Equations of Normal in Different Forms

1. **Point form** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$$

2. **Parametric form** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$ is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$$

3. **Slope form** The equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of slope 'm' is

$$y=mx\pm\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

TRICK(S) FOR PROBLEM SOLVING

- The coordinates of the points of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$$

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$SS_1=T^2$$

where $S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1,\ S_1\equiv\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

and $T\equiv\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

CHORD WITH A GIVEN MID POINT

The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with $P(x_1, y_1)$ as its middle point is given by $T=S_1$ where

$$T\equiv\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1 \text{ and } S_1\equiv\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1.$$

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $T=0$, where $T\equiv\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$.

SOLVED EXAMPLES

69. The locus of the centre of a circle which touches two given circles externally is

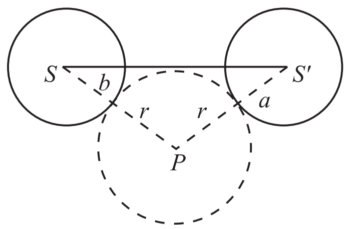
(A) an ellipse (B) a parabola
(C) a hyperbola (C) none of these

Solution: (C)

Let S and S' be the centres and a and b be the radii of the given circles.

Let P be the centre and r be the radius of the circle which touches the given circles externally.

Then $S'P=r+a$ and $SP=r+b$



$\therefore$ $S'P-SP=(r+a)-(r+b)=a-b=$ constant.

Hence the locus of P is a hyperbola whose foci are S and S'.

70. If a circle makes intercepts of length 5 and 3 on two perpendicular lines, then the locus of the centre of the circle is

(A) a parabola (B) an ellipse
(C) a hyperbola (D) none of these

Solution: (C)

Let the two given $\perp$ lines be the coordinate axes and let the equation of variable circle be

$$x^2+y^2+2gx+2fy+c=0 \quad (1)$$

Then, $5=2\sqrt{g^2-c}$ and $3=2\sqrt{f^2-c}$

Squaring and subtracting these, we get

$$4(g^2-c)-4(f^2-c)=25-9$$

$\Rightarrow$ $g^2-f^2=4$ or $(-g)^2-(-f)^2=4$

Hence, locus of the centre $(-g, -f)$ of circle is

$$x^2-y^2=4,$$

which is a rectangular hyperbola.

71. If the line $ax+by+c=0$ is a normal to the hyperbola $xy=1$, then

(A) $a>0, b<0$ (B) $a>0, b>0$
(C) $a<0, b<0$ (D) $a<0, b>0$

Solution: (A, D)

Equation of normal at the point $\left(t, \frac{1}{t}\right)$ to the hyperbola $xy=1$ is

$$xt^3-yt-t^4+1=0$$

Its slope $= t^2 = \dfrac{-a}{b}$

$\therefore\quad \dfrac{-a}{b} > 0 \quad\Rightarrow\quad \dfrac{a}{b} < 0$

$\therefore\quad a > 0, b < 0$ or $a < 0, , b > 0$

72. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents
(A) no locus if $k > 0$
(B) an ellipse if $k < 0$
(C) a point if $k = 0$
(D) a hyperbola if $k > 0$

Solution: (C)

We have $2x^2 + 3y^2 - 8x - 18y + 35 = k$
$\Rightarrow\ 2(x^2 - 4x) + 3(y^2 - 6y) + 35 = k$
$\Rightarrow\ 2[(x-2)^2 - 4] + 3[(y-3)^2 - 9] + 35 = k$
$\Rightarrow\ 2(x-2)^2 + 3(y-3)^2 = k$

For $k = 0$, we get $2(x-2)^2 + 3(y-3)^2 = 0$ which represents the point (2, 3).

73. The eccentricity of the hyperbola

$$\frac{\sqrt{1999}}{3}(x^2 - y^2) = 1 \text{ is}$$

(A) $\sqrt{2}$ (B) 2
(C) $2\sqrt{2}$ (D) $\sqrt{3}$

Solution: (A)

Equation of hyperbola is

$$\frac{x^2}{\frac{3}{\sqrt{1999}}} - \frac{y^2}{\frac{3}{\sqrt{1999}}} = 1$$

Here $\qquad a^2 = b^2 = \dfrac{3}{\sqrt{1999}}$

$\therefore\quad$ Eccentricity is $e = \sqrt{1 + \dfrac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$

74. If e, e' be the eccentricities of two conics $S = 0$ and $S' = 0$ and if $e^2 + e'^2 = 3$, then both S and S' can be
(A) hyperbolas (B) ellipses
(C) parabolas (D) none of these

Solution: (A)

For ellipse or parabola $e^2 + e'^2 \le 2 \ne 3$.
For hyperbola, it is possible ($e > 1, e' > 1$).

75. The locus of the foot of the perpendicular from the centre of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ on a variable tangent is
(A) $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$
(B) $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$
(C) $(x^2 + y^2)^2 = b^2x^2 - a^2y^2$
(D) none of these

Solution: (B)

Equation of any tangent to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (1)$$

in the slope form is $y = mx + \sqrt{a^2m^2 - b^2}$ (2)

slope of tangent $= m$.

$\therefore$ slope of any line $\perp$ to it $= -\dfrac{1}{m}$

Equation of $\perp$ from centre (0, 0) of (1) on (2) is

$$y - 0 = -\frac{1}{m}(x - 0) \text{ or } m = -\frac{x}{y} \qquad (3)$$

The required locus is obtained by eliminating the parameter m between (2) and (3). Substituting for m from (3) in (2), we get

$$y = -\frac{x^2}{y} + \sqrt{a^2 \cdot \frac{x^2}{y^2} - b^2}$$

or $\qquad x^2 + y^2 = \sqrt{a^2x^2 - b^2y^2}$

or $\qquad (x^2 + y^2)^2 = a^2x^2 - b^2y^2$

76. The equation of the hyperbola, referred to its axes as axes of coordinates, given that the distances of one of its vertices from the foci are 9 and 1 units, is

(A) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ (B) $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

(C) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = -1$ (D) none of these

Solution: (A)

Let the equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (1)$$

Its vertices are $A(a, 0)$ and $A'(-a, 0)$ and foci are $S(ae, 0)$ and $S'(-ae, 0)$.

Given : $S'A = 9$ and $SA = 1$
$\Rightarrow\ a + ae = 9$ and $ae - a = 1$
$\Rightarrow\ a(1 + e) = 9$ and $a(e - 1) = 1$

$\Rightarrow\ \dfrac{a(1+e)}{a(e-1)} = \dfrac{9}{1} \ \Rightarrow\ 1 + e = 9e - 9 \ \Rightarrow\ e = \dfrac{5}{4}$

$\because\ a(1+e) = 9, \ \therefore\ a\left(1 + \dfrac{5}{4}\right) = 9 \ \Rightarrow\ a = 4$

Also, $b^2 = a^2(e^2 - 1) = 16\left(\dfrac{25}{16} - 1\right) = 9$

Thus, from (1), equation of hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

77. The eccentricity of the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$ is

(A) 5/4 (B) 4/5
(C) 9/16 (D) 16/9

Solution: (A)

The given hyperbola can be written in the form

$$\frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$$

Here $a^2 = 16$ and $b^2 = 9$.

$$\therefore\quad e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow e = \frac{5}{4}.$$

78. The number of tangents to the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$ through (4, 1) is

(A) 1 (B) 2
(C) 0 (D) 3

Solution: (B)

Since $\left.\frac{x^2}{4} - \frac{y^2}{3} - 1\right]_{(4,1)} = \frac{16}{4} - \frac{1}{3} - 1 > 0$

$\therefore$ the point (4, 1) lies outside the hyperbola, hence the number of tangents through (4, 1) is two.

79. The equation of common tangents to the parabola $y^2 = 8x$ and hyperbola $3x^2 - y^2 = 3$, is

(A) $2x \pm y + 1 = 0$ (B) $2x \pm y - 1 = 0$
(C) $x \pm 2y + 1 = 0$ (D) $x \pm 2y - 1 = 0$

Solution: (A)

The equation of tangent to $y^2 = 8x$ is $y = mx + \frac{2}{m}$

Also, the equation of tangent to $\frac{x^2}{1} - \frac{y^2}{3} = 1$

$\Rightarrow\quad y = mx \pm \sqrt{m^2 - 3}$

On comparing, we get

$m = \pm 2$ or tangent as $2x \pm y + 1 = 0$

80. A tangent to the hyperbola $y = \frac{x+9}{x+5}$ passing through the origin is

(A) $x + 25y = 0$ (B) $5x + y = 0$
(C) $5x - y = 0$ (D) $x - 25y = 0$

Solution: (C)

$$y = \frac{x+9}{x+5} = 1 + \frac{4}{x+5}; \frac{dy}{dx} \text{ at } (x_1, y_1) = -\frac{4}{(x_1+5)^2}$$

Equation of tangent is $y - y_1 = -\frac{4}{(x_1+5)^2}(x - x_1)$

$$y - 1 - \frac{4}{x_1+5} = -\frac{4}{(x_1+5)^2}(x - x_1)$$

Since it passes through origin (0, 0)

$$-1 - \frac{4}{x_1+5} = \frac{4x_1}{(x_1+5)^2}$$

$\Rightarrow\ (x_1 + 5)^2 + 4(x_1 + 5) + 4x_1 = 0$

$\Rightarrow\ x_1^2 + 18x_1 + 45 = 0$

$\Rightarrow\ (x_1 + 15)(x_1 + 3) = 0$

$\Rightarrow\ x_1 = -15$ or $x_1 = -3$

So equation of tangent is

$$y - 1 - \frac{4}{(-15+5)} = -\frac{4}{(-15+5)^2}(x+15)$$

$$\Rightarrow\ y - 1 + \frac{2}{5} = -\frac{1}{25}(x+15)$$

$$\Rightarrow\ y - \frac{3}{5} = -\frac{x}{25} - \frac{3}{5}$$

$\Rightarrow\ x + 25y = 0$

or $$y - 1 - \frac{4}{(-3+5)} = -\frac{4}{(-3+5)^2}(x+3)$$

$\Rightarrow\ y - 1 - 2 = -(x + 3)$ or $x + y = 0$

81. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, and $S(x_4, y_4)$, then

(A) $x_1 + x_2 + x_3 + x_4 = 0$
(B) $y_1 + y_2 + y_3 + y_4 = 2$
(C) $x_1x_2x_3x_4 = 2c^4$
(D) $y_1y_2y_3y_4 = 2c^4$

Solution: (A)

Since $y = \frac{c^2}{x}$ and $x^2 + y^2 = a^2$

$$\Rightarrow\quad x^2 + \frac{c^4}{x^2} = a^2$$

$$\Rightarrow\quad x^4 - a^2x^2 + c^4 = 0$$

This has four roots say x_1, x_2, x_3, x_4

$$\therefore\quad x_1 + x_2 + x_3 + x_4 = 0$$

82. If PQ is a double ordinate of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola. Then the eccentricity e of the hyperbola satisfies

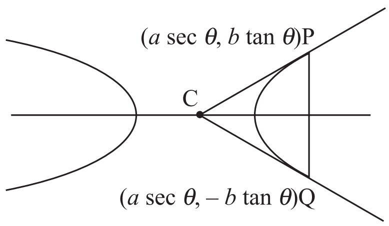
(A) $1 < e < \frac{2}{\sqrt{3}}$ (B) $e = \frac{2}{\sqrt{3}}$
(C) $e = \frac{\sqrt{3}}{2}$ (D) $e > \frac{2}{\sqrt{3}}$

Solution: (D)

Let $P(a\sec\theta, b\tan\theta)$; $Q(a\sec\theta, -b\tan\theta)$ be end points of double ordinate and $C(0, 0)$, is the centre of the hyperbola. Now $PQ = 2b\tan\theta$

$$CQ = CP = \sqrt{a^2 \sec^2\theta + b^2 \tan^2\theta}$$

since $CQ = CP = PQ$



$\therefore$ $4b^2 \tan^2\theta = a^2 \sec^2\theta + b^2 \tan^2\theta$

$\Rightarrow$ $3b^2 \tan^2\theta = a^2 \sec^2\theta$

$\Rightarrow$ $3b^2 \sin^2\theta = a^2$

$\Rightarrow$ $3a^2(e^2 - 1) \sin^2\theta = a^2$

$\Rightarrow$ $3(e^2 - 1)\sin^2\theta = 1$

$\Rightarrow$ $\frac{1}{3(e^2-1)} = \sin^2\theta < 1 \quad (\because \quad \sin^2\theta < 1)$

$\Rightarrow$ $\frac{1}{e^2-1} < 3 \Rightarrow e^2 - 1 > \frac{1}{3}$

83. Centre of the hyperbola $x^2 + 4y^2 + 6xy + 8x - 2y + 7 = 0$ is

(A) (1, 1) (B) (0, 2)
(C) (2, 0) (D) none of these

Solution: (D)

Let centre be (h, k)

$\Rightarrow$ $h + 3k + 4 = 0,\ 3h + 4k - 1 = 0$

$\Rightarrow$ $(h,k) \equiv \left(\frac{19}{5}, -\frac{13}{5}\right)$

84. Equation of the straight line, passing through the point (3, 4) and farthest from the circle $x^2 + y^2 + 8x + 6y + 16 = 0$, is

(A) $x - y + 1 = 0$ (B) $3x + 4y = 25$
(C) $x + y - 7 = 0$ (D) none of these

Solution: (C)

Let $P \equiv (3, 4)$ and C being the centre of circle, where $C \equiv (-4, -3)$. The line which is farthest from the circle is the line perpendicular to CP.

$\therefore$ Required equation is

$$y - 4 = -\left(\frac{3+4}{4+3}\right)(x-3)$$

$\Rightarrow$ $x + y = 7$

85. The area of the triangle formed by the lines $x - y = 0$, $x + y = 0$ and any tangent to the hyperbola $x^2 - y^2 = a^2$ is

(A) $2a^2$ (B) $4a^2$
(C) a^2 (D) none of these.

Solution: (C)

Any tangent at $P(a \sec\theta, b \tan\theta)$ to the hyperbola $x^2 - y^2 = a^2$ is

$$x \sec\theta - y \tan\theta = a \quad (1)$$

Given lines are $x - y = 0$ (2)

and $x + y = 0$ (3)

Solve (1) and (2), (2) and (3), (3) and (1), we get vertices of the triangle as

$$\left(\frac{a}{\sec\theta - \tan\theta}, \frac{a}{\sec\theta - \tan\theta}\right),$$

$$\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-a}{\sec\theta + \tan\theta}\right) \text{ and } (0,0)$$

$\therefore$ Area of the triangle $= \frac{1}{2}|x_1y_2 - x_2y_1|$

$$= \frac{a^2}{2}\left(\frac{-1}{\sec^2\theta - \tan^2\theta} - \frac{1}{\sec^2\theta - \tan^2\theta}\right)$$

$$= \frac{a^2}{2}(-2) = a^2 \text{ (in magnitude)}.$$

86. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is

(A) $9x^2 - 8y^2 + 18x - 9 = 0$
(B) $9x^2 - 8y^2 - 18x + 9 = 0$
(C) $9x^2 - 8y^2 - 18x - 9 = 0$
(D) $9x^2 - 8y^2 + 18x + 9 = 0$

Solution: (D)

The equation of chord of contact at ponit (h, k) is $xh - yk = 9$

Comparing with $x = 9$, we have $h = 1$, $k = 0$

Hence equation of pair of tangents at point (1, 0) is $SS_1 = T^2$

$\Rightarrow$ $(x^2 - y^2 - 9)(1^2 - 0^2 - 9) = (x - 9)^2$

$\Rightarrow$ $-8x^2 + 8y^2 + 72 = x^2 - 18x + 81$

$\Rightarrow$ $9x^2 - 8y^2 - 18x + 9 = 0$

87. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

(A) $3y = 9x + 2$ (B) $y = 2x + 1$
(C) $2y = x + 8$ (D) $y = x + 2$

Solution: (D)

Any tangent to the parabola $y^2 = 8x$ is $y = mx + \frac{2}{m}$.

Clearly (a), (b) and (d) satisfy the equation (by m = 3, 2, 1).

For $xy = -1$

Equation of tangent with slope m, to $xy = -1$ are

$$y - \sqrt{m} = m\left(x + \frac{1}{\sqrt{m}}\right)$$

$$\left(\because \frac{dy}{dx} = \frac{1}{x^2} = m \quad \therefore \quad x = \pm\frac{1}{\sqrt{m}}, y = \mp\sqrt{m}\right)$$

Clearly (d) satisfies these equations. Hence common choice is (d).

88. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2},1\right)$. Its one directrix is the common tangent at the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse in standard form is

(A) $9\left(x-\frac{1}{3}\right)^2 + (y-1)^2 = 1$

(B) $9\left(x-\frac{1}{3}\right)^2 + 12(y-1)^2 = 1$

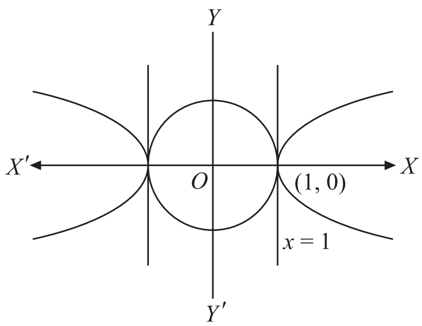
(C) $\frac{\left(x-\frac{1}{3}\right)^2}{4} + \frac{(y-1)^2}{3} = 1$

(D) none of these

Solution: (B)

Clearly, the common tangent to the circle $x^2 + y^2 = 1$ and hyperbola $x^2 - y^2 = 1$ is $x = 1$ [which is nearer to $P(1/2, 1)$].

Given one focus at $P\left(\frac{1}{2},1\right)$.



$\therefore$ equation of the directrix is $x = 1$.

