CHAPTER
15

Indefinite Integration

Chapter Highlights

Indefinite Integral, Properties of Indefinite Integration, Standard Formulae of Integration, Methods of Integration, Some Useful Trigonometric Identities, Method of Substitution, Method of Integration by Parts, Method of Partial Fractions for Rational Functions, Some Special Integrals.

Integration is the inverse process of differentiation. The process of finding a function, whose differential coefficient is known, is called integration.

If the differential coefficient of $F(x)$ is $f(x)$, i.e.,

$$\frac{d}{dx}[F(x)] = f(x),$$

then we say that the **anti-derivative** or **integral** of $f(x)$ is F(x), written as

$$\int f(x)\,dx = f(x),$$

Here $\int \{\}\,dx$ is the notation of integration. $f(x)$ is the integrand, x is the variable of integration and dx denotes the integration with respect to x.

INDEFINITE INTEGRAL

We know that if $\frac{d}{dx}[F(x)] = f(x)$, then $\int f(x)\,dx = F(x)$.

Also, for any arbitrary constant C,

$$\frac{d}{dx}[F(x) + C] = \frac{d}{dx}[F(x)] + 0 = f(x)$$

$$\therefore \quad \int f(x)\,dx = F(x) + C,$$

This shows that $F(x)$ and $F(x) + C$ are both integrals of the same function $f(x)$. Thus, for different values of C, we obtain different integrals of $f(x)$. This implies that the integral of $f(x)$ is not definite. By virtue of this property $F(x)$ is called the indefinite integral of $f(x)$.

NOTE

If a function f (x) is continuous on in interval [a, b], then it has an anti-derivative.

PROPERTIES OF INDEFINITE INTEGRATION

1. $\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$
2. $\int f'(x)\,dx = \int \frac{d}{dx}[f(x)]\,dx = f(x)$
3. $\int k\,f(x)\,dx = \int k\,f(x)\,dx \quad dx$, where k is any constant
4. If $f_1(x), f_2(x), f_3(x)$, ... (finite in number) are functions of x, then
$$\int \left[f_1(x) \pm f_2(x) \pm f_3(x) \ldots\right] dx$$
$$= \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \int f_3(x)\,dx \pm \ldots$$
5. If $\int f(x)\,dx = F(x)$, then
$$\int f(ax \pm b)\,dx = \frac{1}{a}\,F(ax \pm b)$$

STANDARD FORMULAE OF INTEGRATION

The following results are a direct consequence of the definition of an integral.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
2. $\int \frac{1}{x}\,dx = \log|x| + C$
3. $\int e^x\,dx = e_x + C$
4. $\int a^x\,dx = \frac{a^x}{\log_e a} + C$

5. $\int \sin x \, dx = -\cos x + C$

6. $\int \cos x \, dx = \sin x + C$

7. $\int \sec^2 x \, dx = \tan x + C$

8. $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

9. $\int \sec x \tan x \, dx = \sec x + C$

10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

11. $\int \tan x \, dx = -\log|\cos x| + C = \log|\sec x| + C$

12. $\int \cot x \, dx = \log|\sin x| + C$

13. $\int \sec x \, dx = \log|\sec x + \tan x| + C$

14. $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$

15. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C; |x| < 1$

16. $\int \frac{dx}{(1+x^2)} = \tan^{-1} x + C$

17. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C; |x| > 1$

18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

19. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

20. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

IMPORTANT POINTS

In any of the fundamental integration formulae, if *x* is replaced by *ax* + *b*, then the same formulae is applicable but we must divide by coefficient of *x* or derivative of (*ax* + *b*) i.e., *a*. In general, if $\int f(x)dx = \phi(^{x}) + C$, then $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b) + C$.

SOLVED EXAMPLES

1. If $\int f(x)dx = f(x)$, then $\int [f(x)]^2 dx$ is equal to

(A) $\frac{1}{2}[f(x)]^2$ (B) $f(x)^2$

(C) $\frac{1}{3}[f(x)]^3$ (D) none of these

Solution: (A)

We have, $\int f(x)dx = f(x) \Rightarrow \frac{d}{dx}[f(x)] = f(x)$

$$\Rightarrow \int \frac{1}{f(x)} d[f(x)] = \int dx$$

$$\Rightarrow \log[f(x)] = x + \log c$$

$$\Rightarrow f(x) = ce^x$$

$$\therefore \int [f(x)]^2 dx = \int c^2 e^{2x} dx = \frac{c^2 e^{2x}}{2} = \frac{1}{2}[f(x)]^2$$

2. If $I = \int_0^1 \frac{dx}{1+x^{\pi/2}}$, then

(A) $I > \log 2$ (B) $I < \log 2$

(C) $I < \frac{\pi}{4}$ (D) $I > \frac{\pi}{4}$

Solution: (A, C)

As $0 < x < 1 \Rightarrow x^2 < x\pi^{/2} < x$

$$\Rightarrow \frac{1}{1+x} < \frac{1}{1+x^{\pi/2}} < \frac{1}{1+x^2}$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x} < \int_0^1 \frac{dx}{1+x^{\pi/2}} < \int_0^1 \frac{dx}{1+x^2}$$

$$\Rightarrow \log 2 < I < \frac{\pi}{2}$$

3. $\int \frac{dx}{x^2\left(1+x^4\right)^{3/4}}$ is equal to

(A) $-\frac{\left(1+x^4\right)^{1/4}}{x} + C$ (B) $\frac{\left(1+x^4\right)^{1/4}}{x} + C$

(C) $-\frac{\left(1+x^4\right)^{3/4}}{x} + C$ (D) none of these

Solution: (A)

$$\int \frac{dx}{x^2(1+x^4)^{3/4}} = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$$

$$= \frac{-1}{4}\int \frac{dt}{t^{3/4}} \quad \left(\text{Putting } 1+\frac{1}{x^4} = t \Rightarrow \frac{dx}{x^5} = \frac{-1}{4}\,dt\right)$$

$$= \frac{-1}{4}\cdot\frac{t^{1/4}}{1/4} + C = -\left(1+\frac{1}{x^4}\right)^{1/4} + C$$

$$= -\frac{(1+x^4)^{1/4}}{x} + C$$

4. If $f(x) = \int \dfrac{\left(x^2 + \sin^2 x\right)}{1+x^2}\sec^2 x\,dx$ and $f(0) = 0$, then $f(1) =$

(A) $1 - \dfrac{\pi}{4}$ (B) $\dfrac{\pi}{4} - 1$

(C) $\tan 1 - \dfrac{\pi}{4}$ (D) none of these

Solution: (C)

We have, $f(x) = \int \dfrac{\left(x^2 + \sin^2 x\right)}{1+x^2}\sec^2 x\,dx$

$$= \int \frac{x^2 + \left(1 - \cos^2 x\right)}{1+x^2}\cdot \sec^2 x\,dx$$

$$= \int \left(\sec^2 x - \frac{1}{1+x^2}\right) dx$$

$$= \tan x - \tan^{-1} x + C$$

$\because$ $f(0) = 0$, $\therefore$ $C = 0$

Thus, $f(x) = \tan x - \tan^{-1} x$.

Hence, $f(1) = \tan 1 - \tan^{-1} 1 = \tan 1 - \dfrac{\pi}{4}$

5. $\displaystyle\int \frac{\sin^3 x\,dx}{\left(\cos^4 x + 3\cos^2 x + 1\right)\tan^{-1}\left(\sec x + \cos x\right)} =$

(A) $\tan^{-1}(\sec x + \cos x) + c$

(B) $\log\left|\tan^{-1}(\sec x + \cos x)\right| + c$

(C) $\dfrac{1}{\left(\sec x + \cos x\right)^2} + c$

(D) none of these

Solution: (B)

$$I = \int \frac{\sin^3 x\,dx}{\left(\cos^4 x + 3\cos^2 x + 1\right)\tan^{-1}\left(\sec x + \cos x\right)}$$

Let $\tan^{-1}(\sec x + \cos x) = t$

$$\Rightarrow \frac{1}{1 + \left(\sec x + \cos x\right)^2}(\sec x \tan x - \sin x)\,dx = dt$$

$$\Rightarrow \frac{\sin^3 x dx}{\cos^4 x + 3\cos^2 x + \mathrm{I}} = dt$$

$$\therefore \quad I = \int \frac{dt}{t} = \log|t| + c$$

$$= \log\left|\tan^{-1}(\sec x + \cos x)\right| + C$$

6. If $f\left(\dfrac{3x-4}{3x+4}\right) = x + 2$, then $\int f(x)\,dx$ is equal to

(A) $e^{x-2}\ln\left(\dfrac{3x-4}{3x+4}\right) + c$

(B) $-\dfrac{8}{3}\ln|x-1| + \dfrac{2}{3}x + c$

(C) $\dfrac{8}{3}\ln|x-1| + \dfrac{2}{3}x + c$

(D) none of these

Solution: (B)

We have $f\left(\dfrac{3x-4}{3x+4}\right) = x + 2$

Let $\left(\dfrac{3x-4}{3x+4}\right) = t \Rightarrow 3x - 4 = 3xt + 4t$

$$\Rightarrow \quad x = \frac{4t+4}{3(1-t)}$$

$$\therefore \quad f(t) = \frac{4t+4}{3(1-t)} + 2$$

$$\therefore \quad f(x) = \frac{4x+4}{3(1-x)} + 2 \qquad = \frac{4(x-1)+8}{3(1-x)} + 2$$

$$= 2 - \frac{4}{3} - \frac{8}{3(x-1)}$$

$$\therefore \quad \int f(x)\,dx = \frac{2}{3}x - \frac{8}{3}\ln|x-1| + c$$

7. Let $f(x)$ be a polynomial of degree three satisfying $f(0) = -1$ and $f(1) = 0$. Also, 0 is a stationary point of $f(x)$. If $f(x)$ does not have an extremum at $x = 0$, then $\displaystyle\int \frac{f(x)}{x^3 - 1}\,dx$ is equal to

(A) $\dfrac{x^2}{2} + C$ (B) $x + C$

(C) $\dfrac{x^3}{6} + C$ (D) none of these

Solution: (B)

Let $f(x) = ax^3 + bx^2 + cx + d$.

Since $f(0) = -1$ and $f(1) = 0$

$\Rightarrow$ $d = -1$ and $a + b + c + d = 0$

$\Rightarrow$ $d = -1$ and $a + b + c = 1$ ...(1)

Since 0 is a stationary point of $f(x)$, $\therefore f'(0) = 0$

$\Rightarrow$ $3a(0)^2 + 2b(0) + c = 0 \Rightarrow c = 0$

Since $f(x)$ does not have an extremum at $x = 0$,

$\therefore$ $f''(0) = 0 \Rightarrow b = 0$. and $\therefore$ from (1), $a = 1$.

So, $f(x) = x^3 - 1$

$$\therefore \quad \int \frac{f(x)}{x^3 - 1}\,dx = \int 1\,dx = x + C$$

METHODS OF INTEGRATION

Method of Transformation

When the integrand is a trigonometric function, we transform the given function into standard integrals or their algebraic sum by using trigonometric formulae:

SOME USEFUL TRIGONOMETRIC IDENTITIES

NOTE

- $\sin^2 mx = \dfrac{1-\cos 2mx}{2}$
- $\cos^2 mx = \dfrac{1+\cos 2mx}{2}$
- $\sin mx = 2\sin \dfrac{\sin mx}{2} \cos \dfrac{mx}{2}$
- $\sin^3 mx = \dfrac{3\sin mx - \sin 3mx}{4}$
- $\cos^3 mx = \dfrac{3\cos mx + \cos 3mx}{4}$
- $\tan^2 mx = \sec^2 mx - 1$
- $\cot^2 mx = \operatorname{cosec}^2 mx - 1$
- $2\cos A \cos B = \cos(A+B) + \cos(A-B)$
- $2\sin A \cos B = \sin(A+B) + \sin(A-B)$
- $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

SOLVED EXAMPLE

8. If $\int \dfrac{\cos 8x + 1}{\tan 2x - \cot 2x}\, dx = a\cos 8x + C$, then

(A) $a = \dfrac{-1}{16}$ (B) $a = \dfrac{1}{8}$

(C) $a = \dfrac{1}{16}$ (D) $a = \dfrac{-1}{8}$.

Solution: (C)

$$\int \frac{\cos 8x + 1}{\tan 2x - \cot 2x}\, dx$$

$$= \int \frac{2\cos^2 4x}{\sin^2 2x - \cos^2 2x} \cdot \sin 2x \cos 2x\, dx$$

$$= -\int \sin 4x \cos 4x\, dx = \frac{-1}{2}\int \sin 8x\, dx$$

$$= \frac{1}{16}\cos 8x + C. \quad \therefore \quad a = \frac{1}{16}$$

METHOD OF SUBSTITUTION

By suitable substitution, the variable x in $\int f(x)\, dx$ is changed into another variable t so that the integrand $f(x)$ is changed into $F(t)$ which is some standard integral or algebraic sum of standard integrals.

There is no general rule for finding a proper substitution and the best guide in this matter is experience. However, the following suggestions will prove useful.

1. If the integrand is of the form $f'(ax+b)$, then we put $ax + b = t$ and $dx = \dfrac{1}{a}\, dt$

Thus, $$\int f'(ax+b)\, dx = \int f'(t)\frac{dt}{a}$$
$$= \frac{1}{a}\int f'(t)\, dt$$
$$= \frac{f(t)}{a} = \frac{f(ax+b)}{a}.$$

2. When the integrand is of the form $x^{n-1} f'(x_n)$, we put $x^n = t$ and $nx^{n-1}\, dx = dt$.

Thus, $$\int x^{n-1} f'(x^n)\, dx = \int f'(t)\frac{dt}{n}$$
$$= \frac{1}{n}\int f'(t)\, dt$$
$$= \frac{1}{n} f(t) = \frac{1}{n} f(x^n).$$

3. When the integrand is of the form $[f(x)]^n . f'(x)$, we put $f(x) = t$ and $f'(x)\, dx = dt$.

Thus, $$\int [f(x)]^n f'(x)\, dx = \int t^n\, dt = \frac{t^{n+1}}{n+1}$$
$$= \frac{[f(x)]^{n+1}}{n+1}$$

4. When the integrand is of the form $\dfrac{f'(x)}{f(x)}$, we put $f(x) = t$ and $f'(x)\, dx = dt$

Thus, $$\int \frac{f'(x)}{f(x)}\, dx = \int \frac{dt}{t} = \log t = \log f(x)$$

SOLVED EXAMPLES

9. $\int \sqrt{\sec x - 1}\, dx$ is equal to

(A) $2\log\left(\cos\dfrac{x}{2} + \sqrt{\cos^2\dfrac{x}{2} - \dfrac{1}{2}}\right) + C$

(B) $\log\left(\cos\dfrac{x}{2} + \sqrt{\cos^2\dfrac{x}{2} - \dfrac{1}{2}}\right) + C$

(C) $-2\log\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$

(D) none of these

Solution: (C)

$$\int\sqrt{\sec x-1}\;dx=\int\sqrt{\frac{1-\cos x}{\cos x}}\;dx$$

$$=\sqrt{2}\int\frac{\sin\frac{x}{2}}{\sqrt{2\cos^2\frac{x}{2}-1}}\,dx=-2\sqrt{2}\int\frac{dz}{\sqrt{2z^2-1}}$$

$$\left(\text{Putting }\cos\frac{x}{2}=z\Rightarrow\sin\frac{x}{2}\;dx=-2dz\right)$$

$$=-2\int\frac{dz}{\sqrt{z^2-\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$=-2\log\left[z+\sqrt{z^2-\left(\frac{1}{\sqrt{2}}\right)^2}\right]+C$$

$$=-2\log\left(\cos\frac{x}{2}+\sqrt{\cos^2\frac{x}{2}-\frac{1}{2}}\right)+C$$

10. $\int\frac{dx}{\cos x\sqrt{\cos 2x}}$ is equal to

(A) $\sin^{-1}(\tan x)+C$ (B) $\cos^{-1}(\tan x)+C$
(C) $\tan^{-1}(\sin x+C$ (D) none of these

Solution: (A)

$$\int\frac{dx}{\cos x\sqrt{\cos 2x}}=\int\frac{dx}{\cos x\sqrt{\frac{1-\tan^2 x}{1+\tan^2 x}}}$$

$$=\int\frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx=\int\frac{dz}{\sqrt{1-z^2}}$$

[Putting $\tan x=z\Rightarrow\sec^2 x\,dx=dz$]
$=\sin^{-1}z+C=\sin^{-1}(\tan x)+C$

11. $\int\frac{dx}{\cos^3 x\sqrt{\sin 2x}}$ is equal to

(A) $\sqrt{2}\left(\sqrt{\cot x}+\frac{1}{5}\tan^{5/2}x\right)+C$

(B) $\sqrt{2}\left(\sqrt{\tan x}+\frac{1}{5}\tan^{5/2}x\right)+C$

(C) $\sqrt{2}\left(\sqrt{\tan x}+\frac{1}{5}\tan^{5/2}x\right)+C$

(D) none of these

Solution: (B)

$$\int\frac{dx}{\cos^3 x\sqrt{\sin 2x}}=\int\frac{dx}{\cos^3 x\sqrt{\frac{2\tan x}{1+\tan^2 x}}}$$

$$=\int\frac{\sec^4 x}{\sqrt{2\tan x}}\,dx=\frac{1}{\sqrt{2}}\int\frac{(1+z^4).\,2z\,dz}{z}$$

(Putting $\tan x=z^2\Rightarrow\sec^2 x\,dx=2z\,dz$)

$$=\sqrt{2}\left(z+\frac{z^5}{5}\right)+C=\sqrt{2}\left(\sqrt{\tan x}+\frac{1}{5}\tan^{5/2}x\right)+C.$$

12. $\int\frac{\cos x+x\sin x}{x(x+\cos x)}\,dx$ is equal to

(A) $\log\left|\frac{x}{x+\cos x}\right|+C$ (B) $\log\left|\frac{x+\cos x}{x}\right|+C$

(C) $\log\left|\frac{1}{x+\cos x}\right|+C$ (D) $\log|x+\cos x|+C$

Solution: (A)

$$\int\frac{\cos x+x\sin x}{x(x+\cos x)}\,dx=\int\frac{(x+\cos x)-x(1-\sin x)}{x(x+\cos x)}\,dx$$

$$=\int\left(\frac{1}{x}-\frac{1-\sin x}{x+\cos x}\right)dx$$

$$=\log|x|-\log|x+\cos x|+C$$

$$=\log\left|\frac{x}{x+\cos x}\right|+C.$$

13. $\int\sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$ is equal to

(A) $(x+a)\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{ax}+C$

(B) $(x+a)\tan^{-1}\sqrt{\frac{x}{a}}+\sqrt{ax}+C$

(C) $(x+a)\cot^{-1}\sqrt{\frac{x}{a}}-\sqrt{ax}+C$

(D) none of these

Solution: (A)

Put $x=a\tan^2\theta\Rightarrow dx=2a\tan\theta\sec^2\theta\,d\theta$

$$\therefore\quad\int\sin^{-1}\sqrt{\frac{x}{a+x}}=\int\theta.\,2a\tan\theta\sec^2\theta\,d\theta$$

$$=2a\left(\theta.\frac{\tan^2\theta}{2}-\int\frac{\tan^2\theta}{2}\,d\theta\right)$$

$$=a\theta\tan^2\theta-a\int(\sec^2\theta-1)\,d\theta$$

$$=a\theta\tan^2\theta-a(\tan\theta-\theta)+C$$

$$=a\theta\,(1+\tan^2\theta)-a\tan\theta+C$$

$$=(x+a)\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{ax}+C$$

14. $\int \dfrac{dx}{\left(1+\sqrt{x}\right)\sqrt{x-x^2}}$ is equal to

(A) $\dfrac{2\left(\sqrt{x}-1\right)}{\sqrt{1-x}}+C$ (B) $\dfrac{2\left(1-\sqrt{x}\right)}{\sqrt{1-x}}+C$

(C) $\dfrac{\left(\sqrt{x}-1\right)}{\sqrt{1-x}}+C$ (D) none of these

Solution: (A)

$$\int \frac{dx}{\left(1+\sqrt{x}\right)\sqrt{x-x^2}}$$

$$= \int \frac{2\sin\theta\cos\theta\, d\theta}{(1+\sin\theta)\sqrt{\sin^2\theta-\sin^4\theta}}$$

(Putting $x=\sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$)

$$= 2\int \frac{d\theta}{1+\sin\theta} = 2\int \frac{1-\sin\theta}{\cos^2\theta}\, d\theta = 2\,(\tan\theta-\sec\theta)$$

$$= 2\left(\frac{\sqrt{x}}{\sqrt{1-x}}-\frac{1}{\sqrt{1-x}}\right)+C = \frac{2\left(\sqrt{x}-1\right)}{\sqrt{1-x}}+C$$

15. $\int \dfrac{\sin^3 x\, dx}{\left(1+\cos^2 x\right)\sqrt{1+\cos^2 x+\cos^4 x}}$ is equal to

(A) $\sec^{-1}(\sec x+\cos x)+C$
(B) $\sec^{-1}(\sec x-\cos x)+C$
(C) $\sec^{-1}(\cos x-\tan x)+C$
(D) none of these

Solution: (A)

$$\int \frac{\sin^3 x}{\left(1+\cos^2 x\right)\sqrt{1+\cos^2 x+\cos^4 x}}\, dx$$

$$= \int \frac{\sin^3 x}{\cos x\left(\sec x+\cos x\right)\cos x\sqrt{\sec^2 x+1+\cos^2 x}}\, dx$$

$$= \int \frac{\sin^3 x\, dx}{\cos^2 x\left(\sec x+\cos x\right)\sqrt{\left(\sec x+\cos x\right)^2-1}}$$

$$= \int \frac{dz}{z\sqrt{z^2-1}}$$

$$\left(\text{Putting } \sec x+\cos x = z \Rightarrow \frac{\sin^3 x}{\cos^2 x}\, dx = dz\right)$$

$= \boldsymbol{\sec^{-1} z + c} = \sec^{-1}(\sec x+\cos x)+C$

16. $\int \dfrac{x^2-2}{x^3\sqrt{x^2-1}}\, dx$ is equal to

(A) $\dfrac{x^2}{\sqrt{x^2-1}}$ (B) $-\dfrac{x^2}{\sqrt{x^2-1}}$

(C) $\dfrac{\sqrt{x^2-1}}{x^2}$ (D) $-\dfrac{\sqrt{x^2-1}}{x^2}$

Solution: (D)

$$\int \frac{x^2-2}{x^3\sqrt{x^2-1}}\, dx = \int \frac{dx}{x\sqrt{x^2-1}} - 2\int \frac{dx}{x^3\sqrt{x^2-1}}$$

$$= \sec^{-1}x - 2\int \frac{\sec\theta\tan\theta}{\sec^3\theta\tan\theta}\, d\theta$$

(Putting $x=\sec\theta \Rightarrow dx = \sec\theta\tan\theta\, d\theta$)

$$= \sec^{-1}x - 2\int \cos^2\theta\, d\theta$$

$$= \sec^{-1}x - \int \left(1+\cos 2\theta\right)\, d\theta$$

$$= \sec^{-1}x - \left(\theta+\frac{\sin 2\theta}{2}\right)+C$$

$$= \sec^{-1}x - \sec^{-1}x - \frac{\sqrt{x^2-1}}{x^2}+C$$

$$= -\frac{\sqrt{x^2-1}}{x^2}+\boldsymbol{C}$$

17. Integral of $\dfrac{1}{\sqrt{x^2+4}}$ with respect to (x^2+3) is equal to

(A) $\sqrt{x^2+4}+C$ (B) $\dfrac{1}{\sqrt{x^2+4}}+C$

(C) $2\sqrt{x^2+4}+C$ (D) none of these

Solution: (C)

$$\int \frac{d\left(x^2+3\right)}{\sqrt{x^2+4}} = \int \frac{d\left(x^2+3\right)}{\sqrt{\left(x^2+3\right)+1}}$$

$= \int \dfrac{dt}{\sqrt{t+1}}$ [Putting $x^2+3=t \Rightarrow d(x^2+3)=dt$]

$$= 2\sqrt{t+1}+C = 2\sqrt{x^2+4}+C$$

METHOD OF INTEGRATION BY PARTS

The process of integration of the product of two functions is known as integration by parts.

For example, if u and v are two functions of x, then

$$\int (uv)\, dx = u\int v dx - \int\left(\frac{du}{dx}\cdot\int v dx\right) dx$$

In words, integral of the product of two functions = first function × integral of the second – integral of (differential of first × integral of the second function).

TRICK(S) FOR PROBLEM SOLVING

- Choose the first and second function in such a way that the derivative of the first function and the integral of the second function can be easily found.
- In case of integrals of the form $\int f(x) . x^n dx$, take x^n as the first function and $f(x)$ as the second function.
- In case of integrals of the form $\int (\log x)^n \cdot dx$, take 1 as the second function and $(\log x)^n$ as the first function.
- Rule of integration by parts may be used repeatedly, if required.
- If the two functions are of different type, we can choose the first function as the one whose initial comes first in the word "ILATE", where

  I — Inverse Trigonometric function
  L — Logarithmic function
  A — Algebraic function
  T — Trigonometric function
  E — Exponential function.
- In case, both the functions are trigonometric, take that function as second function whose integral is simpler. If both the functions are algebraic, take that function as first function whose derivative is simpler.
- If the integral consists of an inverse trigonometric function of an algebraic expression in x, first simplify the integrand by a suitable trigonometric substitution and then integrate the new integrand.

IMPORTANT POINTS

Integration by parts is useful in evaluating integrals of the type $\int P_n(x) f(x)\, dx$, where $P_n(x)$ is a polynomial of degree *n* and *f* (*x*) is such that it can be easily integrated *n* + 1 times.

