CHAPTER 6

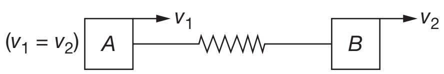
Rigid Body Dynamics

Chapter Highlights

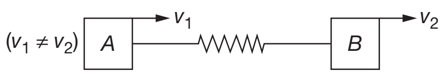
Basic concept of rotational motion, moment of inertia and radius of gyration, value of moment of inertia of simple geometrical object,parallel and perpendicular axis theorem and their application, moment of force, Torque, Angular momentum, conservation of angular momentum and its application, Rigid body rotation and equation of rotational motion

RIGID BODY

Rigid body is defined as a system of particles in which distance between each pair of particles remains constant (with respect to time). Remember, rigid body is a mathematical concept, and any system which satisfies the above condition is said to be rigid as long as it satisfies it.

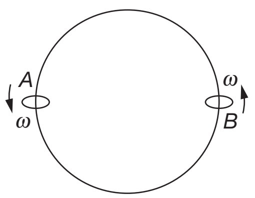


System behaves as a rigid body

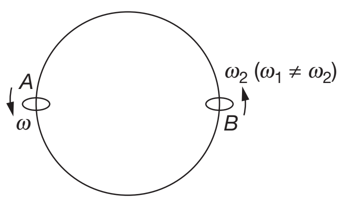


System behaves as a non-rigid body

A and B are beads which move on a circular fixed ring



$A + B$ is a rigid body system
but $A + B$ + ring is non-rigid system



$A + B$ is a non-rigid system

1. If a system is rigid, there is no change in the distance between any pair of particles of the system; shape and size of system remains constant. Hence, we can assume that while a stone or cricket ball are rigid bodies, a balloon or elastic string is non-rigid.
   But any of the above system is rigid as long as relative distance does not change, whether it is a cricket ball or a balloon. But at the moment when the bat hits the cricket ball or if the balloon is squeezed, relative distance changes and now the system behaves like a non-rigid system.
2. For every pair of particles in a rigid body, there is no velocity of separation or approach between the particles, i.e., any relative motion of a point B on a rigid body with respect to another point A on the rigid body will be perpendicular to line joining A to B, hence with respect to any particle A of a rigid body, the motion of any other particle B of that rigid body is in circular motion.

Let velocities of A and B with respect to ground be $\vec{v}_A$ and $\vec{v}_B$ respectively in Fig. 6.1.

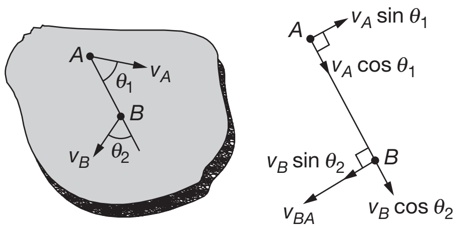


Fig. 6.1

If the above body is rigid $v_A \cos\theta_1 = v_B \cos\theta_2$ (velocity of approach/separation is zero)

v_{BA} = relative velocity of B with respect to A.

$v_{BA} = v_A \sin\theta_1 + v_B \sin\theta_2$ (which is perpendicular to line AB)

B will appear to move in a circle to an observer fixed at A.

With respect to any point of the rigid body, the angular velocity of all other points of the rigid body is same.

For example A, B, C is a rigid system and during any motion sides AB, BC and CA must rotate through the same angle. Hence, all the sides rotate by the same rate.

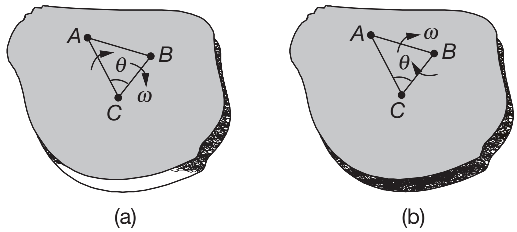
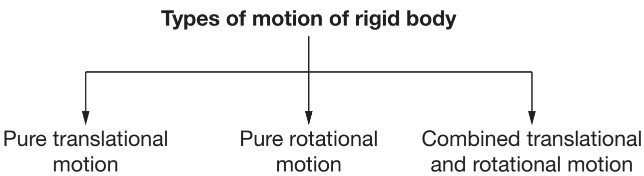


Fig. 6.2

From Fig. 6.2 (a), angular velocity of A and B with respect to C is ω,

From Fig. 6.2 (b), angular velocity of A and C with respect to B is ω,

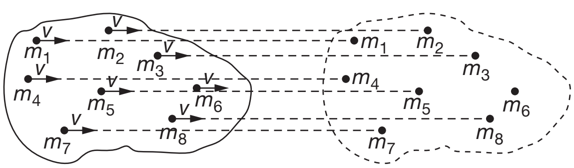


Pure Translational Motion

A body is said to be in pure translational motion, if the displacement of each particle of the system is same during any time interval. During such a motion, all the particles have same displacement ($\vec{s}$), velocity ($\vec{v}$) and acceleration ($\vec{a}$) at an instant.

Consider a system of n particle of mass $m_1, m_2, m_3, \ldots m_n$ undergoing pure translation.

From above definition of translational motion,



$$\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = \ldots \vec{a}_n = \vec{a} \quad \text{(say)}$$

and
$$\vec{v}_1 = \vec{v}_2 = \vec{v}_3 = \ldots \vec{v}_n = \vec{v} \quad \text{(say)}$$

From Newton's laws for a system,

$$\vec{F}_{\text{ext}} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots$$

$$\vec{F}_{\text{ext}} = M\vec{a}$$

Where m = Total mass of the body

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots$$

$$\vec{P} = m\vec{v}$$

Total kinetic energy of body

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \ldots = \frac{1}{2}mv^2$$

Pure Rotational Motion

Figure 6.3 shows a rigid body of arbitrary shape in rotation about a fixed axis, called the axis of rotation. Every point of the body moves in a circle whose centre lies on the axis of rotation, and every point moves through the same angle during a particular time interval. Such a motion is called pure rotation.

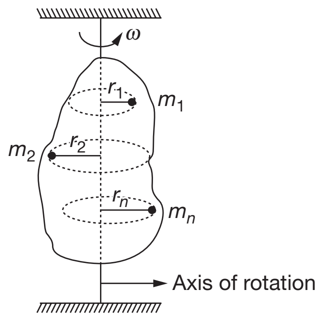


Fig. 6.3

We know that each particle has same angular velocity. (Since the body is rigid.)

So, $v_1 = \omega r_1,\ v_2 = \omega r_2,\ v_3 = \omega r_3 \ldots v_n = \omega r_n$

$$\text{Total kinetic energy} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \ldots$$

$$= \frac{1}{2}[m_1r_1^2 + m_2r_2^2 + \ldots]\omega^2$$

$$= \frac{1}{2}I\omega^2$$

where, $I = m_1r_1^2 + m_2r_2^2 + \ldots$ (called moment of inertia)

ω = angular speed of body.

Combined Translational and Rotational Motion

A body is said to be in combined translation and rotational motion if all points in the body rotates about an axis of rotation and the axis of rotation moves with respect to the ground. Any general motion of a rigid body can be viewed as a combined translational and rotational motion.

SOLVED EXAMPLES

1. A bucket is being lowered down into a well through a rope passing over a fixed pulley of radius 10 cm. Assume that the rope does not slip on the pulley. Find the angular velocity and angular acceleration of the pulley at an instant when the bucket is going down at a speed of 20 cm/s and has an acceleration of 4.0 m/s^2.

Solution:

Since the rope does not slip on the pulley, the linear speed v of the rim of the pulley is same as the speed of the bucket.

The angular velocity of the pulley is then

$$\omega = v/r = \frac{20\,\text{cm/s}}{10\,\text{cm}} = 2 \text{ rad/s}$$

and the angular acceleration of the pulley is

$$\alpha = a/r = \frac{4.0 \text{ m/s}^2}{10 \text{ cm}} = 40 \text{ rad/s}^2.$$

2. A wheel rotates with a constant angular acceleration of 2.0 rad/s^2. If the wheel starts from rest, how many revolutions will it make in the first 10 seconds?

Solution:

The angular displacement in the first 10 seconds is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}(2.0 \text{ rad/s}^2)(10 \text{ s})^2 = 100 \text{ rad.}$$

As the wheel turns by 2π radian in each revolution, the number of revolutions in 10 s in

$$n = \frac{100}{2\pi} = 16.$$

3. The wheel of a motor, accelerated uniformly from rest, rotates through 2.5 radian during the first second. Find the angle rotated during the next second.

Solution:

As the angular acceleration is constant, we have

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2.$$

Thus, $\qquad 2.5 \text{ rad} = \frac{1}{2}\alpha(1\text{s})^2$

$$\alpha = 5 \text{ rad/s}^2 \quad \text{or} \quad \alpha = 5 \text{ rad/s}^2$$

The angle rotated during the first two seconds is

$$= \frac{1}{2} \times (5 \text{ rad/s}^2)(2\text{s})^2 = 10 \text{ rad.}$$

Thus, the angle rotated during the next second is

$$10 \text{ rad} - 2.5 \text{ rad} = 7.5 \text{ rad.}$$

4. Starting from rest, a fan takes five seconds to attain the maximum speed of 400 rpm (revolution per minute). Assuming constant acceleration, find the time taken by the fan in attaining half the maximum speed.

Solution:

Let the angular acceleration be α. According to the question,

$$400 \text{ rpm} = 0 + \alpha 5 \qquad (1)$$

Let t be the time taken in attaining the speed of 200 rpm, which is half the maximum.

Then, $\qquad 200 \text{ rpm} = 0 + \alpha t \qquad (2)$

Dividing (1) by (2), we get,

$$2 = 5t \quad \text{or} \quad t = 2.5 \text{ s.}$$

5. The motor of an engine is rotating about its axis with an angular velocity of 100 rpm. It comes to rest in 15 s, after being switched off. Assuming constant angular deceleration, calculate the number of revolutions made by it before coming to rest.

Solution:

The initial angular velocity = 100 rpm

= $(10\pi/3)$ rad/s.

Final angular velocity = 0.

Time intertval = 15 s.

Let the angular acceleration be α. Using the equation $\omega = \omega_0 + at$, we obtain

$$\alpha = (-2\pi/9) \text{ rad/s}^2$$

The angle rotated by the motor during this motion is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= \left(\frac{10\pi}{3}\frac{\text{rad}}{\text{s}}\right)(15\,\text{s}) - \frac{1}{2}\left(\frac{2\pi}{9}\frac{\text{rad}}{\text{s}^2}\right)(15\,\text{s})^2$$

$$= 25\pi \text{ rad} = 12.5 \text{ revolutions.}$$

Hence the motor rotates through 12.5 revolutions before coming to rest.

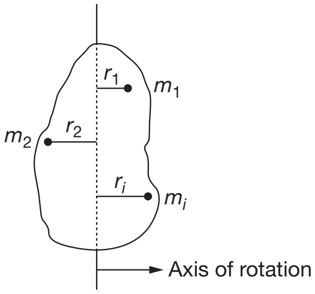
MOMENT OF INERTIA (I) ABOUT AN AXIS

1. Moment of inertia of a system of n particles about an axis is defined as:

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2$$

i.e., $\qquad I = \sum_{i=1}^{n} m_i r_i^2$

where r_i = It is perpendicular distance of mass m_i from axis of rotation



SI units of moment of inertia is kgm^2.

Moment of inertia is a scalar positive quantity.

2. For a continuous system,

$$I = \int r^2 (dm)$$

where dm = mass of a small element

r = perpendicular distance of the mass element dm from the axis

Moment of inertia depends on:

1. Density of the material of body
2. Shape and size of body
3. Axis of rotation

In totality we can say that it depends upon distribution of mass relative to axis of rotation.

NOTE

Moment of inertia does not change if the mass:

- Is shifted parallel to the axis of the rotation, because r_i does not change.
- Is rotated about axis of rotation in a circular path, because r_i does not change.

SOLVED EXAMPLES

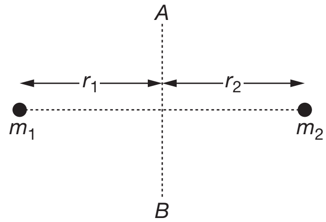
6. Two particles having masses m_1 and m_2 are situated in a plane perpendicular to line AB at a distance of r_1 and r_2, respectively as shown.

(A) Find the moment of inertia of the system about axis AB?

(B) Find the moment of inertia of the system about an axis passing through m_1 and perpendicular to the line joining m_1 and m_2?

(C) Find the moment of inertia of the system about an axis passing through m_1 and m_2?

(D) Find the moment of inertia about axis passing through centre of mass and perpendicular to line joining m_1 and m_2



Solution:

(A) Moment of inertia of particle on left is $I_1 = m_1 r_1^2$.

Moment of inertia of particle on right is $I_2 = m_2 r_2^2$.

Moment of inertia of the system about AB is $I = I_1 + I_2 = m_1 r_2^2 + m_2 r_2^2$

(B) Moment of inertia of particle on left is $I_1 = 0$

Moment of inertia of particle on right is $I_2 = m_2(r_1 + r_2)^2$.

Moment of inertia of the system about AB is $I = I_1 + I_2 = 0 + m_2(r_1 + r_2)^2$

(C) Moment of inertia of particle on left is $I_1 = 0$

Moment of inertia of particle on right is $I_2 = 0$

Moment of inertia of the system about AB is $I = I_1 + I_2 = 0 + 0$

(D) Centre of mass of system $r_{CM} = m_2 \left(\dfrac{r_1 + r_2}{m_1 + m_2}\right) =$

Distance of centre mass from mass m_1

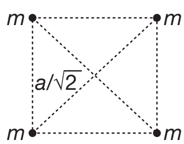
Distance of centre of mass from mass $m_2 = m_1 \left(\dfrac{r_1 + r_2}{m_1 + m_2}\right)$

So moment of inertia about centre of mass =

$$I_{cm} = m_1 \left(m_2 \frac{r_1 + r_2}{m_1 + m_2}\right)^2 + m_2 \left(m_1 \frac{r_1 + r_2}{m_1 + m_2}\right)^2$$

$$I_{cm} = \frac{m_1 m_2}{m_1 + m_2}(r_1 + r_2)^2.$$

7. Four particles each of mass m are kept at the four corners of a square of edge a. Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.



Solution:

The perpendicular distance of every particle from the given line is $a/\sqrt{2}$. The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2}ma^2$. The moment of inertia of the system is, therefore,

$$4 \times \frac{1}{2}ma^2 = 2ma^2.$$

8. Four point masses are connected by a massless rod as shown Fig. 6.4. Find out the moment of inertia of the system about axis CD?

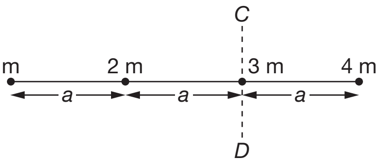


Fig. 6.4

Solution:

$$I_1 = m(2a)^2$$
$$I_2 = 2\,ma^2$$
$$I_3 = 0$$
$$I_4 = 4\,ma^2$$
$$I_{CD} = I_1 + I_2 + I_3 + I_4 = 10\,ma^2.$$

9. Three particles, each of mass m, are situated at the vertices of an equilateral triangle ABC of side L (Fig. 6.5). Find the moment of inertia of the system about

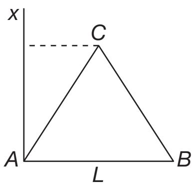


Fig. 6.5

(A) the line AX perpendicular to AB in the plane of ABC.
(B) One of the sides of the triangle ABC
(C) About an axis passing through the centroid and perpendicular to plane of the triangle ABC.

Solution:

(A) Perpendicular distance of A from $AX = 0$
Perpendicular distance of B from $AX = L$
Perpendicular distance of C from $AX = L/2$
Thus, the moment of inertia of the particle at $A = 0$, of the particle at $B - mL^2$, and of the particle at $C = m(L/2)^2$. The moment of inertia of the three-particle system about AX is

$$0 + mL^2 + m(L/2)^2 = \frac{5\,mL^2}{4}$$

Note that the particles on the axis do not contribute to the moment of inertia.

(B) Moment of inertia about the side AC = mass of particle $B \times$ square of perpendicular distance of B from side AC, $I_{AC} = m\left(\frac{\sqrt{3}}{2}L\right)^2 = \frac{3mL^2}{4}$

(C) Distance of centroid from all the particle is $\frac{L}{\sqrt{3}}$, so moment of inertia about an axis and passing through the centroic perpendicular plane of triangle $ABC = I_C = 3m\left(\frac{L}{\sqrt{3}}\right)^2 = mL^2.$

10. Three point masses are located at the corners of an equilibrium triangle of side 1 cm. Masses are of 1, 2, and 3 kg, respectively, and kept as shown in Fig. 6.6. Calculate the moment of inertia of system about an axis passing through 1 kg mass and perpendicular to the plane of triangle?

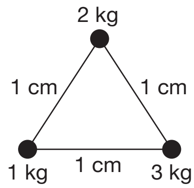


Fig. 6.6

Solution:

Moment of inertia of 2 kg mass about an axis passing through 1 kg mass $I_1 = 2 \times (1 \times 10^{-2})^2 = 2 \times 10^{-4}$

Moment of inertia of 3 kg mass about an axis passing through 1 kg mass $I_2 = 3 \times (1 \times 10^{-2})^2 = 3 \times 10^{-4}$

$$I = I_1 + I_2 = 5 \times 10^{-4}\ \text{kgm}^2.$$

11. Calculate the moment of inertia of a ring having mass M, radius R and having uniform mass distribution about an axis passing through the centre of ring and perpendicular to the plane of ring?

Solution:

$$I = \int (dm)\,r^2$$

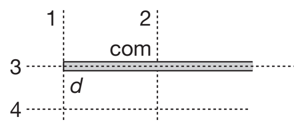
Because each element is equally distanced from the axis so $r = R$

$$= R^2 \int dm = MR^2$$

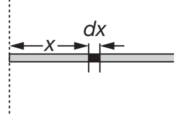
$$I = MR^2$$

Answer will remain same even if the mass is non-uniformly distributed because $\int dm = M$ always.

12. Calculate the moment of inertia of a uniform rod of mass M and length ℓ about an axis 1, 2, 3 and 4.



Solution:



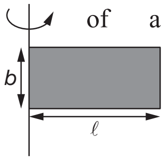
$$(I_1) = \int (dm)\,r^2 = \int_0^{\ell} \left(\frac{M}{\ell}dx\right) x^2 = \frac{M\ell^2}{3}$$

$$(I_2) = \int (dm)\,r^2 = \int_{-\ell/2}^{\ell/2} \left(\frac{M}{\ell}dx\right) x^2 = \frac{M\ell^2}{12}$$

$(I_3) = 0$ (axis 3 passing through the axis of rod)

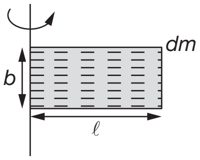
$$(I_4) = d^2 \int (dm) = Md^2.$$

13. Determined the moment of inertia of a uniform rectangular plate of mass m, side b and ℓ about an axis passing through the edge b and in the plane of plate.



Solution:

Each section of dm mass rod in the rectangular plate has moment of inertia about an axis passing through edge b $dI = \dfrac{dm\ell^2}{3}$



So $$I = \int dI = \frac{\ell^2}{3}\int dm = \frac{m\ell^2}{3}.$$

14. Find out the moment of inertia of Fig. 6.7 (A), (B), (C) shown each having mass m, radius R and having uniform mass distribution about an axis passing through the centre and perpendicular to the plane?

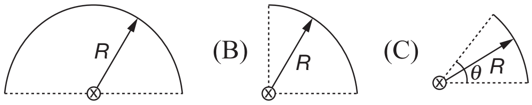


Fig. 6.7

Solution:

mR^2 (in fact, moment of inertia of any part of mass m of a ring of radius R about axis passing through geometrical centre and perpendicular to the plane of the ring is $= mR^2$.)

Moment of inertia of a large object can be calculated by integrating moment of inertia of an element of the object

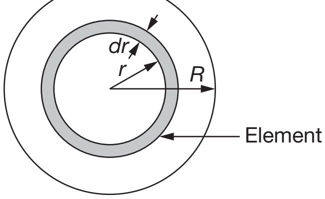
$$I = \int dI_{\text{element}}$$

where dI = moment of inertia of a small element

Element chosen should be such that: either perpendicular distance of axis from each point of the element is same or the moment of inertia of the element about the axis of rotation is known.

15. Determine the moment of Inertia of a uniform disc having mass m, radius R about an axis passing through centre and perpendicular to the plane of disc.

Solution:



$$I = \int dI_{\text{ring}}$$

element – ring $\qquad dI = dmr^2$

$dm = \dfrac{m}{\pi R^2}\, 2\pi r dr$ (here we have used the uniform mass distribution)

$$\therefore \quad I = \int_0^R \frac{m}{\pi R^2}\cdot(2\pi r dr)\cdot r^2 \quad\Rightarrow\quad I = \frac{mR^2}{2}.$$

16. Calculate the moment of inertia of Fig. 6.8 (A), (B), (C) shown, each having mass m, radius R and having uniform mass distribution about an axis perpendicular to the plane and passing through the centre?

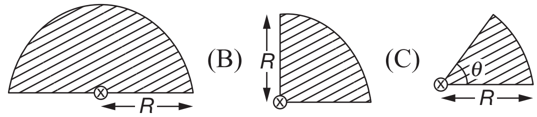


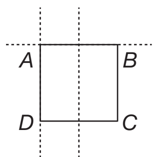
Fig. 6.8

Solution:

$$dI = dm\,\frac{R^2}{2}$$

$$I = \int dI = \frac{R^2}{2}\int dm = \frac{mR^2}{2}.$$

17. Find the moment of inertia of the uniform square plate of side a and mass m about the axis AB.

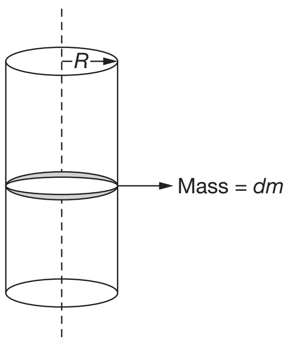


Solution:

$$dI = dm\,\frac{a^2}{3}$$

$$\Rightarrow \qquad I = \int dI = \frac{a^2}{3}\int dm = \frac{ma^2}{3}.$$

18. Calculate the moment of inertia of a uniform hollow cylinder of mass m, radius R and length ℓ about its axis.

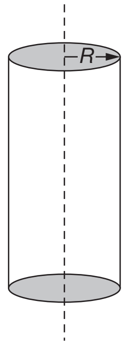


Solution:

Moment of inertia of a uniform hollow cylinder is

$$I = \int (dm)\, R^2$$
$$= mR^2.$$

19. Calculate the moment of inertia of a uniform solid cylinder of mass m, radius R and length ℓ about its axis.



Solution:

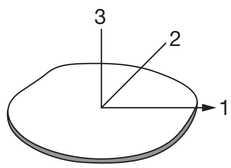
Each segment of cylinder is solid disc so

$$\int dI = \int dm \frac{R^2}{2}$$

$$I = \frac{mR^2}{2}.$$

Two Important Theorems on Moment of Inertia

Perpendicular Axis Theorem [Only Applicable to Plane Laminar Bodies (i.e., for 2-dimensional Objects Only)]



Body is in 1–2 plane

If axis 1 and 2 are in the plane of the body and perpendicular to each other.

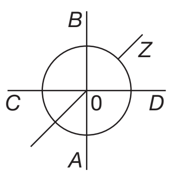
Axis 3 is perpendicular to plane of 1 and 2.

Then, $$I_3 = I_1 + I_2$$

The point of intersection of the three axis need not be centre of mass, it can be any point in the plane of body which lies on the body or even outside it.