$\therefore$ ellipse is

$$\sqrt{\left(x-\frac{1}{2}\right)^2 + (y-1)^2} = \frac{1}{2}(x-1)$$

On simplification, it becomes

$$9\left(x-\frac{1}{3}\right)^2 + 12(y-1)^2 = 1$$

89. The tangents to the hyperbola $x^2 - y^2 = 3$ are parallel to the st. line $2x + y + 8 = 0$ at the following points

(A) (2, 1) or (1, 2) (B) (2, –1) or (–2, 1)
(C) (–1, –2) (D) (–2, –1)

Solution: (B)

Clearly tangents at (2, –1) or (–2, 1) are parallel to

$$2x + y + 8 = 0$$

90. If P and Q are two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose centre is C such that CP is perpendicular to CQ, where $a < b$, then $\frac{1}{CA^2} + \frac{1}{CQ^2}$ is

(A) $\frac{1}{a^2} - \frac{1}{b^2}$ (B) $\frac{1}{a^2} + \frac{1}{b^2}$
(C) $a^2 - b^2$ (D) $a^2 + b^2$

Solution: (A)

Since the line CP passes through the origin i.e. centre, let its equation be $y = mx$. The line CP meets the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in P whose abscissa is given by

$$\frac{x^2}{a^2} - \frac{m^2x^2}{b^2} = 1 \text{ or } x^2 = \frac{a^2b^2}{b^2 - a^2m^2}$$

$$\therefore \quad y^2 = m^2x^2 = \frac{a^2b^2m^2}{b^2 - a^2m^2}$$

$$\therefore \quad CP^2 = x^2 + y^2$$

$$= \frac{a^2b^2 + a^2b^2m^2}{b^2 - a^2m^2}$$

$$= a^2b^2(1 + m^2)$$

Since $CQ \perp CP$

Replace m by $-\frac{1}{m}$, we get

$$CQ^2 = \frac{a^2b^2\left(1+\frac{1}{m^2}\right)}{b^2 - \frac{c^2}{m^2}} = \frac{a^2b^2(m^2+1)}{b^2m^2 - a^2}$$

$$\therefore \quad \frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{b^2 - a^2m^2 + b^2m^2 - a^2}{a^2b^2(1+m^2)}$$

$$= \frac{b^2 - a^2}{a^2b^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

91. Let $P(a \sec\theta, b \tan\theta)$ and $Q(a \sec\phi, b \tan\phi)$, where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P and Q, then k is equal to

(A) $\frac{a^2 + b^2}{a}$ (B) $-\left(\frac{a^2 + b^2}{a}\right)$

(C) $\frac{a^2 + b^2}{b}$ (D) $-\left(\frac{a^2 + b^2}{b}\right)$

Solution: (D)

Given $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$.

The equation of tangent at point P is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$$

Slope of tangent $=\frac{b}{\tan\theta}\times\frac{\sec\theta}{a}=\frac{b}{a}\cdot\frac{1}{\sin\theta}$

Hence, the equation of perpendicular at P is

$$y-b\tan\theta=-\frac{a\sin\theta}{b}(x-a\sec\theta)$$

or $\quad by - b^2\tan\theta = -a\sin\theta\, x + a^2\tan\theta$

or $\quad a\sin\theta\, x + by = (a^2+b^2)\tan\theta \qquad (1)$

Similarly the equation of perpendicular at Q is

$$a\sin\phi\, x + by = (a^2+b^2)\tan\phi \qquad (2)$$

On multiplying (1) by $\sin\phi$ and (2) by $\sin\phi$, we get

$$a\sin\theta\sin\phi\, x + b\sin\phi\, y = (a^2+b^2)\tan\theta\sin\phi$$

$$a\sin\phi\sin\theta\, x + b\sin\theta\, y = (a^2+b^2)\tan\phi\sin\theta$$

On subtraction we get

$by(\sin\phi-\sin\theta)=(a^2+b^2)(\tan\theta\sin\phi-\tan\phi\sin\theta)$

$$\therefore\; y=k=\frac{a^2+b^2}{b}\cdot\frac{\tan\theta\sin\phi-\tan\phi\sin\theta}{\sin\phi-\sin\theta}$$

$$\because\; \theta+\phi=\frac{\pi}{2}\;\Rightarrow\;\phi=\frac{\pi}{2}-\theta$$

$\Rightarrow \sin\phi=\cos\theta$ and $\tan\phi=\cot\theta$

$$\therefore\; y=k=\frac{a^2+b^2}{b}\cdot\frac{\tan\theta\cos\theta-\cos\theta\sin\theta}{\cos\theta-\sin\theta}$$

$$=\frac{a^2+b^2}{b}\left(\frac{\sin\theta-\cos\theta}{\cos\theta-\sin\theta}\right)=-\frac{(a^2+b^2)}{b}.$$

EXERCISES

Single Option Correct Type

1. Let $y=f(x)$ be a parabola, having its axis parallel to y-axis, which is touched by the line $y=x$ at $x=1$, then

(A) $f'(0)=f'(1)$
(B) $2f(0)=1-f'(0)$
(C) $f'(1)=1$
(D) $f(0)+f'(0)+f''(0)=1$

2. A ray of light is coming along the line which is parallel to y-axis and strikes a concave mirror whose intersection with the xy-plane is a parabola $(x-4)^2=4(y+2)$. After reflection, the ray must pass through the point

(A) $(4,-1)$ (B) $(0, 1)$
(C) $(-4, 1)$ (D) none of these

3. If $y+3=m_1(x+2)$ and $y+3=m_2(x+2)$ are two tangents to the parabola $y^2=8x$, then

(A) $m_1+m_2=0$ (B) $m_1m_2=-1$
(C) $m_1m_2=1$ (D) none of these

4. A line bisecting the ordinate PN of a point $P(at^2, 2at)$, $t>0$, on the parabola $y^2=4ax$ is drawn parallel to the axis to meet the curve at Q. If NQ meets the tangent at the vertex at the point T, then the coordinates of T are

(A) $\left(0,\frac{4}{3}at\right)$ (B) $(0, 2at)$
(C) $\left(\frac{1}{4}at^2, at\right)$ (D) $(0, at)$

5. Coordinates of any point on the parabola, whose focus is $\left(\frac{-3}{2},-3\right)$ and the directrix is $2x+5=0$ is given by

(A) $(2t^2+2, 2t-3)$ (B) $(2t^2-2, 2t-3)$
(C) $(2t^2-2, 2t+3)$ (D) none of these

6. The mirror image of the directrix of the parabola $y^2=4(x+1)$ in the line mirror $x+2y=3$ is

(A) $x=-2$ (B) $4y-3x=16$
(C) $3x+4y+16=0$ (D) none of these

7. The centroid of the triangle formed by the feet of the normals from the point (h, k) to the parabola $y^2+4ax=0$, $(a>0)$ lies on

(A) x-axis (B) y-axis
(C) $x=h$ (B) $y=k$

8. If from a point, the two tangents drawn to the parabola $y^2=4ax$ are normals to the parabola $x^2=4by$, then

(A) $a^2>8b^2$ (B) $b^2>8a^2$
(C) $a^2<8b^2$ (D) none of these

9. If the focus of the parabola $(y-\beta)^2=4(x-\alpha)$ always lies between the lines $x+y=1$ and $x+y=3$, then

(A) $1<\alpha+\beta<2$ (B) $0<\alpha+\beta<1$
(C) $0<\alpha+\beta<2$ (D) none of these

10. If the focal distance of an end of the minor axis of any ellipse (referred to its axes as the axes of x and y

respectively) is k and the distance between the foci is $2h$, then its equation is

(A) $\frac{x^2}{k^2}+\frac{y^2}{k^2+h^2}=1$ (B) $\frac{x^2}{k^2}+\frac{y^2}{h^2-k^2}=1$

(C) $\frac{x^2}{k^2}+\frac{y^2}{k^2-h^2}=1$ (D) $\frac{x^2}{k^2}+\frac{y^2}{h^2}=1$

11. An ellipse slides between two lines at right angles to one another. The locus of its centre is

(A) a parabola (B) an ellipse
(C) a circle (D) None of these

12. If $P(a\cos\theta, b\sin\theta)$ is a point on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then '$\theta$' is

(A) angle of OP line from positive direction of x-axis (O is origin)
(B) angle of OQ line from positive direction of x-axis [when Q is $(a\cos\theta, a\sin\theta)$]
(C) it depends on the point P
(D) none of the above

13. If (5, 12) and (24, 7) are the foci of an ellipse passing through the origin, then the eccentricity of the conic is

(A) $\frac{\sqrt{386}}{12}$ (B) $\frac{\sqrt{386}}{13}$

(C) $\frac{\sqrt{386}}{25}$ (D) $\frac{\sqrt{386}}{38}$

14. A tangent to the ellipse $x^2+4y^2=4$ meets the ellipse $x^2+2y^2=6$ at P and Q. The angle between the tangents at P and Q of the ellipse $x^2+2y^2=6$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

15. The locus of the centre of a circle which touches two given circles externally is

(A) an ellipse (B) a parabola
(C) a hyperbola (D) none of these

16. If a circle makes intercepts of length 5 and 3 on two perpendicular lines, then the locus of the centre of the circle is

(A) a parabola (B) an ellipse
(C) a hyperbola (D) none of these

17. The equation $2x^2+3y^2-8x-18y+35=k$ represents

(A) no locus if $k>0$
(B) an ellipse if $k<0$
(C) a point if $k=0$
(D) a hyperbola if $k>0$

18. If PQ is a double ordinate of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola. Then, the eccentricity e of the hyperbola satisfies

(A) $1<e<2/\sqrt{3}$ (B) $e=2/\sqrt{3}$
(C) $e=\sqrt{3}/2$ (D) $e>2/\sqrt{3}$

19. The equation of the diameter which bisects the chord $7x+y-2=0$ of the hyperbola $\frac{x^2}{3}-\frac{y^2}{7}=1$ is

(A) $x+2y=0$ (B) $x-2y=0$
(C) $x-3y=0$ (D) $x+3y=0$

20. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2},1\right)$. Its one directrix is the common tangent at the point P, to the circle $x^2+y^2=1$ and the hyperbola $x^2-y^2=1$. The equation of the ellipse in standard form is

(A) $9\left(x-\frac{1}{3}\right)^2+(y-1)^2=1$

(B) $9\left(x-\frac{1}{3}\right)^2+12(y-1)^2=1$

(C) $\frac{\left(x-\frac{1}{3}\right)^2}{4}+\frac{(y-1)^2}{3}=1$

(D) none of these

21. If P and Q are two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose centre is C such that CP is perpendicular to CQ, where $a<b$, then $\frac{1}{CP^2}+\frac{1}{CQ^2}$ is

(A) $\frac{1}{a^2}-\frac{1}{b^2}$ (B) $\frac{1}{a^2}+\frac{1}{b^2}$

(C) a^2-b^2 (D) a^2+b^2

22. Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta+\phi=\frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h, k) is the point of intersection of the normals at P and Q, then k is equal to

(A) $\frac{a^2+b^2}{a}$ (B) $-\left(\frac{a^2+b^2}{a}\right)$

(C) $\frac{a^2+b^2}{b}$ (D) $-\left(\frac{a^2+b^2}{b}\right)$

23. If b, k are intercepts of a focal chord of the parabola $y^2=4ax$ then k is equal to

(A) $\frac{ab}{b-a}$ (B) $\frac{b}{b-a}$

(C) $\frac{a}{b-a}$ (D) $\frac{ab}{a-b}$

24. The point $(2a, a)$ lies inside the region bounded by the parabola $x^2 = 4y$ and its latus rectum. Then,

(A) $0 \le a \le 1$ (B) $0 < a < 1$
(C) $0 < a \le 1$ (D) none of these

25. The point P on the parabola $y^2 = 4ax$ for which $|PR - PQ|$ is maximum, where $R(-a, 0)$, $Q(0, a)$ is

(A) $(a, 2a)$ (B) $(a, -2a)$
(C) $(4a, 4a)$ (D) $(4a, -4a)$

26. The shortest distance between the parabola $y^2 = 4x$ and the circle $x^2 + y^2 + 6x - 12y + 20 = 0$ is

(A) $4\sqrt{2} - 5$ (B) 0
(C) $3\sqrt{2} + 5$ (D) 1

27. The tangent and normal at the point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meet the x-axis in T and G, respectively, then angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle through P, T, G is

(A) $\tan^{-1}(t^2)$ (B) $\cot^{-1}(t^2)$
(C) $\tan^{-1}(t)$ (B) $\cot^{-1}(t)$

28. If normals are drawn from a point $P(h, k)$ to the parabola $y^2 = 4ax$, then the sum of the intercepts which the normals cut off from the axis of the parabola is

(A) $(h + a)$ (B) $3(h + a)$
(C) $2(h + a)$ (D) none of these

29. If the normal drawn from the point on the axis of the parabola $y^2 = 8ax$ whose distance from the focus is $8a$ and which is not parallel to either axis, makes an angle θ with the axis of x, then θ is equal to

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$
(C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

30. Ordinates of three points A, B, C on the parabola $y^2 = 4ax$ are in G. P. Tangents at A and C intersect on

(A) line through B parallel to x-axis
(B) line through B parallel to y-axis
(C) line through B and vertex of parabola
(D) line through B and focus of parabola

31. The condition that the parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have a common normal other than x-axis (a, b, c being distinct positive real numbers) is

(A) $\frac{b}{a-c} < 2$ (B) $\frac{b}{a-c} > 2$
(C) $\frac{b}{a-c} < 1$ (D) $\frac{b}{a-c} > 1$

32. The shortest distance between the parabolas $y^2 = 4x$ and $y^2 = 2x - 6$ is

(A) 2 (B) $\sqrt{5}$
(C) 3 (D) none of these

33. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2}, 1\right)$. Its one directrix is the common tangent nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse in the standard form is

(A) $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}} + \frac{(y-1)^2}{\frac{1}{12}} = 1$

(B) $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}} + \frac{(y+1)^2}{\frac{1}{12}} = 1$

(C) $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}} - \frac{(y-1)^2}{\frac{1}{12}} = 1$

(D) $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}} - \frac{(y+1)^2}{\frac{1}{12}} = 1$

34. The locus of the middle point of the intercept of the tangents drawn from an external point to the ellipse $x^2 + 2y^2 = 2$ between the coordinate axes is

(A) $\frac{1}{x^2} + \frac{1}{2y^2} = 1$ (B) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$

(C) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (D) $\frac{1}{2x^2} + \frac{1}{y^2} = 1$

35. If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated about centre in its own plane by 90º in clockwise direction then the point $(a \cos\theta, b \sin\theta)$ becomes

(A) $(a \cos\theta, -b \sin\theta)$ (B) $(b \sin\theta, -a \cos\theta)$
(C) $(b \sin\theta, a \cos\theta)$ (D) none of these

36. If two points are taken on minor axis of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the same distance from the centre as the foci, the sum of the squares of the perpendiculars from these points on any tangent to the ellipse, if $a < b$ is

(A) a^2 (B) b^2
(C) $2a^2$ (D) $2b^2$

37. The area of the rectangle formed by the perpendiculars from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent and normal at a point whose eccentric angle is $\frac{\pi}{4}$ is

(A) $\frac{(a^2-b^2)ab}{a^2+b^2}$ (B) $\frac{(a^2+b^2)ab}{a^2-b^2}$

(C) $\frac{a^2-b^2}{ab(a^2+b^2)}$ (D) $\frac{a^2+b^2}{ab(a^2-b^2)}$

38. The points of intersection of the two ellipses
$x^2+2y^2-6x-12y+23=0$ and
$4x^2+2y^2-20x-12y+35=0$

(A) lie on a circle centred at $\left(\frac{8}{3},3\right)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$

(B) lie on a circle centred at $\left(-\frac{8}{3},-3\right)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$

(C) lie on a circle centred at (8, 9), and of radius $\frac{1}{3}\sqrt{\frac{47}{3}}$

(D) are not concyclic

39. If the eccentric angles of the ends of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ are θ_1 and θ_2, then value of $\tan\theta_1\tan\theta_2$ equals

(A) $\frac{e-1}{e+1}$ (B) $\frac{e-1}{e^2+1}$

(C) $\frac{e+1}{e-1}$ (D) $\frac{e^2+1}{e-1}$

40. If the eccentric angle of a point lying in the first quadrant on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be α and the line joining the centre to the point makes an angle β with x-axis then $\alpha-\beta$ will be maximum when $\alpha=$

(A) 0 (B) $\cot^{-1}\sqrt{\frac{a}{b}}$

(C) $\tan^{-1}\sqrt{\frac{a}{b}}$ (D) $\pi/4$

41. If a variable line $x\cos\alpha+y\sin\alpha=p$ which is a chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $(b>a)$ subtends a right angle at the centre of the hyperbola, then it always touches a fixed circle whose radius is

(A) $\frac{ab}{\sqrt{a^2+b^2}}$ (B) $\frac{ab}{\sqrt{b^2-a^2}}$

(C) $\frac{ab}{\sqrt{a^2-b^2}}$ (D) none of these

42. The tangent at a point P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets one of the directrix in F. If PF subtends an angle θ at the corresponding focus, then θ equals

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) $\frac{3\pi}{4}$ (D) π

43. The number of point(s) outside the hyperbola $\frac{x^2}{25}-\frac{y^2}{36}=1$ from where two perpendicular tangents can be drawn to the hyperbola is/are

(A) none (B) 1
(C) 2 (D) infinte

44. The slopes of common tangents to the hyperbolas $\frac{x^2}{9}-\frac{y^2}{16}=1$ and $\frac{y^2}{9}-\frac{x^2}{16}=1$ are

(A) ±2 (B) ±1
(C) $\pm\sqrt{2}$ (D) none of these

45. The equation of a line passing through the centre of a rectangular hyperbola is $x-y=1$. If one of the asymptotes is $3x-4y-6=0$, the equation of other asymptote is

(A) $4x-3y+17=0$ (B) $-4x-3y+17=0$
(C) $-4x+3y+1=0$ (D) $4x+3y+17=0$

46. If the sum of the slopes of the normal from a point 'P' to the hyperbola $xy=c^2$ is equal to λ $(\lambda\in R+)$, then locus of point 'P' is

(A) $x^2=\lambda c^2$ (B) $y^2=\lambda c^2$
(C) $xy=\lambda c^2$ (D) none of these

47. If a ray of light incident along the line $3x+(5-4\sqrt{2})y=15$, gets reflected from the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ then its reflected ray goes along the line

(A) $x\sqrt{2}-y+5=0$ (B) $\sqrt{2}y-x+5=0$

(C) $\sqrt{2}y-x-5=0$ (D) none of these

48. If a hyperbola passing through the origin has $3x-4y-1=0$ and $4x-3y-6=0$ as its asymptotes, then the equations of its transverse and conjugate axis are

(A) $x+y-5=0$, $x+y-1=0$
(B) $x-y+5=0$, $x-y-1=0$
(C) $x+y-5=0$, $x-y-1=0$
(D) none of these

49. All the chords of the hyperbola $3x^2 - y^2 - 2x + 4y = 0$, subtending a right angle at the origin pass through the fixed point

(A) $(1, -2)$ (B) $(-1, 2)$
(C) $(1, 2)$ (D) none of these

50. The point on the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$ which is nearest to the line $3x + 2y + 1 = 0$ is

(A) $(-6, 3)$ (B) $(6, -3)$
(C) $(6, 3)$ (D) none of these

51. The locus of point of intersection of tangents at the end of normal chord of hyperbola $x^2 - y^2 = a^2$ is

(A) $a^2(y^2 - x^2) = 4x^2y^2$ (B) $a^2(y^2 + x^2) = 4x^2y^2$
(C) $y^2 + x^2 = 4a^2x^2$ (C) none of these