SOLVED EXAMPLES

18. If $\int \frac{x^2}{(x\sin x+\cos x)^2}\,dx$

$$= \frac{f(x)}{x\sin x+\cos x} + \tan x + C, \text{ then}$$

(A) $f(x) = \frac{x}{\cos x}$ (B) $f(x) = \frac{\cos x}{x}$

(C) $f(x) = \frac{-x}{\cos x}$ (D) none of these

Solution: (C)

Put $\frac{1}{x\sin x+\cos x} = z$

$$\Rightarrow \frac{-1}{(x\sin x+\cos x)^2}(\sin x + x\cos x - \sin x)\,dx = dz$$

$$\Rightarrow \int \frac{x\cos x}{(x\sin x+\cos x)^2}\,dx = -\,dz,$$

$$\therefore \int \frac{x^2}{(x\sin x+\cos x)^2}\,dx = -z$$

$$\therefore \int \frac{x}{\cos x}\cdot\frac{x\cos x}{(x\sin x+\cos x)^2}\,dx$$

$$= \int \frac{x}{\cos x}\cdot\frac{x\cos x}{(x\sin x+\cos x)^2}\,dx$$

$$= \frac{x}{\cos x}.\,(-z) - \int (-\,z)\left(\frac{\cos x.1 + x\sin x}{\cos^2 x}\right)dx$$

$$= \frac{-\,x}{\cos x\,(x\sin x+\cos x)} + \int \sec^2 x\,dx$$

$$= \frac{-\,x}{\cos x\,(x\sin x+\cos x)} + \tan x + C$$

$$\therefore f(x) = \frac{-\,x}{\cos x}$$

19. If $\int f(x)\,dx = F(x)$, then $\int x^3 f(x^2)\,dx$ is equal to

(A) $\frac{1}{2}\left[x^2 (F(x))^2 - \int (F(x))^2\,dx\right]$

(B) $\frac{1}{2}\left[x^2 . F(x^2) - \int F(x^2)\,d(x^2)\right]$

(C) $\frac{1}{2}\left[x^2 F(x) - \frac{1}{2}\int (F(x))^2\,dx\right]$

(D) none of these

Solution: (B)

$$\int x^3 f(x^2)\,dx = \int x^2 f(x^2)\cdot x\,dx$$

$$= \frac{1}{2}\int z\,f(z)\,dz \qquad \left[\text{Putting } x^2 = z \Rightarrow xdx = \frac{1}{2}dz\right]$$

$$= \frac{1}{2}\left[z\,F(z) - \int 1\cdot F(z)\,dz\right] \quad \left[\because \int f(x)\,dx = F(x)\right]$$

$$= \frac{1}{2} z\,F(z) - \frac{1}{2}\int F(z)\,dz$$

$$= \frac{1}{2}\boldsymbol{x^2 F(x^2)} - \frac{1}{2}\int F(x^2)\,\boldsymbol{d(x^2)}.$$

METHOD OF PARTIAL FRACTIONS FOR RATIONAL FUNCTIONS

Integrals of the type $\int \frac{p(x)}{g(x)}\,dx$ can be integrated by resolving the integrand into partial fractions. We proceed as follows:

- Check degree of $p(x)$ and $g(x)$.
- If degree of $p(x)$ > degree of $g(x)$, then divide $p(x)$ by $g(x)$ till its degree is less, i.e., put in the form

$$\frac{p(x)}{g(x)} = r(x) + \frac{f(x)}{g(x)}$$

where degree of $f(x)$ < degree of $g(x)$.

CASE 1: When the denominator contains non-repeated linear factors. That is

$g(x) = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$.

In such a case write $f(x)$ and $g(x)$ as:

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \dots + \frac{A_n}{x-\alpha_n}$$

where $A_1, A_2, \dots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

CASE 2: When the denominator contains repeated as well as non-repeated linear factors. That is

$g(x) = (x - \alpha_1)^2 (x - \alpha_3) \dots (x - \alpha_n)$.

In such a case express $f(x)$ and $g(x)$ as:

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{(x-\alpha_1)^2} + \frac{A_3}{x-\alpha_3}$$

$$+ \dots + \frac{A_n}{(x-\alpha_n)}$$

where $A_1, A_2, \dots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

CASE 3: When the denominator contains a non repeated quadratic factor which cannot be factorised further:

$g(x) = (ax^2 + bx + c)(x - \alpha_3)(x - \alpha_4) \dots (x - \alpha_n)$

In such a case express $f(x)$ and $g(x)$ as

$$\frac{f(x)}{g(x)} = \frac{A_1 x + A_2}{ax^2 + bx + c} + \frac{A_3}{x-\alpha_3} + \dots + \frac{A_n}{x-\alpha_n}$$

where $A_1, A_2, \dots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

CASE 4: When the denominator contains a repeated quadratic factor which cannot be factorised further. That is

$g(x) = (ax^2 + bx + c)^2 (x - \alpha_5)(x - \alpha_6)$
$\dots (x - \alpha_n)$

In such a case write $f(x)$ and $g(x)$ as

$$\frac{f(x)}{g(x)} = \frac{A_1 x + A_2}{ax^2 + bx + c} + \frac{A_3 x + A_4}{(ax^2 + bx + c)^2}$$

$$+ \frac{A_5}{x-\alpha_5}$$

$$+ \dots + \frac{A_n}{(x-\alpha_n)}$$

where $A_1, A_2, \dots A_n$ are constants to be determined by comparing the coefficients of various powers of x on both sides after taking L.C.M.

NOTE

Corresponding to repeated linear factor $(x - a)^r$ in the denominator, a sum of r partial fractions of the type $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_r}{(x-a)^r}$ is taken.

CASE 5: If the integrand contains only even powers of x

(i) Put $x^2 = z$ in the integrand.

(ii) Resolve the resulting rational expression in z into partial fractions

(iii) Put $z = x^2$ again in the partial fractions and then integrate both sides.

SOLVED EXAMPLE

20. If $f\left(\frac{3x-4}{3x+4}\right) = x + 2$, then $\int f(x)\,dx =$ $\frac{2}{3}x + \frac{k}{3}\log|x-1|$, where k is equal to

(A) –4 (B) 4
(C) –8 (D) 8

Solution: (C)

We have, $f\left(\frac{3x-4}{3x+4}\right) = x + 2$

Let $\frac{3x-4}{3x+4} = u \quad \Rightarrow \quad 3x - 4 = 3xu + 4u$

$$\Rightarrow \quad x = \frac{4u+4}{3(1-u)}$$

$$\therefore \quad f(u) = \frac{4u+4}{3(1-u)} + 2$$

$$\therefore \quad f(x) = \frac{4x+4}{3(1-x)} + 2 = \frac{4(x-1)+8}{3(1-x)} + 2$$

$$= 2 - \frac{4}{3} - \frac{8}{3(x-1)}$$

$$\therefore \quad \int f(x)\,dx = \frac{2}{3}x - \frac{8}{3}\log|x-1|$$

$$\therefore \quad k = -8$$

SOME SPECIAL INTEGRALS

1. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

2. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

3. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

4. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

5. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

6. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

7. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

8. $\int \sqrt{x^2 + a^2}\, dx$

$= \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

9. $\int \sqrt{x^2 - a^2}\, dx$

$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

INTEGRALS OF THE FORM

(A) $\int f\left(a^2 - x^2\right) dx$, (B) $\int f\left(a^2 + x^2\right) dx$,

(C) $\int f\left(x^2 - a^2\right) dx$, (D) $\int f\left(\frac{a-x}{a+x}\right)$ **dx**,

TRICK(S) FOR PROBLEM SOLVING

Integral	**Substitution**
$\int f\left(a^2 - x^2\right) dx$	$x = a \sin\theta$ or $x = a\cos\theta$
$\int f\left(x^2 + a^2\right) dx$	$x = a\tan\theta$ or $x = a\cot\theta$
$\int f\left(x^2 - a^2\right) dx$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
$\int f\left(\frac{a-x}{a+x}\right) dx$ or $\int f\left(\frac{a+x}{a-x}\right) dx$	$x = a\cos 2\theta$

INTEGRALS OF THE FORM $\int \sin^m x \cos^n x\, dx$

WORKING RULE

(i) If the power of $\sin x$ is an odd positive integer, put $\cos x = z$.

(ii) If the power of $\cos x$ is an odd positive integer, put $\sin x = z$.

(iii) If the power of $\sin x$ and $\cos x$ are both odd positive integers, put $\sin x = z$ or $\cos x = z$.

(iv) If the power of $\sin x$ and $\cos x$ are both even positive integers, use De' Moivre's theorem as follows:

Let $\cos x + i\sin x = z$. Then $\cos x - i\sin x = z^{-1}$

Adding these, we get

$z + \frac{1}{z} = 2\cos x$ and $z - \frac{1}{z} = 2i\sin x$

By De' Moivre's theorem, we have

$z^n + \frac{1}{z^n} = 2\cos nx$ and $z^n - \frac{1}{z^n} = 2i\sin nx$ ...(1)

$$\therefore \quad \sin^m x\cos^n x = \frac{1}{(2i)^m}\cdot\frac{1}{2^n}\left(z + \frac{1}{z}\right)^n\left(z - \frac{1}{z}\right)^m$$

$$= \frac{1}{2^{m+n}}\cdot\frac{1}{i^m}\left(z + \frac{1}{z}\right)^n\left(z - \frac{1}{z}\right)^m$$

Now expand each of the factors on the R.H.S. using Binomial theorem. Then group the terms equidistant from the beginning and the end. Thus express all such pairs as the sines or cosines of multiple angles. Further integrate term by term.

Alternative Method:

Simplify the integral as the sum of since and cosines of multiple angles, using

$\sin^2 x = \frac{1 - \cos 2x}{2}$, $\cos^2 x = \frac{1 + \cos 2x}{2}$

$\sin x\cos x = \frac{\sin 2x}{2}$, $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$\cos^3 x = \frac{3\cos x + \cos 3x}{4}$

(v) If the sum of powers of $\sin x$ and $\cos x$ is an even negative integer, put $\tan x = z$ and $dx = \frac{dz}{1 + z^2}$

Simplify the integrand using

$\sin x = \frac{z}{\sqrt{1 + z^2}}$, $\cos x = \frac{1}{\sqrt{1 + z^2}}$

INTEGRALS OF THE FORM

$$\int \tan^m x\sec^n x\, dx \text{ and } \int \cot^m x\,\text{cosec}^n x\, dx$$

WORKING RULE

(i) If the power of sec$x$ is an even positive integer, put tan$x$ = t

(ii) If the power of sec$x$ is an odd positive integer, then look of the power of tan$x$ i.e., m

- If m is an odd positive integer, put sec$x$ = t.
- If m is 0, use integration by parts
- If m an even positive integer, write sec^2x – 1 in place of tan^2x; the integrand reduces to a polynomial in sec$x$; use integration by parts.

Similarly $\int \cot^m x \operatorname{cosec}^n x \, dx$ can be tacked.

INTEGRALS OF THE FORM

$$\text{(A)} \int \frac{dx}{a + b\cos x}, \qquad \text{(B)} \int \frac{dx}{a + b\sin x},$$

$$\text{(C)} \int \frac{dx}{a + b\cos x + c\sin x}$$

WORKING RULE

(i) Put cos$x$ = $\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and sin$x$ = $\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ so that the given integrand becomes a function of tan $\frac{x}{2}$.

(ii) Put tan $\frac{x}{2}$. = $z \Rightarrow \frac{1}{2}$ sec^2 $\frac{x}{2}$. $dx = dz$

(iii) Integrate the resulting rational algebraic function of z

(iv) In the answer, put z = tan $\frac{x}{2}$.

INTEGRALS OF THE FORM

$$\text{(A)} \int \frac{dx}{a + b\cos^2 x}, \quad \text{(B)} \int \frac{dx}{a + b\sin^2 x},$$

$$\text{(C)} \int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x}$$

WORKING RULE

(i) Divide the numerator and denominator by cos^2x.

(ii) In the denominator, replace sec^2x, if any, by 1 + tan^2x.

(iii) Put tan$x$ = $z \Rightarrow$ sec^2x $dx = dz$.

(iv) Integrate the resulting rational algebraic function of z.

(iv) In the answer, put z = tan$x$.

INTEGRALS OF THE FORM

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\, dx$$

WORKING RULE

(i) Put Numerator = λ (denominator) + μ(derivative of denominator)

a cos$x$ + b sin$x$ = λ (c cos$x$ + d sin$x$) + μ (– c sin$x$ + d cos$x$).

(ii) Equate coefficients of sin$x$ and cos$x$ on both sides and find the values of λ and μ.

(iii) Split the given integral into two integrals and evaluate each integral separately, i.e.

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\, dx = \lambda \int 1\, dx + \mu \int \frac{-c\sin x + d\cos x}{a\cos x + b\sin x}\, dx$$

= λx + μ log | a cos$x$ + b sin$x$ |

(iv) Substitute the values of λ and μ found in step 2.

INTEGRALS OF THE FORM

$$\int \frac{a + b\cos x + c\sin x}{e + f\cos x + g\sin x}\, dx$$

WORKING RULE

(i) Put Numerator = l (denominator) + m (derivative of denominator) + n

a + b cos$x$ + c sin$x$ = l (e + f cos$x$ + g sin$x$) + m (– f sin$x$ + g cos$x$) + n

(ii) Equate coefficients of sin$x$, cos$x$ and constant term on both sides and find the values of l, m, n.

(iii) Split the given integral into three integrals and evaluate each integral separately, i.e.,

$$\int \frac{a + b\cos x + c\sin x}{e + f\cos x + g\sin x}\, dx$$

$$= l \int 1 dx + m \int \frac{-f\sin x + g\cos x}{e + f\cos x + g\sin x}\, dx$$

$$+ n \int \frac{dx}{e + f\cos x + g\sin x}$$

= lx + m log | e + f cos$x$ + g sin$x$ |

$$+ n \int \frac{dx}{e + f\cos x + g\sin x}\, dx$$

(iv) Substitute the values of l, m, n found in Step (ii).

INTEGRALS OF THE FORM

$$\int \frac{\cos x + \sin x}{1 - \sin x \cos x}\,dx \text{ and } \int \frac{\cos x - \sin x}{1 - \sin x \cos x}\,dx$$

WORKING RULE

Let $I_1 = \int \frac{\cos x + \sin x}{1 - \sin x \cos x}\,dx$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\,dx = dt$

Also, $2 \sin x \cos x = 1 - (\sin x - \cos x)^2 = 1 - t^2$

$$\therefore \quad I_1 = \int \frac{dt}{1 - \left(\frac{1-t^2}{2}\right)} = \int \frac{2\,dt}{1+t^2} = 2\tan^{-1} t + C$$

$$= 2\tan^{-1}(\sin x - \cos x) + C$$

Let $I_2 = \int \frac{\cos x - \sin x}{1 - \sin x \cos x}\,dx$

Put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\,dx = dt$

Also, $2 \sin x \cos x = (\sin x + \cos x)^2 - 1 = t^2 - 1$

$$\therefore \quad I_2 = \int \frac{dt}{1 - \left(\frac{t^2-1}{2}\right)} = \int \frac{2}{3 - t^2}\,dt$$

$$= -\frac{1}{\sqrt{3}} \int \left(\frac{1}{t - \sqrt{3}} - \frac{1}{t + \sqrt{3}}\right) dt$$

$$= -\frac{1}{\sqrt{3}} \log \left| \frac{t - \sqrt{3}}{t + \sqrt{3}} \right| + C$$

$$= -\frac{1}{\sqrt{3}} \log \left| \frac{\sin x + \cos x - \sqrt{3}}{\sin x + \cos x + \sqrt{3}} \right| + C$$

INTEGRALS OF THE FORM

$$\text{(A)} \int \frac{dx}{ax^2 + bx + c}, \quad \text{(B)} \int \frac{dx}{\sqrt{ax^2 + bx + c}},$$

$$\text{(C)} \int \sqrt{ax^2 + bx + c}\,\mathbf{dx}$$

WORKING RULE

(i) Make the coefficient of x^2 unity by taking the coefficient of x^2 outside the quadratic.

(ii) Complete the square in the terms involving x, i.e. write $ax^2 + bx + c$ in the form $a\,[x \pm \alpha)^2 \pm \beta^2]$.

(iii) The integrand is converted to one of the nine special integrals.

(iv) Integrate the function.

INTEGRALS OF THE FORM

$$\text{(A)} \int \frac{px + q}{ax^2 + bx + c}\,dx,$$

$$\text{(B)} \int \frac{px + q}{\sqrt{ax^2 + bx + c}}\,dx,$$

$$\text{(C)} \int (px + q)\sqrt{ax^2 + bx + c}\;dx$$

INTEGRAL	WORKING RULE
$\int \frac{px+q}{ax^2+bx+c}\,dx$	Put $px + q = \lambda\,(2ax + b) + \mu$ or $px + q = \lambda$ (derivative of quadratic) $+ \mu$. Comparing the coefficient of x and constant term on both sides, we get $p = 2a\lambda$ and $q = b\lambda + \mu \Rightarrow \lambda = \frac{p}{2a}$ and $\mu = \left(q - \frac{bp}{2a}\right)$
$\int \frac{px+q}{ax^2+bx+c}\,dx$	Then the integral becomes $= \frac{p}{2a} \int \frac{2ax+b}{ax^2+bx+c}\,dx + \left(q - \frac{bp}{2a}\right) \int \frac{dx}{ax^2+bx+c}$ $= \frac{p}{2a} \log \mid ax^2 + bx + c \mid + \left(q - \frac{bp}{2a}\right) \int \frac{dx}{ax^2+bx+c}$
$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$	In this case the integral becomes $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx = \frac{p}{2a} \int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx$ $+ \left(q - \frac{bp}{2a}\right) \int \frac{dx}{\sqrt{ax^2+bx+c}}$ $= \frac{p}{a} \sqrt{ax^2+bx+c} + \left(q - \frac{bp}{2a}\right) \int \sqrt{ax^2+bx+c}\;dx$

$\int (px+q)\sqrt{ax^2+bx+c}\,dx$	The integral in this case is converted to $\int (px+q)\sqrt{ax^2+bx+c}\,dx = \frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\,dx$ $+\left(q-\frac{bp}{2a}\right)\int \sqrt{ax^2+bx+c}\,dx$ $= \frac{p}{3a}(ax^2+bx+c)^{3/2}$ $+\left(q-\frac{bp}{2a}\right)\int \sqrt{ax^2+bx+c}\,dx$

INTEGRALS OF THE FORM

$\int \frac{P(x)}{\sqrt{ax^2+bx+c}}$ ***dx*, where P(*x*) is a polynomial in *x* of degree *n*.**

WORKING RULE

Write

$$\int \frac{P(x)}{\sqrt{ax^2+bx+c}}\,dx = (a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1})\sqrt{ax^2+bx+c} + k\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

where k, a_0, a_1, ... a_{n-1} are constants to be determined by differentiating the above relation and equating the coefficients of various powers of x on both sides.

INTEGRALS OF THE FORM

$\int \frac{x^2+1}{x^4+kx^2+1}$ ***dx* or** $\int \frac{x^2-1}{x^4+kx^2+1}$ ***dx*, where *k* is a constant positive, negative or zero.**

WORKING RULE

(i) Divide the numerator and denominator by x^2.

(ii) Put $x - \frac{1}{x} = z$ or $x + \frac{1}{x} = z$

whichever substitution, on differentiation gives, the numerator of the resulting integrand.

(iii) Evaluate the resulting integral in z.

(iv) Express the result in terms of x.

INTEGRALS OF THE FORM

$\int \frac{x^2+a^2}{x^4+kx^2+a^4}dx$, $\int \frac{x^2-a^2}{x^4+kx^2+a^4}dx$, where k is a constant positive, negative or zero.

WORKING RULE

These integrals can be obtained by dividing numerator and denominator by x^2, then putting $x - \frac{a^2}{x}$ and $x + \frac{a^2}{x} = t$ respectively.

INTEGRALS OF THE FORM

$\int \frac{dx}{P\sqrt{Q}}$**, where *P*, *Q* are linear or quadratic functions of *x*.**

TRICK(S) FOR PROBLEM SOLVING

Integral	Substitution
$\int \frac{1}{(ax+b)\sqrt{cx+d}}\,dx$	$cx + d = z^2$
$\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$	$px + q = z^2$
$\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$	$px + q = \frac{1}{z}$
$\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$	$x = -$

INTEGRALS OF THE FORM

$\int \mathbf{f}\left(\mathbf{x}, (\mathbf{ax}+\mathbf{b})^{\pm/\mathbf{n}}\right)$ ***dx*, where α and *n* are integers.**

WORKING RULE

Put $ax + b = z^n$.

INTEGRALS OF THE FORM

$\int f\left(x, (ax+b)^{\alpha/n}, (ax+b)^{\beta/m}\right)$ where α, β, m, n are integers.

WORKING RULE

Put $ax + b = z^k$, where $k = l.c.m.\ (n, m)$

INTEGRALS OF THE FORM

(A) $\int \frac{x^m}{(a+bx)^p}\,dx$ (B) $\int \frac{dx}{x^m (a+bx)^p}$

(C) $\int x^m \left(a+bx^n\right)^p\,dx$

TRICK(S) FOR PROBLEM SOLVING

Integral	Substitution
$\int \frac{x^m}{(a+bx)^p}\,dx$, m is a + *ve* integer	Put $a + bx = z$
$\int \frac{dx}{x^m (a+bx)^p}$, where either ($m$ and p positive integers) or (m and p are fractions, but $m + p$ = integers > 1)	Put $a + bx = zx$
$\int x^m (a+bx^n)^p\,dx$, where m, n, p are rationals.	
(i) p is a + *ve* integer	Apply Binomial theorem to $(a + bx^n)^p$.
(ii) p is a – *ve* integer	Put $x = z^k$ where k = common denominator of m and n.
(iii) $\frac{m+1}{n}$ is an integer	Put $a + bx^n = z^k$, where k = denominator of p.
(iv) $\frac{m+1}{n} + p$ is an integer	Put $a + bx^n = x^n z^k$ where k = denominator of fraction p.

SOLVED EXAMPLE

21. If $\int x^{13/2}\cdot(1+x^{5/2})^{1/2}\,dx$
$= A\,(1 + x^{5/2})^{7/2} + B\,(1 + x^{5/2})^{5/2} + C\,(1 + x^{5/2})^{3/2}$, then

(A) $A = -\frac{4}{35}$, $B = -\frac{8}{25}$, $C = \frac{4}{15}$

(B) $A = \frac{4}{35}$, $B = -\frac{8}{25}$, $C = -\frac{4}{15}$

(C) $A = \frac{4}{35}$, $B = -\frac{8}{25}$, $C = \frac{4}{15}$

(D) none of these

Solution: (C)

$$\int x^{13/2}.\left(1+x^{5/2}\right)^{1/2}\,dx$$

$$= \int x^5 . x^{3/2} . (1+x^{5/2})^{1/2}\,dx = \int x^5 . \frac{4}{5}\, z . z\,dz$$

$$\left[\begin{aligned} &\text{Putting } 1+x^{5/2} = z^2 \Rightarrow \frac{5}{2}x^{3/2}\,dx = 2z\,dz \\ &\qquad\qquad \text{i.e., } x^{3/2}\,dx = \frac{4}{5}\,z\,dz \end{aligned}\right]$$

$$= \frac{4}{5}\int z^2 \cdot (z^2 - 1)^2\,dz = \frac{4}{5}\int z^2\left(z^4 - 2z^2 + 1\right)\,dz$$

$$= \frac{4}{5}\left[\frac{z^7}{7} - \frac{2z^5}{5} + \frac{z^3}{3}\right] + C$$

$$= \frac{4}{35}\,(1 + x^{5/2})^{7/2} - \frac{8}{25}\,(1 + x^{5/2})^{5/2} + \frac{4}{15}\,(1 + x^{5/2})^{3/2} + C$$

$$\therefore \quad A = \frac{4}{35},\ B = -\frac{8}{25} \text{ and } C = \frac{4}{15}$$

22. $\int \frac{1+x}{1+\sqrt[3]{x}}\,dx$ is equal to

(A) $\frac{3}{5}x^{5/3} - \frac{3}{4}x^{4/3} + x + C$

(B) $\frac{3}{5}x^{5/3} + \frac{3}{4}x^{4/3} + x + C$

(C) $\frac{3}{5}x^{5/3} - \frac{3}{4}x^{4/3} - x + C$

(D) none of these

Solution: (A)

Put $x = z^3 \Rightarrow dx = 3z^2\,dz$

$$\therefore \int \frac{1+x}{1+\sqrt[3]{x}}\,dx = \int \frac{(1+z^3)\,3z^2}{1+z}\,dz$$

$$= 3\int z^2(z^2 - z + 1)dz$$

$$= 3\int (z^4 - z^3 + z^2)\,dz$$

$$= 3\int\left(\frac{z^5}{5} - \frac{z^4}{4} + \frac{z^3}{3}\right) + C$$

$$= \boldsymbol{\frac{3}{5}x^{5/3} - \frac{3}{4}x^{4/3} + x + C}$$

23. $\int \frac{\sqrt[2]{x}}{1+\sqrt[4]{x^3}}\,dx$ is equal to

(A) $\frac{4}{3}\left[1+x^{3/4}+\log(1+x^{3/4})\right]+C$

(B) $\frac{4}{3}\left[1+x^{3/4}-\log(1+x^{3/4})\right]+C$

(C) $\frac{4}{3}\left[1+x^{3/4}+\log(1+x^{3/4})\right]+C$

(D) none of these

Solution: (B)

Put $x = z^4 \Rightarrow dx = 4z^3\,dz.$

$$\therefore \int \frac{\sqrt[2]{x}}{1+\sqrt[4]{x^3}}\,dx = \int \frac{z^2.4z^3}{1+z^3}\,dz = 4\int \frac{z^3.z^2}{z^3+1}\,dz$$

$$= \frac{4}{3}\int \frac{(y-1)}{y}\,dy$$

$$\left(\text{Putting } z^3+1=y \Rightarrow z^2dz=\frac{1}{3}dy\right)$$

$$= \frac{4}{3}\,(y-\log y)+C$$

$$= \frac{4}{3}\,[1+x^{3/4}-\log(1+x^{3/4})]+C$$

24. $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}\,dx$ is equal to

(A) $\frac{3}{2}\,x^{2/3}+6\tan^{-1}x^{1/6}+C$

(B) $\frac{3}{2}\,x^{2/3}-6\tan^{-1}x^{1/6}+C$

(C) $\frac{-3}{2}\,x^{2/3}+6\tan^{-1}x^{1/6}+C$

(D) none of these

Solution: (A)

Put $x = z^6 \Rightarrow dx = 6z^5\,dz$

$$\therefore \int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}\,dx = \int \frac{\left(z^6+z^4+z\right)6z^5dz}{z^6\left(1+z^2\right)}$$

$$= 6\int \frac{z^5+z^3+1}{z^2+1}\,dz$$

$$= 6\int\left(z^3+\frac{1}{z^2+1}\right)dz$$

$$= \frac{3}{2}\,z^4+6\tan^{-1}z+C$$

$$= \frac{3}{2}\,x^{2/3}+6\tan^{-1}x^{1/6}+C$$

25. $\int x^{1/3}\,(2+x^{2/3})^{1/4}\,dx$ is equal to

(A) $\frac{2}{3}\,(2+x^{2/3})^{9/4}+\frac{12}{5}\,(2+x^{2/3})^{5/4}+C$

(B) $\frac{2}{3}\,(2+x^{2/3})^{9/4}-\frac{12}{5}\,(2+x^{2/3})^{5/4}+C$

(C) $\frac{1}{3}\,(2+x^{2/3})^{9/4}-\frac{12}{5}\,(2+x^{2/3})^{5/4}+C$

(D) none of these

Solution: (B)

Put $2+x^{2/3} = z^4 \Rightarrow dx = 6z^3.\,x^{1/3}\,dz$

$$\therefore \int x^{1/3}\left(2+x^{2/3}\right)^{1/4}dx = \int x^{1/3}.\;z\,(6z^3.\,x^{1/3})\,dz$$

$$= 6\int z^4\left(z^4-2\right)dz = 6\int\left(\frac{z^9}{9}-\frac{2z^5}{5}\right)+C$$

$$= \frac{2}{3}\,(2+x^{2/3})^{9/4}-\frac{12}{5}\,(2+x^{2/3})^{5/4}+C$$

26. $\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$ is equal to

(A) $2\sqrt{1+\sqrt{x}}-2\log\left(\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right)+C$

