HINTS AND SOLUTIONS

Single Option Correct Type

1. $\because \int_0^1 e^{x^2}(x-\alpha)\,dx = 0$, $\therefore e^{x^2}(x-\alpha)$ must be + ve and – ve both for $x \in (0, 1)$ i.e., $e^x(x-\alpha) = 0$ for one $x \in (0, 1)$

$\therefore \alpha \in (0, 1)$.

The correct option is (C)

2. We have,

$$\frac{dx}{dt} = (\sin^{-1} z)_{z=\sin t} \cdot \frac{d}{dt}(\sin t)$$

$$= \sin^{-1}(\sin t) \cdot \cos t = t \cos t \text{ and}$$

$$\frac{dy}{dt} = \left(\frac{\sin z^2}{z}\right)_{z=\sqrt{t}} \cdot \frac{d}{dt}\left(\sqrt{t}\right) = \frac{\sin t}{\sqrt{t}} \cdot \frac{1}{2\sqrt{t}} = \frac{\sin t}{2t}.$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{2t} \times \frac{1}{t\cos t} = \frac{\tan t}{2t^2}$$

The correct option is (A)

3. $g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$

As $\quad \frac{1}{2} \le f(t) \le 1$ for $t \in [0, 1]$

we get $\quad \frac{1}{2}\int_0^1 dt \le \int_0^1 f(t)\,dt \le \int_0^1 dt$

or $\quad \frac{1}{2} \le \int_0^1 f(t)\,dt \le 1 \qquad (1)$

Also, $\quad 0 \le f(t) \le \frac{1}{2}$ for $t \in (1, 2]$

$\Rightarrow \quad \int_1^2 0\,dt \le \int_1^2 f(t)\,dt \le \int_1^2 \left(\frac{1}{2}\right)dt$

$\Rightarrow \quad 0 \le \int_1^2 f(t)\,dt \le \frac{1}{2} \qquad (2)$

Equation (1) and (2), we get

$$\frac{1}{2} \le \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \le \frac{3}{2}$$

or $\quad \frac{1}{2} \le g(2) \le \frac{3}{2} \Rightarrow 0 < g(2) < 2$

The correct option is (B)

4. $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r}{n^2} \cdot \sec^2 \frac{r^2}{n^2}$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{r}{n} \cdot \sec^2 \frac{r^2}{n^2}\right)\frac{1}{n} = \int_0^1 x \sec^2 x^2\,dx$$

$$= \frac{1}{2}\int_0^1 \sec^2 t\,dt \qquad \text{(Putting } x^2 = t \Rightarrow 2x\,dx = dt\text{)}$$

$$= \frac{1}{2}[\tan t]_0^1 = \frac{1}{2}\tan 1$$

The correct option is (C)

5. $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \sin^{2k} \frac{r\pi}{2n}$

$$= \int_0^1 \sin^{2k} \frac{\pi x}{2}\,dx = \frac{2}{\pi}\int_0^{\pi/2} \sin^{2k} t\,dt$$

$$\left(\text{Putting } \frac{\pi x}{2} = t \Rightarrow dx = \frac{2}{\pi}dt\right)$$

$$= \frac{2}{\pi} \cdot \frac{(2k-1)(2k-3)\cdots 1}{2k(2k-2)\cdots 2} \cdot \frac{\pi}{2}$$

$$= \frac{[(2k-1)(2k-3)(2k-5)\cdots 1][2k\cdot(2k-2)\cdots 2]}{2^k[k(k-1)(k-2)\cdots 1][2k\cdot(2k-2)\cdots 2]}$$

$$= \frac{2k(2k-1)(2k-2)(2k-3)\cdots 2\cdot 1}{2^k[k(k-1)(k-2)\cdots 1]\cdot 2^k[k\cdot(k-1)(k-2)\cdots 1]}$$

$$= \frac{(2k)!}{2^{2k}\cdot(k!)^2}$$

The correct option is (A)

6. We have

$$2\sin\frac{x}{2}\cos x = \sin\frac{3x}{2} - \sin\frac{x}{2}$$

$$2\sin\frac{x}{2}\cos 2x = \sin\frac{5x}{2} - \sin\frac{3x}{2}$$

... ...

... ...

$$2\sin\frac{x}{2}\cos nx = \sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x$$

Adding the above n equations, we get

$$2\sin\frac{x}{2}(\cos x + \cos 2x + \ldots + \cos nx)$$

$$= \sin\left(n+\frac{1}{2}\right)x - \sin\frac{x}{2}$$

$$\Rightarrow \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}} - 1 = 2(\cos x + \cos 2x + \ldots + \cos nx)$$

$$\Rightarrow \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}dx - \int_0^{\pi} 1\cdot dx$$

$$= 2\int_0^{\pi} (\cos x + \cos 2x + ... + \cos nx)\,dx$$

$$\Rightarrow \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}dx - \pi = 0$$

$$\therefore \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}dx = \pi$$

The correct option is (C)

7. Let $y = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$

$$\Rightarrow \log y = \lim_{n\to\infty}\frac{1}{n}\times\log\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right) = \int_0^1 \log(1+x)\,dx$$

$$= [x\log(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}dx$$

$$= \log 2 - \int_0^1 \frac{(1+x)-1}{1+x}dx$$

$$= \log 2 - \int_0^1\left(1-\frac{1}{1+x}\right)dx$$

$$= \log 2 - [x - \log(1+x)]_0^1$$

$$= \log 2 - [(1 - \log 2) - 0]$$

$$= 2\log 2 - \log e = \log\frac{4}{e}$$

$$\therefore \quad y = \frac{4}{e}$$

The correct option is (D)

8. $I = \int_3^{3+3T} f(2x)\,dx = \frac{1}{2}\int_6^{6+6T} f(x)\,dx = 3\int_0^T f(x)\,dx$

$= 3I$

The correct option is (C)

9. Let $f(x) = \frac{\sin 8x\cdot\log(\cot x)}{\cos 2x}$

$$\Rightarrow f\left(\frac{\pi}{2}-x\right) = \frac{\sin 8\left(\frac{\pi}{2}-x\right)\cdot\log\left[\cot\left(\frac{\pi}{2}-x\right)\right]}{\cos 2\left(\frac{\pi}{2}-x\right)}$$

$$= \frac{\sin(4\pi - 8x)\log\tan x}{\cos(\pi - 2x)}$$

$$= \frac{-\sin 8x\cdot\log\cot x}{\cos 2x} = -f(x)$$

$$\therefore \int_0^{\pi/2}\frac{\sin 8x\log\cot x}{\cos 2x}dx = 0$$

The correct option is (D)

10. Since $I_m = \int_1^x (\log x)^m\,dx$

Using ILATE, we have

$$I_m = \left[(\log x)^m\cdot x\right]_1^x - \int_1^x m(\log x)^{m-1}\cdot\frac{1}{x}x\,dx$$

$$\Rightarrow \quad I_m = (\log x)^m\cdot x - mI_{m-1}$$

Since, $I_m = k - lI_{m-1}$ (given)

$\therefore \; k - lI_{m-1} = x(\log x)^m - mI_{m-1}$

$\therefore \quad k = x(\log x)^m$ and $l = m$

The correct option is (B)

11. $\int_{-1}^{1}[x(1+\sin\pi x)+1]\,dx$

$$= \int_{-1}^{0}[x(1+\sin\pi x)+1]\,dx + \int_0^1[x(1+\sin\pi x)+1]\,dx$$

Now $-1 < x < 0 \Rightarrow (1 + \sin\pi x) = 0$

$0 < x < 1 \Rightarrow (1 + \sin\pi x) = 1$

$\Rightarrow \; [x(1 + \sin px) + 1] = 1$

So $\int_{-1}^{1}[x(1+\sin\pi x)+1]\,dx = 2$

The correct option is (A)

12. $\int_0^{41\pi/4}|\cos x|\,dx = \int_0^{10\pi}|\cos x|\,dx + \int_{10\pi}^{41\pi/4}|\cos x|\,dx$

$$= 10\int_0^{\pi}|\cos x|\,dx + \int_{10\pi}^{10\pi+\frac{\pi}{4}}|\cos x|\,dx$$

(Since $|\cos x|$ is a periodic function of period π)

$$= 10\int_0^{\pi}|\cos x|\,dx + \int_0^{\pi/4}|\cos x|\,dx$$

$$= 10\left(\int_0^{\pi/2}\cos x\,dx - \int_{\pi/2}^{\pi}\cos x\,dx\right) + \sin x\Big|_0^{\pi/4}$$

$$= 10\left(\sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^{\pi}\right) + \frac{1}{\sqrt{2}}$$

$$= 10\,(1+1) + \frac{1}{\sqrt{2}} = 20 + \frac{1}{\sqrt{2}}$$

The correct option is (B)

13. We have, $\int_0^y e^{-t^2}\,dt + \int_0^{x^2} \sin^2 t\,dt = 0$

Differentiating with respect to x, we get

$$\left[e^{-t^2}\right]_{t=y}\cdot\frac{dy}{dx} + \left[\sin^2 t\right]_{t=x^2}\cdot\frac{d}{dx}\,(x^2) = 0$$

$$\Rightarrow\quad e^{-y^2}\frac{dy}{dx} + \sin^2 x^2\cdot 2x = 0$$

$$\therefore\quad \frac{dy}{dx} = -\,2x\sin^2 x^2\cdot e^{y^2}$$

The correct option is (B)

14. $I = \int_a^b [f\{g[h(x)]\}]^{-1} f'\{g[h(x)]\}\,g'[h(x)]\,h'(x)\,dx$

Put $\quad h(x) = t \Rightarrow h'(x)\,dx = dt$

$$\therefore\quad I = \int_{h(a)}^{h(b)} \{f[g(t)]\}^{-1} f'[g(t)]\,g'(t)\,dt = 0$$

Since $h(A) = h(b)$ and $\int_a^a f(x)\,dx = 0$

The correct option is (A)

15. Let $I_n = \int_0^\infty x^n e^{-x}\,dx$

$$= \left[x^n\cdot\frac{e^{-x}}{(-1)}\right]_0^\infty - \int_0^\infty nx^{n-1}.\frac{e^{-x}}{(-1)}\,dx$$

$$= n\int_0^\infty x^{n-1} e^{-x}\,dx = nI_{n-1}$$

$\therefore\quad I_n = n\cdot I_{n-1}\qquad$ (1)

Changing n to $n-1$, $I_{n-1} = (n-1)\,I_{n-2}$

Substituting the value of I_{n-1} in (1), we get

$I_n = n\,(n-1)\,I_{n-2}$

Generalising from (1) and (2), we have

$I_n = [n\,(n-1)\ \ldots$ to n factors$]\ I_{n-n}$

$$= n\,!\,I_0 = n\,!\int_0^\infty x^0 e^{-x}\,dx = n\,!\int_0^\infty e^{-x}\,dx$$

$$= n\,!\left(\frac{e^{-x}}{-1}\right)_0^\infty = n\,!$$

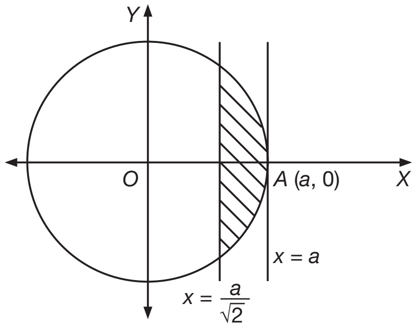
The correct option is (A)

16. Required area $= 2\int_{a/\sqrt{2}}^{a} y\,dx = 2\int_{a/\sqrt{2}}^{a}\sqrt{a^2-x^2}\,dx$

$$= 2\left(\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_{a/\sqrt{2}}^{a}$$

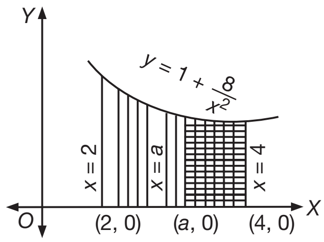
$$= 2\left[\frac{a^2}{2}\cdot\frac{\pi}{2} - \left(\frac{a}{2\sqrt{2}}\cdot\frac{a}{\sqrt{2}} + \frac{a^2}{2}\cdot\frac{\pi}{4}\right)\right]$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$$



The correct option is (B)

17. The area bounded by the curve $y = 1 + \frac{8}{x^2}$, x-axis and the ordinates $x = 2$, $x = 4$ is



$$= \int_2^4 y\,dx = \int_2^4\left(1+\frac{8}{x^2}\right)dx = \left(x - \frac{8}{x}\right)_2^4$$

$$= (4-2) - (2-4) = 4$$

Since $x = a$ divides this area into two equal parts,

$\therefore$ Required area $= 2\int_2^a y\,dx$

$$\therefore\qquad 4 = 2\int_2^a\left(1+\frac{8}{x^2}\right)dx$$

$$\Rightarrow\qquad 2 = \left(x - \frac{8}{x}\right)_2^a = \left(a - \frac{8}{a}\right) - (2-4)$$

$\Rightarrow\ a^2 = 8,\quad \therefore\ a = 2\sqrt{2}$

The correct option is (B)

18. Let $I=\int_4^{10}\frac{(x^2)}{(x^2-28x+196)+(x^2)}dx$

$$=\int_4^{10}\frac{(x^2)}{[(x-14)^2]+(x^2)}dx \qquad (1)$$

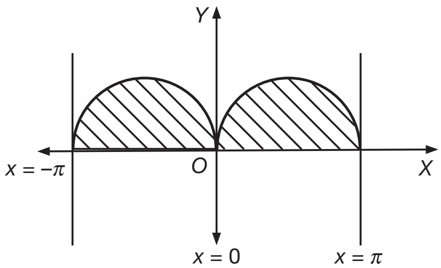
$$=\int_4^{10}\frac{[(14-x)^2]}{(x^2)+[(14-x)^2]}dx \qquad (2)$$

Adding (1) and (2), we get

$$2I=\int_4^{10}1\,dx=6\Rightarrow I=3$$

The correct option is (A)

19. The required area



$$=\int_{-\pi}^{\pi}|\sin x|\,dx=2\int_0^{\pi}\sin x\,dx=-2\,[\cos x]_0^{\pi}$$

$$=-2\,(\cos\pi-\cos 0)=4$$

The correct option is (C)

20. We are given $\int_1^b f(x)\,dx=(b-1)\sin(3b+4)$.

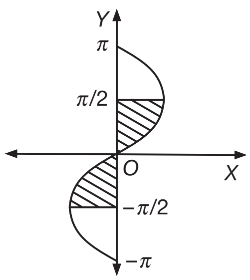
Differentiating with respect to b, we get

$[f(x)]_{x=b}=\sin(3b+4)+(b-1)\cdot 3\cos(3b+4)$

$\therefore\quad f(x)=\sin(3x+4)+3(x-1)\cos(3x+4)$

The correct option is (B)

21. The required area



$$=2\int_0^{\pi/2}x\,dy \quad \text{where } y=\sin^{-1}x \text{ i.e. } x=\sin y$$

$$=2\int_0^{\pi/2}\sin y\,dy=-2\,[\cos y]_0^{\pi/2}=2$$

The correct option is (A)

22. $I_1=\int_{\sin^2 t}^{1+\cos^2 t}(2-x)\,f\{(2-x)\,[2-(2-x)]\}\,dx$

$$=\int_{\sin^2 t}^{1+\cos^2 t}(2-x)\,f[x(2-x)]\,dx$$

$$=2\int_{\sin^2 t}^{1+\cos^2 t}f[x(2-x)]\,dx-\int_{\sin^2 t}^{1+\cos^2 t}xf[x(2-x)]\,dx=2I_2-I_1$$

Therefore, $2I_1=2I_2$ and so $I_1/I_2=1$.

The correct option is (B)

23. Let $I=\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)\,dx\right)$

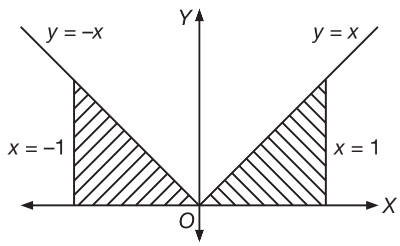
$$\Rightarrow\ I=\int_0^1 f(x)\,dx+\int_0^1 f(1+x)\,dx+\int_0^1 f(2+x)\,dx+\ldots+\int_0^1 f(99+x)\,dx$$

$$\Rightarrow\ I=\int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx+\int_2^3 f(x)\,dx\ \ldots+\int_{99}^{100}f(x)\,dx$$

$$\therefore\ I=\int_0^{100}f(x)\,dx=a\ \text{(given)}$$

The correct option is (B)

24. The required area

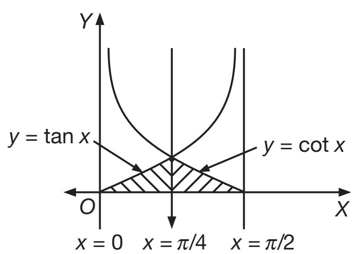


$$=2\int_0^1 y\,dx,\ \text{where } y=x=2\int_0^1 x\,dx=2\left(\frac{x^2}{2}\right)_0^1=1$$

The correct option is (C)

25. Clearly the two curves meet at the point $\left(\frac{\pi}{4},1\right)$

$\therefore$ The required area



$$= \int_0^{\pi/4} \tan x\, dx + \int_{\pi/4}^{\pi/2} \cot x\, dx$$

$$= (\log \sec x)_0^{\pi/4} + (\log \sin x)_{\pi/4}^{\pi/2}$$

$$= \log \sqrt{2} - \log \frac{1}{\sqrt{2}} = 2 \log \sqrt{2} = \log 2$$

The correct option is (B)

26. Since $\log\left(\frac{1+x}{1-x}\right)$ is an odd function,

$$\therefore \int_{-1/2}^{1/2} \log\left(\frac{1+x}{1-x}\right) dx = 0$$

$$\therefore \int_{-1/2}^{1/2} \left[\alpha \log\left(\frac{1+x}{1-x}\right) + \beta \log\left(\frac{1-x}{1+x}\right)^2 + \gamma\right] dx = \gamma \int_{-1/2}^{1/2} dx$$

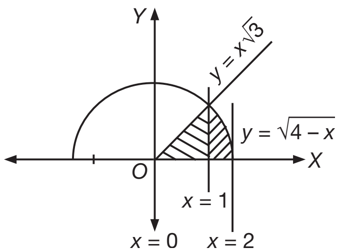
The correct option is (A)

27. The required area

$$= \int_0^1 \sqrt{3}\, x\, dx + \int_1^2 \sqrt{4-x^2}\, dx$$

$$= \frac{\sqrt{3}}{2}(x^2)_0^1 + \left(\frac{x}{2}\sqrt{4-x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_1^2$$

$$= \frac{\sqrt{3}}{2} + 2\cdot\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2\cdot\sin^{-1}\frac{1}{2} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$



The correct option is (B)

28. We have, $\int_0^x (t^2 - 8t + 13)\, dt = x \sin\left(\frac{a}{x}\right)$

$$\Rightarrow \quad \frac{x^3}{3} - 4x^2 + 13x = x \sin\left(\frac{a}{x}\right)$$

$$\Rightarrow \quad \frac{x^2}{3} - 4x + 13 = \sin\left(\frac{a}{x}\right)$$

($x \neq 0$, otherwise RHS is not defined)

$$\Rightarrow \quad x^2 - 12x + 39 = 3 \sin\left(\frac{a}{x}\right)$$

$$\Rightarrow \quad (x-6)^2 + 3 = 3 \sin\left(\frac{a}{x}\right)$$

Now, $-3 \leq 3 \sin\left(\frac{a}{x}\right) \leq 3$

$\Rightarrow \quad -3 \leq (x-6)^2 + 3 \leq 3$

$\Rightarrow \quad (x-6)^2 \leq 0$

$\Rightarrow \quad x - 6 = 0$ or $x = 6$

$\therefore$ $x = 6$ is the only solution.