52. The minimum distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$ is

(A) $\sqrt{7}$ (B) $\sqrt{5}$
(C) $2\sqrt{5}$ (D) none of these

53. The mirror image of the parabola $y^2 = 4x$ in the tangent to the parabola at the point $(1, 2)$ is

(A) $(x + 1)^2 = 4(y - 1)$ (B) $(x - 1)^2 = 4(y - 1)$
(C) $(x + 1)^2 = 4(y + 1)$ (D) none of these

54. A ray of light is coming along the line $y = b$ from the positive direction of x-axis and strikes a concave mirror whose intersection with the xy-plane is a parabola $y^2 = 4ax$. If a and b are positive, then the equation of the reflected ray is

(A) $y - 2at = \frac{2t}{t^2 + 1}(x - at^2)$

(B) $y - 2at = \frac{2t}{t^2 - 1}(x - at^2)$

(C) $y - 2at = \frac{-2t}{t^2 - 1}(x - at^2)$

(D) none of these

55. Form a point A, common tangents are drawn to the circle $x^2 + y^2 = \frac{a^2}{2}$ and parabola $y^2 = 4ax$. The area of the quadrilateral, formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola is

(A) $\frac{9}{4}a^2$ (B) $\frac{15}{4}a^2$

(C) $\frac{21}{4}a^2$ (D) none of these

56. Three normals are drawn from the point $(14, 7)$ to the parabola $y^2 - 16x - 8y = 0$. The coordinates of the feet of the normals are

(A) $(0, 0), (8, -16), (3, -4)$ (B) $(0, 0), (8, 16), (3, -4)$
(C) $(0, 0), (-8, 16), (3, -4)$ (D) none of these

57. Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line drawn from the point P intersects the curve at points Q and R. If the product $PQ. PR$ is independent of the slope of the line, then the curve is a

(A) parabola (B) circle
(C) ellipse (D) none of these

58. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. The angle between the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

59. The maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis is

(A) $\sqrt{3}ab$ (B) $\frac{3\sqrt{3}}{4}ab$

(C) $\frac{5\sqrt{3}}{4}ab$ (D) none of these

60. The tangent at the point 'α' on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the auxiliary circle in two points which subtend a right angle at the centre. The eccentricity of the ellipse is

(A) $\frac{1}{\sqrt{1 + \sin^2\alpha}}$ (B) $\frac{1}{\sqrt{1 + \cos^2\alpha}}$

(C) $\sqrt{1 + \sin^2\alpha}$ (D) none of these

61. If a chord joining two points whose eccentric angles are α, β cut the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at a distance d from the centre, then $\tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} =$

(A) $\frac{d + a}{d - a}$ (B) $\frac{d - a}{d + a}$

(C) $\frac{a - d}{a + d}$ (D) none of these

62. The orbit of the earth is an ellipse with eccentricity $\frac{1}{60}$ with the sum at one focus, the major axis being approximately 186×10^6 miles in length. The shortest and longest distance of the earth from the sun is

(A) 9145×10^4 miles, 9455×10^4 miles
(B) 9147×10^4 miles, 9457×10^4 miles
(C) 9145×10^6 miles, 9455×10^6 miles
(D) none of these

63. PN is the ordinate of any point P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and AA' is its transverse axis. If Q divides AP in the ratio $a^2 : b^2$, then NQ is

(A) $\perp$ to $A'P$ (B) parallel to $A'P$
(C) $\perp$ to OP (D) none of these

64. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2},1\right)$. Its one directrix is the common tan gent, nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse in the standard form is

(A) $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}}+\frac{(y-1)^2}{\frac{1}{12}}=1$

(B) $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{12}}+\frac{(y-1)^2}{\frac{1}{9}}=1$

(C) $\frac{(x-1)^2}{\frac{1}{9}}+\frac{\left(y-\frac{1}{3}\right)^2}{\frac{1}{12}}=1$

(D) none of these

65. A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. The locus of the point which divides the line segment between these two points in the ratio 1 : 2 is

(A) $16x^2 + 10xy + y^2 = 2$
(B) $16x^2 - 10xy + y^2 = 2$
(C) $16x^2 + 10xy + y^2 = 4$
(D) none of these

More than One Option Correct Type

66. If the parabola $x^2 = ay$ makes an intercept of length $\sqrt{40}$ on the line $y - 2x = 1$, then a is equal to

(A) 1 (B) –2
(C) –1 (D) 2

67. The asymptotes of the hyperbola $xy - 3x + 4y + 2 = 0$ are

(A) $x = -4$ (B) $x = 4$
(C) $y = -3$ (D) $y = 3$

68. If the line $ax + by + c = 0$ is a normal to the hyperbola $xy = 1$, then

(A) $a > 0, b < 0$ (B) $a > 0, b > 0$
(C) $a < 0, b < 0$ (D) $a < 0, b > 0$

79. Consider a circle with its centre lying on the focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then, a point of intersection of the circle and the parabola is

(A) $\left(\frac{p}{2}, p\right)$ (B) $\left(\frac{p}{2}, -p\right)$

(C) $\left(-\frac{p}{2}, p\right)$ (D) $\left(-\frac{p}{2}, -\frac{p}{2}\right)$

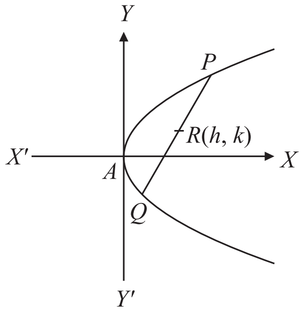
70. P is a point which moves in the xy plane such that the point P is nearer to the centre of a square than any of the sides. The four vertices of the square are $(\pm a, \pm a)$. The region in which P will move is bounded by parts of parabola of which one has the equation

(A) $y^2 = a^2 - 2ax$ (B) $y^2 = a^2 + 2ax$
(C) $x^2 = a^2 - 2ay$ (D) $x^2 = a^2 + 2ay$

Passage Based Questions

Passage 1

Let $R(h, k)$ be the middle point of the chord PQ of the parabola $y^2 = 4ax$.



Equation of PQ is

$$(y-k) = m(x-h) \quad (1)$$

where m is the slope of PQ

$\because$ R lies on the diameter $y = \dfrac{2a}{m}$ bisecting PQ,

$\therefore$ $k = \dfrac{2a}{m} \Leftrightarrow m = \dfrac{2a}{k}$

Subsituting this value of m in (1), we have $y - k = \dfrac{2a}{k}(x-h)$

or, $k(y-k) = 2a(x-h)$ or $ky - 2ax + (2ah - k^2) = 0$ which is the required equation.

The locus of the middle point of chords of the parabola which

71. subtend a constant angle α at the vertex is $(y^2 - 2ax + 8a^2)^2\tan^2\alpha = ka^2(4ax - y^2)$, where $k =$

(A) 4 (B) 8
(C) 16 (D) none of these

72. passes through the focus is

(A) $y^2 = a(x-a)$ (B) $y^2 = 2a(x-a)$
(C) $y^2 = 4a(x-a)$ (D) none of these

73. are such that the focal distances of their extremities are in the ratio 2 : 1, is $9(y^2 - 2ax)^2 = ka^2(2x-a)(4x+a)$, where $k =$

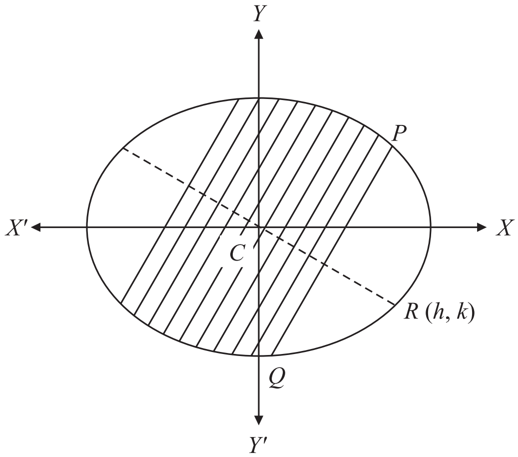
(A) 4 (B) 8
(C) 16 (D) none of these

74. are such that the normals at their extremities meet on the parabola is

(A) $y^2 = a(x+2a)$ (B) $y^2 = a(x-2a)$
(C) $y^2 = 2a(x-2a)$ (D) $y^2 = 2a(x+2a)$

Passage 2

Let $R(h, k)$ be the middle point of one of the chords, say PQ of the system of parallel chords of the ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let m be the slope of these parallel chords.

The locus of the middle point (h, k) is $y = -\dfrac{b^2}{a^2 m}x$, which is called a diameter of the ellipse. Two diameters are said to be conjugate if each bisects all chords parallel to the other. The condition that two diameters $y = m_1x$ and $y = m_2x$ may be conjugate with respect to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $m_1m_2 = -\dfrac{b^2}{a^2}$. Note that the eccentric angles of the extremities of two conjugate semi-diameters differ by a right angle. If CP and CD be any two conjugate semi-diameters of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then

75. the tangents at P and D intersect on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = k$, where $k =$

(A) 1 (B) 2
(C) 4 (D) 16

Assertion-Reason Type

Instructions *In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:*

(A) Assertion(A) is True and Reason(R) is True; Reason(R) is a correct explanation for Assertion(A)
(B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)
(C) Assertion(A) is True, Reason(R) is False
(D) Assertion(A) is False, Reason(R) is True

76. **Assertion:** If PSQ is the focal chord of the parabola $y^2 = 8x$ such that $SP = 6$, then the length SQ is

Reason: The semi-latus rectum of a parabola is the harmonic mean between the segment of any focal chord of a parabola.

77. **Assertion:** The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

Reason: The equation of a hyperbola and its asymptotes differ in constant terms only.

78. **Assertion:** The locus of the centre of the circle described on any focal chord of a parabola $y^2 = 4ax$ as diameter is $y^2 = 2a(x - a)$

Reason: If $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ be the extremities of a focal chord for the parabola $y^2 = 4ax$, then $t_1t_2 = -1$

79. **Assertion:** The angle of intersection between the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2 + y^2 = ab$ is $\tan^{-1}\frac{(b-a)}{\sqrt{ab}}$.

Reason: The point of intersection of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2 + y^2 = ab$ is $\left(\sqrt{\frac{a^2b}{a+b}}, \sqrt{\frac{ab^2}{a+b}}\right)$

Previous Year's Questions

80. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and having its centre at (0, 3), is: **[2002]**

(A) 4 unit (B) 3 unit
(C) $\sqrt{12}$ unit (D) $\frac{7}{2}$ unit

81. The equation of the ellipse whose foci are (±2, 0) and eccentricity is $\frac{1}{2}$ is: **[2002]**

(A) $\frac{x^2}{12}+\frac{y^2}{16}=1$ (B) $\frac{x^2}{16}+\frac{y^2}{12}=1$
(C) $\frac{x^2}{16}+\frac{y^2}{8}=1$ (D) none of these

82. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is : **[2002]**

(A) $\frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$
(B) $\frac{x}{x_1-x_2}+\frac{y}{y_1-y_2}=1$
(C) $\frac{x}{y_1+y_2}+\frac{y}{x_1+x_2}=1$
(D) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$

83. If x_1, x_2, x_3 and y_1, y_2, y_3 are both in G.P. with the same common ratio, then the points (x_1, y_1) (x_2, y_2) and (x_3, y_3) **[2003]**

(A) lie on a straight line
(B) lie on an ellipse
(C) lie on a circle
(D) are vertices of a triangle

84. The foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide. Then the value of b^2 is **[2003]**

(A) 1 (B) 5
(C) 7 (D) 9

85. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is **[2004]**

(A) (2, 4) (B) (2, –4)
(C) $\left(\frac{-9}{8},\frac{9}{2}\right)$ (D) $\left(\frac{9}{8},\frac{9}{2}\right)$

86. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then **[2004]**

(A) $d^2 + (2b + 3c)^2 = 0$ (B) $d^2 + (3b + 2c)^2 = 0$
(C) $d^2 + (2b - 3c)^2 = 0$ (D) $d^2 + (3b - 2c)^2 = 0$

87. The eccentricity of an ellipse, with its centre at the origin, is $\frac{1}{2}$. If one of the directrices is $x = 4$, then the 2 equation of the ellipse is **[2004]**

(A) $3x^2 + 4y^2 = 1$ (B) $3x^2 + 4y^2 = 12$
(C) $4x^2 + 3y^2 = 12$ (D) $4x^2 + 3y^2 = 1$

88. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is **[2005]**

(A) $2ab$ (B) ab
(C) $\sqrt{ab}$ (D) $\frac{a}{b}$

89. The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is **[2005]**

(A) an ellipse (B) a circle
(C) a parabola (D) a hyperbola

90. An ellipse has OB as semi minor axis, F and F' its focii and the angle FBF' is a right angle. Then the eccentricity of the ellipse is **[2005]**

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$
(C) $\frac{1}{4}$ (D) $\frac{1}{\sqrt{3}}$

91. In an ellipse, the distance between its foci is 6 and minor axis is 8. Then its eccentricity is **[2006]**

(A) $\frac{3}{5}$ (B) $\frac{1}{2}$
(C) $\frac{4}{5}$ (D) $\frac{1}{\sqrt{5}}$

92. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant when α. varies? **[2007]**

(A) eccentricity (B) directrix
(C) abscissae of vertices (D) abscissae of foci

93. A focus of an ellipse is at the origin. The directrix is the line $x = 4$ and the eccentricity is $\frac{1}{2}$. Then the length of the semi-major axis is **[2008]**

(A) $\frac{8}{3}$ (B) $\frac{2}{3}$
(C) $\frac{4}{3}$ (D) $\frac{5}{3}$

94. A parabola has the origin as its focus and the line $x = 2$ as the directrix. Then the vertex of the parabola is at **[2008]**

(A) (0, 2) (B) (1, 0)
(C) (0, 1) (D) (2, 0)

95. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn in inscribed in another ellipse that passes through the point (4,0). Then the equation of the ellipse is [2009]

(A) $x^2 + 16y^2 = 16$ (B) $x^2 + 12y^2 = 16$
(C) $4x^2 + 48y^2 = 48$ (D) $4x^2 + 64y^2 = 48$

96. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of the point P is **[2010]**

(A) $2x + 1 = 0$ (B) $x = -1$
(C) $2x - 1 = 0$ (D) $x = 1$

97. **Statement 1:** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse **[2012]**

$2x^2 + y^2 = 4y = 2x + 2\sqrt{3}$

Statement 2: If the line $y = mx + \frac{4\sqrt{3}}{m}, (m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$.

(A) Statement 1 is false, statement 2 is true
(B) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1
(C) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1
(D) Statement 1 is true, statement 2 is false

98. An ellipse is drawn by considering a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y - 2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is the origin and its axes are the coordinate axes, then the equation of the ellipse is **[2012]**

(A) $4x^2 + y^2 = 4$ (B) $x^2 + 4y^2 = 8$
(C) $4x^2 + y^2 = 8$ (D) $x^2 + 4y^2 = 16$

99. Given: A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}x$. [2013]

Statement - I: An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement - II: If the line, $y = mx + \frac{\sqrt{5}}{m} (m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$.

(A) Statement - I is True; Statement -II is true; Statement-II is not a correct explanation for Statement-I
(B) Statement -I is True; Statement -II is False.
(C) Statement -I is False; Statement -II is True
(D) Statement -I is True; Statement -II is True; Statement-II is a correct explanation for Statement-I

100. The circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ with center at (0, 3) has equation **[2013]**

(A) $x^2 + y^2 - 6y + 7 = 0$ (B) $x^2 + y^2 - 6y - 5 = 0$
(C) $x^2 + y^2 - 6y + 5 = 0$ (D) $x^2 + y^2 - 6y - 7 = 0$

101. The locus of the foot of the perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is **[2014]**

(A) $(x^2 - y^2)^2 = 6x^2 + 2y^2$
(B) $(x^2 - y^2)^2 = 6x^2 - 2y^2$
(C) $(x^2 + y^2)^2 = 6x^2 + 2y^2$
(D) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

102. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is **[2014]**

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$
(C) $\frac{1}{8}$ (D) $\frac{2}{3}$

103. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is: **[2015]**

(A) $y^2 = x$ (B) $y^2 = 2x$
(C) $x^2 = 2y$ (D) $x^2 = y$

104. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, is: **[2015]**

(A) 18 (B) $\frac{27}{2}$
(C) 27 (D) $\frac{27}{4}$

105. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is **[2016]**

(A) $\sqrt{3}$ (B) $\frac{4}{3}$
(C) $\frac{4}{\sqrt{3}}$ (D) $\frac{2}{\sqrt{3}}$

106. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the cente C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is **[2016]**

(A) $x^2 + y^2 - 4x + 9y + 18 = 0$
(B) $x^2 + y^2 - 4x + 8y + 12 = 0$
(C) $x^2 + y^2 - x + 4y + 12 = 0$
(D) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

107. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on **[2016]**

(A) A parabola
(B) A circle
(C) An ellipse which is not a circle
(D) A hyperbola

ANSWER KEYS

Single Option Correct Type

1. (B) **2.** (A) **3.** (B) **4.** (A) **5.** (B) **6.** (B) **7.** (A) **8.** (A) **9.** (C) **10.** (C)
11. (C) **12.** (B) **13.** (D) **14.** (D) **15.** (C) **16.** (C) **17.** (C) **18.** (D) **19.** (D) **20.** (B)
21. (A) **22.** (D) **23.** (A) **24.** (B) **25.** (A) **26.** (A) **27.** (C) **28.** (C) **29.** (C) **30.** (B)
31. (B) **32.** (B) **33.** (A) **34.** (C) **35.** (B) **36.** (C) **37.** (A) **38.** (A) **39.** (A) **40.** (C)
41. (B) **42.** (B) **43.** (A) **44.** (B) **45.** (D) **46.** (A) **47.** (C) **48.** (C) **49.** (A) **50.** (B)
51. (A) **52.** (B) **53.** (B) **54.** (B) **55.** (B) **56.** (B) **57.** (B) **58.** (A) **59.** (B) **60.** (A)
61. (B) **62.** (A) **63.** (A) **64.** (A) **65.** (A)

More than One Option Correct Type

66. (A, B) **67.** (A, D) **68.** (A, D) **69.** (A, B) **70.** (A, B, C, D)

Passage Based Questions

71. (C) **72.** (B) **73.** (A) **74.** (D) **75.** (B)

Assertion-Reason Type

76. (A) **77.** (A) **78.** (A) **79.** (A)

Previous Year's Questions

80. (A) **81.** (B) **82.** (A) **83.** (A) **84.** (C) **85.** (D) **86.** (A) **87.** (B) **88.** (A) **89.** (D)
90. (A) **91.** (A) **92.** (A) **93.** (B) **94.** (B) **95.** (B) **96.** (B) **97.** (D) **98.** (A) **99.** (D)
100. (C) **101.** (A) **102.** (C) **103.** (C) **104.** (B) **105.** (D) **106.** (B) **107.** (A)

HINTS AND SOLUTIONS

Single Option Correct Type

1. The general equation of a parabola having its axis parallel to y-axis is

$y = ax^2 + bx + c$ (1)

Since the line $y = x$ touches the parabola at $x = 1$, therefore slope of the tangent at (1, 1) = 1

i.e., $\left(\frac{dy}{dx}\right)_{(1,1)} = 1$ or $2a + b = 1$.