SOLVED EXAMPLES

20. Find the moment of inertia of a uniform ring of mass m and radius R about a diameter.



Solution:

Let AB and CD be two mutually perpendicular diameters of the ring. Take them as X and Y-axes and the line perpendicular to the plane of the ring through the centre as the Z-axis. The moment of inertia of the ring about the Z-axis is $I = mR^2$. As the ring is uniform, all of its diameters are equivalent and so $I_x = I_y$. From perpendicular axes theorem,

$$I_z = I_x + I_y.$$

Hence $$I_x = \frac{I_z}{2} = \frac{mR^2}{2}.$$

Similarly, the moment of inertia of a uniform disc about a diameter is $mR^2/4$.

21. Two uniform identical rods each of mass m and length ℓ are joined to form a cross as shown in Fig. 6.9. Find the moment of inertia of the cross about a bisector as shown dotted in the Fig. 6.9.

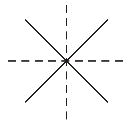


Fig. 6.9

Solution:

Consider the line perpendicular to the plane of the Fig. 6.9 through the centre of the cross. The moment of inertia of each rod about this line is $\frac{m\ell^2}{12}$ and hence the moment of inertia of the cross is $\frac{m\ell^2}{6}$. The moment of inertia of the cross about the two bisector are equal by symmetry and according to the theorem of perpendicular axes, the moment of inertia of the cross about the bisector is $\frac{m\ell^2}{12}$.

22. In Fig. 6.10 find moment of inertia of a plate having mass m, length ℓ and width b about axis 1, 2, 3 and 4. Assume that mass is uniformly distributed.

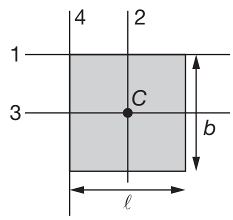


Fig. 6.10

Solution:

Moment of inertia of the plate about axis 1 (by taking rods perpendicular to axis 1)

$$I_1 = mb^2/3$$

Moment of inertia of the plate about axis 2 (by taking rods perpendicular to axis 2)

$$I_2 = m\ell^2/12$$

Moment of inertia of the plate about axis 3 (by taking rods perpendicular to axis 3)

$$I_3 = mb^2/12$$

Moment of inertia of the plate about axis 4 (by taking rods perpendicular to axis 4)

$$I_4 = m\ell^2/3.$$

23. In Fig. 6.11 find the moment of inertia of square plate having mass m and sides a. About an axis 2 passing through point C (centre of mass) and in the plane of plate.

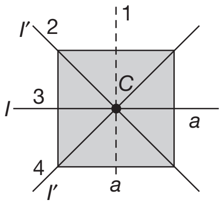


Fig. 6.11

Solution:

Using perpendicular axis theorems $I_C = I_4 + I_2 = 2I'$

Using perpendicular theorems $I_C = I_3 + I_1 = I + I = 2I$

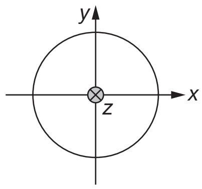
$$2I' = 2I$$

$$I' = I$$

$$IC = 2I = \frac{ma^2}{6}$$

$$I' = \frac{ma^2}{12}.$$

24. Find the moment of inertia of a uniform disc of mass m and radius R about a diameter.



Solution:

Consider x and y, two mutually perpendicular diameters of the ring.

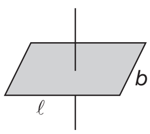
$$I_x + I_y = I_z$$

$$I_x = I_y \text{ (due to symmetry)}$$

$$I_z = \frac{mR^2}{2}$$

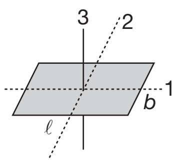
$$\Rightarrow \quad I_x = I_y = \frac{mR^2}{4}.$$

25. Find the moment of inertia of a uniform rectangular plate of mass m, edges of length ℓ and b about its axis passing through centre and perpendicular to it.



Solution:

Using perpendicular axis theorem $I_3 = I_1 + I_2$

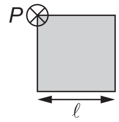


$$I_1 = \frac{mb^2}{12}$$

$$I_2 = \frac{m\ell^2}{12}$$

$$I_3 = \frac{m(\ell^2 + b^2)}{12}.$$

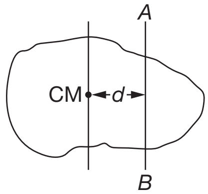
26. Find the moment of inertia of a uniform square plate of mass m, edge of length ℓ about its axis passing through P and perpendicular to it.



Solution:

$$I_P = \frac{m\ell^2}{6} + \frac{m\ell^2}{2} = \frac{2m\ell^2}{3}.$$

Parallel Axis Theorem (Applicable to Planar as Well as 3-Dimensional Objects)



If I_{AB} = Moment of inertia of the object about axis AB

I_{cm} = Moment of inertia of the object about an axis passing through centre of mass and parallel to axis AB

m = total mass of object

d = perpendicular distance between axis AB about which moment of inertia is to be calculated and the one passing through the centre of mass and parallel to it.

$$I_{AB} = I_{cm} + md^2$$

SOLVED EXAMPLES

27. Find out relation between I_1 and I_2.

I_1 and I_2 moment of inertia of a rigid body mass m about an axis as shown in Fig. 6.12.

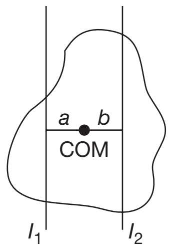


Fig. 6.12

Solution:

Using parallel axis theorem,

$$I_1 = I_C + ma^2 \quad (1)$$

$$I_2 = I_C + mb^2 \quad (2)$$

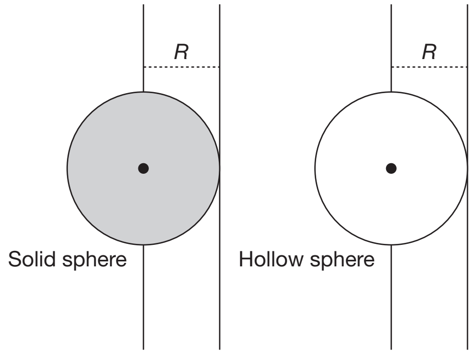
From (1) and (2)

$$I_1 - I_2 = m(a^2 - b^2).$$

28. Find the moment of inertia of a uniform sphere of mass m and radius R about a tangent if the spheres

(A) solid (B) hollow

Solution:



(A) Using parallel axis theorem

$I = I_{CM} + md^2$

for solid sphere,

$I_{CM} = \frac{2}{5} mR^2, d = R$

$I = \frac{7}{5} mR^2$

(B) Using parallel axis theorem

$I = I_{CM} + md^2$

for hollow sphere,

$I_{CM} = \frac{2}{3} mR^2, d = R$

$I = \frac{5}{3} mR^2.$

29. Find the moment of inertia of a solid cylinder of mass m and radius R about a line parallel to the axis of the cylinder and on the surface of the cylinder.

Solution:

The moment of inertia of the cylinder about its axis = $\frac{mR^2}{2}$.

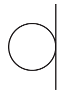
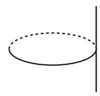
Using parallel axes theorem,

$$I = I_0 + mR^2 = \frac{mR^2}{2} + mR^2 = \frac{3}{2} mR^2.$$

Similarly, the moment of inertia of a solid sphere about a tangent is

$$\frac{2}{5} mR^2 + mR^2 = \frac{7}{5} mR^2.$$

30. Find out the moment of inertia of a ring having uniform mass distribution of mass m and radius R about an axis which is tangent to the ring and (A) in the plane of the ring (B) perpendicular to the plane of the ring.

(A)  (B) 

Solution:

(A) Moment of inertia about an axis passing through centre of ring and plane of the ring

$I_1 = \frac{mR^2}{2}$

Using parallel axis theorem,

$I' = I_1 + mR^2 = \frac{3mR^2}{2}.$

(B) Moment of inertia about an axis passing through centre of ring and perpendicular to plane of the ring

$I_C = mR^2$

Using parallel axis theorem,

$I'' = I_C + mR^2 = 2mR^2.$

31. Calculate the moment of inertia of a rectangular frame formed by uniform rods having mass m each as shown in Fig. 6.13 about an axis passing through its centre and perpendicular to the plane of frame? Also find moment of inertia about an axis passing through PQ?

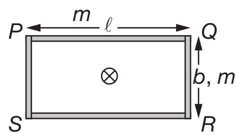


Fig. 6.13

Solution:

(A) Moment of inertia about an axis passing through its centre and perpendicular to the plane of frame

$$I_C = I_1 + I_2 + I_3 + I_4$$

$$I_1 = I_3, I_2 = I_4$$

$$I_C = 2I_1 + 2I_2$$

$$I_1 = \frac{m\ell^2}{12} + m\left(\frac{b}{2}\right)^2$$

$$I_2 = \frac{mb^2}{12} + m\left(\frac{\ell}{2}\right)^2$$

so, $$I_C = \frac{2m}{3}(\ell^2 + b^2).$$

(B) MI about axis PQ of rod PQ $I_1 = 0$

MI about axis PQ of rod PS $I_2 = \frac{mb^2}{2}$

MI about axis PQ of rod QR $I_3 = \frac{mb^2}{2}$

MI about axis PQ of rod SR $I_4 = mb^2$

$$I = I_1 + I_2 + I_3 + I_4 = \frac{5mb^2}{3}.$$

32. Find out the moment of inertia of a semi-circular disc about an axis passing through its centre of mass and perpendicular to the plane?

Solution:

Moment of inertia of a semi circular disc about an axis passing through centre and perpendicular to plane of disc, $I = \dfrac{mR^2}{2}$

Using parallel axis theorem, $I = I_{CM} + md^2$, d is the perpendicular distance between two parallel axes passing through centre C and COM.

$$I = \frac{mR^2}{2},\ d = \frac{4R}{3\pi}$$

$$\frac{mR^2}{2} = I_{CM} + m\left(\frac{4R}{3\pi}\right)^2$$

$$I_{CM} = \left[\frac{mR^2}{2} - m\left(\frac{4R}{3\pi}\right)^2\right].$$

33. Find the moment of inertia of the two uniform joint rods having mass m each about point P as shown in Fig. 6.14. Using parallel axis theorem,

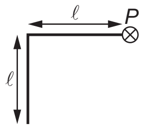
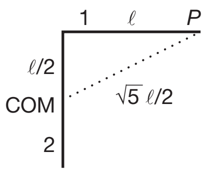


Fig. 6.14

Solution:



Moment of inertia of rod 1 about axis P, $I_1 = \dfrac{m\ell^2}{3}$

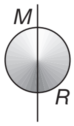
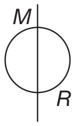
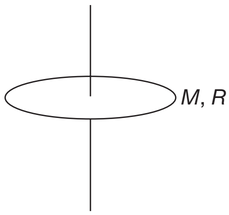
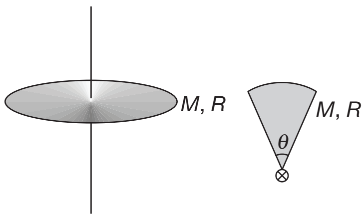
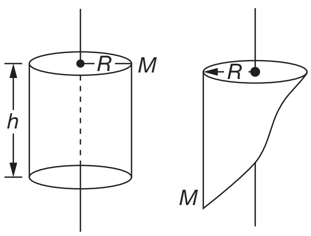
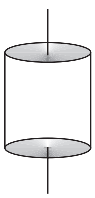
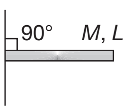
Moment of inertia of rod 2 about axis P, $I_2 = \dfrac{m\ell^2}{12} + m\left(\sqrt{5}\,\dfrac{\ell}{2}\right)^2$

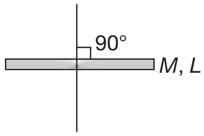
So momentum of inertia of a system about axis P,

$$I = I_1 + I_2 = \frac{m\ell^2}{3} + \frac{m\ell^2}{12} + m\left(\sqrt{5}\,\frac{\ell}{2}\right)^2$$

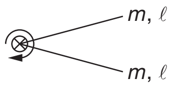
$$I = \frac{m\ell^2}{3}.$$

List of Some Useful Formulae

Object	Moment of Inertia
Solid Sphere 	$\frac{2}{5}MR^2$ (Uniform)
Hollow Sphere 	$\frac{2}{3}MR^2$ (Uniform)
Ring 	MR^2 (Uniform or non-uniform)
Disc 	$\frac{MR^2}{2}$ (Uniform)
Hollow cylinder 	MR^2 (Uniform or non-uniform)
Solid cylinder 	$\frac{MR^2}{2}$ (Uniform)
	$\frac{ML^2}{3}$ (Uniform)

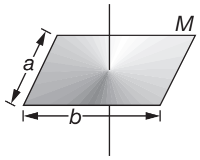


$\frac{ML^2}{12}$ (Uniform)



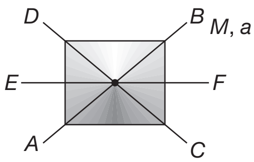
$\frac{2m\ell^2}{3}$ (Uniform)

Rectangular Plate



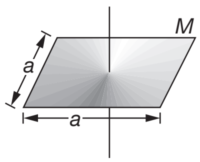
$I = \frac{M(a^2+b^2)}{12}$ (Uniform)

Square Plate



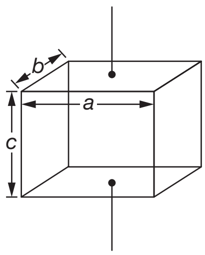
$I_{AB} = I_{CD} = I_{DF} = \frac{Ma^2}{12}$ (Uniform)

Square Plate



$\frac{Ma^2}{6}$ (Uniform)

Cuboid



$\frac{M(a^2+b^2)}{12}$ (Uniform)

RADIUS OF GYRATION

We define a new parameter, the radius of gyration (K), on the basis of the measure in which the mass of rigid body is distributed with respect to the axis of rotation. It is related to the moment of inertia and total mass of the body.

$$I = MK^2$$

where I = Moment of inertia of a body
M = Mass of a body
K = Radius of gyration

$$K = \sqrt{\frac{I}{M}}$$

Length K is the geometrical property of the body and axis of rotation.

SI unit of K is meter.

SOLVED EXAMPLES

34. Find the radius of gyration of a solid uniform sphere of radius R about its tangent.

Solution:

$$I = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2 = mK^2$$

$$\Rightarrow \quad K = \sqrt{\frac{7}{5}}R.$$

35. Find the radius of gyration of a hollow uniform sphere of radius R about its tangent.

Solution:

Moment of inertia of a hollow sphere about a tengent,

$$I = \frac{5}{3}mR^2$$

$$mK^2 = \frac{5}{3}mR^2$$

$$K = \sqrt{\frac{5}{3}}R.$$

MOMENT OF INERTIA OF BODIES WITH CUT

SOLVED EXAMPLES

36. A uniform disc of radius R has a round disc of radius $R/3$ cut as shown in Fig. 6.15. The mass of the remaining (shaded) portion of the disc equals M. Find the moment of inertia of such a disc relative to the axis passing through geometrical centre of original disc and perpendicular to the plane of the disc.

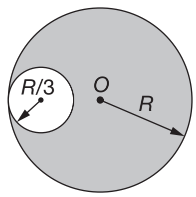


Fig. 6.15

Solution:

Let the mass per unit area of the material of disc be σ. Now the empty space can be considered as having density $-\sigma$ and σ.

Now $I_0 = I_\sigma + I_{-\sigma}$

$I_\sigma = (\sigma \pi R^2)R^2/2$ = Moment of inertia of σ about o

$$I_{-\sigma} = \frac{-\sigma\pi(R/3)^2(R/3)^2}{2} + [-\sigma\pi(R/3)^2]\,(2R/3)^2$$

= Moment of inertia of $-\sigma$ about o

$$\therefore \quad I_0 = \frac{4}{9} MR^2.$$

37. Find the moment of inertia of a uniform disc of radius R_1 having an empty symmetric annular region of radius R_2 in between, about an axis passing through geometrical centre and perpendicular to the disc.

Solution:

$$\rho = \frac{M}{\pi(R_1^2 - R_2^2)} \Rightarrow I = \rho \times \left(\frac{\pi R_1^4 - \pi R_2^4}{2}\right)$$

$$I = \frac{M\left(R_1{}^2 + R_2{}^2\right)}{2}.$$

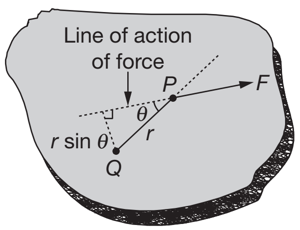
TORQUE

Torque represents the capability of a force to produce change in the rotational motion of the body.

Torque About a Point

Torque of force $\vec{F}$ about a point

$$\vec{\tau} = \vec{r} \times \vec{F}$$



Where $\vec{F}$ = force applied

P = point of application of force

Q = point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point of application of force with respect to the point about which we need to determine the torque.

$$|\vec{\tau}| = rF \sin\theta = r_\perp F = rF_\perp$$

Where θ = angle between the direction of force and the position of vector P with respect to Q.

$r_\perp = r \sin\theta$ = perpendicular distance of line of action of force from point Q; it is also called force arm.

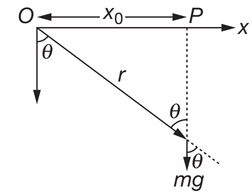
$F_\perp = F \sin\theta$ = component of $\vec{F}$ perpendicular to $\vec{r}$

SI unit of torque is Nm

Torque is a vector quantity and its direction is determined using right-hand thumb rule and its always perpendicular to the plane of rotation of the body.

SOLVED EXAMPLES

38. A particle of mass m is released in vertical plane from a point P at $x = x_0$. On the x-axis it falls vertically along the y-axis. Find the torque τ acting on the particle at a time t about origin.



Solution:

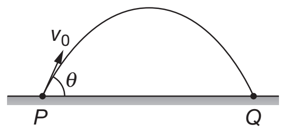
Torque is produced by the force of gravity.

$$\vec{\tau} = rF \sin\theta \hat{k}$$

or $$\tau = r_\perp F = x_0 mg$$

$$= rmg \frac{x_0}{r} = mgx_0 \hat{k}.$$

39. A particle having mass m is projected with a velocity v_0 from a point P on a horizontal ground making an angle θ with horizontal. Find out the torque about the point of projection acting on the particle when it is at its maximum height?

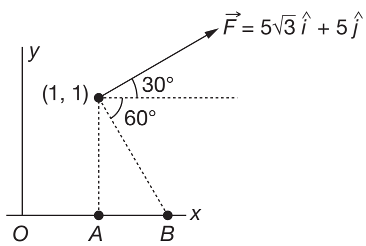


Solution:

$$\tau = rF\sin\theta = \frac{R}{2}mg = \frac{v_0{}^2 \sin 2\theta}{2g} mg$$

$$\Rightarrow \quad \tau = \frac{mv_0{}^2 \sin 2\theta}{2}.$$

40. Find the torque about points O and A.



Solution:

Torque about point O,

$$\vec{\tau} = \vec{r}_0 \times \vec{F},\ \vec{r}_0 = \hat{i} + \hat{j},\ \vec{F} = 5\sqrt{3}\,\hat{i} + 5\,\hat{j}$$

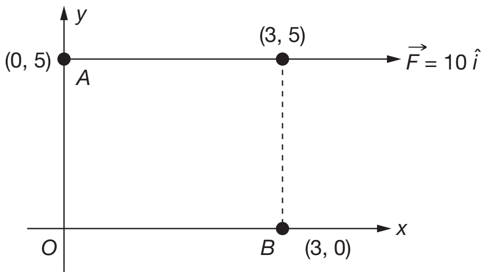
$$\vec{\tau} = (\hat{i} + \hat{j}) \times (5\sqrt{3}\,\hat{i} + 5\,\hat{j}) = 5(1 - \sqrt{3})\,\hat{k}$$

Torque about point A,

$$\vec{\tau} = \vec{r}_a \times \vec{F},\ \vec{r}_a = \hat{j}\text{'}\ \vec{F} = 5\sqrt{3}\,\hat{i} + 5\,\hat{j}$$

$$\vec{\tau} = \hat{j} \times (5\sqrt{3}\,\hat{i} + 5\,\hat{j}) = 5(-\sqrt{3})\,\hat{k}.$$

41. Find out the torque about point A, O and B



Solution:

Torque about point A,

$$\vec{\tau}_A = \vec{r}_A \times \vec{F},\ \vec{r}_A = 3\hat{i},\ \vec{F} = 10\hat{i}$$

$$\vec{\tau}_A = 3\hat{i} \times 10\hat{i} = 0$$

Torque about point B,

$$\vec{\tau}_B = \vec{r}_B \times \vec{F},\ \vec{r}_B = 5\,\hat{j},\ \vec{F} = 10\hat{i}$$

$$\vec{\tau}_B = 5\,\hat{j} \times 10\hat{i} = -50\,\hat{k}$$

Torque about point O,

$$\vec{\tau}_O = \vec{r}_O \times \vec{F},\ \vec{r}_O = 3\hat{i} + 5\,\hat{j},\ \vec{F} = 10\hat{i}$$

$$\vec{\tau}_O = (3\hat{i} + 5\,\hat{j}) \times 10\hat{i} = -50\,\hat{k}.$$

Torque about an Axis

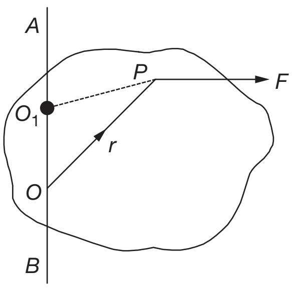


Fig. 6.16

The torque of a force $\vec{F}$ about an axis AB is defined as the component of torque of $\vec{F}$ about any point O on the axis AB, along the axis AB.

In Fig. 6.16 torque of $\vec{F}$ about O is $\vec{\tau}_0 = \vec{r} \times \vec{F}$

The torque of $\vec{F}$ about AB, τ_{AB} is component of $\vec{\tau}_0$ along line AB.

There are four cases of torque of a force about an axis.

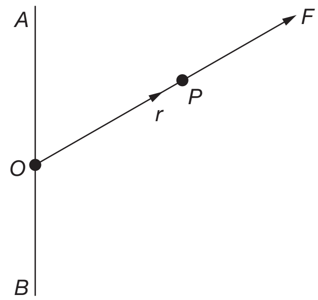
Case I: Force is parallel to the axis of rotation, $\vec{F} \parallel \overrightarrow{AB}$

AB is the axis of rotation about which torque is required

$\vec{r} \times \vec{F}$ is perpendicular to $\vec{F}$, but $\vec{F} \parallel \overrightarrow{AB}$, hence $\vec{r} \times \vec{F}$ is perpendicular to $\overrightarrow{AB}$.

The component of $\vec{r} \times \vec{F}$ along $\overrightarrow{AB}$ is, therefore, zero.

Case II: The line of force intersects the axis of rotation (F intersects AB)



$\vec{F}$ intersects AB along $\vec{r}$, then $\vec{F}$ and $\vec{r}$ are along the same line. The torque about O is $\vec{r} \times \vec{F} = 0$. Hence component of this torque along line AB is also zero.

Case III: $\vec{F}$ perpendicular to $\overrightarrow{AB}$, but $\vec{F}$ and AB do not intersect.

In the three dimensions, two lines may be perpendicular without intersecting each other.

Two nonparallel and nonintersecting lines are called skew lines.