The correct option is (B)

29. We are given

$$\int_1^a [f(x) - 2]\, dx = \frac{2}{3}\left[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}\right]$$

Differentiating w.r.t a, we get

$$f(a) - 2 = \frac{2}{3}\left(\frac{3}{2}\sqrt{2a}\cdot 2 - 3\right)$$

$\Rightarrow \quad f(a) = 2\sqrt{2a}$, $a \geq 1$ $\therefore f(x) = 2\sqrt{2x}$, $x \geq 1$

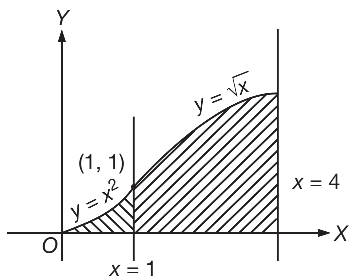
The correct option is (A)

30. The required area

$$= \int_0^1 x^2\, dx + \int_1^4 \sqrt{x}\, dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{2x^{3/2}}{3}\right)_1^4$$

$$= \frac{1}{3} + \frac{2}{3}(8-1) = 5$$



The correct option is (D)

31. The required area

$$= 2\int_0^1 \sqrt{1-x}\, dx + 2\cdot\left(\frac{1}{4}\cdot\pi\cdot 1^2\right)$$

$$= -2\cdot\frac{2(1-x)^{3/2}}{3}\Bigg|_0^1 + \frac{\pi}{2} = \frac{\pi}{2} + \frac{4}{3}$$

The correct option is (C)

32. We have, $f(x) = \int_0^1 \frac{dt}{1 + |x-t|}$

$$= \int_0^x \frac{dt}{1+x-t} + \int_x^1 \frac{dt}{1-x+t}$$

$$= \log(1+x) + \log(2-x)$$

$\therefore \quad f'(x) = \dfrac{1}{1+x} - \dfrac{1}{2-x}$

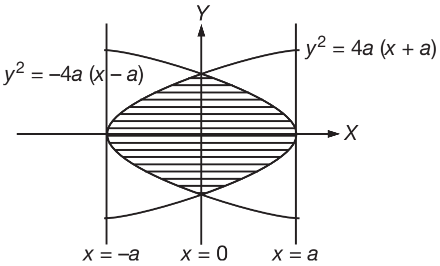
$\Rightarrow f'\left(\dfrac{1}{2}\right) = \dfrac{1}{1+1/2} - \dfrac{1}{2-1/2} = 0$

The correct option is (D)

33. The required area

$$= 4\int_0^a \sqrt{4a(a-x)}\; dx$$

$$= 4 \cdot 2\sqrt{a} \cdot \left[\frac{-2(a-x)^{3/2}}{3}\right]_0^a = \frac{16}{3}a^2$$



The correct option is (A)

34. $I_1 - I_2 = \displaystyle\int_0^{\pi/2} (\cos\theta - \sin 2\theta)\, f\,(\sin\theta + \cos^2\theta)\, d\theta$

Put $\sin\theta + \cos^2\theta = t$

$\Rightarrow \quad [\cos\theta + 2\cos\theta(-\sin\theta)]\, d\theta = dt$

$\Rightarrow \quad (\cos\theta - \sin 2\theta)d\theta = dt$

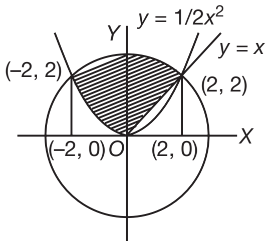
Also, if $\quad \theta = 0,\ t = 0 + 1^2 = 1$

and if $\quad \theta = \dfrac{\pi}{2},\ t = 1 + 0 = 1$

$\therefore \quad I_1 - I_2 = \displaystyle\int_1^1 f(t)\,dt = 0 \Rightarrow I_1 = I_2$

The correct option is (A)

35. The required area



$$= \int_{-2}^{2} \sqrt{8 - x^2} - \int_{-2}^{0} \frac{1}{2}x^2\, dx - \int_0^2 x\, dx$$

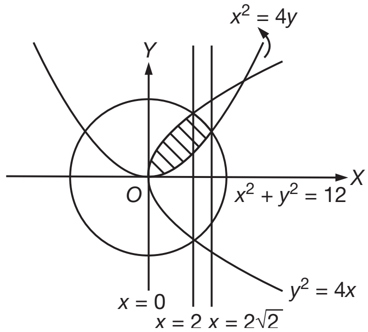
$$= 2\int_0^2 \sqrt{8 - x^2}\,dx - \left(\frac{x^3}{6}\right)_{-2}^{0} - \left(\frac{x^2}{2}\right)_0^2$$

$$= 2\left(\frac{x}{2}\sqrt{8 - x^2} + \frac{8}{2}\sin^{-1}\frac{x}{2\sqrt{2}}\right)_0^2 - \frac{4}{3} - 2$$

$$= 2\left(2 + 4\cdot\frac{\pi}{4}\right) - \frac{10}{3} = \frac{2}{3} + 2\pi$$

The correct option is (A)

36. The required area



$$= \int_0^2 2\sqrt{x}\, dx + \int_2^{2\sqrt{2}} \sqrt{12 - x^2}\, dx - \int_0^{2\sqrt{2}} \frac{x^2}{4}\, dx$$

$$= \left.\frac{4x^{3/2}}{3}\right|_0^2 + \left(\frac{x}{2}\sqrt{12 - x^2} + \frac{12}{2}\sin^{-1}\frac{x}{2\sqrt{3}}\right)_2^{2\sqrt{2}} - \frac{1}{4}\left(\frac{x^3}{3}\right)_0^{2\sqrt{2}}$$

$$= 4\left(\frac{\sqrt{2}}{3} + \frac{3}{2}\sin^{-1}\frac{1}{3}\right)$$

The correct option is (B)

37. We have, $\phi(t) = \displaystyle\int_0^t f(t-y)\, g(y)\, dt$

$$= \int_0^t e^{t-y}\cdot y\, dt = e^t \int_0^t e^{-y}\cdot y\, dy$$

$$= e^t\left[(-ye^{-y})\Big|_0^t - \int_0^t 1\cdot(-e^{-y})\,dy\right]$$

$$= e^t(-te^{-t} - e^{-t} + 1) = (1 - e^{-t}(1+t))e^t$$

$$= e^t - (1+t)$$

The correct option is (A)

38. According to the property of greatest integer function, we have

$$(x) - (-x) = -1 \quad \forall\, x \in Z \qquad (1)$$

Let $\qquad I = \displaystyle\int_{-1}^{1} f(x)\,dx$

where $\qquad f(x) = \dfrac{\sin^2 x}{\left(\dfrac{x}{\sqrt{2}}\right) + \dfrac{1}{2}}$

Now, $f(-x) = \dfrac{\sin^2 x}{\left(-\frac{x}{\sqrt{2}}\right) + \frac{1}{2}}$

$\Rightarrow \quad f(-x) = \dfrac{\sin^2 x}{-1 - \left(\frac{x}{\sqrt{2}}\right) + \frac{1}{2}}$ [using (1)]

$\because \dfrac{x}{\sqrt{2}}$ is not an integer in (–1, 0) and (0, 1)

$\Rightarrow \quad f(-x) = \dfrac{\sin^2 x}{-\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{2}} = -f(x)$

$\Rightarrow$ $f(x)$ is an odd function in x

$\therefore \quad I = \int_{-1}^{1} f(x)\,dx = 0$

The correct option is (B)

39. Since $\int_0^1 t\, f''(t)\,dt = 0,$

$\therefore \quad \left| t\, f'(t)\right|_0^1 - \int_0^1 1\cdot f'(t)\,dt = 0$

$\Rightarrow \quad f'(1) - \int_0^1 f'(t)\,dt = 0$

$\Rightarrow \quad f'(1) - \left| f(t)\right|_0^1 = 0$

$\Rightarrow \quad f'(1) - f(1) + f(0) = 0$

$\Rightarrow \quad f(1) = f'(1) + f(0) = 1 + 1 = 2$

The correct option is (B)

40. We have, $\dfrac{d}{dx}\phi(x) = \dfrac{e^{\sin x}}{x}, x > 0$

Integrating both sides, we get

$$f(x) = \int \frac{e^{\sin x}}{x}\,dx \quad (1)$$

Also, $\int_1^4 \dfrac{3}{x}\sin x^3 dx = \int_1^4 \dfrac{3x^2}{x^3}\sin x^3\,dx = \phi(k) - \phi(1)$

Let $x^3 = t \Rightarrow 3x^2\,dx = dt$

$\Rightarrow \quad \int_1^{64} \dfrac{e^{\sin t}}{t}\,dt = \phi(k) - \phi(1)$

$\Rightarrow \quad \phi(t)\big|_1^{64} = \phi(k) - \phi(1)$

$\Rightarrow \quad \phi(64) - \phi(1) = \phi(k) - \phi(1)$

$\Rightarrow \quad k = 64$

The correct option is (C)

41. We know that if $f(x + m\pi) = f(x)$ for all integral values of m, then

$$\int_0^{n\pi} f(x)\,dx = n\int_0^{\pi} f(x)\,dx$$

Let $g(x) = f(\cos^2 x)$, then $g(x + m\pi) = f[\cos^2(x + m\pi)]$
$= f(\cos^2 x) = g(x)$.

Hence $\int_0^{3\pi} f(\cos^2 x)\,dx = 3\int_0^{\pi} f(\cos^2 x)\,dx.$

$\therefore \quad I_1 = 3I_2$

The correct option is (C)

42. $u_{10} = \int_0^{\pi/2} x^{10}\sin x\,dx$

$= \left(-x^{10}\cos x\right)_0^{\pi/2} + \int_0^{\pi/2} 10x^9\cos x dx$

$= 10\left(x^9 \sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} 9x^8 \sin x\,dx\right)$

$= 10\left[\left(\frac{\pi}{2}\right)^9 - 9\int_0^{\pi/2} x^8 \sin x\,dx\right]$

$= 10\left(\frac{\pi}{2}\right)^9 - 90\,u_8$

$\therefore u_{10} + 90u_8 = 10\left(\frac{\pi}{2}\right)^9$

The correct option is (B)

43. Since, $f(x) = \dfrac{1}{1-x}, x \neq 1$

Also, $f[f(x)] = \dfrac{1}{1-f(x)} = \dfrac{1}{1-\frac{1}{1-x}}$

$\Rightarrow \quad f[f(x)] = \dfrac{x-1}{x}, x \neq 0, 1$

Again, $f\{f[f(x)]\} = f\left(\dfrac{x-1}{x}\right) = \dfrac{1}{1-\frac{x-1}{x}} = x$

Thus, $f\{f[f(x)]\} = x, x \neq 0, 1$

Clearly, $f\{f[f(x)]\}$ is discontinuous at $x = 0$ and $x = 1$

$\therefore \quad a = 0, b = 1$

Now, $I = \int_a^b \dfrac{f(x)}{f(x) + f(1-x)}dx$

$$\Rightarrow \quad I = \int_0^1 \frac{f(x)}{f(x)+f(1-x)}dx \quad (1)$$

$$\text{Also,} \quad I = \int_0^1 \frac{f(1-x)}{f(1-x)+f(x)}\,dx \quad (2)$$

Adding (1) and (2), we have

$$2I = \int_0^1 \frac{f(x)}{f(x)+f(1-x)}dx + \int_0^1 \frac{f(1-x)}{f(1-x)+f(x)}dx$$

$\Rightarrow \quad 2I = 1$

$\therefore \quad I = \frac{1}{2}$

The correct option is (B)

44. By the given condition

$$\int_1^k (8x^2 - x^5)\,dx = \frac{16}{3}$$

$$\Rightarrow \quad \left(\frac{8x^3}{3} - \frac{x^6}{6}\right)_1^k = \frac{16}{3}$$

$$\Rightarrow \quad \frac{8}{3}(k^3 - 1) - \frac{1}{6}(k^6 - 1) = \frac{16}{3}$$

$$\Rightarrow \quad \left(\frac{8k^3}{3} - \frac{k^6}{6}\right) - \left(\frac{8}{3} - \frac{1}{6}\right) = \frac{16}{3}$$

$\Rightarrow \quad 16k^3 - k^6 - 15 = 32$

$\Rightarrow \quad k^6 - 16k^3 + 47 = 0 \Rightarrow k^3 = 8 \pm \sqrt{17}$

$\therefore \quad k = \left(8 \pm \sqrt{17}\right)^{1/3}$

The correct option is (D)

45. We have, $\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt$

Differentiating with respect to x, we get

$1 \cdot f(x) = 1 - 1 \cdot x f(x) \Rightarrow f(x) = \frac{1}{1+x}$

$\therefore \quad f(1) = \frac{1}{2}$

The correct option is (A)

46. $$\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi + 2 - t}\,dt = \frac{1}{2}\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{1 + \left(2\pi - \frac{t}{2}\right)}\,dt$$

$$= 2 \cdot \frac{1}{2}\int_0^1 \frac{\sin(2\pi - u)}{1+u}\,du$$

$\left(\text{Putting } 2\pi - \frac{t}{2} = u \text{ so that } dt = -2\,du\right)$

$$= -\int_0^1 \frac{\sin u}{1+u}\,du = -\int_0^1 \frac{\sin t}{1+t}\,dt = -\alpha$$

The correct option is (D)

47. Given $\int_1^2 e^{x^2}\,dx = a$. Let $I = \int_e^{e^4} \sqrt{(\ln x)}\,dx$

Put $\ln x = t^2$, $\therefore x = e^{t^2}$, $\Rightarrow dx = 2te^{t^2}$

then $I = \int_1^2 t \cdot 2e^{t^2}\,dt = 2\int_1^2 t^2 e^{t^2}\,dt$

$$= \int_1^2 t \cdot (2te^{t^2})\,dt = \left[t \cdot e^{t^2}\right]_1^2 - \int_1^2 e^{t^2}\,dt \quad \left(\because \int_1^2 e^{x^2}\,dx = a\right)$$

$= 2e^4 - e^1 - a$

$= 2e^4 - e - a.$

The correct option is (D)

48. Let $I = \int_{-\pi/2}^{\pi/2} \left[\left(\frac{x}{\pi}\right) + 0.5\right] dx$

Let $\frac{x}{\pi} = t \Rightarrow dx = \pi\,dt$

$$I = \int_{-1/2}^{1/2} [(t) + 0.5]\pi\,dt$$

$$= \pi\left\{\int_{-1/2}^{0} -\frac{1}{2}\,dt + \int_0^{1/2} \frac{1}{2}\,dt\right\} = \pi\left\{-\frac{1}{4} + \frac{1}{4}\right\} = 0$$

The correct option is (C)

49. We have,

$$\frac{1}{\sqrt{a}}\int_1^a \left(\frac{3}{2}\sqrt{x} + 1 - \frac{1}{\sqrt{x}}\right)dx < 4$$

$$\Rightarrow \frac{1}{\sqrt{a}}\left(a\sqrt{a} - 1 + a - 1 - 2\sqrt{a} + 2\right) < 4$$

$\Rightarrow a + \sqrt{a} - 2 < 4$

$\Rightarrow t^2 + t - 6 < 0$, where $t = \sqrt{a}$

$\Rightarrow (t + 3)(t - 2) < 0$

$\Rightarrow t \in (-3, 2) \Rightarrow \sqrt{a} \in (-3, 2)$

$\Rightarrow \sqrt{a} \in [0, 2) \Rightarrow a \in (0, 4)$

(The given inequality is defined for $a \neq 0$)

The correct option is (A)

50. We have, $\sin x < x$ for $x > 0$

$\Rightarrow \sin(\cos x) < \cos x$ for $0 < x < \pi/2$

$$\Rightarrow \int_0^{\pi/2} \sin(\cos x)\,dx < \int_0^{\pi/2} \cos x\,dx$$

$\therefore \quad I_3 > I_2 \quad (1)$

Now $\cos x < \cos \alpha$ if $x > \alpha$ and $x, \alpha \in \left[0, \frac{\pi}{2}\right]$

$\therefore \quad x > \sin x$

$\Rightarrow \quad \cos x < \cos(\sin x)$

$$\Rightarrow \int_0^{\pi/2} \cos x\, dx < \int_0^{\pi/2} \cos(\sin x)\, dx$$

$\therefore \quad I_3 < I_1 \qquad (2)$

$\therefore$ From (1) and (2) $I_1 > I_3 > I_2$

The correct option is (A)

51. We have, $\int_1^c (8x^2 - x^5)\, dx = \frac{16}{3}$

If $c = -1$, $\left| \int_{-1}^{1} 8x^2 dx - \int_{-1}^{1} x^5 dx \right| = \frac{16}{3}$

$$\Rightarrow \frac{16}{3} = \frac{16}{3} \quad \therefore c = -1$$

The correct option is (D)

52. We have, $\int_0^a f(x)\, dx = \frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$

Differentiating w.r.t a, we get

$$f(a) = a + \frac{1}{2}(\sin a + a\cos a) - \frac{\pi}{2}\sin a$$

Put $a = \frac{\pi}{2}; f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{1}{2} - \frac{\pi}{2} = \frac{1}{2}$

The correct option is (B)

53. When $1 < x < e^3$, $\left(\frac{\log x}{3}\right) = 0$

and when $e^3 < x < e^6$, $\left(\frac{\log x}{3}\right) = 1$.

$$\therefore \int_1^{e^6} \left(\frac{\log x}{3}\right) dx = \int_1^{e^3} \left(\frac{\log x}{3}\right) dx + \int_{e^3}^{e^6} \left(\frac{\log x}{3}\right) dx$$

$$= \int_1^{e^3} 0\, dx + \int_{e^3}^{e^6} 1\, dx = (e^6 - e^3)$$

The correct option is (B)

54. $\int_a^b |\sin x|\, dx = 8 \Rightarrow b - a = 4\pi$

$\because$ period of $|\sin x|$ is π and $\int_0^{\pi} |\sin x|\, dx = 2$.

Also, $|\cos x|$ is periodic with period π and $\int_0^{\pi/2} |\cos x| = 1$

So, $\int_a^{a+b} |\cos x|\, dx = \frac{9}{2} \Rightarrow a + b = \frac{9\pi}{2}$

$$\Rightarrow \quad b = \frac{17\pi}{4} \text{ and } a = \frac{\pi}{4}$$

The correct option is (C)

55. Let $x - [x] = t$. In the interval $\left[0, \frac{\pi}{6}\right]$, $[x] = 0$

$\therefore \quad x = t$

$$\therefore \int_0^{\pi/6} \sec^2 x\, d(x - [x]) = \int_0^{\pi/6} \sec^2 t\, dt = \tan t\Big|_0^{\pi/6} = \frac{1}{\sqrt{3}}.$$

The correct option is (B)

56. Since $x + |x| \geq x - [x]$ for all $x \geq 0$

and $x + |x| = 0 \leq 0 \leq x - [x]$ for all $x < 0$.

$$\therefore \int_{-2}^{2} f(x)\, dx = \int_{-2}^{0} (x - [x])\, dx + \int_0^2 (x + |x|)\, dx$$

$$= 2 \cdot \frac{1}{2} + \int_0^2 2x\, dx = 1 + 4 = 5.$$

The correct option is (C)

57. $\left| \int_{10}^{19} \frac{\sin x\, dx}{1 + x^8} \right| \; 2 \cdot \frac{1}{2} + \int_0^2 2x\, dx \leq \int_{10}^{19} \left| \frac{\sin x}{1 + x^8} \right| dx$

$$\leq \int_{10}^{19} \frac{1}{1 + x^8}\, dx < \int_{10}^{19} \frac{dx}{x^8}$$

$$= \frac{-1}{7}\left[x^{-7}\right]_{10}^{19} = \frac{1}{7}10^{-7} - \frac{1}{7}19^{-7} < \frac{1}{7}10^{-7} < 10^{-7}$$

The correct option is (C)

58. Let $I = \int_0^1 \frac{dx}{2e^x - 1} = \int_0^1 \left(\frac{2e^x}{2e^x - 1} - 1\right) dx$

$$= \left[\log(2e^x - 1) - x\right]_0^1 = \log(2e - 1) - 1.$$

$\therefore \quad p = 1, q = 2, r = 1.$

The correct option is (B)

59. $\int_0^{\pi} (\tan^{-1} x)\, dx = \int_0^{\tan 1} (0)\, dx + \int_{\tan 1}^{\pi} (1)\, dx$

$$= 0 + \pi - \tan 1 = \pi - \tan 1$$

The correct option is (D)

60. $f(x) = \int_0^x 2|t|dt \Rightarrow f'(x) = 2|x|$

Since, it is given that the tangents of the curve are parallel to the bisector of the first quadrant angle i.e. a line which is inclined at an angle of 45° with positive x-axis.

$\therefore \quad f'(x) = 1$

$\Rightarrow \quad 2|x| = 1 \Rightarrow x = \pm \frac{1}{2}$

For $\quad x = \frac{1}{2}, y = f(x) = \int_0^{1/2} 2tdt = \frac{1}{4}$,

and for $\quad x = -\frac{1}{2}, y = \int_0^{1/2} -2tdt = -\frac{1}{4}$

The equations of the tangents at $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{4}\right)$ are $y = x - \frac{1}{4}$ and $y = x + \frac{1}{4}$ respectively.