Also, $x = ax^2 + bx + c$ must have equal roots

$\Rightarrow$ $(b - 1)^2 = 4ac$.

Since, (1, 1) lies on (1),

$\therefore$ $a + b + c = 1$.

Now, $2a + b = 1$ and $a + b + c = 1 \Rightarrow a - c = 0$ or $a = c$.

$\therefore$ $a + b + c = 1 \Rightarrow 2c + b = 1$

$\Rightarrow$ $2f(0) + f'(0) = 1$ ($\because f(0) = c$ and $f'(0) = b$)

2. The equation of axis of the parabola is $x - 4 = 0$ which is parallel to y-axis. So, the ray of light is parallel to the axis of the parabola. We know that any ray parallel to the axis of a parabola passes through the focus after reflection.

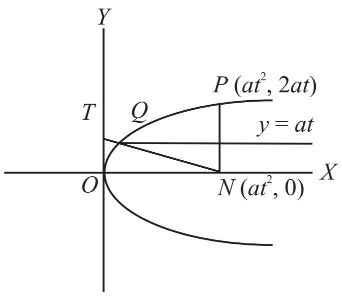
$\therefore$ The ray must pass through the point (4, –1).

3. Clearly, the two tangents, having slopes m_1 and m_2, meet on the line $x = -2$, which is the directrix of the parabola $y^2 = 8x$, therefore the two tangents must be at right angles, i.e., $m_1m_2 = -1$.

4. Equation of the line parallel to the axis and bisecting the ordinate PN of the point $P(at^2, 2at)$ is $y = at$ which meets the parabola $y^2 = 4ax$ at the point $Q\left(\frac{1}{4}at^2, at\right)$.

Coordinates of N are $(at^2, 0)$.

Equation of NQ is $y = \dfrac{0 - at}{at^2 - \frac{1}{4}at^2}(x - at^2)$,



which meets the tangent at the vertex, $x = 0$, at the point

$$y = \frac{4}{3}at.$$

5. The equation of the parabola is

$$\left(x + \frac{3}{2}\right)^2 + (y + 3)^2 = \left(\frac{2x + 5}{2}\right)^2$$

$$\Rightarrow\ 4\left[x^2 + \frac{9}{4} + 3x\right] + 4[y^2 + 9 + 6y] = (4x^2 + 25 + 20x)$$

$\Rightarrow$ $(y^2 + 6y + 9) - 2(x + 2) = 0$

or, $(y + 3)^2 = 2(x + 2)$.

Clearly, $x = 2t^2 - 2$ and $y = 2t - 3$ satisfy it for all t.

6. Directirx of $y^2 = 4(x + 1)$ is $x = -2$

Any point on $x = -2$ is $(-2, k)$

Now, mirror image (x, y) of $(-2, k)$ in the line $x + 2y = 3$ is given by

$$\frac{x + 2}{1} = \frac{y - k}{2} = -2\left(\frac{-2 + 2k - 3}{5}\right)$$

$$\Rightarrow\ x = \frac{10 - 4k}{5} - 2\ \Rightarrow\ x = -\frac{4k}{5} \quad (1)$$

Also, $$y = \frac{20 - 8k}{5} \quad (2)$$

From (1) and (2)

$$y = 4 + \frac{3}{5}\left(\frac{5x}{4}\right)$$

or $4y = 16 + 3x$ is the equation of the mirror image of the directrix.

7. Coordinates of any point on the parabola $y^2 = -4ax$ are $(-at^2, 2at)$.

Equation of the normal at $(-at^2, 2at)$ is

$y - xt = 2at + at^3$

If the normal passes through the point (h, k), then

$k - th = 2at + at^3$

or, $at^3 + (2a + h)t - k = 0$,

which is a cubic equation whose three roots t_1, t_2, t_3 are the parameters of the feet of the three normals.

$\therefore$ Sum of the roots = $t_1 + t_2 + t_3 = -\dfrac{\text{Coefficient of } t^2}{\text{Coefficient of } t^3} = 0$

$\therefore$ Centroid of the triangle formed by the feet of the normals

$$= \left(-\frac{a}{3}(t_1^2 + t_2^2 + t_3^2), \frac{2a}{3}(t_1 + t_2 + t_3)\right)$$

$$= \left(-\frac{a}{3}(t_1^2 + t_2^2 + t_3^2), 0\right)$$

which, clearly, lies on the x-axis.

8. The coordinates of any point on the parabola $x^2 = 4by$ are $(2bt, bt^2)$.

For the parabola $x^2 = 4by$, $\dfrac{dy}{dx} = \dfrac{x}{2b}$.

Slope of the normal at $(2bt, bt^2) = -\dfrac{2b}{2bt} = -\dfrac{1}{t}$

$\therefore$ Equation of normal is $y - bt^2 = -\frac{1}{t}(x - 2bt)$

or, $y = -\frac{x}{t} + 2b + bt^2$

It will touch the parabola $y^2 = 4ax$ if

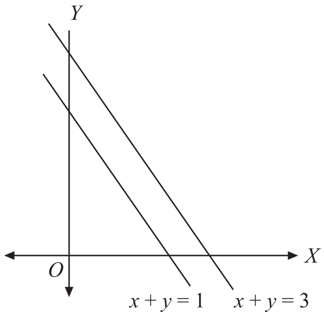
$2b + bt^2 = \frac{a}{-1/t}$ $\left(\because \ c = \frac{a}{m}\right)$

$\Rightarrow$ $bt^2 + at + 2b = 0$

For distinct real roots, discriminant > 0

$\Rightarrow$ $a^2 - 8b^2 = 0$ or $a^2 > 8b^2$

9. The coordinates of the focus of the given parabola are $(\alpha + 1, \beta)$.



Clearly, focus must lie to the opposite side of the origin w.r.t. the line $x + y - 1 = 0$ and same side as origin with respect to the line $x + y - 3 = 0$. Hence, $a + b > 0$ and $a + b < 2$.

10. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let e be the eccentricity of the ellipse.

Since distance between foci = $2h$

$\therefore$ $2ae = 2h \Rightarrow ae = h$ (1)

Focal distance of one end of minor axis say $(0, b)$ is k.

$\therefore$ $a + e(0) = k \Rightarrow a = k$ (2)

From (1) and (2)

$b^2 = a^2(1 - e^2) = k^2 - h^2$.

$\therefore$ The equation of the ellipse is

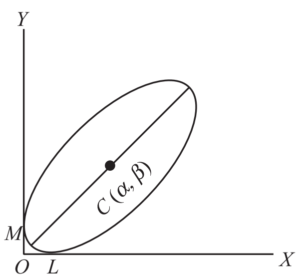
$\frac{x^2}{k^2} + \frac{y^2}{k^2 - h^2} = 1$.

11. Let the two given lines be taken as the coordinate axes.

Let $C(a, b)$ be the centre of the ellipse in any position. Here the position of centre C changes as the ellipse slides. Let a and b be the semi-major and semi-minor axes of the ellipse.

Equation of the director circle of the ellipse is

$(x - a)^2 + (y - b)^2 = a^2 + b^2$ (1)



Since OX and OY are mutually perpendicular tangents to sliding ellipse for all its positions, therefore, $O(0, 0)$ will lie on circle (1)

$\therefore$ $\alpha^2 + \beta^2 = a^2 + b^2$.

Hence, locus of $C(a, b)$ is $x^2 + y^2 = a^2 + b^2$.

12. $P(a\cos\theta, b\sin\theta)$, then θ is angle of a corresponding pont on auxilliary circle $x^2 + y^2 = a^2$ i.e., $(a\cos\theta, a\sin\theta)$.

13. Let $S(5, 12)$, $S'(24, 7)$ be the two foci.

$P(0, 0)$ is a point on the conic.

$SP = \sqrt{25 + 144} = 13$

$S'P = \sqrt{576 + 46} = \sqrt{625} = 25$

$SS' = \sqrt{(24 - 5)^2 + (7 - 12)^2}$

$= \sqrt{19^2 + 5^2} = \sqrt{386}$

If the conic is an ellipse,

then, $SP + S'P = 2a$ and $SS' = 2ae$

$\therefore$ $e = \frac{SS'}{SP + S'P} = \frac{\sqrt{386}}{13 + 25} = \frac{\sqrt{386}}{38}$

14. Given ellipses are $x^2 + 4y^2 = 4$

i.e., $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$ (1)

and, $x^2 + 2y^2 = 6$ i.e., $\frac{x^2}{(\sqrt{6})^2} + \frac{y^2}{(\sqrt{3})^2} = 1$ (2)

Let $R(\alpha, \beta)$ be the point of intersection of the tangents to ellipse (2) at P and Q. then PQ will be chord of contact of R.

$\therefore$ its equation is

$\frac{\alpha x}{6} + \frac{\beta y}{3} = 1$

i.e., $\alpha x + 2y\beta = 6$

or, $y = -\frac{\alpha}{2\beta}x + \frac{3}{\beta}$ (3)

Since (3) touches (1)

$\therefore$ $\left(\frac{3}{\beta}\right)^2 = 2^2 \cdot \frac{\alpha^2}{4\beta^2} + 1^2$ $[c^2 = a^2m^2 + b^2]$

$\Rightarrow$ $\frac{9}{\beta^2} = \frac{\alpha^2}{\beta^2} + 1 = \frac{\alpha^2 + \beta^2}{\beta^2}$

$\Rightarrow$ $\alpha^2 + \beta^2 = 9$

$\therefore$ locus of (α, β) is

$x^2 + y^2 = 9 = (\sqrt{6})^2 + (\sqrt{3})^2$

i.e., director circle.

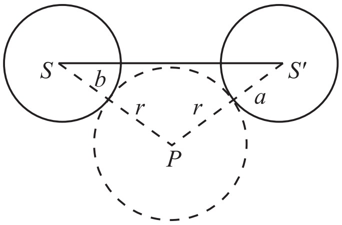
$\therefore$ tangent at P, Q meet at right angles.

15. Let S and S' be the centres and a and b be the radii of the given circles.

Let P be the centre and r be the radius of the circle which touches the given circles externally.

Then, $S'P = r + a$ and $SP = r + b$

$\therefore$ $S'P - SP = (r + a) - (r + b) = a - b$ = constant.



Hence, the locus of P is a hyperbola whose foci are S and S'.

16. Let the two given ⊥ lines be the coordinate axes and let the equation of variable circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

Then, $5 = 2\sqrt{g^2 - c}$ and $3 = 2\sqrt{f^2 - c}$.

Squaring and subtracting these, we get

$4(g^2 - c) - 4(f^2 - c) = 25 - 9$

$\Rightarrow g^2 - f^2 = 4$ or $(-g)^2 - (-f)^2 = 4$.

Hence, locus of the centre $(-g, -f)$ of circle is

$x^2 - y^2 = 4$,

which is a rectangular hyperbola.

17. We have, $2x^2 + 3y^2 - 8x - 18y + 35 = k$

$\Rightarrow 2(x^2 - 4x) + 3(y^2 - 6y) + 35 = k$

$\Rightarrow 2[(x - 2)^2 - 4] + 3[(y - 3)^2 - 9] + 35 = k$

$\Rightarrow 2(x - 2)^2 + 3(y - 3)^2 = k$

For $k = 0$, we get $2(x - 2)^2 + 3(y - 3)^2 = 0$ which represents the point (2, 3).

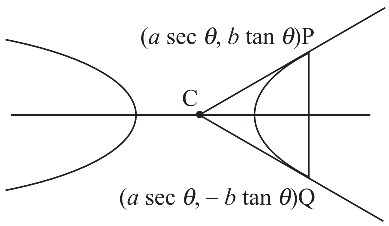
18. Let $P(a\sec\theta, b\tan\theta)$; $Q(a\sec\theta, -b\tan\theta)$ be end points of double ordinate and $C(0, 0)$, is the centre of the hyperbola. Now, $PQ = 2b\tan\theta$

$CQ = CP = \sqrt{a^2\sec^2\theta + b^2\tan^2\theta}$

since $CQ = CP = PQ$

$\therefore \quad 4b^2\tan^2\theta = a^2\sec^2\theta + b^2\tan^2\theta$

$\Rightarrow 3b^2\tan^2\theta = a^2\sec^2\theta$



$\Rightarrow 3b^2\sin^2\theta = a^2$

$\Rightarrow 3a^2(e^2 - 1)\sin^2\theta = a^2$

$\Rightarrow 3(e^2 - 1)\sin^2\theta = 1$

$\Rightarrow \dfrac{1}{3(e^2 - 1)} = \sin^2\theta < 1 \qquad (\because \sin^2\theta < 1)$

$\Rightarrow \dfrac{1}{e^2 - 1} < 3 \Rightarrow e^2 - 1 > \dfrac{1}{3}$

19. Given hyperbola: $\dfrac{x^2}{3} - \dfrac{y^2}{7} = 1$ (1)

Given chord: $7x + y - 2 = 0$ (2)

Its slope $= -7$

Let $P(\alpha, \beta)$ be the mid point of any of the parallel chords whose slope is -7.

$\therefore$ equation of the chord in terms of mid-point $P(\alpha, \beta)$ is $T = S_1$

i.e., $\dfrac{x\alpha}{3} - \dfrac{y\beta}{7} - 1 = \dfrac{\alpha^2}{3} - \dfrac{\beta^2}{7} - 1$

i.e., $\dfrac{\alpha x}{3} - \dfrac{\beta y}{7} = \dfrac{\alpha^2}{3} - \dfrac{\beta^2}{7}$

Its slope $= \dfrac{7}{3}\dfrac{\alpha}{\beta} = -7 \Rightarrow a + 3b = 0$

$\therefore$ locus of (a, b) is $x + 3y = 0$.

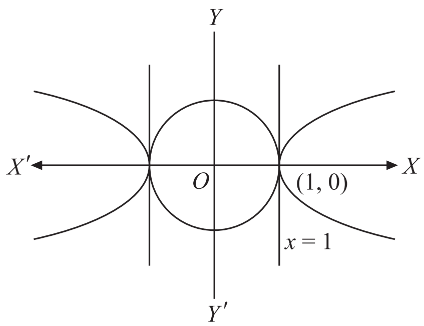
20. Clearly, the common tangent to the circle $x^2 + y^2 = 1$ and hyperbola $x^2 - y^2 = 1$ is $x = 1$ [which is nearer to $P(1/2, 1)$].

Given, one focus at $P\left(\dfrac{1}{2}, 1\right)$.

$\therefore$ equation of the directrix is $x = 1$.

$\therefore$ ellipse is

$$\sqrt{\left(x - \frac{1}{2}\right)^2 + (y - 1)^2} = \frac{1}{2}(x - 1)$$



On simplification, it becomes

$$9\left(x - \frac{1}{3}\right)^2 + 12(y - 1)^2 = 1.$$

21. Since the line CP passes through the origin, i.e., centre, let its equation be $y = mx$. The line CP meets the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ in P whose abscissa is given by

$\dfrac{x^2}{a^2} - \dfrac{m^2x^2}{b^2} = 1$ or $x^2 = \dfrac{a^2b^2}{b^2 - a^2m^2}$

$\therefore \quad y^2 = m^2x^2 = \dfrac{a^2b^2m^2}{b^2 - a^2m^2}$

$\therefore \quad CP^2 = x^2 + y^2$

$= \dfrac{a^2b^2 + a^2b^2m^2}{b^2 - a^2m^2}$

$= a^2b^2(1 + m^2)$

Since $CQ \perp CP$

Replace m by $-\dfrac{1}{m}$, we get

$$CQ^2 = \frac{a^2b^2\left(1+\frac{1}{m^2}\right)}{b^2 - \frac{c^2}{m^2}} = \frac{a^2b^2(m^2+1)}{b^2m^2 - a^2}$$

$$\therefore \quad \frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{b^2 - a^2m^2 + b^2m^2 - a^2}{a^2b^2(1+m^2)}$$

$$= \frac{b^2 - a^2}{a^2b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

22. Given, $P(a \sec\theta, b \tan\theta)$ and $Q(a \sec\phi, b \tan\phi)$.

The equation of tangent at point P is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

Slope of tangent $= \frac{b}{\tan\theta} \times \frac{\sec\theta}{a} = \frac{b}{a} \cdot \frac{1}{\sin\theta}$

Hence, the equation of perpendicular at P is

$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

or, $by - b^2 \tan\theta = -a \sin\theta\, x + a^2 \tan\theta$

or, $a \sin\theta\, x + by = (a^2 + b^2) \tan\theta$ (1)

Similarly, the equation of perpendicular at Q is

$a \sin\phi\, x + by = (a^2 + b^2) \tan\phi$ (2)

On multiplying (1) by $\sin\phi$ and (2) by $\sin\theta$, we get

$a \sin\theta \sin\phi\, x + b \sin\phi\, y = (a^2 + b^2) \tan\theta \sin\phi$

$a \sin\phi \sin\theta\, x + b \sin\theta\, y = (a^2 + b^2) \tan\phi \sin\theta$

On subtraction we get

$by (\sin\phi - \sin\theta) = (a^2 + b^2)(\tan\theta \sin\phi - \tan\phi \sin\theta)$

$$\therefore \quad y = k = \frac{a^2+b^2}{b} \cdot \frac{\tan\theta\sin\phi - \tan\phi\sin\theta}{\sin\phi - \sin\theta}$$

$$\because \quad \theta + \phi = \frac{\pi}{2} \quad \Rightarrow \quad \phi = \frac{\pi}{2} - \theta$$

$\Rightarrow$ $\sin\phi = \cos\theta$ and $\tan\phi = \cot\theta$

$$\therefore \quad y = k = \frac{a^2+b^2}{b} \cdot \frac{\tan\theta\cos\theta - \cos\theta\sin\theta}{\cos\theta - \sin\theta}$$

$$= \frac{a^2+b^2}{b}\left(\frac{\sin\theta - \cos\theta}{\cos\theta - \sin\theta}\right) = -\frac{(a^2+b^2)}{b}$$

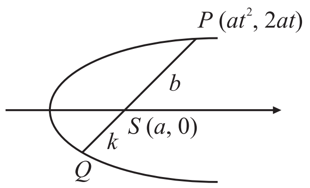
23. If coordinates of one end of focal chord are $P(at^2, 2at)$ then the coordinates of other end will be $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

$$\therefore \quad SP = \sqrt{(at^2 - a)^2 + (2at - 0)^2}$$

$= a(t^2 + 1) = b$ (given) (1)

and, $SQ = \sqrt{\left(\frac{a}{t^2} - a\right)^2 + \left(\frac{-2a}{t} - 0\right)^2}$

$= \sqrt{a\left(\frac{1}{t^2} + 1\right)} = k$ (given) (2)



Dividing (1) by (2), we get $t^2 = \frac{b}{k}$.

