic energy = work done by external agent + work done by gravity

$\frac{1}{2}mv^2=-3+\text{work done by gravity}$

2 = –3 + work done by gravity

Work done by gravity = 5 J

Potential = –5 J/kg

The correct option is (C)

11. Potential energy = 2 (total energy) = $2E_0$

The correct option is (C)

12. Gravitational potential inside the sphere is

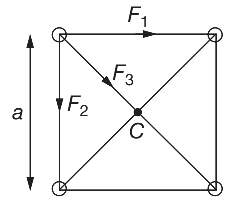
$V=-\frac{GM}{2a^3}\left(3a^2-r^2\right)$ and outside sphere $v=-\frac{GM}{r}$

The correct option is (C)

13. F_{net} (on each particle) towards C $=\frac{GM^2}{a^2}\sqrt{2}+\frac{GM^2}{2a^2}$

required centripetal force = F_{net}

$\frac{Mv^2}{a/\sqrt{2}}=\frac{GM^2}{2a^2}\left(2\sqrt{2}+1\right)$



Given that $\frac{a}{\sqrt{2}}=R$, $v=\sqrt{\frac{GM}{R}\left(\frac{2\sqrt{2}+1}{4}\right)}$

The correct option is (C)

14. $\frac{T_2}{T_1}=\sqrt{\frac{g}{g'}}=\sqrt{\frac{g}{g/4}}=2$

The correct option is (D)

15. $V=-\frac{GM}{2R^2}\left(3R^2-r^2\right)$

at center $r = 0$

$\Rightarrow \quad V=-\frac{3GM}{2R}$

as R decreases V, will decreases.

The correct option is (B)

16. $\text{PE}_1=-\frac{GMm}{R}$, $\text{PE}_2=-\frac{GMm}{\left(R+\frac{R}{5}\right)}=-\frac{5GMm}{6R}$

Increase in potential energy = $\text{PE}_2-\text{PE}_1=\frac{GMm}{R}\left(1-\frac{5}{6}\right)$

$=\frac{GMm}{6R}=\frac{mgR}{6} \quad \left(\because GM=gR^2\right)$

The correct option is (C)

17. $T^2 \propto R^3$, $\frac{T_1}{T_2}=\left(\frac{R}{R/2}\right)^{\frac{3}{2}}=2^{\frac{3}{2}}$

$T_2=\frac{T}{2^{\frac{3}{2}}}=\frac{365}{2^{\frac{3}{2}}}=129$ days.

The correct option is (B)

18. $F=m\omega^2R=\frac{m(4\pi^2)}{T^2}R=kR^{-\frac{5}{2}}$

$T^2 \propto R^{\frac{7}{2}}$

The correct option is (B)

19. In both cases reading will be zero because of weightlessness in space.

Hence, $W_1=W_2$.

The correct option is (A)

20. Applying conservation of angular momentum about the sun

$mvr=mVR$

$V=\frac{vr}{R}$

The correct option is (A)

21. Orbital velocity $= \sqrt{\dfrac{g_0 R^2}{R+h}}$ where R is radius of earth.

If $h = 0$, $v_0 = \sqrt{\dfrac{g_0 R^2}{R}} = \sqrt{g_0 R}$

If $h = \dfrac{R}{2}$, $v = \sqrt{\dfrac{g_0 R^2}{R + \dfrac{R}{2}}} = \sqrt{\dfrac{2 g_0 R}{3}} = \sqrt{\dfrac{2}{3}}\, v_0$.

The correct option is (B)

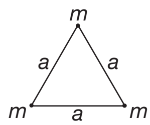
22. $\varepsilon_x = -\dfrac{\partial V}{\partial x} = ky \qquad \varepsilon_y = -\dfrac{\partial V}{\partial y} = kx$

$\therefore \quad \vec{\varepsilon} = k(y\hat{i} + x\hat{j})$ and conservative

The correct option is (B)

23. $\text{PE} = -\dfrac{Gm \times m}{a} - \dfrac{Gm \times m}{a} - \dfrac{Gm \times m}{a}$

$= -\dfrac{3Gm^2}{a}$



The correct option is (A)