(B) $4\sqrt{1+\sqrt{x}}+2\log\left(\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right)+C$

(C) $4\sqrt{1+\sqrt{x}}+2\log\left(\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right)+C$

(D) none of these

Solution: (C)

Put $1+\sqrt{x} = z^2 \Rightarrow dx = 4z\sqrt{x}\,dz$

$$\therefore \int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = \int \frac{z}{x}\left(4z\sqrt{x}\right)dz$$

$$= 4\int \frac{z^2}{z^2-1}\,dz = 4\int\left(1+\frac{1}{z^2-1}\right)dz$$

$$= 4\left(z+\frac{1}{2}\log\frac{z-1}{z+1}\right)+C$$

$$= 4\sqrt{1+\sqrt{x}}+2\log\left(\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right)+C$$

27. $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx$ is equal to

(A) $12\left(\frac{\left(1+\sqrt[4]{x}\right)^{7/3}}{7}+\frac{\left(1+\sqrt[4]{x}\right)^{4/3}}{4}\right)+C$

(B) $12\left[\dfrac{\left(1+\sqrt[4]{x}\right)^{7/3}}{7}-\dfrac{\left(1+\sqrt[4]{x}\right)^{4/3}}{4}\right]+C$

(C) $6\left[\dfrac{\left(1+\sqrt[4]{x}\right)^{7/3}}{7}-\dfrac{\left(1+\sqrt[4]{x}\right)^{4/3}}{4}\right]+C$

(D) none of these

Solution: (B)

Put $1 + x^{1/4} = z^3 \Rightarrow dx = 12z^2 x^{3/4}\, dz$

$$\therefore \quad \int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\, dx = \int \frac{z}{x^{1/2}} \cdot \left(12z^2 x^{3/4}\right) dz$$

$$= 12 \int z^3 . x^{1/4}\, dz = 12 \int z^3\left(z^3-1\right) dz$$

$$= 12\left(\frac{z^7}{7}-\frac{z^4}{4}\right)+C$$

$$= 12\left(\frac{(1+x^{1/4})^{7/3}}{7}-\frac{(1+x^{1/4})^{4/3}}{4}\right)+C$$

28. $\int \sqrt[3]{x}\ \sqrt[7]{1+\sqrt[3]{x^4}}\ dx$ is equal to

(A) $\dfrac{21}{32}\left(1+\sqrt[3]{x^4}\right)^{8/7}+C$

(B) $\dfrac{32}{21}\left(1+\sqrt[3]{x^4}\right)^{8/7}+C$

(C) $\dfrac{7}{32}\left(1+\sqrt[3]{x^4}\right)^{8/7}+C$

(D) none of these

Solution: (A)

Put $1 + x^{4/3} = z^7 \Rightarrow x^{1/3}\, dx = \dfrac{21}{4} z^6\, dz$

$$\therefore \quad \int \sqrt[3]{x}\ \sqrt[7]{1+\sqrt[3]{x^4}}\ dx = \int z\left(\frac{21}{4}z^6\right) dz$$

$$= \frac{21}{4}\cdot\frac{z^8}{8}+C$$

$$= \frac{21}{32}\left(1+\sqrt[3]{x^4}\right)^{8/7}+C$$

29. $\int \dfrac{dx}{x\left(1+\sqrt[3]{x}\right)^2}$ is equal to

(A) $3\left(\log\dfrac{x^{1/3}}{1+x^{1/3}}+\dfrac{1}{1+\sqrt[3]{x}}\right)+C$

(B) $3\left(\mathit{log}\dfrac{\mathit{1}+\sqrt[3]{x}}{\sqrt[3]{x}}+\dfrac{\mathit{1}}{\mathit{1}+\sqrt[3]{x}}\right)+C$

(C) $3\left(\log\dfrac{1+\sqrt[3]{x}}{\sqrt[3]{x}}-\dfrac{1}{1+\sqrt[3]{x}}\right)+C$

(D) none of these

Solution: (A)

Put $x = z^3 \Rightarrow dx = 3z^2\, dz$

$$\therefore \quad \int \frac{dx}{x\left(1+\sqrt[3]{x}\right)^2} = \int \frac{3z^2\, dz}{z^3\left(1+z\right)^2}$$

$$= 3\int \frac{dz}{z\left(z+1\right)^2}$$

$$= 3\int\left[\frac{1}{z}-\frac{1}{1+z}-\frac{1}{\left(1+z\right)^2}\right] dz$$

(by partial fractions)

$$= 3\left[\log z-\log\left(1+z\right)+\frac{1}{1+z}\right]+C$$

$$= 3\left[\log\frac{x^{1/3}}{1+x^{1/3}}+\frac{1}{1+x^{1/3}}\right]+C$$

30. $\int \dfrac{dx}{x\sqrt{1-x^3}} =$

(A) $\dfrac{1}{3}\log\left|\dfrac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+c$

(B) $\dfrac{1}{3}\log\left|\dfrac{\sqrt{1-x^2}+1}{\sqrt{1-x^2}-1}\right|+c$

(C) $\dfrac{1}{3}\log\left|\dfrac{1}{\sqrt{1-x^3}}\right|+c$

(D) $\dfrac{1}{3}\log\left|1-x^3\right|+c$

Solution: (A)

$I = \int x^{-1}\left(1-x^3\right)^{-1/2} dx$

Let $1 - x^3 = t^2 \Rightarrow -3x^2dx = 2t\, dt$

$$\Rightarrow \quad x^{-1}dx = -\frac{2}{3}\frac{tdt}{x^3} = -\frac{2}{3}\frac{tdt}{1-t^2}$$

$$\therefore \quad I = \int\left(t^{-1}\right)\left(\frac{-2}{3}\right)\frac{tdt}{1-t^2} = -\frac{2}{3}\int\frac{dt}{1-t^2}$$

$$= \frac{2}{3}\int\frac{dt}{t^2-1} = \frac{2}{3}\cdot\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|+c$$

$$= \frac{1}{3}\ln\left(\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|\right)+c\,.$$

31. The value of $\int \frac{dx}{x^n\left(1+x^n\right)^{1-n}}$, $n \in N$, is

(A) $\frac{1}{1-n}\left(1+\frac{1}{x^n}\right)^{1-\frac{1}{n}}+c$

(B) $\frac{1}{1+n}\left(1-\frac{1}{x^n}\right)^{1-\frac{1}{n}}+c$

(C) $-\frac{1}{1-n}\left(1-\frac{1}{x^n}\right)^{1-\frac{1}{n}}+c$

(D) $-\frac{1}{n-1}\left(1+\frac{1}{x^n}\right)^{1-\frac{1}{n}}+c$

Solution: (D)

$$I = \int \frac{dx}{x^n\left(1+x^n\right)^{1/n}} = \int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)^{1/n}}$$

Let $1 + \frac{1}{x^n} = t \Rightarrow \frac{-n}{x^{n+1}}\,dx = dt$

$$\therefore\ I \quad = -\frac{1}{n}\int \frac{dt}{t^{1/n}} = -\frac{1}{n}\left(\frac{t^{-1/n+1}}{-\frac{1}{n}+1}\right)+c$$

$$= -\frac{\left(1+\frac{1}{x^n}\right)^{1-\frac{1}{n}}}{n-1}+c\,.$$

32. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx$ is equal to

(A) $(1 + x^{1/3})^{3/2} + C$
(B) $-(1 + x^{1/3})^{3/2} + C$
(C) $2\,(1 + x^{1/3})^{3/2} + C$
(D) none of these

Solution: (C)

$$\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx = \int x^{\frac{-2}{3}}\left(1+x^{\frac{1}{3}}\right)^{\frac{1}{2}}dx$$

Put $1 + x^{1/3} = z^2$ $\left[\begin{array}{l}\text{Here } m = \frac{-2}{3}, n = \frac{1}{3}, p = \frac{1}{2}\\ \therefore \frac{m+1}{n} = 1\,(\text{an integer})\end{array}\right]$

$\Rightarrow x^{-2/3}\,dx = 6z\,dz$

$$\therefore \int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx = \int z.6z\,dz = 2z^3 + C$$

$= \mathbf{2\,(1 + x^{1/3})^{3/2} + C}$

33. If $\int \frac{dx}{x\sqrt{1-x^3}} = a\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + C$, then $a =$

(A) 1/3 (B) 2/3
(C) –1/3 (D) – 2/3

Solution: (A)

Put $1 - x^3 = t^2 \Rightarrow -3x^2\,dx = 2t\,dt$

$$\therefore \int \frac{dx}{x\sqrt{1-x^3}} = \int \frac{x^2}{x^3\sqrt{1-x^3}}\,dx = \frac{2}{3}\int \frac{dt}{t^2-1}$$

$$= \frac{1}{3}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + C.$$

$\therefore\ a = \frac{1}{3}$

34. $\int \frac{dx}{x^{1/2}\left(1+x^2\right)^{5/4}}$ is equal to

(A) $\frac{-2\sqrt{x}}{\sqrt[4]{1+x^2}}+C$ (B) $\frac{2\sqrt{x}}{\sqrt[4]{1+x^2}}+C$

(C) $\frac{-\sqrt{x}}{\sqrt[4]{1+x^2}}+C$ (D) $\frac{\sqrt{x}}{\sqrt[4]{1+x^2}}+C$

Solution: (B)

Put $1 + x^2 = x^2z^4 \Rightarrow x^2 = \frac{1}{z^4-1}$

$$\Rightarrow\ dx = \frac{-2z^3}{x\left(z^4-1\right)^2}\,dz$$

$$\therefore \int \frac{dx}{x^{1/2}\left(1+x^2\right)^{5/4}}$$

$$= \int \frac{1}{x^{1/2}\left(x^2z^4\right)^{5/4}}\left(\frac{-2z^3}{x\left(z^4-1\right)}\right)dz$$

$$= -2\int \frac{dz}{x^4\left(z^4-1\right)^2z^2} = -2\int \frac{dz}{z^2} = \frac{2}{z} + C$$

$$= \frac{2\sqrt{x}}{\sqrt[4]{1+x^2}} + C$$

INTEGRALS OF THE FORM $\int e^x \left[f(x)+f'(x)\right]$ DX

WORKING RULE

(i) Split the integral into two integrals.

(ii) Integrate only the first integral by parts, i.e.,

$$\int e^x \left[f(x)+f'(x)\right]dx$$

$$= \int e^x f(x)\,dx + \int e^x f'(x)\,dx$$

$$= \left[f(x).e^x - \int f'(x).e^x\,dx\right] + \int e^x f'(x)\,dx$$

$$= e^x f(x) + C$$

INTEGRALS OF THE FORM

where the initial integrand reappears after integrating by parts.

WORKING RULE

(i) Apply the method of integration by parts twice.

(ii) On integrating by parts second time, we will obtain the given integrand again. Put it equal to I.

(iii) Transpose and collect terms involving I on one side and evaluate I.

INTEGRALS OF THE FORM: $\int e^{ax}\sin bx\,dx$ AND $\int e^{ax}\cos bx\,dx$.

$$\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

$$\int e^{ax}\sin(bx+c)\,dx$$

$$= \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right] + k$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left[(bx+c) - \tan^{-1}\left(\frac{b}{a}\right)\right] + k$$

$$\int e^{ax}\cos(bx+c)dx$$

$$= \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) - b\sin(bx+c)\right] + k$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left[(bx+c) - \tan^{-1}\left(\frac{b}{a}\right)\right] + k$$

REDUCTION FORMULA

Any formula which reduces the given integral depending on the index n > 0, called the order of the integral, to an integral of the same type with smaller index, is called *reduction formula* for the first integral.

SOME USEFUL REDUCTION FORMULAE

(i) $$\int \sin^n x\,dx = \frac{-\cos x.\sin^{n-1}x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

(ii) $$\int \cos^n x\,dx = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

(iii) $$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$$

(iv) $$\int \cot^n x\,dx = \frac{-1}{n-1}\cot^{n-1}x - \int \cot^{n-2}x\,dx$$

(v) $$\int \sec^n x\,dx = \frac{1}{(n-1)}$$

$$\left[\sec^{n-2}x.\tan x + (n-2)\int \sec^{n-2}x\,dx\right]$$

(vi) $$\int \text{cosec}^n x\,dx = \frac{1}{(n-1)}$$

$$\left[-\text{cosec}^{n-2}x\cot x + (n-2)\int \text{cosec}^{n-2}x\,dx\right]$$

(vii) $$\int \sin^p x\cos^q x\,dx = -\frac{\sin^{q+1}x.\cos^{p-1}x}{p+q}$$

$$+\frac{p-1}{p+q}\int \sin^{p-2}x.\cos^q x\,dx$$

(viii) $$\int \sin^p x\cos^q x\,dx = \frac{\sin^{p+1}x.\cos^{q-1}x}{p+q}$$

$$+\frac{p-1}{p+q}\int \sin^p x.\cos^{q-2}x\,dx$$

(ix) $$\int \frac{dx}{(x^2+k)^n}N = \frac{x}{k(2n-2)(x^2+k)^{n-1}}$$

$$+\frac{(2n-3)}{k(2n-2)}\int \frac{dx}{(x^2+k)^{n-1}}$$

SOLVED EXAMPLE

35. If $I_n = \int \tan^n x\, dx$, then $I_0 + I_1 + 2\,(I_2 + ... + I_8) + I_9 + I_{10}$ is equal to

(A) $\left(\dfrac{\tan x}{1} + \dfrac{\tan^2 x}{2} + ... + \dfrac{\tan^9 x}{9}\right)$

(B) $-\left(\dfrac{\tan x}{1} + \dfrac{\tan^2 x}{2} + ... + \dfrac{\tan^9 x}{9}\right)$

(C) $\left(\dfrac{\cot x}{1} + \dfrac{\cot^2 x}{2} + ... + \dfrac{\cot^9 x}{9}\right)$

(D) $-\left(\dfrac{\cot x}{1} + \dfrac{\cot^2 x}{2} + ... + \dfrac{\cot^9 x}{9}\right)$

Solution: (A)

We have,

$$I_n = \int \tan^n x\, dx = \frac{\tan^{n-1}x}{n-1}\ \tan^2 x\, dx$$

$$= \int \tan^{n-2} x\ (\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x\ \sec^2 x\, dx - \int \tan^{n-2} x\ dx$$

$$= \frac{\tan^{n-1}x}{n-1} - I_{n-2}$$

$$\therefore\ I_n + I_{n-2} = \frac{\tan^{n-1}x}{n-1},\ n \geq 2$$

$$\therefore\ I_0 + I_1 + 2\,(I_2 + I_3 + ... + I_8) + I_9 + I_{10}$$

$$= (I_2 + I_0) + (I_3 + I_1) + (I_4 + I_2) + (I_5 + I_3) + (I_6 + I_4)$$
$$+ (I_7 + I_5) + (I_8 + I_6) + (I_9 + I_7) + (I_{10} + I_8)$$

$$= \left(\frac{\tan x}{9} + \frac{\tan^2 x}{2} + ... + \frac{\tan^9 x}{9}\right)$$

CAUTION

Every antiderivative may not be expressible in terms of elementary function such as polynomial function, trigonometric function, log logarithmic function, exponential function etc. We say that such antiderivatives or integrals cannot be found. For example,

$\int e^{x^2} dx$, $\int \sqrt{\sin x}\, dx$, $\int \dfrac{\sin x}{x} dx$, $\int \sin x^2 dx$,

$\int x \tan x\, dx$ etc.

EXERCISES

Single Option Correct Type

1. The equation of a curve passing through origin is given by $y = \int x^3 \cos x^4\ dx$. If the equation of the curve is written in the form $x = g(y)$, then

(A) $g(y) = \sqrt[3]{\sin^{-1}(4y)}$

(B) $g(y) = \sqrt{\sin^{-1}(4y)}$

(C) $g(y) = \sqrt[4]{\sin^{-1}(4y)}$

(D) none of these

2. If $\phi(x) = \int \dfrac{dx}{\sin^{1/2} x \cos^{7/2} x^2}$, then $\phi\left(\dfrac{\pi}{4}\right) - \phi(0) =$

(A) $\dfrac{12}{5}$ (B) $\dfrac{9}{5}$

(C) $\dfrac{6}{5}$ (D) 0

3. If $\phi(x) = \lim\limits_{n\to\infty} \dfrac{x^n - x^{-n}}{x^n + x^{-n}}$, $0 < x < 1$, $n \in N$, then $\int \sin^{-1}x\ \phi(x)\, dx$ is equal to

(A) $x \sin^{-1}x + \sqrt{1 - x^2} + C$

(B) $-\left(x \sin^{-1} x + \sqrt{1 - x^2}\right) + C$

(C) $x \sin^{-1} x - \sqrt{1 - x^2} + C$

(D) none of these.

4. $\int \dfrac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$ is equal to

(A) $\dfrac{1}{2}\sin 2x + c$ (B) $-\dfrac{1}{2}\sin 2x + c$

(C) $-\dfrac{1}{2}\sin x + c$ (D) $-\sin^2 x + c$

5. If $\int \frac{\left(\sqrt{x}\right)^5}{\left(\sqrt{x}\right)^7 + x^6} dx = a \text{ In}\left(\frac{x^k}{x^k+1}\right) + c$, the values of a and k respectively are

(A) $\frac{5}{2}$ and $\frac{2}{5}$ (B) $\frac{2}{5}$ and $\frac{5}{2}$

(C) $\frac{5}{2}$ and 2 (D) none of there

6. $\int x[f(x^2)g''(x^2) - f''(x^2)g(x^2)]dx$

(A) $f(x^2)\, g'(x^2) - g(x^2)\, f'(x^2) + c$

(B) $\frac{1}{2}[f(x^2)g(x^2)f'(x^2)] + c$

(C) $\frac{1}{2}[f(x^2)\, g'(x^2) - g(x^2)f'(x^2)] + c$

(D) none of the above

7. The anti-derivative of $\frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}$ is

(A) $\frac{\sin 2x}{2} + \cos x + c$

(B) $-\frac{\sin 2x}{2} + \sin x + c$

(C) $-\frac{\sin 2x}{2} - \sin x + c$

(D) $\frac{\sin 2x}{2} - \cos x + c$

8. If $\int \tan^4 x\, dxn = K\tan^3 x + L\tan x + f(x)$, then

(A) $K = \frac{1}{3}, L = -1, f(x) = x + C$

(B) $K = 1, L = -1, f(x) = -x + C$

(C) $K = -1, L = 1, f(x) = 2x + C$

(D) $K = \frac{1}{2}, L = \frac{1}{3}, f(x) = 3x + C$

9. $\int \frac{1}{\left[(x-1)^3\,(x+2)^5\right]^{1/4}} dx$ is equal to

(A) $\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$ (B) $\frac{4}{3}\left(\frac{x+2}{x-1}\right)^{1/4} + c$

(C) $\frac{1}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$ (D) $\frac{1}{3}\left(\frac{x+2}{x-1}\right)^{1/4} + c$

10. $\int \left(\frac{\ln x - 1}{(\ln x)^2 + 1}\right)^2 dx$ is equal to

(A) $\frac{x}{x^2+1} + c$ (B) $\frac{\ln x}{(\ln x)^2 + 1} + c$

(C) $\frac{x}{(\ln x)^2 + 1} + c$ (D) $e^x\left(\frac{x}{x^2+1}\right) + c$

11. The value of $\int \frac{ax^2 - b}{x\sqrt{c^2x^2 - (ax^2+b)^2}} dx$ is equal to

(A) $\sin^{-1}\left(\frac{\left(ax + \frac{b}{x}\right)}{c}\right) + k$

(B) $\sin^{-1}\left(\frac{ax^2 + \frac{b}{x^2}}{c}\right) + k$

(C) $\cos^{-1}\left(\frac{ax + \frac{b}{x}}{c}\right) + k$

(d) $\cos^{-1}\left(\frac{\left(ax^2 + \frac{b}{x^2}\right)}{c}\right) + k$

12. The value of $\int \frac{\sec x\, dx}{\sqrt{\sin(2x+\theta) + \sin\theta}}$ is

(A) $\sqrt{(\tan x + \tan\theta)\sec\theta} + c$

(B) $\sqrt{2(\tan x + \tan\theta)\sec\theta} + c$

(C) $\sqrt{2(\sin x + \tan\theta)\sec\theta} + c$

(D) none of these

13. $\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx$ is equal to

(A) $\sin^{-1}(\sin x + \cos x) + c$

(B) $\sin^{-1}\left[\frac{1}{3}(\sin x + \cos x)\right] + c$

(C) $\cos^{-1}(\sin x + \cos x) + c$

(D) none of these

14. If $f(x) = \int \frac{x^2 dx}{(1+x^2)\left(1+\sqrt{1+x^2}\right)}$ and $f(0) = 0$, then the value of $f(1)$ is

(A) $\log\left(1+\sqrt{2}\right)$ (B) $\log\left(1+\sqrt{2}\right)-\frac{\pi}{4}$

(C) $\log\left(1+\sqrt{2}\right)+\frac{\pi}{2}$ (D) none of these

15. If $\int \frac{(x+1)}{x(1+xe^x)^2}\,dx = \log|1-f(x)| + f(x) + C$, then $f(x) =$

(A) $\frac{1}{x+e^x}$ (B) $\frac{1}{1+xe^x}$

(C) $\frac{1}{(1+xe^x)^2}$ (D) $\frac{1}{(x+e^x)^2}$

16. If $\int f(x)\sin x\cos x\,dx = \frac{1}{2(b^2-a^2)}\log[f(x)] + C$, then $f(x)$ is equal to

(A) $\frac{1}{a^2\sin^2 x + b^2\cos^2 x}$

(B) $\frac{1}{a^2\sin^2 x - b^2\cos^2 x}$

(C) $\frac{1}{a^2\cos^2 x + b^2\sin^2 x}$

(D) $\frac{1}{a^2\cos^2 x - b^2\sin^2 x}$

17. $\int \frac{dx}{(x+a)^{8/7}(x-b)^{6/7}}$ is equal to

(A) $\left(\frac{7}{a+b}\right)\left(\frac{x+a}{x-b}\right)^{1/7} + c$

(B) $\left(\frac{7}{a+b}\right)\left(\frac{x-b}{x+a}\right)^{1/7} + c$

(C) $\frac{6}{a+b}\left(\frac{x-b}{x+a}\right)^{1/6} + c$

(D) $\frac{6}{a+b}\left(\frac{x+a}{x-b}\right)^{1/6} + c$

18. $\int \sin(\log x)\,dx = f(x)\,[\sin g(x) - \cos h(x)] + c$, then

(A) $\lim_{x\to 2} f(x) = 2$ (B) $g(e^3) = -3$

(C) $h(e^5) = -4$ (D) $\lim_{x\to 1}\frac{g(x)}{h(x)} = 1$

19. $\int \frac{\sqrt{x}}{\sqrt{x^3+4}}\,dx$ equals

(A) $\frac{2}{3}\ln\left(\frac{2}{\sqrt{x^3}-\sqrt{x^3-4}}\right) + C$

(B) $\frac{2}{3}\ln\left(\frac{2}{\sqrt{x^3}-\sqrt{x^3-4}}\right) + C$

(C) $\frac{2}{3}\ln\left(\frac{2}{\sqrt{x^3}-\sqrt{x^3-4}}\right) + C$

(D) none of these

20. $\int \frac{[f(x).\phi(x) - \phi(x).\phi(x)]}{f(x).\phi(x)}\log\frac{f(x)}{\phi(x)}\,dx$ is equal to

(A) $\log\frac{\phi(x)}{f(x)} + k$

(B) $\frac{1}{2}\left[\log\frac{\phi(x)}{f(x)}\right]^2 + k$

(C) $\frac{\phi(x)}{f(x)}\log\frac{\phi(x)}{f(x)} + k$

(D) none of these

21. If $I = \int \frac{1}{2p}\sqrt{\frac{p-1}{p+1}}\,dp = f(p) + c$, then $f(p)$ is equal to

(A) $\frac{1}{2}\ln\left(p-\sqrt{p^2-1}\right)$

(B) $\left(\frac{1}{2}\cos^{-1}p + \frac{1}{2}\sec^{-1}p\right)$

(C) $\frac{1}{2}\ln\sqrt{p+\sqrt{p^2-1}} - \frac{1}{2}\sec^{-1}p$

(D) none of these

22. If $\int \frac{1}{x+x^5}\,dx = f(x) + c$, then $\int \frac{x^4}{x+x^5}\,dx$ is equal to

(A) $\log|x| + f(x) + c$ (B) $\log|x| - f(x) + c$

(C) $xf(x) + c$ (D) none of these

23. If $l^r(x)$ means log log log ...x, the log being repeated r times, then

$\int [xl(x)l^2(x)l^3(x)\ldots l^r(x)]^{-1}\,dx$ is equal to

(A) $l^{r+1}(x) + C$ (B) $\frac{l^{r+1}(x)}{r+1} + C$

(C) $l^r(x) + C$ (D) none of these

24. $\int \frac{(x^2-2)\,dx}{(x^4+5x^2+4)\tan^{-1}\left(\frac{x^2+2}{x}\right)}$ is

(A) $\log\left|\tan^{-1}\sqrt{x+2}\right|+C$

(B) $\log\left|\tan^{-1}\left(x+\frac{2}{x}\right)\right|+C$

(C) $\sin^{-1}\left(\frac{x+2}{x}\right)+C$

(D) $\tan^{-1}\left(\frac{x+2}{x}\right)+C$

25. The value of $\int e^x \frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}dx$ is

(A) $e^x\frac{\sqrt{1-x^n}}{1-x^n}+C$ (B) $e^x\frac{\sqrt{1+x^{2n}}}{1-x^{2n}}+C$

(C) $e^x\frac{\sqrt{1-x^{2n}}}{1-x^{2n}}+C$ (D) $e^x\frac{\sqrt{1-x^{2n}}}{1-x^n}+C$

26. $\int\sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}\,dx$ is equal to

(A) $\frac{2}{3}\sin^{-1}(\cos^{3/2}x)+c$ (B) $\frac{2}{3}\sin^{-1}(\cos^{3/2}x)+c$