The correct option is (C)

61. Since, $f(x) + f(y) = f(x + y)$ (1)

Replace y by $-x$

$\Rightarrow \quad f(x) + f(-x) = f(x - x)$

$\Rightarrow \quad f(x) + f(-x) = f(0)$ (2)

Also, using (1), we have

$f(0) + f(0) = f(0 + 0) = f(0)$

$\Rightarrow \quad f(0) = 0$

$\therefore \quad f(-x) = -f(x)$ [using (2)]

$\Rightarrow \quad \int_{-2}^{2} f(x)dx = 0$

The correct option is (B)

62. $\int_0^{\pi} |1 + 2\cos x|dx$

$= \int_0^{2\pi/2} |1 + 2\cos x|dx + \int_{2\pi/3}^{\pi} |1 + 2\cos x|dx$

$= \int_0^{2\pi/2} |(1 + 2\cos x)dx - \int_{2\pi/3}^{\pi} (1 + 2\cos x)dx$

$= (x + 2\sin x)\Big|_0^{2\pi/3} - (x + 2\sin x)\Big|_{2\pi/3}^{\pi}$

$= \frac{\pi}{3} + 2\sqrt{3}$

The correct option is (D)

63. Let $I_n = \int_0^{\infty} x^n e^{-ax} = \left(x^n \cdot \frac{e^{-ax}}{-a}\right)_0^{\infty} - \int_0^{\infty} nx^{n-1} \cdot \frac{e^{-ax}}{-a} dx$

$= -\frac{1}{a} \lim_{x\to\infty} \frac{x^n}{e^{ax}} + \frac{n}{a} I_{n-1}$

$\therefore \quad I_n = \frac{n}{a} I_{n-1} \qquad \left(\because \lim_{x\to\infty} \frac{x^n}{e^{ax}} = 0\right)$

$= \frac{n}{a} \cdot \frac{n-1}{a} I_{n-2} = \frac{n(n-1)(n-2)}{a^3} I_{n-3}$

....

....

$= \frac{n!}{a^n} \int_0^{\infty} e^{-4x}dx = \frac{n!}{a^{n+1}}$

The correct option is (C)

64. According to question $f'(1) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$f'(2) = \tan \frac{\pi}{3} = \sqrt{3}$ and $f'(3) = \tan \frac{\pi}{4} = 1$

so, $\int_1^3 f''(x)f'(x)dx + \int_2^3 f''(x)dx = \left(\frac{\{f'(x)\}^2}{2}\right)_1^3 + [f'(x)]_2^3$

$= \frac{1}{2}\left(1 - \frac{1}{3}\right) + \left(1 - \sqrt{3}\right)$

$= \frac{4}{3} - \sqrt{3}$

The correct option is (C)

65. $f(x) = \frac{e^x}{1 + e^x} \Rightarrow f(-x) = \frac{e^{-x}}{1 + e^{-x}} = \frac{1}{e^x + 1}$

$\therefore f(x) + f(-x) = 1, \ \forall x$

Now, $I_1 = \int_{f(-a)}^{f(a)} xg[x(1 - x)]dx$

$= \int_{f(-a)}^{f(a)} (1 - x)g[x(1 - x)]dx = I_2 - I_1 \Rightarrow 2I_1 = I_2$

The correct option is (D)

66. We have,

$I = \int_0^1 \frac{8\log(1 + x)}{1 + x^2} dx$

Put $x = \tan\theta$

$\Rightarrow \quad I = \int_0^{\pi/4} 8 \cdot \frac{\log(1 + \tan\theta)}{\sec^2\theta} \sec^2\theta \, d\theta$

$\Rightarrow \quad = 8\int_0^{\pi/4} \log(1 + \tan\theta)d\theta$

$= 8\int_0^{\pi/4} \log\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right)d\theta$

$$= 8\int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= 8(\log 2)\cdot\frac{\pi}{4} - I$$

$\Rightarrow \quad 2I = 2\pi \log 2$

$\Rightarrow \quad I = \pi \log 2$

The correct option is (B)

67. We have,

$$f(x) = \int_0^x \sqrt{t}\sin t\, dt$$

$\Rightarrow \quad f'(x) = \sqrt{x}\sin x$

For maximum or minimum value of $f(x)$, $f'(x) = 0$

$\Rightarrow \quad x = n^{4\sqrt{2}}, n = Z$

We observe that

+ve	–ve	+ve
0 to π	π to 2π	2π to 5π/2

$f'(x)$ changes its sign from +ve to –ve in the neighbourhood of π and –ve to +ve in the neighbourhood of 2π.

Hence, $f(x)$ has local maximum at $x = \pi$ and local minima at $x = 2\pi$.

The correct option is (A)

68. The equation of the tangent to $x = y^2$ having slope 1 is $y = x + 1/4$.

Hence shortest distance $= \left|\frac{1-1/4}{\sqrt{2}}\right|$

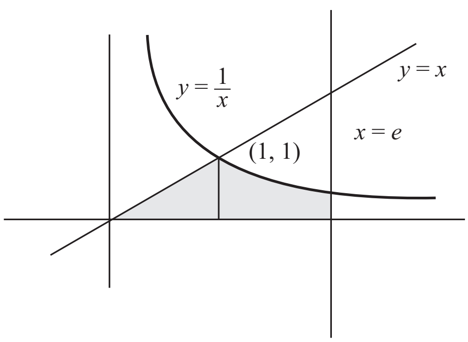
$$= \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8} \text{ units}$$

The correct option is (C)

69. Required area $= \frac{1}{2}\times 1\times 1 + \int_1^e \frac{1}{x}dx$

$$= \frac{1}{2} + (\ln x)_1^e$$

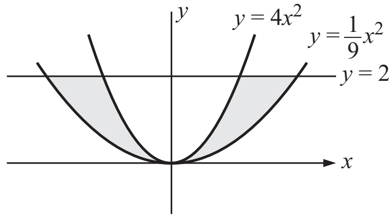
$$= \frac{1}{2} + 1 - 0 = \frac{3}{2} \text{ sq. units}$$



The correct option is (D)

70. $y = 4x^2$

$y = \frac{1}{9}x^2$



$$\text{Area} = 2\int_0^2\left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right)dy$$

$$= 2\left(\frac{5y}{2}\frac{\sqrt{y}}{3/2}\right)_0^2 = 2\cdot\frac{5}{3}2\sqrt{2} = \frac{20\sqrt{2}}{3}$$

The correct option is (C)

71. $1 < x < 3$

$\Rightarrow \quad 1 < x^3 < 27$

$\Rightarrow \quad 4 < x^3 + 3 < 30$

$\Rightarrow \quad 2 < \sqrt{x^3+3} < \sqrt{30}$

$$\int_1^3 2dx \le \int_1^3 \sqrt{x^3+3dx} < \int_1^3 \sqrt{30}\,dx$$

$\Rightarrow \quad 4 \le I \le 2\sqrt{30}$

The correct option is (D)

72. $\lim_{n\to\infty}\frac{1}{n}\left[1 + \frac{n^2}{n^2+1^2} + \frac{n^2}{n^2+2^2} + \ldots + \frac{n^2}{n^2+(n-1)^2}\right]$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{n^2}{n^2+r^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{1+\frac{r^2}{n^2}}$$

$$= \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^1 = \frac{\pi}{4}$$

The correct option is (C)

73. $I = \int_0^1 \frac{\sin x}{\sqrt{x}}dx < \int_0^1 \frac{x}{\sqrt{x}}dx = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}$

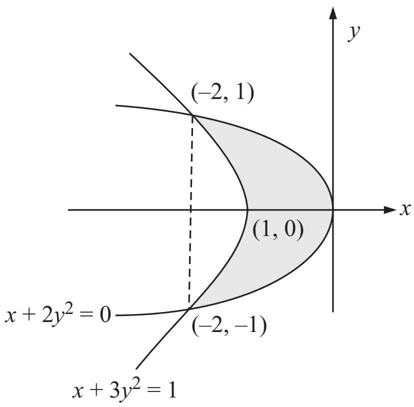
$\Rightarrow \quad I < \frac{2}{3}$

$$J = \int_0^1 \frac{\cos x}{\sqrt{x}}dx < \int_0^1 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_0^1 = 2$$

$\therefore \quad J \le 2.$

The correct option is (B)

74. Solving the equations we get the points of intersection (–2, 1) and (–2, –1)



The bounded region is shown as shaded region.

The required area = $2\int_0^1 (1-3y^2)-(-2y^2)$

$= 2\int_0^1 (1-y^2)dy = 2\left(y-\frac{y^3}{3}\right)_0^1$

$= 2\times\frac{2}{3} = \frac{4}{3}$

The correct option is (D)

75. Let $I=\int_0^{\pi}[\cot x]dx$ (1)

$= \int_0^{\pi}[\cot(\pi-x)]dx = \int_0^{\pi}[-\cot x]dx$ (2)

Adding (1) and (2)

$2I = \int_0^{\pi}[\cot x]dx + \int_0^{\pi}[-\cot x]dx$

$= \int_0^{\pi}(-1)dx \quad \left[\because \begin{array}{l}(x)+(-x)=-1 \text{ if } x\notin Z\\ \qquad\qquad = 0 \text{ if } x\in Z\end{array}\right]$

$= (-x)_0^{\pi} = -\pi$

$\therefore \quad I = -\frac{\pi}{2}$

The correct option is (D)

76. Given $\int_1^2 e^{x^2}dx = a$, let $I=\int_e^{e^4}\sqrt{\ln(x)}\,dx$

Put $\ln(x) = t^2 \Rightarrow \frac{1}{x}dx = 2tdt$

$\therefore \quad I = \int_1^2 e^{t^2}\cdot 2t^2\,dt = \left(t\,e^{t^2}\right)_1^2 - \int_1^2 e^{t^2}dt$

$= 2e^4 - e - a.$

The correct option is (D)

77. $\int_{-2}^{2}[x^2]\,dx = 2\int_0^2[x^2]\,dx$ [$\because$ integrand is even]

$= 2\left[\int_0^1[x^2]\,dx + \int_1^{\sqrt{2}}[x^2]\,dx + \int_{\sqrt{2}}^{\sqrt{3}}[x^2]\,dx + \int_{\sqrt{3}}^{2}[x^2]\,dx\right]$

$\left[\because [x^2] = 0 \text{ if } 0\le x<1; 1 \text{ if } 1\le x<\sqrt{2};\ 2 \text{ if } \sqrt{2}\le x<\sqrt{3}; 3 \text{ if } \sqrt{3}\le x<2\right]$

$= 2\left[\int_0^1 0\,dx + \int_1^{\sqrt{2}}1\,dx + \int_{\sqrt{2}}^{\sqrt{3}}2\,dx + \int_{\sqrt{3}}^{2}3\,dx\right]$

$= 2\ (x)\big|_1^{\sqrt{2}} + 4\,(x)\big|_{\sqrt{2}}^{\sqrt{3}} + 6\,(x)\big|_{\sqrt{3}}^{2}$

$= \left(10 - 2\sqrt{3} - 2\sqrt{2}\right).$

The correct option is (A)

78. From the given equation, we have

$\int_{-1}^{x} f(t)\,dt - x\,f'''(3)$

$= \left(\frac{x^4}{4}-\frac{1}{4}\right) - f'(1)\frac{x^3}{3} + f''(2)\left(\frac{9}{2}-\frac{x^2}{2}\right)$

Differentiating w.r.t. x, we get

$f(x) - f'''(3) = x^3 - x^2 f'(1) - x f''(2)$

Differentiating again, we have

$f'(x) = 3x^2 - 2xf'(1) - f''(2)$

$\therefore \quad f'(4) = 48 - 8f'(1) - f''(2).$

The correct option is (A)

79. For $-2\le x\le -1$, we have $1-x\ge 2$

and $1-x > 1+x$

$\therefore$ max $\{(1-x), (1+x), 2\} = 1-x$.

For $-1<x<1$, we have $0<1-x<2$ and $0<1+x<2$

$\therefore$ max $\{(1-x), (1+x), 2\} = 2$.

For $1\le x\le 2$, we have $1+x\ge 2$ and $1+x>1-x$

$\therefore$ max $\{(1-x), (1+x), 2\} = 1+x$.

$\therefore \int_{-2}^{2}\max\{(1-x),(1+x),2\}\,dx$

$= \int_{-2}^{-1}(1-x)\,dx + \int_{-1}^{1}2\,dx + \int_1^2(1+x)\,dx$

$= \left[x-\frac{x^2}{2}\right]_{-2}^{-1} + [2x]_{-1}^{1} + \left[x+\frac{x^2}{2}\right]_1^2 = 9.$

The correct option is (C)

80. We have, $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x\,f(x(3-x))\,dx$

$$= \int_{\sec^2 z}^{2-\tan^2 z} (3-x)\, f((3-x)\{3-(3-x)\})\, dx$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right]$$

$$= \int_{\sec^2 z}^{2-\tan^2 z} (3-x)\, f(x(3-x))$$

$$= 3\int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x))\, dx - \int_{\sec^2 z}^{2-\tan^2 z} x\, f(x(3-x))\, dx$$

$$= 3\, I_2 - I_1$$

$\therefore\ 2\, I_1 = 3\, I_2$ and so $I_1/I_2 = \dfrac{3}{2}$.

The correct option is (A)

81. $\displaystyle\int_1^2 \sqrt{(2x+3)(3x^2+4)}\, dx$

$$\leq \sqrt{\int_1^2 (2x+3)\, dx \cdot \int_1^2 (3x^2+4)\, dx}$$

$$\left[\because \left|\int_a^b f(x)\cdot g(x)\, dx\right| \leq \sqrt{\int_a^b f^2(x)\, dx \cdot \int_a^b g^2(x)\, dx}\right]$$

$$= \sqrt{[x^2+3x]_1^2 \cdot [x^3+4x]_1^2} = \sqrt{6\times 11} = \sqrt{66}.$$

The correct option is (B)

82. Let $f(x) = 2x^3 - 9x^2 + 12x + 4$, then $f(x)$ is a decreasing function on the interval [1, 2].

$\therefore\quad 8 = f(2) < f(x) < f(1) = 9.$

$$\therefore\quad \frac{1}{3} < \frac{1}{\sqrt{2x^3-9x^2+12x+4}} < \frac{1}{\sqrt{8}}$$

$$\Rightarrow\quad \frac{1}{3}\int_1^2 dx < \int_1^2 \frac{dx}{\sqrt{2x^3-9x^2+12x+4}} < \frac{1}{\sqrt{8}}\int_1^2 dx$$

$$\Rightarrow\quad \frac{1}{4} < \frac{1}{3} < I < \frac{1}{\sqrt{8}} < 1$$

Hence, $\dfrac{1}{4} < I < 1$.

The correct option is (C)

83. $$I_{2,n} - I_{2,n-1} = \int_0^{\pi/2} \frac{(\sin^2 nx - \sin^2(n-1)x)}{\sin^2 x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n-1)x \sin x}{\sin^2 x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x}\, dx = I_{1,n}$$

$\therefore\quad I_{2,n+1} - I_{2,n} = I_{1,n+1}$.

The correct option is (B)

84. $$\int_0^{2[x]} (x-[x])\, dx = \int_0^{2[x]\cdot 1} (x-[x])\, dx$$

$$= 2\,[x] \int_0^1 (x-[x])\, dx$$

[$\because x - [x]$ is a periodic function of period 1]

$$= 2\,[x]\left(\left.\frac{x^2}{2}\right|_0^1 - \int_0^1 [x]\, dx\right)$$

$$= 2\,[x]\left(\frac{1}{2} - 0\right) = [x].$$

The correct option is (A)

85. We have,

$$\int_0^x f(t)\, dt = \int_x^1 t^2 f(t)\, dt + \frac{x^{16}}{8} + \frac{x^6}{3} + k \qquad (1)$$

For $x = 1$, $\displaystyle\int_x^1 f(t)\, dt = 0 + \frac{1}{8} + \frac{1}{3} + k = \frac{11}{24} + k \qquad (2)$

Differentiating both sides of (1), w.r.t. x, we get

$f(x) = -x^2 f(x) + 2x^{15} + 2x^5$

$$\Rightarrow\quad f(x) = \frac{2(x^{15}+x^5)}{1+x^2}$$

$$\therefore\quad \int_0^1 f(t)\, dt = 2\int_0^1 \frac{(t^{15}+t^5)}{1+t^2} = \frac{11}{24} + k \qquad \text{(using (2))}$$

$$\Rightarrow\quad 2\int_0^1 (t^{13} - t^{11} + t^9 - t^7 + t^5)\, dt = \frac{11}{24} + k$$

$$\Rightarrow\quad 2\left(\frac{1}{14} - \frac{1}{12} + \frac{1}{10} - \frac{1}{8} + \frac{1}{6}\right) = \frac{11}{24} + k$$

$$\Rightarrow\quad k = -\frac{167}{840}.$$

The correct option is (B)

86. We have, $f(x) = \dfrac{x-1}{x+1}$

$$\Rightarrow\quad f^2(x) = f(f(x)) = f\left(\frac{x-1}{x+1}\right) = \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1} = -\frac{1}{x}$$

$$\Rightarrow\quad f^4(x) = f^2(f^2(x)) = f^2\left(-\frac{1}{x}\right) = \frac{-1}{\frac{-1}{x}} = x$$

$\therefore \quad \phi(x) = f^{1998}(x) = f^2(f^{1996}(x)) = f^2(x)$

$$\left[\because f^{1996}(x) = \frac{(f^4(f^4(f^4...f^4)(x)}{499\text{ times}} = x\right]$$

$\Rightarrow \quad \phi(x) = -\frac{1}{x}.$

$$\therefore \int_{1/e}^{1} \phi(x)\,dx = \int_{1/e}^{1}\left(-\frac{1}{x}\right)dx = (\log_e x)\Big|_{1/e}^{1}$$

$$= -(\log_e 1 - \log_e 1/e) = -(0+1) = -1.$$

The correct option is (B)

87. We have, $g(2) = \int_0^2 f(t)\,dt$

$$= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \qquad \text{(i)}$$

Now, $\frac{1}{2} \le f(t) \le 1$, for $t \in [0, 1]$

and, $0 \le f(t) \le \frac{1}{2}$, for $t \in [1, 2]$

$\Rightarrow \quad \frac{1}{2}(1-0) \le \int_0^1 f(t)\,dt \le 1(1-0)$

and, $0\,(2-1) \le \int_1^2 f(t)\,dt \le \frac{1}{2}(2-1)$

$$[\because m \le f(x) \le M \text{ for } x \in [a, b]$$

$\Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)\,dt \le 1$ and $0 \le \int_1^2 f(t)\,dt \le \frac{1}{2}$

$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \le \frac{3}{2}$

$\Rightarrow \quad \frac{1}{2} \le \int_0^2 f(t)\,dt \le \frac{3}{2}$ or $\frac{1}{2} \le g(2) \le \frac{3}{2}.$

The correct option is (C)

88. $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \sin^{2k}\frac{r\pi}{2n}$

$$= \int_0^1 \sin^{2k}\frac{\pi x}{2}\,dx = \frac{2}{\pi}\int_0^{\pi/2} \sin^{2k} t\,dt$$

$$\left[\text{Putting } \frac{\pi x}{2} = t \Rightarrow dx = \frac{2}{\pi}dt\right]$$

$$= \frac{2}{\pi}\cdot\frac{(2k-1)(2k-3)\cdots\cdot 1}{2k\,(2k-2)\cdots\cdot 2}\cdot\frac{\pi}{2}$$

$$= \frac{[(2k-1)(2k-3)(2k-5)\cdots 1][2k\cdot(2k-2)\cdots\cdot 2]}{2^k\,[k\,(k-1)(k-2)\cdots 1][2k\cdot(2k-2)\cdots 2]}$$

$$= \frac{2k\,(2k-1)(2k-2)(2k-3)\cdots 2\cdot 1}{2^k\,[k\,(k-1)(k-2)\cdots 1]\cdot 2^k\,[k\cdot(k-1)(k-2)\cdots 1]}$$

$$= \frac{(2k)!}{2^{2k}\cdot(k!)^2}.$$

The correct option is (A)

89. We have, $\lim_{x\to 0}\left\{1 + \frac{f(x)}{x^3}\right\}^{1/x} = e$

$\Rightarrow \quad \lim_{x\to 0}\frac{f(x)}{x^3} = 0$ and $\lim_{x\to 0}\frac{f(x)}{x^4} = 1$

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + ...$, then

$\lim_{x\to 0}\frac{f(x)}{x^3} = 0 \Rightarrow a_0 = a_1 = a_2 = a_3 = 0.$

Also, $\lim_{x\to 0}\frac{f(x)}{x^4} = 1$

$\Rightarrow \quad \lim_{x\to 0} a_4 + a_5x + a_6x^2 + ... = 1$

$\Rightarrow \quad a_4 = 1$

Since $f(x)$ is a polynomial of least degree satisfying the given condition, therefore,

$a_5 = a_6 = 0.$

Thus, $f(x) = x^4$.

We have to find the area of the smaller region enclosed by $y = x^4$, $x^2 + y^2 = 2$ and x-axis.