24. $F_{net} = \sqrt{F^2 + F^2 + 2F^2 \cos 60^\circ}$, $F = \dfrac{GM^2}{4R^2}$, $F_{net} = \dfrac{\sqrt{3}GM^2}{4R^2}$

The correct option is (D)

25. $T^2 \alpha R^3$, $\dfrac{T_2}{T_1} = \left(\dfrac{R_2}{R_1}\right)^{3/2}$

$\Rightarrow \quad 8 = \left(\dfrac{R_2}{R_1}\right)^{3/2}$, $\dfrac{R_2}{R_1} = \mathbf{4}$

The correct option is (C)

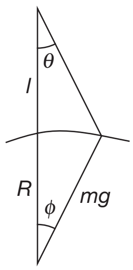
26. $F_{\text{restoring}} = -mg(\theta + \varphi)$

$a = -g\left(\dfrac{x}{l} + \dfrac{x}{R}\right)$

$l = R_e$

$a = -g\left(\dfrac{2}{R_e}\right)x$

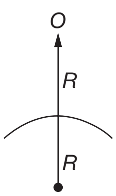
$T = 2\pi\sqrt{\dfrac{R}{2g}}$



The correct option is (B)

27. $-\dfrac{GMm}{2R} = -\dfrac{GMm}{R} + \dfrac{1}{2}mv^2$,

$\dfrac{1}{2}mv^2 = \dfrac{GMm}{2R}$, $V = \sqrt{\dfrac{GM}{R}}$



The correct option is (B)

28. If we assume the earth as a sphere of uniform density, then it can be treated as point mass placed at its centre. In this case, acceleration due to gravity $g = 0$ at the centre. It is not so, if the earth is considered as a sphere of non-uniform density. In that case, value of g will be different points and cannot be zero at any point.

The correct option is (D)

29. As observed from the earth, the sum appears to move in an approximate circular orbit. The gravitational force of attraction between the earth and the sun always follows inverse square law.

Due to relative motion between the earth and mercury, the orbit of mercury, as observed from the earth, will not be approximately circular, since the major gravitational force on mercury is due to the sun.

The correct option is (C)

30. As the earth is revolving around the sun in a circular motion due to gravitational attraction, the force of attraction will be of radial nature, i.e., angle between position vector r and force F is zero. So, torque $= |t| = |r \times F| = rF\sin 0^\circ = 0$.

The correct option is (A)

31. As the total energy of the earth satellite bounded system is negative $\left(\dfrac{-GM}{2a}\right)$ where a is radius of the satellite and M is mass of the earth.

Due to the viscous force acting on satellite, energy decreases continuously and radius of the orbit or height decreases gradually.

The correct option is (C)

32. As observed from the sun, two types of forces are acting on the moon one is due to gravitational attraction between the sun and the moon and the other is due to gravitational attraction between the earth and the moon. Hence, total force on the moon is not central.

The correct option is (B)

33. Asteroids are also being acted upon by central gravitational forces. Hence, they are moving in circular orbits like planets and obey Kepler's laws.

The correct option is (D)

34. Gravitational mass of proton is equivalent to its inertial mass and is independent of neighbouring heavy objects.

The correct option is (D)

35. Force on B due to $A = F_{BA} = \dfrac{G(2Mm)}{(AB)^2}$ towards BA

Force on B due to $C = F_{BC} = \dfrac{GMm}{(BC)^2}$ towards BC

As, $(BC) = 2AB$

$\Rightarrow \quad F_{BC} = \dfrac{GMm}{(2AB)^2} = \dfrac{GMm}{4(AB)^2} < F_{BA}$

Hence, m will move towards BA (i.e., 2M)

The correct option is (C)

More than One Option Correct Type

36. The correct option is (A) and (B)

37. $g = \frac{GM}{R^2} = \frac{G\left[\frac{4}{3}\pi R^3\rho\right]}{R^2}, g \propto R, \quad \frac{g_1}{g_2} = \frac{R_1}{R_2} = 2$

$\frac{g_1}{g_3} = \frac{R_1}{R_3} = 3, \quad v = \sqrt{2gR} \propto \sqrt{R^2} \propto R,$