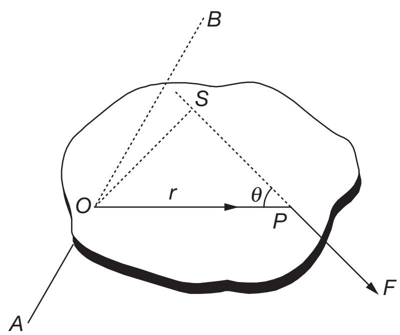


Fig. 6.17

Fig. 6.17 shows the plane through the point of application of force P that is perpendicular to the axis of rotation AB. Suppose the plane intersects the axis at point O force F is in this plane (since F is perpendicular to AB). Taking the origin at O,

$$\text{Torque} = \vec{r} \times \vec{F} = \overrightarrow{OP} \times \vec{F}.$$

Thus, $\qquad$ torque $= rF \sin\theta = F(OS)$

where OS is perpendicular from O to the line of action of force $\vec{F}$. The line OS is also perpendicular to the axis of rotation. It is thus the length of the common perpendicular to the force and the axis of rotation.

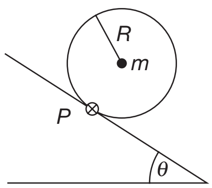
The direction of $\vec{\tau} = \overrightarrow{OP} \times \vec{F}$ is along the axis AB because $\overrightarrow{AB} \perp \overrightarrow{OP}$ and $\overrightarrow{AB} \perp \vec{F}$. The torque about AB is, therefore, equal to the magnitude of $\vec{\tau}$, that is, $F.(OS)$.

Thus, the torque of F about AB = magnitude of the force $F \times$ length of the common perpendicular to the force and the axis. The common perpendicular OS is called the lever arm or moment arm of this torque.

Case IV: $\vec{F}$ and $\overrightarrow{AB}$ are skew but not perpendicular. Here we resolve $\vec{F}$ into two components, one is parallel to axis and other is perpendicular to axis. Torque of the parallel part is zero and that of the perpendicular part may be found form the result of **case (III)**.

SOLVED EXAMPLES

42. Find the torque of weight about the axis passing through point P.



Solution:

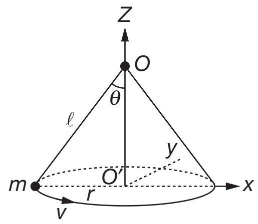
$$\vec{\tau} = \vec{r} \times \vec{F}, \vec{r} = R, \vec{F} = mg\sin\theta$$

r and F both are perpendicular so torque about point $P = mgR\sin\theta$.

43. A bob of mass m is suspended at point O by string of length ℓ. Bob is moving in a horizontal circle. Find out

(A) Torque of gravity and tension about point O and O'.

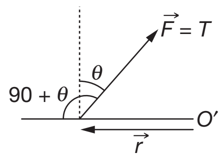
(B) Net torque about axis OO'.



Solution:

(A) Torque about point O

Torque of tension (T), $\tau_{ten} = 0$ (tension is passing through point O)



Torque of gravity $\tau_{mg} = \ell mg \sin\theta$

Torque about point O'

Torque of gravity $\tau_{mg} = mgr$ $r = \ell \sin\theta$

Torque of tension $\tau_{mg} = \ell m\, g\sin\theta$ (along negative $\hat{j}$)

Torque of tension $\tau_{ten} = Tr\sin(90 + \theta)$

$(T\cos\theta = mg)$

$\tau_{ten} = Tr\cos\theta$

$\tau_{ten} = \dfrac{mg}{\cos\theta}(\ell\sin\theta)\cos\theta = mg\ell\sin\theta$ (along positive $\hat{j}$)

(B) Torque about axis OO'

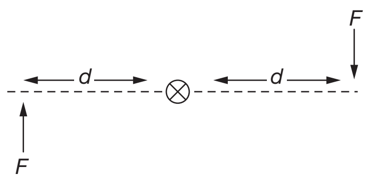
Torque of gravity about axis OO' $\tau_{mg} = 0$ (force mg is parallel to axis OO')

Torque of tension about axis OO' $\tau_{ten} = 0$ (force T is passing through the axis OO')

Net torque about axis OO' $\tau_{net} = 0$.

Force Couple

A pair of forces each of same magnitude and acting in opposite direction is called a force couple. Torque due to couple = magnitude of one force × distance between their lines of action.

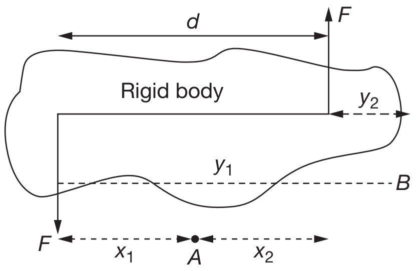


Magnitude of torque $= \tau = F(2d)$

A couple does not exert a net force on an object even though it exerts a torque.

Net torque due to a force couple is same about any point.

$$\text{Torque about } A = x_1F + x_2F$$
$$= F(x_1 + x_2) = Fd$$



Torque about $B = y_1F - y_2F$

$$= F(y_1 - y_2) = Fd$$

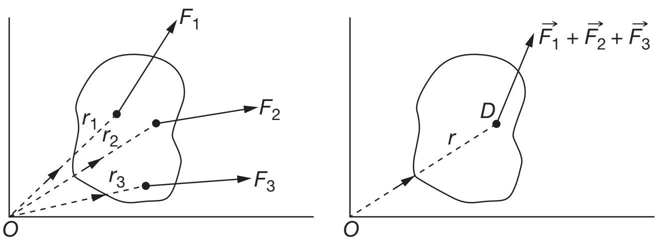
If net force acting on a system is zero, torque is same about any point.

A consequence is that, if $F_{net} = 0$ and $\tau_{net} = 0$ about one point, then $\tau_{net} = 0$ about any point.

Point of Application of Force

Point of application of force is the point at which, if net force is assumed to be acting, then it will produce same translational as well as rotational effect as produced earlier.

We can also define point of application of force as a point about which torque of all the forces is zero.



Consider three forces $\vec{F}_1, \vec{F}_2, \vec{F}_3$ acting on a body if D is point of application of force then torque of $\vec{F}_1 + \vec{F}_2 + \vec{F}_3$ acting at a point D about O is same as the original torque about O

$$\left[\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3\right] = \vec{r} \times (\vec{F}_1 + \vec{F}_2 + \vec{F}_3)$$

SOLVED EXAMPLES

44. Determine the point of application of force, when forces of 20 N and 30 N are acting on the rod as shown in Fig. 6.18.

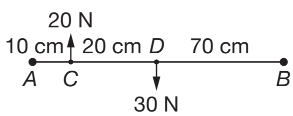
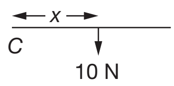


Fig. 6.18

Solution:



Net force acting on the rod

$F_{rel} = 10$ N

Net torque acting on the rod about point C

$$\tau_c = (20 \times 0) + (30 \times 20) = 600 \qquad \text{clockwise}$$

Let the point of application be at a distance x from point C

$$600 = 10\,x \Rightarrow x = 60 \text{ cm}$$

$\therefore$ 70 cm from A is point of application.

45. Determine the point of application of force, when forces are acting on the rod as shown in Fig. 6.19.

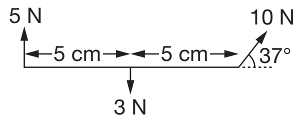
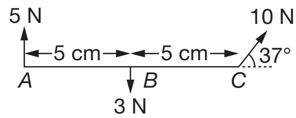


Fig. 6.19

Solution:



Torque of B about A $\tau_1 = 3\text{ N} \times 5 = 15$ N cm (clockwise)

Torque of C about A $\tau_2 = 6\text{ N} \times 10 = 60$ N cm (anti-clockwise)

Resultant force perpendicular to the rod $F = 8$ N

$$\tau_1 + \tau_2 = Fx \quad (x = \text{distance from point } A)$$

$$-15 + 60 = 8x$$

$$x = 45/8 = 5.625 \text{ cm}$$

5.625 cm right on the rod from the point where 5 N force is acting.

NOTE

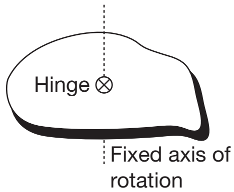
- Point of application of gravitational force is known as the centre of gravity.
- Centre of gravity coincides with the centre of mass if value of $\vec{g}$ is assumed to be constant.
- Concept of point of application of force is imaginary, and in some cases it can lie outside the body.

Rotation About a Fixed Axis

If I_{Hinge} = moment of inertia about the axis of rotation (This axis passes through the hinge, hence the name I_{Hinge}).

$(\vec{\tau}_{ext})$ = resultant external torque acting on the body about axis of rotation

α = angular acceleration of the body.



$$(\vec{\tau}_{ext})_{Hinge} = I_{Hinge}\,\vec{\alpha}$$

$$\text{Rotational kinetic energy} = \frac{1}{2}\cdot I\cdot\omega^2$$

$$\vec{P} = m\vec{v}_{CM}$$

$$\vec{F}_{external} = m\vec{a}_{CM}$$

Net external force acting on the body has two components tangential and centripetal.

$$\Rightarrow \quad F_C = ma_C = m\frac{v^2}{r_{CM}} = m\omega^2 r_{CM}$$

$$\Rightarrow \quad F_t = ma_t = m\alpha\, r_{CM}$$

SOLVED EXAMPLES

46. A wheel of radius R and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination θ as shown in Fig. 6.20. A string is wrapped round the wheel and its free end supports a block of mass m which can slide on the plane. Initially, the wheel is rotating at a speed ω in a direction such that the block slides up the plane. How far will the block move before stopping?

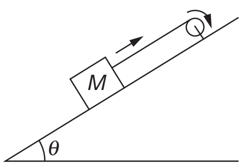


Fig. 6.20

Solution:

Suppose the deceleration of the block is a. The linear deceleration of the rim of the wheel is also a. The angular deceleration of the wheel is $\alpha = a/R$. If the tension in the string is T, the equations of motion are as follows:

$$mg\sin\theta - T = ma \quad \text{and} \quad Tr = I\alpha = Ia/R.$$

Eliminating T from these equations,

$$mg\sin\theta - I\frac{a}{R^2} = Ma$$

giving,
$$a = \frac{mg\,R^2\sin\theta}{I + mR^2}$$

The initial velocity of the block up the incline is $v = \omega r$. Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{\omega^2 r^2 (I + mr^2)}{2mr^2\sin\theta} = \frac{(I + mr^2)\omega^2}{2mg\sin\theta}.$$

47. The pulley shown in Fig. 6.21 has a moment of inertia I about its axis and its radius is R. Find the magnitude of the acceleration of the two blocks.

Assume that the string is light and does not slip on the pulley.

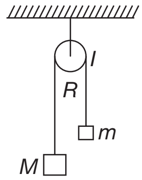


Fig. 6.21

Solution:

Suppose the tension in the left string is T_1 and that in the right string in T_2. Suppose the block of mass M goes down with an acceleration α and the other block moves up with the same acceleration. This is also the tangential acceleration of the rim of the wheel as the string does not slip over the rim. The angular acceleration of the wheel is, therefore, $\alpha = a/R$. The equations of motion for the mass M, the mass m and the pulley are as follows:

$$Mg - T_1 = Ma \qquad (1)$$

$$T_2 - mg = ma \qquad (2)$$

$$T_1R - T_2R = I\alpha = I\alpha/R \qquad (3)$$

Putting T_1 and T_2 from (1) and (2) into (3),

$$[(Mg - a) - m(g + a)]\,R = I\frac{a}{R}$$

which gives

$$a = \frac{(M - m)gR^3}{I + (M + m)R^2}.$$

48. A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H.

(A) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest?

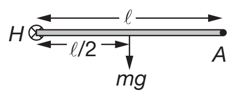
(B) Calculate the acceleration (tangential and radial) of point A at this moment.

(C) Calculate net hinge force acting at this moment.

(D) Find α and ω when rod becomes vertical.

(E) Find hinge force when rod becomes vertical.

Solution:



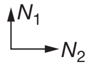
(A) $\tau_H = I_H\,\alpha$

$$mg\cdot\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha \quad \Rightarrow \quad \alpha = \frac{3g}{2\ell}$$

(B) $a_{tA} = \alpha\ell = \dfrac{3g}{2\ell} \cdot \ell = \dfrac{3g}{2}$

$a_{CA} = \omega^2 r = 0 \cdot \ell = 0 \quad (\because\ \omega = 0$ just after release)

(C) Suppose hinge exerts normal reaction in component form as shown



In vertical direction,

$F_{ext} = ma_{CM}$

$\Rightarrow mg - N_1 = m \cdot \dfrac{3g}{4}$ (We get the value of a_{CM} from previous example)

$\Rightarrow N_1 = \dfrac{mg}{4}$

In horizontal direction,

$F_{ext} = ma_{CM} \Rightarrow N_2 = 0$ ($\because$ a_{CM} in horizontal $= 0$ as $\omega = 0$ just after release).

(D) Torque = 0 when rod becomes vertical.

so $\alpha = 0$

using energy conservation, $\dfrac{mg\ell}{2} = \dfrac{1}{2}I\omega^2 \left(I = \dfrac{m\ell^2}{3}\right)$

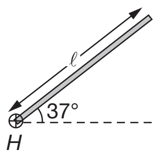
$\omega = \sqrt{\dfrac{3g}{\ell}}$

(E) When rod becomes vertical

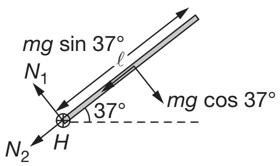
$\alpha = 0,\ \omega = \sqrt{\dfrac{3g}{\ell}}$

$\Rightarrow \quad F_H - mg = \dfrac{m\omega^2\ell}{2} \Rightarrow F_H = \dfrac{5mg}{2}.$

49. A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H. Find angular acceleration α of the rod just after it is released from initial position making an angle of 37° with horizontal from rest? Find the force exerted by the hinge just after the rod is released from rest.



Solution:



Torque above hinge = $\tau_H = I\alpha$

$$mg\cos 37 \frac{\ell}{2} = \frac{m\ell^2}{3}\alpha$$

$$\alpha = 6g/5\ell$$

$$a_t = \alpha\frac{\ell}{2} = \frac{3g}{5}$$

$$mg\cos 37° - N_1 = ma_t$$

$$N_1 = \frac{mg}{5}$$

Angular velocity of rod is zero. So, $N_2 = mg\sin 37° = 3\ mg/5$

$$N = \sqrt{N_1^2 + N_2^2} = \sqrt{\left(\frac{mg}{5}\right)^2 + \left(\frac{3mg}{5}\right)^2} = \frac{mg\sqrt{10}}{5}$$

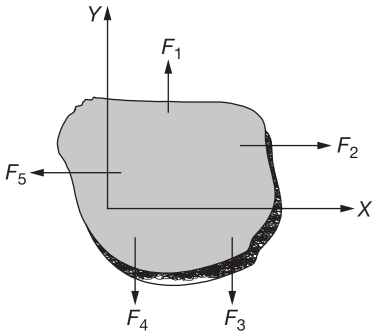
$$6g/5\ell,\ \frac{mg\sqrt{10}}{5}.$$

EQUILIBRIUM

A system is in mechanical equilibrium if it is in translational as well as rotational equilibrium.

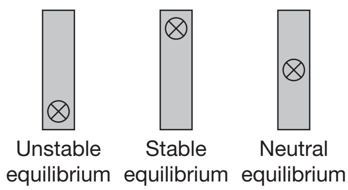
For this, $\overrightarrow{F_{net}} = 0$

$\overrightarrow{\tau_{net}} = 0$ (about every point)



From (6.3), if $\overrightarrow{F_{net}} = 0$ then $\overrightarrow{\tau_{net}}$ is same about every point

Hence necessary and sufficient condition for equilibrium is $\overrightarrow{F_{net}} = 0$, $\overrightarrow{\tau_{net}} = 0$ about any one point, which we can choose as per our convenience. ($\overrightarrow{\tau_{net}}$ will automatically be zero about every point)



The equilibrium of a body is called **stable** if the body tries to regain its equilibrium position after being slightly displaced and released. It is called **unstable** if it gets further displaced after being slightly displaced and released. If it can stay in equilibrium even after being slightly displaced and released, it is said to be in **neutral** equilibrium.

SOLVED EXAMPLES

50. Two small kids weighing 10 kg and 15 kg are trying to balance a see-saw of total length 5.0 m, with the fulcrum at the centre. If one of the kids is sitting at an end, where should the other sit?

Solution:

It is clear that the 10 kg kid should sit at the end and the 15 kg kid should sit closer to the centre. Suppose his distance from the centre is x. As the kids are in equilibrium, the normal force between a kid and the see-saw equals the weight of that kid. Considering the rotational equilibrium of the see-saw, the torque of the forces acting on it should add to zero. The forces are

(A) (15 kg) g downward by the 15 kg kid
(B) (10 kg) g downward by the 10 kg kid
(C) weight of the see-saw
(D) the normal force by the fulcrum

Taking torques about the fulcrum,

$$(15\text{ kg})g\,x = (10\text{ kg})g\,(2.5\text{ m})$$

or $$x = 1.7\text{ m}.$$

51. A uniform ladder of mass m = 10 kg leans against a smooth vertical wall making an angle $\theta = 53°$ with it. The other ends rests on a rough horizontal floor. Find the normal force and the friction force that the floor exerts on the ladder.

Solution:

The forces acting on the ladder are shown in Fig. 6.22. They are

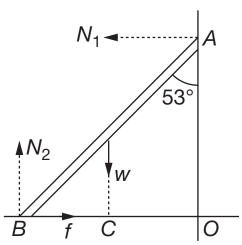


Fig. 6.22

(A) its weight W.
(B) normal force N_1 by the vertical wall.
(C) normal force N_2 by the floor.
(D) frictional force f by the floor.

Taking horizontal and vertical components,

$$N_1 = f \quad (1)$$

and $$N_2 = mg \quad (2)$$

Taking torque about B,

$$N_1(AO) = mg(CB)$$

or, $$N_1(AB)\cos\theta = mg\frac{AB}{2}\sin\theta$$

or $$N_1\frac{3}{5} = \frac{W}{2}\frac{4}{5}$$

or, $$N_1 = \frac{2}{3}W \quad (3)$$

The normal force by the floor is

$$N_2 = W = (10\text{ kg})\,(9.8\text{ m/s}^2) = 98\text{ N}.$$

The frictional force is

$$f = N_1 = \frac{2}{3}W = 65\text{ N}.$$

52. The ladder shown in Fig. 6.23 has negligible mass and rests on a frictionless floor. A crossbar connects the two legs of the ladder at the centre as shown. The angle between the two legs is 60°. The fat person sitting on the ladder has a mass of 80 kg. Find the constant forces exerted by the floor on each leg and the tension in the crossbar.

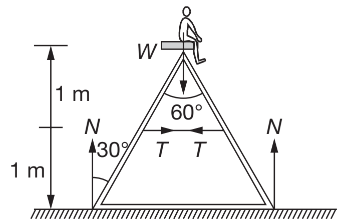


Fig. 6.23

Solution:

The forces acting on different parts are shown in Fig. 6.23. Consider the vertical equilibrium of 'the ladder plus the person' system. The forces acting on this system are its weight (80 kg)g and the contact force N + N = 2 N due to the floor. Thus

$$2\text{ N} = (80\text{ kg})\text{ g}$$

or $$\text{N} = (40\text{ kg})\,(9.8\text{ m/s}^2) = 392\text{ N}.$$

Next consider the equilibrium of the left leg of the ladder. Taking torques of the forces acting on it about the upper end,

$$\text{N}\,(2\text{m})\tan 30° = T(1\text{ m})$$

or $$T = \text{N}\frac{2}{\sqrt{3}} = (392\text{ N}) \times \frac{2}{\sqrt{3}} = 450\text{ N}.$$

53. A uniform rod of length ℓ, mass m is hung from two strings of equal length from a ceiling as shown in Fig. 6.24. Determine the tensions in the strings?

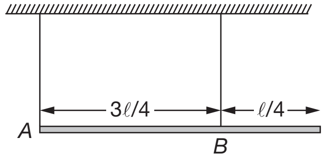


Fig. 6.24

Solution:

$$T_A + T_B = mg \qquad (1)$$

Torque about point A is zero

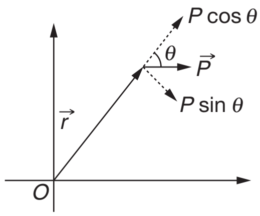
So, $$T_B \times \frac{3\ell}{4} = \text{mg}\ \frac{\ell}{2} \qquad (2)$$

From Equations (1) and (2),

$$T_A = \text{mg}/3,\ T_B = 2\text{mg}/3.$$

ANGULAR MOMENTUM ($\vec{L}$)

Angular Momentum of a Particle About a Point



$$\vec{L} = \vec{r} \times \vec{P}$$

$\Rightarrow$ $$L = rP\sin\theta$$

or $$|\vec{L}| = r_\perp \times P$$

or $$|\vec{L}| = P_\perp \times r$$

Where $\vec{P}$ = momentum of particle

$\vec{r}$ = position of vector of particle with respect to point O about which angular momentum is to be calculated.

θ = angle between vectors $\vec{r}$ and $\vec{P}$

$r_\perp$ = perpendicular distance of line of motion of particle from point O.

$P_\perp$ = component of momentum perpendicular to $\vec{r}$.

SI unit of angular momentum is kgm^2/sec.

SOLVED EXAMPLES

54. A particle is projected at time $t = 0$ from a point P with a speed v_0 at an angle of 45° to the horizontal. Find the magnitude and the direction of the angular momentum of the particle about the point P at time $t = v_0/g$.

Solution:

Let us take the origin at P, X-axis along the horizontal and Y-axis along the vertically upward direction as shown in Fig. 6.25.

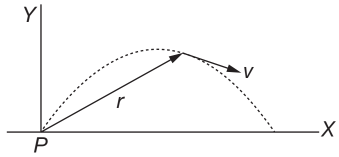


Fig. 6.25

For horizontal motion during the time 0 to t,

$$v_x = v_0 \cos 45° = v_0/\sqrt{2}$$

and $$x = b_x t = \frac{v_0}{\sqrt{2}} \cdot \frac{v_0}{g} = \frac{v_0^2}{\sqrt{2}\,g}$$

For vertical motion,

$$v_y = v_0 \sin 45° = \frac{v_0}{\sqrt{2}} - v_0 = \frac{(1-\sqrt{2})}{\sqrt{2}} v_0$$

and $$y = (v_0 \sin 45°)\, t - \frac{1}{2} gt^2$$

$$= \frac{v_0^2}{\sqrt{2}\,g} - \frac{v_0^2}{2\,g} = \frac{v_0^2}{2\,g}(\sqrt{2}-1).$$

The angular momentum of the particle at time t about the origin is

$$L = \vec{r} \times \vec{p} = m\,\vec{r} \times \vec{v}$$

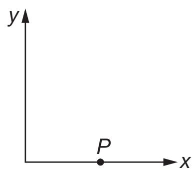
$$= m(\hat{i}\,x + \hat{j}\,y) \times (\hat{i}\,v_x + \hat{j}\,v_y) = m\,(\hat{k}\,xv_u - \hat{k}\,yv_x)$$

$$= m\,\hat{k}\left[\left(\frac{v_0^2}{\sqrt{2}\,g}\right)\frac{v_0}{\sqrt{2}}(1-\sqrt{2}) - \frac{v_0^2}{2g}(\sqrt{2}-1)\frac{v_0}{\sqrt{2}}\right]$$

$$= -\hat{k}\,\frac{mv_0^3}{2\sqrt{2}\,g}.$$

Thus, the angular momentum of the particle is $\frac{mv_0^3}{2\sqrt{2}\,g}$ in the negative Z-direction i.e., perpendicular to the plane of motion, going into the plane.