(C) $\frac{2}{3}\cos^{-1}(\cos^{3/2}x)+c$ (D) none of the above

27. $\int\cos^{-3/7}x\sin^{-11/7}x\,dx=$

(A) $\log|\sin^{4/7}x|+c$ (B) $\frac{4}{7}\tan^{4/7}x+c$

(C) $\frac{-7}{4}\tan^{-4/7}x+c$ (D) $\log|\cos^{3/7}x|+c$

(e) $\frac{7}{4}\tan^{-4/7}x+c$

28. If the integral $\int\frac{5\tan x}{\tan x-2}dx=x+a\ln|\sin x-2\cos x|+k$, then a is equal to:

(A) -1 (B) -2

(C) 1 (D) 2

29. The integral $\int\frac{dx}{(a^2-b^2x^2)^{3/2}}$, equals:

(A) $\frac{x}{\sqrt{a^2-b^2x^2}}+C$ (B) $\frac{x}{a^2\sqrt{a^2-b^2x^2}}+C$

(C) $\frac{ax}{\sqrt{a^2-b^2x^2}}+C$ (D) $\frac{x}{a^2\sqrt{a^2-b^2x^2}}+C$

30. The value of $\sqrt{2}\int\frac{\sin x\,dx}{\sin\left(x-\frac{\pi}{4}\right)}$ is

(A) $x+\ln\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$

(B) $x-\ln\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

(C) $x+\ln\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

(D) $x-\ln\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$

31. For a natural number n, the value of the integral $\int(x^{3n}+x^{2n}+x^n)(2x^{2n}+3x^n+6)^{1/n}\,dx$ is

(A) $\frac{1}{6n}(2x^{3n}+3x^{2n}+6x^n)^{1/n}+C$

(B) $\frac{1}{6n}(2x^{3n}+3x^{2n}+6x^n)^{1/n+1}+C$

(C) $\frac{1}{6(n+1)}(2x^{3n}+3x^{2n}+6x^n)^{1/n+1}+C$

(D) none of these

32. If $f\left(\frac{3x-4}{3x+4}\right)=x+2$, then $\int f(x)\,dx$ is equal to

(A) $ex-{}^{2}\ln\left|\frac{3x-4}{3x+4}\right|+c$

(B) $-\frac{8}{3}\ln|x-1|+\frac{2}{3}x+c$

(C) $\frac{8}{3}\ln|x-1|+\frac{x}{3}+c$

(D) none of these

33. $\int\frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx$ is equal to

(A) $(1+x^{1/3})^{3/2}+C$ (B) $-(1+x^{1/3})^{3/2}+C$

(C) $2(1+x^{1/3})^{3/2}+C$ (D) none of these

34. $\int x\{f(x^2)g''(x^2)-f''(x^2)g(x^2)\}\,dx=$

(A) $f(x^2)\,g'(x^2)-g(x^2)\,f'(x^2)+c$

(B) $\frac{1}{2}\{f(x^2)g(x^2)f'(x^2)\}+c$

(C) $\frac{1}{2}\{f(x^2)\,g'(x^2) - g(x^2)f'(x^2)\} + c$

(D) none of these

35. $\int f(x)\sin x\cos x\,dx = \frac{1}{2(b^2-a^2)}\log(f(x)) + C$, then $f(x)$ is equal to

(A) $\frac{1}{a^2\sin^2 x + b^2\cos^2 x}$

(B) $\frac{1}{a^2\sin^2 x - b^2\cos^2 x}$

(C) $\frac{1}{a^2\cos^2 x + b^2\sin^2 x}$

(D) $\frac{1}{a^2\cos^2 x - b^2\sin^2 x}$

36. If $f(x) = \tan^{-1}x + \ln\sqrt{1+x} - \ln\sqrt{1-x}$, then the integral of $\frac{1}{2}f'(x)$ w.r.t. x^4 is

(A) $\ln\,|1 - x^4| + C$ (B) $-\ln|1 - x^4| + C$
(C) $\ln|x^4 - 1| + C$ (D) $\ln|1 + x^4| + C$

37. $\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx =$

(A) $e^x\frac{\sqrt{1+x}}{\sqrt{1-x^2}} + C$ (B) $e^x\frac{\sqrt{1-x}}{\sqrt{1+x}} + C$

(C) $e^x\frac{\sqrt{1+x}}{\sqrt{1-x}} + C$ (D) none of these

38. $\int \left\{\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right\}\ln x\,dx =$

(A) $\left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + C$

(B) $\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x + C$

(C) $\left(\frac{x}{e}\right)^x - 2\left(\frac{e}{x}\right)^x + C$

(D) none of these

39. $\int (x^3a + x^2a + xa)\,(2x^2a + 3xa + 6)^{1/}a\,dx =$

(A) $\frac{1}{6(a+1)}\left(2x^{3a} + 3x^{2a} + 6x^a\right)^{1-\frac{1}{a}} + C$

(B) $\frac{1}{6(a+1)}\left(2x^{3a} + 3x^{2a} + 6x^a\right)^{1+\frac{1}{a}} + C$

(C) $\frac{1}{3(a+1)}\left(2x^{3a} + 3x^{2a} + 6x^a\right)^{1+\frac{1}{a}} + C$

(D) none of these

40. If $y(x-y)^2 = x$, then $\int \frac{dx}{x-3y} =$

(A) $\frac{1}{2}\ln\left|(x-y)^2+1\right| + C$

(B) $\frac{1}{2}\ln\left|(x-y)^2-1\right| + C$

(C) $\frac{1}{4}\ln\left|(x-y)^2+1\right| + C$

(D) none of these.

41. $\int \frac{\cos x}{1-\sin x\cos x}dx = \tan^{-1}(\sin x - \cos x)$

$+\frac{k}{\sqrt{3}}\ln\left|\frac{\sin x + \cos x - \sqrt{3}}{\sin x + \cos x - \sqrt{3}}\right| + C$, where $k =$

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$

(C) -1 (D) 1

42. $\int \frac{\cos\left(x + \frac{\pi}{4}\right)}{2+\sin 2x}dx$

(A) $\sqrt{2}\tan^{-1}(\sin x - \cos x) + C$

(B) $\frac{1}{\sqrt{2}}\tan^{-1}(\sin x - \cos x) + C$

(C) $\frac{1}{\sqrt{2}}\tan^{-1}(\sin x + \cos x) + C$

(D) $\sqrt{2}\tan^{-1}(\sin x + \cos x) + C$

43. $\int \frac{dx}{1-\cos^4 x} = -\frac{1}{2\tan x} + \frac{k}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$, where $k =$

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$

(C) -1 (D) 1

44. $\int \frac{\sqrt{\cot x} - \sqrt{\tan x}}{1+3\sin 2x}dx$

$= k\tan^{-1}\left(\frac{\sqrt{\tan x} + \sqrt{\cot x}}{2}\right) + C$,

where $k =$

(A) 1 (B) -1
(C) 2 (D) -2

45. $\int \dfrac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\,dx = k \sin x\,(1 + \cos x) + C$, where $k =$

(A) 2 (B) -2

(C) 1 (D) -1

46. $\int \dfrac{\sec x\,dx}{\sqrt{\sin(2x+a) + \sin a}} = k\sqrt{\tan x + \tan a} + C$, where $k =$

(A) $\sqrt{\dfrac{2}{\cos a}}$ (B) $\sqrt{2\cos a}$

(C) $\sqrt{\cos a}$ (D) $\sqrt{\dfrac{1}{\cos a}}$

47. $\int \sqrt{x + \sqrt{x^2+2}}\,dx$

$= \dfrac{1}{3}\left(\sqrt{x^2+2}+x\right)^{3/2} + k\left(\sqrt{x^2+2}-x\right)^{1/2} + C$,

where $k =$

(A) 2 (B) $\sqrt{2}$

(C) -2 (D) $-\sqrt{2}$

48. $\int \dfrac{x^4 - 1}{x^2\sqrt{x^4 + x^2 + 1}}\,dx =$

(A) $\dfrac{\sqrt{x^4+x^2+1}}{x} + C$

(B) $\dfrac{x}{\sqrt{x^4+x^2+1}} + C$

(C) $-\dfrac{\sqrt{x^4+x^2+1}}{x} + C$

(D) none of these

49. $\int \dfrac{x^2 - 1}{(x^2+1)\sqrt{1+x^4}}\,dx = k\cos^{-1}\left(\dfrac{\sqrt{2}\,x}{x^2+1}\right) + C$, where $k =$

(A) $\dfrac{1}{2}$ (B) 2

(C) $\dfrac{1}{\sqrt{2}}$ (D) $\sqrt{2}$

50. $\int \dfrac{dx}{\left(1+\sqrt{x}\right)\sqrt{x - x^2}} = k\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right) + C$, where $k =$

(A) 1 (B) 2

(C) 3 (D) 4

51. $\int \dfrac{dx}{\cos^3 x\sqrt{\sin 2x}} =$

(A) $\sqrt{2}\left(\tan^{1/2} x + \dfrac{1}{5}\tan^{5/2} x\right) + C$

(B) $\sqrt{2}\left(\cot^{1/2} x + \dfrac{1}{5}\cot^{5/2} x\right) + C$

(C) $\sqrt{2}\left(\tan^{1/2} x - \dfrac{1}{5}\tan^{5/2} x\right) + C$

(D) none of these

52. $\int \dfrac{1 + x^4}{(1 - x^4)^{3/2}}\,dx =$

(A) $\dfrac{1}{\sqrt{x^2 - \dfrac{1}{x^2}}} + c$ (B) $\dfrac{1}{\sqrt{\dfrac{1}{x^2} - x^2}} + c$

(C) $\dfrac{1}{\sqrt{x^2 + \dfrac{1}{x^2}}} + c$ (D) none of these

53. $\int \dfrac{(x^2 - 1)}{(x^4 + 3x^2 + 1)\tan^{-1}\left(x + \dfrac{1}{x}\right)}\,dx =$

(A) $\log\left|\tan^{-1}\left(x + \dfrac{1}{x}\right)\right| + c$

(B) $\log\left|\cot^{-1}\left(x + \dfrac{1}{x}\right)\right| + c$

(C) $2\log\left|\tan^{-1}\left(x + \dfrac{1}{x}\right)\right| + c$

(D) none of these

54. $\int \sqrt{\dfrac{\cos x - \cos^3 x}{1 - \cos^3 x}}\,dx =$

(A) $\dfrac{2}{3}\sin^{-1}\left(\cos^{3/2} x\right) + c$

(B) $-\dfrac{2}{3}\sin^{-1}\left(\cos^{3/2} x\right) + c$

(C) $\dfrac{3}{2}\sin^{-1}\left(\cos^{3/2} x\right) + c$

(D) none of these

55. $\int \dfrac{dx}{(x-1)^{3/4}(x+2)^{5/4}} =$

(A) $\dfrac{4}{3}\left(\dfrac{x-1}{x+2}\right)^{1/4} + c$ (B) $\dfrac{3}{4}\left(\dfrac{x-1}{x+2}\right)^{1/4} + c$

(C) $\dfrac{4}{3}\left(\dfrac{x+2}{x-1}\right)^{1/4} + c$ (D) none of these

56. $\int \dfrac{\cos 7x - \cos 8x}{1 + 2\cos 5x}\,dx =$

(A) $\dfrac{\sin 2x}{2}+\dfrac{\sin 3x}{3}+c$

(B) $-\dfrac{\sin 2x}{2}-\dfrac{\sin 3x}{3}+c$

(C) $\dfrac{\sin 2x}{2}-\dfrac{\sin 3x}{3}+c$

(D) none of these.

57. If $In = \int x^n\sqrt{a^2-x^2}\,dx$, then $(n+2)In-(n-1)a^2\,In_{-2}=$

(A) $x^n\sqrt{a^2-x^2}$ (B) $-x^{n-1}\sqrt{a^2-x^2}$

(C) $-xn^{-1}(a^2-x^2)^{3/2}$ (D) none of these

58. $\displaystyle\int\frac{(1-\cot^{n-2}x)\,dx}{\tan x+\cot x\bullet\cot^{n-2}x}=$

(A) $\dfrac{1}{n}\log|\sin^n x-\cos^n x|+c$

(B) $\dfrac{1}{n}\log|\sin^n x+\cos^n x|+c$

(C) $\dfrac{1}{n-1}\log|\sin^n x+\cos^n x|+c$

(D) none of these

59. $\int e^{\tan x}(\sec x-\sin x)\,dx=$

(A) $e^{\tan}x\sin x+c$ (B) $-e^{\tan}x\sin x+c$

(C) $-e^{\tan}x\cos x+c$ (D) $e^{\tan}x\cos x+c$

60. $\displaystyle\int\frac{(x-1)\,dx}{(x+1)\sqrt{x^3+x^2+x}}=k\tan^{-1}\sqrt{\frac{x^2+x+1}{x}}+c,$

where $k=$

(A) 1 (B) 2

(C) 4 (D) none of these

61. $\displaystyle\int\frac{dx}{\tan x+\cot x+\sec x+\operatorname{cosec} x}=$

(A) $\dfrac{1}{2}(\sin x-\cos x+x)+c$

(B) $\dfrac{1}{2}(\sin x-\cos x-x)+c$

(C) $\dfrac{1}{2}(\sin x+\cos x+x)+c$

(D) none of these

62. If $\int f(x)\sin x\cos x\,dx=\dfrac{1}{2(b^2-a^2)}\log f(x)+c$, then $f(x)$ is equal to

(A) $\dfrac{1}{a^2\sin^2 x-b^2\cos^2 x}$

(B) $\dfrac{1}{a^2\cos^2 x+b^2\sin^2 x}$

(C) $\dfrac{1}{a^2\sin^2 x+b^2\cos^2 x}$

(D) none of these

63. $\displaystyle\int\frac{(\sin x-\cos x)\,dx}{(\sin x+\cos x)\sqrt{\sin x\cos x+\sin^2 x\cos^2 x}}=$

(A) $\operatorname{cosec}^{-1}(1+\sin 2x)+c$

(B) $-\operatorname{cosec}^{-1}(1+\sin 2x)+c$

(C) $\sec^{-1}(1+\sin 2x)+c$

(D) $-\sec^{-1}(1+\sin 2x)+c$

64. $\displaystyle\int\frac{dx}{(2ax+x^2)^{3/2}}=$

(A) $\dfrac{1}{a^2}\dfrac{x+a}{\sqrt{x^2+2ax}}+c$

(B) $\dfrac{1}{a^2}\dfrac{x-a}{\sqrt{x^2+2ax}}+c$

(C) $-\dfrac{1}{a^2}\cdot\dfrac{x+a}{\sqrt{x^2+2ax}}+c$

(D) none of these

65. $\displaystyle\int\frac{\sin^3\theta/2}{\cos\theta/2\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}\,d\theta$

$=\tan^{-1}\sqrt{k}+C,$ where $k=$

(A) $\cos\theta+\sec\theta+1$ (B) $\cos\theta-\sec\theta+1$

(C) $\cos\theta+\sec\theta-1$ (D) none of these

66. $\displaystyle\int\frac{(2\sin\theta+\sin 2\theta)\,d\theta}{(\cos\theta-1)\sqrt{\cos\theta+\cos^2\theta+\cos^3\theta}}$

$=-\dfrac{2}{3}\log\left|\dfrac{k-\sqrt{3}}{k+\sqrt{3}}\right|+c,$ where $k=$

(A) $\sqrt{\cos\theta+\sec\theta+1}$

(B) $\cos\theta+\sec\theta+1$

(C) $\sqrt{\cos\theta+\sec\theta-1}$

(D) $\cos\theta+\sec\theta-1$

67. If $[\cdot]$ and $\{\cdot\}$ denote the greatest integer and fractional part, respectively, then $\int[\{[\{[x]\}]\}]\,dx$ is equal to

(A) C (B) $x + C$

(C) $\frac{x^2}{2} + C$ (D) cannot be integrated

68. If $f(x) = \lim\limits_{n\to\infty}\left[2x + 4x^3 + ... + 2nx^{2n-1}\right], 0 < x < 1,$ then $\int f(x)\, dx =$

(A) $\frac{1}{\sqrt{1-x^2}} + C$ (B) $\sqrt{1-x^2} + C$

(C) $(1 - x^2) + C$ (D) $\frac{1}{1-x^2} + C$

69. If $Im, n = \int \cos^m x \sin nx\, dx$, then $7I_{4,3} - 4I_{3,2} =$

(A) $-\cos 3x \cos^4 x + C$

(B) $\cos 3x \cos^4 x + C$

(C) $-\frac{1}{3}\cos 3x \cos^4 x + C$

(D) none of these

More than One Option Correct Type

70. $\int \frac{dx}{(1+\sqrt{x})^8} = \frac{-1}{3(1+\sqrt{x})^{k_1}} + \frac{2}{7(1+\sqrt{x})^{k_2}} + C,$

where

(A) $k_1 = 6$ (B) $k_2 = 7$

(C) $k_1 = -6$ (D) $k_2 = -7$

71. If $\int f(x) \sin 2x\, dx = \frac{ln|f(x)|}{b^2 - a^2} + C$, then $f(x)$ is equal to

(A) $\frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$

(B) $\frac{1}{a^2 \sin^2 x - b^2 \cos^2 x}$

(C) $\frac{1}{a^2 \cos^2 x - b^2 \sin^2 x}$

(D) $\frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$

72. If $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7 + x^6}\, dx = a \log\left(\frac{x^k}{1+x^k}\right) + C$, then

(A) $a = \frac{2}{5}$ (B) $a = -\frac{2}{5}$

(C) $k = \frac{5}{2}$ (D) $k = -\frac{5}{2}$

73. If $P = \int e^{ax} \cos bx\, dx$ and $Q = \int e^{ax} \sin bx\, dx$, then

(A) $P = \frac{e^{ax}}{\sqrt{a^2+b^2}} \cos\left(bx - \tan^{-1}\frac{b}{a}\right)$

(B) $Q = \frac{e^{ax}}{\sqrt{a^2+b^2}} \sin\left(bx - \tan^{-1}\frac{b}{a}\right)$

(C) $(P^2 + Q^2)(a^2 + b^2) = e^2ax$

(D) $\tan^{-1}\left(\frac{Q}{P}\right) + \tan^{-1}\frac{b}{a} = bx$

74. $\int \frac{x^4+1}{x^6+1}\, dx = \tan^{-1}k_1 - \frac{2}{3}\tan^{-1} k_2 + C$, where

(A) $k_1 = x + \frac{1}{x}$ (B) $k_2 = x^3$

(C) $k_1 = x - \frac{1}{x}$ (D) $k_2 = x^4$

75. If $\int \frac{x \log(x + \sqrt{1+x^2})}{\sqrt{1+x^2}}\, dx$

$= A\sqrt{1+x^2} \log\left(x + \sqrt{1+x^2}\right) + Bx + C$, then

(A) $A = -1$ (B) $B = -1$

(C) $A = 1$ (D) none of these

76. If $\int \frac{3\cot 3x - \cot x}{\tan x - 3\tan 3x}\, dx = Ax + B \log\left|\frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x}\right| +$ C, then

(A) $A = 1$ (B) $B = -\sqrt{3}$

(C) $B = -\frac{1}{\sqrt{3}}$ (D) none of these

77. Let $f(x) = \frac{x+2}{2x+3}$, if $\int \left(\frac{f(x)}{x^2}\right)^{1/2} dx$

$= \frac{1}{\sqrt{2}} g\left(\frac{1+\sqrt{2f(x)}}{1-\sqrt{2f(x)}}\right) - \sqrt{\frac{2}{3}}\, h\left(\frac{\sqrt{3f(x)}+\sqrt{2}}{\sqrt{3f(x)}-\sqrt{2}}\right) + C$,

then

(A) $g(x) = \log|x|$ (B) $h(x) = \log|x|$

(C) $g(x) = \tan^{-1}x$ (D) $h(x) = \tan^{-1}x$

78. If $f(x) = \lim\limits_{n\to\infty} e^{x \tan\left(\frac{1}{n}\right) \log\left(\frac{1}{n}\right)}$ and $\int \frac{f(x)}{\sqrt[3]{\sin^{11} x \cos x}}\, dx =$ $g(x) + C$, then

(A) $g\left(\frac{\pi}{4}\right) = \frac{15}{8}$

(B) $g(x)$ is continuous for all x

(C) $g\left(\frac{\pi}{4}\right) = -\frac{15}{8}$

(D) $g(x)$ is not differentiable at infinitely many points

79. If for all $x \in [-1, 0)$, $\int (\cos^{-1} x + \cos^{-1}\sqrt{1-x^2})\,dx$

$= Ax + f(x)\sin^{-1}x - 2\sqrt{1-x^2} + C$, then

(A) $A = \frac{\pi}{4}$ (B) $A = \frac{\pi}{2}$

(C) $f(x) = x$ (D) $f(x) = -2x$

80. If $\int \frac{x^2+20}{(x\sin x + 5\cos x)^2}\,dx = -\frac{x}{A\cos x} + B$, then

(A) $A = x\sin x + 5\cos x$

(B) $B = \cot x$

(C) $A = -(x\sin x + 5\cos x)$

(D) $B = \tan x$

Passage Based Questions

Passage 1

The integral of the form $\int x^m(a+bx^n)^p\,dx$, where m, n, p are rational numbers, is expressed through elementary functions only in the following cases:

I: p is a positive integer

In this case, $(a + bxn)p$ can be expanded by the binomial theorem and the integrand can be expressed as a sum of rational powers of xn in the form $a_0 = a_1xn + \ldots + apxnp$, which can be integrated easily.

II: p is a negative integer

Put $x = zk$, where k is the common denominator of the fractions m and n.

III: $\frac{m+1}{n}$ is an integer

Put $a + bxn = zk$, where k is the denominator of the fraction p.

IV: $\frac{m+1}{n} + p$ is an integer

Put $a + bxn = xn\,zk$, where k is the denominator of the fraction p.

81. $\int x^{1/3}(2+x^{2/3})^{1/4}\,dx$ is equal to

(A) $\frac{2}{3}\left(2+x^{2/3}\right)^{9/4} + \frac{12}{5}\left(2+x^{2/3}\right)^{5/4} + C$

(B) $\frac{2}{3}\left(2+x^{2/3}\right)^{9/4} - \frac{12}{5}\left(2+x^{2/3}\right)^{5/4} + C$

(C) $\frac{1}{3}\left(2+x^{2/3}\right)^{9/4} - \frac{12}{5}\left(2+x^{2/3}\right)^{5/4} + C$

(D) none of these

82. $\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$ is equal to

(A) $2\sqrt{1+\sqrt{x}} - 2\log\left(\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right) + C$

(B) $4\sqrt{1+\sqrt{x}} + 2\log\left(\frac{\sqrt{1+\sqrt{x}}+1}{\sqrt{1+\sqrt{x}}-1}\right) + C$

(C) $4\sqrt{1+\sqrt{x}} + 2\log\left(\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right) + C$

(D) none of these

83. $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx$ is equal to

(A) $12\left(\frac{\left(1+\sqrt[4]{x}\right)^{7/3}}{7} + \frac{\left(1+\sqrt[4]{x}\right)^{4/3}}{4}\right) + C$

(B) $12\left(\frac{\left(1+\sqrt[4]{x}\right)^{7/3}}{7} - \frac{\left(1+\sqrt[4]{x}\right)^{4/3}}{4}\right) + C$

(C) $6\left(\frac{\left(1+\sqrt[4]{x}\right)^{7/3}}{7} - \frac{\left(1+\sqrt[4]{x}\right)^{4/3}}{4}\right) + C$

(D) none of these

84. $\int \sqrt[3]{x}\,\sqrt[7]{1+\sqrt[3]{x^4}}\,dx$ is equal to

(A) $\frac{21}{32}\left(1+\sqrt[3]{x^4}\right)^{8/7} + C$

(B) $\frac{32}{21}\left(1+\sqrt[3]{x^4}\right)^{8/7} + C$

(C) $\frac{7}{32}\left(1+\sqrt[3]{x^4}\right)^{8/7} + C$

(D) none of these

Passage 2

If the integrand is a rational function of fractional powers of an independent variable x

i.e., $f\left\{(ax+b), (ax+b)^{\frac{p_1}{q_1}}, (ax+b)^{\frac{p_2}{q_2}} \ldots (ax+b)^{\frac{p_k}{q_k}}\right\}$,

where pi, $qi \in Z$ and $qi \neq 0$ $\forall$ $1 \leq i \leq k$, we take the L.C.M. of all the denominators of radicals, i.e., qi's and then substitute

$x = t^{(\text{lcm of all } qi\text{'s})}$, $\forall$ $1 \leq i \leq k$

The above integration reduces the fractional powers of x to integral powers of t, thereby transforming the integrand into a rational function of t.

Passage 3

A standard technique for solving problems is to reduce the problem into a similar problem but with lesser complexity. For example, in order to evaluate $\int \sin^n x\, dx$, we may convert it into a relation which requires the solution of $\int \sin^{n-1} x\, dx$ instead. By repeated conversion, we may reduce it ultimately to an integral that requires the evaluation of $\int \sin^2 x\, dx$. Integrals of the type $\int \sin^m x \cos^n x\, dx$ are evaluated by reducing the powers m and n to small values like 0, 1, 2, etc. A host of other integrals may also be evaluated easily using this technique. Since this process connects the given integral with another of the same type but with a reduced power, the relation obtained is called a reduction formula.

85. If $I_n = \int \tan^n x\, dx$, then $I_0 + I_1 + 2(I_2 + \ldots + I_8) + I_9 + I_{10}$ is equal to

(A) $\left(\frac{\tan x}{1} + \frac{\tan^2 x}{2} + \ldots + \frac{\tan^9 x}{9}\right)$

(B) $-\left(\frac{\tan x}{1} + \frac{\tan^2 x}{2} + \ldots + \frac{\tan^9 x}{9}\right)$

(C) $\left(\frac{\cot x}{1} + \frac{\cot^2 x}{2} + \ldots + \frac{\cot^9 x}{9}\right)$

(D) $-\left(\frac{\cot x}{1} + \frac{\cot^2 x}{2} + \ldots + \frac{\cot^9 x}{9}\right)$

86. If $I_n = \int \cos nx \operatorname{cosec} x\, dx$, then $I_n - I_{n-2} =$

(A) $\frac{\cos(n-1)x}{n-1}$ (B) $\frac{2\cos(n-1)x}{n-1}$

(C) $\frac{2\sin(n-1)x}{n-1}$ (D) none of these

87. If $I_n = \int \frac{x^n}{\sqrt{x^2+a^2}}\, dx \ (n \geq 2)$, then

$$I_n = \frac{x^{n-1}\sqrt{x^2+a^2}}{n} + k\, I_{n-2}, \text{ where } k =$$

(A) $\frac{a^2(1-n)}{n} I_{n-2}$ (B) $\frac{a^2(n-1)}{n} I_{n-2}$

(C) $\frac{a^2(n+1)}{n} I_{n-2}$ (D) none of these

88. If $I_{m,n} = \int \frac{\sin^m x}{\cos^n x}\, dx$, then

$$I_{m,n} = \frac{\sin^{m-1} x}{(n-1)\cos^{n-1} x} + k\, I_{m-2,n-2}, \text{ where } k =$$

(A) $\frac{m-1}{n-1}$ (B) $\frac{1-m}{n-1}$

(C) $\frac{m-1}{n}$ (D) $\frac{m}{n-1}$

Match the Column Type

89.

Column–I	Column–II
(A) $\int \sqrt{\sec x - 1}\, dx$	1. $\sin^{-1}(\tan x)$
(B) $\int \frac{dx}{\cos x\sqrt{\cos 2x}}$	2. $\sec^{-1}(\sec x + \cos x)$
(C) $\int \frac{dx}{\cos^3 x\sqrt{\sin 2x}}$	3. $-2\log\left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right)$
(D) $\int \frac{\sin^3 x\, dx}{(1+\cos^2 x)\sqrt{1+\cos^2 x+\cos^4 x}}$	4. $\sqrt{2}\left(\sqrt{\tan x} + \frac{1}{5}\tan^{5/2} x\right)$

90.