Clearly, from the figure, required area

$$= 2\int_0^1\left(\sqrt{2-x^2} - x^4\right)dx$$

$$= 2\left[\frac{x}{2}\sqrt{2-x^2} + \frac{(\sqrt{2})^2}{2}\sin^{-1}\frac{x}{\sqrt{2}} - \frac{x^5}{5}\right]_0^1$$

$$= 2\left(\frac{1}{2} + \sin^{-1}\frac{1}{\sqrt{2}} - \frac{1}{5}\right) = 2\left(\frac{\pi}{4} + \frac{3}{10}\right)$$

$$= \left(\frac{\pi}{2} + \frac{3}{5}\right)\text{sq. units.}$$

The correct option is (B)

90. $\int_0^{41\pi/4} |\cos x|\,dx = \int_0^{10\pi} |\cos x|\,dx + \int_{10\pi}^{41\pi/4} |\cos x|\,dx$

$$= 10\int_0^{\pi} |\cos x|\,dx + \int_{10\pi}^{10\pi+\frac{\pi}{4}} |\cos x|\,dx$$

[Since | cos x| is a periodic function of period π]

$$= 10\int_0^{\pi} |\cos x|\,dx + \int_0^{\pi/4} |\cos x|\,dx$$

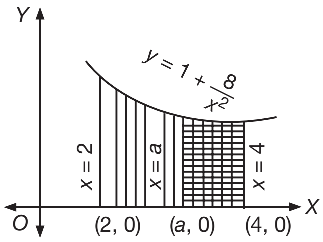
$$= 10\left(\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx\right) + \sin x\Big|_0^{\pi/4}$$

$$= 10\left(\sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^{\pi}\right) + \frac{1}{\sqrt{2}}$$

$$= 10\,(1 + 1) + \frac{1}{\sqrt{2}} = 20 + \frac{1}{\sqrt{2}}.$$

The correct option is (B)

91. The area bounded by the curve $y = 1 + \dfrac{8}{x^2}$, x-axis and the ordinates $x = 2$, $x = 4$ is



$$= \int_2^4 y\,dx = \int_2^4\left(1+\frac{8}{x^2}\right)dx = \left[x - \frac{8}{x}\right]_2^4$$

$= (4 - 2) - (2 - 4) = 4.$

Since $x = a$ divides this area into two equal parts,

$\therefore \quad 4 = 2\displaystyle\int_2^a\left(1+\frac{8}{x^2}\right)dx$

$\Rightarrow \quad 2 = \left[x - \dfrac{8}{x}\right]_2^a = \left(a - \dfrac{8}{a}\right) - (2 - 4)$

$\Rightarrow \quad a^2 = 8, \therefore a = 2.\sqrt{2}$

The correct option is (B)

92. Let $I = \displaystyle\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)\,dx\right)$

$\Rightarrow \quad I = \displaystyle\int_0^1 f(x)\,dx + \int_0^1 f(1+x)\,dx + \int_0^1 f(2+x)\,dx + \ldots + \int_0^1 f(99+x)\,dx$

$\Rightarrow \quad I = \displaystyle\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx \ldots + \int_{99}^{100} f(x)\,dx$

$\therefore \quad I = \displaystyle\int_0^{100} f(x)\,dx = a$ {given}

The correct option is (B)

93. We are given

$$\int_1^a [f(x) - 2]\,dx = \frac{2}{3}\left[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}\right].$$

Differentiating w.r.t. a, we get

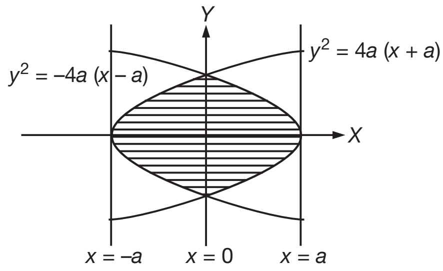
$$f(a) - 2 = \frac{2}{3}\left[\frac{3}{2}\sqrt{2a}\cdot 2 - 3\right].$$

$\Rightarrow \quad f(a) = 2\sqrt{2a}$, $a \ge 1$

$\therefore \quad f(x) = 2\sqrt{2x}$, $x \ge 1$.

The correct option is (A)

94. The required area



$$= 4\int_0^a \sqrt{4a\,(a-x)}\;dx$$

$$= 4\cdot 2\sqrt{a}\cdot\left[\frac{-2\,(a-x)^{3/2}}{3}\right]_0^a = \frac{16}{3}a^2.$$

The correct option is (A)

95. $\displaystyle\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi + 2 - t}\,dt = \frac{1}{2}\int_{4\pi-2}^{4\pi}\frac{\sin t/2}{1+\left(2\pi - \frac{t}{2}\right)}\,dt$

$$= 2\cdot\frac{1}{2}\int_0^1\frac{\sin\,(2\pi - u)}{1+u}\,du$$

$\left[\text{Putting } 2\pi - \dfrac{t}{2} = u \text{ so that } dt = -2\,du\right]$

$$= -\int_0^1 \frac{\sin u}{1+u}\,du = -\int_0^1\frac{\sin t}{1+t}\,dt = -\alpha$$

The correct option is (D)

96. We have, $\sin x < x$ for $x > 0$

$\Rightarrow \sin(\cos x) < \cos x$ for $0 < x < \pi/2$

$\Rightarrow \displaystyle\int_0^{\pi/2}\sin(\cos x)\,dx < \int_0^{\pi/2}\cos x\,dx$

$\therefore \quad I_3 > I_2 \qquad (1)$

Now, $\cos x < \cos\alpha$ if $x > \alpha$ and $x, \alpha \in \left[0, \dfrac{\pi}{2}\right]$

$\therefore \quad x > \sin x$

$\Rightarrow \cos x < \cos(\sin x)$

$\Rightarrow \int_0^{\pi/2} \cos x\, dx < \int_0^{\pi/2} \cos(\sin x)\, dx$

$\therefore \quad I_3 < I_1$ (2)

$\therefore$ from (1) and (2) $I_1 > I_3 > I_2$.

The correct option is (A)

97. $a_n + a_{n+2} = \int_0^{\pi/4} (\cot^n x + \cot^{n+2} x)\, dx,\ n \geq 2$

$$= \int_0^{\pi/4} \cot^n x \operatorname{cosec}^2 x\, dx$$

$$= \left[\frac{-\cot^{n+1} x}{n+1}\right]_0^{\pi/4} = \frac{-1}{n+1}$$

$\therefore a_2 + a_4, a_3 + a_5, a_4 + a_6$ are in H.P.

98. Since, $f(x) + f(y) = f(x+y)$ (1)

Replace y by $-x$

$\Rightarrow \quad f(x) + f(-x) = f(x-x)$

$\Rightarrow \quad f(x) + f(-x) = f(0)$ (2)

Also, using (1), we have

$f(0) + f(0) = f(0+0) = f(0)$

$\Rightarrow \quad f(0) = 0$

$\therefore \quad f(-x) = -f(x)$ {using (2)}

$\Rightarrow \quad \int_{-2}^{2} f(x)\, dx = 0$

The correct option is (B)

99. Let $I_n = \int_0^{\infty} x^n e^{-ax}\, dx$

$$= \left[x^n \cdot \frac{e^{-ax}}{-a}\right]_0^{\infty} - \int_0^{\infty} nx^{n-1} \cdot \frac{e^{-ax}}{-a}\, dx$$

$$= -\frac{1}{a} \lim_{x\to\infty} \frac{x^n}{e^{ax}} + \frac{n}{a} I_{n-1}$$

$\therefore \quad I_n = \frac{n}{a} I_{n-1}$ $\left[\because \lim_{x\to\infty} \frac{x^n}{e^{ax}} = 0\right]$

$$= \frac{n}{a} \cdot \frac{n-1}{a} I_{n-2}$$

$$= \frac{n(n-1)(n-2)}{a^3} I_{n-3}$$

....

....

$$= \frac{n!}{a^n} \int_0^{\infty} e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$

The correct option is (C)

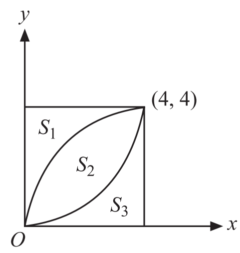
100. Let $A = \int_{\pi/4}^{\beta} f(x)\, dx = \beta \sin \beta + \frac{\pi}{4} \cos \beta + \sqrt{2}\beta$.

Then, $\frac{dA}{d\beta} = f(\beta) = \sin \beta + \beta \cos \beta - \frac{\pi}{4} \sin \beta + \sqrt{2}$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{4} + \sqrt{2}$$

The correct option is (B)

101.



$S_3 = \int_0^4 \frac{x^2}{4}\, dx = \frac{64}{12}$, $S_1 = \int_0^4 \frac{y^2}{4}\, dy = \frac{64}{12}$,

$S_2 = 16 - S_1 - S_3 = 16 - \frac{128}{12} = \frac{64}{12}$

The correct option is (B)

102. $\int_1^a [x] f'(x)\, dx$

$$= \int_1^2 f'(x)\, dx + 2\int_2^3 f'(x)\, dx + 3\int_3^4 f'(x)\, dx + \ldots +$$

$$\int_{[a]-1}^{[a]} ([a]-1) f'(x)\, dx + [a]\int_{[a]}^{a} f'(x)\, dx$$

$= (f(2) - f(1)) + 2(f(3) - f(2)) + 3(f(4) - f(3)) + \ldots$
$+ [a]\,(f(a) - f([a]))$

$= [a] f(a) - \{f(1) + f(2) + f(3) + \ldots + f([a])\}$

The correct option is (C)

103. We have, $I_1 = \int_0^a [x]\, dx$

$$= \int_0^1 0\, dx + \int_1^2 1\, dx + \int_2^3 2\, dx + \cdots + \int_{a-1}^{a} (a-1)\, dx$$

$= 1 + 2 + \cdots + (a-1) = \frac{a(a-1)}{2}$ (1)

$$I_2 = \int_0^a \{x\}\, dx = \int_0^a (x - [x])\, dx = \int_0^a x\, dx - \int_0^a [x]\, dx$$

$= \left.\frac{x^2}{2}\right|_0^a - \frac{a(a-1)}{2} = \frac{a^2}{2} - \frac{a(a-1)}{2} = \frac{a}{2}$ (2)

From (1) and (2), we have

$\frac{I_1}{I_2} = (a-1) \cdot \therefore I_1 = (a-1)\, I_2.$

The correct option is (B)

104. Let $f(x) = \dfrac{1}{1+x^2+2x^5}, x \in [0, 1]$

In the interval [0, 1], $f(x)$ is strictly decreasing, therefore, we have,

$f(1) \le f(x) \le f(0)$, i.e., $\frac{1}{4} \le f(x) \le 1$

$$\therefore \quad (1-0)\frac{1}{4} \le \int_0^1 f(x)\,dx \le (1-0)\,1$$

i.e., $\frac{1}{4} \le \int_0^1 f(x)\,dx \le 1$

The correct option is (A)

105. We have,

$1 + x^2 + 2x^5 \ge 1 + x^2$

and $1 + x^2 + 2x^5 \le 1 + x^2 + 2x^5 = 1 + 3x^2$

$$\therefore \quad \frac{1}{1+3x^2} \le \frac{1}{1+x^2+2x^5} \le \frac{1}{1+x^2}$$

$$\Rightarrow \int_0^1 \frac{dx}{1+3x^2} \le \int_0^1 \frac{dx}{1+x^2+2x^5} \le \int_0^1 \frac{dx}{1+x^2}$$

$$\Rightarrow \left[\frac{\tan^{-1}\sqrt{3}x}{\sqrt{3}}\right]_0^1 \le \int_0^1 \frac{dx}{1+x^2+2x^5} \le [\tan^{-1}x]_0^1$$

$$\Rightarrow \frac{\pi}{3\sqrt{3}} \le \int_0^1 \frac{dx}{1+x^2+2x^5} \le \frac{\pi}{4}$$

The correct option is (C)

106. We have,

$$I = \int_0^{\sin^2 x} \sin^{-1}\left(\sqrt{t}\right) dt + \int_0^{\cos^2 x} \cos^{-1}\left(\sqrt{t}\right) dt$$

$$= \left[t \sin^{-1}\left(\sqrt{t}\right)\right]_0^{\sin^2 x} - \int_0^{\sin^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt$$

$$+ \left[t \cos^{-1}\left(\sqrt{t}\right)\right]_0^{\cos^2 x} - \int_0^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt$$

$$= x\,(\sin^2 x + \cos^2 x) + \int_{\sin^2 x}^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt$$

$$= x\sin^2 x + \int_{\sin^2 x}^{0} \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt + x\cos^2 x + \int_0^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt$$

$$= x\,(\sin^2 x + \cos^2 x) + \int_{\sin^2 x}^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt$$

Putting $t = \sin^2\theta$ and $dt = 2\sin\theta\cos\theta\,d\theta$, we get,

$$\int \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt = \int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\,2\sin\theta\cos\theta\,d\theta$$

$$= \int \sin^2\theta\,d\theta = \int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}$$

Also, when $t = \sin^2 x$, $\theta = x$ and when $t = \cos^2 x$, $\theta = \pi/2 - x$

$$\therefore \quad I = x + \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_x^{\pi/2 - x}$$

$$= x + \left(\frac{\pi}{4} - \frac{x}{2} - \frac{\sin 2x}{4}\right) - \left(\frac{x}{2} - \frac{\sin 2x}{4}\right)$$

$$= x + \frac{\pi}{4} - x = \frac{\pi}{4}$$

The correct option is (A)

107. Differentiating the given equation w.r.t. x, we get

$$f(x) + f'''(3) = x^3 + x^2 f'(1) + x f''(2) \qquad (1)$$

Differentiating successively w.r.t. x, we get

$$f'(x) = 3x^2 + 2x f'(1) + f''(2) \qquad (2)$$

$$f''(x) = 6x + 2 f'(1) \qquad (3)$$

$$f'''(x) = 6 \qquad (4)$$

Putting x = 1, 2 and 3 in equations (2), (3) and (4) respectively, we get

$$f'(1) = 3 + 2f'(1) + f''(2), f''(2) = 12 + 2f'(1)$$

and, $f'''(3) = 6$

Solving, we have

$$f'(1) = -5, f''(2) = 2, f'''(3) = 6$$

Putting the values in equation (1), we have

$$f(x) = x^3 - 5x^2 + 2x - 6.$$

The correct option is (D)

108. $$\lim_{x\to 0} \frac{\int_0^{x+y} e^{\sin^2 t}dt - \int_0^{y} e^{\sin^2 t}dt}{x} = \lim_{x\to 0} \frac{\int_y^{0} e^{\sin^2 t}dt + \int_0^{x+y} e^{\sin^2 t}dt}{x}$$

$$= \lim_{x\to 0} \frac{\int_y^{x+y} e^{\sin^2 t}dt}{x} = \lim_{x\to 0} \frac{e^{\sin^2(x+y)} \cdot \frac{d}{dx}(x+y) - e^{\sin^2 y} \cdot \frac{dy}{dx}}{1}$$

$$= \lim_{x\to 0} \frac{e^{\sin^2(x+y)} \bullet 1 - e^{\sin^2 y} \bullet 0}{1} = e^{\sin 2y}$$

The correct option is (A)

109. $$\int_0^5 \frac{\tan^{-1}(x-[x])}{1+(x-[x])^2}\,dx$$

$$= \int_0^5 \frac{\tan^{-1}(x-[x])}{1+(x-[x])^2}\,dx$$

$$= \int_0^1 \frac{\tan^{-1} x}{1+x^2}\,dx + \int_1^2 \frac{\tan^{-1}(x-1)}{1+(x-1)^2}\,dx + \ldots + \int_4^5 \frac{\tan^{-1}(x-4)}{1+(x-4)^2}\,dx$$

$$= \int_0^1 \frac{\tan^{-1} x}{1+x^2}\,dx + \int_0^1 \frac{\tan^{-1} t}{1+t^2}\,dt + \ldots + \int_0^1 \frac{\tan^{-1} t}{1+t^2}\,dt$$

(Putting $x - 1 = t$) (Putting $x - 4 = t$)

$$= 5\int_0^1 \frac{\tan^{-1} x}{1+x^2}\,dx = 5\int_0^{\pi/4} u\,du \qquad [\text{Putting } \tan^{-1} x = u]$$

$$= 5\left[\frac{u^2}{2}\right]_0^{\pi/2} = \frac{5\pi^2}{32}$$

The correct option is (C)

110. We have,

$$\int_0^x \left(x - [x] - \frac{1}{2}\right)dx = \int_0^{[x]+\{x\}} \left(\{x\} - \frac{1}{2}\right)dx$$

$$= \int_0^{[x]} \left(\{x\} - \frac{1}{2}\right)dx + \int_{[x]}^{[x]+\{x\}} \left(\{x\} - \frac{1}{2}\right)dx$$

$$= [x]\int_0^1 \left(\{x\} - \frac{1}{2}\right)dx + \int_0^{\{x\}} \left(\{x\} - \frac{1}{2}\right)dx$$

[$\because$ $\{x\}$ has period 1]

$$= [x]\int_0^1 \left(x - \frac{1}{2}\right)dx + \int_0^{\{x\}} \left(x - \frac{1}{2}\right)dx$$

$$= [x]\left[\frac{x^2}{2} - \frac{x}{2}\right]_0^1 + \left[\frac{x^2}{2} - \frac{x}{2}\right]_0^{\{x\}}$$

$$= [x]\left(\frac{1}{2} - \frac{1}{2}\right) + \frac{\{x\}(\{x\}-1)}{2}$$

$$= \frac{1}{2}\{x\}(\{x\}-1).$$

The correct option is (A)

111. We have,

$$\int_0^{k\pi} \sin\left[\frac{2x}{\pi}\right]dx$$

$$= \int_0^{\pi/2} \sin 0\,dx + \int_{\pi/2}^{2\pi/2} \sin 1\,dx + \int_{2\pi/2}^{3\pi/2} \sin 2\,dx + \ldots$$

$$+ \int_{(2k-1)\pi/2}^{2k\pi/2} \sin(2k-1)\,dx$$

$$= \frac{\pi}{2}[\sin 1 + \sin 2 + \sin 3 + \ldots + \sin(2k-1)]$$

$$= \frac{\frac{\pi}{2}\left[\sin\frac{1}{2}\sin 1 + \sin\frac{1}{2}\sin 2 + \sin\frac{1}{2}\sin 3 + \ldots + \sin\frac{1}{2}\sin(2k-1)\right]}{\sin\frac{1}{2}}$$

$$= \frac{\frac{\pi}{2}\left[\cos\frac{1}{2} - \cos\frac{3}{2} + \cos\frac{3}{2} - \cos\frac{5}{2} + \ldots + \cos\left(2k - \frac{3}{2}\right) - \cos\left(2k + \frac{1}{2}\right)\right]}{2\sin\frac{1}{2}}$$

$$= \frac{\frac{\pi}{2}\left(\cos\frac{1}{2} - \cos\left(2k + \frac{1}{2}\right)\right)}{2\sin\frac{1}{2}}$$

$$= \frac{\pi}{2}\cdot\frac{\sin k\,.\,\sin\left(k + \frac{1}{2}\right)}{\sin\left(\frac{1}{2}\right)}$$

$\therefore$ $A = \pi/2$.