55. A particle of mass m starts moving from origin with a constant velocity $u\hat{i}$. Find out its angular momentum about origin at this moment. What will be the answer later on? What will be the answer if the speed increases?

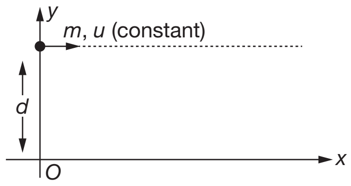


Solution:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = r\hat{i} \times mu\hat{i} = 0.$$

56. A particle of mass m starts moving from point (o, d) with a constant velocity $u\hat{i}$. Find out its angular momentum about origin at this moment. What will be the answer at the later time?

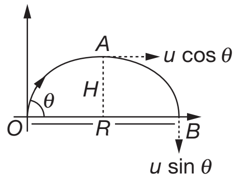


Solution:

$$\vec{L} = -m\,d\,u\,\hat{k}.$$

57. A particle of mass m is projected on horizontal ground with an initial velocity of u making an angle θ with horizontal. Find out the angular momentum of particle about the point of projection when

(A) it just starts its motion.
(B) it is at highest point of path.
(C) it just strikes the ground.



Solution:

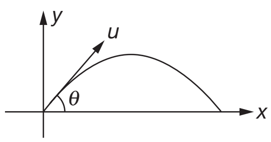
(A) Angular momentum about point O is zero.
(B) Angular momentum about point A.

$$\vec{L} = \vec{r} \times \vec{p}$$
$$L = H \times mu\cos\theta$$
$$L = mu\cos\theta\,\frac{u^2\sin^2\theta}{2g}.$$

(C) Angular momentum about point B.

$$L = R \times mu\sin\theta$$
$$mu\sin\theta\,\frac{u^2\sin 2\theta}{g}.$$

58. A particle of mass m is projected on horizontal ground with an initial velocity of u making an angle θ with horizontal. Find out the angular momentum at any time t of particle p about:

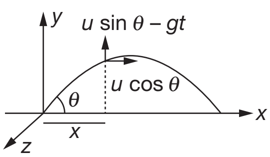


(A) y-axis
(B) z-axis

Solution:

(A) Velocity components are parallel to the y-axis. so, $L = 0$

(B) $\tau = \dfrac{dL}{dt}$

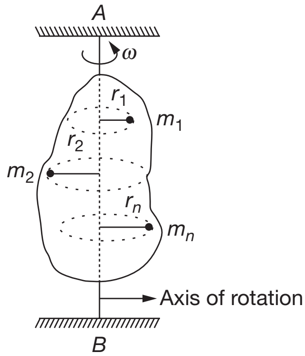


$$-1/2\,mu\cos\theta \cdot gt^2$$
$$-mgx = \frac{dL}{dt}$$
$$-mgx\,dt = dL$$
$$\int_0^t -mgx\,dt = \int_0^L dL$$

angular momentum about the z-axis is

$$L = -1/2\,mu\cos\theta \cdot gt^2.$$

Angular Momentum of a Rigid Body Rotating about Fixed Axis



Angular momentum of a rigid body about the fixed axis AB is

$$L_{AB} = L_1 + L_2 + L_3 + \ldots + L_n$$
$$L_1 = m_1 r_1 \omega r_1,\ L_2 = m_2 r_2 \omega r_2,\ L_3 = m_3 r_3 \omega r_3,\ L_n = m_n r_n \omega r_n$$
$$L_{AB} = m_1 r_1 \omega_{r1} + m_2 r_2 \omega r_2 + m_3 r_3 \omega r_3 \ldots + m_n\,r_n \omega r_n$$
$$L_{AB} = \sum_{n=1}^{n=n} m_n (r_n)^2 \times \omega$$
$$\Rightarrow \qquad \left[\sum_{n=1}^{n=n} m_n (r_n)^2 = I_H\right]$$
$$L_{AB} = I_H\,\omega$$
$$\Rightarrow \qquad L_H = I_H\,\omega$$

L_H = angular momentum of object about axis of rotation.
I_H = moment of inertia of rigid body about axis of rotation.
ω = angular velocity of the object.

SOLVED EXAMPLES

59. Two small balls A and B, each of mass m, are attached rigidly to the ends of a light rod of length d. The structure rotates about the perpendicular bisector of the rod at an angular speed ω. Calculate the angular momentum of the individual balls and of the system about the axis of rotation.

Solution:

Consider the situation shown in Fig. 6.26. The velocity of the ball A with respect to the centre O is $v = \dfrac{\omega d}{2}$.

The angular momentum of the ball with respect to the axis is

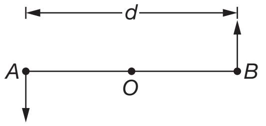


Fig. 6.26

$L_1 = mvr = m\left(\frac{\omega\, d}{2}\right)\left(\frac{d}{2}\right) = \frac{1}{4}\; m\omega d^2$. The angular momentum L_2 of the second ball will be same. The angular momentum of the system is equal to sum of these two angular momenta i.e., $L = 1/2\; m\omega d^2$.

60. Two particles of mass m each are attached to a light rod of length d, one at its centre and the other at a free end. The rod is fixed at the other end and is rotated in a plane at an angular speed ω. Calculate the angular momentum of the particle at the end with respect to the particle at the centre.

Solution:

The situation is shown in Fig. 6.27. The velocity of the particle A with respect to the fixed end O is $v_A = \omega(d/2)$ and that of B with respect to O is $v_B = \omega d$. Hence the velocity of B with respect to A is $v_B - v_A = \omega(d/2)$. The angular momentum of B with respect to A is, therefore,

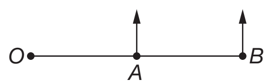


Fig. 6.27

$$L = mvr = m\omega\left(\frac{d}{2}\right)\frac{d}{2} = \frac{1}{4}\; m\omega d^2$$

along the direction perpendicular to the plane of rotation.

61. A uniform circular disc of mass 200 g and radius 4.0 cm is rotated about one of its diameter at an angular speed of 10 rad/s. Find the kinetic energy of the disc and its angular momentum about the axis of rotation.

Solution:

The moment of inertia of the circular disc about its diameter is

$$I = \frac{1}{4}Mr^2 = \frac{1}{4}(0.200 \text{ kg})\,(0.04 \text{ m})^2 = 8.0 \times 10^{-5} \text{ kg/m}^2.$$

The kinetic energy is

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}\;(8.0 \times 10^{-5} \text{ kg/m}^2)\,(100 \text{ rad}^2/\text{s}^2)$$

$$= 4.0 \times 10^{-3} \text{ J}$$

The angular momentum about the axis of rotation is

$$L = I\omega = (8.0 \times 10^{-5} \text{ kg/m}^2)\,(10 \text{ rad/s})$$

$$= 8.0 \times 10^{-4} \text{ kg-m}^2/\text{s} = 8.0 \times 10^{-4} \text{ J/s}.$$

Conservation of Angular Momentum

Newton's 2nd Law In Rotation

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

where $\vec{\tau}$ and $\vec{L}$ are about the same axis.

Angular momentum of a particle or a system remains constant if $\tau_{ext} = 0$ about the axis of rotation.

Even if net angular momentum is not constant, one of its components of an angular momentum about an axis remains constant if component of torque about that axis is zero

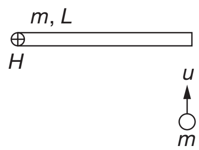
Impulse of Torque

$$\int \tau dt = \Delta J$$

$\Delta J \rightarrow$ Charge in angular momentum.

SOLVED EXAMPLES

62. A uniform rod of mass m and length L can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod, strikes the rod in-elastically at its free end. Find out the angular velocity of the rod just after collision?



Solution:

Angular momentum is conserved about H because no external force is present in horizontal plane which is producing torque about H.

$$mul = \left(\frac{m\ell^2}{3} + m\ell^2\right)\omega \quad\Rightarrow\quad \omega = \frac{3a}{4\ell}.$$

63. A uniform rod of mass m and length a lies on a smooth horizontal plane. A particle of mass m moving at a speed v perpendicular to the length of the rod strikes it at a distance $a/4$ from the centre and stops after the collision. Find (a) the velocity of the centre of the rod and (b) the angular velocity of the rod about its centre just after the collision.

Solution:

The situation is shown in Fig. 6.28. Consider the rod and the particle together as the system. As there is no external resultant force, the linear momentum of the system will remains constant. Also there is no resultant external torque on the system and so the angular momentum of the system about any line will remain constant. Suppose the velocity of the centre of the rod is v and the angular velocity about the centre is ω.

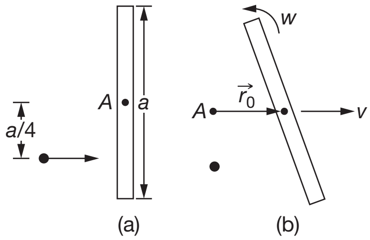


Fig. 6.28

(A) The linear momentum before the collision is mv and that after the collision is Mv. Thus,

$$mv = Mv, \quad \text{or} \quad v = \frac{m}{M}\,v.$$

(B) Let A be the centre of the rod when it is at rest. Let AB be the line perpendicular to the plane of the Fig. 6.28. Consider the angular momentum of 'the rod plus the particle' system about AB. Initially, the rod is at rest. The angular momentum of the particle about AB is

$$L = mv(a/4)$$

After the collision, the particle comes to rest. The angular momentum of the rod about A is

$$\vec{L} = \vec{L}_{cm} + M\vec{r}_0 \times \vec{v}$$

As $\quad \vec{r}_0 \parallel \vec{v}, \quad \vec{r}_0 \times \vec{v} = 0$

Thus, $\quad \vec{L} = \vec{L}_{cm}$

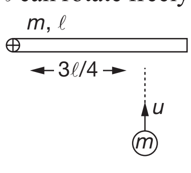
Hence the angular momentum of the rod about AB is

$$L = I\omega = \frac{M\ell^2}{12}\,\omega.$$

Thus, $\quad \dfrac{mva}{4} = \dfrac{Ma^2}{12}\,\omega$

or, $\quad \omega = \dfrac{3mv}{Ma}.$

64. A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod, strikes the rod and sticks to it at a distance of $3\ell/4$ from hinge point. Find out the angular velocity of the rod just after collision?



Solution:

Angular momentum about hinge

$$L_i = L_f$$

$$mu\left(\frac{3\ell}{4}\right) = \left(\frac{m\ell^2}{3} + m\left(\frac{3\ell}{4}\right)^2\right)\omega$$

$$\omega = \frac{36u}{43\ell}.$$

COMBINED TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY

The general motion of a rigid body can be thought of as a sum of two independent motions. A translation of some point of the body plus a rotation about this point. A most convenient choice of the point is the centre of mass of the body as it greatly simplifies the calculations.

Consider a fan inside a train, and an observer a on the platform.

It the fan is switched off while the train moves, the motion of fan is pure translation as each point on the fan undergoes same translation in any time interval.

It fan is switched on while the train is at rest the motion of fan is pure rotation about is axle; as each point on the axle is at rest, while other points revolve about it with equal angular velocity.

If the fan is switched on while the train is moving, the motion of fan to the observer on the platform is neither pure translation nor pure rotation. This motion is an example of general motion of a rigid body.

Now if there is an observer B inside the train, the motion of fan will appear to him as pure rotation.

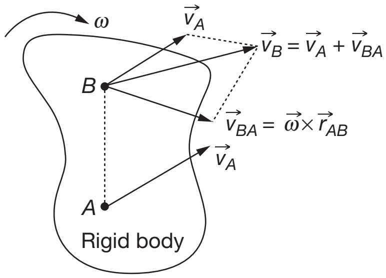
Hence we can see that the general motion of fan with regard to observer A can be resolved into pure rotation of fan as observed by observer B plus pure translation of observer B (with respect to observer A)

Such a resolution of general motion of a rigid body into pure rotation and pure translation is not restricted to just the fan inside the train, but is possible for motion of any rigid system.

Kinematics of General Motion of a Rigid Body

For a rigid body as stated earlier value of angular displacement (θ), angular velocity (ω), angular acceleration (α) is same for all points on the rigid body about any other point on the rigid body.

Hence if we know velocity of any one point (say A) on the rigid body and angular velocity of any point on the rigid body about any other point on the rigid body (say ω), velocity of each point on the rigid body can be calculated.



Since distance AB is fixed

$$\vec{v}_{BA} \perp \overrightarrow{AB}$$

we know that

$$\omega = \frac{v_{BA\perp}}{r_{BA}}$$

$$v_{BA\perp} = v_{BA} = \omega r_{BA}$$

in vector form

$$\vec{v}_{BA} = \vec{\omega} \times \vec{r}_{BA}$$

Now from relative velocity:

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

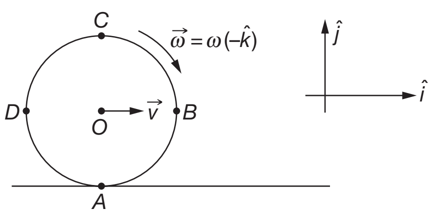
$$\vec{v}_B = \vec{v}_A + \vec{v}_{BA}$$

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{BA}$$

similarly $\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{BA}$ [for any rigid system]

SOLVED EXAMPLE

65. Consider the general motion of a wheel (radius r) which can be viewed on pure translation of its center O (with the velocity v) and pure rotation about O (with angular velocity ω)



Find out $\vec{v}_{AO}, \vec{v}_{BO}, \vec{v}_{CO}, \vec{v}_{DO}$ and $\vec{v}_A, \vec{v}_B, \vec{v}_C, \vec{v}_D$

Solution:

$$\vec{v}_{AO} = (\vec{\omega} \times \vec{r}_{AO})$$

$$\vec{v}_{AO} = \left(\omega(-\hat{k}) \times \overrightarrow{OA}\right)$$

$$\vec{v}_{AO} = \left(\omega(-\hat{k}) \times r(-\hat{j})\right)$$

$$\vec{v}_{AO} = -\omega r \hat{i}$$

similarly

$$\vec{v}_{BO} = \omega r(-\hat{j})$$

$$\vec{v}_{CO} = \omega r(\hat{i})$$

$$\vec{v}_{DO} = \omega r(\hat{j})$$

$$\vec{v}_A = \vec{v}_O + \vec{v}_{AO} = v\hat{i} - \omega r\hat{i}$$

similarly

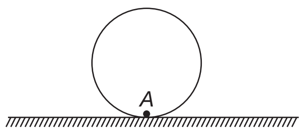
$$\vec{v}_B = \vec{v}_O + \vec{v}_{BO} = v\hat{i} - \omega r\hat{j}$$

$$\vec{v}_C = \vec{v}_O + \vec{v}_{CO} = v\hat{i} + \omega r\hat{i}$$

$$\vec{v}_D = \vec{v}_O + \vec{v}_{DO} = v\hat{i} + \omega r\hat{j}.$$

Pure Rolling (or Rolling without Sliding)

Pure rolling is a special case of general rotation of a rigid body with circular cross section (e.g. wheel, disc, ring, sphere) moving on some surface. Here, there is no relative motion between the rolling body and the surface of contact, at the point of contact



Here contact point is A and contact surface is horizontal ground. For pure rolling velocity of A with respect to ground = 0 $\Rightarrow v_A = 0$.

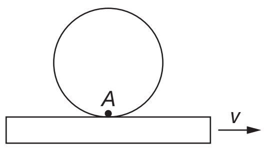


Fig. 6.29

From Fig. 6.29, for pure rolling, velocity of A with respect to plank is zero $\Rightarrow v_A = v$.

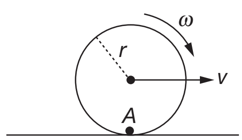


Fig. 6.30

From Fig. 6.30 for, pure rolling, velocity of A with respect to ground is zero.

$\Rightarrow \qquad v - \omega r = 0$

$$v = \omega r$$

Similarly $\qquad a = \alpha r$

SOLVED EXAMPLE

66. A wheel of radius r rolls (rolling without sleeping) on a level road as shown in Fig. 6.31.

Find out velocity of point A and B.

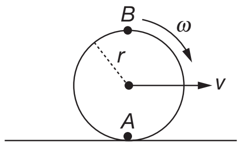


Fig. 6.31

Solution:

Contact surface is in rest for pure rolling velocity of point A is zero.

so $\qquad v = \omega r$

velocity of point $B = v + \omega r = 2v$.

Dynamics of General Motion of a Rigid Body

This motion can be viewed as translation of centre of mass and rotation about an axis passing through centre of mass (COM).

If $\quad I_{CM}$ = moment of inertia about this axis passing through COM

τ_{cm} = net torque about this axis passing through COM

$\vec{a}_{CM}$ = acceleration of COM

$\vec{v}_{CM}$ = velocity of COM

$\vec{F}_{ext}$ = net external force acting on the system

$\vec{P}_{system}$ = linear momentum of system

$\vec{L}_{CM}$ = Angular momentum about centre of mass

$\vec{r}_{CM}$ = position vector of COM with respect to point A

then

1. $\vec{\tau}_{cm} = I_{cm}\vec{\alpha}$
2. $\vec{F}_{ext} = M\vec{a}_{cm}$
3. $\vec{P}_{system} = M\vec{v}_{cm}$
4. Total KE $= \dfrac{1}{2}Mv_{cm^2} + \dfrac{1}{2}I_{cm}\,\omega^2$
5. $\vec{L}_{CM} = I_{CM}\,\vec{\omega}$
6. Angular momentum about point $A = \vec{L}$ about C.M. + $\vec{L}$ of C.M. about A

$$\vec{L}_A = I_{cm}\vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}$$

$\dfrac{dL_A}{dt} = \dfrac{d}{dt}(I_{cm}\vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}) \neq I_A\dfrac{d\vec{\omega}}{dt}$. Notice that torque equation can be applied to a rigid body in a general motion only and only about an axis through centre of mass.

SOLVED EXAMPLES

67. A uniform sphere of mass 200 g rolls without slipping on a plane surface so that its centre moves at a speed of 2.00 cm/s. Find out its kinetic energy.

Solution:

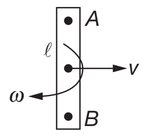
As the sphere rolls without slipping on the plane surface, its angular speed about the centre is $\omega = \dfrac{v_{cm}}{r}$. The kinetic energy is

$$K = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}mv_{cm}^2 = \frac{1}{2}\cdot\frac{2}{5}\cdot mr^2\omega^2 + \frac{1}{2}mv_{cm}^2$$

$$= \frac{1}{5}mv_{cm}^2 + \frac{1}{2}mv_{cm}^2 = \frac{7}{10}Mv_{cm}^2$$

$$= \frac{7}{10}(0.200\text{ kg})(0.02\text{ m/s})^2 = 5.6 \times 10^{-5}\text{ J.}$$

68. Uniform and smooth rod of length ℓ is moving with a velocity of centre v and angular velocity ω on smooth horizontal surface. Findout velocity of points A and B.



Solution:

Velocity of point A with respect to centre is $\omega\dfrac{\ell}{2}$

Velocity of point A with respect to ground $v_A = v + \omega\dfrac{\ell}{2}$

Velocity of point B with respect to centre is $-\omega\dfrac{\ell}{2}$

Velocity of point B with respect to ground $v_A = v - \omega\dfrac{\ell}{2}$.

69. A force F acts tangentially at the highest point of a sphere of mass m kept on a rough horizontal plane. If the sphere rolls without slipping, find the acceleration of the centre (C) and points A and B of the sphere.

Solution:

The situation is shown in Fig. 6.32. As the force F rotates the sphere, the point of contact has a tendency to slip towards left so that the static friction on the sphere acts towards right. Let r be the radius of the sphere and a be the linear acceleration of the centre of the sphere. The angular acceleration about the centre of the sphere is $\alpha = a/r$, as there is no slipping.

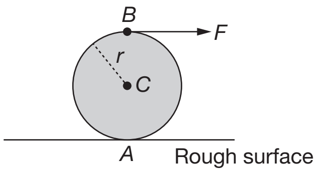


Fig. 6.32

For the linear motion of the centre,

$$F + f = ma \quad (1)$$

and for the rotational motion about the centre,

$$Fr - fr = I\alpha = \left(\frac{2}{5}mr^2\right)\left(\frac{a}{r}\right)$$

or, $$F - f = \frac{2}{5}ma, \quad (2)$$

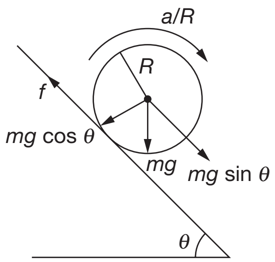
From (1) and (2),

$$2F = \frac{7}{5}ma \quad \text{or} \quad a = \frac{10\,F}{7\,m}.$$

Acceleration of point A is zero.

Acceleration of point B is $2a = 2\left(\frac{10F}{7}\right)$.

70. A circular rigid body of mass m, radius R and radius of gyration (k) rolls without slipping on an inclined plane of an inclination θ. Find the linear acceleration of the rigid body and force of friction on it. What must be the minimum value of coefficient of friction so that rigid body can roll without sliding?



Solution:

If a is the acceleration of the centre of mass of the rigid body and f is the force of friction between sphere and the plane, the equation of translatory and rotatory motion of the rigid body will be.

$$mg\sin\theta - f = ma \quad \text{(Translatory motion)}$$

$$fR = I\,\alpha \quad \text{(Rotatory motion)}$$

$$f = \frac{I\alpha}{R}$$

$I = mk^2$, due to pure rolling $a = \alpha R$

$$mg\sin\theta - \frac{I\alpha}{R} = m\alpha R$$

$$mg\sin\theta = m\alpha R + \frac{I\alpha}{R}$$

$$mg\sin\theta = m\alpha R + \frac{mk^2\alpha}{R}$$

$$mg\sin\theta = ma + \frac{mk^2\alpha}{R}$$

$$mg\sin\theta = a\left[\frac{R^2 + k^2}{R^2}\right]$$

$$a = \frac{g\sin\theta}{\left[\frac{R^2+k^2}{R^2}\right]}$$

$$a = \frac{g\sin\theta}{\left(1 + \frac{k^2}{R^2}\right)}$$

$$f = \frac{I\alpha}{R}$$

$$f = \frac{mk^2 a}{R^2} \Rightarrow \frac{mg\,k^2\sin\theta}{R^2 + k^2}$$

$$f \le \mu N$$

$$\frac{mk^2}{R^2}a \le \mu \le mg\cos\theta$$

$$R^2\,\frac{k^2}{R^2} \times \frac{g\sin\theta}{(k^2 + R^2)} \le \mu g\cos\theta$$

$$\mu \ge \frac{\tan\theta}{\left[1 + \frac{R^2}{k^2}\right]}$$

$$\mu_{\min} = \frac{\tan\theta}{\left[1 + \frac{R^2}{k^2}\right]}.$$

NOTE

From above example if rigid bodies are solid cylinder, hollow cylinder, solid sphere and hollow sphere.

- Increasing order of acceleration.

  $a_{\text{solid sphere}} > a_{\text{hollow sphere}} > a_{\text{solid cylinder}} > a_{\text{hollow cylinder}}$
- Increasing order of required friction force for pure rolling.

  $f_{\text{hollow cylinder}} > f_{\text{hollow sphere}} > f_{\text{solid cylinder}} > f_{\text{solid sphere}}$
- Increasing order of required minimum friction coefficient for pure rolling.