$\frac{v_1}{v_2} = \frac{R_1}{R_2} = 2 \quad \frac{v_1}{v_3} = \frac{R_1}{R_3} = 3$

The correct option is (A), (B) and (D)

38. Density ratio = 1 : 8

$\therefore$ volume ratio = 8 : 1

Ratio of radii = 2 : 1

$g \propto \frac{1}{R^2}$, so gravitational acceleration is 1 : 4 and

$\frac{v_1}{v_2} = \sqrt{\frac{g_1R_1}{g_2R_2}} = \frac{1}{\sqrt{2}}$

The correct option is (B) and (D)

39. $F = \frac{-K}{r}$

where K is proportionally constant.

$\frac{mV^2}{r} = \frac{-K}{r}$ i.e., $V \propto r^0$

$T = \frac{2\pi r}{V} = \frac{2\pi r}{\sqrt{\frac{K}{m}}}$ i.e., $T \propto r$.

So choices (a) and (d) are wrong and choices (b) and (c) are correct.

The correct option is (B) and (C)

40. Gravitational field intensity inside the spherical shell is zero.

Gravitational force is always directed towards the source mass.

Force on P act only due to m_1 because point P is outside the spherical shell of mass m_1.

So choice (a), (b) and (d) are correct.

The correct option is (A), (B) and (D)

41. $F_A = \frac{C_2(M_1 + M_2)m}{r_1^2}, \ F_B = \frac{GM_1m}{r_2^2}, F_C = 0$

The correct option is (A), (C) and (D)

42. The gravitational force due to these masses on a mass at O is equal and opposite.

So, the resultant force is zero hence, the resultant field is zero.

Also, any point on y-z plane which is equidistant from two cavities will have zero field intensity hence constant potential.

The correct option is (A), (C) and (D)

43. At the distance of r the mechanical energy of the system is given by

$\frac{1}{2}4mv^2 + \frac{1}{2}mv^2 - \frac{4Gm^2}{r} = 0, K_{\text{total}} = \frac{4Gm^2}{r}$

The correct option is (A), (C) and (D)

Passage Based Questions

Passage 1

44. The correct option is (C)

45. The correct option is (A)

46. The correct option is (B)

Passage 2

47. The correct option is (D)

48. The correct option is (C)

49. The correct option is (C)

Passage 3

50. Say the density of planet on a function of its radial distance is $\rho = \rho_0 r^a \ R \geq r > 0$ consider the Gaussian surface as shown. Let man inside the Gaussian surface be m. Then accordingly,

$\frac{Gm}{r^2}$ = content $R \geq r > 0$

So $m \propto r^2$ (1)

and $m = \int_0^r \rho_0 r^a \left(4\pi r^2 dr\right)$

$= 4\pi\rho_0 \int_0^r r^{a+2} dr = \frac{4\pi\rho_0}{a+b} r^{a+3}$ (2)

From Equations (1) and (2) $a + 3 = 2 \Rightarrow 1 \ a = -1$

The correct option is (A)

51. The correct option is (B)

52. The correct option is (C)

Match the Column Type

53. (A) → (1, 3); (B) → (1); (C) → (2, 4); (D) → (2,3)

54. Mass of original sphere = M

Radius of original sphere = R

Mass of cavity = $M/8$

Radius of cavity = $R/2$ $[M \propto V \propto r^3]$

Mass of substituted (introduced) sphere = $2M$ [1/8th volume 16 time denser]

Radius of substituted (introduced) sphere = $R/2$

At any point, net potential (applying superposition)

= V_0 (potential due to original sphere) – V_C (potential due to cavity) + V_S (potential due to substituted sphere)

⇒ (A) → 3; (B) → 1; (C) → 4; (D) → 2

55. (A) → (1, 3, 4, 5); (B) → 3; (C) → 3; (D) → (1, 2, 4)

Integer Type

58. Minimum kinetic energy needed to escape the particle

$$= \frac{1}{2}mv_e^2 = \frac{1}{2}m\frac{2GM}{R} = \frac{GMm}{R}.$$

Here, M = mass of earth;

m = mass of particle

Kinetic energy given to the particle $= \dfrac{GMm}{3R}$

Let the maximum height attained by the particle be h.