Column–I	Column–II
(A) $\int \dfrac{dx}{\left(1+\sqrt{x}\right)\sqrt{x-x^2}}$	1. $\dfrac{2\left(\sqrt{x}-1\right)}{\sqrt{1-x}}$
(B) $\int \left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2} \dfrac{dx}{x}$	2. $\dfrac{1}{\sqrt{2}} \sec^{-1}\left(\dfrac{x^2+1}{\sqrt{2}\,x}\right)$
(C) $\int x\sqrt{\dfrac{1+x}{1-x}}\,dx$	3. $2\cot^{-1}\sqrt{x} - 2\,ln\left\lvert\dfrac{1+\sqrt{1-x}}{\sqrt{x}}\right\rvert$
(D) $\int \dfrac{(x^2-1)}{(x^2+1)\sqrt{x^4+1}}$	4. $-\left(1+\dfrac{x}{2}\right)\sqrt{1-x^2} - \dfrac{1}{2}\cos^{-1} x$

91.

Column–I	Column–II
(A) $\int \dfrac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}\,dx$	1. $-\dfrac{2}{3}\sin^{-1}(\cos^{3/2} x)$
(B) $\int \dfrac{\cos 5x + \cos 4x}{1-2\cos 3x}\,dx$	2. $-\dfrac{1}{2}\sin 2x$
(C) $\int \sqrt{\dfrac{\cos x - \cos^3 x}{1-\cos^3 x}}\,dx$	3. $\tan^{-1}(\tan x - \cot x)$
(D) $\int \dfrac{dx}{\sin^6 x + \cos^6 x}$	4. $-\dfrac{\sin 2x}{2} - \sin x$

Assertion-Reason Type

Instructions: *In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:*

(A) Assertion(A) is True and Reason(R) is True; Reason(R) is a correct explanation for Assertion(A)
(B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)
(C) Assertion(A) is True, Reason(R) is False
(D) Assertion(A) is False, Reason(R) is True

92. Assertion:

$$\int e^{(x\sin x + \cos x)}\left(\frac{x^2\cos^2 x - x\sin x - \cos x}{x^2}\right)dx$$

$$= e^{(x\sin x + \cos x)} \cdot \frac{\cos x}{x} + C$$

Reason: $\int e^{g(x)}\{f(x)\,g'(x) + f'(x)\}\,dx$

$= eg(x)f(x)$

93. Assertion: $\int \dfrac{\sin^2 x}{a+b\cos x}\,dx = \dfrac{1}{b^2}(ax - b\sin x)$

$$-\frac{2\sqrt{a^2-b^2}}{b^2}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

Reason: $\int \dfrac{dx}{a+b\cos x}$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

94. **Assertion:** $\int \sqrt{2+\tan^2 x}\, dx = \tan^{-1}\left(\frac{\tan x}{\sqrt{2+\tan^2 x}}\right)$

$+\frac{1}{2} ln\left|\frac{\tan^2 x + \tan x + 2}{\tan^2 x - \tan x + 2}\right|$

Reason: $\int \frac{2}{(1-x^2)(1+x^2)} dx$

$= \tan^{-1} x + \frac{1}{2} ln\left|\frac{x+1}{x-1}\right|$

95. **Assertion:** $\int x^{x+1} \log x\,(1+\log x)\, dx$

$= x^x\,(x\log x - 1) + C$

Reason: $\int x^x (1+\log x)\, dx = x^x$

Previous Year's Questions

96. $\int \frac{dx}{x(x^n+1)}$ is equal to : **[2002]**

(A) $\frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right)+c$

(B) $\frac{1}{n}\log\left(\frac{x^n+1}{x^n}\right)+c$

(C) $\log\left(\frac{x^n}{x^n+1}\right)+c$

(D) none of these

97. The coefficient of the middle term in the binomial expansion in powers of x of $(1+\alpha x)^4$ and of $(1-\alpha x)$ is the same if α equals **[2004]**

(A) $-\frac{5}{3}$ (B) $\frac{3}{5}$

(C) $-\frac{3}{10}$ (D) $\frac{10}{3}$

98. $\int\left\{\frac{(\log x - 1)}{(1+(\log x)^2}\right\}dx$ is equal to **[2005]**

(A) $\frac{\log x}{(\log x)^2+1}+C$

(B) $\frac{x}{x^2+1}+C$

(C) $\frac{xe^x}{1+x^2}+C$

(D) $\frac{x}{(\log x)^2+1}+C$

99. $\int \frac{dx}{\cos x + \sqrt{3}\sin x}$ equals **[2005]**

(A) $\frac{1}{2}\text{logtan}\left(\frac{x}{2}+\frac{\pi}{12}\right)+\text{c}$

(B) $\frac{1}{2}\text{logtan}\left(\frac{x}{2}-\frac{\pi}{12}\right)+\text{c}$

(C) $\text{logtan}\left(\frac{}{2}\quad\frac{}{12}\right)+\text{c}$

(D) $\text{logtan}\left(\frac{x}{2}-\frac{\pi}{12}\right)+\text{c}$

100. The value of $\frac{\sqrt{2}\int \sin x dx}{\sin\left(x-\frac{\pi}{4}\right)}$ is **[2008]**

(A) $x+\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$ x

(B) $x-\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

(C) $x+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

(D) $x-\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$

101. If $\frac{dy}{dx} = y+3; y>-3$ and y (0) = 2, then y (ln2) is equal to **[2011]**

(A) 5
(B) 13
(C) 2
(D) 7

102. If the integral **[2012]**

$$\int \frac{5\tan x}{\tan x - 2}dx = x + a\ \ln|\sin x - 2\cos x| + k,$$

then a is equal to

(A) –1 (B) –2
(C) 1 (D) 2

103. If $\int f(x)dx = \Psi(x)$ then $\int x^5 f(x^3)dx$ is equal to **[2013]**

(A) $\frac{1}{3}x^3\Psi(x^3) - 3\int x^3\Psi(x^3)dx + C$

(B) $\frac{1}{3}x^3\Psi(x^3) - \int x^2\Psi(x^3)dx + C$

(C) $\frac{1}{3}\left[x^3\Psi(x^3) - \int x^3\Psi(x^3)dx\right] + C$

(D) $\frac{1}{3}\left[x^3\Psi(x^3) - \int x^2\Psi(x^3)dx\right] + C$

104. The integral $\int\left(1 + x - \frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to **[2014]**

(A) $(x-1)e^{x+\frac{1}{x}} + c$ (B) $xe^{x+\frac{1}{x}} + c$

(C) $(x+1)e^{x+\frac{1}{x}} + c$ (D) $-xe^{x+\frac{1}{x}} + c$

105. The integral $\int\frac{dx}{x^2\left(x^4+1\right)^{3/4}}$ equals: **[2015]**

(A) $\left(x^4+1\right)^{1/4} + c$ (B) $-\left(x^4+1\right)^{1/4} + c$

(C) $-\left(\frac{x^4+1}{x^4}\right)^{1/4} + c$ (D) $\left(\frac{x^4+1}{x^4}\right)^{1/4} + c$

106. The integral $\int\frac{2x^{12}+5x^9}{(x^5+x^3+1)}$ is equals to: **[2016]**

(A) $\frac{-x^{10}}{2(x^5+x^3+1)} + C$

(B) $\frac{-x^5}{(x^5+x^3+1)^2} + C$

(C) $\frac{-x^{10}}{2(x^5+x^3+1)^2} + C$

(D) $\frac{-x^5}{2(x^5+x^3+1)^2} + C$

ANSWER KEYS

Single Option Correct Type

1. (C) **2.** (A) **3.** (B) **4.** (B) **5.** (B) **6.** (C) **7.** (C) **8.** (A) **9.** (A) **10.** (C)
11. (A) **12.** (B) **13.** (B) **14.** (B) **15.** (B) 16. (A) **17.** (B) **18.** (D) **19.** (A) **20.** (B)
21. (C) **22.** (B) **23.** (A) **24.** (B) **25.** (D) 26. (C) **27.** (C) **28.** (D) **29.** (B) **30.** (C)
31. (C) **32.** (B) **33.** (A) **34.** (C) **35.** (C) 36. (A) **37.** (B) **38.** (C) **39.** (A) **40.** (B)
41. (B) **42.** (A) **43.** (C) **44.** (A) **45.** (B) **46.** (D) **47.** (A) **48.** (D) **49.** (A) **50.** (C)
51. (B) **52.** (A) **53.** (B) **54.** (A) **55.** (B) **56.** (A) **57.** (C) **58.** (B) **59.** (D) **60.** (B)
61. (B) **62.** (C) **63.** (B) **64.** (C) **65.** (A) **66.** (A) **67.** (A) **68.** (D) **69.** (A)

More than One Option Correct Type

70. (A), and (B) **71.** (A) and (D) **72.** (A), and (C) **73**. (A),(B), (C) and (D)
74. (B), and (C) **75.** (B), and (C) **76.** (A), and (C) **77.** (A), and (B)
78. (C), and (D) **79.** (B), and (D) **80.** (A), and (D)

Passage Based Questions

81. (B) **82.** (C) **83.** (b) **84**. (A) **85.** (A) **86.** (B) **87.** (A) **88.** (B)

Match the Column Type

89. (A) → 3; (B) → 1; (C) → 4; (D) → 2 **90.** (A) → 1; (B) → 3; (C) → 4; (D) → 2
91. (A) → 2; (B) → 4; (C) → 1; (D) → 3

Assertion-Reason Type

92. (A) **93.** (A) **94**. (A) **95.** (A)

Previous Year's Questions

96. (A) **97.** (C) **98.** (D) **99.** (A) **100.** (C) **101.** (D) **102.** (D) **103.** (B) **104.** (B) **105.** (C) **106.** (C)

HINTS AND SOLUTIONS

Single Option Correct Type

1. We have, $y = \int x^3 \cos x^4 \, dx = \frac{1}{4}\int \cos t \, dt$

$$\left(\text{Putting } x^4 = t \Rightarrow x^3 \, dx = \frac{1}{4} dt\right)$$

$= \frac{1}{4} \sin t + C = \frac{1}{4} \sin x^4 + C.$

Since the curve passes through (0, 0), ∴ $C = 0$.

∴ $y = \frac{1}{4} \sin x^4 \Rightarrow x^4 = \sin^{-1}(4y) \Rightarrow x = \sqrt[4]{\sin^{-1}(4y)}$

The correct option is (C)

2. Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

∴ $f(x) = \int \frac{dx}{\frac{\sin^{1/2} x}{\cos^{1/2} x} \cdot \cos^4 x} = \int \frac{(1+\tan^2 x)\sec^2 x}{\sqrt{\tan x}} dx$

$= \int \frac{(1+t^2)}{\sqrt{t}} dt = \int (t^{-1/2} + t^{3/2}) \, dt$

$= 2t^{1/2} + \frac{2}{5} t^{5/2} = 2\sqrt{\tan x} + \frac{2}{5}(\tan x)^{5/2}$

∴ $\phi\left(\frac{\pi}{4}\right) - \phi(0) = 2 + \frac{2}{5} = \frac{12}{5}$

The correct option is (A)

3. We have, $\phi(x) = \lim_{n \to \infty} \frac{x^{2n} - 1}{x^{2n} + 1} = -1$ as $0 < x < 1$.

∴ $\int \sin^{-1} x \cdot f(x) \, dx = -\int \sin^{-1} x \, dx$

$= -\left(x \sin^{-1} x + \sqrt{1 - x^2}\right) + c$

The correct option is (B)

4. $I = \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$

$= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - 2\sin^2 x \cos^2 x} dx$

$= \int \frac{(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x)}{1 - 2\sin^2 x \cos^2 x} dx$

$= \int \frac{1(\sin^2 x - \cos^2 x)[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x]}{1 - 2\sin^2 x \cos^2 x} dx$

$= \int \frac{(\sin^2 x - \cos^2 x)(1 - 2\sin^2 x \cos^2 x)}{1 - 2\sin^2 x \cos^2 x} dx$

$= -\int \cos 2x \, dx = -\frac{1}{2} \sin 2x + c$

The correct option is (B)

5. $I = \int \frac{dx}{(\sqrt{x})^2 + (\sqrt{x})^7} = \int \frac{dx}{(\sqrt{x})^7 \left(1 + \frac{1}{(\sqrt{x})^5}\right)}$

Put $\frac{1}{(\sqrt{x})^5} = y \Rightarrow \frac{dy}{dx} = -\frac{5}{2(\sqrt{x})^7}$

∴ $I = \int \frac{-2dy}{5(1+y)} = -\frac{2}{5} \ln|1 + y| + c$

$= +\frac{2}{5} \ln\left(\frac{(\sqrt{x})^5}{(\sqrt{x})^5 + 1}\right) + C$

∴ $a = \frac{2}{5}, k = \frac{5}{2}$

The correct option is (B)

6. Put $x^2 = t$

$\Rightarrow I = \frac{1}{2} \int [f(t) g''(t) - g(t) f''(t)] \, dt$

$$= \frac{1}{2}\left[\int f(t)g''(t)\,dt - \int g(t)f''(t)\,dt\right] + c$$

$$= \frac{1}{2}\left[f(t)g'(t) - \int f'(t)\,g'(t)\,dt - g(t)\,f'(t) + \int g'(t)\,f'(t)\,dt\right] + c$$

$$\therefore \quad I = \frac{1}{2}\,[f(t)g'(t) - g(t)f'(t)] + c$$

$$= \frac{1}{2}\,[f(x^2)g'(x^2) - g(x^2)f'(x^2)] + c$$

The correct option is (C)

7. The given anti-derivative = $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\,dx$

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}}{1 - 2\left(2\cos^2\frac{3x}{2} - 1\right)}\,dx$$

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}}{3 - 4\cos^2\frac{3x}{2}}\,dx$$

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}\cos\frac{3x}{2}}{3\cos\frac{3x}{2} - 4\cos^3\frac{3x}{2}}\,dx$$

(Multiplying and dividing by cos $\frac{3x}{2}$)

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{3x}{2}\cos\frac{x}{2}}{-\cos\frac{9x}{2}}\,dx$$

($\cos 3x = 4\cos^3 x - 3\cos x$)

$$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2}\,dx$$

$$= -\int (\cos 2x + \cos x)\,dx$$

$$= -\frac{\sin 2x}{2} - \sin x + c$$

The correct option is (C)

8. Let $I = \int \tan^4 x\,dx$

$$= \int \tan^2 x(\sec^2 x - 1)\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$

$$= \int \tan^2 x\,d(\tan x) - \int (\sec^2 x - 1)\,dx,$$

where $t = \tan x$

$$= \frac{\tan^3 x}{3} - \tan x + x + C$$

$$= K\tan^3 x + L\tan x + f(x)$$

$$\Rightarrow \quad K = \frac{1}{3};\ L = -1;\ f(x) = x + C$$

The correct option is (A)

9. $\int \frac{1}{\left[(x-1)^3(x+2)^5\right]^{1/4}}\,dx = \int \frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}\,dx$

$$= \frac{1}{3}\int \frac{1}{t^{3/4}}\,dt \quad \left[\because\ \frac{x-1}{x+2} = t \Rightarrow \frac{3}{(x+2)^2}\,dx = dt\right]$$

$$= \frac{1}{3}\left(\frac{t^{1/4}}{1/4}\right) + c = \frac{4}{3}t^{1/4} + c$$

$$= \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$$

The correct option is (A)

10. Put $\ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$$\therefore \quad I = \int e^t\left(\frac{t-1}{t^2+1}\right)^2 dt = \int e^t\left(\frac{1}{t^2+1} - \frac{2t}{(t^2+1)^2}\right)dt$$

$$= \frac{e^t}{t^2+1} + c = \frac{x}{(\ln x)^2 + 1} + c.$$

The correct option is (C)

11. $I = \int \frac{a - \frac{b}{x^2}}{\sqrt{c^2 - \left(\frac{ax^2+b}{x}\right)^2}}\,dx = \int \frac{\left(a - \frac{b}{x^2}\right)}{\sqrt{c^2 - \left(ax + \frac{b}{x}\right)^2}}\,dx$

Put $ax + \frac{b}{x} = t \Rightarrow \left(a - \frac{b}{x^2}\right)dx = dt$

$$\therefore \quad I = \int \frac{dt}{\sqrt{c^2 - t^2}} = \sin^{-1}\left(\frac{ax + \frac{b}{x}}{c}\right) + k$$

The correct option is (A)

12. $\int \frac{\sec x\,dx}{\sqrt{\sin(2x+\theta) + \sin\theta}} = \int \frac{\sec x\,dx}{\sqrt{2\sin(x+\theta)\cos x}}$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec^{3/2} x\,dx}{\sqrt{\sin x\cos\theta + \sin\theta\cos x}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\sec^2 x\, dx}{\sqrt{\tan x \cos\theta + \sin\theta}} \qquad (t = \tan x)$$

$$= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{t \cos\theta + \sin\theta}}$$

$$= \frac{2}{\sqrt{2}} \frac{\sqrt{t \cos\theta + \sin\theta}}{\cos\theta} + c$$

$$= \sqrt{2\,(\tan x \sec\theta + \tan\theta \sec\theta)} + c$$

The correct option is (B)

13. Put $\cos x + \sin x = z$

$\Rightarrow$ $(-\sin x + \cos x)\, dx = dz$

Also, $1 + \sin 2x = z^2$

$$\therefore \quad I = \int \frac{dz}{\sqrt{8 - (z^2 - 1)}}$$

$$= \frac{dz}{\sqrt{9 - z^2}} = \sin^{-1} \frac{z}{3} + c$$

$$= \sin^{-1}\left(\frac{\cos x + \sin x}{3}\right) + c$$

The correct option is (B)

14. $f(x) = \int \frac{x^2 dx}{(1+x^2)\left(1+\sqrt{1+x^2}\right)}$

Let $x = \tan\theta$ $\Rightarrow$ $dx = \sec^2\theta\, d\theta = (1 + x^2).d\theta$

$$\therefore \quad f(x) = \int \frac{x^2 dx}{(1+x^2)\left(1+\sqrt{1+x^2}\right)}$$

$$= \int \frac{\tan^2\theta \sec^2\theta d\theta}{\sec^2\theta(1+\sec\theta)}$$

$$= \int \frac{\tan^2\theta\, d\theta}{1+\sec\theta}\theta = \int \frac{\sin^2\theta\, d\theta}{\cos\theta(1+\cos\theta)}$$

$$= \int \frac{1-\cos^2\theta\, d\theta}{\cos\theta(1+\cos\theta)}$$

$$= \int \frac{(1-\cos\theta)\, d\theta}{\cos\theta} = \int \sec\theta d\theta - \int d\theta$$

$$= \log\left(x + \sqrt{1+x^2}\right) - \tan^{-1} x + c$$

$$f(0) = \log\left(0 + \sqrt{1+0}\right) - \tan^{-1}(0) + c$$

$\Rightarrow$ $0 = \log 1 - 0 + c \Rightarrow c = 0$

$$\therefore \quad f(1) = \log\left(1 + \sqrt{1+1^2}\right) - \tan^{-1}(1)$$

$$= \log\left(1+\sqrt{2}\right) - \frac{\pi}{4}$$

The correct option is (B)

15. Put $t = xe^x$. $\Rightarrow$ $dt = e^x(1 + x)\, dx$

Thus, $\int \frac{(x+1)\, dx}{x(1+xe^x)^2} = \int \frac{e^x(1+x)\, dx}{xe^x(1+xe^x)^2}$

$$= \int \frac{dt}{t(1+t)^2}$$

$$= \int \left(\frac{1}{t} - \frac{1}{1+t} - \frac{1}{(1+t)^2}\right) dt$$

$$= \log|t| - \log|(1+t)| + \frac{1}{1+t} + C$$

$$= \log\left|\frac{t}{1+t}\right| + \frac{1}{1+t} + C$$

$$= \log\left|\left(1 - \frac{1}{1+t}\right)\right| + \frac{1}{1+t} + C.$$

The correct option is (B)

16. $\int f(x) \sin x \cos x\, dx$

$$= \frac{1}{2(b^2 - a^2)} \log(f(x)) + C$$

$$\Rightarrow \quad f(x) \sin x \cos x = \frac{1}{2(b^2 - a^2)} \bullet \frac{1}{f(x)} f'(x)$$

[by differentiating both the sides]

$$\Rightarrow \quad 2(b^2 - a^2) \sin x \cos x = \frac{f'(x)}{(f(x))^2}$$

$$\Rightarrow \quad \int (2b^2 \sin x \cos x - 2a^2 \sin x \cos x)\, dx$$

$$= \int \frac{f'(x)}{(f(x))^2} dx$$

[by integrating both the sides]

$$\Rightarrow \quad -b^2 \cos^2 x - a^2 \sin^2 x = -\frac{1}{f(x)}$$

$$\Rightarrow \quad f(x) = \frac{1}{(a^2 \sin^2 x + b^2 \cos^2 x)}$$

The correct option is (A)

17. Let $I = \int \frac{dx}{(x+a)^{8/7}\,(x-b)^{6/7}}$

$$= \int \frac{dx}{(x+a)^2 \left(\frac{x-b}{x+a}\right)^{6/7}}$$

Put $\frac{x-b}{x+a} = y \Rightarrow \frac{a+b}{(x+a)^2}dx = dy$

$$\therefore \quad I = \frac{1}{a+b}\int \frac{dy}{y^{6/7}} = \frac{7}{a+b}\cdot y^{1/7}$$

$$= \frac{7}{a+b}\left(\frac{x-b}{x+a}\right)^{1/7} + c$$

The correct option is (B)

18. $I = \int \sin(\log x).1\,dx$

$$= x\sin(\log x) - \int \frac{\cos(\log x)}{x}\cdot x\,dx$$

$$= x\sin(\log x) - \int \cos(\log x)\bullet 1dx$$

$$= x\sin(\log x) - [x\cos(\log x + \int \sin(\log x)\,dx]$$

$$\therefore \quad 2I = x\sin(\log x) - x\cos(\log x)$$

$$\Rightarrow \quad I = \frac{x}{2}[\sin(\log x) - \cos(\log x)]$$

So, $f(x) = \frac{x}{2} \Rightarrow \lim_{x\to 2} f(x) = 1$

Also, $\lim_{x\to 1}\frac{g(x)}{h(x)} = 1$

The correct option is (D)

19. $I = \int \frac{\sqrt{x}}{\sqrt{x^3+4}}dx$

Let $x^{3/2} = 2\tan\theta \Rightarrow \frac{3}{2}x^{1/2} = 2\sec^2\theta\frac{d\theta}{dx}$

$$\Rightarrow x^{1/2}dx = \frac{4}{3}\sec^2\theta\,d\theta$$

$$\therefore \quad I = \int \frac{\frac{4}{3}\sec^2\theta d\theta}{\sqrt{4\tan^2\theta+4}} = \frac{2}{3}\int \sec\theta d\theta$$

$$= \frac{2}{3}\ln(\sec\theta + \tan\theta) + c$$

$$= \frac{2}{3}\ln\left(\sqrt{\frac{x^3-4}{4}} + \frac{x^{3/2}}{2}\right) + c$$

$$= \frac{2}{3}\ln\left(\frac{\sqrt{x^3}+\sqrt{x^3-4}}{2}\right) + c$$

$$= \frac{2}{3}\ln\left[\frac{x^3-(x^3-4)}{2\left(\sqrt{x^3}-\sqrt{x^3-4}\right)}\right] + c$$

$$= \frac{2}{3}\ln\left(\frac{2}{\sqrt{x^3}-\sqrt{x^3-4}}\right) + c$$

The correct option is (A)

20. $I = \int \frac{f(x)\phi'(x) - f'(x)\phi(x)}{f(x)\,\phi(x)}\log\left[\frac{\phi(x)}{f(x)}\right]dx$

Put $\log\frac{\phi(x)}{f(x)} = t$

$$\Rightarrow \frac{f(x)}{\phi(x)}\bullet\frac{f(x)\,\phi'(x) - f'(x)\,\phi(x)}{(f(x))^2}dx = dt,$$

we get $I = \int t\,dt = \frac{1}{2}t^2 + k$

$$= \frac{1}{2}\left(\log\frac{\phi(x)}{f(x)}\right)^2 + k,\ k\in R$$

The correct option is (B)

21. Let $I = \int \frac{1}{2p}\sqrt{\frac{p-1}{p+1}}dp$

$$= \frac{1}{2}\int \frac{p-1}{p\sqrt{(p+1)(p-1)}}dp$$

$$= \frac{1}{2}\int \frac{pdp}{p\sqrt{p^2-1}} - \frac{1}{2}\int \frac{dp}{p\sqrt{p^2-1}}$$

$$= \frac{1}{2}\log_e\left(p+\sqrt{p^2-1}\right) - \frac{1}{2}\sec^{-1}p.$$

The correct option is (C)

22. Let $I_1 = \int \frac{1}{x+x^5}dx$ and $I_2 = \int \frac{x^4}{x+x^5}dx$

Now, $I_1 + I_2 = \int \frac{dx}{x}$

$$\Rightarrow \quad I_1 + I_2 = \log|x| + c$$

$$\Rightarrow \quad I_2 = \log|x| - f(x) + c$$

The correct option is (B)

23. Putting $l^{r+1}(x) = t$

and $\frac{1}{x\,l(x)\,l^2(x)\ldots l^r(x)}dx = dt$

we get, $\int \frac{1}{x\,l^2(x)\,l^3(x)\ldots l^r(x)}dx$

$$= \int 1\cdot dt = t + C = l^{r+1}(x) + C$$

The correct option is (A)

24. Put $\dfrac{x^2+2}{x} = y$. $\therefore\ dy = \left(1-\dfrac{2}{x^2}\right)dx$

$$\therefore \int \frac{(x^2-2)\,dx}{(x^4+5x^2+4)\tan^{-1}\left(\frac{x^2+2}{x}\right)}$$

$$= \int \frac{x^2\left(1-\frac{2}{x^2}\right)dx}{[(x^2+2)^2+x^2]\tan^{-1}\left(\frac{x^2+2}{x}\right)}$$

$$= \int \frac{dy}{(y^2+1)\tan^{-1} y} \qquad \left(\text{Putting } x+\frac{2}{x} = t\right)$$

$$= \log|\tan^{-1}y| = \log\left|\tan^{-1}\left(x+\frac{2}{x}\right)\right| + C$$

The correct option is (B)

25. $$\int e^x\left(\frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\right)dx$$

$$= \int e^x\left[\frac{\sqrt{1-x^{2n}}}{1-x^n}+\frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right]$$

$$= e^x\frac{\sqrt{1-x^{2n}}}{1-x^n}+C$$

The correct option is (D)

26. $$I = \int\sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}\,dx = \int\frac{\sqrt{\cos x}\,.\sin x}{\sqrt{1-(\cos^{3/2} x)^2}}\,dx$$