  $\mu_{\text{hollow cylinder}} > \mu_{\text{hollow sphere}} > \mu_{\text{solid cylinder}} > \mu_{\text{solid sphere}}$

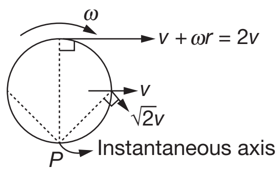
Instantaneous Axis of Rotation

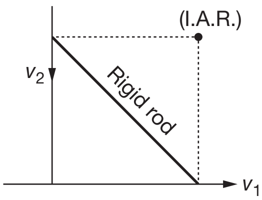
It is the axis about which the combined translational and rotational motion appears as pure rotational motion.

The combined effect of translation of centre of mass and rotation about an axis through the centre of mass is equivalent to a pure rotation with the same angular speed about a stationary axis ; this axis is called instantaneous axis of rotation. It is defined for an instant and its position changes with time.

For example, in pure rolling, the point of contact with the surface is the instantaneous axis of rotation.

Geometrical construction of instantaneous axis of rotation (I.A.R). Draw velocity vector at any two points on the rigid body. The I.A.R. is the point of intersection of the perpendicular drawn on them.





In case of pure rolling the lower point is instantaneously axis of rotation.

The motion of body in pure rolling can therefore by analysed as pure rotation about this axis.

Consequently

$$\tau_P = I_P \alpha$$

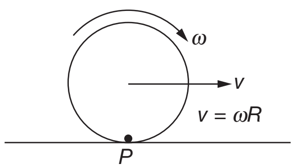
$$\alpha_P = I_P \omega$$

$$\text{KE} = 1/2\ I_P\ \omega^2$$

Where I_P is moment of inertial instantaneous axis of rotation passing through P.

SOLVED EXAMPLES

71. Prove that kinetic energy = $1/2\ I_P\ \omega^2$



Solution:

$$\text{KE} = \frac{1}{2}\ I_{cm}\ \omega^2 + \frac{1}{2}\ mv_{cm}^2$$

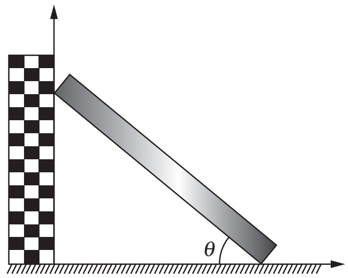
$$= \frac{1}{2}\ I_{cm}\ \omega^2 + \frac{1}{2}\ m\omega^2 R^2$$

$$\frac{1}{2}(I_{cm} + mR^2)\ \omega^2$$

$$\Rightarrow \quad \frac{1}{2}\left(I_{\text{contact point}}\right)\omega^2$$

Notice that pure rolling of uniform object equation of torque can also be applied about the contact point.

72. A uniform bar of length ℓ and mass m stands vertically touching a vertical wall (y-axis). When slightly displaced, its lower end begins to slide along the floor (x-axis). Obtain an expression for the angular velocity (ω) of the bar as a function of θ. Neglect friction everywhere.



Solution:

The position of instantaneous axis of rotation (IAOR) is shown in Fig. 6.33.

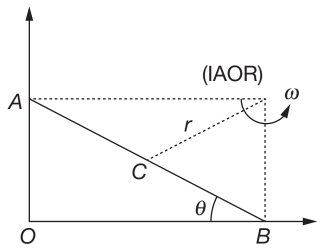


Fig. 6.33

$$C = \left(\frac{\ell}{2}\cos\theta,\ \frac{\ell}{2}\sin\theta\right)$$

$$r = \frac{\ell}{2} = \text{half of the diagonal}$$

All surfaces are smooth. Therefore, mechanical energy will remain conserved.

∴ Decrease in gravitational potential energy of bar = increase in rotational kinetic energy of bar about IAOR.

$$\therefore \quad mg\frac{\ell}{2}\ (1 - \sin\theta) = \frac{1}{2}I\omega^2 \qquad (1)$$

Here, $$I = \frac{m\ell^2}{12} + mr^2 \quad \text{(about IAOR)}$$

or $$I = \frac{m\ell^2}{12} + \frac{m\ell^2}{4} = \frac{m\ell^2}{3}$$

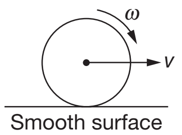
Substituting in Equation (1), we have

$$mg\frac{\ell}{2}\ (1 - \sin\ \theta) = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2$$

or $$\omega = \sqrt{\frac{3g\ (1-\sin\theta)}{\ell}}.$$

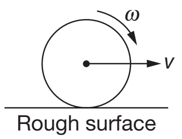
The Nature of Friction in the following Cases Assume Body is Perfectly Rigid

1. $v = \omega R$



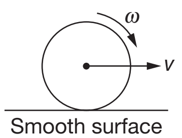
No friction and pure rolling.

2. $v = \omega R$



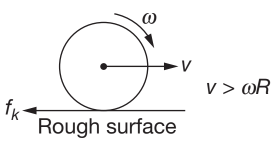
No friction and pure rolling (If the body is not perfectly rigid, then there is a small friction acting in this case which is called rolling friction).

3. $v > \omega R$ or $v < \omega R$



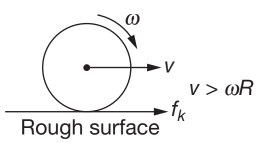
No friction force but not pure rolling.

4. $v > \omega R$



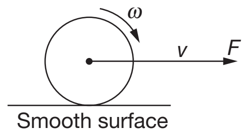
There is relative motion at point of contact so kinetic friction, $f_k = \mu N$ will act in backward direction. This kinetic friction decrease v and increase ω. So after some time $v = \omega R$ and pure rolling will resume as in case (2).

5. $v < \omega R$



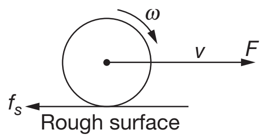
There is relative motion at point of contact so kinetic friction, $f_k = \mu N$ will act in forward direction. This kinetic friction increases v and decreases ω. So after some time $v = \omega R$ and pure rolling will resume as in case (2).

6. $v = \omega R$ (initial)



No friction and no pure rolling.

7. $v = \omega R$ (initial)



Static friction whose value can lie between zero and $\mu_s N$ will act in backward direction. If coefficient of friction is sufficiently high, then f_s compensates for increasing v due to F by increasing ω, and body may continue in pure rolling which increases v as well as ω.

SOLVED EXAMPLES

73. A rigid body of mass m and radius r rolls without slipping on a rough surface. A force is acting on a rigid body x distance from the centre as shown in Fig. 6.34. Find the value of x so that static friction is zero.

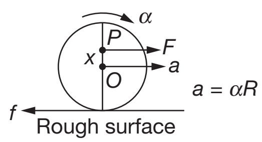


Fig. 6.34

Solution:

Torque about centre of mass

$$Fx = I_{cm}\ \alpha \quad (1)$$

$$F = ma \quad (2)$$

From Equations (1) and (2),

$$max = I_{cm}\ \alpha \quad (a = \alpha R)$$

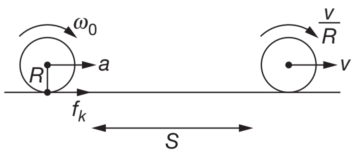
$$x = \frac{I_{cm}}{mR}.$$

NOTE

For pure rolling if any friction is required then friction force will be statics friction. It may be zero, backward direction or forward direction depending on value of x. If F is below point P then friction force will act in backward direction or if it is above point P, friction force will act in forward direction.

74. A cylinder is given angular velocity ω_0 and kept on a horizontal rough surface the initial velocity is zero. Find distance travelled by the cylinder before it performs pure rolling and work done by friction force.

Solution:



$$\mu Mg\,R = \frac{MR^2\alpha}{2}$$

$$\alpha = \frac{2\mu g}{R} \quad (1)$$

Initial velocity $u = 0$

$$v^2 = u^2 + 2as$$

$$v^2 = 2as \quad (2)$$

$$f_K = Ma$$

$$\mu Mg = Ma$$

$$a = \mu g \quad (3)$$

$$\omega = \omega_0 - \alpha t$$

from Equation (1)

$$\omega = \omega_0 - \frac{2\mu g}{R}t$$

$$v = u + at$$

from Equation (3)

$$v = \mu gt$$

$$\omega = \omega_0 - \frac{2v}{R}$$

$$\omega = \omega_0 - 2\omega$$

$$\omega = \frac{\omega_0}{3}$$

from Equation (2)

$$\left(\frac{\omega_0 R}{3}\right)^2 = (2as) = 2\mu gs$$

$$s = \left(\frac{\omega_0{}^2 R^2}{18\,\mu g}\right)$$

Work done by the friction force

$$w = (-f_k\,R\mathrm{d}\theta + f_k\Delta s)$$

$$-\,\mu mg\,R\,\Delta\theta + \frac{\mu mg \times \omega_0^2 R^2}{18\,\mu g}$$

$$\Delta\theta = \omega_0 \times t - \frac{1}{2}\,\alpha t^2$$

$$= \omega_0 \times \left(\frac{\omega_0 R}{3\mu g}\right) - \frac{1}{2} \times \frac{2\mu g}{R}\left(\frac{\omega_0 R}{3\mu g}\right)^2$$

$$\frac{\omega_0^2 R}{3\mu g} - \frac{\omega_0^2 R}{9\mu g}$$

$$\frac{2\,\omega_0^2 R}{9\mu g}$$

$$-\,\mu mg \times R\,\frac{2\,\omega_0^2 R}{9\mu g} + \mu mg \times \frac{\omega_0^2 R^2}{18\mu g}$$

$$-\,\frac{2m\omega_0^2 R^2}{9} + \frac{m\,\omega_0^2 R^2}{18}$$

$$\frac{-3m\,\omega_0^2 R^2}{18} = -\frac{m\,\omega_0^2 R^2}{6}$$

Alternative Solution:

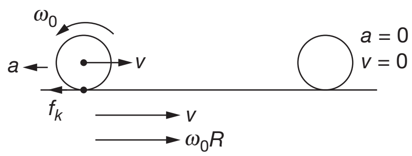
Using work-energy theorem

$$w_g + w_a + w_{f_k} = \Delta K$$

$$w_{f_k} = \left[\frac{1}{2}m\left(\frac{\omega_0 R}{3}\right)^2 + \frac{1}{2}\frac{mR^2}{2} \times \left(\frac{\omega_0}{3}\right)^2\right]$$

$$-\left[\frac{1}{2}\frac{mR^2}{2} \times \omega_0^2\right] = \left(-\frac{m\omega_0{}^2 R^2}{6}\right).$$

75. A hollow sphere is projected horizontally along a rough surface with speed v and angular velocity ω_0. Find out the ratio $\frac{v}{\omega_0}$ so that the sphere stops moving after some time.



Solution:

Torque about lowest point of sphere.

$$f_k \times R = I\alpha$$

$$\mu mg \times R = \frac{2}{3}mR^2\alpha$$

$\alpha = \frac{3\mu g}{2R}$ angular acceleration in opposition direction of angular velocity.

$\omega = \omega_0 - \alpha t$ (final angular velocity $\omega = 0$)

$$\omega_0 = \frac{3\mu g}{2R} \times t$$

$$t = \frac{\omega_0 \times 2R}{3\mu g}$$

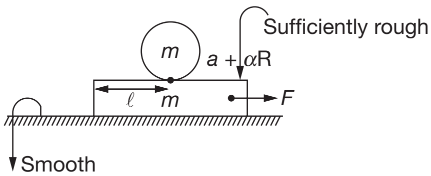
Acceleration ‘$a = \mu g$’

$$v_f = v - at \quad \text{(final velocity } v_f = 0)$$
$$v = \mu g \times t$$
$$t = \frac{v}{\mu g}$$

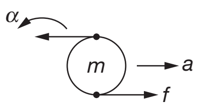
To stop the sphere time at which v and ω are zero, should be same.

$$\frac{v}{\mu g} = \frac{2\omega_0 R}{3\mu g} = \frac{v}{\omega_0} = \frac{2R}{3}.$$

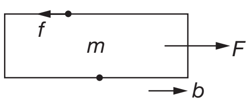
Rolling on Moving Surface



Friction on the plate backward or on cylinder friction forward causes cylinder to move forward.



Because of pure rolling static friction f.

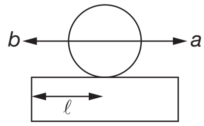


$$fR = \frac{mR^2}{2}\alpha$$
$$\alpha = \frac{2f}{mR}$$
$$f = ma$$
$$F - f = mb$$
$$F = m(a + b)$$
$$a = \frac{\alpha R}{2}$$

At contact point,

$$b = a + \alpha R$$
$$b = \frac{3\alpha R}{2}$$
$$b = 3a$$
$$F = 4ma$$
$$a = \frac{F}{4m}$$
$$b = \frac{3F}{4m}$$

with respect to plate distance is covered = ℓ
and acceleration with respect to plate $(b - a)$



$$\ell = \frac{1}{2}(b - a)\,t^2$$
$$\ell = \frac{1}{2} \times 2at^2 = t = \sqrt{\frac{a \times \ell}{F}} = 2\sqrt{\frac{m\ell}{F}}$$

SOLVED EXAMPLES

76. A cylinder is released from rest from the top of an incline of inclination θ and length ℓ. If the cylinder rolls without slipping, what will be its speed when it reaches the bottom?

Solution:

Let the mass of the cylinder be m and its radius r. Suppose the linear speed of the cylinder when it reaches the bottom is v. As the cylinder rolls without slipping, its angular speed about its axis is $\omega = v/r$. The kinetic energy at the bottom will be

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 + \frac{1}{2}mv^2$$
$$= \frac{1}{4}mv^2 + \frac{1}{2}mv^2 = \frac{3}{4}mv^2.$$

This should be equal to the loss of potential energy $mg\,\ell\sin\theta$. Thus,

$$\frac{3}{4}mv^2 = mg\,\ell\sin\theta$$

or
$$v = \sqrt{\frac{4}{3}g\ell\sin\theta}.$$

77. Figure 6.35 shows two cylinders of radii r_1 and r_2 having moments of inertia I_1 and I_2 about their respective axes. Initially, the cylinders rotate about their axes with angular speed ω_1 and ω_2 as shown in the Fig. 6.35. The cylinders are moved closer to touch each other keeping the axes parallel. The cylinders first slip over each other at the contact but the slipping finally ceases due to the friction between them. Find the angular speeds of the cylinders after the slipping ceases.

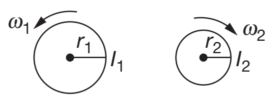


Fig. 6.35

Solution:

When slipping ceases, the linear speeds of the points of contact of the two cylinders will be equal. If ω_1' and ω_2' be the respective angular speeds, we have

$$\omega_1' r_1 \text{ and } \omega_2' r_2 \quad (1)$$

The change in the angular speed is brought about by the frictional force which acts as long as the slipping exists. If this force f acts for a time t, the torque on the first cylinder is fr_1 and that on the second is fr_2. Assuming $\omega_1 > \omega_2$, the corresponding angular impulses are $-fr_1 t$ and $fr_2 t$, we, therefore, have

$$-fr_1 t = I_1(\omega_1' - \omega_1)$$

and $$fr_2 t = I_2(\omega_2' - \omega_2)$$

or, $$-\frac{I_1}{r_1}(\omega_1' - \omega_1) = \frac{I_2}{r_2}(\omega_2' - \omega_2) \quad (2)$$

Solving (1) and (2)

$$\omega_1' = \frac{I_1\omega_1 r_2 + I_2\omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_2$$

and $$\omega_2' = \frac{I_1\omega_1 r_2 + I_2\omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_1.$$

78. A cylinder of mass m is suspended through two strings wrapped around it as shown in Fig. 6.36. Find (a) the tension T in the string and (b) the speed of the cylinder as it falls through a distance h.

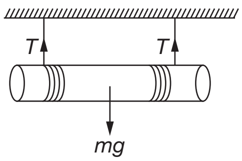


Fig. 6.36

Solution:

The portion of the strings between the ceiling and the cylinder is at rest. Hence the points of the cylinder where the strings leave it are at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of the cylinder falls with an acceleration a. The angular acceleration of the cylinder about its axis is $\alpha = a/R$, as the cylinder does not slip over the strings.

The equation of motion for the centre of mass of the cylinder is

$$mg - 2T = ma \quad (1)$$

and for the motion about the centre of mass, it is

$$2Tr = \left(\frac{1}{2}mr^2\alpha\right) = \frac{1}{2}mra \quad (2)$$

or $$2T = \frac{1}{2} = ma.$$

From (1) and (2),

$$a = \frac{2}{3}g \quad \text{and} \quad T = \frac{mg}{6}.$$

As the centre of the cylinder starts moving from rest, the velocity after it has fallen through a distance h is given by

$$v^2 = 2\left(\frac{2}{3}g\right)h \quad \text{or} \quad v = \sqrt{\frac{4gh}{3}}.$$

79. A sphere of mass M and radius r shown in Fig. 6.37 slips on a rough horizontal plane. At some instant it has translational velocity v_0 and rotational velocity about the centre $\frac{v_0}{2r}$. Find the translational velocity after the sphere starts pure rolling.

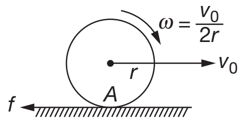


Fig. 6.37

Solution:

Velocity of the centre $= v_0$ and the angular velocity about the centre $= \frac{v_0}{2r}$. Thus $v_0 > \omega_0 r$. The sphere slips forward and thus the friction by the plane on the sphere will act backward. As the friction is kinetic, its value is $\mu N = \mu Mg$ and the sphere will be decelerated by $a_{cm} = f/M$. Hence,

$$v(t) = v_0 - \frac{f}{M}t. \quad (1)$$

This friction will also have a torque $\Gamma = fr$ about the centre. This torque is clockwise and in the direction of ω_0. Hence the angular acceleration about the centre will be

$$\alpha = f\frac{r}{(2/5)Mr^2} = \frac{5f}{2Mr}$$

and the clockwise angular velocity at time t will be

$$\omega(t) = \omega_0 + \frac{5f}{2Mr}t = \frac{v_0}{2r} + \frac{5f}{2Mr}t.$$

Pure rolling starts when $v(t) = r\omega(t)$

i.e., $$v(t) = \frac{v_0}{2} + \frac{5f}{2M}t. \quad (2)$$

Eliminating t from (1) and (2),

$$\frac{5}{2}v(t) + v(t) = \frac{5}{2}v_0 + \frac{v_0}{2}$$

or, $$v(t) = \frac{2}{7} \times 3v_0 = \frac{6}{7}v_0.$$

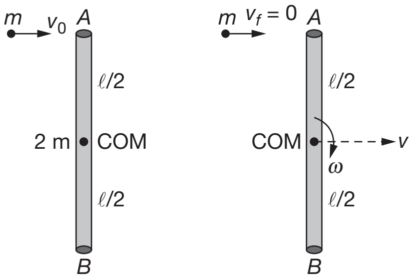
Thus, the sphere rolls with translational velocity $\frac{6v_0}{7}$ in the forward direction.

80. A rod AB of mass 2 m and length ℓ is lying on a horizontal frictionless surface. A particle of mass m traveling along the surface hits the end A of the rod with a velocity v_0 in a direction perpendicular to AB. The collision is elastic. After the collision, the particle comes to rest. Find out after collision:

(A) Velocity of centre of mass of rod
(B) Angular velocity

Solution:

(A) Let just after collision the sped of COM of rod be v and angular velocity about COM be ω.



Before collision After collision

External force on the system (rod + mass) in horizontal plane is zero.
Apply conservation of linear momentum in x direction

$$mv_0 = 2mv \qquad (1)$$

Net torque on the system about any point is zero
Apply conservation of angular momentum about COM of rod.

$$mv_0 \frac{\ell}{2} = I\omega$$

$$\Rightarrow \qquad mv_0 \frac{\ell}{2} = \frac{2m\ell^2}{12}\omega$$

$$mv_0 = m\omega\frac{\ell}{3} \qquad (2)$$

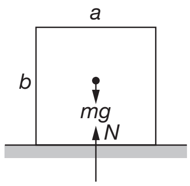
From Equation (1), velocity of centre of mass $v = \dfrac{v_0}{2}$

From Equation (2), angular velocity $\omega = \dfrac{3v_0}{\ell}$.

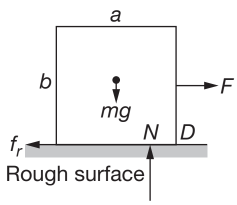
TOPPLING

In many situations, an external force is applied to a body to cause it to slide along a surface. In certain cases, the body may tip over before sliding ensues. This is known as topping.

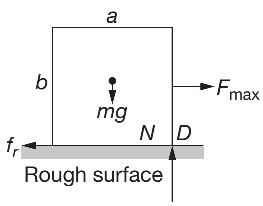
1. There is a no horizontal force so pressure at bottom is uniform and normal and is collinear with mg.



2. If a force is applied at COM, pressure is not uniform. Normal shifts right so that torque of N can counter balance torque of friction.



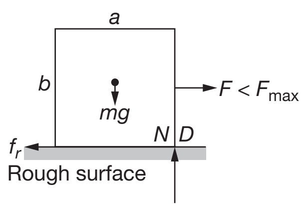
3. If F is continuously increased, N keeps shifting towards right until it reaches the right most point D.
Here we have assumed that the surface is sufficiently rough so that there is no sliding it. F is increased to F_{max}.
If force is increased any further, then torque of N can not counter balance torque of friction f_r and body will topple.
The value of force now is the max value for which toppling will not occur F_{max}.



$$F_{max} = f_r$$
$$N = mg$$
$$f_r \cdot b/2 = N \cdot a/2$$
$$\Rightarrow \qquad f_r = Na/b = mg\, a/b,\ F_{max} = mg\, a/b$$

4. If surface is not sufficiently rough and the body slides before F is increased to F_{max} = mg a/b, then body will slide before toppling. Once body starts sliding friction becomes constant and hence no topping. This is the case if

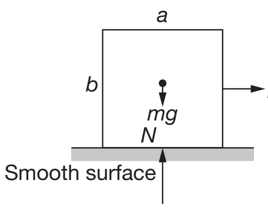
$$F_{max} > f_{limit} \quad \Rightarrow \quad mg\, a/b > \mu mg$$
$$\mu < a/b$$



Condition for toppling when $\mu \geq a/b$. In this case body will topple if $F > mg\, a/b$.
But if $\mu < a/b$, body will not topple if any value of F is applied to the COM.

SOLVED EXAMPLES

81. Find out minimum value of F for toppling.



Solution:

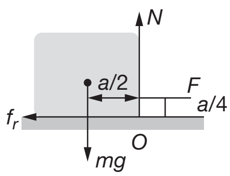
Never topples.

82. A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly below the centre of the face, at a height $\frac{a}{4}$ above the base.

(A) What is the minimum value of F for which the cube begins to tip about an edge?
(B) What is the minimum value of μ_s so that toppling occurs?
(C) If $\mu = \mu_{min}$, find minimum force for topping.
(D) Minimum μ_s so that F_{min} can cause toppling.

Solution:

(A) In the limiting case normal reaction will pass through O. The cube will tip about O if torque of Fabout O exceeds the torque of mg.



Hence, $F\left(\frac{a}{4}\right) > mg\left(\frac{a}{2}\right)$

or $F > 2\,mg$

therefore, minimum value of F is 2 mg.

(B) In this case since it is not acting at COM, toppling can occur even after body started sliding because of increasing the torque of F about COM. Hence $\mu_{min} = 0$,

(C) Now body is sliding before toppling, O is not IAR, torque equation can not be applied across it. It can now be applied about COM.