The correct option is (C)

112. We have,

$$f(x) = \int_0^x (1+t^3)^{-1/2}\,dt$$

$$\Rightarrow \quad f(g(x)) = \int_0^{g(x)} (1+t^3)^{-1/2}\,dt$$

$$\Rightarrow \quad x = \int_0^{g(x)} (1+t^3)^{-1/2}\,dt$$

[g is inverse of $f \Rightarrow f\{g(x)\} = x$]

Differentiating w.r.t. x, we have

$$1 = (1+g^3)^{-1/2}\cdot g'$$

i.e., $(g')^2 = 1 + g^3$

Differentiating again w.r.t. x, we have

$$2g'g'' = 3g^2g'$$

$$\Rightarrow \quad \frac{g''}{g^2} = \frac{3}{2}.$$

The correct option is (B)

113. $$I_n = \int_0^{\pi/2} \cos^n x \cos nx\,dx$$

$$= \left[\cos^n x \cdot \frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} n\cos^{n-1}$$

$$= 0 + \int_0^{\pi/2} \cos^{n-1} x \sin x \sin nx\,dx$$

$$= \int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x - \int_0^{\pi/2} \cos^n x \cos nx \, dx$$

[Using the identity
$\cos(n-1)x = \cos nx \cos x + \sin nx \sin x$
i.e., $\sin nx \sin x = \cos(n-1)x - \cos nx \cos x$]

$$= \int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x - \int_0^{\pi/2} \cos^n x \cos nx \, dx$$

$$= I_{n-1} - I_n$$

i.e., $\dfrac{I_n}{I_{n-1}} = \dfrac{1}{2} \Rightarrow I_1, I_2, I_3$ are in GP

The correct option is (B)

114. We have,

$$I_{n+1} = \int_0^1 (1-x^a)^{n+1} \, dx$$

$$= \left[x(1-x^a)^{n+1}\right]_0^1 + (n+1)a \int_0^1 x^a (1-x^a)^n \, dx$$

$$= (n+1)a \int_0^1 (x^a - 1 + 1)(1-x^a)^n dx$$

$$= (n+1)a \int_0^1 (1-x^a)^n dx - (n+1)a \int_0^1 (1-x^a)^{n+1} \, dx$$

$$= (n+1)aI_n - (n+1)\, a\, I_{n+1}$$

$$\Rightarrow \frac{I_n}{I_{n+1}} = 1 + \frac{1}{(n+1)a}$$

$\therefore \quad k = (n+1)a.$

The correct option is (A)

115. Let $S = \lim\limits_{n\to\infty} \dfrac{(n!)^{1/n}}{n}$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{n}{n}\right]^{1/n}$$

Taking log on both sides, we get

$$\log S = \lim_{n\to\infty} \frac{1}{n}\left[\log\frac{1}{n} + \log\frac{2}{n} + \ldots + \log\frac{n}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(\frac{r}{n}\right) = \int_0^1 \log x \, dx$$

$$= \left[x \log x - x\right]_0^1 = -1$$

$\therefore \; S = e^{-1}.$

The correct option is (B)

116. We have,

$$\lim_{n\to\infty} \frac{1 + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n-1}}{\sqrt[3]{n^4}}$$

$$= \lim_{n\to\infty} \sum_{1}^{n-1} \frac{\sqrt[3]{r}}{\sqrt[3]{n^4}} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \sqrt[3]{\frac{r}{n}}$$

(adding one term will not affect the limit)

$$= \int_0^1 \sqrt[3]{x} = \left[\frac{x^{4/3}}{4/3}\right]_0^1 = \frac{3}{4}.$$

The correct option is (C)

117. We have,

$$\phi(4) = \int_0^4 g(t)dt = \int_0^1 g(t)dt + \int_1^3 g(t)dt + \int_3^4 g(t)dt$$

But

$$\frac{-1}{2} \cdot 1 \le \int_0^1 g(t)dt \le 0.1$$

$$\frac{1}{2} \cdot 2 \le \int_1^3 g(t)dt \le 0.2$$

$$\int_3^4 g(t)dt \le 1.1$$

Adding the above inequalities, we get $\phi(4) \le 3$.

The correct option is (C)

118. We have,

$$\int_1^4 (\{x\})^{[x]} dx$$

$$= \int_1^4 (x-[x])^{[x]} dx$$

$$= \int_1^2 (x-[x])^{[x]} dx + \int_2^3 (x-[x])^{[x]} dx + \int_3^4 (x-[x])^{[x]} dx$$

$$= \int_1^2 (x-1)^1 dx + \int_2^3 (x-2)^2 dx + \int_3^4 (x-3)^3 dx$$

$$= \left[\frac{(x-1)^2}{2}\right]_1^2 + \left[\frac{(x-2)^3}{3}\right]_2^3 + \left[\frac{(x-3)^4}{4}\right]_3^4$$

$$= \left(\frac{1}{2} - 0\right) + \left(\frac{1}{3} - 0\right) + \left(\frac{1}{4} - 0\right) = \frac{13}{12}.$$

The correct option is (C)

119. $I = \displaystyle\int_0^2 [x + [x + [x]]]dx$

$$= \int_0^2 [x + 2[x]]dx$$

($\because$ [x + Integer] = [x] + Integer $\Rightarrow$ [x + [x]] = [x] + [x])

$= \int_0^2 [x] + 2[x]dx = \int_0^2 3[x]dx$

$= 3\left\{\int_0^1 [x]dx + \int_1^2 [x]dx\right\}$

$= 3\left\{\int_0^1 0.dx + \int_1^2 1dx\right\}$

$= 3 \{(x)_1^2\} = 3 (2 - 1) = 3.$

The correct option is (C)

120. Let $I = \int_{\sqrt{(3a^2+b^2)/4}}^{\sqrt{(a^2+b^2)/2}} \frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}} dx$

Put $x^2 = a^2\cos^2 t + b^2\sin^2 t$

$\Rightarrow$ $2\,x\,dx = [2a^2\cos t\,(-\sin t) + 2\,b^2\sin t\,(\cos t)]\,dt$

$\Rightarrow$ $x\,dx = \frac{1}{2}(b^2 - a^2)\sin 2t \cdot dt$

For $x^2 = \frac{a^2+b^2}{2} = a^2\cos^2 t + b^2\sin^2 t$

$\Rightarrow$ $a^2 + b^2 = 2(1 - \sin^2 t)\,a^2 + 2b^2\sin^2 t$

or, $(a^2 + b^2) = 2a^2 + 2(b^2 - a^2)\sin^2 t$

$\Rightarrow$ $\sin^2 t = \frac{1}{2} \Rightarrow \cos 2t = 0 \Rightarrow t = \pi/4$

For $x^2 = \frac{3a^2+b^2}{4} = a^2\cos^2 t + b^2\sin^2 t$

$\Rightarrow$ $3a^2 + b^2 = 4a^2 + 4(b^2 - a^2)\sin^2 t$

$\Rightarrow$ $\sin^2 t = \frac{1}{4} \Rightarrow \cos 2t = \frac{1}{2} \Rightarrow t = \frac{\pi}{4}$

$\therefore$ $I = \int_{\pi/6}^{\pi/4} \frac{1}{2}\frac{(b^2-a^2)\sin 2t\,.dt}{\sqrt{(b^2-a^2)\sin^2 t\,(b^2-a)\cos^2 t}}$

$= \int_{\pi/6}^{\pi/4} dt = (t)_{\pi/6}^{\pi/4} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$

The correct option is (D)

121. $S = \lim_{n\to\infty}\left(\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdot\sin\frac{3\pi}{2n}\cdots\sin\frac{(n-1)\pi}{n}\right)^{1/n}$

$\Rightarrow \log S = \lim_{n\to\infty}\frac{1}{n}\log\left(\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdots\sin\frac{(n-1)\pi}{n}\right)$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2(n-1)}\log\sin\frac{r\pi}{2n}$

$= \int_0^2 \log\sin\left(\frac{\pi}{2}x\right)dx$

$= \frac{2}{\pi}\int_0^{\pi}\log(\sin t)\,dt$ $\left(\text{Put, } \frac{\pi}{2}x = t\right)$

$= \frac{2}{\pi}\cdot 2\int_0^{\pi/2}\log(\sin t)\,dt$

$= \frac{4}{\pi}\left\{-\frac{\pi}{2}\log 2\right\}$

$\left\{\text{using, } \int_0^{\pi/2}\log(\sin t)\,dt = -\frac{\pi}{2}\log 2\right\}$

$= -2\log 2$

$\therefore$ $\log S = \log\left(\frac{1}{4}\right) \Rightarrow S = \frac{1}{4}.$

The correct option is (A)

122. Let $I = \int_{-2\pi}^{5\pi}\cot^{-1}(\tan x)\,dx$

$= 7\int_0^{\pi}\cot^{-1}\left(\cot(\pi/2 - x)\right)dx$ (1)

($\because$ Period is π)

Since $\cot^{-1}(\cot x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi + x, & \pi/2 < x < \pi \end{cases}$

$\therefore$ $I = 7\left\{\int_0^{\pi/2}\left(\frac{\pi}{2} - x\right)dx + \int_{\pi/2}^{\pi}\left(\pi + \frac{\pi}{2} - x\right)dx\right\}$

$= 7\left\{\left(\frac{\pi}{2}x - \frac{x^2}{2}\right)_0^{\pi/2} + \left(\frac{3\pi}{2}x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right\}$

$= 7\left\{\left(\frac{\pi^2}{4} - \frac{\pi^2}{8}\right) + \left(\frac{3\pi^2}{2} - \frac{\pi^2}{2} - \frac{3\pi^2}{4} + \frac{\pi^2}{8}\right)\right\}$

$= \frac{7\pi^2}{2}.$

The correct option is (B)

123. Let $I = \int_0^{\sqrt{3}}\frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Now, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & \text{if } -1 \le x \le 1 \\ \pi - 2\tan^{-1}x, & \text{if } x > 1 \end{cases}$

$\therefore$ $I = \int_0^1 \frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

$+ \int_1^{\sqrt{3}}\frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

$= \int_0^1 \frac{2\tan^{-1}x}{1+x^2}dx + \int_1^{\sqrt{3}}\frac{\pi - 2\tan^{-1}x}{1+x^2}dx$

$= 2\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx + \pi\int_1^{\sqrt{3}}\frac{1}{1+x^2}dx - 2\int_1^{\sqrt{3}}\frac{\tan^{-1}x}{1+x^2}dx$

$$= 2\int_0^{\pi/4} t\,dt + \pi(\tan^{-1}x)_1^{\sqrt{3}} - 2\int_{\pi/4}^{\pi/3} t\,dt,$$

(Put $\tan^{-1}x = t$)

$$= 2\left\{\frac{t^2}{2}\right\}_0^{\pi/4} + \pi\{\tan^{-1}\sqrt{3} - \tan^{-1}1\} - 2\left\{\frac{t^2}{2}\right\}_{\pi/4}^{\pi/3}$$

$$= \frac{\pi^2}{16} + \pi\left\{\frac{\pi}{3} - \frac{\pi}{4}\right\} - \left\{\frac{\pi^2}{9} - \frac{\pi^2}{16}\right\} = \frac{7}{72}\pi^2.$$

The correct option is (A)

124. Let $f(x) = \dfrac{x}{x^2+16}$

$\therefore\quad f'(x) = \dfrac{(x^2+16)\cdot 1 - x\cdot 2x}{(x^2+16)^2}$

$= \dfrac{16-x^2}{(x^2+16)^2} \geq 0$

$\Rightarrow\quad 16 \geq x^2 \Rightarrow x^2 \leq 16 \Rightarrow -4 \leq x \leq 4$

$\therefore$ $f(x)$ is monotonic increasing in $[-4, 4]$. Since $[0, 1], \subseteq [-4, 4]$

$\therefore$ $f(x)$ is monotonic increasing in $[0, 1]$

$\therefore\quad M = \dfrac{1}{1+16} = \dfrac{1}{17}$ and $m = \dfrac{0}{0+16} = 0$

$\therefore\quad m(1-0) \leq \displaystyle\int_0^1 f(x)\,dx \leq M(1-0)$

$\Rightarrow\quad 0(1-0) \leq \displaystyle\int_0^1 \frac{x}{x^2+16}dx \leq \frac{1}{17}(1-0)$

$\Rightarrow\quad 0 \leq \displaystyle\int_0^1 \frac{x\,dx}{x^2+16} \leq \frac{1}{17}$

$\therefore$ The smallest such interval is $\left[0, \dfrac{1}{17}\right]$.

The correct option is (C)

125. Let $f(x) = \dfrac{1}{\sqrt{4-x^2-x^3}}$

Since $4 - x^2 \geq 4 - x^2 - x^3 \geq 4 - 2x^2 > 1\ \forall x \in [0, 1]$

$\therefore\quad \sqrt{4-x^2} \geq \sqrt{4-x^2-x^3} \geq \sqrt{4-2x^2} > 1\ \forall\ x \in [0, 1]$

$\Rightarrow\quad \dfrac{1}{\sqrt{4-x^2}} \leq \dfrac{1}{\sqrt{4-x^2-x^3}} \leq \dfrac{1}{\sqrt{4-2x^2}}\ \forall\ x \in [0, 1]$

$\Rightarrow\quad \displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}} \leq \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leq \int_0^1 \frac{dx}{\sqrt{2-2x^2}}$

$\Rightarrow\quad \left|\sin^{-1}\dfrac{x}{2}\right|_0^1 \leq \displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leq \frac{1}{\sqrt{2}}\left|\sin^{-1}\frac{x}{\sqrt{2}}\right|_0^1$

$\dfrac{\pi}{6} \leq \displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leq \frac{\pi}{4\sqrt{2}}.$

The correct option is (B)

126. Let, $\quad I_n = \displaystyle\int_0^1 e^x(x-1)^n\,dx$

Put $x - 1 = t \Rightarrow dx = dt$

$\therefore\quad I_n = \displaystyle\int_{-1}^0 e^{t+1}\cdot t^n\,dt$

$= e\left[\left|t^n e^t\right|_{-1}^0 - \displaystyle\int_{-1}^0 n t^{n-1} e^t\,dt\right]$

$= e\left[0 - (-1)^n e^{-1} - n\displaystyle\int_{-1}^0 t^{n-1} e^t\,dt\right]$

$= (-1)^{n+1} - ne\displaystyle\int_{-1}^0 t^{n-1} e^t\,dt = (-1)^{n+1} - nI_{n-1}$

Putting $n = 1$, we get

$I_1 = (-1)^2 - I_0 = 1 - (e-1) = 2 - e$

$I_2 = (-1)^{2+1} - 2I_1 = 2e - 5$

$I_3 = (-1)^{3+1} - 3I_2 = 1 - 3(2e-5) = 16 - 6e$

Hence, $n = 3$.

The correct option is (C)

127. We have,

$f'(x) = x(x^2 - 3x + 2) = x(x-1)(x-2)$

Clearly, $f'(x) \leq 0$ in $1 \leq x \leq 2$ and $f'(x) \geq 0$ in $2 \leq x \leq 3$.

$\therefore$ $f'(x)$ is monotonic decreasing in $[1, 2]$ and monotonic increasing in $[2, 3]$.

$\therefore$ Min. $f(x) = f(2) = \displaystyle\int_1^2 x(x^2-3x+2)\,dx$

$= \left|\dfrac{x^4}{4} - x^3 + x^2\right|_1^2 = \dfrac{-1}{4}$

Max. $\quad f(x) = $ the greatest among $(f(1), f(3))$

Now, $\quad f(1) = \displaystyle\int_1^1 x(x^2-3x+2)\,dx = 0$

$f(3) = \displaystyle\int_1^3 x(x^2-3x+2)\,dx$

$= \left.\dfrac{x^4}{4} - x^3 + 2\right|_1^3 = 2.\ \therefore$ Max. $f(x) = 2$

Hence, Range $= \left[\dfrac{-1}{4}, 2\right]$.

The correct option is (A)

128. Since $\left|\sqrt{2x-\frac{5}{2}}-\sqrt{\frac{5}{2}}\right|^2$

$$= 2x-\frac{5}{2}+\frac{5}{2}-2\sqrt{\left(2x-\frac{5}{2}\right)\frac{5}{2}} = 2x-\sqrt{(4x-5)5}$$

$\therefore$ Given Integral

$$= \int_2^3 \left|\sqrt{2x-\frac{5}{2}}-\sqrt{\frac{5}{2}}\right|dx + \int_2^3 \left|\sqrt{2x-\frac{5}{2}}+\sqrt{\frac{5}{2}}\right|dx$$

$$= 2\int_2^{5/2}\sqrt{\frac{5}{2}}\,dx + 2\int_{5/2}^3 \sqrt{2x-5/2}\,dx$$

$$\left[\because \sqrt{2x-\frac{5}{2}} > \sqrt{\frac{5}{2}} \Rightarrow 2x-\frac{5}{2} > \frac{5}{2} \Rightarrow 2x > 5 \Rightarrow x > \frac{5}{2}\right]$$

$$= 2\sqrt{\frac{5}{2}}\left(\frac{5}{2}-2\right)+2\left|\frac{(2x-5/2)^{3/2}}{\frac{3}{2}\cdot 2}\right|_{5/2}^3$$

$$= \sqrt{\frac{5}{2}}(1)+\frac{2}{3}\left[(6-5/2)^{3/2}-(5/2)^{3/2}\right]$$

$$= \sqrt{\frac{5}{2}}+\frac{2}{3}\left(\frac{7}{2}\sqrt{\frac{7}{2}}\right)-\frac{2}{3}\frac{5\sqrt{5}}{2\sqrt{2}}$$

$$= \frac{\sqrt{5}}{\sqrt{2}}+\frac{7\sqrt{7}}{3\sqrt{2}}-\frac{5\sqrt{5}}{3\sqrt{2}}$$

$$= \frac{3\sqrt{5}+7\sqrt{7}-5\sqrt{5}}{3\sqrt{2}} = \frac{7\sqrt{7}-2\sqrt{5}}{3\sqrt{2}}.$$

The correct option is (B)

129. Since, $\int_{-a}^{a} x f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx$

$$\therefore \int_{-a}^{a}\frac{f(x)}{b^{g(x)}+1}dx = \int_0^a \frac{f(x)}{b^{g(x)}+1}+\int_0^a\frac{f(-x)}{b^{g(-x)}+1}$$

$$= \int_0^a \frac{f(x)}{b^{g(x)}+1}dx+\int_0^a\frac{f(x)}{b^{-g(x)}+1}$$

$$= \int_0^a f(x)dx, \text{ which is independent of } g.$$

The correct option is (B)

130. We have,

$$f(x) = \int_1^x \frac{\ln t}{1+t}dt,\ x>0 \qquad (1)$$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = \int_1^{1/x}\frac{\ln t}{1+t}dt$$

Put $\quad y=\frac{1}{t} \Rightarrow dt = \frac{-1}{y^2}dy$

$$\therefore\ f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln\left(\frac{1}{y}\right)}{1+\frac{1}{y}}\left(\frac{-1}{y^2}\right)dy$$

$$= \int_1^x \frac{\ln y}{y(1+y)}dy$$

$$= \int_1^x \frac{\ln t}{(1+t)t}dt \qquad (2)$$

From (1) and (2),

$$f(x)+f\left(\frac{1}{x}\right) = \int_1^x \frac{\left(1+\frac{1}{t}\right)\ln t}{1+t}dt$$

$$= \int_1^x \frac{\ln t}{t}dt = \frac{(\ln x)^2}{2}$$

$$\Rightarrow \quad f(e)+f\left(\frac{1}{e}\right) = \frac{(\ln e)^2}{2} = \frac{1}{2}.$$

The correct option is (C)

131. $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta$

$$= \text{Real part of } \int_0^{2\pi} e^{\cos\theta}\{\cos(\sin\theta)+i\sin(\sin\theta)\}\,d\theta$$

$$= \text{Real part of } \int_0^{2\pi} e^{\cos\theta}\,e^{i\sin\theta}$$

$$= \text{Real part of } \int_0^{2\pi} e^{\cos\theta+i\sin\theta}\,d\theta$$

$$= \text{Real part of } \int_0^{2\pi} e^{e^{i\theta}}\,d\theta$$

$$= \text{Real part of } \int_0^{2\pi}\left[1+e^{i\theta}+\frac{e^{2i\theta}}{2!}+\frac{e^{3i\theta}}{3!}+\ldots\right]d\theta$$

$$= \text{Real part of } \int_0^{2\pi}\left[1+(\cos\theta+i\sin\theta) + \frac{1}{2!}(\cos 2\theta+i\sin 2\theta)+\ldots\right]d\theta$$

$$= \int_0^{2\pi}\left[1+\cos\theta+\frac{1}{2!}\cos 2\theta+\ldots\right]d\theta$$

$$= \left[\theta+\sin\theta+\frac{\sin 2\theta}{2.2!}+\ldots\right]_0^{2\pi} = 2\pi.$$

The correct option is (C)

132. Given $f(x) + f(x + 6) = f(x + 3) + f(x + 9)$

Put $x = x + 3$, we get

$$f(x + 3) + f(x + 9) = f(x + 6) + f(x + 12)$$

$\Rightarrow \quad f(x) = f(x + 12)$

Let $g(x) = \int\limits_x^{x+12} f(t)\,dt \Rightarrow g'(x) = f(x + 12) - f(x) = 0$

$\Rightarrow g(x)$ is a constant function.