From conservation of mechanical energy

$$\frac{GMm}{3R} + \left(-\frac{GMm}{R}\right) = -\frac{GMm}{R+h}$$

$$\Rightarrow \quad R + h = \frac{3R}{2}$$

$$\Rightarrow \quad h = \frac{R}{2} \Rightarrow n = 1.$$

59. As gravitational force provides centripetal force

$$\frac{mv^2}{r} = \frac{GMm}{r^n}$$

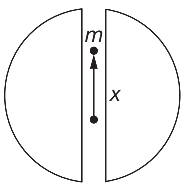
i.e., $v^2 = GMr^{-n+1}$

So that $T = \dfrac{2\pi r}{v} = 2\pi r\sqrt{\dfrac{r^{n-1}}{GM}}$

$\therefore \quad T^2 \propto r^{n+1}$

$n + 1 = 4$

60. Force on the particle



$$F = -\left(\frac{GMm}{R^3}\right)x$$

or, $F \propto -x$

Hence, the particle will perform simple harmonic (SHM) with

Time period $T = 2\pi\sqrt{\dfrac{R}{g}}$

The amplitude is R.

$\therefore$ Velocity at the centre $= \omega A = \sqrt{\dfrac{g}{R}}\,R = \sqrt{gR}$

$n = 1.$

61. Assuming that the rocket starts from rest, velocity acquired is given by

$$v = u\log_e\left(\frac{m_0}{m}\right) = 2.3026u\log_{10}\left(\frac{m_0}{m}\right)$$

$$\therefore \quad 11.2 = 2.3026 \times 2\log_{10}\left(\frac{m_0}{m}\right)$$

$$\log_{10}\left(\frac{m_0}{m}\right) = \frac{11.2}{2.3026 \times 2} = 2.4332$$

$$\frac{m_0}{m} = 270.$$

62. Escape velocity $= \sqrt{2gR}$, where g is acceleration due to gravity on surface of earth and R the radius of earth.

Orbital velocity $= \dfrac{1}{2}v_e = \dfrac{1}{2}\sqrt{2gR} = \sqrt{\dfrac{gR}{2}}$

If h is the height of satellite above earth $\dfrac{mv_0^2}{R+h} = \dfrac{GMm}{(R+h)^2}$

$$v_0^2 = \frac{GM}{R+h} = \frac{gR^2}{(R+h)}$$

$$\therefore \quad \left(\frac{1}{2}v_e\right)^2 = \frac{gR^2}{R+h}$$

$$\frac{gR}{2} = \frac{gR^2}{R+h}$$

$$R + h = 2R$$

$$h = R$$

$$n = 1.$$

Previous Years' Questions

63. Escape velocity $v_e = \sqrt{2gR}$

$\therefore$ Kinetic energy

$$= \frac{1}{2}mv_e^2 = \frac{1}{2}m \times 2gR = mgR$$

The correct option is (C)

64. Escape velocity $= \sqrt{2gR} = \sqrt{\dfrac{2GM_e}{R}}$

Escape velocity does not depend on mass of body which escapes or it depends on m^0.

The correct option is (A)

65. Let the spheres collide after time t, when the smaller sphere covered distance x_1 and bigger sphere covered distance x_2.

The gravitational force acting between two spheres depends on the distance which is a variable quantity.

The gravitational force, $F(x) = \dfrac{GM \times 5M}{(12R - x)^2}$

Acceleration of smaller body, $a_1(x) = \dfrac{G \times 5M}{(12R - x)^2}$

Acceleration of bigger body, $a_2(x) = \dfrac{GM}{(12R - x)^2}$

From equation of motion,

$x_1 = \dfrac{1}{2}a_1(x)t^2$ and $x_2 = \dfrac{1}{2}a_2(x)t^2$

$\Rightarrow \dfrac{x_1}{x_2} = \dfrac{a_1(x)}{a_2(x)} = 5 \Rightarrow x_1 = 5x_2$

we know that $x_1 + x_2 = 9R$

$x_1 + \dfrac{x_1}{5} = 9R$

$\therefore\ x_1 = \dfrac{45R}{6} = 7.5R$

Therefore the two spheres collide when the smaller sphere covered the distance of $7.5R$.