Putting in (1), $\frac{b}{k} + 1 = \frac{b}{a} \Rightarrow k = \frac{ab}{b-a}$.

24. Let $S \equiv x^2 - 4y$

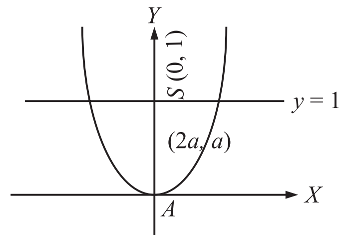
Since the point $(2a, a)$ lies inside the parabola,

$\therefore$ $S]_{(2a,\, a)} = 4a^2 - 4a < 0$

i.e., $4a(a - 1) < 0$

or, $a(a - 1) < 0$ (1)

Also, the vertex $A(0, 0)$ and the point $(2a, a)$ are on the same side of the line $y = 1$ (the equation of latus-rectum)



So, $a - 1 < 0$ i.e., $a < 1$ (2)

From (1) and (2), we have $a(a - 1) < 0$

or, $0 < a < 1$.

25. We know any side of the triangle is more than the difference of remaining two sides, such that $|PR - PQ| \le RQ$.

$\Rightarrow$ The required piont P will be the point of intersection of the line RQ with parabola which is **$(a, 2a)$** as RQ is a tangent to the parabola.

26. Normal at a point $(m^2, -2m)$ on the parabola $y^2 = 4x$ is given by $y = mx - 2m - m^3$. If this is normal to the circle also, then it will pass through centre of the circle so

$6 = -3m - 2m - m^3 \Rightarrow m = -1$

Since shortest distance between parabola and circle will occur along common normal, shortest distance is $4\sqrt{2} - 5$.

27. Tangent at P is $ty = x + at^2$, which meets axis at $T(-at^2, 0)$. Normal at P is $tx + y = 2at + at^3$, which meets axis at $G(2a + at^2, 0)$

$\therefore$ $\angle TPG = \frac{\pi}{2}$, so TG is diameter of the circle.

Equation of the circle is

$(x + at^2)(x - 2a - at^2) + (y - 0)(y - 0) = 0$

$\Rightarrow$ $x^2 + y^2 - 2ax - at^2(2a + at^2) = 0$

Tangent to above circle at $P(at^2, 2at)$ is

$xat^2 + y \,.\, 2at - a(x + at^2) - a^2t^2(1 + t^2) = 0$

or, $(t^2 - 1)x + 2ty - a(2 + t^2) = 0$. It has slope $= \frac{1-t^2}{2t}$

$\therefore$ Angle θ between two tangents is given by

$$\tan\theta = \frac{\frac{1}{t} - \frac{1-t^2}{2t}}{1 + \frac{1}{t} \cdot \frac{1-t^2}{2t}} = \frac{t(1+t^2)}{t^2+1} \Rightarrow \theta = \tan^{-1}(t)$$

28. Equation of normal is $y = mx - 2am - am^3$

Put $y = 0$, we get

$x_1 = 2a + am_1^2$

$x_2 = 2a + am_2^2$

and, $x_3 = 2a + am_3^2$

where x_1, x_2, x_3 are intercepts on the axis of the parabola. The normal passes through (h, k).

$\therefore \quad am^3 + (2a - h)m + k = 0$

Now, $\quad m_1 + m_2 + m_3 = 0$

$m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a - h}{a}$

$\Rightarrow \quad m_1^2 + m_2^2 + m_3^2 = (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1)$

$= -2\dfrac{(2a - h)}{a}$

$\Rightarrow \quad x_1 + x_2 + x_3 = 6a - 2(2a - h) = 2(h + a)$

29. The focus of the parabola $y^2 = 8ax$ is $(2a, 0)$.

So, the coordinates of the point on the axis of the parabola at a distance $8a$ from the focus is $(10a, 0)$. Equation of a normal to the parabola $y^2 = 8ax$ is

$y = mx - 4am - 2am^3$

Since it passes through $(10a, 0)$,

$\therefore \quad 0 = 10am - 4am - 2am^3$

$\Rightarrow \quad 2am(3 - m^2) = 0 \Rightarrow m^2 = 3 \qquad (\because m \neq 0)$

$\Rightarrow \quad m = \pm\sqrt{3} = \tan\left(\pm\dfrac{\pi}{3}\right)$

30. Let the three points A, B and C on the parabola are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, respectively.

Also, $y_2^2 = y_1y_3 \qquad$ {given}

If T be the point of intersection of tangents at y_1 and y_3 then T is

$T\left(\dfrac{y_1y_3}{4a}, \dfrac{y_1 + y_3}{2}\right) = T\left(\dfrac{y_2^2}{4a}, \dfrac{y_1 + y_3}{2}\right)$

or, $\quad T\left(x_2, \dfrac{y_1 + y_3}{2}\right) \qquad (\because \; y_2^2 = 4ax_2)$

This point lies on $x = x_2$ which is a line through $B(x_2, y_2)$ parallel to y-axis.

31. $y^2 = 4ax$, Normal: $y = mx - 2am - am^3 \qquad (1)$

$y^2 = 4c(x - b)$, Normal: $y = m(x - b) - 2cm - cm^3 \qquad (2)$

Since the two parabolas have a common normal, therefore (1) and (2) must be identical

After comparing the coefficients we get

$m = \pm\sqrt{\dfrac{2(a - c) - b}{(c - a)}} \qquad \therefore \; -2 - \dfrac{b}{c - a} > 0 \;\Rightarrow\; \dfrac{b}{a - c} > 2$

TRICK(S) FOR PROBLEM SOLVING

Shortest distance between two curves occurred along the common normal

32. Normal to $y^2 = 4x$ at $(m^2, 2m)$ is $y + mx - 2m - m^3 = 0$

Normal to $y^2 = 2(x - 3)$ at $(1/2t^2 + 3, t)$ is

$y + t(x - 3) - t - \dfrac{1}{2}t^3 = 0$

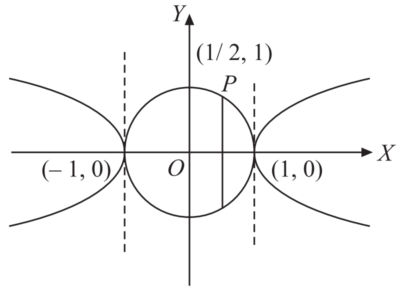
Both are same if $-2m - m^3 = -4m - 1/2m^3$

$\Rightarrow \quad m = 0, \pm 2$

So, points will be $(4, 4)$ and $(5, 2)$

Hence, shortest distance will be $= \sqrt{1 + 4} = \sqrt{5}$.

33. These are two common tangents to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. These are $x = 1$ and $x = -1$



Out of these, $x = 1$ is nearer to the point $P(1/2, 1)$. Thus, a directrix of the required ellipse is $x = 1$.

If $Q(x, y)$ is any pont on the ellipse, then its distance from the focus is $QP = \sqrt{\left(x - \dfrac{1}{2}\right)^2 + (y - 1)^2}$ and its distance from the directrix $x = 1$ is $|x - 1|$

By definition of ellipse, $QP = e|x - 1|$

$\Rightarrow \quad \sqrt{\left(x - \dfrac{1}{2}\right)^2 + (y - 1)^2} = \dfrac{1}{2}|x - 1|$

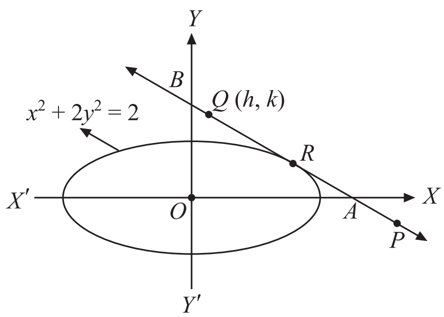
$\Rightarrow \quad 3x^2 - 2x + 4y^2 - 8y + 4 = 0$

or, $\quad \dfrac{\left(x - \dfrac{1}{3}\right)^2}{1/9} + \dfrac{(y - 1)^2}{1/12} = 1.$

34. Let the point of contact be

$R \equiv \left(\sqrt{2}\cos\theta, \sin\theta\right)$

Equation of tangent AB is



$\dfrac{x}{\sqrt{2}}\cos\theta + y\sin\theta = 1$

$\Rightarrow \quad A \equiv \left(\sqrt{2}\sec\theta, 0\right); B \equiv (0, \operatorname{cosec}\theta)$

Let the middle point Q of AB be (h, k)

$\Rightarrow \quad h = \dfrac{\sec\theta}{\sqrt{2}}, k = \dfrac{\operatorname{cosec}\theta}{2}$

$\Rightarrow \quad \cos\theta = \dfrac{1}{h\sqrt{2}}, \sin\theta = \dfrac{1}{2k}$

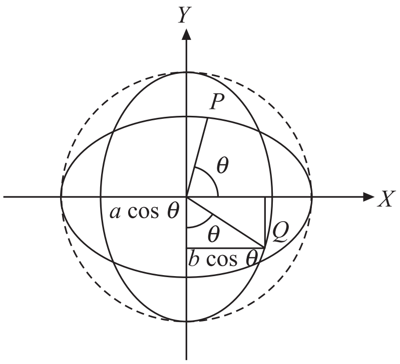
$\Rightarrow \quad \dfrac{1}{2h^2} + \dfrac{1}{4k^2} = 1$

$\therefore$ Required locus is $\dfrac{1}{2x^2} + \dfrac{1}{4y^2} = 1.$

Trick: The locus of mid-points of the portion of tangents to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ intercepted between axes is $a^2y^2 + b^2x^2 = 4x^2y^2$

i.e., $\dfrac{a^2}{4x^2} + \dfrac{b^2}{4y^2} = 1$ or $\dfrac{1}{2x^2} + \dfrac{1}{4y^2} = 1$.

35. Point P goes to Q. Its direction with respect to x-axis is q in original position. In new position y-axis will play the role of major axis so its inclination with negative direction of y-axis will be same. So, new coordinates will be $(b \sin\theta, -a \cos\theta)$.

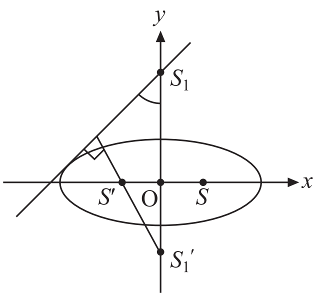


36. Given, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\therefore \quad OS = ae = a\sqrt{1 - \dfrac{b^2}{a^2}} = \sqrt{a^2 - b^2}$

So, two points on the minor axis are,

$S_1\left(0, \sqrt{a^2 - b^2}\right), S_2\left(0, -\sqrt{a^2 - b^2}\right)$



Let tangent to the ellipse by $y = mx + c = mx + \sqrt{a^2m^2 + b^2}$ where m is parameter. Now, sum of the squares of $\perp$'s on this tangent from the points S_1 and S'_1 is

$$\left(\frac{\sqrt{a^2 - b^2} - \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}\right)^2 + \left(-\frac{\sqrt{a^2 - b^2} - \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}\right)^2$$

$$= 2\left(\frac{a^2 - b^2 + a^2m^2 + b^2}{1 + m^2}\right) = \frac{2a^2(1 + m^2)}{1 + m^2} = 2a^2$$

37. Equation of the tangent at $\dfrac{\pi}{4}$ is

$$\frac{x\left(\frac{1}{\sqrt{2}}\right)}{a} + \frac{y\left(\frac{1}{\sqrt{2}}\right)}{b} = 1 \text{ i.e. } \frac{x}{a} + \frac{y}{b} - \sqrt{2} = 0 \qquad (1)$$

Equation of the normal at $\dfrac{\pi}{4}$ is

$$\frac{x}{b} - \frac{y}{a} = \frac{a}{b\sqrt{2}} - \frac{b}{a\sqrt{2}} \qquad (2)$$

p_1 = length of the perpendicular from the centre to the

$$\text{tangent} = \left|\frac{-\sqrt{2}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\right| = \frac{\sqrt{2}ab}{\sqrt{a^2 + b^2}}$$

p_2 = length of the perpendicular from the centre to the

$$\text{normal} = \frac{\frac{a}{b\sqrt{2}} - \frac{b}{a\sqrt{2}}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \left|\frac{a^2 - b^2}{\sqrt{2}\sqrt{a^2 + b^2}}\right|.$$

Area of the rectangle $= p_1 p_2 = \dfrac{ab(a^2 - b^2)}{a^2 + b^2}$.

38. If $S_1 = 0$ and $S_2 = 0$ are the equations,

then, $\lambda S_1 + S_2 = 0$ is a second degree curve passing through the points of intersection of $S_1 = 0$ and $S_2 = 0$.

$\Rightarrow \quad (\lambda + 4)x^2 + 2(\lambda + 1)y^2 - 2(3\lambda + 10)x$

$- 12(\lambda + 1)y + (23\lambda + 35) = 0 \qquad (1)$

For it to be a circle, choose l such that the coefficients of x^2 and y^2 are equal:

$\Rightarrow \quad \lambda + 4 = 2\lambda + 2$

$\therefore \quad \lambda = 2$

This gives the equation of the circle as

$6(x^2 + y^2) - 32x - 36y + 81 = 0 \qquad \{(\text{using } (1))\}$

$\Rightarrow \quad x^2 + y^2 - \dfrac{16}{3}x - 6y + \dfrac{27}{2} = 0.$

Its centre is $C\left(\dfrac{8}{3}, 3\right)$ and radius is

$$r = \sqrt{\frac{64}{9} + 9 - \frac{27}{2}} = \frac{1}{3}\sqrt{\frac{47}{2}}.$$

39. Equation of the chord joining the points $(a\cos\theta_1, b\sin\theta_1)$ and $(a\cos\theta_2, b\sin\theta_2)$ is given by

$$\frac{x}{a}\cos\frac{\theta_1 + \theta_2}{2} + \frac{y}{b}\sin\frac{\theta_1 + \theta_2}{2} = \cos\frac{\theta_1 - \theta_2}{2}$$

which passes through $(ae, 0)$, we have

$$e\cos\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\frac{\theta_1 - \theta_2}{2}$$

$$\Rightarrow \quad e\left[\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right]$$

$$= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$$

$$\Rightarrow \quad e\left[1 - \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\right] = 1 + \tan\frac{\theta_2}{2}\tan\frac{\theta_2}{2}$$

$\Rightarrow \; (e-1) = (e+1)\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}$

$\Rightarrow \; \tan\frac{\theta_1}{2}\cdot\tan\frac{\theta_2}{2} = \frac{e-1}{e+1}$ which is (a)

40. The coordinates of the given point P are (a cosα, b sinα).

$\therefore \; \tan\beta = \frac{b\sin\alpha - 0}{a\cos\alpha - 0} = \frac{b}{a}\tan\alpha$

$\therefore \; \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\cdot\tan\beta}$

$= \frac{\tan\alpha(a-b)}{a+b\tan^2\alpha} = l$ (say)

$\frac{dl}{d\alpha} = \frac{(a+b\tan^2\alpha)(a-b)\sec^2\alpha - (a-b)\tan\alpha\cdot 2b\tan\alpha\sec^2\alpha}{(a+b\tan^2\alpha)^2}$

$= \frac{(a-b)\sec^2\alpha(a-b\tan^2\alpha)}{(a+b\tan^2\alpha)^2}$

For extremum, $\frac{dl}{d\alpha} = 0 \;\Rightarrow\; a - b\tan^2 a = 0$

$\Rightarrow \; \tan\alpha = \sqrt{\frac{a}{b}}$.

41. Since $x\cos\alpha + y\sin\alpha = p$ subtends a right angle at the centre (0, 0), therefore

making equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ homogeneous with the help of $x\cos\alpha + y\sin\alpha = p$

we get $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2$

i.e., $x^2\left(\frac{1}{a^2} - \frac{\cos^2\alpha}{p^2}\right) + y^2\left(-\frac{1}{b^2} - \frac{\sin^2\alpha}{p^2}\right) + \frac{-2\sin\alpha\cos\alpha\, xy}{p^2} = 0$

Coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow \; \frac{1}{a^2} - \frac{\cos^2\alpha}{p^2} - \frac{1}{b^2} - \frac{\sin^2\alpha}{p^2} = 0$

$\Rightarrow \; \frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{p^2} \;\Rightarrow\; p = \frac{ab}{\sqrt{b^2-a^2}}$.

Since p is also the length of the perpendicular from (0, 0) to the line $x\cos\alpha + y\sin\alpha = p$

$\therefore$ Radius of the circle $= p = \frac{ab}{\sqrt{b^2-a^2}}$.

42. Let the directrix be $x = a/e$ and focus be $S(ae, 0)$. Let $P(a\sec\theta, b\tan\theta)$ be any point on the curve. Equation of tangent at P is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. Let F be the intersection point of tangent and the directrix, then $F = \left(\frac{a}{e}, \frac{b(\sec\theta - e)}{e\tan\theta}\right)$

$\Rightarrow \; m_{SF} = \frac{b(\sec\theta - e)}{-e\tan\theta(a^2-1)}, \; m_{PS} = \frac{b\tan\theta}{a(\sec\theta - e)}$

$\Rightarrow \; m_{SF}\times m_{PS} = -1$.

43. Director circle is the locus of point of intersection of perpendicular tangents drawn to a curve:

i.e., $x^2 + y^2 = a^2 - b^2$

director circle of given hyperbola is

$x^2 + y^2 = -11$,

which is not possible.

44. Given two hyperbolas are

$\frac{x^2}{9} - \frac{y^2}{16} = 1$ (1)

and, $\frac{y^2}{9} - \frac{x^2}{16} = 1$ (2)

Equation of tangent to (1), having slope m is

$y = mx \pm \sqrt{9m^2 - 16}$ (3)

Eliminating y using equation (2) and (3), we get

$16\left(mx \pm \sqrt{9m^2-16}\right)^2 - 9x^2 = 144$

$\Rightarrow \; (16m^2 - 9)x^2 \pm 32m\sqrt{9m^2-16}x + (144m^2 - 400) = 0$ (4)

For it to be a tangent, we must have $D = 0$

$\therefore \; \left(32m\sqrt{9m^2-16}\right)^2 = 4(16m^2-9)(144m^2-400)$

$\Rightarrow \; m^2 = 1 \Rightarrow m = \pm 1$

45. Since the asymptotes of rectangular hyperbola are mutually perpendicular, the other asymptote should be $4x + 3y + \lambda = 0$. Also, intersection point of asymptotes is also the centre of the hyperbola. Thus, intersection point of $4x + 3y + \lambda = 0$ and $3x - 4y - 6 = 0$

i.e., $\left(\frac{18-4\lambda}{25}, \frac{-12\lambda - 96}{100}\right)$ should lie on the line $x - y - 1 = 0$.