The correct option is (C)

133. Given

$$f(x) = x + x\int_0^1 y^2 f(y)\,dy - x^2\int_0^1 y\,f(y)\,dy$$

$$= x\left(1 + \int_0^1 y^2 f(y)\,dy\right) - x^2\left(\int_0^1 y\,f(y)\,dy\right)$$

$\Rightarrow f(x)$ is a quadratic expression;

$\Rightarrow f(x) = ax + bx^2$ or $f(y) = ay + by^2$ (1)

where, $a = 1 + \int\limits_0^1 y^2\, f(y)\,dy$

$$= 1 + \int_0^1 y^2(ay + by^2)\,dy$$

$$= 1 + \left(\frac{ay^4}{4} + \frac{by^5}{5}\right)_0^1 = 1 + \left(\frac{a}{4} + \frac{b}{5}\right)$$

$\Rightarrow \quad 20a = 20 + 5a + 4b$ or $15a - 4b = 20$ (2)

and, $b = \int\limits_0^1 y\,f(y)\,dy = \int\limits_0^1 y\cdot(ay + by^2)\,dy$

$$= \left(\frac{ay^3}{3} + \frac{by^4}{4}\right)_0^1 = \frac{a}{3} + \frac{b}{4}$$

$\Rightarrow \quad 12b = 4a + 3b$ or $9b - 4a = 0$ (3)

From (2) and (3),

$$a = \frac{180}{119},\ b = \frac{80}{119}$$

$\therefore$ Equation (1) reduces to

$$f(x) = \frac{80x^2 + 180x}{119}$$

$$\therefore \quad f'(x) = \frac{160x + 180}{119} = 0 \Rightarrow x = \frac{-9}{8}$$

and, $f''(x) = \dfrac{160}{119} > 0 \Rightarrow f(x)$ attains minimum at $x = \dfrac{-9}{8}$

The correct option is (D)

134. Given limit $= \lim\limits_{n\to\infty} \dfrac{\sum\limits_{r=1}^{n} r^2 \cdot \sum\limits_{r=1}^{n} r^3}{\sum\limits_{r=1}^{n} r^6}$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}\sum\limits_{r=1}^{n}\left(\frac{r}{n}\right)^2 \bullet \frac{1}{n}\sum\limits_{r=1}^{n}\left(\frac{r}{n}\right)^3}{\frac{1}{n}\sum\limits_{r=1}^{n}\left(\frac{r}{n}\right)^6}$$

$$= \frac{\int_0^1 x^2\,dx \cdot \int_0^1 x^3\,dx}{\int_0^1 x^6\,dx} = \frac{7}{12}$$

The correct option is (A)

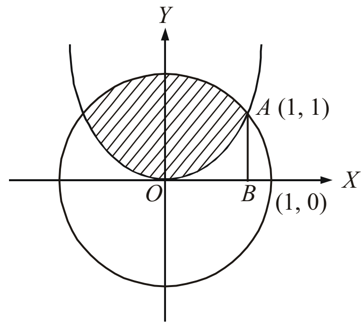
135. Since, $\lim\limits_{x\to 0}\left[1 + \dfrac{f(x)}{x^3}\right]^{1/x}$ exists, so $\lim\limits_{x\to 0}\dfrac{f(x)}{x^3} = 0$

$\therefore f(x) = a_4x^4 + a_5x^5 + \ldots + a_nx^n,\ a_n \neq 0,\ n \geq 4$

Since, $f(x)$ is of least degree $\Rightarrow f(x) = a_4x^4$

Again, $\lim\limits_{x\to 0}\left[1 + \dfrac{f(x)}{x^3}\right]^{1/x} = e \Rightarrow a_4 = 1 \Rightarrow f(x) = x^4$

The graph of $y = x^4$ and $x^2 + y^2 = 2$ are shown in the figure



$\therefore$ The required area

$$= 2\int_0^1 \left(\sqrt{2 - x^2} - x^4\right)dx = \frac{\pi}{2} - \frac{3}{5}.$$

The correct option is (B)

136. For $0 < x < 1$, we have $\dfrac{1}{2}x^2 < x^2 < x$

i.e., $-x^2 > -x$ so that $e^{-x2} > e^{-x}$

Therefore, $\int\limits_0^1 e^{-x^2}\cos^2 x\,dx > \int e^{-x}\cos^2 x\,dx$

Also, $\cos^2 x \leq 1$, therefore

$$\int_0^1 e^{-x^2}\cos^2 x\,dx \leq \int_0^1 e^{-x^2}dx < \int_0^1 e^{-x^2/2}dx = I_4$$

Hence, I_4 is the greatest integral.

The correct option is (D)

137. We have, $\int\limits_0^x \sin^2\dfrac{t}{2}\,dt = \dfrac{1}{2}\int\limits_0^x (1 - \cos t)\,dt$

$$= a^2x^2 - \frac{3x}{2} + \frac{1}{2} + \frac{1}{a^2}$$

$$\Rightarrow \quad \frac{x}{2} - \frac{(\sin x)}{2} = \left(ax - \frac{1}{a}\right)^2 + \frac{x}{2}$$

$$\Rightarrow \quad 2\left(ax - \frac{1}{a}\right)^2 + 1 = -\sin x$$

Since $ax - \frac{1}{a} \geq 0$ for all x, so the only possiblity is

$$ax = \frac{1}{a} \text{ and } \sin x = -1$$

$$\Rightarrow \quad x = 2n\pi - \frac{\pi}{2}\text{. Therefore,}$$

$$a = \pm\frac{1}{\sqrt{x}} = \pm\frac{1}{\sqrt{2n\pi - \frac{\pi}{2}}},\ n \in N.$$

The correct option is (C)

138. Put $\log\left(x + \sqrt{x^2+1}\right) = t$

$$\Rightarrow \ x + \sqrt{x^2+1} = e^t$$

$$\Rightarrow \quad \sqrt{x^2+1} = e^{2t} - 2x\,e^t + x^2$$

$$\therefore \quad x = \frac{e^{2t}-1}{2e^t} = \frac{e^t - e^{-t}}{2}$$

$$\Rightarrow \quad dx = \left(\frac{e^t + e^{-t}}{2}\right)dt$$

When $x \to 0$, $t \to 0$ and when $x \to \infty$, $t \to \infty$

$$\therefore \quad I = \int_0^\infty \left(\frac{e^t + e^{-t}}{2\cdot e^{nt}}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{(1-n)t} + e^{-(1+n)t}\right]dt$$

$$= \frac{1}{2}\lim_{a\to\infty}\left\{\frac{e^{(1-n)t}}{(1-n)} - \frac{e^{-(1+n)t}}{(1+n)}\right\}_0^a$$

$$= \frac{1}{2}\lim_{a\to\infty}\left\{\frac{1}{(1-n)e^{(n-1)a}} - \frac{1}{(1+n)e^{(n+1)a}} - \frac{1}{1-n} + \frac{1}{1+n}\right\}$$

$$= \frac{1}{2}\left\{0 - 0 + \frac{1}{n-1} + \frac{1}{n+1}\right\}$$

$$= \frac{1}{2}\left\{\frac{2n}{n^2-1}\right\} = \frac{n}{n^2-1}$$

The correct option is (B)

139. Let $I(\alpha) = \int_0^1 \left(\frac{x^{\cos\alpha} - 1}{\ln x}\right)dx$

$$\therefore \quad I'(\alpha) = \int_0^1 \frac{x^{\cos\alpha}\ \ln x(-\sin\alpha)}{\ln x}\,dx$$

$$= (-\sin\alpha)\int_0^1 x^{\cos\alpha}\,dx$$

$$= (-\sin\alpha)\left\{\frac{x^{\cos\alpha+1}}{\cos\alpha+1}\right\}_0^1$$

$$= \frac{-\sin\alpha}{1+\cos\alpha}$$

Integrating both sides w.r.t. α, we have

$I(\alpha) = \ln(1 + \cos\alpha) + c$

$$\therefore \int_0^1 \frac{x^{\cos\alpha}-1}{\ln x}\,dx = \ln(1+\cos\alpha) + c$$

$$\int_0^1 \frac{x^0 - 1}{\ln x}\,dx = \ln 1 + c \text{ (Putting } \alpha = \pi/2)$$

$\Rightarrow 0 = 0 + c$ or $c = 0$. Hence, $I(\alpha) = \ln(1 + \cos\alpha)$.

The correct option is (A)

140. We have,

$$f'(x) \geq f^3(x) + \frac{1}{f(x)}$$

$$\Rightarrow \quad \frac{f(x)\,f'(x)}{1+f^4(x)} \geq 1$$

Integrating on the interval (a, b), we get

$$\int_a^b \frac{f(x)\,f'(x)}{1+f^4(x)}\,dx \geq \int_a^b dx$$

$$\Rightarrow \frac{1}{2}\tan^{-1} f^2(x)\Big|_a^b \geq b - a$$

$$\Rightarrow b - a \leq \frac{1}{2}\left[\lim_{x\to b^-}\tan^{-1} f^2(x) - \lim_{x\to a^+}\tan^{-1} f^2(x)\right]$$

$$= \frac{\pi}{24}.$$

The correct option is (C)

141. $\int_0^1 (\{2x\}-1)(\{3x\}-1)\,dx$

$$= \int_0^{1/3} (\{2x\}-1)(\{3x\}-1)\,dx$$

$$+ \int_{1/3}^{2/3} (\{2x\}-1)(\{3x\}-1)\,dx + \int_{2/3}^{1} (\{2x\}-1)(\{3x\}-1)\,dx$$

$$= \int_0^{1/3} (2x-1)(3x-1)\,dx + \int_{1/3}^{1/2} (2x-1)(3x-2)\,dx$$

$$+ \int_{1/2}^{2/3} (2x-2)(3x-2)\,dx + \int_{2/3}^{1} (2x-2)(3x-2)\,dx$$

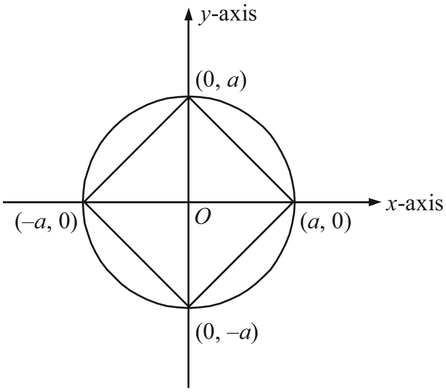
$$= \int_0^{1/3} (6x^2-5x+1)\,dx + \int_{1/3}^{1/2} (6x^2-7x+2)\,dx$$

$$+ \int_{1/2}^{3/2} (6x^2-10x+4)\,dx + \int_{2/3}^{1} (6x^2-12x+6)\,dx$$

$$= \frac{19}{72}.$$

The correct option is (A)

142. The graphs $|x|+|y|=a$ and $|x|^2+|y|^2=a^2$ are as shown in the figure



From the figure, it is clear that when powers of $|x|$ and $|y|$ both are reduced to half the straight lines get stretched inside.

Thus, the required area

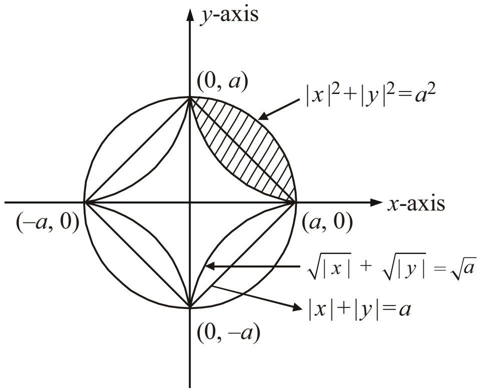
$= 4$ [shaded area in the first quadrant]

$$= 4\left[\frac{\pi a^2}{4} - \int_0^a \left(\sqrt{a}-\sqrt{x}\right)^2 dx\right] = \left(\pi - \frac{2}{3}\right)a^2$$

[In the first quadrant $x, y > 0$, therefore ,

$\sqrt{|x|} + \sqrt{|y|} = \sqrt{a} \Rightarrow \sqrt{x} + \sqrt{y} = \sqrt{a}$

$. \Rightarrow y = \left(\sqrt{a}-\sqrt{x}\right)^2$]



The correct option is (B)

More than One Option Correct Type

143. Let $I_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\,dx$

Assuming, $I = I_{n+1} - 2I_n + I_{n-1}$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin^2(n+1)x - 2\sin^2 nx + \sin^2(n-1)x}{\sin^2 x}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin(2n+1)x\sin x - \sin(2n-1)x\sin x}{\sin x}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin^2 x}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{2\sin x\cos 2nx}{\sin x}\,dx$$

$$= 2\int_0^{\pi/2} \cos 2nx dx$$

$$\Rightarrow \quad I = \frac{1}{n}(\sin 2nx)_0^{\pi/2} = \frac{1}{n}(\sin n\pi - 0) = 0$$

$\therefore \quad I_{n+1} + I_{n-1} = 2I_n$ for $n \geq 1$

$\Rightarrow I_1, I_2, I_3$ are in AP

So, option (C) is true.

Now, $I_1 = \int_0^{\pi/2} 1\cdot dx = \frac{\pi}{2}$ and, $I_2 = \int_0^{\pi/2} \frac{\sin^2 2x}{\sin^2 x}$

$$\Rightarrow \quad I_2 = 4\int_0^{\pi/2} \cos^2 x dx = 4 \times \frac{\pi}{4} = \pi$$

$$\therefore \quad I_2 - I_1 = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

Then, $I_1, I_2, I_3, \ldots$ are in AP with first term $\frac{\pi}{2}$ and common difference $\frac{\pi}{2}$.

$\therefore I_n = \frac{\pi}{2} + (n-1)\frac{\pi}{2} = \frac{n\pi}{2}$

Thus, option (A) is true.

Now $\sin(I_{16}) = \sin\left(\frac{16\pi}{2}\right) = \sin 8\pi$

Hence, option (D) is also true.

The correct option is (A, C, D)

144. We have

$$A_{n+1} - A_n = \int_0^{\pi/2} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x} dx$$

$$= \int_0^{\pi/2} \frac{2\cos 2nx \sin x}{\sin x} dx$$

$$= 2\int_0^{\pi/2} \cos 2n\pi \, dx = 0$$

Again, $B_{n+1} - B_n = \int_0^{\pi/2} \frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x} dx$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x \sin x}{\sin^2 x} dx = A_{n+1}$$

The correct option is (A) and (D)

145. $g(x+\pi) = \int_0^{x+\pi} \cos 4t \, dt = g(x) + \int_0^{\pi} \cos 4t \, dt$

$= g(x) + g(\pi)$

Here $g(\pi) = \int_0^{\pi} \cos 4t \, dt = 0$

The correct option is (B) and (C)

146. Let $[x] = l$ then $x = l + k$, $0 \le k < 1$.

$$\therefore \int_0^x [x]\, dx = \int_0^{l+k} [x]\, dx$$

$$= \int_0^1 [x]\, dx + \int_1^2 [x]\, dx + \cdots + \int_{l-1}^{l} [x]\, dx + \int_l^{l+k} [x]\, dx$$

$$= 0 + 1 + 2 + \cdots + (l-1) + l \cdot k$$

$$= \frac{1}{2}(l-1)\, l + lk$$

$$= \frac{1}{2}[x]\,([x]-1) + [x](x - [x])$$

$$= [x]\left(\frac{1}{2}([x]-1) + (x - [x])\right)$$

$\therefore A = [x] - 1$ and $B = x - [x]$.

The correct option is (A) and (B)

147. $\int_1^4 |x-3|\, dx = \int_1^3 |x-3|\, dx + \int_1^4 |x-3|\, dx$

$$= -\int_1^3 (x-3)\, dx + \int_1^4 (x-3)\, dx$$

$$= -\left(\frac{x^2}{2} - 3x\right)\Bigg|_1^3 + -\left(\frac{x^2}{2} - 3x\right)\Bigg|_3^4$$

$$= 2 + \frac{1}{2} = \frac{5}{2}.$$

Thus, $2A + B = \frac{5}{2}$.

The correct option is (B), (C), (D)

148. $\int_a^b |\sin x|\, dx = 8 \Rightarrow b - a = 4\pi$

$\because$ period of $|\sin x|$ is p and $\int_0^{\pi} |\sin x|\, dx = 2.$

Also, $|\cos x|$ is periodic with period p and $\int_0^{\pi/2} |\cos x| = 1$

So, $\int_a^{a+b} |\cos x|\, dx = \frac{9}{2}$

$\Rightarrow \quad a + b = \frac{9\pi}{2}$

$\Rightarrow \quad b = \frac{17\pi}{4}$ and $a = \frac{\pi}{4}$

The correct option is (C) and (D)

149. Let $I = \int_0^1 \frac{dx}{2e^x - 1}$

$$= \int_0^1 \left(\frac{2e^x}{2e^x - 1} - 1\right) dx$$

$$= \left[\log(2e^x - 1) - x\right]_0^1 = \log(2e - 1) - 1.$$

$\therefore \quad p = 1, q = 2, r = 1.$

The correct option is (A), (B) and (C)

150. We have,

$$f(x) = e^x + x\int_0^1 f(y)\,dy + e^x \int_0^1 yf(y)\,dy$$

$$= e^x\left(1 + \int_0^1 yf(y)\,dy\right) + e^x \int_0^1 f(y)\,dy$$

$$= ae^x + bx$$

where a and b are constants, given by

$$a = 1 + \int_0^1 yf(y)\,dy = 1 + \int_0^1 y(ae^y + by)\,dy$$

$$= 1 + a\left[(y-1)e^y\right]_0^1 + \left[\frac{by^3}{3}\right]_0^1$$

$$= 1 + a + \frac{b}{3}$$

and, $b = \int_0^1 f(y)\,dy$

$$= \int_0^1 (ae^y + by)\,dy$$

$$= \left[ae^y + \frac{by^2}{3}\right]_0^1 = a(e-1) + \frac{b}{2}$$

Solving, we have $b = -3$ and $a = \dfrac{-3}{2(e-1)}$

$\therefore\ f(x) = \dfrac{-3e^x}{2(e-1)} - 3x \therefore a = \dfrac{-3}{2(e-1)}$ and $b = -3$.

The correct option is (A) and (B)

151. $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx$

$$= \int_0^{\pi/4} f(\sin 2x)\sin x\,dx + \int_{\pi/4}^{\pi/2} f(\sin 2x)\sin x\,dx$$

$$= \int_0^{\pi/4} f(\sin 2x)\sin x\,dx + \int_0^{\pi/4} f\left(\sin 2\left(\frac{\pi}{4}+z\right)\right)\sin(\pi/4+z)\,dz$$

[Putting $x = \dfrac{\pi}{4} + z \Rightarrow dx = dz$]

$$= \int_0^{\pi/4} f\left(\sin 2\left(\frac{\pi}{4}-x\right)\right)\sin\left(\frac{\pi}{4}-x\right)dx$$

$$+ \frac{1}{\sqrt{2}}\int_0^{\pi/4} f(\cos 2x)(\sin x + \cos x)\,dx$$

$$\left[\because \int_a^b f(z)\,dz = \int_a^b f(x)\,dx\right]$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} f(\cos 2x),(\cos x + \sin x)\,dx$$

$$+ \frac{1}{\sqrt{2}}\int_0^{\pi/4} f(\cos 2x),(\cos x - \sin x)\,dx$$

$$= 2\cdot\frac{1}{\sqrt{2}}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$$

Also, $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx$

$$= \int_0^{\pi/2} f\big(\sin 2(\pi/2 - x)\big)\sin(\pi/2 - x)\,dx$$

$$= \int_0^{\pi/2} f(\sin 2x)\cos x\,dx$$

The correct option is (A) and (B)

152. $\left|\int_{10}^{19} \dfrac{\sin x}{1+x^8}dx\right| \le \int_{10}^{19} \dfrac{|\sin x|}{1+x^8}dx$

$$\left[\because |f(x)| \le \int |f(x)|\,dx\right]$$

$\le \int_{10}^{19} \dfrac{dx}{1+x^8}$ $\qquad [\because\ |\sin x| \le 1]$

$< \int_{10}^{19} \dfrac{dx}{x^8}$ $\qquad \left[\because 1+x^8 > x^8 \Rightarrow \dfrac{1}{1+x^8} < \dfrac{1}{x^8}\right]$

$< \int_{10}^{19} \dfrac{dx}{10^8}$ $\qquad \left[\because x > 10 \Rightarrow \dfrac{1}{x} < \dfrac{1}{10}\right]$

$$= \frac{1}{10^8}(19-10)$$

$= 9\times 10^{-8} < 10 \times 10^{-8} < 10^{-7}$

Again, $\because\ 10^7 > 10^6 \Rightarrow 10^{-7} < 10^{-6}$

$\therefore$ given integral is $< 10^{-6}$

The correct option is (A) and (C)

153. $\int_{-1/2}^{1/2} \sqrt{\left(\dfrac{x+1}{x-1}\right)^2 + \left(\dfrac{x-1}{x+1}\right)^2 - 2}\ dx$

$$= \int_{-1/2}^{1/2} \sqrt{\left(\frac{x+1}{x-1} - \frac{x-1}{x+1}\right)^2}\ dx$$

$$= \int_{-1/2}^{1/2} \sqrt{\left(\frac{(x+1)^2-(x-1)^2}{(x^2-1)}\right)^2}\ dx$$

$$= \int_{-1/2}^{1/2} \sqrt{\left(\frac{4x}{x^2-1}\right)^2}\ dx = \int_{-1/2}^{1/2} \left|\frac{4x}{x^2-1}\right| dx$$

$$= 4.2\int_0^{1/2} \left|\frac{x}{x^2-1}\right| dx = -8\int_0^{1/2} \frac{x\,dx}{x^2-1}$$

$$= -4\ \left|\log(x^2-1)\right|_0^{1/2} = -4\left[\log\left|\frac{1}{4}-1\right| - \log|-1|\right]$$

$$= -4\left[\log\frac{3}{4} - 0\right] = 4\log\frac{4}{3} = \log\frac{256}{81} = -\log\frac{81}{256}$$