The correct option is (C)

66. The escape velocity is independent of angle of projection. Hence, it will remain same, ie, 11 kms^{-1}.

The correct option is (C)

67. For a satellite

centripetal force = gravitational force

$$\therefore\ \frac{mv_0^2}{(R+x)} = \frac{GMm}{(R+x)^2}$$

or $\quad v_0^2 = \dfrac{GM}{(R+x)} = \dfrac{gR^2}{(R+x)} \qquad \left[\because\ g = \dfrac{GM}{R^2}\right]$

or $\quad v_0 = \sqrt{\dfrac{gR^2}{R+x}}$

The correct option is (D)

68. For a satellite

centripetal force = gravitational force

$$\therefore\ mR\omega^2 = \frac{GmM_e}{R^2}$$

where $R = r_e + h$

or $\quad \omega = \sqrt{\dfrac{GM_e}{R^3}} = \sqrt{\dfrac{GM_e}{(r_e + h)^3}}$

$$\therefore\ T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{(r_e+h)^3}{GM_e}}$$

$\therefore$ T is independent of mass (m) of satellite.

The correct option is (A)

69. Force on object $= \dfrac{GMm}{x^2}$ at x from centre of earth.

$\therefore$ Work done $= \dfrac{GMm}{x^2}dx$

$\therefore$ Work done $= GMm\displaystyle\int_R^{2R}\frac{dx}{x^2}$

$\therefore$ Potential energy gained

$$= GMm\left[-\frac{1}{x}\right]_R^{2R} = \frac{GMm \times 1}{2R} \qquad \left[\because\ g = \frac{GM}{R^2}\right]$$

$\therefore$ Gain in PE $= \dfrac{1}{2}mR\left(\dfrac{GM}{R^2}\right) = \dfrac{1}{2}mgR$

The correct option is (B)

70. For motion of a planet in circular orbit,

centripetal force = gravitational force

$\therefore\ mR\omega^2 = \dfrac{GMm}{R^n}$ or $\omega = \sqrt{\dfrac{GM}{R^{n+1}}}$

$$\therefore\ T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^{n+1}}{GM}} = \frac{2\pi}{\sqrt{GM}}R^{\left(\frac{n+1}{2}\right)}$$

$\therefore\ T$ is proportional to $R^{\left(\frac{n+1}{2}\right)}$

The correct option is (A)

71. Acceleration due to gravity at height h

$$g_h = g\left(1 - \frac{2h}{g}\right) \qquad (1)$$

and depth d

$$g_d = g\left(1 - \frac{d}{R}\right) \qquad (2)$$

From Equations (1) and (2),

$$g\left(1 - \frac{2h}{R}\right) = g\left(1 - \frac{d}{g}\right)$$

$\Rightarrow\ 2h = d$

The correct option is (C)

72. Acceleration due to gravity $g = \dfrac{GM}{R^2}, \quad M = \left(\dfrac{4}{3}\pi R^3\right)\rho$

$$\therefore\ g = \frac{4G}{3}\frac{\pi R^3}{R^2}\rho$$

$\Rightarrow\ g = \left(\dfrac{4G\pi R}{3}\right)\rho \qquad$ (ρ = average density)

$\Rightarrow\ g \propto \rho$ or $\rho \propto g$

The correct option is (B)

73. At height, $g_h = g\left(1 - \dfrac{2h}{R}\right)$ where $h << R$

or $\quad g = g_h = \dfrac{2hg}{R}$ or $\Delta g_h = \dfrac{2hg}{R} \qquad (1)$

At depth, $g_d = g\left(1 - \dfrac{d}{R}\right)$

where $d << R$

or $\quad g - g_d = \dfrac{dg}{R}$

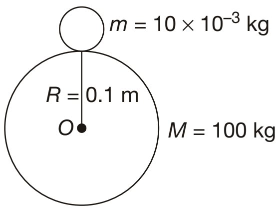
or $\quad \Delta\, g_d = \dfrac{dg}{R}$

when $\Delta\, g_h = \Delta\, g_d$

$\dfrac{2hg}{R} = \dfrac{dg}{R}$ or $d = 2h$

The correct option is (A)