$\therefore \; \frac{18-4\lambda}{25} - \frac{12\lambda - 96}{100} - 1 = 0$

$\Rightarrow \; \lambda = 17$.

Hence, the equation of other asymptote is $4x + 3y + 17 = 0$

46. Equation of normal at any point $(ct, c/t)$ is $ct^4 - xt^3 + ty - c = 0$

$\Rightarrow$ Slope of normal $= t^2$

Let $P(h, k)$ be the point through which the normal is passing. Then, $ct^4 - ht^3 + tk - c = 0$

$\Rightarrow \; St_i = h/c$ and $St_it_j = 0$

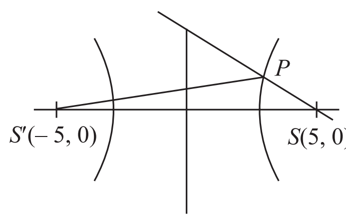
Hence, sum of the slopes of the normal

$= \sum t_i^2 = \left(\sum t_i\right)^2 = h^2 = c^2\lambda$

Therefore, required locus is $x^2 = \lambda c^2$

47. We have, for the given hyperbola $9 = 16(e^2 - 1) \Rightarrow e = \frac{5}{4}$

Since (5, 0) satisfies the equation of the line $3x + \left(5 - 4\sqrt{2}\right)y = 15$, so the reflected ray must pass through (–5, 0) and $P = (4\sqrt{2}, 3)$



$\therefore$ equation of $S'P$ is

$\sqrt{2}y = x + 5.$

TRICK(S) FOR PROBLEM SOLVING

The transverse axis is the bisector of the angle between the aymptotes containing the origin and the conjugate axis is the other bisector.

48. The equations of the bisectors of the angles between the asymptotes are

$$\frac{3x-4y-1}{5} = \pm\frac{4x-3y-6}{5}$$

So, the equations of transverse and conjugate axis are $x + y - 5 = 0$ and $x - y - 1 = 0$.

49. Let $ax + by = 1$ be the chord (1)

Making the equation of hyperbola homogeneous using (1), we get

$3x^2 - y^2 + (-2x + 4y)(ax + by) = 0$

or, $(3 - 2a)x^2 + (-1 + 4b)y^2 + (-2b + 4a)xy = 0$

Since the angle subtended at the origin is a right angle, so, coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow$ $(3 - 2a) + (-1 + 4b) = 0 \Rightarrow a = 2b + 1$

$\therefore$ The chords are $(2b + 1)x + by - 1 = 0$

or, $b(2 + y) + (x - 1) = 0,$

which, clearly, pass through the fixed point $(1, -2)$.

50. The coordinates of any point on the hyperbola are $\left(\sqrt{24}\sec\theta, \sqrt{18}\tan\theta\right)$.

Equation of tangent at this point is

$$\frac{x\sec\theta}{\sqrt{24}} - \frac{y\tan g\theta}{\sqrt{18}} = 1 \quad (1)$$

The point is nearest to the line

$3x + 2y + 1 = 0$ (2)

If (1) and (2) are parallel

$$\Rightarrow \frac{\sec\theta}{\sqrt{24}}\cdot\frac{\sqrt{18}}{\tan\theta} = -\frac{3}{2} \Rightarrow \sin\theta = -\frac{1}{\sqrt{3}}$$

Thus, the point is $(6, -3)$.

51. Let the point be (x_1, y_1),

Equation of chord of contact of tangents drawn from the point (x_1, y_1) to the hyperbola $x^2 - y^2 = a^2$ is

$xx_1 - yy_1 = a^2$ (1)

Equation of normal chord is

$$\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a \quad (2)$$

Since (1) and (2) are identical, comparing coefficients in (1) and (2), we get

$$\frac{x_1}{1/\sec\theta} = \frac{-y_1}{1/\tan\theta} = \frac{a^2}{2a}$$

$$\Rightarrow \sec\theta = \frac{a}{2x_1} \text{ and } \tan\theta = \frac{-a}{2y_1}$$

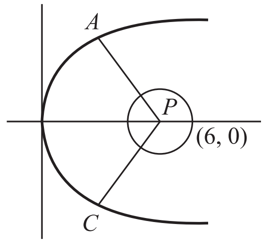
$\sec^2\theta - \tan^2\theta = 1$

$$\Rightarrow \frac{a^2}{4x_1^2} - \frac{a^2}{4y_1^2} = 1$$

$\therefore$ The required locus is $a^2(y^2 - x^2) = 4x^2y^2$.

52. Centre and radius of the given circle is $P(6, 0)$ and, respectively.

Equation of normal for $y^2 = 4x$ at $(t^2, 2t)$ is $y = -tx + 2t + t^3$. The normal must pass though $(6, 0)$ in order that it gives minimum distance between the two curves.



$\therefore$ $0 = t^3 - 4t \Rightarrow t = 0$ or $t = \pm 2$

$\therefore$ A (4, 4) and C (4, –4)

$PA = PC = \sqrt{20} = 2\sqrt{5}$

$\therefore$ required minimum distance $= 2\sqrt{5} - \sqrt{5} = \sqrt{5}$.

53. Any point on the given parabola is $(t^2, 2t)$. The equation of the tangent at (1, 2) is $x - y + 1 = 0$

The image (h, k) of the point $(t^2, 2t)$ in $x - y + 1 = 0$ is given by

$$= \frac{h-t^2}{1} = \frac{k-2t}{-1} = -\frac{2(t^2-2t+1)}{1+1}$$

$\therefore$ $h = t^2 - t^2 + 2t - 1 = 2t - 1$

and, $k = 2t + t^2 - 2t + 1 = t^2 + 1$

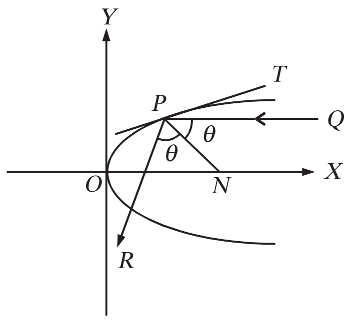
Eliminating t from $h = 2t - 1$ and $k = t^2 + 1$.

we get $(h + 1)^2 = 4(k - 1)$

The required equation of reflection is $(x + 1)^2 = 4(y - 1)$.

54. Given parabola is $y^2 = 4ax$ (1)

Let QP and PR be the incident and reflected rays, respectively. Let PT be the tangent to the parabola at P and PN be the normal to the parabola at P.



Let $\angle QPN = \theta$, then $\angle RPN = \theta$.

Let S be the focus of the parabola. Then, $S \equiv (a, 0)$.

Let $P \equiv (at^2, 2at)$.

Equation of tangent PT is

$yt = x + at^2$. Its slope $= \frac{1}{t}$.

$\therefore$ Slope of normal $PN = -t$.

Slope of $PQ = 0$. Let slope of $PR = m$.

Equating the two values of $\tan\theta$, we get

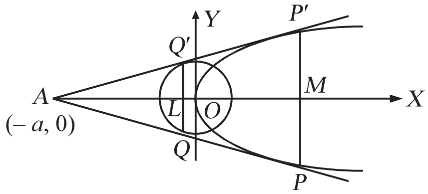
$$\frac{0+t}{1+0(-t)} = \frac{-t-m}{1-m} \Rightarrow t = \frac{-t-m}{1-m}$$

$$\Rightarrow t - t^2m = -t - m \Rightarrow m = \frac{2t}{t^2-1}.$$

$\therefore$ Equation of PR is $y - 2at = \frac{2t}{t^2-1}(x - at^2)$.

55. Given circle is $x^2 + y^2 = \frac{a^2}{2}$ (1)

and given parabola is $y^2 = 4ax$ (2)



Let PQ be a common tangent to the circle and the parabola.

Let $P \equiv (at^2, 2at)$.

Now, equation of PQ which is a tangent to the parabola at P is

$ty = x + at^2$ or $x - ty + at^2 = 0$ (3)

Since PQ is also a tangent to circle (1)

$$\therefore \frac{|0 - 0 \cdot t + at^2|}{\sqrt{1+t^2}} = \frac{a}{\sqrt{2}}$$

$$\Rightarrow 2t^4 - t^2 - 1 = 0 \Rightarrow t^2 = 1, \frac{-1}{2}.$$

$\therefore$ $t^2 = 1 \Rightarrow t = \pm 1$.

Hence, there will be two points $P'(a, 2a)$ and $P(a, -2a)$ on the parabola, the tangents at which will also be tangents to the circle.

Now, equations of tangents to the parabola at $P'(a, 2a)$ and $P(a, -2a)$ will be

$y = x + a$ (4)

and, $-y = x + a$ (5)

Solving (4) and (5), we get $x = -a, y = 0$

$\therefore$ $A \equiv (-a, 0)$.

$\therefore$ Equation of QQ' will be $-ax + y \times 0 = \frac{a^2}{2}$

or, $2x = -a$ (6)

Equation of PP' is $y \times 0 = 2a(x - a)$ or $x = a$ (7)

Solving (4) and (6), we get $Q' \equiv \left(\frac{-a}{2}, \frac{a}{2}\right)$.

Solving (5) and (6), we get $Q \equiv \left(-\frac{a}{2}, \frac{-a}{2}\right)$

Solving (4) and (7), we get $P' \equiv (a, 2a)$.

Solving (5) and (6), we get $P \equiv (a, -2a)$.

Clearly, $QPP'Q'$ is a trapezium, therefore its area

$$= \frac{1}{2}(QQ' + PP') \cdot LM = \frac{1}{2}(a + 4a)\left(\frac{a}{2} + a\right)$$

$$= \frac{1}{2} \cdot 5a \cdot 3\frac{a}{2} = \frac{15}{4}a^2.$$

56. Given parabola is $y^2 - 16x - 8y = 0$ (1)

Let the coordinates of the feet of the normal from (14, 7) be $P(\alpha, \beta)$.

Now, equation of the tangent at $P(\alpha, \beta)$ to parabola (1) is $y\beta - 8(x + \alpha) - 4(y + \beta) = 0$

or, $(\beta - 4)y = 8x + 8\alpha + 4\beta$ (2)

Its slope $= \frac{8}{\beta - 4}$.

Equation of normal to parabola (1) at (a, b) is

$$y - \beta = \frac{4-\beta}{8}(x - \alpha).$$

It passes through (14, 7),

$\therefore$ $7 - \beta = \frac{4-\beta}{8}(14 - \alpha)$ or $\alpha = \frac{6\beta}{\beta - 4}$ (3)

Also, (α, β) lies on parabola (1), $\therefore$ $\beta^2 - 16\alpha - 8\beta = 0$ (4)

Putting the value of a from (3) in (4), we get

$$\beta^2 - \frac{96\beta}{\beta-4} - 8\beta = 0 \Rightarrow \beta(\beta^2 - 12\beta - 64) = 0$$

$\Rightarrow \beta(\beta - 16)(\beta + 4) = 0$.

$\therefore$ $\beta = 0, 16, -4$.

From (3), when $\beta = 0$, $\alpha = 0$,

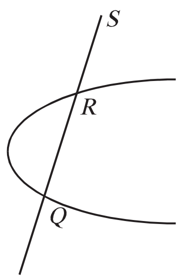
when $\beta = 16$, $\alpha = 8$ and when $\beta = -4$, $\alpha = 3$.

Hence, the feet of the normals are (0, 0) , (8, 16) and (3, – 4).

57. Given curve is $ax^2 + 2hxy + by^2 = 1$ (1)

Let $P \equiv (\alpha, \beta)$.

Let line PS make an angle θ with the positive direction of x-axis. Coordinates of any point on line PS may be taken as $(\alpha + r\cos\theta, \beta + r\sin\theta)$.



If point $(\alpha + r\cos\theta, \beta + r\sin\theta)$ lies on curve (1), then

$a(\alpha + r\cos\theta)^2 + 2h(\alpha + r\cos\theta)(b + r\sin\theta) + b(b + r\sin\theta)^2 = 1$

$\Rightarrow$ $(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)r^2$

$+ 2(a\alpha\cos q + h\beta\cos\theta + h\alpha\sin\theta + b\beta\sin\theta)r$

$+ a\alpha^2 + 2h\alpha\beta + b\beta^2 - 1 = 0$ (2)

Equation (2) will give two real values of r say r_1 and r_2 and corresponding to these two values of r we will get two points Q and R on curve (1).

Also, $PQ = |r_1|$ and $PR = |r_2|$

Now, $PQ \times PR = |r_1||r_2| = |r_1 r_2|$

$$= \left|\frac{a\alpha^2 + 2h\alpha\beta + b\beta^2 - 1}{a\cos^2\theta + b\sin^2\theta + h\sin 2\theta}\right|$$

From (3), it is clear that $PQ \times PR$ will be independent of slope of line PS, i.e., independent of θ, if $a = b$ and $h = 0$.

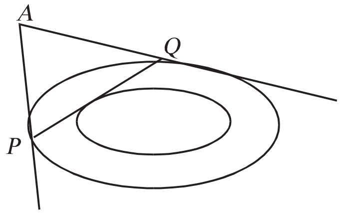
Thus, equation of curve will become $a(x^2 + y^2) = 1$, which is a circle.

58. We can write the ellipse $x^2 + 4y^2 = 4$ as

$$\frac{x^2}{4} + y^2 = 1 \qquad (1)$$

Equation of any tangent to the ellipse (1) can be written as

$$\frac{x}{2}\cos\theta + y\sin\theta = 1 \qquad (2)$$



Equation of the second ellipse can be written as

$$\frac{x^2}{6} + \frac{y^2}{3} = 1 \qquad (3)$$

Suppose, the tangents at P and Q meet in $A(h, k)$. Equation of the chord of contact of the tangents through $A(h, k)$ is

$$\frac{hx}{6} + \frac{ky}{3} = 1 \qquad (4)$$

Since (4) and (2) represent the same line

$$\therefore \quad \frac{h/6}{\frac{\cos\theta}{2}} = \frac{k/3}{\sin\theta} = \frac{1}{1}$$

$\Rightarrow \quad h = 3\cos\theta$ and $k = 3\sin\theta$.

Thus, coordinates of A are $(3\cos\theta, 3\sin\theta)$.

The joint equation of the tangents at A is given by $T^2 = SS_1$

i.e., $$\left(\frac{hx}{6} + \frac{ky}{3} - 1\right)^2 = \left(\frac{x^2}{6} + \frac{y^2}{3} - 1\right)\left(\frac{h^2}{6} + \frac{k^2}{3} - 1\right) \qquad (5)$$

Let a = coefficient of x^2 in (5)

$$= \frac{h^2}{36} - \frac{1}{6}\left(\frac{h^2}{6} + \frac{k^2}{3} - 1\right) = \frac{-k^2}{18} + \frac{1}{6}$$

and, b = coefficient of y^2 in (5)

$$= \frac{-k^2}{9} - \frac{1}{3}\left(\frac{h^2}{6} + \frac{k^2}{3} - 1\right) = \frac{-h^2}{18} + \frac{1}{3}.$$

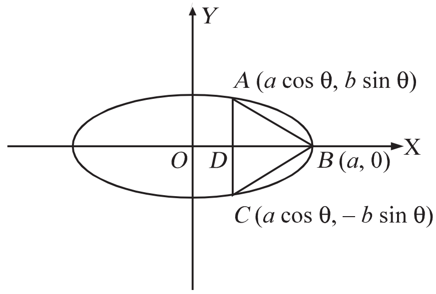
We have, $$a + b = \frac{-1}{18}(h^2 + k^2) + \frac{1}{6} + \frac{1}{3}$$

$$= \frac{-1}{18}(9\cos^2\theta + 9\sin^2\theta) + \frac{1}{2}$$

$$= \frac{-1}{18}(9) + \frac{1}{2} = 0.$$

Thus, (5) represents two lines which are at right angles to each other.

59. The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Let $A \equiv (a\cos\theta, -b\sin\theta)$

Then, $C \equiv (a\cos\theta, -b\sin\theta)$

$$\Delta = \text{Area of } \Delta ABC = \frac{1}{2} \times AC \times BD = AD \times BD$$

$$= b\sin\theta(a - a\cos\theta)$$

$$= \frac{1}{2}ab(2\sin\theta - \sin 2\theta)$$

Now, $$\frac{d\Delta}{d\theta} = \frac{1}{2}ab(2\cos\theta - 2\cos 2\theta) = 0 \qquad (1)$$

$\Rightarrow \quad \cos 2\theta = \cos\theta \Rightarrow 2\cos^2\theta - \cos\theta - 1 = 0$

$\Rightarrow \quad (2\cos\theta + 1)(\cos\theta - 1) = 0$

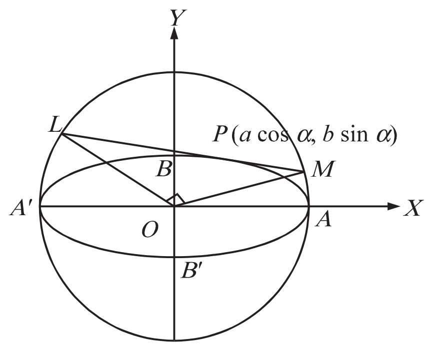
$\Rightarrow \quad \cos\theta = \frac{-1}{2}$ or $\cos\theta = 1$

If $\theta = 0$, $\Delta = 0$, which is not possible.

$\therefore \quad \theta = 2\pi/3$.

$\therefore \quad \Delta_{max} = \frac{3\sqrt{3}}{4}ab$ [substitiuting the value of θ in (1)]

60. Given ellipse is $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$



Its auxiliary circle is $x^2 + y^2 = a^2$ (2)

Let $P \equiv (a\cos\alpha, b\sin\alpha)$

Equation of tangent to the ellipse at $P(a\cos\alpha, b\sin\alpha)$ is

$$\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b} = 1 \qquad (3)$$

Making equation (2) homogeneous with the help of (3), we get

$$x^2 + y^2 - a^2\left(\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b}\right)^2 = 0$$

$$\Rightarrow\ (1-\cos^2\alpha)x^2+\left(1-\frac{a^2}{b^2}\sin^2\alpha\right)y^2 \qquad (4)$$

$$-2\frac{a}{b}\cos\alpha\sin\alpha\, xy = 0$$

(4) is the joint equation of OL and OM.