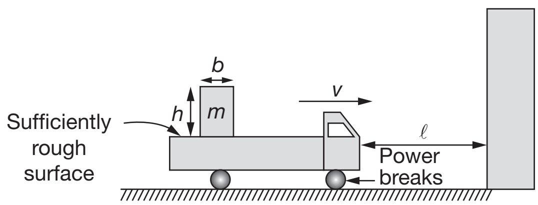
$$F \times \frac{a}{4} = N \times \frac{a}{2} \quad (1)$$

$$N = mg \quad (2)$$

From (1) and (2),
$F = 2$ mg

(D) $F > 2$ mg (1)
(from sol. (A))
$N =$ mg (2)
$F = \mu_s N = \mu_s mg$ (3)
From (1) and (2)
$\mu_s = 2$.

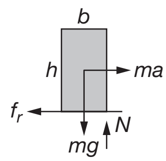
83. Find minimum value of ℓ so that truck can avoid the dead end, without toppling the block kept on it.



Solution:

$$ma\,\frac{h}{2} \leq mg\,\frac{b}{2}$$

$$a \leq \frac{b}{h}\,g$$



Final velocity of truck is zero. So that

$$0 = v^2 - 2\left(\frac{b}{h}g\right)\ell$$

$$\ell = \frac{h}{2b}\,\frac{v^2}{g}.$$

BRAIN MAP I

SYSTEM OF PARTICLES

1. LOCATION OF CM

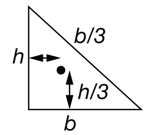
- For a system of particles

$$X_{CM} = \frac{\sum M_i X_i}{\sum M_i}, \quad Y_{CM} = \frac{\sum M_i Y_i}{\sum M_i}, \quad Z_{CM} = \frac{\sum M_i Z_i}{\sum M_i}$$

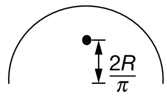
- For a continuous mass system

$$X_{CM} = \frac{\int x dm}{\int dm}, \quad Y_{CM} = \frac{\int y dm}{\int dm}, \quad Z_{CM} = \frac{\int z dm}{\int dm}$$

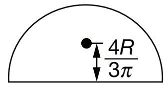
2. LOCATION OF CM FOR UNSYMMETRICAL BODIES



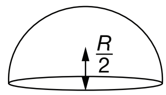
Right-angled triangular lamina



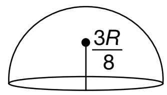
Semicircular ring



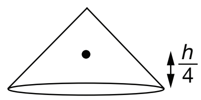
Semicircular disc



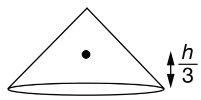
Hemispherical shell



Hemisphere



Right circular cone (solid)



Right circular cone (hollow)

3. MOTION OF CM

- **Velocity of CM,**

$$\vec{v}_{cm} = \frac{\sum M_i \vec{v}_i}{\sum M_i}$$

- **Acceleration of CM,**

$$\vec{a}_{cm} = \frac{\sum M_i \vec{a}_i}{\sum M_i}$$

4. NEWTON'S LAW

- Newton's second law of motion applicable for the system of particles:

$$\sum \vec{F}_{ext} = (\sum M_i)\vec{a}_{CM}$$

- If net external force on a system of body is zero and initially its center of mass were at rest then center of mass will remain at rest.

BRAIN MAP 2

MOTION OF RIGID BODIES

1. In case of translational motion of a rigid body, displacement, velocity and acceleration of each particle of rigid body will be same. For rigid body, we apply equation of motion as

$$\vec{F}_{\text{ext}} = M\vec{a}_{CM}$$

2. Kinematic relation for the rotational motion of rigid body

$$\omega = \frac{d\theta}{dt}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt} = \frac{\omega d\omega}{d\theta}$$

3. Equations for rigid body

$$\tau = I\alpha$$

$$\text{KE} = \frac{1}{2}I\omega^2$$

Angular momentum, $L = I\omega$

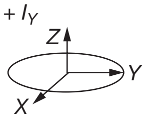
Power of torque $P = \tau\omega$

4. Moment of inertia

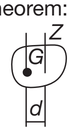
- For point mass system
  $I = \Sigma m_i r_i^2$
- For continuous mass system
  $I = \int dmr^2$

5. Theorems related to moment of inertia

- Perpendicular axes theorem:
  $I_Z = I_X + I_Y$



- Parallel axes theorem:
  $I_Z = I_G + Md^2$



6. Rolling motion

- Equation of motion for translation
  $\vec{F}_{\text{ext}} = M\vec{a}_{cm}$
  For rotation, $\tau_{cm} = I_{cm}\,\alpha$
- KE of rolling body =
  $\frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\,\omega^2$
- Angular momentum of rolling body = $Mvr_{cm} \pm I\,\omega_{cm}$

7. Important points of rolling motion

- In case of rolling with slipping frictional force may take any value between zero and $\mu_s N$.
- In case of rolling with slipping frictional force is $\mu_k N$.
- Relation between acceleration of center of mass and angular acceleration can be written in case of pure rolling motion.
- Pure rolling motion can be treated as rotation about point of contact also.

EXERCISES

More than One Option Correct Type

1. Let a_r and a_t represent radial and tangential acceleration. The motion of a particle may be circular if

(A) $a_r = a_t = 0$ (B) $a_r = 0$ and $a_t \neq 0$
(C) $a_r \neq 0$ and $a_t = 0$ (D) $a_r \neq 0$ and $a_t \neq 0$

2. A particle moves on a straight line with a uniform velocity. Its angular momentum

(A) is always zero.
(B) is zero about a point on the straight line.
(C) is not zero about a point away from the straight line.
(D) about any given point remains constant.

3. A horizontal disc rotates freely about a vertical axis through its centre. A ring, having the same mass and radius as the disc, is now gently placed on the disc. After some time, the two rotate with a common angular velocity

(A) some friction exists between the disc and the ring.
(B) the angular momentum of the disc plus ring is conserved.
(C) the final common angular velocity is $\frac{2}{3}$rd of the initial angular velocity of the disc.
(D) $\frac{2}{3}$ rd of the initial kinetic energy changes to heat.

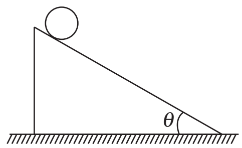
4. A mass m of radius r is rolling horizontally without any slip with a linear speed v. It then rolls up to a height given by $\frac{3}{4}\frac{v^2}{g}$

(A) the body is identified to be a disc or a solid cylinder.
(B) the body is a solid sphere.
(C) MOI of the body about instantaneous axis of rotation is $\frac{1}{2}mr^2$.
(D) MOI of the body about instantaneous axis of rotation is $\frac{2}{5}mr^2$.

5. A hollow cylinder, a spherical shell, a solid cylinder and a solid sphere are allowed to roll on an inclined rough surface of coefficient of friction μ and inclination θ. The correct statements are

(A) if cylindrical shell can roll on inclined plane, all other objects will also roll.
(B) if all the objects are rolling and have same mass, the KE of all the objects will be same at the bottom of inclined plane.
(C) work done by the frictional force will be zero, if objects are rolling.
(D) frictional force will be equal for all the objects, if mass is same.

6. A solid cylinder rolls without slipping down the rough inclined surface, as shown. Choose the correct alternative (s).



(A) If θ is decreased, the force of friction will decrease.
(B) Frictional force is dissipative.
(C) Frictional force is necessarily equal to $\mu mg\cos\theta$.
(D) Frictional force favours rotational motion but opposes translational motion.

7. Three marbles roll down on three different smooth tracks of same vertical height. Track (i) is inclined at 75° to the ground, track (ii) is inclined at 60° to the ground and track (iii) is inclined at 30° to ground. These marbles reach the ground with respective velocities u_1, u_2 and u_3

(A) u_1 is greatest of all (B) u_3 is least of all
(C) $u_1 = u_2$ (D) $u_2 = u_3$

8. A man of 50 kg is riding a bicycle. Man changes angular speed of wheel of radius 0.5 m by 4 rad/s in 1 s. (Assume pure rolling)

(A) The angular acceleration of wheel is 4 rad/s^2
(B) Net horizontal force on the man is 100 N
(C) Net force on the man is 100 N
(D) Net force on the man is $\sqrt{26}\times 100$ N

9. A ring rolls without slipping on the ground. Its centre C moves with a constant speed u. P is any point on the ring. The speed of P with respect to the ground is v.

(A) $0 \leq v \leq 2u$
(B) $v = u$, if CP is horizontal
(C) $v = u$, if CP makes an angle of 30° with the horizontal and P is below the horizontal level of C
(D) $v = \sqrt{2}u$, if CP is horizontal

10. A particle moves on a straight line with a uniform velocity. Its angular momentum

(A) is always zero.
(B) is zero about a point on the straight line.
(C) is not zero about a point away from the straight line.
(D) about any given point remains constant.

Passage Based Questions

Passage 1

A thin uniform rod of mass m and length ℓ rotates with constant angular velocity ω about the vertical axis passing through the rod's suspension point O. In doing so, the rod describes a conical surface with a half aperture angle Q (see Fig. 6.38).

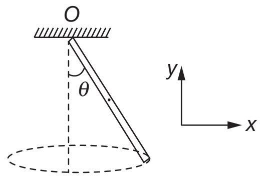


Fig. 6.38

11. Torque about point O by centrifugal force is

(A) $\frac{mgl}{2}\sin\theta$ (B) $\frac{m\omega^2 l}{2}\cos\theta$

(C) $mgl\sin\theta$ (D) $\frac{m\omega^2 l}{2}\sin\theta$

12. Component of reaction force at O in y-direction is

(A) mg (B) $\frac{mg}{2}\sin\theta$

(C) $mg\cos\theta$ (D) $\frac{mg}{\cos\theta}$

13. Component of reaction force at O in –ve x-direction is

(A) $m\omega^2 l$ (B) $\frac{m\omega^2 l}{2}$

(C) $\frac{m\omega^2 l}{2}\sin\theta$ (D) $\frac{m\omega^2 l}{2}\cos\theta$

14. Magnitude of angular momentum of rod about axis of rotation is

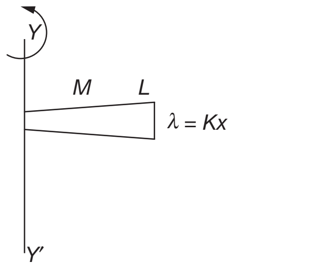
(A) $\frac{ml^2\omega}{12}\sin^2\theta$ (B) $\frac{ml^2\omega}{12}$

(C) $\frac{ml^2\omega}{3}\sin^2\theta$ (D) $\frac{ml^2\omega}{3}$

Passage 2

Moment of inertia is a physical term which opposes the change in rotational motion. Moment of inertia depends on distribution of mass, shape of the body as well as distance from the rotational axis. Moment of linear momentum is called angular momentum. If no external torque acts on the system, then angular momentum of the system remains conserved. Geometrical meaning of angular momentum relates to the real velocity.

15. Mass m is distributed over the rod of length ℓ. If linear mass density (λ) linearly increases with length as $\lambda = Kx$. The MI of the rod about one end perpendicular to rod i.e., (YY')



(A) $\frac{m\ell^2}{3}$ (B) $\frac{m\ell^2}{12}$

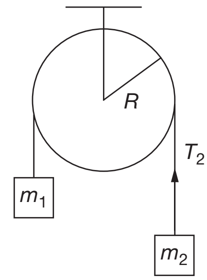
(C) $\frac{2}{3}m\ell^2$ (D) $\frac{KL^4}{4}$

16. A particle of mass m is moving along the line $y = 3x + 5$ with speed v. The magnitude of angular momentum about origin is

(A) $\sqrt{\frac{5}{2}}mv$ (B) $\frac{5}{2}mv$

(C) $\frac{1}{2}mv$ (D) $\frac{1}{\sqrt{3}}mv$

17. Acceleration of block of mass m_1 is (given moment of inertia of pulley is I and string does not slip over the pulley).

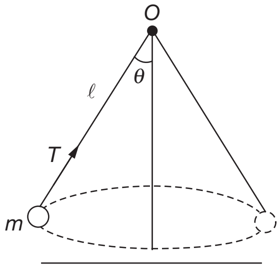


(A) $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$ (B) $\frac{(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}$

(C) $\frac{\left(m_1 - m_2 + \frac{I}{R^2}\right)g}{m_1 + m_2}$ (D) $\frac{\left(m_1 - m_2 + \frac{I}{R^2}\right)g}{\left(m_1 + m_2 - \frac{I}{R^2}\right)}$

Passage 3

A particle of mass m tied to a string of length ℓ is rotated with speed v in horizontal plane in the form of a conical pendulum. Where the string makes angle θ with vertical. The angular momentum of the mass about point of suspension has constant magnitude mvl. But the direction of angular momentum about point O changes as torque about the point O due to its weight (mg) acts on the pendulum.



18. The direction of infinitesimal change in angular momentum of the pendulum at any instant about point O is
(A) parallel to instantaneous velocity.
(B) perpendicular to instantaneous velocity.
(C) perpendicular to length of the string.
(D) None of these as angular momentum is constant.

19. The angle between the net torque and angular momentum about O is
(A) 0° (B) θ
(C) 90° (D) 180°

20. Rate of change of angular momentum about point O is
(A) mvl (B) zero
(C) $mgl \cos \theta$ (D) $mgl \sin \theta$

21. The magnitude of force ($\vec{F}$) acting on particle at $t = 1$ s is
(A) $8\sqrt{10}$ N (B) $16\sqrt{2}$ N
(C) 16 N (D) 32 N

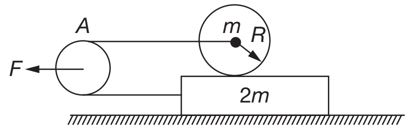
22. The magnitude of torque with respect to origin, acting on the particle at $t = 1$ s will be
(A) Zero (B) 48 N/m
(C) 64 N/m (D) 80 N/m

23. The momentum of particle at an instant when its y co-ordinate is double of the x co-ordinate is
(A) $-16\hat{i}$ (B) $-48\hat{j}$
(C) $-8\hat{i} - 12\hat{j}$ (D) $-16\hat{i} - 48\hat{j}$

24. Work done by force acting on it in first two seconds will be
(A) 32 J (B) 256 J
(C) 320 J (D) 288 J

Passage 4

A solid cylinder of mass m and radius R is kept at rest on a plank of mass 2 m lying on a smooth horizontal surface. Massless and inextensible string connecting cylinder to the plank is passing over a massless pulley. The friction between the cylinder and the plank is sufficient to prevent slipping. Pulley A is pulled with a constant horizontal force F.



25. Acceleration of cylinder with respect to earth is
(A) $\frac{5F}{21\text{ m}}$ (B) $\frac{F}{7\text{ m}}$ (C) $\frac{3F}{7\text{ m}}$ (D) $\frac{2F}{7\text{ m}}$

26. Acceleration of plank with respect to earth is
(A) $\frac{5F}{21\text{ m}}$ (B) $\frac{F}{7\text{ m}}$ (C) $\frac{3F}{7\text{ m}}$ (D) $\frac{2F}{7\text{ m}}$

27. Magnitude of friction force acting on the plank is
(A) $\frac{F}{7}$ (B) $\frac{F}{14}$ (C) $\frac{F}{21}$ (D) $\frac{2F}{7}$

Passage 5

A massive disc of radius R is mounted on a light axle with the axle held horizontal, the disc is made to spin with angular speed ω_0. It is then lowered gently on to a level table and released as soon as its rim makes contact with the table top. The disc begins to move along the table, skidding and picking up speed as it goes.

28. While the disc skids, its translational and angular acceleration are related as
(A) $a = R\,\alpha$ (B) $a = \frac{1}{4} R\,\alpha$
(C) $a = \frac{1}{2} R\,\alpha$ (D) $a = 2R\,\alpha$

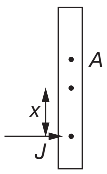
29. If μ_k is the coefficient of kinetic friction between the rim of the disc and table top, disc will continue to skid for a time
(A) $\frac{R\omega_0}{3\mu_k g}$ (B) $\frac{R\omega_0}{2\mu_k g}$ (C) $\frac{R\omega_0}{\mu_k g}$ (D) $\frac{3R\omega_0}{\mu_k g}$

30. When the disc stops skidding and begins to roll without slipping, its speed will be
(A) $R\omega_0$ (B) $\frac{1}{3}R\omega_0$ (C) $\frac{1}{2}R\omega_0$ (D) $\frac{1}{4}R\omega_0$

31. If initial kinetic energy of the disc is K, then the kinetic energy of the disc after it starts pure rolling will be
(A) $\frac{2}{3}K$ (B) $\frac{1}{2}K$ (C) $\frac{1}{3}K$ (D) $\frac{3}{4}K$

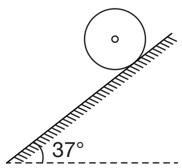
Match the Column Type

32. A uniform rod of mass m and length ℓ is placed in gravity-free space and linear impulse J is given to rod at a distance $x = l/4$ from centre and perpendicular to rod. Point A is at a distance $l/3$ from centre.



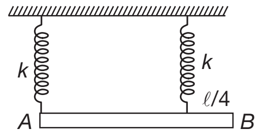
Column-I	Column-I
(A) Speed of centre of rod is	(1) Zero
(B) Speed of point A is	(2) $\frac{5}{2}\frac{J}{m}$
(C) Speed of upper end of rod is	(3) $\frac{J}{m}$
(D) Speed of lower end of rod is	(4) $\frac{J}{2m}$
	(5) $\frac{J}{5m}$

33. A hollow sphere, ring, disc and solid sphere each of mass 1 kg and radius 1 m is released from rest on an identical inclined plane of inclination 37°. (tan 37° = 3/4 and $g = 10$ m/s^2). The co-efficient of friction between body and surface is μ. Then match the column.



Column-I	Column-II
(A) If $\mu = 1/5$, then friction force on the body is 1.6 N for	(1) Ring
(B) If $\mu = 0.3$, then there is no slipping for	(2) Disc
(C) If $\mu = 0.2$, then pure rolling does not take place for	(3) Hollow sphere
(D) If $\mu = 0.5$, then work done by friction force zero on the	(4) Solid sphere

34. A uniform rod of mass m and length l is suspended horizontally in equilibrium by two springs of spring constant k each. First spring is connected at the end A and the second spring is connected l/4 away from the other end B. The second spring is now cut. Just after that the acceleration of centre of mass of the rod is a_{cm}, that of end A is a_A and angular acceleration α.



Column-I	Column-II
(A) $12\,\lvert a_{cm}\rvert$	(1) $9g$
(B) $\lvert a_A\rvert$	(2) g
(C) $\left\lvert\frac{l\alpha}{2}\right\rvert$	(3) $\frac{g}{3}$
	(4) $8g$

35.

Column-I	Column-II
(A) Particle moving with speed v_0 strikes to a rod placed on a smooth table and sticks to it.	(1) Kinetic energy will change.
(B) A thin rod of mass m and length ℓ inclined at an angle θ with horizontal is dropped on a smooth horizontal plane without any angular velocity. Its tip does not rebound after impact.	(2) Momentum is conserved.
(C) A solid sphere of mass m and radius R is rolling with velocity v_0 along a horizontal plane. It suddenly encounters an obstacle.	(3) Angular momentum is conserved about any point.
(D) Two cylinders of radii r_1 and r_2 rotating about their axis with angular speed ω_1 and ω_2 moved closer to touch each other keeping their axis parallel. Cylinders first slip over each other at the contact point but slipping ceases after some time due to friction.	(4) Angular momentum is conserved just before and after impact only about point of impact.

36. A uniform disc of mass 10 kg, radius 1 m is placed on a rough horizontal surface. The co-efficient of friction between the disc and the surface is 0.2. A horizontal time varying force is applied at the centre of the disc at $t = 0$ whose variation with time is shown in Fig. 6.39.

After 8 s, the force falls down to zero instantly.

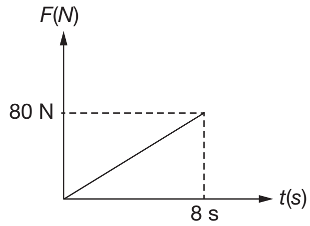


Fig. 6.39

Column-I	**Column-II**
(A) Disc rolls without slipping at	(1) $t = 5$ s
(B) Disc rolls with slipping at	(2) $t = 7$ s
(C) Friction force is less than applied force at	(3) $t = (7)5$ s
(D) Friction force is zero at	(4) $t = 9$ s

Assertion-Reason Type

37. **Assertion:** A projectile gets exploded at its highest point. The centre of mass will fall at a point which is farther than the point where the projectile would have fallen in unexploded condition.

Reason: The weight of the projectile is the external force.

(A) A (B) B (C) C (D) D

38. **Assertion:** The velocity of centre of mass of body at the bottom of an inclined plane of given height, is more when it slides down the plane, compared to, when it rolling down the same plane.

Reason: In rolling down, a body acquires both kinetic energy of translation and rotation.

(A) A (B) B (C) C (D) D

39. **Assertion:** In case of pure rolling of a disc on a plank, work done by friction force on the disc must be zero.

Reason: In case of pure rolling, point of contact is relatively at rest with respect to the surface.

(A) A (B) B (C) C (D) D

40. **Assertion:** Two balls are thrown simultaneously in air. The acceleration of centre of mass of the two balls while in air depends on the masses of the two balls.

Reason: The acceleration of centre of mass is given by $\vec{a} = \dfrac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2}$

(A) A (B) B (C) C (D) D

41. **Assertion:** A uniform sphere is placed on a smooth horizontal surface and a horizontal force F is applied on it at a distance of h from centre, then acceleration of centre of mass is independent of h.

Reason: Acceleration of centre of mass depends on force and mass.

(A) A (B) B (C) C (D) D

42. **Assertion:** A cyclist cannot take circular turn on smooth horizontal surface.

Reason: Component of reaction force can provide centripetal force when cyclist takes circular turn.

(A) A (B) B (C) C (D) D

43. **Assertion:** Net external torque (τ_{ext}) on a system of particles is equal to rate of change of angular momentum $\dfrac{d\vec{L}}{dt}$ if τ_{ext} and $\vec{L}$ are measured with respect to any fixed point in an inertial frame.

Reason: If a body is in rotation equilibrium, then net torque on a body about any fixed point is zero.

(A) A (B) B (C) C (D) D

44. **Assertion:** Two identical spheres A and B are free to move and rotate about their centres. They are given the same impulse. The line of action of impulses passes through their centres. Spheres A and B have same kinetic energy.

Reason: In above assertion, their linear momentum and mass are equal.

(A) A (B) B (C) C (D) D

45. **Assertion:** In case of pure rolling of a disc on a plank, work done by friction force on the disc must be zero.

Reason: In case of pure rolling, point of contact is relatively at rest with respect to the surface.

(A) A (B) B (C) C (D) D

46. **Assertion:** The work done by friction force on a rigid body which is in pure rolling without slipping must be zero.

Reason: Instantaneous velocity of point of contact with respect to the surface is zero in pure rolling.

(A) A (B) B (C) C (D) D

47. **Assertion:** Speed of any point on rigid body executing rolling motion can be calculated by expression $v = r\omega$, where r is distance of point from instantaneous centre of rotation.

Reason: Rolling motion of rigid body can be considered as a rotational motion about instantaneous centre of rotation.

(A) A (B) B (C) C (D) D

Integer Type

48. A cylinder of mass m has a length ℓ that is $\sqrt{3}$ times its radius R. What is the ratio of its moment of inertia about its own axis and that of an axis passing through its centre and perpendicular to its axis?

49. A uniform rod of length 1 m and mass 2 kg is suspended on two vertical inextensible strings as shown in Fig. 6.40. Calculate tension T (in newton) in the string at the instant, when right string snaps ($g = 10$ m/s^2).