The correct option is (B), (C) and (D)

154. $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$

$= \left| \frac{1}{(1+x^2)^n} . x \right|_0^1 - \int_0^1 -n(1+x^2)^{n-1} . 2x . x\, dx$

$= \frac{1}{2^n} + n \int_0^1 \frac{2x^2}{(1+x^2)^{n+1}} dx$

$= \frac{1}{2^n} + 2n \int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}} dx$

$= \frac{1}{2^n} + 2n \int_0^1 \frac{dx}{(1+x^2)^n} - 2n \int_0^1 \frac{dx}{(1+x^2)^{n+1}}$

$$(2n-1) I_n + \frac{1}{2^n} = 2n\, I_{n+1}$$

$\therefore 2I_2 = I_1 + \frac{1}{2}$

$\Rightarrow I_2 = \frac{I_1}{2} + \frac{1}{4} = \frac{\pi}{8} + \frac{1}{4}$

$$\left[\because I_1 = \int_0^1 \frac{dx}{1+x^2} = \left|\tan^{-1} x\right|_0^1 = \frac{\pi}{4}\right]$$

The correct option is (B) and (C)

155. We have,

$g(x+2) = \int_0^{x+2} f(t)\, dt$

$= \int_0^2 f(t)\, dt + \int_2^{x+2} f(t)\, dt$

$= g(2) + \int_0^x f(t)\, dt$

$$\left[\because f \text{ is periodic with period 2 } \therefore \int_a^{a+t} f(t)\, dt = \int_0^t f(t)\, dt\right]$$

$\therefore g(x+2) = g(2) + g(x)$

Also, $g(2) = \int_0^2 f(t)\, dt = \int_0^1 f(t)\, dt + \int_1^2 f(t)\, dt$

$= \int_0^1 f(t)\, dt + \int_{-1}^0 f(t)\, dt = \int_{-1}^1 f(t)\, dt = 0$

[$\because f(x)$ is odd]

$\therefore g(x)$ is periodic with period 2

$\therefore g(2n) = 0$

The correct option is (B) and (C)

156. If e_1 and e_2 be the eccentricity of hyperbola and its conjugate, then

$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1 \Rightarrow e_2 = \frac{e_1}{\sqrt{e_1^2 - 1}}$

$\therefore f(e) = \frac{e}{\sqrt{e^2-1}} \Rightarrow ff(e) = e$

$$\therefore \underbrace{f\, f\, f \dots f}_{n \text{ times}}(e) = \begin{cases} \frac{e}{\sqrt{e^2-1}}, & \text{if } n \text{ is odd} \\ e, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \int_1^3 f\, f\, f(e)\, de = \begin{cases} 2\sqrt{2} & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even} \end{cases}$$

The correct option is (B) and (C)

157. We have,

$$A_n = \int_0^{\pi/4} (\tan x)^n\, dx$$

$$\Rightarrow \quad A_{n-2} = \int_0^{\pi/4} (\tan x)^{n-2}\, dx$$

$$\therefore \quad A_n + A_{n-2} = \int_0^{\pi/4} (\tan x)^{n-2} (\tan^2 x + 1)\, dx$$

$$= \int_0^{\pi/4} (\tan x)^{n-2} \sec^2 x\, dx$$

Let $\tan x = t$, so that $\sec^2 x\, dx = dt$

$$\therefore \quad A_n + A_{n-2} = \int_0^1 t^{n-2}\, dt$$

$$= \left(\frac{t^{n-1}}{n-1}\right)_0^1 = \frac{1}{n-1}$$

Now, since $0 \le x \le \frac{\pi}{4} \therefore 0 \le \tan x \le 1.$

$\Rightarrow \tan^{n+2} x < \tan^n x < \tan^{n-2} x$

$\Rightarrow \int_0^{\pi/4} \tan^{n+2} x\, dx < \int_0^{\pi/4} \tan^n x\, dx < \int_0^{\pi/4} \tan^{n-2} x\, dx$

$\Rightarrow A_{n+2} < A_n < A_{n-2}$

$\Rightarrow A_n + A_{n+2} < 2A_n < A_n + A_{n-2}$

$\therefore \quad \frac{1}{n+1} < 2A_n < \frac{1}{n-1}$

or, $\frac{1}{2(n+1)} < A_n < \frac{1}{2(n-1)}$

The correct option is (A) and (B)

Passage Based Questions

Passage 1

158. $f(x) = \begin{cases} 0, & \text{if } x = 0,1,2,...,m \\ 1, & \text{if } r-1 < r < r, r = 1, 2,...; m \end{cases}$

$\Rightarrow f$ is bounded and has only $m + 1$ points of finite discontinuity at 0, 1, 2, ..., m. Since the points of discontinuity of f on $[0, m]$ are finite in number, therefore, f is integrable on $[0, m]$.

$$\int_0^m f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + ... + \int_{m-1}^m f(x)\,dx$$

$$= \int_0^1 1\,dx + \int_1^2 1\,dx + ... + \int_{m-1}^m 1\,dx$$

$$= (1-0) + (2-1) + ... + (m-(m-1))$$

$$= 1 + 1 + ... + 1 = m$$

The correct option is (B)

159. $f(x) = 1$, when $\frac{1}{2} < x \le 1$

$= \frac{1}{2}$, when $\frac{1}{2^2} < x \le \frac{1}{2}$

$= \frac{1}{2^2}$, when $\frac{1}{2^3} < x \le \frac{1}{2^2}$

$\vdots \quad \vdots \quad \vdots$

$= \frac{1}{2^{n-1}}$, when $\frac{1}{2^n} < x \le \frac{1}{2^{n-1}}$

$\vdots \quad \vdots \quad \vdots$

$= 0$, when $x = 0$

Thus, we notice that f is bounded and continuous on [0, 1] except at the points $0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...$

The set of points of discontinuity of f on [0, 1] is

$$\left\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} ...\right\}$$

which has only one limit point 0. Since the set of points of discontinuity of f on [0, 1] has a finite number of limit points, f is integrable on [0, 1]

Now,

$$\int_{1/2^n}^1 f(x)\,dx = \int_{1/2}^1 f(x)\,dx + \int_{1/2^2}^{1/2} f(x)\,dx + \int_{1/2^3}^{1/2^2} f(x)\,dx + ... + \int_{1/2^n}^{1/2^{n-1}} f(x)\,dx$$

$$= \int_{1/2}^1 1\,dx + \int_{1/2^2}^{1/2} \frac{1}{2}\,dx + \int_{1/2^3}^{1/2^2} \frac{1}{2^2}\,dx ... + \int_{1/2^n}^{1/2^{n-1}} \frac{1}{2^{n-1}}\,dx$$

$$= \left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2} - \frac{1}{2^3}\right) + ... + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^3}\right) ... + \frac{1}{2^{n-1}}\left(\frac{1}{2^n}\right)$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 ... + \left(\frac{1}{2^2}\right)^{n-1}\right]$$

$$= \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2^2}\right)}{1 - \frac{1}{2^2}} = \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

Proceeding to the limit when $n \to \infty$, we get $\int_0^1 f(x)\,dx = \frac{2}{3}$

The correct option is (C)

160. $f(x) = 1$, when $\frac{1}{2} < x \le 1$

$= \frac{1}{2}$, when $\frac{1}{3} < x \le \frac{1}{2}$

$= \frac{1}{3}$, when $\frac{1}{4} < x \le \frac{1}{3}$

$\vdots$

$= \frac{1}{n}$, when $\frac{1}{n+1} < x \le \frac{1}{n}$

$= 0$, when $x = 0$

Thus, we notice that f is bounded and continuous on [0, 1] except at the points $0, 1, \frac{1}{2}, \frac{1}{3}, ...$

The set of points of discontinuity of f on [0, 1] is

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\right\}$$

which has only one limit point 0. Since the set of points of discontinuity of f on [0, 1] has a finite number of limit points, therefore, f is integrable on [0, 1]

Now, $$\int_{1/(n+1)}^1 f(x)\,dx = \int_{1/2}^1 f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \int_{1/4}^{1/3} f(x)\,dx + ... + \int_{1/(n+1)}^{1/n} f(x)\,dx$$

$$= \int_{1/2}^1 1\,dx + \int_{1/3}^{1/2} \frac{1}{2}\,dx + \int_{1/4}^{1/3} \frac{1}{3}\,dx + ... \int_{1/(n+1)}^{1/n} \frac{1}{n}\,dx$$

$$= \left(1-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}\right)+\frac{1}{3}\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots\frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\right)-\left(\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}\right)$$

$$= \frac{\pi^2}{6}-\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= \frac{\pi^2}{6}-\left(1-\frac{1}{n+1}\right)$$

Proceeding to the limit as $n \to \infty$, we get

$$\int_0^1 f(x)\,dx = \frac{\pi^2}{6} - 1.$$

The correct option is (A)

Passage 2

161. Let $f(x) = \dfrac{1}{1+x^2}$ and $g(x) = \sin px$, then f, g are continuous on [0, 1] and hence integrable on [0, 1]

Also, $g(x) = \sin \pi x \geq 0$ on [0, 1]

Since f is decreasing on [0, 1], $\inf f = f(1) = \dfrac{1}{2}$

and $\sup f = f(0) = 1$

$$\int_0^1 f(x)g(x)\,dx = \mu \int_0^1 g(x)\,dx$$

i.e., $\displaystyle\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx = \mu \int_0^1 \sin \pi x\,dx$

But, $\displaystyle\int_0^1 \sin \pi x\,dx = -\left.\frac{\cos \pi x}{\pi}\right]_0^1 = \frac{2}{\pi}$

$$\therefore \quad \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx = \mu\,\frac{2}{\pi}. \qquad (1)$$

Since f is continuous on [0, 1], it attains every value between its bounds $\dfrac{1}{2}$ and 1.

$\therefore \mu \in \left[\dfrac{1}{2}, 1\right] \Rightarrow \exists$ a number $c \in [0, 1]$ such that $f(c) = \mu$.

From (1), $f(c) = \mu = \dfrac{2}{\pi}\displaystyle\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx$

But $0 \leq c \leq 1$ and f is decreasing on [0, 1]

$\Rightarrow f(0) = f(c) \geq f(1)$

$\Rightarrow \dfrac{1}{2} \leq \dfrac{\pi}{2}\displaystyle\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \leq 1 \quad \therefore \quad \frac{1}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \leq \frac{2}{\pi}$

The correct option is (A)

162. Let $f(x) = \dfrac{1}{\sin x}$ and $g(x) = x$, then f, g are continuous on $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$ and hence integrable on $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$.

Also, $g(x) = x > 0$ on $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$. Since f is decreasing on $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$, $\inf f = f\left(\dfrac{\pi}{2}\right) = 1$ and $\sup f = f\left(\dfrac{\pi}{6}\right) = 2$

$\therefore$ There exists $\mu \in [1, 2]$ such that

$$\int_{\pi/6}^{\pi/2} f(x)g(x)\,dx = \mu \int_{\pi/6}^{\pi/2} g(x)\,dx$$

i.e., $\displaystyle\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx = \mu \int_{\pi/6}^{\pi/2} x\,dx$

But, $\displaystyle\int_{\pi/6}^{\pi/2} x\,dx = \left.\frac{x^2}{2}\right]_{\pi/6}^{\pi/2} = \frac{1}{2}\left(\frac{\pi^2}{4}-\frac{\pi^2}{36}\right) = \frac{\pi^2}{9}$

$$\therefore \quad \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx = \mu.\,\frac{\pi^2}{9}. \qquad (1)$$

Since f is continuous on $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$, it attains every value between its bounds 1 and 2.

$\therefore \quad \mu \in [1, 2] \Rightarrow \exists$ a number $c \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$ such that $f(C) = \mu$

From (1), $f(c) = \mu = \dfrac{9}{\pi^2}\displaystyle\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx$

But, $\dfrac{\pi}{6} \leq c \leq \dfrac{\pi}{2}$ and f is decreasing on $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$.

$$\Rightarrow f\left(\frac{\pi}{6}\right) \geq f(c) \geq f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow 1 \leq \frac{9}{\pi^2}\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \leq 2$$

$$\therefore \quad \frac{\pi^2}{9} \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \leq \frac{2\pi^2}{9}$$

The correct option is (B)

163. Take $f(x) = \dfrac{1}{\sqrt{1+x^2}}$ and $g(x) = x^2$.

The correct option is (C)

Passage 3

164. Let $y = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$

$$\Rightarrow \log y = \lim_{n\to\infty}\frac{1}{n}\times\log\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$= \int_0^1 \log(1+x)\,dx$$

$$= [x\log(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}\,dx$$

$$= \log 2 - \int_0^1 \frac{(1+x)-1}{1+x}\,dx$$

$$= \log 2 - \int_0^1\left(1-\frac{1}{1+x}\right)dx$$

$$= \log 2 - [x-\log(1+x)]_0^1$$

$$= \log 2 - [(1-\log 2)-0]$$

$$= 2\log 2 - \log e = \log\frac{4}{e}\cdot$$

$$\therefore \quad y = \frac{4}{e}$$

The correct option is (D)

Passage 4

165. The area A of the region is given by

$$A = \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \frac{1}{2}\int_0^{1}\left\{a\cos^3 t(3b\sin^2 t\cos t) - b\sin^3 t(-3a\cos^2 t\sin)\right\}dt$$

$$= \frac{3ab}{2}\int_0^{2\pi}\cos^2 t\sin^2 t\,dt$$

$$= \frac{3ab}{2}\times 4\int_0^{2\pi}\cos^2 t\sin^2 t\,dt$$

$$= 6ab\times\frac{1\cdot 1}{4\cdot 2}\times\frac{\pi}{2} = \frac{3ab\pi}{8}$$

The correct option is (A)

Match the Column Type

166. I. $\int_0^{\infty}\left[\frac{2}{e^x}\right]dx = \int_0^{\ln 2}\left[\frac{2}{e^x}\right]dx + \int_{\ln 2}^{\infty}\left[\frac{2}{e^x}\right]dx$

$$= \int_0^{\ln 2} 1dx + 0 = ln\ 2$$

The correct option is (B)

II. $\int_0^{1.5}[x^2]\,dx = \int_0^{1}[x^2]\,dx + \int_1^{\sqrt{2}}[x^2]\,dx + \int_{\sqrt{2}}^{1.5}[x^2]\,dx$

$$\left[\because [x^2] = \begin{array}{l} 0, 0\le x<1 \\ 1, 1\le x<\sqrt{2} \\ 2, \sqrt{2}\le x<1.5\end{array}\right]$$

$$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\cdot dx + \int_{\sqrt{2}}^{1.5} 2\,dx$$

$$= (x)\Big|_1^{\sqrt{2}} + 2(x)\Big|_{\sqrt{2}}^{3/2}$$

$$= \sqrt{2} - 1 + 2\left(\frac{3}{2} - \sqrt{2}\right) = 2 - \sqrt{2}\,.$$

The correct option is (C)

III. Let $P = \frac{\sin x}{2+\cos x}$

$$\Rightarrow \frac{dP}{dx} = \frac{(2+\cos x)\cdot\cos x - \sin x\cdot(-\sin x)}{(2+\cos x)^2}$$

$$= \frac{2\cos x + 1}{(2+\cos x)^2}\,.$$

Integrating both sides with respect to x between the limits 0 and $\frac{\pi}{2}$,

we get $[P]_0^{\pi/2} = \int_0^{\pi/2}\frac{2\cos x+1}{(2+\cos x)^2}\,dx$

i.e., $\int_0^{\pi/2}\frac{2\cos x+1}{(2+\cos x)^2}\,dx = \left[\frac{\sin x}{2+\cos x}\right]_0^{\pi/2} = \left(\frac{1}{2} - 0\right)$

$$= \frac{1}{2}\,.$$

The correct option is (D)

IV. Let $I = \int_3^4 \frac{[x^2]}{[x^2 - 14x + 49] + [x^2]} dx \quad (1)$

$$= \int_3^4 \frac{[x^2]}{[(7-x)^2] + [x^2]} dx$$

$$= \int_3^4 \frac{\left[(7-x)^2\right]}{\left[(7-(7-x))^2\right] + \left[(7-x)^2\right]} dx$$

$$\left[\text{Using} \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

$$= \int_3^4 \frac{\left[(7-x)^2\right]}{[x^2] + \left[(7-x)^2\right]} dx \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_3^4 1\,dx = 1. \qquad \therefore I = \frac{1}{2}.$$

The correct option is (D)

167. I. $\int_0^2 x^3 \sqrt{2x - x^2}\, dx$

$$= \int_0^{\pi/2} 8 \sin^6\theta \sqrt{4\sin^2\theta - 4\sin^4\theta}\ (4 \sin\theta \cos\theta)\, d\theta$$

$$[\text{Putting } x = 2\sin^2\theta \Rightarrow dx = 4 \sin\theta \cos\theta\, d\theta]$$

$$= 64 \int_0^{\pi/2} \sin^8\theta \cdot \cos^2\theta\, d\theta = 64 \cdot \frac{7\cdot 5\cdot 3\cdot 1\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{7\pi}{8}.$$

The correct option is (B)

II. $\int_{\pi/2}^{3\pi/2} [\sin x]\, dx = \int_{\pi/2}^{\pi} [\sin x]\, dx + \int_{\pi}^{3\pi/2} [\sin x]\, dx$

$$= \int_{\pi/2}^{\pi} 0\, dx + \int_{\pi}^{3\pi/2} (-1)\, dx = -[x]_{\pi}^{3\pi/2}$$

$$= -\left(\frac{3\pi}{2} - \pi\right) = \frac{-\pi}{2}.$$

The correct option is (C)

III. Since $\sin 2px$ is positive for $0 < x \le \frac{1}{2}$ and negative for $\frac{1}{2} < x < 1$.

$$\therefore |\sin 2px| = \begin{bmatrix} \sin 2\pi x, \text{ for } 0 < x \le \frac{1}{2} \\ -\sin 2\pi x, \text{ for } \frac{1}{2} < x < 1 \end{bmatrix}$$

$$\therefore \int_0^1 |\sin 2\pi x|\, dx = \int_0^1 \sin 2\pi x\, dx + \int_{1/2}^1 (-\sin 2\pi x)\, dx$$

$$= \left[\frac{-\cos 2\pi x}{2\pi}\right]_0^{1/2} + \left[\frac{\cos 2\pi x}{2\pi}\right]_{1/2}^1$$

$$= \frac{2}{\pi}.$$

The correct option is (D)

IV. We have,

$$2 \sin\frac{x}{2} \cos x = \sin\frac{3x}{2} - \sin\frac{x}{2}$$

$$2 \sin\frac{x}{2} \cos 2x = \sin\frac{5x}{2} - \sin\frac{3x}{2}$$

... ...