74.



$$U_i = -\frac{GMm}{r}$$

$$U_i = -\frac{6.67\times10^{-11}\times100\times10^{-2}}{0.1}$$

$$U_i = -\frac{6.67\times10^{-11}}{0.1}$$

$$= -6.67\times10^{-10}\ \text{J}$$

We know

$\therefore\ W = \Delta U$

$\quad = U_f - U_i \qquad (\because U_f = 0)$

$\therefore\ W = U_i = 6.67\times10^{-10}$ J

The correct option is (D)

75. Mass of planet, $M_p = 10M_e$, where M_e is mass of earth.

Radius of planet, $R_p = \dfrac{R_e}{10}$, where R_e is radius of earth.

Escape speed is given by,

$$v = \sqrt{\frac{2GM}{R}}$$

So, for planet $v_p = \sqrt{\dfrac{2G\times M_p}{R_p}} = \sqrt{\dfrac{100\times 2GM_e}{R_e}}$

$= 10\times v_e$

$= 10\times 11\ \text{kms}^{-1} = 110\ \text{kms}^{-1}$

The correct option is (C)

76. Gravitational flux through a closed surface is given by

$\sqrt{E_g \vec{dS}} = -4\pi GM$

where, M = mass enclosed in the closed surface

This relationship is valid when $|E_g| \propto \dfrac{1}{r^2}$.

The correct option is (B)

77. We know that $\dfrac{g'}{g} = \dfrac{R^2}{(R+h)^2}$

$$\therefore\ \frac{g/9}{g} = \left[\frac{R}{R+h}\right]^2$$

$$\therefore\ \frac{R}{R+h} = \frac{1}{3}$$

$\therefore\ h = 2R$

The correct option is (D)

78. $E = mgR = 6.4\times10^{10}$ J

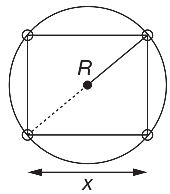
The correct option is (D)

79. $i = \dfrac{i_g\,(G+S)}{S} = i_g\left(1+\dfrac{G}{S}\right)$

The resistance of ammeter is $\dfrac{GS}{G+S}$.

The correct option is (C)

80. Side length of square



$2x^2 = 4R^2$

$x = \sqrt{2}\,R$

Net force on angular point

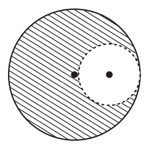
$$= \sqrt{2}\frac{GM^2}{x^2} + \frac{GM^2}{(2R)^2} = \frac{\sqrt{2}\,GM^2}{2R^2} + \frac{GM^2}{4R^2}$$

$$\frac{GM^2}{R^2}\left(\frac{\sqrt{2}}{2}+\frac{1}{4}\right) = \frac{Mv^2}{R}$$

$$V = \frac{1}{2}\sqrt{\frac{GM}{R}\left(1+2\sqrt{2}\right)}$$

The correct option is (D)

81. Potential due to complete solid sphere at centre of cavity



$$= -\frac{GM}{2R^3}\left[3R^2 - r^2\right]$$

$$= -\frac{GM}{2R^3}\left[3R^2 - \frac{R^2}{4}\right] \qquad \left(r = \frac{R}{2}\right)$$

$$= -\frac{11GM}{8R}.$$

Potential due to cavity mass at its centre

$$= -\frac{3GM'}{2R'} = \frac{-3G\left(\frac{M}{8}\right)}{2\left(\frac{R}{2}\right)} = -\frac{3GM}{8R}$$

Net potential at centre of cavity $= -\dfrac{11GM}{8R} - \left(-\dfrac{3GM}{8R}\right)$

$$= -\frac{8GM}{8R} = -\frac{GM}{R}$$

The correct option is (A)