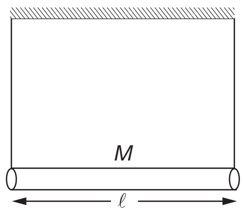


Fig. 6.40

50. A disc is rotating with an angular velocity ω_0. A constant retarding torque is applied on it to stop the disc. The angular velocity becomes $\omega_0/2$ after n rotations. If the disc will rotate $\frac{n}{x}$ rotation, more before coming to rest then the value of x is?

51. A wheel is rotating at 900 rpm about its axis. When power is cut off it comes to rest in 1 minute. The angular retardation is $\frac{\pi}{n}$ rad/s^2, then the value of n is.

52. The centre of a wheel rolling on a plane surface moves with a speed v_0. A particle on the rim of the wheel at the same level as that centre will be moving at speed $\sqrt{n}\, v_0$ then the value of n is.

53. An equilateral triangle ABC has its centre at O as shown in Fig. 6.41. Three forces 10 N, 5 N and F are acting along the sides AB, BC and AC. Magnitude of F so that the net torque about O is zero is.

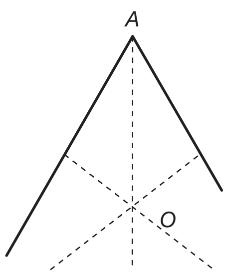
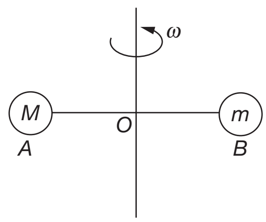


Fig. 6.41

54. A thick-walled hollow sphere has outer radius R. It rolls down on an inclined plane without slipping and its speed at bottom is v_0. Now the incline is waxed so that the friction becomes zero. The sphere is observed to slide down without rolling and the speed now is $(5\, v_0/4)$. The radius of gyration of the hollow sphere about the axis through its centre is $\frac{nR}{4}$. Then the value of n is.

55. A string is wrapped several times round a solid cylinder of mass m and then the end of the string is held stationary while the cylinder is released from rest with no initial motion. The acceleration of the cylinder will be $\frac{ng}{3}$, then the value of n is.

56. Two balls of mass $M = 9$ g and $m = 3$ g are attached by massless threads AO and OB. The length AB is 1 m. They are set in rotational motion in a horizontal plane about a vertical axis at O with constant angular velocity ω. The ratio of length AO and OB $\left(\frac{OB}{AO}\right)$ for which the tension in threads are same will be.



57. A uniform solid disc of mass 1 kg and radius 1 m is kept on a rough horizontal surface. Two forces of magnitude 2 N and 4 N have been applied on the disc as shown in Fig. 6.42. Linear acceleration of the centre of mass of the disc is.

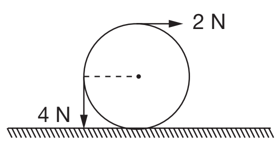


Fig. 6.42

Previous Years' Questions

58. Two identical particles move towards each other with velocity $2v$ and v, respectively. The velocity of the centre of mass is **[2002]**

(A) v (B) $\frac{v}{3}$ (C) $\frac{v}{2}$ (D) Zero

59. The initial angular velocity of a circular disc of mass M is. Then two small spheres of mass m are attached gently to diametrically opposite points on the edge of the disc. What is the final angular velocity of the disc? **[2002]**

(A) $\left(\frac{M+m}{M}\right)\omega_1$ (B) $\left(\frac{M+m}{m}\right)\omega_1$

(C) $\left(\frac{M}{M+4m}\right)\omega_1$ (D) $\left(\frac{M}{M+2m}\right)\omega_1$

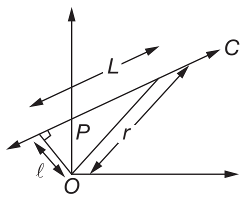
60. A solid sphere, a hollow sphere and a ring are released from top of an inclined plane (frictionless) so that they slide down the plane. Then maximum acceleration down the plane is for (no rolling) **[2002]**

(A) Solid sphere (B) Hollow sphere
(C) Ring (D) All same

61. The moment of inertia of a circular wire of mass m and radius R about its diameter is **[2002]**

(A) $mR^2/2$ (B) mR^2
(C) $2mR^2$ (D) $mR^2/4$

62. A particle of mass m moves along line PC with velocity v as shown. What is the angular momentum of the particle about P? **[2002]**



(A) mvL (B) $mv\ell$ (C) mvr (D) Zero

63. Let $\vec{F}$ be the force acting on a particle having position vector $\vec{r}$, and $\vec{\tau}$ be the torque of this force about the origin. Then **[2003]**

(A) $\vec{r}\cdot\vec{T}=0$ and $\vec{F}\cdot\vec{\tau}\neq 0$
(B) $\vec{r}\cdot\vec{T}\neq 0$ and $\vec{F}\cdot\vec{\tau}=0$
(C) $\vec{r}\cdot\vec{F}\neq 0$ and $\vec{F}\cdot\vec{\tau}\neq 0$
(D) $\vec{r}\cdot\vec{\tau}=0$ $\vec{F}\cdot\vec{\tau}=0$

64. A circular disc X of radius R is made from an iron plate of thickness t, and another disc Y of radius $4R$ is made from an iron plate of thickness $t/4$. Then the relation between the moment of inertia I_X and I_Y is **[2003]**

(A) $I_Y = 32I_X$ (B) $I_Y = 16I_X$
(C) $I_Y = I_X$ (D) $I_Y = 64I_X$

65. A particle performing uniform circular motion has angular frequency is doubled and its kinetic energy halved, then the new angular momentum is **[2003]**

(A) $\frac{L}{4}$ (B) $2L$ (C) $4L$ (D) $\frac{L}{2}$

66. A uniform chain of length 2 m is kept on a table such that a length of 60 cm hangs freely from the edge of the table. The total mass of the chain is 4 kg. What is the work done in pulling the entire chain on the table? Take $g = 10\text{ m/s}^2$ **[2004]**

(A) 12 J (B) 3.6 J (C) 7.2 J (D) 1200 J

67. A solid sphere is rotating in free space. If the radius of the sphere is increased keeping mass same which one of the following will not be affected? **[2004]**

(A) Angular velocity
(B) Angular momentum
(C) Moment of inertia
(D) Rotational kinetic energy

68. One solid sphere A and another hollow sphere B are of same mass and outer radii. Their moment of inertia about their diameters are respectively I_A and I_B, such that d_A and d_B are the densities **[2004]**

(A) $I_A < I_B$ (B) $I_A > I_B$

(C) $I_A = I_B$ (D) $\frac{I_A}{I_B} = \frac{d_A}{d_B}$

69. Which of the following statement is false for a particle moving in a circle with a constant angular speed? **[2004]**

(A) The velocity vector is tangent to the circle.
(B) The acceleration vector is tangent to the circle.
(C) The acceleration vector points towards the centre of the circle.
(D) The velocity and acceleration vectors are perpendicular to each other.

70. A body A of mass M while falling vertically downwards under gravity breaks into two parts … a body B of mass $M/3$ and a body C of mass $2M/3$. The centre of mass of bodies B and C taken together shifts compared to that of body A towards **[2005]**

(A) body B.
(B) body C.
(C) does not shift.
(D) depends on height of breaking.

71. A T shaped object with dimensions shown in Fig. 6.43, is lying on a smooth floor. A force $\vec{F}$ is applied at the point P parallel to AB, such that the object has only the translational motion without rotation. Find the location of P with respect to C. **[2005]**

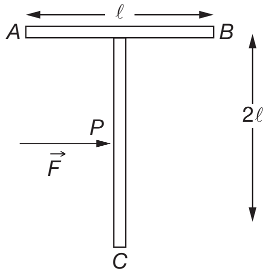


Fig. 6.43

(A) ℓ (B) $\frac{4}{3}\ell$ (C) $\frac{3}{2}\ell$ (D) $\frac{2}{3}\ell$

72. An annular ring with inner and outer radii R_1 and R_2 is rolling without slipping with a uniform angular speed. The ratio of the forces experienced by two particles situated on the inner and outer parts of the ring is **[2005]**

(A) $\frac{R_1}{R_2}$ (B) 1 (C) $\left(\frac{R_1}{R_2}\right)^2$ (D) $\frac{R_2}{R_1}$

[**Note:** The particles should be of same mass]

73. The moment of inertia of a uniform semi-circular disc of mass m and radius R about a line perpendicular to the plane of the disc through the centre is **[2005]**

(A) $\frac{1}{2}mR^2$ (B) mR^2 (C) $\frac{2}{5}mR^2$ (D) $\frac{1}{4}mR^2$

74. Consider a two particle system with particles having masses m_1 and m_2. If the first particle is pushed towards the centre of mass through a distance d, by what distance should the second particle be moved, so as to keep the centre of mass at the same position? **[2006]**

(A) $\frac{m_1}{m_2}d$ (B) d

(C) $\frac{m_2}{m_1}d$ (D) $\frac{m_1}{m_1+m_2}d$

75. A thin circular ring of mass m and radius R is rotating about its axis with a constant angular velocity ω. Two objects each of mass M are attached gently to the opposite ends of a diameter of the ring. The ring now rotates with an angular velocity of **[2006]**

(A) $\frac{\omega m}{m+M}$ (B) $\frac{\omega m}{m+2M}$

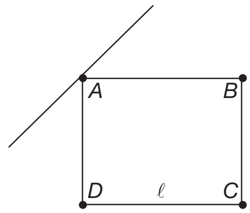
(C) $\frac{\omega(m+2M)}{m}$ (D) $\frac{\omega(m-2M)}{m+2M}$

76. A force of $-F\hat{k}$ acts on O, the origin of the coordinates system. The torque about the point $(1,-1)$ is **[2006]**

(A) $F(\hat{i}+\hat{j})$ (B) $-F(\hat{i}-\hat{j})$

(C) $F(\hat{i}-\hat{j})$ (D) $-F(\hat{i}+\hat{j})$

77. Four point masses, each of value m, are placed at the corners of a square ABCD of side ℓ. The moment of inertia of this system about an axis passing through A and parallel to BD is **[2006]**



(A) $3m\ell^2$ (B) $m\ell^2$

(C) $2m\ell^2$ (D) $\sqrt{3}m\ell^2$

78. A circular disc of radius R is removed from a bigger circular disc of radius $2R$ such that the circumferences of the discs coincide. The centre of mass of the new disc is a/R from the centre of the bigger disc. The value of a is **[2007]**

(A) $\frac{1}{2}$ (B) $\frac{1}{6}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$

79. A round uniform body of radius R, mass m and moment of inertia I rolls down (without slipping) an inclined plane making an angle with the horizontal. Then its acceleration is **[2007]**

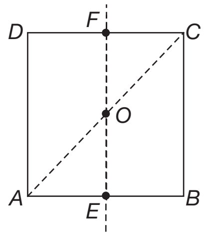
(A) $\frac{g\sin\theta}{1+\frac{mR^2}{I}}$ (B) $\frac{g\sin\theta}{1-\frac{I}{mR^2}}$

(C) $\frac{g\sin\theta}{1-\frac{mR^2}{I}}$ (D) $\frac{g\sin\theta}{1+\frac{I}{mR}}$

80. The angular momentum of a particle rotating with a central force is constant due to **[2007]**

(A) constant linear momentum.
(B) zero torque.
(C) constant torque.
(D) constant force.

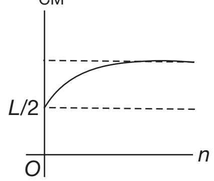
81. For the given uniform square lamina $ABCD$, whose centre is O, **[2007]**

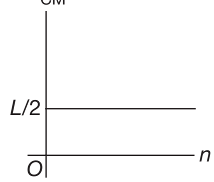


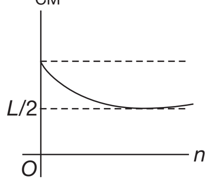
(A) $I_{AD} = 3I_{EF}$ (B) $I_{AC} = I_{EF}$

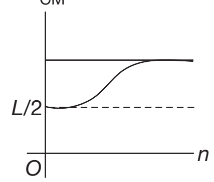
(C) $I_{AC} = \sqrt{2}I_{EF}$ (D) $\sqrt{2}I_{AC} = I_{EF}$

82. A thin rod of length L is lying along the x-axis with its ends at $x = 0$ and $x = L$. Its linear density (mass/length) varies with x as $k\left(\frac{x}{L}\right)^n$, where n can be zero or any positive number. If the position x_{CM} of the centre of mass of the rod is plotted against n, which of the following graphs best approximates the dependence of x_{CM} on n? **[2008]**

(A)


(B)


(C)


(D)


83. Consider a uniform square plate of side a and mass m. The moment of inertia of this plate about an axis perpendicular to its plane and passing through one of its corners is **[2008]**

(A) $\frac{5}{6}ma^2$ (B) $\frac{1}{12}ma^2$

(C) $\frac{7}{12}ma^2$ (D) $\frac{2}{3}ma^2$

84. A thin uniform rod of length ℓ and mass m is swinging freely about a horizontal axis passing through its end. Its maximum angular speed is ω. Its centre of mass rises to a maximum height of **[2009]**

(A) $\frac{1}{3}\frac{\ell^2\omega^2}{g}$ (B) $\frac{1}{6}\frac{\ell\omega}{g}$

(C) $\frac{1}{2}\frac{\ell^2\omega^2}{g}$ (D) $\frac{1}{6}\frac{\ell^2\omega^2}{g}$

85. A point P moves in counter-clockwise direction on a circular path as shown in Fig. 6.44. The movement of P is such that it sweeps out a length $s = t^3 + 5$, where s is in metres and t is in seconds. The radius of the path is 20 m. The acceleration of P when $t = 2$ is nearly **[2010]**

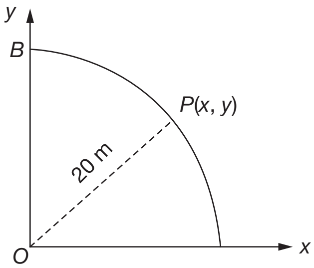
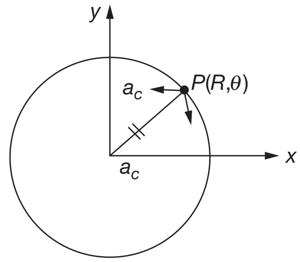


Fig. 6.44

(A) 13 m/s^2 (B) 12 m/s^2

(C) 7.2 m/s^2 (D) 14 m/s^2

86. For a particle in uniform circular motion, the acceleration $\vec{a}$ at a point $P(R,\theta)$ on the circle of radius R is (here θ is measured from the x-axis) **[2010]**



(A) $-\frac{v^2}{R}\cos\theta\hat{i} + \frac{v^2}{R}\sin\theta\hat{j}$

(B) $-\frac{v^2}{R}\sin\theta\hat{i} + \frac{v^2}{R}\cos\theta\hat{j}$

(C) $-\frac{v^2}{R}\cos\theta\hat{i} - \frac{v^2}{R}\sin\theta\hat{j}$

(D) $\frac{v^2}{R}\hat{i} + \frac{v^2}{R}\hat{j}$

87. A small particle of mass m is projected at an angle θ with the x-axis with an initial velocity v_0 in the x-y plane as shown in Fig. 6.45. At a time $t < (v_0 \sin\theta/g)$, the angular momentum of the particle is **[2010]**

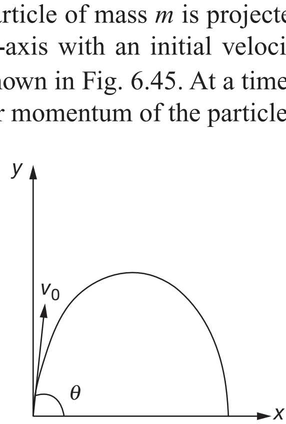


Fig. 6.45

(A) $-mgv_0 t^2 \cos\theta \hat{j}$ (B) $mgv_0 t \cos\theta \hat{k}$

(C) $-\frac{1}{2} mgv_0 t^2 \cos\theta \hat{k}$ (D) $\frac{1}{2} mgv_0 t^2 \cos\theta \hat{i}$

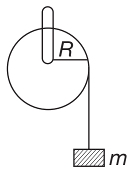
88. A hoop of radius R and mass m rotating with an angular velocity ω_0 is placed on a rough horizontal surface. The initial velocity of the centre of the hoop is zero. What will be the velocity if the centre of the loop ceases to slip? **[2013]**

(A) $\frac{r\omega_0}{3}$ (B) $\frac{r\omega_0}{2}$ (C) $r\omega_0$ (D) $\frac{r\omega_0}{4}$

89. A bob of mass m attached to an inextensible string of length ℓ is suspended from a vertical support. The bob rotates in a horizontal circle with an angular speed ω rad/s about the vertical. About the point of suspension **[2014]**

(A) Angular momentum is conserved.
(B) Angular momentum changes in magnitude but not in direction.
(C) Angular momentum changes in direction but not in magnitude.
(D) Angular momentum changes both in direction and magnitude.

90. A mass m is supported by a massless string wound around a uniform hollow cylinder of mass m and radius R. If the string does not slip on the cylinder, with what acceleration will the mass fall on release? **[2014]**



(A) $\frac{2g}{3}$ (B) $\frac{g}{2}$

(C) $\frac{5g}{6}$ (D) g

91. From a solid sphere of mass M and radius R a cube of maximum possible volume is cut. Moment of inertia of cube about an axis passing through its center and perpendicular to one of its face is: **[2015]**

(A) $\frac{MR^2}{16\sqrt{2}\pi}$ (B) $\frac{4MR^2}{9\sqrt{3}\pi}$

(C) $\frac{4MR^2}{3\sqrt{3}\pi}$ (D) $\frac{MR^2}{32\sqrt{2}\pi}$

92. A particle of mass m is moving along the side of a square of side a, with a uniform speed v in the x–y plane as shown in Fig. 6.46.

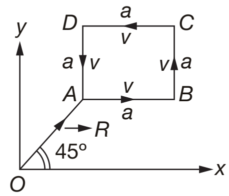


Fig. 6.46

Which of the following statements is **false** for the angular momentum $\vec{L}$ about the origin? **[2016]**

(A) $\vec{L} = mv\left[\frac{R}{\sqrt{2}} - a\right]\hat{k}$ when the particle is moving from C to D.

(B) $\vec{L} = mv\left[\frac{R}{\sqrt{2}} + a\right]\hat{k}$ when the particle is moving from B to C.

(C) $\vec{L} = \frac{mv}{\sqrt{2}} R\,\hat{k}$ when the particle is moving from D to A.

(D) $\vec{L} = -\frac{mv}{\sqrt{2}} R\,\hat{k}$ when the particle is moving from A to B.

93. A roller is made by joining two cones at their vertices O. It is kept on two rails AB and CD which are placed asymmetrically (see Fig. 6.47), with its axis perpendicular to CD and its centre O at the centre of line joining AB and CD (see Fig. 6.47). It is given a light push so that it starts rolling with its centre O moving parallel to CD in the direction shown. As it moves, the roller will tend to: **[2016]**

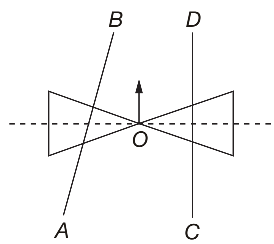


Fig. 6.47

(A) Turn right
(B) Go straight
(C) Turn left and right alternately
(D) Turn left

ANSWER KEYS

More than One Option Correct Type

1. (C) and (D) **2.** (B), (C) and (D) **3.** (A), (B) and (D) **4.** (A) and (C) **5.** (A), (B) and (C)
6. (A) and (D) **7.** (C) and (D) **8.** (A) and (B) **9.** (A), (C) and (D) **10.** (B), (C) and (D)

Passage Based Questions

Passage 1

11. (A) **12.** (A) **13.** (B) **14.** (B)

Passage 2

15. (D) **16.** (A) **17.** (B)

Passage 3

18. (A) **19.** (C) **20.** (D) **21.** (A) **22.** (D) **23.** (B) **24.** (C)

Passage 4

25. (C) **26.** (D) **27.** (B)

Passage 5

28. (C) **29.** (A) **30.** (B) **31.** (C)

Match the Column Type

32. (A) → 3; (B) → 1; (C) → 4; (D) → 2
33. (A) → 1, 2, 3, 4; (B) → 2, 3, 4; (C) → 1, 2, 3, 4; (D) → 1, 2, 3, 4
34. (A) → 4; (B) → 3; (C) → 2
35. (A) → 1, 2, 3; (B) → 1, 4; (C) → 1, 4; (D) → 1
36. (A) → 1, 4; (B) → 2, 3; (C) → 1, 2, 3; (D) → 4

Assertion-Reason Type

37. (D) **38.** (B) **39.** (D) **40.** (D) **41.** (A) **42.** (B) **43.** (B) **44.** (A) **45.** (D) **46.** (D)
47. (A)

Integer Type

48. 1 **49.** 5 N **50.** 3 **51.** 2 **52.** 2 **53.** 8 N **54.** 3 **55.** 2 **56.** 3 **57.** 0

Previous Years' Questions

58. (C) **59.** (C) **60.** (D) **61.** (A) **62.** (D) **63.** (D) **64.** (D) **65.** (A) **66.** (B) **67.** (B)
68. (A) **69.** (B) **70.** (C) **71.** (B) **72.** (A) **73.** (A) **74.** (A) **75.** (B) **76.** (A) **77.** (A)
78. (D) **79.** (D) **80.** (B) **81.** (B) **82.** (A) **83.** (D) **84.** (D) **85.** (D) **86.** (C) **87.** (C)
88. (B) **89.** (C) **90.** (B) **91.** (B) **92.** (A) and (C) **93.** (D)

HINTS AND SOLUTIONS

More than One Option Correct Type

1. The correct option is (C) and (D)
2. The correct option is (B), (C) and (D)
3. By conservation of angular momentum,

$$I_1\omega_i = (I_1 + I_2)\omega_f$$

$$\omega_f = \omega_1/3$$

$$E_i = \frac{1}{2}I_1\omega_i^2$$

$$E_f = \frac{1}{2}(I_1 + I_2)\omega_f^2$$

Ratio of the heat produced to initial kinetic energy

$= \frac{E_i - E_f}{E_i} = \frac{2}{3}$

The correct option is (A), (B) and (D)

4. KE on horizontal level $= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\frac{Iv^2}{r^2}$

PE $mgh = mg \times \frac{3v^2}{4g}$

Loss of KE = gain of PE

$\frac{1}{2}mv^2 + \frac{1}{2}\frac{Iv^2}{r^2} = mg \times \frac{3v^2}{4g}$

$\therefore \quad \frac{I}{r^2} = \frac{m}{2}$ i.e., $I = \frac{1}{2}mr^2$

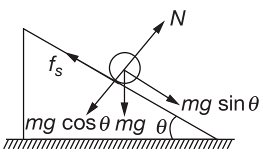
$I' = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2$

The correct option is (A) and (C)

5. Frictional force needed to roll hollow cylinder will be maximum. Under rolling work done by friction is zero.

The correct option is (A), (B) and (C)

6.



For linear motion, $mg\sin\theta - f_s = ma$ (1)

For rotational motion, $f_s R = \frac{mR^2}{2} \times \frac{a}{R}, \; f_s = \frac{ma}{2}$ (2)

Solving (1) and (2), we get $f_s = \frac{mg\sin\theta}{3}$

Also, f_s is opposing translational motion but supporting rotational motion

The correct option is (A) and (D)

7. Since the vertical distance travelled by all the marbles is same, their velocities will also be same because $v = \sqrt{2gh}$

The correct option is (C) and (D)

8. $\alpha = 4$ rad/s^2

Let acceleration of the (cycle + mass) be a where $a = 4 \times 0.5$ m/s^2 = 2 m/s^2

Since acceleration of the man is in horizontal direction and equal to 2 m/s^2 and net vertical force is zero.