Since $\angle LOM = 90°$, $\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow\ 1-\cos^2\alpha+1-\frac{a^2}{b^2}\sin^2\alpha = 0$$

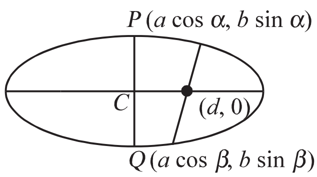
$$\Rightarrow\ \sin^2\alpha\left(\frac{a^2}{b^2}-1\right)=1$$

or, $\sin^2\alpha\left(\dfrac{1}{1-e^2}-1\right)=1$ $\qquad [\because\ b^2 = a^2(1-e^2)]$

$\Rightarrow\ e^2\sin^2\alpha = 1-e^2$ or $e^2(1+\sin^2\alpha) = 1$

$$\therefore\ e=\frac{1}{\sqrt{1+\sin^2\alpha}}$$

61. Let $P \equiv (a\cos\alpha, b\sin\alpha)$ and $Q \equiv (a\cos\beta, b\sin\beta)$



The equation of the chord PQ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right) \qquad (2)$$

Since it cuts the major axis of the ellipse at a distance d from the centre, $\therefore$ it must pass through the point $(d, 0)$, i.e.,

$$\frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow\ \frac{d-a}{d+a}=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}\right)}$$

[By componendo and dividendo]

$$=\frac{2\sin\alpha/2\sin\beta/2}{2\cos\alpha/2\cos\beta/2}=\tan\alpha/2\times\tan\beta/2.$$

$$\therefore\ \tan\alpha/2\cdot\tan\beta/2=\frac{d-a}{d+a}$$

62. Let the orbit of the earth be the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1)$$

Length of major axis $= 2a = 186 \times 10^6$ miles (given)

$\Rightarrow\ a = 93 \times 10^6$ miles.

Also, eccentricity $e = \dfrac{1}{60}$ (given).

Let the sun be at the focus $S(ae, 0)$. Then, the earth will be at shortest and longest distance from the sun when the earth is at the extremities of the major axis which are respectively nearest and farthest from this focus S.

$\therefore$ Shortest distance of the earth from the sun

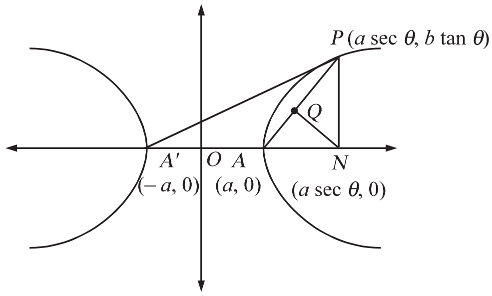
$= SA$, where S is $(ae, 0)$ and A is $(a, 0)$

$= a - ae = (93 \times 10^6)\left(1-\dfrac{1}{60}\right) = 9145 \times 10^4$ miles and longest distance of the earth from the sun

$= SA'$, where S is $(ae, 0)$ and A' is $(-a, 0)$

$= a + ae = (93 \times 10^6)\left(1+\dfrac{1}{60}\right) = 9455 \times 10^4$ miles.

63. Let $P \equiv (a\sec\theta, b\tan\theta)$



Then, $N \equiv (a\sec\theta, 0)$.

Since Q divides AP in the ratio $a^2 : b^2$,

$\therefore$ coordinates of Q are

$$=\left(\frac{ab^2+a^2\sec\theta}{a^2+b^2},\ \frac{a^2b\tan\theta}{a^2+b^2}\right).$$

Slope of $\quad A'P = \dfrac{b\tan\theta}{a(\sec\theta+1)}$

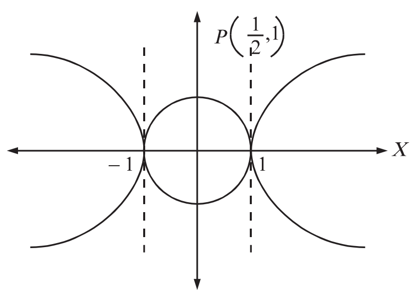
Slope of $\quad QN = \dfrac{a^2b\tan\theta}{ab^2+a^3\sec\theta-a^3\sec\theta-ab^2\sec\theta}$

$$=\frac{a^2b\tan\theta}{ab^2(1-\sec\theta)}.$$

$\therefore$ Slope of $A'P \times$ slope of $QN = \dfrac{a^2b^2\tan^2\theta}{-a^2b^2\tan^2\theta} = -1.$

$\therefore\ QN$ is $\perp$ to $A'P$.

64. There are two common tangents to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = -1$. These are $x = 1$ and $x = -1$. Out of these two, $x = 1$ is nearer to the point $P\left(\dfrac{1}{2}, 1\right)$. Thus, a directrix of the required ellipse is $x = 1$.



If $Q(x, y)$ is any point on the ellipse, then its distance from the focus is $QP = \sqrt{\left(x-\frac{1}{2}\right)^2 + (y-1)^2}$ and its distance from the directrix $x = 1$ is $|x-1|$. By definition of ellipse,

$QP = e\,|x-1|$

$$\Rightarrow \sqrt{\left(x-\frac{1}{2}\right)^2 + (y-1)^2} = \frac{1}{2}|x-1|$$

$$\Rightarrow x^2 - x + \frac{1}{4} + y^2 - 2y + 1 = \frac{1}{4}(x^2 - 2x + 1)$$

$$\Rightarrow 3x^2 - 2x + 4y^2 - 8y + 4 = 0$$

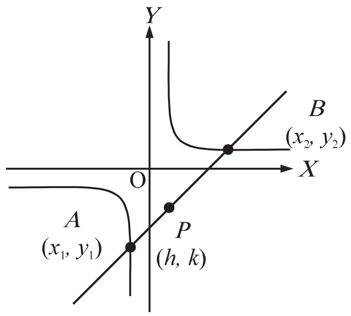
$$\Rightarrow 3\left(x^2 - \frac{2}{3}x\right) + 4(y-1)^2 = 0$$

$$\Rightarrow 3\left[\left(x-\frac{1}{3}\right)^2 - \frac{1}{9}\right] + 4(y-1)^2 = 0$$

$$\Rightarrow 3\left(x-\frac{1}{3}\right)^2 + 4(y-1)^2 = \frac{1}{3}$$

$$\Rightarrow \frac{\left(x-\frac{1}{3}\right)^2}{1/9} + \frac{(y-1)^2}{1/12} = 1.$$

65. Let $P(h, k)$ be any point on the locus. Equation of the line through P and having slope 4 is



$y - k = 4(x - h)$ (1)

Suppose, this line meets $xy = 1$ (2)

in $A(x_1, y_1)$ and $B(x_2, y_2)$.

Eliminating y from (1) and (2), we get

$$\frac{1}{x} - k = 4(x - h)$$

$\Rightarrow 4x^2 - (4h - k)x - 1 = 0$ (3)

Since x_1 and x_2 are the roots of (3)

$\therefore\ x_1 + x_2 = \frac{4h-k}{4}$ (4)

and, $x_1x_2 = \frac{-1}{4}$ (5)

Also, $\left(\frac{8h+k}{2}\right)\left(-\frac{2h+k}{2}\right) = -\frac{1}{4} = h$ or $2x_1 + x_2 = 3h$ (6)

From (4) and (6), we get

$$x_1 = 3h - \frac{(4h-k)}{4} = \frac{8h+k}{4}$$

and, $x_2 = 3h - \frac{(8h+k)}{2} = \frac{-(2h+k)}{2}$

Putting these values in (5), we get

$$\left(\frac{8h+k}{2}\right)\left(-\frac{2h+k}{2}\right) = -\frac{1}{4}$$

$\Rightarrow (8h + k)(2h + k) = 2$ or $16h^2 + 10hK + k^2 = 2$.

Thus, equation of required locus is

$16x^2 + 10xy + y^2 = 2$.

More than One Option Correct Type

66. Solving the two equation $x^2 = ay$ and $y - 2x = 1$, we get

$x^2 = a(2x + 1)$ or $x^2 - 2ax - a = 0$.

$\therefore\ x_1 + x_2 = 2a$ and $x_1x_2 = -a$.

So, the given line cuts the parabola at two points (x_1, y_1) and (x_2, y_2).

Now, $\left(\sqrt{40}\right)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ [Given]

$$\Rightarrow 40 = (x_1 - x_2)^2 + \left(\frac{x_1^2}{a} - \frac{x_2^2}{a}\right)^2$$

$$= (x_1 - x_2)^2\left[1 + \frac{(x_1 + x_2)^2}{a^2}\right]$$

$$= [(x_1 + x_2)^2 - 4x_1x_2]\left(\frac{4a^2}{a^2} + 1\right)$$

$= 5(4a^2 + 4a)$.

$\therefore\ a^2 + a - 2 = 0$ or $(a + 2)(a - 1) = 0$

$\therefore\ a = 1, -2$.

67. Since the equation of a hyperbola and its asymptotes differ in constant terms only, therefore, the equations of asymptotes of the given hyperbola are given by

$xy - 3x + 4y + k = 0$

where k is a constant to be determined by the condition that $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

i.e., $0 + 2 \times 2 \times \left(\frac{-3}{2}\right) \times -0 - 0 - k \times \left(\frac{1}{2}\right)^2 = 0$

$\Rightarrow k = -12$.

$\therefore$ Asymptotes of the given hyperbola are

$xy - 3x + 4y - 12 = 0$ or $(x + 4)(y - 3) = 0$

i.e., $x = -4$ and $y = 3$.

68. Equation of normal at the point $\left(t, \frac{1}{t}\right)$ to the hyperbola $xy = 1$ is

$xt^3 - yt - t^4 + 1 = 0$.

Its slope $= t^2 = \frac{-a}{b}$

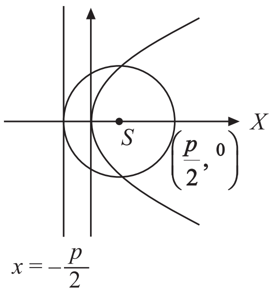
$\therefore \frac{-a}{b} > 0 \Rightarrow \frac{a}{b} < 0$

$\therefore a > 0, b < 0$ or $a < 0, b > 0$.

69. The equation of circle will be

$$\left(x - \frac{p}{2}\right)^2 + y^2 = (p)^2 \Rightarrow x^2 + y^2 - px - \frac{3p^2}{4} = 0$$

Solving it with $y^2 = 2px$, we get



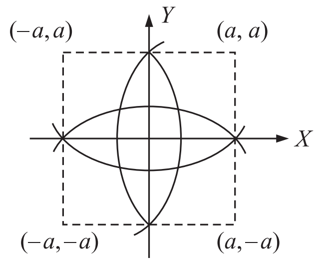
$$x^2 + px - \frac{3p^2}{4} \Rightarrow x = -\frac{3p}{2} \text{ or } \frac{p}{2}$$

$\therefore x > 0, \quad \therefore x = \frac{p}{2}$ and then, $y = \pm p$.

70. If $P \equiv (x, y)$, then

$$\sqrt{x^2 + y^2} < |x - a|, \sqrt{x^2 + y^2} < |a + x|$$

$$\sqrt{x^2 + y^2} < |a - y|, \sqrt{x^2 + y^2} < |a + y|$$



$\therefore$ The region is bounded by the curves

$x^2 + y^2 = (a - x)^2$, $x^2 + y^2 = (a + x)^2$

$x^2 + y^2 = (a - y)^2$, $x^2 + y^2 = (a + y)^2$

or, $y^2 = a^2 - 2ax$, $y^2 = a^2 + 2ax$,

$x^2 = a^2 - 2ay$ $x^2 = a^2 + 2ay$

Passage Based Questions

71. I. Given, $x = t^2 + t + 1$ and $y = t^2 - t + 1$

$\Rightarrow x + y = 2(t^2 + 1)$ and $x - y = 2t$.

Eliminating t, we get

$$x + y = \frac{1}{2}(x - y)^2 + 2$$

or, $x^2 - 2xy + y^2 - 2x - 2y + 4 = 0$

Here, $a = 1, h = -1, b = 1, g = -1, f = -1, c = 4$.

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 1 \times 1 \times 4 + 2(-1)(-1)(-1) - 1 \times 1 - 1 \times 1 - 4 \times 1$

$= 4 - 2 - 1 - 1 - 4 = -4 \neq 0$.

Also, $h^2 - ab = (-1)^2 - 1 \times 1 = 0$.

Hence, the given curve represents a parabola.

II. Let $P(x_1, y_1)$ be the point of contact of the two given parabolas

$y^2 = 4a(x - 2)$ (1)

and, $x^2 = 4a(y - 3)$ (2)

Equation of tangent at P to (1) is

$yy_1 = 2a(x + x_1) - 8a$

or, $2ax - y_1 y + (2ax_1 - 8a) = 0$ (3)

Equation of tangent at P to (2) is

$xx_1 = 2a(y + y_1) - 12a$

or, $x_1 x - 2ay - (2ay_1 - 12a) = 0$ (4)

Since (3) and (4) represent the same line,

$$\therefore \frac{2a}{x_1} = \frac{-y_1}{-2a} \Rightarrow x_1 y_1 = 4a^2.$$

$\therefore$ point of contact (x_1, y_1) lies on the curve $xy = 4a^2$. which is a hyperbola.

72. III. $\frac{x\cos\theta}{3\sqrt{3}} + y\sin\theta = 1$

Sum of intercepts $= 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta = f(\theta)$, (say)

$$f'(\theta) = \frac{3\sqrt{3}\sin^3\theta - \cos^3\theta}{\sin^2\theta\cos^2\theta}.$$

At $\theta = \frac{\pi}{6}$, $f(\theta)$ is minimum.

IV. Equation of tangent is

$\frac{x}{a} \cdot \frac{1}{2} + \frac{y}{b} \cdot \frac{\sqrt{3}}{2} = 1$ (1)

Also, equation of tangent at the point $(a\cos\theta, b\sin\theta)$ is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ (2)

Since (1) and (2) are identical, we get

$$\cos\theta = \frac{1}{2} \text{ and } \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

73. I. Equations of normals to $y^2 = 4ax$ and $x^2 = 4by$ are given by

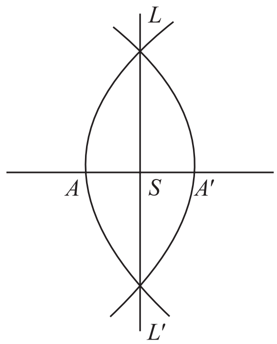
$$y = mx - 2am - am^3 \text{ and } y = mx + 2b + \frac{b}{m^2}.$$

For common normals, $2b + \frac{b}{m^2} + 2am + am^3 = 0$

$\Rightarrow \quad am^5 + 2am^3 + 2bm^2 + b = 0$

So, a maximum of 5 normals are possible.

II. L and L' are the ends of latus rectum. S bisects LL'. As A' is perpendicular bisector of LL', where $AS = \frac{1}{4}LL' = A'S$.



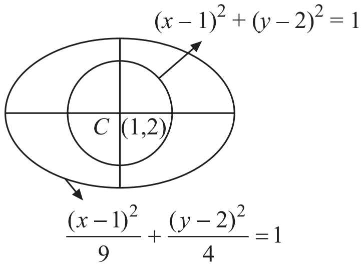
Clearly, two parabolas are possible.

III. Since $3 \times 3^2 + 5 \times 5^2 - 32 > 0$, the point (3, 5) lies outside the ellipes $3x^2 + 5y^2 = 32$.

Also, $25 \times 3^2 + 9 \times 5^2 - 450 = 0$, $\therefore$ the point (3, 5) lies on the ellipse $25x^2 + 9y^2 = 450$. So, the required number of tangents is 3.

IV. Clearly, from the figure, the circle and the ellipse do not meet at any point,

$\therefore$ length of the common chord = 0



74. I. If P and Q be $(at_2^2, 2at_2)$ and $(at_2^2, 2at_2)$, , then T is $\{at_1t_2, a(t_1 + t_2)\}$.

Any tangent to the parabola is $y = mx + \frac{a}{m}$

or, $\quad m^2x - my + a = 0$

Now, $p_1 = \frac{m^2at_1^2 - 2at_1m + a}{\sqrt{m^4 + m^2}} = \frac{a(mt_1 - 1)^2}{m\sqrt{m^2+1}}$ (1)

Similarly, $p_3 = \frac{a(mt_2 - 1)^2}{m\sqrt{m^2 - 1}}$ (2)

Also, $p_2 = \frac{m^2at_1t_2 - a(t_1 + t_2)m + a}{\sqrt{m^4 + m^2}}$

$\Rightarrow \quad p_2 = \frac{a}{m\sqrt{m^2+1}}\{(mt_1 - 1)(mt_2 - 1)\}$ (3)

Using (1), (2) and (3), we have

$p_2^2 = p_1p_3$

$\Rightarrow \quad p_1, p_2$ and p_3 are in G. P.

II. Let the equation of the hyperbola be

$H(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$

then, $A(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$ is the equation of asymptotes

and $C(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$ is the equation of its conjugate hyperbola.

Now, $H(\alpha, \beta) + C(\alpha, \beta)$

$= \left(\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - 1\right) + \left(\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} + 1\right)$

$= 2\left(\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2}\right) = 2A(\alpha, \beta)$

$\Rightarrow \quad H(\alpha, \beta), A(\alpha, \beta)$ and $C(\alpha, \beta)$ are in A. P.

III. Let the equation of the parabola be $y^2 = 4ax$. P is $(at_1^2, 2at_1)$ and Q is $(at_2^2, 2at_2)$.

The point of intersection of the tangents is $(at_1t_2, a(t_1 + t_2))$

$y_1 = 2at_1$ and $y_2 = 2at_2$

$y_3 = a(t_1 + t_2) = at_1 + at_2$

$= \frac{y_1}{2} + \frac{y_2}{2}$

$\Rightarrow \quad y_1 + y_2 = 2y_3$

$\therefore \quad y_1, y_3, y_2$ are in A. P.

75. I. Any point on the parabola is $(x, x^2 + 7x + 2)$

Its distance from the line $y = 3x - 3$ is given by

$P = \left|\frac{3x - (x^2 + 7x + 2) - 3}{\sqrt{9+1}}\right|$

$= \left|\frac{x^2 + 4x + 5}{\sqrt{10}}\right|$

$= \frac{x^2 + 4x + 5}{\sqrt{10}}$ (as $x^2 + 4x + 5 > 0$ for all $x \in R$)

$\frac{dP}{dx} = 0 \quad \Rightarrow \quad x = -2$. So, the required point is (–2, –8).