Put $\cos^{3/2}x = t \Rightarrow \dfrac{3}{2}\sqrt{\cos x}\,(-\sin x)\,dx = dt$

$$\Rightarrow\quad I = \int\frac{-\frac{2}{3}dt}{\sqrt{1-t^2}} \qquad = -\frac{2}{3}\sin^{-1}t + c$$

$$= -\frac{2}{3}\sin^{-1}(\cos^{3/2}x) + c$$

$$= -\frac{2}{3}\left(\frac{\pi}{2}-\cos^{-1}(\cos^{3/2}x)\right)+c$$

$$= \frac{2}{3}\cos^{-1}(\cos^{3/2}x) + c$$

The correct option is (C)

27. $m+n = -\dfrac{3}{7}+\left(\dfrac{-11}{7}\right) = -2$ (–ve integer)

$$I = \int\cos^{-3/7}x\left(\sin^{(-2+3/7)}x\right)dx$$

$$= \int\cos^{-3/7}x\sin^{-2}x\sin^{3/7}x\,dx$$

$$= \int\frac{\operatorname{cosec}^2 x}{\left(\frac{\cos^{3/7}x}{\sin^{3/7}x}\right)}dx = \int\frac{\operatorname{cosec}^2 x\,dx}{\cot^{3/7}x}$$

Put $\cot x = t \Rightarrow -\operatorname{cosec}^2 x\,dx = dt$

$$I \quad = -\int\frac{dt}{t^{3/7}} = -\frac{t^{-\frac{3}{7}+1}}{-\frac{3}{7}+1}+c$$

$$= -\frac{7}{4}t^{4/7}+c = -\frac{7}{4}\cot^{4/7}x+c$$

$$= -\frac{7}{4}\tan^{-4/7}x+c.$$

The correct option is (C)

28. $$\int\frac{5\tan x}{\tan x-2}dx = \int\frac{5\tan x}{\sin x-2\cos x}dx$$

$$= \int\frac{(\sin x-2\cos x)+2(\cos x+2\sin x)}{(\sin x-2\cos x)}dx$$

$$= \int dx + 2\int\frac{\cos x+2\sin x}{\sin x-2\cos x}dx$$

$$= x + 2\ln|(\sin x-2\cos x)| + k$$

$\Rightarrow\quad a = 2$

The correct option is (D)

29. Let $I = \displaystyle\int\frac{dx}{(a^2-b^2x^2)^{3/2}}$

Let $bx = a\sin\theta \ \Rightarrow\ b\,dx = a\cos\theta\,d\theta$

$$\Rightarrow\quad I = \int\frac{a\cos\theta\,d\theta}{b(a^2-a^2\sin^2\theta)^{3/2}} = \frac{1}{a^2b}\int\sec^2\theta\,d\theta$$

$$= \frac{\tan\theta}{a^2b}+c = \frac{bx}{a^2b\sqrt{a^2-b^2x^2}}+c$$

The correct option is (B)

30. $$\sqrt{2}\int\frac{\sin x\,dx}{\sin\left(x-\frac{\pi}{4}\right)} = \sqrt{2}\int\frac{\sin\left(x-\frac{\pi}{4}+\frac{\pi}{4}\right)dx}{\sin\left(x-\frac{\pi}{4}\right)}$$

$$= \sqrt{2}\int\left(\cos\frac{\pi}{4}+\cot\left(x-\frac{\pi}{4}\right)\sin\frac{\pi}{4}\right)dx$$

$$= \int dx + \int \cot\left(x - \frac{\pi}{4}\right) dx$$

$$= x + \ln\left|\sin\left(x - \frac{\pi}{4}\right)\right| + c.$$

The correct option is (C)

31. $I = \int (x^{3n} + x^{2n} + x^n)(2x^{2n} + 3x^n + 6)^{1/n} dx$

$$= \int (x^{3n} + x^{2n} + x^n)\frac{(2x^{3n} + 3x^{2n} + 6x^n)^{1/n}}{x} dx$$

$$= \int (x^{3n-1} + x^{2n-1} + x^{n-1})$$

$$(2x^{3n} + 3x^{2n} + 6x^n)^{1/n} dx$$

$$= \frac{1}{6}\int (2x^{3n} + 3x^{2n} + 6x^n)^{1/n}$$

$$(6x^{3n-1} + 6x^{2n-1} + 6x^{n-1}) dx$$

$= \frac{1}{6n}\int t^{1/n}\, dt$, where $t = 2x^{3n} + 3x^{2n} + 6x^n$

$$= \frac{1}{6n} \cdot \frac{t^{\frac{1}{n}+1}}{\frac{1}{n}+1}$$

$$= \frac{1}{6(n+1)}(2x^{3n} + 3x^{2n} + 6x^n)^{\frac{1}{n}+1} + C$$

The correct option is (C)

32. We have, $f\left(\frac{3x-4}{3x+4}\right) = x + 2$

Let $\frac{3x-4}{3x+4} = t \Rightarrow 3x - 4 = 3xt + 4t$

$$\Rightarrow \quad x = \frac{4t+4}{3(1-t)}$$

$$\therefore \quad f(t) = \frac{4t+4}{3(1-t)} + 2$$

$$\therefore \quad f(x) = \frac{4x+4}{3(1-x)} + 2 \qquad = \frac{4(x-1)+8}{3(1-x)} + 2$$

$$= 2 - \frac{4}{3} - \frac{8}{3(x-1)}$$

$$\therefore \quad \int f(x)\, dx = \frac{2}{3}x - \frac{8}{3}\ln|x-1| + c$$

The correct option is (B)

33. $I = \int x^{-1}(1-x^3)^{-1/2} dx$

Let $1 - x^3 = t^2 \Rightarrow -3x^2 dx = 2t\, dt$

$$\Rightarrow \quad x^{-1}dx = -\frac{2}{3}\frac{tdt}{x^3} = -\frac{2}{3}\frac{tdt}{1-t^2}$$

$$\therefore \quad I = \int (t^{-1})\left(\frac{-2}{3}\right)\frac{tdt}{1-t^2} = -\frac{2}{3}\int \frac{dt}{1-t^2}$$

$$= \frac{2}{3}\int \frac{dt}{t^2-1} = \frac{2}{3}\cdot\frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + c$$

$$= \frac{1}{3}\ln\left(\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|\right) + c$$

The correct option is (A)

34. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\, dx = \int x^{\frac{-2}{3}}\left(1 + x^{\frac{1}{3}}\right)^{\frac{1}{2}} dx$

Put $1 + x^{1/3} = z^2$ $\left[\begin{array}{l}\text{Here } m = \frac{-2}{3}, n = \frac{1}{3}, p = \frac{1}{2} \\ \therefore \frac{m+1}{n} = 1 \text{ (an integer)}\end{array}\right]$

$$\Rightarrow \quad x^{-2/3}\, dx = 6z\, dz$$

$$\therefore \quad \int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\, dx \quad = \int z.6z\, dz = 2z^3 + C.$$

$$= 2(1 + x^{1/3})^{3/2} + C$$

The correct option is (C)

35. Put $x^2 = t$

$$\Rightarrow I = \frac{1}{2}\int \{f(t)g''(t) - g(t)f''(t)\}dt$$

$$= \frac{1}{2}\left\{\int f(t)g''(t)dt - \int g(t)f''(t)dt\right\} + c$$

$$= \frac{1}{2}\left\{f(t)g'(t) - \int f'(t)g'(t)dt - g(t)f'(t)\right.$$

$$\left. + \int g'(t)f'(t)dt\right\} + c$$

$$\therefore \quad I = \frac{1}{2}\{f(t)g'(t) - g(t)f'(t)\} + c$$

$$= \frac{1}{2}\{f(x^2)g'(x^2) - g(x^2)f'(x^2)\} + c$$

The correct option is (C)

36. $\int f(x)\sin x \cos x\, dx$

$$= \frac{1}{2(b^2-a^2)}\log(f(x)) + C$$

$$\Rightarrow \quad f(x)\sin x \cos x = \frac{1}{2(b^2-a^2)} \bullet \frac{1}{f(x)}f'(x)$$

[by differentiating both the sides]

$$\Rightarrow \quad 2(b^2 - a^2)\sin x \cos x = \frac{f'(x)}{(f(x))^2}$$

$$\Rightarrow \quad \int (2b^2 \sin x \cos x - 2a^2 \sin x \cos x) dx$$

$$= \int \frac{f'(x)}{(f(x))^2} dx$$

Integrating both the sides, we get

$$\Rightarrow \quad -b^2\cos^2 x - a^2\sin^2 x = -\frac{1}{f(x)}$$

$$\Rightarrow \quad f(x) = \frac{1}{(a^2\sin^2 x + b^2\cos^2 x)}$$

The correct option is (A)

37. We have,

$$f(x) = \tan^{-1}x + \ln\sqrt{1+x} - \ln\sqrt{1-x}$$

$$= \tan^{-1}x + \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)$$

$$= \frac{1}{1+x^2} + \frac{1}{1-x^2} = \frac{2}{1-x^4}$$

$$\therefore \quad \int \frac{f'(x)}{2}\, d(x^4) = \int \frac{d(x^4)}{1-x^4} = -\int \frac{d\left(1-x^4\right)}{1-x^4}$$

$$= -\ln|1-x^4| + C$$

The correct option is (B)

38. We have,

$$I = \int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}\,dx = \int \frac{e^x(1+1-x^2)\,dx}{(1-x)\sqrt{1-x^2}}$$

$$= \int e^x\left[\frac{1}{(1-x)\sqrt{1-x^2}} + \frac{\sqrt{1+x}}{\sqrt{1-x}}\right]dx$$

Now, $\frac{d}{dx}\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right)$

$$= \frac{\sqrt{1-x}\left(\sqrt{1+x}\right)' - \sqrt{1+x}\left(\sqrt{1-x}\right)'}{1-x}$$

$$= \frac{\frac{\sqrt{1-x}}{2\sqrt{1+x}} + \frac{\sqrt{1+x}}{2\sqrt{1-x}}}{1-x}$$

$$= \frac{1-x+1+x}{2(1-x)\sqrt{1-x^2}} = \frac{1}{(1-x)\sqrt{1-x^2}}$$

$$\therefore I = \int e^x\left[\frac{d}{dx}\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right) + \frac{\sqrt{1+x}}{\sqrt{1-x}}\right]dx$$

$$= e^x\sqrt{\frac{1+x}{1-x}} + C.$$

The correct option is (C)

39. We have,

$$I = \int\left\{\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right\}\ln x\,dx$$

Putting $\left(\frac{x}{e}\right)^x = t$

$$\Rightarrow \quad \left[x\left(\frac{x}{e}\right)^{x-1}\frac{1}{e} + \left(\frac{x}{e}\right)^x \ln\left(\frac{x}{e}\right)\right]dx = \mathrm{dt}$$

$$\Rightarrow \quad \left(\frac{x}{e}\right)^x\left[1 + \ln\left(\frac{x}{e}\right)\right]dx = \mathrm{dt}$$

$$\Rightarrow \quad \ln x\,dx = \frac{dt}{t}$$

Therefore, the given integral reduces to

$$I = \int\left(t + \frac{1}{t}\right)\frac{dt}{t} = \int\left(1 + \frac{1}{t^2}\right)dt = t - \frac{1}{t} + C$$

$$= \left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + C$$

The correct option is (A)

40. We have,

$$I = \int\left(x^{3a} + x^{2a} + x^a\right)\left(2x^{2a} + 3x^a + 6\right)^{1/a}dx$$

$$= \int x\left(x^{3a-1} + x^{2a-1} + x^{a-1}\right)\left(2x^{2a} + 3x^a + 6\right)^{1/a}dx$$

$$= \int\left(x^{3a-1} + x^{2a-1} + x^{a-1}\right)\left(2x^{3a} + 3x^{2a} + 6x^a\right)^{1/a}dx$$

Putting $2x^{3a} + 3x^{2a} + 6x^a = u$

$\Rightarrow \quad (6a\,x^{3a-1} + 6a\,x^{2a-1} + 6a\,x^{a-1})\,dx = du$

$$\therefore \quad I = \int u^{1/a}\frac{du}{6a} = \frac{1}{6a}\,\frac{u^{\frac{1}{a}+1}}{\frac{1}{a}+1} + C$$

$$= \frac{1}{6(a+1)}\left(2x^{3a} + 3x^{2a} + 6x^a\right)^{1+\frac{1}{a}} + C$$

The correct option is (B)

41. Put $x - y = t$ ...(1)

According to the given condition

$y(x-y)^2 = x$

i.e., $yt^2 = x$ ...(2)

Solving equations (1) and (2), we get

$$x = \frac{t^3}{t^2-1} \text{ and } y = \frac{t}{t^2-1}$$

$$\therefore \int \frac{dx}{x-3y} = \int \frac{1}{\frac{t^3}{t^2-1} - \frac{3t}{t^2-1}}\,d\left(\frac{t^3}{t^2-1}\right)$$

$$= \int \frac{t^2-1}{t(t^2-3)} \bullet \frac{(t^2-1)3t^2 - t^3(2t)}{(t^2-1)^2}\,dt$$

$$= \int \frac{1}{t(t^2-3)} \bullet \frac{t^2(t^2-3)}{t^2-1}\,dt$$

$$= \int \frac{t\,dt}{t^2-1} = \frac{1}{2}\int \frac{dt}{z} \quad \left[\begin{array}{l}\text{Putting } t^2-1=z\\ \text{and } 2tdt = dz\end{array}\right]$$

$$= \frac{1}{2}\ln|z| + C = \frac{1}{2}\ln\left|t^2-1\right| + C$$

$$= \frac{1}{2}\ln\left|(x-y)^2-1\right| + C$$

The correct option is (B)

42. $I = \frac{1}{2}\int \frac{(\cos x+\sin x)+(\cos x-\sin x)}{1-\sin x\cos x}\,dx$

$$= \frac{1}{2}\left[\int \frac{\cos x+\sin x}{1-\sin x\cos x}\,dx + \int \frac{\cos x-\sin x}{1-\sin x\cos x}\,dx\right]$$

$$= \frac{1}{2}\left[I_1 + I_2\right]$$

$$I_1 = \int \frac{\sin x+\cos x}{1-\sin x\cos x}\,dx$$

$$\begin{array}{ll}\text{Putting } \sin x - \cos x & = t\\ \Rightarrow (\cos x + \sin x)dx & = dt\\ \text{and, } 2\sin x\cos x & = 1-(\sin x-\cos x)^2\\ & = 1-t^2\end{array}$$

$$\therefore I_1 = \int \frac{dt}{1-\left(\frac{1-t^2}{2}\right)} = \int \frac{2dt}{1+t^2} = 2\tan^{-1} t + C$$

$$= 2\tan^{-1}(\sin x - \cos x) + C$$

Let $I_2 = \int \frac{\cos x - \sin x}{1-\sin x\cos x}\,dx$

Putting $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\,dx = dt$

and, $2\sin x\cos x = (\sin x + \cos x)^2 - 1 = t^2 - 1$

$$\therefore I_2 = \int \frac{dt}{1-\left(\frac{t^2-1}{2}\right)} = \int \frac{2dt}{3t^2}$$

$$= \frac{-1}{\sqrt{3}}\int\left(\frac{1}{t-\sqrt{3}} - \frac{1}{t+\sqrt{3}}\right)dt$$

$$= \frac{-1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + C$$

$$= \frac{-1}{\sqrt{3}}\ln\left|\frac{\sin x+\cos x-\sqrt{3}}{\sin x+\cos x+\sqrt{3}}\right| + C. \qquad \therefore k = -\frac{1}{2}$$

The correct option is (A)

43. $\int \frac{\cos\left(x+\frac{\pi}{4}\right)}{2+\sin 2x}\,dx = \int \frac{\cos x - \sin x}{\sqrt{2}\,(2+2\sin x\cos x)}\,dx$

$$= \frac{1}{\sqrt{2}}\int \frac{d(\sin x+\cos x)}{(\sin x+\cos x)^2-1}\,dx$$

Putting $\sin x + \cos x = t$, we have

$$I = \frac{1}{\sqrt{2}}\int \frac{dt}{1+t^2} = \frac{1}{\sqrt{2}}\tan^{-1} t + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}(\sin x + \cos x) + C$$

The correct option is (C)

44. Let $I = \int \frac{dx}{1-\cos^4 x}$

$$= \int \frac{dx}{(1-\cos^2 x)(1+\cos^2 x)}$$

$$= \int \frac{dx}{\left(1-\frac{1+\cos 2x}{2}\right)\left(1+\frac{1+\cos 2x}{2}\right)}$$

$$= \int \frac{4\,dx}{(1-\cos 2x)(3+\cos 2x)}$$

Putting $\tan x = t \Rightarrow dx = \frac{dt}{1+t^2}$ and $\cos 2x = \frac{1-t^2}{1+t^2}$

$$\therefore I = \int \frac{4}{\left(1-\frac{1-t^2}{1+t^2}\right)\left(3+\frac{1-t^2}{1+t^2}\right)}\cdot\frac{dt}{1+t^2}$$

$$= \int \frac{4(1+t^2)\,dt}{2t^2(2t^2+4)} = \int \frac{(1+t^2)\,dt}{t^2(t^2+2)}$$

$$= \frac{1}{2}\int\left(\frac{1}{t^2}+\frac{1}{t^2+2}\right)dt$$

$$= \frac{1}{2}\int\left[\frac{-1}{t}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\right]+C$$

$$= \frac{-1}{2\tan x}+\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)+C$$

The correct option is (A)

45. Given integral $= \int \frac{(1-\tan x)\,dx}{\sqrt{\tan x}\,(1+6\sin x\cos x)}$

$$= \int \frac{(1-\tan x)\sec^2 x\,dx}{\sqrt{\tan x}\,(\sec^2 x+6\tan x)}$$

Putting $\tan x = t^2 \Rightarrow \sec^2 x\,dx = 2t\,dt$

$$\therefore I = \int \frac{(1-t^2)2t\,dt}{t(1+t^4+6t^2)} = 2\int \frac{\left(\frac{1}{t^2}-1\right)dt}{t^2+\frac{1}{t^2}+6}$$

$$= -2\int \frac{d\left(t+\frac{1}{t}\right)}{\left(t+\frac{1}{t}\right)^2+4} = -2\int \frac{dy}{y^2+4}$$

$$\left[\text{Putting } t+\frac{1}{t}=y\right]$$

$$= -2\cdot\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)+C$$

$$= -\tan^{-1}\left(\frac{t^2+1}{2t}\right)+C$$

$$= -\tan^{-1}\left(\frac{\tan x+1}{2\sqrt{\tan x}}\right)+C$$

$$= -\tan^{-1}\left(\frac{\sqrt{\tan x}+\sqrt{\cot x}}{2}\right)+C. \qquad \therefore\ k=-1.$$

The correct option is (B)

46. (d). $\int \frac{\cos 5x+\cos 4x}{1-2\cos 3x}\,dx$

$$= \int \frac{\sin 3x\,(\cos 5x+\cos 4x)}{\sin 3x-\sin 6x}\,dx$$

$$= \int \frac{\sin 3x \bullet 2\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\cos\frac{9x}{2}\sin\frac{3x}{2}}\,dx$$

$$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2}\,dx$$

$$= -\int (\cos 2x+\cos x)\,dx$$

$$= -\frac{\sin 2x}{2}-\sin x+C = -\sin x\,(1+\cos x)+C$$

47. $\int \frac{\sec x\,dx}{\sqrt{\sin(2x+a)+\sin a}}$

$$= \int \frac{\sec x\,dx}{\sqrt{\sin 2x\cos a+(1+\cos 2x)\sin a}}$$

$$= \int \frac{\sec x\,dx}{\sqrt{2\cos x\,(\sin x\cos a+\cos x\sin a)}}$$

$$= \int \frac{\sec x\,dx}{\sqrt{2\cos^2 x\cos a\,(\tan x+\tan x)}}$$

$$= \frac{1}{\sqrt{2\cos a}}\int \frac{\sec^2 x\,dx}{\sqrt{\tan x+\tan a}}$$

Putting $\tan x+\tan a = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore I = \frac{1}{\sqrt{2\cos a}}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{\sqrt{2\cos a}}\,2\sqrt{t}+C$$

$$= \sqrt{\frac{2}{\cos a}}\,\sqrt{\tan x+\tan a}+C$$

The correct option is (A)

48. Let $I = \int \sqrt{x+\sqrt{x^2+2}}\,dx$

Put $\sqrt{x^2+2}+x = t^2$ ...(1)

$$\Rightarrow \left(1+\frac{x}{\sqrt{x^2+2}}\right)dx = 2t\,dt$$

$$\Rightarrow dx = \frac{\sqrt{x^2+2}}{x+\sqrt{x^2+2}}\cdot 2t\,dt = \frac{2\sqrt{x^2+2}}{t}\,dt$$

Now, rationalizing L.H.S. of equation (1), we get

$$\sqrt{x^2+2}-x = \frac{2}{t^2} \quad ...(2)$$

Adding equations (1) and (2), we get

$$\sqrt{x^2+2} = \frac{1}{2}\left(t^2+\frac{2}{t^2}\right) = \frac{t^4+2}{2t^2}$$

Thus, $dx = \frac{t^4+2}{t^3}\,dt$

Hence, $I = \int t\cdot\frac{t^4+2}{t^3}\,dt = \int\left(t^2+\frac{2}{t^2}\right)dt$

$$= \frac{t^3}{3}-\frac{2}{t}+C$$

$$= \frac{1}{3}\left(\sqrt{x^2+2}+x\right)^{3/2}$$

$$-\sqrt{2}\left(\sqrt{x^2+2}-x\right)^{1/2}+C.$$

$\therefore\ k = -\sqrt{2}$

The correct option is (D)

49. Let $I = \int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\,dx$

$$= \int \frac{(x^2-1)(x^2+1)}{x^3\sqrt{x^2+\frac{1}{x^2}+1}}\,dx$$

$$= \int \frac{\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)}{\sqrt{\left(x+\frac{1}{x}\right)^2-1}}\,dx$$

Putting $\left(x+\frac{1}{x}\right)^2-1 = t \Rightarrow 2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)dx = dt,$

$$\therefore\ I = \int \frac{dt}{2\sqrt{t}} = \sqrt{t}+C$$

$$= \sqrt{x^2+\frac{1}{x^2}+1}+C = .\ \frac{\sqrt{x^4+x^2+1}}{x}+C.$$

The correct option is (A)

50. Let $I = \int \frac{x^2-1}{(x^2+1)\sqrt{1+x^4}}\,dx$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)x^2}{x\left(1+\frac{1}{x}\right)x\sqrt{x^2+\frac{1}{x^2}}}\,dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(1+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-2}}$$

Putting $x+\frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$

$$\therefore I = \int \frac{dt}{t\sqrt{t^2-2}} = \int \frac{dt}{t^2\sqrt{1-\frac{2}{t^2}}}$$

Putting $\frac{\sqrt{2}}{t} = 4$ and $\frac{\sqrt{2}}{t^2}\,dt$ = du, we have

$$I = \frac{-1}{\sqrt{2}}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{\sqrt{2}}\cos^{-1}u + C$$

$$= \frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{\sqrt{2}}{x+\frac{1}{x}}\right)+C$$

$$= \frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{\sqrt{2}\,x}{x^2+1}\right)+C. \qquad \therefore\; k = \frac{1}{\sqrt{2}}.$$

The correct option is (C)

51. Putting $x = \cos^2\theta$ and $dx = -2\cos\theta\sin\theta\,d\theta$, we have

$$I = \int \frac{-2\cos\theta\sin\theta}{(1+\cos\theta)\cos\theta\sin\theta}\,d\theta$$

$$= -\int \sec^2\left(\frac{\theta}{2}\right)d\theta = \frac{-\tan\theta/2}{1/2}+C$$

$$= -2\left(\frac{1-\cos\theta}{1+\cos\theta}\right)+C = 2\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)+C.$$

$\therefore\; k = 2.$

The correct option is (B)

52. We have,

$$I = \int \frac{dx}{\cos^3 x\sqrt{\sin 2x}}$$

$$= \frac{1}{\sqrt{2}}\int (\sin x)^{-1/2}(\cos x)^{-7/2}\,dx$$

Putting $\tan x = t \;\Rightarrow\; dx = \frac{dt}{1+t^2}$,

Also, $\sin x = \frac{t}{\sqrt{1+t^2}}$ and $\cos x = \frac{1}{\sqrt{1+t^2}}$

$$\therefore I = \frac{1}{\sqrt{2}}\int \frac{t^{-1/2}}{(1+t^2)^{-1/4}}\bullet\frac{1}{(1+t^2)^{-7/4}}\bullet\frac{dt}{1+t^2}$$

$$= \frac{1}{\sqrt{2}}\int t^{-1/2}(1+t^2)\,dt$$

$$= \frac{1}{\sqrt{2}}\left(2\sqrt{t}+\frac{2}{5}t^{5/2}\right)+C$$

$$= \sqrt{2}\left(\tan^{1/2}x+\frac{1}{5}\tan^{5/2}x\right)+C.$$

The correct option is (A)

53. Let $I = \int \frac{1+x^4}{(1-x^4)^{3/2}}\,dx = \int \frac{x+\frac{1}{x^3}}{\left(\frac{1}{x^2}-x^2\right)^{3/2}}\,dx$

Put $\frac{1}{x^2}-x^2 = z \Rightarrow -2\left(\frac{1}{x^3}+x\right)dx = dz$

$$\therefore I = \frac{-1}{2}\int \frac{dz}{z^{3/2}} = \frac{1}{\sqrt{z}}+c = \frac{1}{\sqrt{\frac{1}{x^2}-x^2}}+c$$

The correct option is (B)

53. $I = \int \frac{(x^2-1)\,dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)}$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x^2+3+\frac{1}{x^2}\right)\tan^{-1}\left(x+\frac{1}{x}\right)}$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left\{\left(x+\frac{1}{x}\right)^2+1\right\}\tan^{-1}\left(x+\frac{1}{x}\right)}$$

$$\left[\text{Put } x+\frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt\right]$$

$$= \int \frac{dt}{(t^2+1)\tan^{-1}t}$$

$$\left[\text{Put } \tan^{-1}t = u \Rightarrow \frac{dt}{1+t^2} = du\right]$$

$= \log|u| + c$

$= \log|\tan^{-1}t| = \log\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right| + c$

The correct option is (A)

55. Let $I = \int \sqrt{\dfrac{\cos x - \cos^3 x}{1 - \cos^3 x}}\, dx$

$$= \int \frac{\sqrt{\cos x}\left(\sqrt{1-\cos^2 x}\right)}{\sqrt{1-\left(\cos^{3/2} x\right)^2}} = \int \frac{\sqrt{\cos x} \bullet \sin x\, dx}{\sqrt{1-\left(\cos^{3/2} x\right)^2}}$$

$$= \frac{-2}{3} \int \frac{dt}{\sqrt{1-t^2}} = \frac{-2}{3} \sin^{-1}(t) + c$$

$$\left[\text{Putting } \cos^{3/2} x = t \Rightarrow \frac{3}{2} \cos^{1/2} x\,(-\sin x)\, dx = dt\right]$$

$$= \frac{-2}{3} \sin^{-1}(\cos^{3/2} x) + c$$

The correct option is (B)