$\therefore$ net force on the man = net horizontal force on the man i.e. $F = 50 \times 2$ N = 100 N

The correct option is (A) and (B)

9. The correct option is (A), (C) and (D)

10. The correct option is (B), (C) and (D)

Passage Based Questions

Passage 1

11. The correct option is (A)

12. The correct option is (A)

13. The correct option is (B)

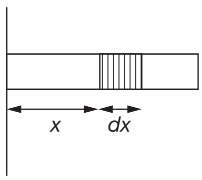
14. The correct option is (B)

Passage 2

15. $I = \int dm\,x^2 = \int_0^2 (\lambda dx)x^2 = k\int_0^L x^3 dx$

$I = \frac{KL^4}{4}$

The correct option is (D)



16. $L = mvr = mv\frac{5}{\sqrt{1^2+3^2}} = \sqrt{\frac{5}{2}}mv$

The correct option is (A)

17. $m_1 g - T_1 = m_1 a$ (1)

$T_2 - m_2 g = m_2 a$ (2)

$(T_1 - T_2)R = \frac{Ia}{R}$ (3)

Solving above equation $\frac{(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}$

The correct option is (B)

Passage 3

18. Torque acts parallel to velocity, so change in momentum is parallel to $\vec{v}$.

The correct option is (A)

19. $\vec{\tau}$ is changing only the direction of angular momentum and not its magnitude.

So $\vec{\tau}$ is perpendicular $\vec{L}$

The correct option is (C)

20. $\vec{\tau} = \vec{l} \times m\vec{g} = mgl\sin\theta = \frac{d\vec{L}}{dt}$

The correct option is (D)

The position vector of a body of mass $m = 4$ kg is given as $\vec{r} = \hat{i}(t^2 - 4t) + \hat{j}(-t^3)$, where $\vec{r}$ is in metres and t in seconds.

21. $\vec{r} = \hat{i}(t^2 - 4t) + \hat{j}(-t^3)$

$\vec{v} = (2t - 4)\hat{i} - 3t^2\hat{j}$

$\vec{a} = 2\hat{i} - 6t\hat{j}$

$\vec{F} = m\vec{a}$

$\vec{F} = 8\hat{i} - 24t\hat{j}$

at $t = 1s$

$\vec{F} = 8(\hat{i} - 3\hat{j})$

$|\vec{F}| = 8\sqrt{10}$ N

The correct option is (A)

22. At $t = 1s$

$\vec{r} = -3\hat{i} - \hat{j}$

$\vec{F} = 8\hat{i} - 24\hat{j}$

$\therefore \quad \vec{\tau} = \vec{r} \times \vec{F}$

$$= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -1 & 0 \\ 8 & -24 & 0 \end{bmatrix}$$

$\vec{\tau} = \hat{k}(72 + 8)$

$\vec{\tau} = 80\hat{k}$

$|\vec{\tau}| = 80$ N/m

The correct option is (D)

23. $y = 2x$

$\Rightarrow \quad -t^3 = 2(t^2 - 4t)$

$t^2 + 2t - 8 = 0$

$\Rightarrow \quad t = 2s$

at $\quad t = 2s$

$\vec{v} = (2 \times 2 - 4)\hat{i} - 3 \times 2^2\hat{j}$

$\vec{v} = -12\hat{j}$

$\vec{p} = m\vec{v} = -48\hat{j}$

The correct option is (B)

24. $t_1 = 0, \ t_2 = 2s$

Displacement $(\vec{s}) = \vec{r}(t_2) - \vec{r}(t_1)$

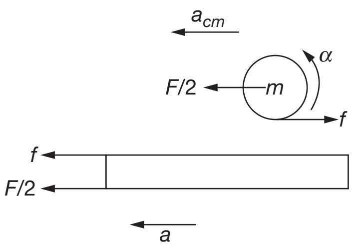
$\vec{s} = -12\hat{j} - (-4\hat{i}) = 4\hat{i} - 12\hat{j}$

$\vec{F} = 8\hat{i} - 24\hat{j}$

$W = \vec{F} \cdot \vec{S} = 32 + 288 = 320$ J

The correct option is (C)

Passage 4



25. $\dfrac{F}{2} - f = m\,a_{cm} \quad (1)$

$fR = \dfrac{1}{2}mR^2\alpha \quad (2)$

$\dfrac{F}{2} + f = 2ma \quad (3)$

For pure rolling

$a_{cm} - R\alpha = a \quad (4)$

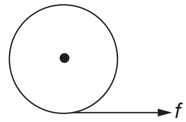
On solving $f = \dfrac{F}{14}, \ a_{cm} = \dfrac{3F}{7m}, \ a = \dfrac{2F}{7m}$

The correct option is (C)

26. The correct option is (D)

27. The correct option is (B)

Passage 5



28. $f = ma \quad (1)$

$f \cdot R = I\alpha = \dfrac{mR^2\alpha}{2} \quad (2)$

dividing equation (1) by (2), $a = \dfrac{1}{2}R\alpha$

The correct option is (C)

29. $v = u + at = \dfrac{f}{m}t \quad (1)$

$\omega = \omega_0 - \alpha t = \omega_0 - \dfrac{2f}{mR}t \quad (2)$

After time t, $v = R\omega$

$\dfrac{f}{m}t = R\left[\omega_0 - \dfrac{2f}{mR}t\right]$

Since $f = \mu_k mg$

$\mu_k gt = R\omega_0 - 2\mu_k gt \Rightarrow t = \dfrac{R\omega_0}{3\mu_k g}$

The correct option is (A)

30. On putting the value of t in equation (1), we get $v = \dfrac{1}{3}R\omega_0$

The correct option is (B)

31. Initial kinetic energy, $K = \dfrac{1}{2}I\omega_0^2 = \dfrac{1}{2}\dfrac{mR^2\omega_0^2}{2}$

Final kinetic energy, $K' = \dfrac{1}{2}\dfrac{mR^2}{2}\dfrac{v^2}{R^2} + \dfrac{1}{2}mv^2$

Thus $K' = \dfrac{K}{3}$

The correct option is (C)

Match the Column Type

32. $J\dfrac{l}{4} = \dfrac{ml^2}{12}\omega \Rightarrow \omega = \dfrac{3J}{ml}, \ v_{cm} = \dfrac{J}{m}$

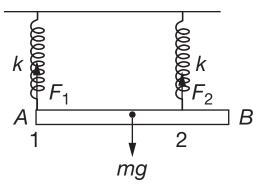
$v_A = v_{cm} - \dfrac{l}{3}\omega = 0, \ v_{\text{upper}} = \left|v_{cm} - \dfrac{l}{2}\omega\right| = \dfrac{J}{2\,m},$

$v_{\text{lower}} = \left|v_{cm} + \dfrac{l}{2}\omega\right| = \dfrac{5}{2}\dfrac{J}{m}$

(A) → 3; (B) → 1; (C) → 4; (D) → 2

33. (A) → (1, 2, 3, 4); (B) → (2, 3, 4); (C) → (1, 2, 3, 4); (D) → (1, 2, 3, 4)

34. Torque about 2



$F_1\dfrac{3l}{4} = \dfrac{mgl}{4},$

As the spring 2 is cut,

$mg - \dfrac{mg}{3} = ma_{cm},$

$a_{cm} = \frac{2g}{3}$, $12a_{cm} = 8g$

Torque about centre of mass,

$\frac{mg}{3}\frac{l}{2} = \frac{ml^2}{12}\alpha$, $\frac{l\alpha}{2} = g$, $a_A = a_{\text{cm}} - \frac{l}{2}\alpha = \frac{g}{3}$

(A) → 4; (B) → 3; (C) → 2

35. (A) → 1, 2, 3; (B) → 1, 4; (C) → 1, 4; (D) → 1

36. $F - F_s = ma$

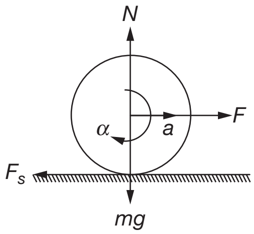
$F_s R = I\alpha$, $a = R\alpha$

Also, $F_s \le \mu mg$

∴ For pure rolling,

$F_{\max} = 60\text{ N}$

This value of applied force will be achieved at $t = 6$ s



Till $t = 6$ s

$a = \frac{2}{3}t$, $\alpha = \frac{2}{3}t$

∴ at $t = 6$ s, $v = 12$ m/s, $\omega = 12$ rad/s.

Between $t = 6$ s to $t = 8$ s

$10t - 20 = 10a$

$20R = I\alpha$

At $t = 8$ s,

$v = 22$ m/s, $\omega = 20$ rad/s.

After this, disc will again start pure rolling.

Conserving angular momentum about point of contact,

$m(22)R + I(20) = mvR + I\omega$ and $v = R\omega$

$v = \frac{64}{3}$ m/s; $\frac{64}{3} = 22 - 2t$

$\Rightarrow t = \frac{1}{3}$s

(A) → 1, 4; (B) → 2, 3; (C) → 1, 2, 3; (D) → 4

Integer Type

48. $I_{\text{axis}} = \frac{mR^2}{2}$, $I_1 = m\left(\frac{R^2}{2} + \frac{l^2}{12}\right) = m\left[\frac{R^2}{4} + \frac{(\sqrt{3}R)^2}{12}\right]$

$= m\left[\frac{R^2}{4} + \frac{R^2}{4}\right] = \frac{MR^2}{2}$

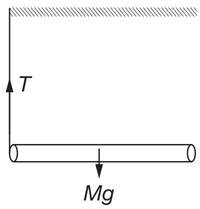
so $\frac{I_1}{I_{\text{axis}}} = 1$

49. $Mg - T = Ma_y$ (1)

$T\left(\frac{L}{2}\right) = \frac{ML^2}{12}\alpha$ (2)

and $ay = \frac{L}{2}\alpha$ (3)

on solving $T = \frac{Mg}{4} = 5\text{ N}$



50. Since $\omega^2 - \omega_0^{\,2} = 2\alpha\theta$, where $\theta = 2\pi n$

∴ $2\alpha = \frac{\omega_0^{\,2} - \left(\frac{\omega_0}{2}\right)^2}{2\pi n}$ and $0 = \left(\frac{\omega_0^{\,2}}{2}\right)^2 - (2\alpha)(2\pi n')$

∴ $n' = \frac{n}{3}$, $x = 3$

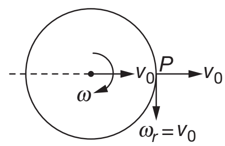
51. $\omega_0 = 900 \times \frac{2\pi}{60} = 30\,\pi$, $0 = \omega_0 - \alpha \times t$

$\Rightarrow \alpha = \frac{\omega_0}{t} = \frac{30\pi}{60} = \frac{\pi}{2}$

$n = 2$

52. $v_p = \sqrt{v_0^2 + v_0^2} = \sqrt{2}\,v_0$

$n = 2$



53. $Fd - 5d - 3d = 0$; $F = 8$ N

54. Applying conservation of energy

$Mgh = \frac{1}{2}Mv_0^2 + \frac{1}{2}I\omega_0^2 = \frac{1}{2}M\left(\frac{5v_0}{4}\right)^2 \Rightarrow I = \frac{9MR^2}{16}$

$Mx^2 = I \Rightarrow x = \frac{3R}{4}$

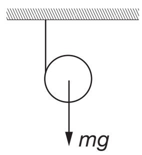
$n = 3$

55. $mg - T = ma$

$TR = I\alpha = \frac{mR^2}{2}\alpha$ $(a = R\alpha)$;

$a = \frac{2g}{3}$

$n = 2$



56. $T_1 = T_2$

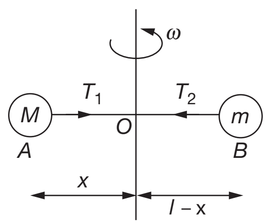
$\Rightarrow M\omega^2 x = m\omega^2(l - x)$

$x = \frac{ml}{M + m}$

$l - x = \frac{Ml}{M + m}$

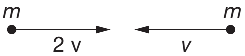
$\frac{AO}{OB} = \frac{x}{l - x} = \frac{m}{M} = \frac{3}{9} = \frac{1}{3}$

$\frac{OB}{AO} = 3$



57. Taking torque about contact point, $\tau = 4 \times R - 2 \times 2R = 0$

Previous Years' Questions

58. 

$$v_c = \frac{m_1 v_1}{m_1 + m_2} = \frac{m(2v) + m(-v)}{m+m}$$

$$\therefore \quad v_c = \left(\frac{v}{2}\right)$$

The correct option is (C)

59. $I_1\omega_1 = I_2\omega_2$

$$\Rightarrow \quad \frac{1}{2}MR^2(\omega_1) = \left(\frac{1}{2}MR^2 + 2mR^2\right)\omega_2$$

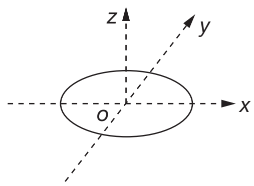
$$\therefore \quad \omega_2 = \left(\frac{M}{M+4m}\right)\omega_1$$

The correct option is (C)

60. This is the case of sliding, hence acceleration of all the bodies is same ($g \sin \theta$)

The correct option is (D)

61. From symmetry $I_x = I_y$



$$\Rightarrow \quad I_x + I_y = I_z$$

$$\Rightarrow 2I_x = MR^2$$

$$\therefore \quad I_x = \frac{MR^2}{2}$$

The correct option is (A)

62. Line of action of momentum passes through the axis of rotation.

The correct option is (D)

63. $|\vec{\tau}| = |\vec{r} \times \vec{F}| \Rightarrow \vec{\tau} \perp \vec{r}$ and $\vec{\tau} \perp \vec{F}$

The correct option is (D)

64. $F = \frac{m}{v} \Rightarrow M_x = f(\pi R^2 t)$ and $M_y = f\left(\pi(rR)^2 \frac{t}{4}\right)$

$$\Rightarrow I_x = \frac{1}{2}M_y R^2 = \frac{1}{2}\left(P\pi R^2 t\right)R^2 = \frac{\pi d}{2} + R^4 \qquad (1)$$

So, $\frac{I_x}{I_y} = \frac{(t)R^4}{\left(\frac{t}{4}\right)(4R)^4} = \frac{1}{64} \Rightarrow I_y = 64\, I_x$

The correct option is (D)

65. $K = \frac{1}{2}I\omega^2 = \frac{1}{2}L\omega \qquad \because \ (L = I\omega)$

$\frac{K}{K'} = \frac{L \times \omega}{L' \times \omega'}$; (Given $K' = \frac{K}{2}$)

$$\Rightarrow \frac{K}{K/2} = \frac{L\omega}{L(2\omega)} \Rightarrow L' = \frac{L}{4}$$

The correct option is (A)

66. $W = mgh = 1.2 \times 10 \times 0.3 = 3.6$ J

The correct option is (B)

67. Angular momentum will remain the same since external torque is zero.

The correct option is (B)

68. MOI of a solid sphere $A \Rightarrow I_A = \frac{2}{5}MR^2$

MOI of a hollow $B \Rightarrow I_B = \frac{2}{3}MR^2$

clearly $I_A < I_B$ $I_A < I_B$

The correct option is (A)

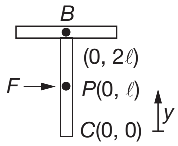
69. The acceleration vector is along the radius of circle.

The correct option is (B)

70. COM continues its original path.

The correct option is (C)

71. To have linear motion $\vec{F}$ has to be applied at COM.



$$\Rightarrow \quad y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

$$= \frac{m(2\ell) + (2m)\ell}{3m} = \frac{4\ell}{3}$$

The correct option is (B)

72. There will be only centrifugal acceleration. If all particles as angular speed is constant.

$F_1 = mR_1\omega^2$ and $F_2 = mR_2\omega^2$

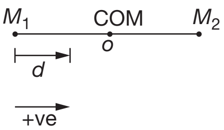
$$\Rightarrow \quad \frac{F_1}{F_2} = \frac{R_1}{R_2}$$

The correct option is (A)

73. Disc may be assumed as combination of two semicircular parts. If I be the MOI of the uniform semicircular disc.

$$\Rightarrow \quad 2I = \frac{2Mr^2}{2} \quad \Rightarrow I = \frac{1}{2}Mr^2$$

The correct option is (A)

74. 

$$m_1(\Delta x_1) + m_2(\Delta x_2) = 0$$

$$\Rightarrow \Delta x_2 = \frac{m_1}{m_2}(\Delta x_1)$$

$$\therefore \quad \Delta x_2 = \frac{m_1}{m_2}(d) \leftarrow$$

The correct option is (A)

75. Conservation of angular momentum.

$$I\omega = I'\omega'$$

$$\Rightarrow \quad mR^2\omega = (mR^2 + 2mR^2)\ \omega'$$

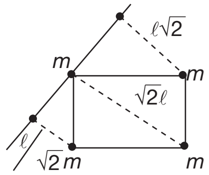
$$\omega' = \left(\frac{m}{m+2m}\right)\omega$$

The correct option is (B)

76. $\vec{\tau} = \vec{r} \times \vec{f} = (\hat{i} - \hat{j}) \times (-F\hat{k}) = F(\hat{i} + \hat{j})$

The correct option is (A)

77. $I = m\left(\sqrt{2}\ell\right)^2 + 2m\left(\frac{\ell}{\sqrt{2}}\right)^2$

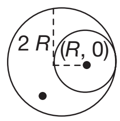


$\therefore \quad I = 3m\ell^2$

The correct option is (A)

78. Let mass per unit R be 'σ'.

$$x_{cm} = \frac{(4\pi\sigma R^2)(0) + (-\pi\sigma R^2)(R)}{4\pi\sigma R^2 - \pi\sigma R^2}$$



$\Rightarrow \quad x_{cm} = -\frac{R}{3}$

$\Rightarrow \quad \alpha = \frac{1}{3}$

The correct option is (D)

79. By formula $\Rightarrow \quad a = \dfrac{g\sin\theta}{1 + \dfrac{I}{MR^2}}$

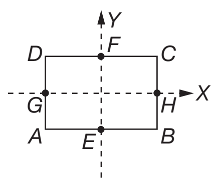
The correct option is (D)

80. Central forces act along centre of mass and therefore torque about centre of mass is zero.

The correct option is (B)

81. From $\perp^r$ axis theorem.

$I_{EF} + I_{GH} = I_z; (I_{EF} = I_{GH})$



$\Rightarrow \quad 2I_{EF} = I_2$

$\therefore \quad I_{EF} = \frac{I_2}{2}$

Again, $I_{AC} + I_{BD} = I_z; \; (I_{AC} = I_{BD})$

$\Rightarrow \quad 2I_{AC} = I_2$

$\therefore \quad I_{AC} = \frac{I_z}{2}$

Hence, $I_{EF} = I_{AC}$

The correct option is (B)

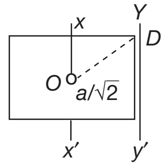
82. For $n = 0; x = k \Rightarrow x_{cm} = \frac{L}{2}$ and hence (c) is ruled out.

For $n > 0$; rate of increase of linear man density with increase in x increases and hence COM will shift towards $x = L$, (end of rod), as n increases and hence (b) is ruled out.

With increase in n, COM shifts towards $x = L$ such that first the shifting in at a higher rate and then the rate decreases with the value of n. These characteristics are represented by (a).

The correct option is (A)

83. $I_{yy'} = I_{xx'} + m\left(\frac{a}{\sqrt{2}}\right)^2$



$\Rightarrow \quad I_{yy'} = \frac{Ma^2}{6} + \frac{Ma^2}{2}$

$\therefore \quad I_{yy'} = \frac{2}{3}Ma^2$

The correct option is (D)

84. $mgh = \frac{1}{2}I\omega^2$

$\Rightarrow \quad mgh = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2$

$\Rightarrow \quad h = \frac{I^2\omega^2}{6g}$

The correct option is (D)

85. $v = \frac{ds}{dt} = 3t^2$

$a_t = \frac{dv}{dt} = 6t$; at $t = 2$ s, $a_t = (6 \times 2) = 12$ m/s^2

$a_r = \frac{v^2}{R} = \left(\frac{144}{20}\right)$ m/s^2

$\therefore \quad a_n = \sqrt{at^2 + ar^2} = 14$ m/s^2

The correct option is (D)

86. $\vec{a} = \frac{v^2}{R}$ towards centre of circle.

$\therefore \quad \vec{a} = \frac{v^2}{R}(-\cos\hat{i} - \sin\theta\,\hat{j}) = -\frac{v^2}{R}\cos\theta\hat{i} - \frac{v^2}{R}\sin\theta\,\hat{j}$

The correct option is (C)

87. $\vec{L} = m(\vec{r} \times \vec{v}) = m\left[(v_0\cos\theta)t\,\hat{i} + \left(v_0\sin\theta\, t - \frac{1}{2}gt^2\hat{j}\right)\right]x$

$\left[v_0\cos\theta\hat{i} + (v_0\sin\theta - gt)\hat{j}\right]$

$\therefore \quad \vec{L} = -\frac{1}{2}mgv_0t^2\cos\theta + (\hat{k})$

The correct option is (C)

88. Conservation of angular momentum about any fixed point on the surface.

$mr^2\omega_0 = 2m^2\omega \Rightarrow \omega = \frac{\omega_0}{2}$

$\therefore \quad v_{cm} = (\omega r) = \frac{r\omega_0}{2}$

The correct option is (B)

89. Conservation of angular momentum :

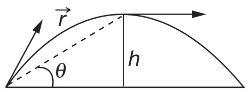
$I_1\omega_1 = I_2\omega_2$

$\Rightarrow \quad (MR^2)\omega_1 = (MR^2 + 2mr^2)\omega_2 \quad (\therefore \quad \omega_1 = \omega)$

$$\Rightarrow \quad \omega_2 = \left(\frac{M\omega}{M+2m}\right)$$

The correct option is (C)

90. $h = \dfrac{v^2 \sin^2\theta}{2g} = \dfrac{v^2 \sin^2 45^\circ}{2g} = \dfrac{v^2}{4g}$



$v_h = v\cos\theta = v\cos 45^\circ = \dfrac{v}{\sqrt{2}}$

$|\vec{L}| = |\vec{r} \times m\vec{v}| = h(mv); \quad (\because \ 4 \sin\theta = h)$

$\therefore \quad L = \dfrac{mv^3}{4\sqrt{2g}}$

The correct option is (B)

91. For maximum volume of req. cube

$\sqrt{3}a = 2R \qquad [a \rightarrow \text{side of cube}]$

$a = \dfrac{2R}{\sqrt{3}}$

Mass of removed cube

$m = \rho \times a^3 = \dfrac{3M}{4\pi R^3} \times \dfrac{8R^3}{3\sqrt{3}} = \dfrac{2M}{\sqrt{3}\pi}$

$I = \dfrac{m}{6}a^2 = \dfrac{4MR^2}{9\sqrt{3}\pi}$

The correct option is (B)

92. For a particle moving with constant velocity $\vec{L} = mvr_\perp \hat{n}$ where $\hat{n}$ is unit vector in the direction of $\vec{r} \times \vec{P}$.

When particle is moving from C to D, $r_\perp$ is $\left(\dfrac{R}{\sqrt{2}} + a\right)$, so (1) is false.

When particle is moving from D to A, angular

$\vec{L} = \dfrac{mv}{\sqrt{2}} R(-\hat{k})$, so (3) is also false.

The correct option is (A) and (C)

93. Distance covered by point of contact with line AB will decrease. Distance covered by point of contact with line CD will be more, so the roller will turn left to full fill the condition.

The correct option is (C)