82. $g' = \dfrac{GM}{(R+h)^2}$, acceleration due to gravity at height h

$$\Rightarrow\ \frac{g}{9} = \frac{GM}{R^2}\cdot\frac{R^2}{(R+h)^2}$$

$$= g\left(\frac{R}{R+h}\right)^2$$

$$\Rightarrow\ \frac{1}{9} = \left(\frac{R}{R+h}\right)^2$$

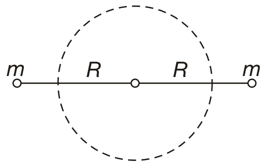
$\Rightarrow \quad \frac{R}{R+h} = \frac{1}{3}$

$\Rightarrow \quad 3R = R + h$

$\Rightarrow \quad 2R = h$

The correct option is (A)

83. Gravitational force provides necessary centripetal force

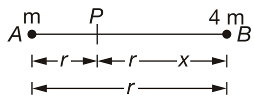


$$\frac{Gm^2}{(2R)^2} = \frac{mv^2}{R}$$

$$\Rightarrow \quad v = \sqrt{\frac{Gm}{4R}}$$

The correct option is (B)

84. Let gravitation field be zero at P as shown in below.



$$\therefore \quad \frac{Gm}{x^2} = \frac{G(4m)}{(r-x)^2}$$

$$\Rightarrow \quad 4x^2 = (r-x)^2$$

$$\Rightarrow \quad 2x = r - x$$

$$\Rightarrow \quad x = \frac{r}{3}$$

$$\therefore \quad V_p = -\frac{Gm}{x} - \frac{G(4m)}{r-x}$$

The correct option is (C)

85. $E = mgR = 6.4 \times 10^{10}$ J

The correct option is (D)

86. $i = \frac{i_g\,(G+S)}{S} = i_g\left(1 + \frac{G}{S}\right)$

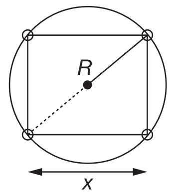
The resistance of ammeter is $\frac{GS}{G+S}$.

The correct option is (C)

87. Side length of square

$2x^2 = 4R^2$

$x = \sqrt{2}\,R$



Net force on angular point

$$= \sqrt{2}\frac{GM^2}{x^2} + \frac{GM^2}{(2R)^2} = \frac{\sqrt{2}\,GM^2}{2R^2} + \frac{GM^2}{4R^2}$$

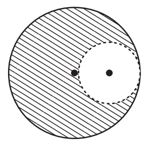
$$F_C = \frac{Mv^2}{R}$$

$$\frac{GM^2}{R^2}\left(\frac{\sqrt{2}}{2} + \frac{1}{4}\right) = \frac{Mv^2}{R}$$

$$V = \frac{1}{2}\sqrt{\frac{GM}{R}\left(1 + 2\sqrt{2}\right)}$$

The correct option is (D)

88. Potential due to complete solid sphere at centre of cavity



$$= -\frac{GM}{2R^3}\left[3R^2 - r^2\right]$$

$$= -\frac{GM}{2R^3}\left[3R^2 - \frac{R^2}{4}\right] \qquad \left(r = \frac{R}{2}\right)$$

$$= -\frac{11GM}{8R}$$

Potential due to cavity mass at its centre

$$= -\frac{3GM'}{2R'} = \frac{-3G\left(\frac{M}{8}\right)}{2\left(\frac{R}{2}\right)} = -\frac{3GM}{8R}$$

Net potential at centre of cavity $= -\frac{11GM}{8R} - \left(-\frac{3GM}{8R}\right)$

$$= -\frac{8GM}{8R} = -\frac{GM}{R}$$

The correct option is (A)

89. The orbital velocity of a satellite is given by

$$v_0 = \sqrt{\frac{GM}{R+h}} = \sqrt{\frac{GM}{R}} \qquad (\because \; h << R)$$

$$= \sqrt{gR} \qquad \left[\because \; g = \frac{GM}{R^2}\right]$$

The escape speed of the earth is given by

$V_e = \sqrt{2gR}$

$\therefore$ Increase in orbital speed

$$= \sqrt{2gR} - \sqrt{gR} = \sqrt{gR}\left(\sqrt{2} - 1\right)$$

The correct option is (C)