56. Let $I = \int \dfrac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$

$$= \int \frac{dx}{(x+2)^2 \left(\dfrac{(x-1)}{(x+2)}\right)^{3/4}}$$

Let $\dfrac{x-1}{x+2} = t \Rightarrow \dfrac{3}{(x+2)^2} dx = dt$

$$\therefore\ I = \int \frac{dt}{3t^{3/4}} + \frac{1}{3} \int t^{-3/4} dt$$

$$= \frac{1}{3} \frac{t^{1/4}}{1/4} + c = \frac{4}{3} \left(\frac{x-1}{x+2}\right)^{1/4} + c$$

The correct option is (A)

57. $\int \dfrac{\cos 7x - \cos 8x}{1 + 2\cos 5x}\, dx$

$$= \int \frac{2\sin\left(\dfrac{15}{2} x\right) \bullet \sin\left(\dfrac{x}{2}\right)}{1 + 2\cos 5x}\, dx$$

Multiplying and dividing by $\sin \dfrac{5x}{2}$

$$= \int \frac{2\sin \dfrac{15}{2} x \bullet \sin \dfrac{x}{2} \bullet \dfrac{5x}{2}}{\sin \dfrac{5x}{2} + 2\sin \dfrac{5x}{2} \bullet \cos 5x}\, dx$$

$$= \int \frac{2\sin \dfrac{15x}{2} \bullet \sin \dfrac{x}{2} \bullet \sin \dfrac{5x}{2}}{\sin \dfrac{5x}{2} + \sin \dfrac{15x}{2} \bullet \sin \dfrac{5x}{2}}\, dx$$

$$= \int \frac{2\sin \dfrac{15x}{2} \bullet \sin \dfrac{x}{2} \bullet \sin \dfrac{5x}{2}}{\sin \dfrac{15x}{2}}\, dx$$

$$= \int 2\sin \frac{x}{2} \bullet \sin \frac{5x}{2}\, dx$$

$$= \int \left(\cos 2x - \cos 3x\right) dx = \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + c$$

The correct option is (C)

58. Let $I = \int \dfrac{(1 - \cot^{n-2} x)\, dx}{\tan x + \cot x \bullet \cot^{n-2} x}$

$$= \int \frac{(\sin^{n-2} x - \cos^{n-2} x)\, dx}{\sin^{n-2} x \tan x + \cos^{n-2} x \bullet \cot x}$$

$$= \int \frac{\sin x \cos x\, (\sin^{n-2} x - \cos^{n-2} x)\, dx}{\sin^n x + \cos^n x}$$

Put $\sin^n x + \cos^n x = t$

$\Rightarrow\ n(\sin^{n-1} x \cdot \cos x - \cos^{n-1} x \cdot \sin x)\, dx = dt$

$$\therefore\ I = \frac{1}{n} \int \frac{dt}{t}$$

$$= \frac{1}{n} \log|t| + c = \frac{1}{n} \log|\sin^n x + \cos^n x| + c$$

The correct option is (B)

59. Let $I = \int e^{\tan x}(\sec x - \sin x)\, dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$ or $dx = \dfrac{dt}{1+t^2}$

$$I = \int e^t \left\{\sqrt{1+t^2} - \frac{t}{\sqrt{1+t^2}}\right\} \frac{dt}{1+t^2}$$

$$= \int e^t \left\{\frac{1}{\sqrt{1+t^2}} - \frac{t}{(1+t^2)^{3/2}}\right\} dt$$

$$= \frac{e^t}{\sqrt{1+t^2}} + c$$

$$\left[\text{using } \int e^t \left[f(t) + f'(t)\right] dt = e^t f(t)\right]$$

$= e^{\tan x} \cdot \cos x + c.$

The correct option is (D)

60. Let $I = \int \dfrac{(x-1)\, dx}{(x+1)\sqrt{x^3 + x^2 + x}}$

$$= \int \frac{(x^2 - 1)\, dx}{(x+1)^2 \sqrt{x^3 + x^2 + x}}$$

$$= \int \frac{x^2\left(1 - \dfrac{1}{x^2}\right) dx}{(x^2 + 2x + 1)\sqrt{x^3 + x^2 + x}}$$

$$= \int \frac{x^2\left(1 - \dfrac{1}{x^2}\right) dx}{x\left(x + 2 + \dfrac{1}{x}\right) \cdot x \sqrt{x + 1 + \dfrac{1}{x}}}$$

Put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{(t+2)\sqrt{t+1}}$$

Put $1 + t = z^2 \Rightarrow dt = 2z\,dz$

$$= \int \frac{2z\,dz}{(z^2+1)\sqrt{z^2}} \qquad = 2\int \frac{dz}{z^2+1}$$

$$= 2\tan^{-1}(z) + c = 2\tan^{-1}\left(\sqrt{1+t}\right) + c$$

$$= 2\tan^{-1}\sqrt{\frac{x^2+x+1}{x}} + c$$

$\therefore\ k = 2.$

The correct option is (B)

61. Let $I = \int \frac{dt}{\tan x + \cot x + \sec x + \operatorname{cosec} x}$

$$= \int \frac{(\sin x \cos x)\,dx}{1 + \sin x + \cos x}$$

$$= \int \frac{\sin x\,dx}{\sec x + \tan x + 1}$$

Multiplying and dividing by $(1 + \tan x - \sec x)$, we get

$$= \int \frac{\sin x\,(1 + \tan x - \sec x)\,dx}{(1 + \tan x)^2 - \sec^2 x}$$

$$= \int \frac{\sin x\,(1 + \tan x - \sec x)}{2\tan x}\,dx$$

$$= \frac{1}{2}\int \cos x\,(1 + \tan x - \sec x)\,dx$$

$$= \frac{1}{2}\int (\cos x + \sin x - 1)\,dx$$

$$= \frac{1}{2}(\sin x - \cos x - x) + c$$

The correct option is (B)

62. Since

$$\int f(x)\sin x \cos x\,dx = \frac{1}{2(b^2 - a^2)}\log(f(x)) + c.$$

Differentiating both sides w.r.t. x, we get

$$f(x)\sin x \cos x = \frac{1}{2(b^2 - a^2)} \bullet \frac{1}{f(x)} \bullet f'(x)$$

$$\Rightarrow 2(b^2 - a^2)\sin x \cos x = \frac{f'(x)}{[f(x)]^2}$$

$$\Rightarrow 2b^2 \sin x \cos x - 2a^2 \sin x \cos x = \frac{f'(x)}{[f(x)]^2}$$

Integrating both sides w.r.t. x, we get,

$$\Rightarrow -b^2\cos^2 x - a^2\sin^2 x = -\frac{1}{f(x)}$$

$$\Rightarrow f(x) = \frac{1}{a^2\sin^2 x + b^2\cos^2 x}$$

The correct option is (C)

63. $\int \frac{(\sin x - \cos x)\,dx}{(\sin x + \cos x)\sqrt{\sin x \cos x + \sin^2 x \cos^2 x}}$

$$= \int \frac{(\sin x - \cos x)\,dx}{(\sin x + \cos x)\sqrt{\sin x \cos x\,(1 + \sin x \cos x)}}$$

[Put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\,dx = dt$]

$$= \int \frac{-dt}{t\sqrt{\frac{1}{2}(t^2-1)^2 + \frac{1}{4}(t^2-1)^2}}$$

$$= -\int \frac{2\,dt}{t\sqrt{t^2-1}} \qquad = -\operatorname{cosec}^{-1}(t^2) + c$$

$$= -\operatorname{cosec}^{-1}(1 + \sin 2x) + c$$

The correct option is (B)

64. Let $I = \int \frac{dx}{(2ax + x^2)^{3/2}} = \int \frac{dx}{\left\{(x+a)^2 - a^2\right\}^{3/2}}$

Put $x + a = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\,d\theta$

$$I = \int \frac{a\sec\theta\tan\theta\,d\theta}{\left\{a^2(\mathrm{dec}^2\theta - 1)\right\}^{3/2}}$$

$$= \int \frac{a\sec\theta\tan\theta\,d\theta}{a^3\sec^3\theta}$$

$$= \frac{1}{a^2}\int \frac{1}{\cos\theta} \bullet \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{a^2}\left(-\frac{1}{\sin\theta}\right) + c = \frac{-1}{a^2}\,\frac{x+a}{\sqrt{x^2+2ax}} + c$$

The correct option is (C)

65. $\int \frac{\sin^3(\theta/2)}{\cos\theta/2\sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}.d\theta$

$$= \frac{1}{2}\int \frac{2\sin\theta/2 \bullet \cos\theta/2 \bullet 2\sin^2\theta/2}{2\cos^2\frac{\theta}{2}\sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}\,d\theta$$

$$= \frac{1}{2}\int \frac{\sin\theta\,(1 - \cos\theta)}{(1 + \cos\theta)\sqrt{\cos^3\theta + \cos^2\theta + \cos\theta}}\,d\theta$$

Put $\cos\theta = t \Rightarrow -\sin\theta\,d\theta = dt$

$$= \frac{1}{2}\int \frac{(t-1)\,dt}{(t+1)\sqrt{t^3 + t^2 + t}}$$

$$= \frac{1}{2}\int \frac{t^2 - 1}{(t+1)^2\sqrt{t^3 + t^2 + t}}\,dt$$

$$= \frac{1}{2}\int \frac{\left(1-\frac{1}{t^2}\right)}{\left(t+\frac{1}{t}+2\right)\sqrt{t+1+\frac{1}{t}}}\,dt$$

Put $t + 1 + \frac{1}{t} = u^2 \Rightarrow \left(1-\frac{1}{t^2}\right) dt = 2udu$

$$= \frac{1}{2}\int \frac{2udu}{(u^2+1)\bullet u} = \int \frac{du}{u^2+1} = \tan^{-1}(u) + c$$

$$= \tan^{-1}\left(t+1+\frac{1}{t}\right)^{3/2} + c$$

$$= \tan^{-1}(\cos\theta + \sec\theta + 1)^{1/2} + c$$

$\therefore\ k = \cos\theta + \sec\theta + 1$

The correct option is (A)

66. $$\int \frac{(2\sin\theta + \sin 2\theta)\,d\theta}{(\cos\theta - 1)\sqrt{\cos\theta + \cos^2\theta + \cos^3\theta}}$$

Put $\cos\theta = x^2 \Rightarrow -\sin\theta\, d\theta = 2xdx$

$$= 2\int \frac{(1+x^2)}{(1-x^2)}\bullet\frac{2xdx}{\sqrt{x^2+x^4+x^6}}$$

$$= 4\int \frac{\left(1+1/x^2\right)dx}{\left(\frac{1}{x}-x\right)\sqrt{\left(\frac{1}{x}-x\right)^2+3}}$$

Put $\frac{1}{x} - x = t \Rightarrow \left(\frac{-1}{x^2}-1\right)dx = dt$

$$= -4\int \frac{dt}{t\sqrt{t^2+3}}$$

Put $t^2 + 3 = u^2 \Rightarrow 2tdt = 2udu$

$$= 4\int \frac{-u\,du}{u\,(u^2-3)} = -4\int \frac{du}{u^2-3}$$

$$= \frac{-2}{\sqrt{3}}\log\left|\frac{4-\sqrt{3}}{4+\sqrt{3}}\right| + c$$

$$= \frac{-2}{\sqrt{3}}\log\left|\frac{\sqrt{t^2+3}-\sqrt{3}}{\sqrt{t^2+3}+\sqrt{3}}\right| + c$$

$$= \frac{-2}{\sqrt{3}}\log\left|\frac{\sqrt{x^2+\frac{1}{x^2}+1}-\sqrt{3}}{\sqrt{x^2+\frac{1}{x^2}+1}+\sqrt{3}}\right| + c$$

$$= \frac{-2}{\sqrt{3}}\log\left|\frac{\sqrt{\cos\theta+\sec\theta+1}-\sqrt{3}}{\sqrt{\cos\theta+\sec\theta+1}+\sqrt{3}}\right| + c$$

$\therefore\ k = \sqrt{\cos\theta + \sec\theta + 1}$

The correct option is (A)

67. $[x] = n$ where n is an integer

$\{[x]\} = \{n\} = 0$

$\therefore\ \int [\{[\{[x]\}]\}]\,dx = \int o\,dx =$ **constant** $= C$

The correct option is (A)

68. Let $g(x) = 1 + x^2 + x^4 + \ldots + x^{2n} = \frac{x^{2n+2}-1}{x^2-1}$

So, $2x + 4x^3 + \ldots + 2nx^{2n-1} = h(x)$

$$= g'(x) = \frac{2x\,\{nx^{2n+2} - (n+1)\,x^{2n} + 1\}}{(x^2-1)^2}$$

Now, $f(x) = \lim\limits_{n\to\infty} h(x) = \frac{2x}{(x^2-1)^2}$ as $0 < x < 1$

Thus, $\int f(x)\,dx = \int \frac{2x}{(x^2-1)^2}\,dx$

$$= -\frac{1}{x^2-1} = \frac{1}{1-x^2} + C$$

The correct option is (D)

69. We have,

$$I_{4,3} = \frac{-\cos 3x\cos^4 x}{3} - \frac{4}{3}\int \cos^3 x \sin x \cos 3x\,dx$$

But $\sin x \cos 3x = -\sin 2x + \sin 3x \cos x$,

$$\therefore I_{4,3} = \frac{-\cos 3x\cos^4 x}{3} + \frac{4}{3}\int \cos^3 x \sin 2x\,dx$$

$$-\frac{4}{3}\int \cos^4 x \sin 3x\,dx + C$$

$$= \frac{-\cos 3x\cos^4 x}{3} + \frac{4}{3}I_{3,2} - \frac{4}{3}I_{4,3} + C$$

Therefore,

$$\frac{7}{3}I_{4,3} - \frac{4}{3}I_{3,2} = \frac{-\cos 3x\cos^4 x}{3} + C$$

or, $7\,I_{4,3} - 4\,I_{3,2} = -\cos 3x \cos^4 x + C$

The correct option is (A)

More than One Option Correct Type

70. $$I = \int \frac{dx}{(1+\sqrt{x})^8}$$

$$= \int \frac{2\sqrt{x}}{(1+\sqrt{x})^8}\cdot\frac{dx}{2\sqrt{x}}$$

$$= 2\int \frac{(t-1)}{t^8}\,dt \qquad \left[\begin{array}{l}\text{Putting } 1+\sqrt{x}=t \\ \Rightarrow \qquad \dfrac{dx}{2\sqrt{x}} = dt\end{array}\right]$$

$$= 2\int (t^{-7} - t^{-8})\,dt$$

$$= 2\left(\frac{t^{-6}}{-6} - \frac{t^{-7}}{-7}\right) + C$$

$$= \frac{-1}{3\,(1+\sqrt{x})^6} + \frac{2}{7\,(1+\sqrt{x})^7} + C$$

The correct option is (A, B)

71. We have,

$$\int f(x)\sin 2x\,dx = \frac{ln\,|\,f(x)\,|}{b^2 - a^2} + C$$

Differentiating w.r.t. x, we have

$$f(x)\sin 2x = \frac{1}{b^2 - a^2}\cdot\frac{f'(x)}{f(x)}$$

$$\Rightarrow \int \frac{df}{f^2} = b^2\int 2\sin x\cos x\,dx - a^2\int 2\sin x\cos x\,dx$$

$$\Rightarrow \frac{-1}{f(x)} = (b^2\sin^2 x + a^2\cos^2 x) \text{ or } (-b^2\cos^2 x - a^2\sin^2 x)$$

$$\therefore\ f(x) = \frac{1}{b^2\cos^2 x + a^2\sin^2 x}$$

$$\text{or, } \frac{1}{b^2\cos^2 x + a^2\sin^2 x}$$

The correct option is (A, D)

72. Let $I = \int \dfrac{(\sqrt{x})^5}{(\sqrt{x})^7 + x^6}\,dx$

$$= \int \frac{dx}{(\sqrt{x})^2 + (\sqrt{x})^7}$$

$$= \int \frac{dx}{x^{7/2}\left\{1 + \dfrac{1}{x^{5/2}}\right\}} \qquad \left[\begin{array}{l}\text{Put } \quad \dfrac{1}{x^{5/2}} = y \\ \Rightarrow \dfrac{-5}{2}\,\dfrac{dx}{x^{7/2}} = dy\end{array}\right]$$

$$= \frac{-2}{5}\int \frac{dy}{1+y} \qquad \frac{-2}{5}\log|1+y| + C$$

$$= \frac{2}{5}\log\left|\frac{1}{1+y}\right| + C$$

$$= \frac{2}{5}\log\left|\frac{x^{5/2}}{x^{5/2}+1}\right| + C$$

$$\therefore\ a = \frac{2}{5} \quad \text{and} \quad k = \frac{5}{2}$$

The correct option is (A, C)

73. We have,

$$P = \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{r}\cos(bx - Q) \quad \text{...(1)}$$

$$Q = \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{r}\sin(bx - Q) \quad \text{...(2)}$$

where $r = \sqrt{a^2 + b^2}$ and $Q = \tan^{-1}\left(\dfrac{b}{a}\right)$

Squaring and adding (1) and (2), we have

$$P^2 + Q^2 = \frac{e^{2ax}}{r^2}\left[\cos^2(bx - Q) + \sin^2(bx - Q)\right]$$

$(P^2 + Q^2)\,r^2 = e^2ax$

$\Rightarrow (P^2 + Q^2)\,(a^2 + b^2) = e^2ax$

Again, dividing (2) by (1), we get

$$\frac{Q}{P} = \tan(bx - Q)$$

$$\Rightarrow \tan^{-1}\left(\frac{Q}{P}\right) = bx - \tan^{-1}\left(\frac{b}{a}\right)$$

$$\text{or, } \int \frac{x^4+1}{x^6+1}\,dx\ \tan^{-1}\left(\frac{Q}{P}\right) + \tan^{-1}\left(\frac{b}{a}\right) = bx$$

The correct option is (A, B, C, D)

74. $I = \int \dfrac{x^4+1}{x^6+1}\,dx = \int \dfrac{(x^2+1)^2 - 2x^2}{(x^2+1)\,(x^4 - x^2 + 1)}\,dx$

$$= \int \frac{\left(1 + \dfrac{1}{x^2}\right)dx}{\left(x^2 + \dfrac{1}{x^2} - 1\right)} - 2\int \frac{x^2\,dx}{(x^3)^2 + 1}$$

$$= \int \frac{\left(1 + \dfrac{1}{x^2}\right)dx}{\left(x - \dfrac{1}{x}\right)^2 + 1} - 2\int \frac{x^2\,dx}{(x^3)^2 + 1}$$

[In first integral put $x - \dfrac{1}{x} = t$ and in second put $x^3 = u$]

$$= \int \frac{dt}{t^2+1} - \frac{2}{3}\int \frac{du}{u^2+1}$$

$$= \tan^{-1}(t) - \frac{2}{3}\tan^{-1}(u) + C$$

$$= \tan^{-1}\left(x - \frac{1}{x}\right) - \frac{2}{3}\tan^{-1}(x^3) + C$$

$$\therefore\ k_1 = x - \frac{1}{x} \text{ and } k_2 = x^3$$

The correct option is (B, C)

75. Put $\sqrt{x^2+1} = z$

$$\Rightarrow \frac{1}{2\sqrt{x^2+1}}\cdot 2x\,dx = dz$$

$$\Rightarrow \frac{x}{\sqrt{x^2+1}}\,dx = dz$$

$$\therefore\ I = \int 1.\log(\sqrt{z^2-1}+z)\,dz$$

$$= \log\left(z+\sqrt{z^2-1}\right).z - \int \frac{\left(1+\frac{2z}{2\sqrt{z^2-1}}\right).z}{z+\sqrt{z^2-1}}\,dz$$

$$= \log\left(z+\sqrt{z^2-1}\right).z - \int \frac{z\,dz}{\sqrt{z^2-1}}$$

$$= \log\left(z+\sqrt{z^2-1}\right).z - \sqrt{z^2-1}+C$$

$$= \sqrt{x^2+1}\,\log\left(x+\sqrt{x^2+1}\right) - x + C$$

$\therefore\ A = 1, B = -1$

The correct option is (B, C)

76. $$\frac{3\cot 3x-\cot x}{\tan x-3\tan 3x} = \frac{\frac{3}{\tan 3x}-\frac{1}{\tan x}}{\tan x-3\tan 3x}$$

$$= \frac{3\frac{\tan x}{\tan 3x}-1}{\tan^2 x\left(1-\frac{3\tan 3x}{\tan x}\right)}$$

$$= \frac{-8}{3-\tan^2 x}+3$$

$$\therefore\ I = \int\left(\frac{-8}{3-\tan^2 x}+3\right)dx$$

$$= \int\frac{-8}{3-\tan^2 x}\,dx+3x+C$$

Let $t = \tan x \Rightarrow dt = \sec^2 x\,dx$

$$\Rightarrow dx = \frac{dt}{\sec^2 x} = \frac{dt}{1+t^2}$$

$$\therefore\ I = \int\frac{-8\,dt}{(1+t^2)(3-t^2)}+3x+C$$

$$= -2\int\left(\frac{1}{3-t^2}+\frac{1}{1+t^2}\right)dx+3x+C$$

$$= -2.\frac{1}{2\sqrt{3}}.\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| - 2\tan^{-1}(t)+3x+C$$

$$= \frac{-1}{\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|+x+C$$

$$\Rightarrow A = 1,\quad B = \frac{-1}{\sqrt{3}}$$

The correct option is (A, C)

77. Let $I = \int\left(\frac{f(x)}{x^2}\right)^{1/2}dx = \int\left(\frac{x+2}{2x+3}\right)^{1/2}\frac{dx}{x}$

Put $\frac{x+2}{2x+3} = y^2 \Rightarrow x = \frac{3y^2-2}{1-2y^2}$ and $dx = \frac{-2y\,dy}{(1-2y^2)^2}$

$$\therefore\ I = -\int y.\frac{2y}{(1-2y^2)^2}.\frac{1-2y^2}{3y^2-2}\,dy$$

$$= 2\int\frac{y^2dy}{(2y^2-1)(3y^2-2)}$$

$$= 2\int\left[\frac{2}{3y^2-2}-\frac{1}{2y^2-1}\right]dy$$

$$= \frac{4}{3}\int\frac{dy}{y^2-\frac{2}{3}}-\int\frac{dy}{y^2-\frac{1}{2}}$$

$$= \frac{4}{3}\,\frac{1}{2\sqrt{\frac{2}{3}}}\,ln\left|\frac{y-\sqrt{2/3}}{y+\sqrt{2/3}}\right|-\frac{1}{2\times\frac{1}{\sqrt{2}}}\,ln\left|\frac{y-\sqrt{\frac{1}{2}}}{y+\sqrt{\frac{1}{2}}}\right|+C$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}y}{1-\sqrt{2}y}\right|-\sqrt{\frac{2}{3}}\log\left|\frac{\sqrt{3}y+\sqrt{2}}{\sqrt{3}y-\sqrt{2}}\right|+C$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2f(x)}}{1-\sqrt{2f(x)}}\right|$$

$$-\sqrt{\frac{2}{3}}\log\left|\frac{\sqrt{3f(x)}+\sqrt{2}}{\sqrt{3f(x)}-\sqrt{2}}\right|+C$$

Thus, $g(x) = \log|x|$ and $h(x) = \log|x|$

The correct option is (A, B)

78. $f(x) = \lim\limits_{n\to\infty} e^{\tan\left(\frac{1}{n}\right)\log\left(\frac{1}{n}\right)}$

But $\lim\limits_{n\to\infty}\tan\left(\frac{1}{n}\right)\log\left(\frac{1}{n}\right) = \lim\limits_{n\to\infty}\left[-\frac{\log n}{n}\,\frac{\tan\frac{1}{n}}{\frac{1}{n}}\right] = 0$

So, $f(x) = e^0 = 1$

Hence, $\int\frac{f(x)}{\sqrt[3]{\sin^{11}x\cos x}}\,dx = \int\frac{1}{\sqrt[3]{\sin^{11}x\cos x}}\,dx$

$$= \int\frac{1+t^2}{t^{11/3}}\,dt$$

[Putting $\tan x = t \Rightarrow \sec^2 x\, dx = dt$]

$$= \int (t^{-11/3} + t^{-5/3})\, dt = \frac{-3}{8} t^{-8/3} - \frac{3}{2} t^{-2/3} + C$$

$$= \frac{-3}{8} \frac{(1 + 4\tan^2 x)}{\tan^2 x \sqrt[3]{\tan^2 x}} + C$$

Thus, $g(x) = \dfrac{-3}{8} \dfrac{(1 + 4\tan^2 x)}{\tan^2 x \sqrt[3]{\tan^2 x}}$ and $g\left(\dfrac{\pi}{4}\right) = \dfrac{-15}{8}$

Clearly, g is not defined at $x = 0$ and odd multiple of $\dfrac{\pi}{2}$.