II. Let (x_1, y_1) be a point on $y = x + 2$.

$\therefore \quad y_1 = x_1 + 2$

Equation of the line perpendicular to the given line through (x_1, y_1) is

$y - (x_1 + 2) = -(x - x_1)$ i.e., $y = -x + 2(x_1 + 1)$

If this line is a tangent to $y^2 = 8x$, $c = \frac{a}{m}$ gives

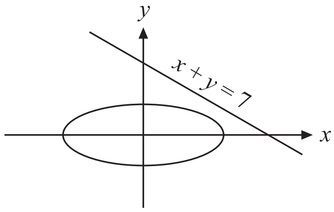
$2(x_1 + 1) = \frac{2}{-1}$ i.e., $x_1 + 1 = -1 \Rightarrow x_1 = -2$

Hence, $y_1 = 0$

$\therefore \quad$ The required point is (–2, 0).

III. Given equation of ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$. Slope of the tangent at any point $P(x_1, y_1)$ to $\frac{x^2}{6}+\frac{y^2}{3}=1$ is given by

$$2x+4y\frac{dy}{dx}=0$$



$\Rightarrow \; x = 2y$

$\because \; \frac{dy}{dx}=\frac{-x}{2y}=-1.$

Putting $x = 2y$ in the equation of ellipse we have $y = 1$. Evidently, the point lies in the first quadrant

$\therefore \; y = 1$ and $x = 2$

Hence, required point is (2, 1)

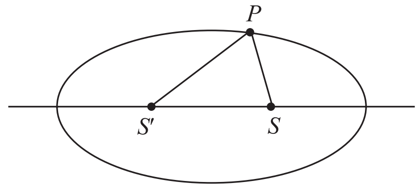
IV. The coordinates of the point P are $\left(\frac{5}{2},\frac{3\sqrt{3}}{2}\right)$.

Since $e=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$,

so, the coordinates of the foci are $S(4, 0)$ and $S'(-4, 0)$ and $SS' = 8$.

Also, $SP = a - ex_1 = 5-\frac{4}{5}\times\frac{5}{2}=3=3$

and, $S'P = a + ex_1 = 7$.



Therefore, the coordinates of incentre (x_1, y_1) are

$$x_1=\frac{7\times4+3\times-4+8\times\frac{5}{2}}{7+3+8}=2$$

$$y_1=\frac{7\times0+3\times0+8\times\frac{3\sqrt{3}}{2}}{7+3+8}=\frac{2}{\sqrt{3}}$$

Assertion-Reason Type

76. Since the semi-latus rectum of a parabola is the harmonic mean between the segment of any focal chord of a parabola, therefore SP, 4, SQ are in H. P.

$$\Rightarrow \; 4=2\frac{SP\cdot SQ}{SP+SQ} \;\Rightarrow\; 4=2\cdot\frac{6\cdot SQ}{6+SQ} \;\Rightarrow\; SQ=3$$

77. Given, equation of hyperbola

$2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ and eqation of asymptotes

$2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ (1)

which is the equation of a pair of straight lines. We know that the standard equation of a pair of straight lines is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. comparing equation (1) with standard equation, we get $a = 2$, $b = 2$, $h=\frac{5}{2}$, $g = 2$, $f=\frac{5}{2}$ and $c = l$.

We also know that the condition for a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

Therefore, $4\lambda+25-\frac{25}{2}-8-\frac{25}{4}\lambda=0$ or $-\frac{9\lambda}{4}+\frac{9}{2}=0$ or $\lambda = 2$. Substituting value of l in equation (i), we get

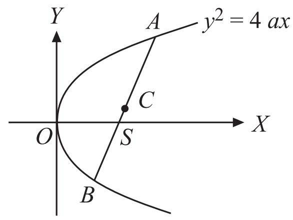
$2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

78. If $A(at_1^2, at_1), B(at_2^2, 2at_2)$, be the extremities of a focal chord for the parabola $y^2 = 4ax$, then $t_1t_2 = -1$.

We want to find locus of point $C(\alpha, \beta)$ where C is the centre of the circle having AB as the diameter.

$$\Rightarrow \; a=\frac{a}{2}(t_1^2+t_2^2) \;;\; \beta=a(t_1+t_2)$$

To eliminate t_1, t_2



$$\beta^2 = a^2(t_1^2+t_2^2+2t_1t_2)=a^2\left(\frac{2\alpha}{a}-2\right)$$

$\Rightarrow \; \beta^2 = 2a(\alpha - a)$ i.e., $y^2 = 2a(x - a)$

79. Let us find the point of intersection of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (1)$$

and circle

$$x^2 + y^2 = ab \quad (2)$$

We have, $\frac{x^2}{a^2}+\frac{1}{b^2}(ab-x^2)=1$

$$\Rightarrow \; x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=1-\frac{a}{b}=\frac{b-a}{b}$$

$$\Rightarrow \; x^2\frac{(b^2-a^2)}{a^2b^2}=\frac{b-a}{b} \;\Rightarrow\; x^2=\frac{a^2b}{a+b} \quad (3)$$

and, $y^2 = ab - x^2$

$\Rightarrow \quad y^2 = ab - \dfrac{a^2b}{a+b}$ (using (1) and (3))

$\Rightarrow \quad y^2 = \dfrac{ab(a+b-a)}{a+b} = \dfrac{ab^2}{a+b}$

$\therefore$ Intersection point is $\left(\sqrt{\dfrac{a^2b}{a+b}}, \sqrt{\dfrac{ab^2}{a+b}}\right)$

The slope of tangent to ellipse at any point (x_1, y_1) is

$m_1 = -\dfrac{b^2x_1}{a^2y_1}$

$\Rightarrow \quad m_1 = -\dfrac{b^2}{a^2}\dfrac{\sqrt{\dfrac{a^2b}{a+b}}}{\sqrt{\dfrac{ab^2}{a+b}}} = -\dfrac{b^2}{a^2}\sqrt{\dfrac{a}{b}}$

and slope of tangent to the circle is

$m_2 = -\dfrac{x_1}{y_1} = -\sqrt{\dfrac{a}{b}}$

$\therefore \quad \theta = \tan^{-1}\left(\dfrac{-\sqrt{\dfrac{a}{b}} + \dfrac{b^2}{a^2}\sqrt{\dfrac{a}{b}}}{1 + \dfrac{b^2}{a^2}, \dfrac{a}{b}}\right) = \tan^{-1}\dfrac{(b-a)}{\sqrt{ab}}$

Previous Year's Questions

80. The equation of an ellipse is

$\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

Here $a = 4$, $b = 3$

$\therefore$ Foci of the above ellipse are $(\pm\sqrt{7}, 0)$

$\therefore$ Radius of required circle

$= \sqrt{(\sqrt{7} - 0)^2 + (0-3)^2}$

$= \sqrt{(\sqrt{7} + 9} = \sqrt{16} = 4$ unit

81. Key Idea :The foci of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is given by $(\pm ae, 0)$

Since, $e = \dfrac{1}{2}, ae = 2$

$\Rightarrow \quad a = 4$

$b^2 = a^2(1 - e^2)$

$\therefore \quad = 16\left(1 - \dfrac{1}{4}\right) = 12$

Thus, the equation of an ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$.

82. The mid-point of the chord is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

The equation of the chord in terms of its mid-point is $T = S_1$

i.e., $x\left(\dfrac{y_1 + y_2}{2}\right) + y\left(\dfrac{x_1 + x_2}{2}\right) = 2\left(\dfrac{x_1 + x_2}{2}\right)\left(\dfrac{y_1 + y_2}{2}\right)$

$\Rightarrow \quad x(y_1 + y_2) + y(x_1 + x_2) = (x_1 + x_2)(y_1 + y_2)$

$\Rightarrow \quad \dfrac{x}{x_1 + x_2} + \dfrac{y}{y_1 + y_2} = 1$

83. is correct answer.

84. $\dfrac{x^2}{\left(\dfrac{12}{5}\right)^2} - \dfrac{y^2}{\left(\dfrac{9}{5}\right)^2} = 1$

Eccentricity of hyperbola , $e_1 = \dfrac{5}{4}$

Now, $ae_2 = \sqrt{1 - \dfrac{b^2}{16}} \times 4 = 3$

$\Rightarrow \quad b^2 = 7$

Hence, (C) is the correct answer.

85. Any point on the given parabola is of the form $\left(\dfrac{9}{2}t^2, 9t\right)$

Differentiating $y^2 = 18x$ w.r.t. x

$\dfrac{dy}{dx} = \dfrac{9}{y} = \dfrac{1}{t} = 2$ (given) $\Rightarrow \quad t = \dfrac{1}{2}$

$\Rightarrow \quad$ Point is $\left(\dfrac{9}{8}, \dfrac{9}{2}\right)$

86. Points of intersection of given parabolas are (0, 0) and $(4a, 4a)$.

And the equation of line passing through these points is $y = x$

On comparing this line with the given line $2bx + 3cy + 4d = 0$, we get $d = 0$ and $2b + 3c = 0 \quad \Rightarrow \quad (2b + 3c)^2 + d^2 = 0$.

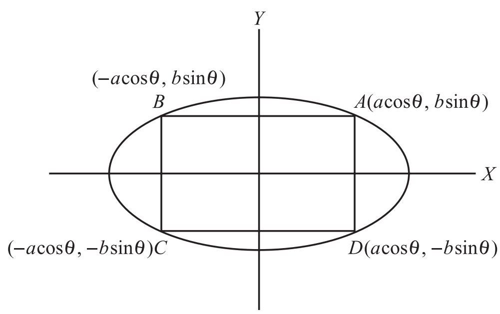
87. The equation of directrix is $x = \dfrac{a}{e} = 4$. So, $a = 2b^2 = a^2(1 - e^2) \Rightarrow b^2 = 3$

Hence the equation of ellipse is $3x^2 + 4y^2 = 12$.

88. Area of rectangle $ABCD = (2a\cos\theta)(2b\sin\theta) = 2ab\sin 2\theta$

$\Rightarrow \quad$ Area of greatest rectangle is equal to $2ab$

When $\sin 2\theta = 1$



89. Tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$y = mx \pm \sqrt{a^2m^2 - b^2}$

Given that $y = \alpha x + \beta$ is the tangent of hyperbola

$\Rightarrow \quad m = \alpha$ and $a^2m^2 - b^2 = \beta^2$

$\therefore \quad a^2\alpha^2 - b^2 = \beta^2$

Locus is $a^2x^2 - y^2 = b^2$ which is hyperbola.

90. $\because \quad \angle FBF' = 90°$

$\therefore \quad \left(\sqrt{a^2e^2 + b^2}\right)^2 + \left(\sqrt{a^2e^2 + b^2}\right)^2 = 2(ae)^2$

$\Rightarrow \quad 2(a^2e^2 + b^2) = 4a^2e2$

$\Rightarrow \quad e^2 = \frac{b^2}{a^2}$

Also, $e^2 = 1 - \frac{b^2}{a^2} = 1 - e^2$

$\Rightarrow \quad 2e^2 = 1 \quad \Rightarrow \quad e = \frac{1}{\sqrt{2}}$.

91. We have $a^2 = \cos^2\alpha$ and $b^2 = \sin^2\alpha$

So, the coordinates of foci are (±ae, 0)

$\therefore \quad b^2 = a^2(e^2 - 1) \quad \Rightarrow \quad e = \sec\alpha$

Hence, abscissae of foci remain constant when α varies.

92. Major axis of length $2a$ is along x-axis.0

Now, $\frac{a}{e} - ae = 4$

$\Rightarrow \quad a\left(2 - \frac{1}{2}\right) = 4$

$\Rightarrow \quad a = \frac{8}{3}$

93. We have

$2ae = 6 \quad \Rightarrow \quad ae = 3$

$2b = 8 \quad \Rightarrow \quad b = 4$

$\therefore \quad b^2 = a^2(1 - e^2)$

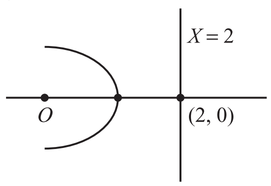
$\Rightarrow \quad 16 = a^2 - a^2e^2$

$\Rightarrow \quad a^2 = 16 + 9 = 25$

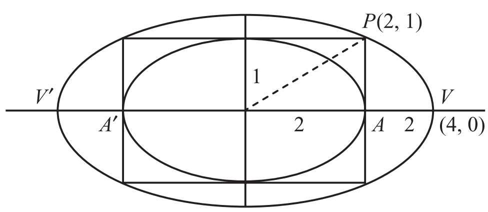
$\Rightarrow \quad a = 5$

$\therefore \quad e = \frac{3}{a} = \frac{3}{5}$

94. Vertex is (1, 0)



95.



Given ellipse $x^2 + 4y^2 = 4 \quad \Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{1} = 1 \quad \Rightarrow \quad a = 2, b = 1 \quad \Rightarrow \quad p = (2, 1)$

Required Ellipse has equation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{x^2}{4^2} + \frac{y^2}{b^2} = 1$

Now, (2, 1) lies on it

$\Rightarrow \quad \frac{4}{16} + \frac{1}{b^2} = 1 \quad \Rightarrow \quad \frac{1}{b^2} = 1 - \frac{1}{4} = \frac{3}{4} \quad \Rightarrow \quad b^2 = \frac{4}{3}$

$\therefore \quad \frac{x^2}{16} + \frac{y^2}{\left(\frac{4}{3}\right)} = 1 \quad \Rightarrow \quad \frac{x^2}{16} + \frac{3y^2}{4} = 1$

$\Rightarrow \quad x^2 + 12y^2 = 16$

96. The locus of perpendicular tangents is directrix

i.e, $x = -a$; $x = -1$

97. $y^2 = 16\sqrt{3}x$

$\frac{x^2}{2} + \frac{y^2}{4} = 1$

$y = mx + \frac{4\sqrt{3}}{m}$ is tangent to parabola

Which is tangent to ellipse $\Rightarrow c^2 = a^2m^2 + b^2$

$\Rightarrow \quad \frac{48}{m^2} = 2m^2 + 4$

$\Rightarrow \quad m^4 + 2m^2 = 24$

$\Rightarrow \quad m^4 = 4$

98. Semi minor axis $b = 2$

Semi major axis $a = 4$

Equation of ellipse $= \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \quad \frac{x^2}{16} + \frac{y^2}{4} = 1$

$\Rightarrow \quad x^2 + 4y^2 = 16.$

99. Let the tangent to the parabola be $y = mx + \frac{\sqrt{5}}{m}(m \neq 0)$

Now, its distance from the center of the circle must be equal to the radius of the circle.

Therefore,

$\left|\frac{\sqrt{5}}{m}\right| = \frac{\sqrt{5}}{\sqrt{2}}\sqrt{1 + m^2} \quad \Rightarrow \quad (1 + m^2)m^2 = 2$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

$\Rightarrow (m^2 - 1)(m^2 + 2) = 0 \Rightarrow m = \pm 1$

And so, the common tangents are

$y = x + \sqrt{5}$ and $y = -x - \sqrt{5}$.

100. Foci $\equiv (\pm ae, 0)$

We have, $a^2e^2 = a^2 - b^2 = 7$

Now, the equation of circle is

$(x - 0)^2 + (y - 3)^2 = \left(\sqrt{7} - 0\right)^2 + (0 - 3)^2$

$\Rightarrow \quad x^2 + y^2 - 6y - 7 = 0.$

101. Let the foot of perpendicular be $P(h, k)$

Equation of tangent with slope m passing $P(h, k)$ is

$$y = mx \pm \sqrt{6m^2 + 2} \text{ where } m = -\frac{h}{k}$$

$$\Rightarrow \quad \sqrt{\frac{6h^2}{k^2} + 2} = \frac{h^2 + k^2}{k}$$

$6h^2 + 2k^2 = (h^2 + k^2)^2$

So required locus is $6x^2 + 2y^2 = (x^2 + y^2)^2$.

102. Equation of tangent at $A(t^2, 2t)$

$yt = x + t^2$ is tangent to $x^2 + 32y = 0$ at B

$$\Rightarrow \quad x^2 + 32\left(\frac{x}{t} + t\right) = 0$$

$$\Rightarrow \quad x^2 + \frac{32}{t}x + 32t = 0$$

$$\Rightarrow \quad \left(\frac{32}{t}\right)^2 - 4(32t) = 0$$

$$\Rightarrow \quad 32\left(\frac{32}{t^2} - 4t\right) = 0$$

$\Rightarrow \quad t^3 = 8 \quad \Rightarrow \quad t = 2.$

$\Rightarrow \quad$ Slope of tangent is $\frac{1}{t} = \frac{1}{2}$.

103. $h = \frac{4t}{4} = t$

And, $k = \frac{2t^2}{4} = \frac{t^2}{2}$

$\Rightarrow x^2 = 2y$

104. $a = 3, b = \sqrt{5}$

$$e = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$$

Foci = $(\pm 2, 0)$

Tangent at $P \quad \Rightarrow \quad \frac{2x}{9} + \frac{5y}{3.5} = 1$

$$\Rightarrow \quad \frac{2x}{9} + \frac{y}{3} = 1$$

$\Rightarrow \quad 2x + 3y = 9$

$\therefore \quad$ Area of quadrilateral

= 4 × (area of triangle QCR)

$$= \left(\frac{1}{2} \times \frac{9}{2} \times 3\right) \times 4 = 27$$

105. Given

$$\frac{2b^2}{a} = 8 \qquad (1)$$

$\Rightarrow \quad 2b = ae \qquad (2)$

We know

$b^2 = a^2(e^2 - 1) \qquad (3)$

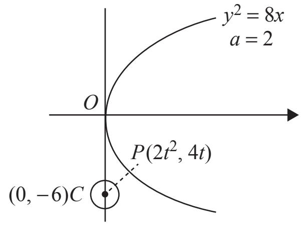
Putting $\frac{b}{a} = \frac{e}{2}$ from (2) in (3), we get

$$\frac{e^2}{4} = e^2 - 1$$

$\Rightarrow \quad 4 = 3e^2$

$$\Rightarrow \quad e = \frac{2}{\sqrt{3}}$$

106. Circle and parabola are as shown:



Minimum distance occurs along common normal.

Let the equation of normal to parabola

$y + tx = 2.2.t + 2t^3$

Since it passes through (0, –6)

$\therefore \quad -6 = 4t + 2t^3 \Rightarrow t^3 + 2t + 3 = 0$

$\Rightarrow \quad t = -1$ (only real value)

$\therefore \quad$ coordinates of point P are (2, –4).

$\therefore \quad CP = \sqrt{4 + 4} = 2\sqrt{2}$

$\therefore \quad$ equation of circle is

$$(x - 2)^2 + (y + 4)^2 = (2\sqrt{2})^2$$

$\Rightarrow \quad x^2 + y^2 - 4x + 8y + 12 = 0$

107. Consider line L at a distance of 6 units below x axis

$\Rightarrow \quad PC = PQ$

$\Rightarrow \quad P$ lies on a parabola, for which C is focus and L is directrix