The correct option is (A, C)

79. $\cos^{-1} \sqrt{1 - x^2} = -\sin^{-1} x,\ x < 0$

$$\therefore \int (\cos^{-1} x + \cos^{-1} \sqrt{1 - x^2})\, dx$$

$$= \int (\cos^{-1} x - \sin^{-1} x)\, dx$$

$$= \int \left(\frac{\pi}{2} - 2\sin^{-1} x\right) dx = \frac{\pi}{2} x \cdot$$

$$-2x \sin^{-1} x + \int \frac{2x}{\sqrt{1 - x^2}}\, dx$$

$$= \frac{\pi}{2} x - 2x \sin^{-1} x - 2\sqrt{1 - x^2} + C$$

$\therefore\ A = \dfrac{\pi}{2}$ and $f(x) = -2x$

The correct option is (B, D)

80. Given integral can be written as

$$I = \int \frac{x^8 (x^2 + 20)}{(x^5 \sin x + 5x^4 \cos x)^2}\, dx$$

$$= \int \frac{(x^5 + 20x^3) \cos x}{(x^5 \sin x + 5x^4 \cos x)^2} \left(\frac{x^5}{\cos x}\right) dx$$

Since $\dfrac{d}{dx} [x^5 \sin x + 5x^4 \cos x] = (x^5 + 20x^3) \cos x$,

Therefore, integrating by parts, we get

$$I = -\frac{x^5}{\cos x\, (x^5 \sin x + 5x^4 \cos x)} + \int \frac{5x^4 \cos x + x^5 \sin x}{(x^5 \sin x + 5x^4 \cos x)} \frac{dx}{\sec^2 x}$$

$$= -\frac{x}{\cos x\, (x \sin x + 5\cos x)} + \tan x$$

$\therefore\ A = x \sin x + 5\cos x$ and $B = \tan x$

The correct option is (A, D)

Passage Based Questions

81. Put $2 + x^{2/3} = z^4 \Rightarrow dx = 6z^3 . x^{1/3}\, dz$

$$\therefore\ \int x^{1/3} \left(2 + x^{2/3}\right)^{1/4} dx = \int x^{1/3} .\ z (6z^3 . x^{1/3})\, dz$$

$$= 6 \int z^4 \left(z^4 - 2\right) dz = 6 \int \left(\frac{z^9}{9} - \frac{2z^5}{5}\right) + C$$

$$= \frac{2}{3} (2 + x^{2/3})^{9/4} - \frac{12}{5} (2 + x^{2/3})^{5/4} + C.$$

The correct option is (B)

82. Put $1 + \sqrt{x} = z^2 \Rightarrow dx = 4z \sqrt{x}\, dz$

$$\therefore\ \int \frac{\sqrt{1 + \sqrt{x}}}{x}\, dx = \int \frac{z}{x} \left(4z \sqrt{x}\right) dz$$

$$= 4 \int \frac{z^2}{z^2 - 1}\, dz = 4 \int \left(1 + \frac{1}{z^2 - 1}\right) dz$$

$$= 4 \left(z + \frac{1}{2} \log \frac{z - 1}{z + 1}\right) + C$$

$$= 4 \sqrt{1 + \sqrt{x}} + 2\log \left(\frac{\sqrt{1 + \sqrt{x}} - 1}{\sqrt{1 + \sqrt{x}} + 1}\right) + C.$$

The correct option is (C)

83. Put $1 + x^{1/4} = z^3 \Rightarrow dx = 12 z^2 x^{3/4}\, dz$

$$\therefore\ \int \frac{\sqrt[3]{1 + \sqrt[4]{x}}}{\sqrt{x}}\, dx = \int \frac{z}{x^{1/2}} \cdot (12 z^2 x^{3/4})\, dz$$

$$= 12 \int z^3 . x^{1/4}\, dz = 12 \int z^3 \left(z^3 - 1\right) dz$$

$$= 12 \left(\frac{z^7}{7} - \frac{z^4}{4}\right) + C$$

$$= 12 \left(\frac{(1 + x^{1/4})^{7/3}}{7} - \frac{(1 + x^{1/4})^{4/3}}{4}\right) + C.$$

The correct option is (B)

84. Put $1 + x^{4/3} = z^7 \Rightarrow x^{1/3}\, dx = \dfrac{21}{4} z^6\, dz$

$$\therefore\ \int \sqrt[3]{x}\ \sqrt[7]{1 + \sqrt[3]{x^4}}\, dx = \int z \left(\frac{21}{4} z^6\right) dz$$

$$= \frac{21}{4} \cdot \frac{z^8}{8} + C$$

$$= \frac{21}{32} \left(1 + \sqrt[3]{x^4}\right)^{8/7} + C.$$

The correct option is (A)

85. We have,

$$In = \int \tan^n x\, dx = \int \tan^{n-2} x \cdot \tan^2 x\, dx$$

$$= \int \tan^{n-2} x\ (\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x\ \sec^2 x\, dx - \int \tan^{n-2} x\ dx$$

$$= \frac{\tan^{n-1} x}{n-1} - In_{-2}$$

$$\therefore\ In + In_{-2} = \frac{\tan^{n-1} x}{n-1},\ n \geq 2$$

$$\therefore\ I_0 + I_1 + 2\,(I_2 + I_3 + \ldots + I_8) + I_9 + I_{10}$$

$$= (I_2 + I_0) + (I_3 + I_1) + (I_4 + I_2) + (I_5 + I_3) + (I_6 + I_4)$$

$$+ (I_7 + I_5) + (I_8 + I_6) + (I_9 + I_7) + (I_{10} + I_8)$$

$$= \left(\frac{\tan x}{1} + \frac{\tan^9 x}{2} + \ldots + \frac{\tan^9 x}{9}\right).$$

The correct option is (A)

86. $$In - In_{-2} = \int \frac{(\cos nx - \cos(n-2)x)}{\sin x}\, dx$$

$$= \int \frac{-2\sin(n-1)x \sin x}{\sin x}\, dx$$

$$= \frac{2\cos(n-1)x}{n-1}$$

The correct option is (B)

87. Let $$In = \int \frac{x^n dx}{\sqrt{x^2 + a^2}} = \int x^{n-1} \cdot \frac{x}{\sqrt{x^2 + a^2}}\, dx$$

Integrating by parts, we have

$$In = x^{n-1}\sqrt{x^2 + a^2} - \int (n-1)\, x^{n-2}\sqrt{x^2 + a^2}\, dx$$

$$= x^{n-1}\sqrt{x^2 + a^2} - (n-1)\int x^{n-2} \bullet \frac{x^2 + a^2}{\sqrt{x^2 + a^2}}\, dx$$

$$= x^{n-1}\sqrt{x^2 + a^2} - (n-1)\int \frac{x^n}{\sqrt{x^2 + a^2}}\, dx$$

$$-(n-1)\,a^2 \int \frac{x^{n-2}}{\sqrt{x^2 + a^2}}\, dx$$

$$= x^{n-1}\sqrt{x^2 + a^2} - (n-1)\, I_n - (n-1)\,a^2\, I_{n-2}$$

i.e., $n\, In = x^{n-1}\sqrt{x^2 + a^2} - (n-1)\,a^2\, I_{n-2}$

i.e., $In = \dfrac{x^{n-1}\sqrt{x^2 + a^2}}{n} - \dfrac{a^2(n-1)}{n}\, I_{n-2}$.

The correct option is (A)

88. Let $$Im_{,}n = \int \frac{\sin^m x}{\cos^n x}\, dx = \int \sin^{m-1} x \cdot \frac{\sin x}{\cos^n x}\, dx$$

Integrating by parts, we have

$$Im_{,}n = \sin^{m-1} x\ \frac{1}{(n-1)\cos^{n-1} x} - \frac{m-1}{n-1}$$

$$\int \frac{\sin^{m-2} x \cos x}{\cos^{n-1} x}\, dx$$

$$= \frac{\sin^{m-1} x}{(n-1)\cos^{n-1} x} - \frac{m-1}{n-1}\int \frac{\sin^{m-2} x}{\cos^{n-2} x}\, dx$$

$$= \frac{\sin^{m-1} x}{(n-1)\cos^{n-1} x} - \frac{m-1}{n-1}\, I_{m-2,n-2}$$

The correct option is (B)

Match the Column Type

89. $$\int \sqrt{\sec x - 1}\ dx = \int \sqrt{\frac{1 - \cos x}{\cos x}}\ dx$$

$$= \sqrt{2}\ \int \frac{\sin \frac{x}{2}}{\sqrt{2\cos^2 \frac{x}{2} - 1}}\ dx = -2\ \sqrt{2}\ \int \frac{dz}{\sqrt{2z^2 - 1}}$$

$$\left[\text{Putting } \cos \frac{x}{2} = z \Rightarrow \sin \frac{x}{2}\ dx = -2dz\right]$$

$$= -2 \int \frac{dz}{\sqrt{z^2 - \left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -2 \log\left[z + \sqrt{z^2 - \left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$$= -2 \log\left(\cos \frac{x}{2} + \sqrt{\cos^2 \frac{x}{2} - \frac{1}{2}}\right)$$

The correct option is (C)

II (a). $$\int \frac{dx}{\cos x \sqrt{\cos 2x}} = \int \frac{dx}{\cos x \sqrt{\dfrac{1 - \tan^2 x}{1 + \tan^2 x}}}$$

$$= \int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}}\ dx = \int \frac{dz}{\sqrt{1 - z^2}}$$

[Putting $\tan x = z \Rightarrow \sec^2 x\, dx = dz$]

$= \sin^{-1} z = \sin^{-1} (\tan x)$

The correct option is (A)

III $$\int \frac{dx}{\cos^3 x \sqrt{\sin 2x}} = \int \frac{dx}{\cos^3 x \sqrt{\dfrac{2\tan x}{1+\tan^2 x}}}$$

$$= \int \frac{\sec^4 x}{\sqrt{2\tan x}}\, dx = \frac{1}{\sqrt{2}} \int \frac{\left(1+z^4\right).\, 2z\, dz}{z}$$

[Putting $\tan x = z^2 \Rightarrow \sec^2 x\, dx = 2z\, dz$]

$$= \sqrt{\mathbf{2}}\left(z + \frac{z^5}{5}\right) = \sqrt{\mathbf{2}}\left(\sqrt{\tan x} + \frac{1}{5}\tan^{5/2} x\right)$$

The correct option is (D)

IV $$\int \frac{\sin^3 x}{\left(1+\cos^2 x\right)\sqrt{1+\cos^2 x+\cos^4 x}}\, dx$$

$$= \int \frac{\sin^3 x}{\cos x\left(\sec x+\cos x\right)\cos x\sqrt{\sec^2 x+1+\cos^2 x}}\, dx$$

$$= \int \frac{\sin^3 x\, dx}{\cos^2 x\left(\sec x+\cos x\right)\sqrt{\left(\sec x+\cos x\right)^2-1}}$$

$$= \int \frac{dz}{z\sqrt{z^2-1}}$$

$$\left[\text{Putting } \sec x+\cos x = z \Rightarrow \frac{\sin^3 x}{\cos^2 x}\, dx = dz\right]$$

$= \sec^{-1} z = \sec^{-1} (\sec x + \cos x)$.

The correct option is (B)

90. I $$\int \frac{dx}{\left(1+\sqrt{x}\right)\sqrt{x-x^2}}$$

$$= \int \frac{2\sin\theta\cos\theta\, d\theta}{(1+\sin\theta)\sqrt{\sin^2\theta-\sin^4\theta}}$$

[Putting $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$]

$$= 2\int \frac{d\theta}{1+\sin\theta} = 2\int \frac{1-\sin\theta}{\cos^2\theta}\, d\theta = 2\,(\tan\theta - \sec\theta)$$

$$= 2\left(\frac{\sqrt{x}}{\sqrt{1-x}} - \frac{1}{\sqrt{1-x}}\right) = \frac{2\left(\sqrt{x}-1\right)}{\sqrt{1-x}}.$$

The correct option is (A)

II Let $$I = \int \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2} \frac{dx}{x}$$

$$= \int \left(\frac{1-\cos 2\theta}{1+\cos 2\theta}\right)^{1/2} \frac{d\left(\cos^2 2\theta\right)}{\cos^2 2\theta}$$

Putting $x = \cos^2 2\theta$ and $dx = 2\cos 2\theta\,(-2\sin 2\theta)\, d\theta$, we have,

$$I = \int \left(\frac{1-\cos 2\theta}{1+\cos 2\theta}\right)^{1/2} \bullet \frac{-4\cos 2\theta\sin 2\theta\, d\theta}{\cos^2 2\theta}$$

$$= \int \frac{\sin\theta}{\cos\theta} \bullet \frac{-8\sin\theta\cos\theta\, d\theta}{\cos 2\theta} = -8\int \frac{\sin^2\theta}{\cos 2\theta}\, d\theta$$

$$= 4\int \left(\frac{\cos 2\theta - 1}{\cos 2\theta}\right) d\theta = 4\int (1-\sec 2\theta)\, d\theta$$

$$= 4\theta - 2\, ln\,|\sec 2\theta + \tan 2\theta|$$

$$= 2\cot^{-1}\sqrt{x} - 2\, ln\left|\frac{1}{\sqrt{x}} + \sqrt{\frac{1}{x}-1}\right|$$

$$= 2\cot^{-1}\sqrt{x} - 2\, ln\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right|.$$

The correct option is (C)

III Let $$I = \int x\sqrt{\frac{1+x}{1-x}}\, dx$$

Putting $x = \cos 2\theta$ and $dx = -2\sin 2\theta\, d\theta$, we have,

$$I = \int \cos 2\theta\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}} \cdot (-2\sin^2\theta)\, d\theta$$

$$= \int \cos 2\theta \bullet \frac{\cos\theta}{\sin\theta} \bullet (-4\sin\theta\cos\theta)\, d\theta$$

$$= -4\int \cos 2\theta\cos^2\theta\, d\theta$$

$$= -2\int \cos 2\theta(1+\cos 2\theta)\, d\theta$$

$$= -2\int \left(\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= -\sin 2\theta - \theta - \frac{\sin 4\theta}{4}$$

$$= -\sin 2\theta - \theta - \frac{\sin 2\theta\cos 2\theta}{2}$$

$$= -\sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x - \frac{1}{2}x\sqrt{1-x^2}\,.$$

The correct option is (D)

IV $$\int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}\, dx$$

$$= \int \frac{x^2\left(1-\dfrac{1}{x^2}\right)}{x^2\left(x+\dfrac{1}{x}\right)\sqrt{x^2+\dfrac{1}{x^2}}}\, dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-2}}$$

$$\left[\text{Putting } x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt\right]$$

$$= \int \frac{dt}{t\sqrt{t^2-2}} = \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{t}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}\,x}\right).$$

The correct option is (B)

91. **I** $\displaystyle\int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx$

$$= \int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{1-2\sin^2 x\cos^2 x}dx$$

$$= \int \frac{(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)(\sin^4 x+\cos^4 x)}{1-2\sin^2 x\cos^2 x}dx$$

$$= \int \frac{1.(\sin^2 x-\cos^2 x)\,[(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x]}{1-2\sin^2 x\cos^2 x}dx$$

$$= \int \frac{(\sin^2 x-\cos^2 x)(1-2\sin^2 x\cos^2 x)}{1-2\sin^2 x\cos^2 x}dx$$

$$= -\int \cos 2x\,dx = -\frac{1}{2}\sin 2x$$

The correct option is (B)

II $\displaystyle\int \frac{\cos 5x+\cos 4x}{1-2\cos 3x}dx$

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}}{1-2\left\{2\cos^2\frac{3x}{2}-1\right\}}dx$$

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}}{3-4\cos^2\frac{3x}{2}}dx$$

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}\cos\frac{3x}{2}}{3\cos\frac{3x}{2}-4\cos^3\frac{3x}{2}}dx$$

[Multiplying and dividing by $\cos\frac{3x}{2}$]

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{3x}{2}\cos\frac{x}{2}}{-\cos\frac{9x}{2}}dx$$

[$\cos 3x = 4\cos^3 x - 3\cos x$]

$$= -\int 2\cos\frac{3x}{2}\cos\frac{x}{2}dx$$

$$= -\int (\cos 2x+\cos x)\,dx = -\frac{\sin 2x}{2} - \sin x$$

The correct option is (D)

III $\displaystyle I = \int \sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}\,dx = \int \frac{\sqrt{\cos x}\,.\sin x}{\sqrt{1-(\cos^{3/2} x)^2}}\,dx$

Put $\cos^{3/2}x = t \Rightarrow \frac{3}{2}\sqrt{\cos x}\,(-\sin x)\,dx = dt$

$$\Rightarrow I = \int \frac{-\frac{2}{3}dt}{\sqrt{1-t^2}}$$

$$= -\frac{2}{3}\sin^{-1} t = -\frac{2}{3}\sin^{-1}(\cos^{3/2} x)$$

The correct option is (A)

IV $\displaystyle I = \int \frac{dx}{\sin^6 x+\cos^6 x} = \int \frac{dx}{(\sin^2 x)^3+(\cos^2 x)^3}$

$$= \int \frac{dx}{(\sin^2 x+\cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x+\cos^2 x)}$$

$$= \int \frac{dx}{1-3\sin^2 x\cos^2 x} = \int \frac{\sec^4 x}{\sec^4 x-3\tan^2 x}dx$$

$$= \int \frac{1+\tan^2 x}{(1+\tan^2 x)^2-3\tan^3 x}\sec^2 x\,dx$$

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we have

$$I = \int \frac{1+t^2}{(1+t^2)^2-3t^2}dt$$

$$= \int \frac{1+t^2}{t^4-t^2+1}dt$$

$$= \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-1}dt$$

$$= \int \frac{d\left(t-\frac{1}{t}\right)}{\left(t-\frac{1}{t}\right)^2+1}$$

$$= \tan^{-1}\left(t - \frac{1}{t}\right) = \tan^{-1}(\tan x - \cot x).$$

The correct option is (C)

Assertion-Reason Type

92. We have,

$$\int e^{g(x)} \{f(x)\, g'(x) + f'(x)\}\, dx$$

$$= \int e^{g(x)} g'(x)\, f(x)\, dx + \int e^{g(x)} + f'(x)\, dx$$

$$= e^{g(x)}\, f(x) - \int e^{g(x)} f'(x)\, dx + \int e^{g(x)} f'(x)\, dx$$

$$= e^{g(x)}\, f(x)$$

$$\therefore \int e^{(x\sin x + \cos x)} \left(\frac{x^2\cos^2 x - x\sin x - \cos x}{x^2}\right) dx$$

$$= \int e^{(x\sin x + \cos x)} \left(\cos^2 x + \frac{-x\sin x - \cos x}{x^2}\right) dx$$

$$= \int e^{(x\sin x + \cos x)} \left\{x\cos x\left(\frac{\cos x}{x}\right) + \left(\frac{\cos x}{x}\right)'\right\} dx$$

$$= \int e^{g(x)} \left(g'(x) f(x) + f'(x)\right) dx = eg(x)f(x) + C$$

$$= e^{(x\sin x + \cos x)} \cdot \frac{\cos x}{x} + C$$

The correct option is (A)

93. We have,

$$I = \int \frac{\sin^2 x}{a + b\cos x}\, dx = \int \frac{1 - \cos^2 x}{a + b\cos x}\, dx$$

Let $1 - \cos^2 x = \lambda(a^2 - b^2\cos^2 x) + \mu$

Comparing the coefficients, we get

$$\lambda = \frac{1}{b^2} \text{ and } \mu = \frac{b^2 - a^2}{b^2}$$

$$\therefore I = \frac{1}{b^2}\int (a - b\cos x)\, dx - \frac{a^2 - b^2}{b^2}\int \frac{dx}{a + b\cos x}$$

$$= \frac{1}{b^2}(ax - b\sin x) - \left(\frac{a^2 - b^2}{b^2}\right) I_1$$

To evaluate I_1, put $\tan\frac{x}{2} = t,\ dx = \frac{2dt}{1 + t^2}$

and $\cos x = \frac{1 - t^2}{1 + t^2}$

$$\therefore I_1 = \int \frac{\frac{2dt}{1 + t^2}}{a + b\left(\frac{1 - t^2}{1 + t^2}\right)} = \int \frac{2dt}{(a + b) + (a - b)t^2}$$

$$= \frac{2}{a - b}\int \frac{dt}{\frac{a + b}{a - b} + t^2}$$

$$= \frac{2}{a - b} \cdot \sqrt{\frac{a - b}{a + b}} \tan^{-1}\left(t\sqrt{\frac{a - b}{a + b}}\right) + C$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a - b}{a + b}} \tan\frac{x}{2}\right),$$

Hence,

$$I = \frac{1}{b^2}(ax - b\sin x) - \frac{\sqrt[2]{a^2 - b^2}}{b^2} \tan^{-1}\left(\sqrt{\frac{a - b}{a + b}} \tan\frac{x}{2}\right)$$

The correct option is (A)

94. Let $I = \int \sqrt{2 + \tan^2 x}\, dx$

Put $\tan x = \sqrt{2}\tan\theta$

$\Rightarrow \sec^2 x\, dx = \sqrt{2}\sec^2\theta\, d\theta$

or, $dx = \frac{\sqrt{2}\sec^2\theta}{1 + 2\tan^2\theta}\, d\theta$

$$\therefore I = \int \frac{2\sec^3\theta}{1 + 2\tan^2\theta}\, d\theta = \int \frac{2\sec^3\theta}{1 + 2\tan^2\theta}\, d\theta$$

$$= \int \frac{2\cos\theta\, d\theta}{\cos^2\theta\,(\cos^2\theta + 2\sin^2\theta)}$$

$$= 2\int \frac{dt}{(1 + t^2)(1 + t^2)} \quad [\text{Putting } \sin\theta = t]$$

$$\text{Now, } \frac{2}{(1 - t^2)(1 + t^2)} = \frac{1}{1 - t^2} + \frac{1}{1 + t^2}$$

$$= \frac{1}{2}\left(\frac{1}{1 - t} + \frac{1}{1 + t}\right) + \frac{1}{1 + t^2}$$

Hence,

$$I = \int \frac{dt}{1 + t^2} + \frac{1}{2}\int \frac{dt}{1 - t} + \frac{1}{2}\int \frac{dt}{1 + t}$$

$$= \tan^{-1} t + \frac{1}{2} ln\left|\frac{t + 1}{t - 1}\right|$$

$$= \tan^{-1}(\sin\theta) + \frac{1}{2}\, ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|$$

Putting $\sin\theta = \dfrac{\tan x}{\sqrt{2+\tan^2 x}}$, we have

$$I = \tan^{-1}\left(\frac{\tan x}{\sqrt{2+\tan^2 x}}\right) + \frac{1}{2}\, ln\left|\frac{\tan^2 x + \tan x + 2}{\tan^2 x - \tan x + 2}\right|$$

The correct option is (A)

95. Let $I = \int x^{x+1} \log x\,(1+\log x)\, dx$

Now, $\dfrac{d}{dx}(x^x) = xx\,(1 + \log x)$

$$\Rightarrow \int x^x(1+\log x)\, dx = xx$$

$$\therefore I = \int (x\log x)\,(x^x(1+\log x))\, dx$$

$$= (x\log x)\cdot x^x - \int\left(x\cdot\frac{1}{x} + \log x\cdot 1\right)\cdot x^x\, dx$$

$$= x^{x+1}\log x - \int x^x(1+\log x)\, dx$$

$= xx^{+1}\log x - xx + C = xx\,(x\log x - 1) + C$

The correct option is (A)

Previous Year's Questions

96. Let $I = \int \dfrac{dx}{x(x^n+1)}$

Substitution $x^n + 1 = t$

$\Rightarrow nx^{n-1}dx = dt$

$$\therefore\quad I = \frac{1}{n}\int\frac{dt}{t(t-1)} = \frac{1}{n}\int\left(\frac{1}{t-1} - \frac{1}{t}\right)dt$$

$$= \frac{1}{n}\log\left(\frac{t-1}{t}\right) + c = \frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right) + c$$

97. Coefficient of Middle term in $(1 + \alpha x)^4 = t_3 = {}^4C_2 \cdot \alpha^2$ Coefficient of Middle term in $(1 - \alpha x)^6 = t_4 = {}^6C_3\,(-\alpha)^3$

Given that ${}^4C_2\alpha^2 = -{}^6C_3.\alpha^3$

$\Rightarrow -6 = 20\,\alpha$

$\Rightarrow \alpha = \dfrac{-3}{10}$

98. $\int \dfrac{(\log x - 1)^2}{(1+(\log x)^2)^2}dx$

$$= \int\left[\frac{1}{\left(1+(\log x)^2\right)} - \frac{2\log x}{\left(1+(\log x)^2\right)^2}\right]dx$$

$$= \int\left[\frac{e^t}{1+t^2} - \frac{2te^t}{(1+t^2)^2}\right]dt \quad \text{put } \log x = t \Rightarrow dx = e^t\, dt$$

$$\int e^t\left[\frac{1}{1+t^2} - \frac{2t}{(1+t^2)^2}\right]dt$$

$$= \frac{e^t}{1+t^2} + c = \frac{x}{1+(\log x)^2} + c$$

99. The integral $\int \dfrac{dx}{\cos x + \sqrt{3}\sin x}$

$$= \frac{1}{2}\int \sec\left(x - \frac{\pi}{3}\right)dx$$

$$= \frac{1}{2}\log\tan\left(\frac{x}{2} - \frac{\pi}{6} + \frac{\pi}{4}\right) + c$$

$$= \frac{1}{2}\log\tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + c$$

100. The integral

$$\sqrt{2}\int\frac{\sin x dx}{\sin\left(x-\frac{\pi}{4}\right)} = \sqrt{2}\int\frac{\sin\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right)dx}{\sin\left(x - \frac{\pi}{4}\right)}$$

$$= \sqrt{2}\int\left(\cos\frac{\pi}{4} + \cot\left(x - \frac{\pi}{4}\right)\sin\frac{\pi}{4}\right)dx$$

$$= \int dx + \int \cot\left(x - \frac{\pi}{4}\right)dx$$

$$= x + \ln\left|\sin\left(x - \frac{\pi}{4}\right)\right| + c$$

101. $\dfrac{dy}{dx} = y + 3 \Rightarrow \dfrac{dy}{y+3} = dx$

Ln (y + 3) = x + c
X = 0 ⇒ y = 2
⇒ ln 5 = 0 + c

C = ln 5
Ln (y + 3) = x + ln 5
y + 3 = ex + ln 5 ⇒ y + 3 = eln2 + ln5
y + 3 = 10 ⇒ y = 7

102. $\int \frac{5\tan x}{\tan x - 2} dx = \int \frac{5\sin x}{\sin x - 2\cos x} dx$

$$\Rightarrow \int \left[\frac{2(\cos x + 2\sin x) + (\sin x - 2\cos x)}{\sin x - 2\cos x}\right] dx$$

$$= 2\int \left(\frac{\cos x + 2\sin x}{\sin x - 2\cos x}\right) dx + \int dx + k$$

$$= 2\log|\sin x - 2\cos x| + x + k \therefore a = 2$$

103. $\int f(x)dx = \Psi(x)$

Let $x^3 = t$

$3x^2 dx = dt$,

then integration by parts implies that

$$\int x^5 f(x^3)dx = \frac{1}{3}\int tf(t)dt$$

$$= \frac{1}{3}\left[t\int f(t)dt - \int \{1\cdot \int f(t)dt\}dt\right]$$

$$= \frac{1}{3}x^3\Psi(x^3) - \int x^2\Psi(x^3)dx + C$$

104. $\int \left(1 + x - \frac{1}{x}\right) e^{\left(x+\frac{1}{x}\right)} dx$

$$= \int e^{\left(x+\frac{1}{x}\right)} dx + \int x\left(1 - \frac{1}{x^2}\right) e^{\left(x+\frac{1}{x}\right)} dx$$

$$= \int e^{\left(x+\frac{1}{x}\right)} dx + xe^{\left(x+\frac{1}{x}\right)} - \int e^{\left(x+\frac{1}{x}\right)} dx$$

$$= xe^{\left(x+\frac{1}{x}\right)} + c$$

105. $I = \int \frac{dx}{x^5\left(1 + \frac{1}{x^4}\right)^{3/4}}$

Substituting $1 + \frac{1}{x^4} = t$

$$\Rightarrow \frac{-4}{x^5} dx = dt$$

$$\therefore I = -\frac{1}{4}\int \frac{1}{t^{3/4}} dt$$

$$= -\frac{1}{4} \times 4t^{1/4} + c = -\left(1 + \frac{1}{x^4}\right)^{1/4} + c$$

106. The given integral can be written as

$$\int \frac{\left(\frac{2}{x^3} + \frac{5}{x^6}\right) dx}{\left(1 + \frac{1}{x^2} + \frac{1}{x^5}\right)^3}$$

Let 1 $\frac{1}{x^2} + \frac{1}{x^5}$ = t ⇒ dt = $-\left(\frac{2}{x^3} + \frac{5}{x^6}\right) dx$

$$= \int \frac{-dt}{t^3} = \frac{1}{2t^2} + c$$