... ...

$$2 \sin\frac{x}{2} \cos nx = \sin\left(n + \frac{1}{2}\right)x - \sin\left(n - \frac{1}{2}\right)x$$

Adding the above n equations, we get

$$2\sin\frac{x}{2}(\cos x + \cos 2x + \ldots + \cos nx)$$

$$= \sin\left(n + \frac{1}{2}\right)x - \sin\frac{x}{2}$$

$$\Rightarrow \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\frac{x}{2}} - 1 = 2(\cos x + \cos 2x + \ldots + \cos nx)$$

$$\Rightarrow \int_0^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\frac{x}{2}} dx - \int_0^{\pi} 1\cdot dx$$

$$= 2\int_0^{\pi} (\cos x + \cos 2x + \ldots + \cos nx)\, dx$$

$$\Rightarrow \int_0^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\frac{x}{2}} dx - \pi = 0$$

$$\therefore \int_0^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\frac{x}{2}} dx = \pi$$

The correct option is (A)

168. I. Let $I = \int_{-1}^{3} \{|x-2| + [x]\}\, dx$

$$\Rightarrow \quad I = \int_{-1}^{3} |x-2|\, dx + \int_{-1}^{3} [x]\, dx$$

Now, $I_1 = \int_{-1}^{3} |x-2|\, dx$

$$\Rightarrow \quad I_1 = \int_{-1}^{2} |x-2|\,dx + \int_{2}^{3} |x-2|\,dx$$

$$\Rightarrow \quad I_1 = -\int_{-1}^{2} (x-2)\,dx + \int_{2}^{3} (x-2)\,dx$$

$$\Rightarrow \quad I_1 = -\left(\frac{x^2}{2} - 2x\right)\Bigg|_{-1}^{2} + \left(\frac{x^2}{2} - 2x\right)\Bigg|_{2}^{3}$$

$$\Rightarrow \quad I_1 = -\left(\frac{4}{2} - 4\right) + \left(\frac{1}{2} + 2\right) + \left(\frac{9}{2} - 6\right) - \left(\frac{4}{2} - 4\right)$$

$$\Rightarrow \quad I_1 = 2 + \frac{5}{2} - \frac{3}{2} + 2 = 5$$

Also, $I_2 = \int_{-1}^{3} [x]\,dx$

$$\Rightarrow \quad I_2 = \int_{-1}^{0} [x]\,dx + \int_{0}^{1} [x]\,dx + \int_{1}^{2} [x]\,dx + \int_{2}^{3} [x]\,dx$$

$$\Rightarrow \quad I_2 = \int_{-1}^{0} (-1)\,dx + \int_{0}^{1} (0)\,dx + \int_{1}^{2} (1)\,dx + \int_{2}^{3} (2)\,dx$$

$$\Rightarrow \quad I_2 = (-1)(1) + 0 + (1)(1) + 2(1) = 2$$

$$\therefore \quad I = I_1 + I_2 = 5 + 2 = 7$$

The correct option is (C)

II. According to the property of greatest integer function, we have

$$[x] - [-x] = -1 \quad \forall\, x \in Z \qquad (1)$$

Let $I = \int_{-1}^{1} f(x)\,dx$

where, $f(x) = \dfrac{\sin^2 x}{\left[\dfrac{x}{\sqrt{2}}\right] + \dfrac{1}{2}}$

Now, $f(-x) = \dfrac{\sin^2 x}{\left[-\dfrac{x}{\sqrt{2}}\right] + \dfrac{1}{2}}$

$$\Rightarrow \quad f(-x) = \frac{\sin^2 x}{-1 - \left[\dfrac{x}{\sqrt{2}}\right] + \dfrac{1}{2}} \qquad \{\text{using (1)}\}$$

$\because \dfrac{x}{\sqrt{2}}$ is not an integer in (–1, 0) and (0, 1)

$$\Rightarrow \quad f(-x) = \frac{\sin^2 x}{-\left[\dfrac{x}{\sqrt{2}}\right] - \dfrac{1}{2}} = -f(x)$$

$\Rightarrow f(x)$ is an odd function in x

$$I = \int_{-1}^{1} f(x)\,dx = 0$$

The correct option is (D)

III. We have, $\int_{1}^{b} (b - 4x)\,dx \geq 6 - 5b$

$$\Rightarrow \left[bx - 2x^2\right]_1^b \geq 6 - 5b \Rightarrow b^2 - 2b^2 - b + 2 \geq 6 - 5b$$

$$\Rightarrow -b^2 + 4b - 4 \geq 0 \Rightarrow b^2 - 4b + 4 \leq 0$$

$$\Rightarrow (b-2)^2 \leq 0 \Rightarrow b - 2 = 0$$

$[\because (b-2)^2$ cannot be negative$]$

$\Rightarrow b = 2.$

The correct option is (A)

IV. We know that if $f(x + m\pi) = f(x)$ for all integral values of m, then

$$\int_{0}^{n\pi} f(x)\,dx = n\int_{0}^{\pi} f(x)\,dx$$

Let $g(x) = f(\cos^2 x)$, then $g(x + m\pi) = [\cos^2(x + m\pi)]$
$= f(\cos^2 x) = g(x)$.

Hence, $\int_{0}^{3\pi} f(\cos^2 x)\,dx = 3\int_{0}^{\pi} f(\cos^2 x)\,dx.$

$$\therefore \quad I_1 = 3\,I_2.$$

The correct option is (B)

Assertion-Reason Type

169. We have, $2f(x) + 3f\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 2$

$$\Rightarrow 2\,f\left(\frac{1}{x}\right) + 3f(x) = x - 2$$

Solving the above two equations, we get

$$f(x) = \frac{-2}{5x} + \frac{3x}{5} - \frac{2}{5}$$

$$\therefore \int_{1}^{2} f(x)\,dx = \int_{1}^{2}\left(\frac{-2}{5x} + \frac{3x}{5} - \frac{2}{5}\right)dx$$

$$= \left(\frac{-2}{5}\log x + \frac{3x^2}{10} - \frac{2}{5}x\right)\Bigg|_1^2$$

$$= \left(\frac{-2}{5}\log 2 + \frac{6}{5} - \frac{4}{5}\right) - \left(\frac{3}{10} - \frac{2}{5}\right)$$

$$= \left(\frac{-2}{5}\log 2 + \frac{1}{2}\right).$$

The correct option is (A)

170. Let $I_n = \int_0^\infty x^n e^{-x}\, dx$

$$= \left[x^n \cdot \frac{e^{-x}}{(-1)}\right]_0^\infty - \int_0^\infty n x^{n-1} \cdot \frac{e^{-x}}{(-1)}\, dx$$

$$= n\int_0^\infty x^{n-1} e^{-x}\, dx = nI_{n-1}$$

$\therefore\quad I_n = n.\, I_{n-1}$ (1)

Changing n to $n-1$, $I_{n-1} = (n-1)\, I_{n-2}$

Substituting the value of I_{n-1} in (1), we get

$I_n = n\,(n-1)\, I_{n-2}$

Generalizing from (1) and (2), we have

$I_n = [n\,(n-1)$ ... to n factors$]\, I_{n-n}$

$$= n\,!\, I_0 = n\,! \int_0^\infty x^0 e^{-x}\, dx = n\,! \int_0^\infty e^{-x}\, dx$$

$$= n\,! \left[\frac{e^{-x}}{-1}\right]_0^\infty = n\,!.$$

The correct option is (A)

171. Since, $f(x) = \dfrac{1}{1-x}$, $x \neq 1$

Also, $f(f(x)) = \dfrac{1}{1-f(x)} = \dfrac{1}{1-\dfrac{1}{1-x}}$

$\Rightarrow\quad f(f(x)) = \dfrac{x-1}{x}$, $x \neq 0, 1$

Again, $f(f(f(x))) = f\left(\dfrac{x-1}{x}\right) = \dfrac{1}{1-\dfrac{x-1}{x}} = x$

Thus, $f(f(f(x))) = x$, $x \neq 0, 1$

Clearly, $f(f(f(x)))$ is discontinuous at $x = 0$ and $x = 1$

$\therefore\quad a = 0, b = 1$

Now, $I = \displaystyle\int_a^b \frac{f(x)}{f(x) + f(1-x)}\, dx$

$\Rightarrow\quad I = \displaystyle\int_0^1 \frac{f(x)}{f(x) + f(1-x)}\, dx$ (1)

Also, $I = \displaystyle\int_0^1 \frac{f(1-x)}{f(1-x) + f(x)}\, dx$ (2)

Adding (1) and (2), we have

$$2I = \int_0^1 \frac{f(x)}{f(x) + f(1-x)}\, dx + \int_0^1 \frac{f(1-x)}{f(1-x) + f(x)}\, dx$$

$\Rightarrow\quad 2I = 1$

$\therefore\quad I = \dfrac{1}{2}$

The correct option is (A)

172. Let $I = \displaystyle\int_0^{\pi/6} \frac{\sqrt{3\cos 2x - 1}}{\cos x}\, dx$

$$= \int_0^{\pi/6} \frac{\sqrt{2 - 6\sin^2 x}}{\cos^2 x} . \cos x\, dx$$

$$= \int_0^{\pi/6} \frac{\sqrt{1 - 3\sin^2 x}}{1 - \sin^2 x} . \cos x\, dx$$

Put $\sin x = \dfrac{1}{\sqrt{3}} \sin\theta \Rightarrow \cos x\, dx = \dfrac{1}{\sqrt{3}} \cos\theta\, d\theta$

$$\therefore\quad I = \int_0^{\pi/3} \frac{\sqrt{2}\sqrt{1-\sin^2\theta}}{1 - \dfrac{\sin^2\theta}{3}} . \frac{1}{\sqrt{3}} \cos\theta\, d\theta$$

$$= \int_0^{\pi/3} 1\, d\theta - \sqrt{6} \int_0^{\pi/3} \frac{2}{3 - \left(\dfrac{1-\cos 2\theta}{2}\right)}\, d\theta$$

$$= \sqrt{6}\left(\frac{\pi}{3}\right) - 4\sqrt{6} \int_0^{\pi/3} \frac{d\theta}{5 + \cos 2\theta}$$

Putting $\tan\theta = t \Rightarrow d\theta = \dfrac{dt}{1+t^2}$ and $\cos 2\theta = , \dfrac{1-t^2}{1+t^2}$ we have,

$$I = \frac{\sqrt{2}\pi}{\sqrt{3}} - 4\sqrt{6} \int_0^{\sqrt{3}} \frac{dt}{6 + 4t^2}$$

$$= \frac{\sqrt{2}\pi}{\sqrt{3}} - \sqrt{6} . \frac{1}{\sqrt{3/2}} \left[\tan^{-1}\left(\frac{t}{\sqrt{3/2}}\right)\right]_0^{\sqrt{3}}$$

$$= \frac{\sqrt{2}\pi}{\sqrt{3}} - 2\tan^{-1}\sqrt{2}\ .$$

The correct option is (A)

173. We have,

$$\int_a^{b+nT} f(x)\, dx = \int_a^{b+nT} f(x)\, dx + \int_{a+nT}^{a+nT} f(x)\, dx$$

$$= \int_0^{nT} f(x)\, dx + \int_a^b f(x)\, dx$$

$$= n\int_0^T f(x)\, dx + \int_a^b f(x)\, dx$$

Now, $I = \displaystyle\int_{1/3}^{11/2} \{x\}\, dx = \int_{1/3}^{\frac{1}{2}+5} \{x\}\, dx$

$$= \int_{1/3}^{\frac{1}{3}+5} f(x)\,dx + \int_{\frac{1}{3}+5}^{\frac{1}{2}+5} \{x\}\,dx \qquad = 5\int_0^1 \{x\}\,dx + \int_{1/3}^{1/2} x\,dx$$

($\because$ period of $\{x\}$ is 1)

$$= 5\int_0^1 x\,dx + \int_{1/3}^{1/2} x\,dx$$

$$= \left.\frac{5x^2}{2}\right|_0^1 + \left.\frac{x^2}{2}\right|_{1/3}^{1/2}$$

$$= \frac{5}{2} + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5\cdot 37}{2\cdot 36} = \frac{185}{72}$$

The correct option is (A)

174. We have,

$$f(x) + f\left(x + \frac{1}{2}\right) = 1 \qquad (1)$$

Replacing x by $x + \frac{1}{2}$, Equation (1) reduces to

$$f\left(x + \frac{1}{2}\right) + f(x+1) = 1 \qquad (2)$$

Subtracting Equation (1) from Equation (2), we get

$f(x + 1) - f(x) = 0$ i.e., $f(x + 1) = f(x)$

$\Rightarrow f$ is periodic having period 1

$$\therefore \quad I = \int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_0^1 f(x)\,dx$$

$$= \int_{-1/2}^{1/2} f(x)\,dx + \int_{-1/2}^{1/2} f\left(t + \frac{1}{2}\right)dt$$

$\left[\because f \text{ has period } 1, \therefore \int_0^1 f(x)\,dx = \int_{-1/2}^{1/2} f(x)\,dx\right.$

and putting $x = t + \frac{1}{2}$ in the second integral$\left.\right]$

$$= \int_{-1/2}^{1/2}\left[f(t) + f\left(t + \frac{1}{2}\right)\right]dt = \int_{-1/2}^{1/2} 1\,dt$$

$$= [t]_{-1/2}^{1/2} = 1$$

The correct option is (A)

175. We have,

$2\sin x\,[\cos x + \cos 3x + \ldots + \cos (2k - 1)\,x]$

$= 2\sin x\cos x + 2\sin x\cos 3x + \ldots + 2\sin x\cos (2k - 1)\,x$

$= \sin 2x + \sin 4x - \sin 2x + \sin 6x - \sin 4x + \ldots$

$+ \sin 2kx - \sin (2k - 2)\,x$

$= \sin 2kx$

$$\therefore I = \int_0^{\pi/2} \sin 2kx\cot x\,dx = \int_0^{\pi/2} \frac{\sin 2kx}{\sin x}\cos x\,dx$$

$$= 2\int_0^{\pi/2} [\cos x + \cos 3x + \ldots + \cos(2k-1)x]\cos x\,dx$$

$$= \int_0^{\pi/2} [1 + \cos 2x + \cos 2x + \cos 4x + \ldots + \cos(2k-2)x + \cos 2kx\,dx]$$

$$= \int_0^{\pi/2} [1 + 2\cos 2x + 2\cos 4x + \ldots + 2\cos(2k-2)x + \cos 2kx\,dx]$$

$$= \left[x + \frac{2\sin 2x}{2} + \frac{2\sin 4x}{4} + \ldots + \frac{2\sin(2k-2)x}{2k-2} + \frac{\sin 2kx}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + 0 + 0 + \ldots + 0 + 0 = \frac{\pi}{2}.$$

The correct option is (A)

176. We have,

$$f(x) = \int_0^x g(t)\,dt = \int_0^x g(-t)\,dt \qquad (g \text{ is even})$$

$$= -\int_0^{-x} g(u)\,du \qquad [\text{Putting } -t = u]$$

$$= -f(-x)$$

$\therefore f$ is an odd function

Now, $g(0) - g(x) = f(5) - f(x + 5)$ [using $f(x + 5) = g(x)$]

$$= \int_0^5 g(t)\,dt - \int_0^{x+5} g(t)\,dt$$

$$= \int_{x+5}^5 g(t)\,dt = \int_{x+5}^5 g(-t)\,dt \qquad (g \text{ is even})$$

$$= \int_{x+5}^5 f(-t+5)\,dt \qquad [\text{using } f(x+5) = g(x)]$$

$$= -\int_{-x}^0 f(z)\,dz \qquad [\text{Putting } -t+5 = z]$$

$$= \int_{-x}^0 f(-z)\,dz \qquad [f \text{ is odd}]$$

$$= -\int_x^0 f(y)\,dy \qquad [\text{Putting } -z = y]$$

$$= \int_0^x f(y)\,dy = \int_0^x f(t)\,dt$$

The correct option is (A)

177. $\int_0^a f(x)g(x)h(x)\ dx$

$= \int_0^a f(a-x)g(a-x)h(a-x)dx$

$= -\int_0^a f(x)g(x)\left(\dfrac{3h(x)-5}{4}\right)dx$

$= \dfrac{-3}{4}\int_0^a f(x)g(x)h(x)dx + \dfrac{5}{4}\int_0^a f(x)g(x)dx$

Therefore, $\dfrac{7}{4}\int_0^a f(x)g(x)h(x)dx = \dfrac{5}{4}\int_0^a f(x)g(x)dx = 0$

$\left[\because \int_0^a f(x)g(x)dx = \int_0^a f(a-x)g(a-x)dx\right.$

$\left. = \int_0^a f(x)\{-g(x)\}dx \Rightarrow \int_0^a f(x)g(x)dx = 0\right]$

The correct option is (A)

Previous Year's Questions

178. $\because |\sin x|$ is a periodic function with period π.

Therefore, $\int_0^{10\pi} |\sin x|\,dx = 10\int_0^{\pi} |\sin x|\,dx$

$= 10\int_0^{\pi} \sin x\,dx$

$= 10[-\cos x]_0^{\pi}$

$= 10[-\cos\pi + \cos\theta]$

$= 10[-1+1] = 20$

The correct option is (A)

179. $\because I_n = \int_0^{\pi/4} \tan^n x\,dx$

We have $I_{n-2} = \int_0^{\pi/4} \tan^{n+2} x\,dx$

Now $I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x\,(1+\tan^2 x)dx$

$= \int_0^{\pi/4} \sec^2 x \tan^n x\,dx$

Substituting $\tan x = t$, then

$\sec^2 x\,dx = dt$

$\therefore\ I_n + I_{n+2} = \int_0^1 t^n dt$

$= \left[\dfrac{t^{n+1}}{n+1}\right]_0^1 = \dfrac{1}{n+1}$

Hence $\lim\limits_{n\to\infty}\left[I_n + I_{n-2}\right] = \lim\limits_{n\to\infty} \dfrac{n}{n+1}$

$= \lim\limits_{n\to\infty} \dfrac{1}{1+\dfrac{1}{n}}$

$= 1$

The correct option is (B)

180. The integral $\int_0^2 [x^2]dx$

$= \int_0^1 [x^2]\,dx + \int_0^{\sqrt{2}} [x^2]\,dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]\,dx + \int_{\sqrt{3}}^2 [x^2]\,dx$

$= \int_0^1 0\,dx + \int_0^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^2 3\,dx$

$= [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{\sqrt{3}} + [3x]_{\sqrt{3}}^2$

$= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3} = 5 - \sqrt{3} - \sqrt{2}$

The correct option is (D)

181. Key Idea :

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, & f(-x) = f(x) \\ 0, & f(-x) = -f(x) \end{cases}$$

If $I = \int_{-\pi}^{\pi} \dfrac{2x(1+\sin x)}{1+\cos^2 x}dx$, then

$I = \int_{-\pi}^{\pi} \dfrac{2x}{1+\cos^2 x}dx + \int_{-\pi}^{\pi} \dfrac{2x\sin x}{1+\cos^2 x}dx$

$= 0 + 4\int_0^{\pi} \dfrac{x\sin x}{1+\cos^2 x}dx \quad \left(\because \dfrac{2x}{1+\cos^2 x} \text{ is an odd function}\right)$

$\therefore I = 4\int_0^{\pi} \dfrac{x\sin x}{1+\cos^2 x}dx$

$\Rightarrow I = 4\int_0^{\pi} \dfrac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$

$\Rightarrow I = 4\int_0^{\pi} \dfrac{\pi\sin x}{1+\cos^2 x}dx - 4\int_0^{\pi} \dfrac{x\sin x}{1+\cos^2 x}dx$

$\Rightarrow I = 4\pi\int_0^{\pi} \dfrac{\sin x}{1+\cos^2 x}dx - I$

$\Rightarrow I = 2\pi\int_0^{\pi} \dfrac{\sin x}{1+\cos^2 x}dx$

Substitution $\cos x = t$ implies that $-\sin x dx = dt$

$\therefore I = -2\pi\int_1^{-1} \dfrac{1}{1+t^2}dt$

$= 2\pi[\tan^{-1} t]_{-1}^{1}$

$= 2\pi\left[\dfrac{\pi}{4} + \dfrac{\pi}{4}\right] = 2\pi \cdot \dfrac{\pi}{2} = \pi^2$

The correct option is (B)

182. Let, $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}\,dx}{\sqrt{\cos x} + \sqrt{\sin x}}$ (1)

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\cos(\pi/2 - x)} + \sqrt{\sin(\pi/2 - x)}}\,dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad (2)$$

On adding Equations (1) and (2), we write

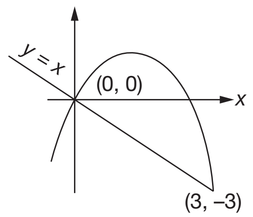
$$2I = \int_0^{\pi/2} 1\,dx \Rightarrow I = \frac{\pi}{4}$$

The correct option is (A)

183. The equations of given curve and a line are

$y = 2x - x^2$ (1)

and, $y = -x$ (2)



On solving Equations (1) and (2), we get the points of intersection of curves which are (0. 0) and (3, -3).

So the required area $= \int_0^3 \{(2x - x)^2 - (-x)\}\,dx$

$$= \int_0^3 (3x - x^2)\,dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]^3$$

$$= \frac{27}{2} - \frac{27}{3} = \frac{27}{2} - 9$$

$$= \frac{9}{2}\text{sq unit}$$

The correct option is (A)

184. The integral

$$F(t) = \int_0^t f(t - y) f(y)\,dy$$

$$= \int_0^t f(y) f(t - y)\,dy$$

$$= \int_0^t e^y (t - y)\,dy$$

$$= x^t - (1 + t)$$

The correct option is (B)

185. We have $\int_a^b x\, f(x)\,dx$

$$= \int_a^b (a + b - x) f(a + b - x)\,dx\,.$$

The correct option is (B)

186. $\lim_{x \to 0} \frac{\tan(x^2)}{x \sin x}$

$$= \lim_{x \to 0} \frac{\tan(x^2)}{x^2\left(\frac{\sin x}{x}\right)} = 1$$

The correct option is (C)

187. $\int_0^1 x(1 - x)^n\,dx = \int_0^1 x^n (1 - x)$

$$\int_0^1 (x^n - x^{n+1}) = \frac{1}{n+1} - \frac{1}{n+2}\,.$$

The correct option is (C)

188. Given $F'(x) = \frac{e^{\sin x}}{x}$ which implies that

$$\int_a^b F'(x)\,dx = F(b) - F(a)\,.$$

Now, $\int_1^4 3\frac{e^{\sin x^3}}{x}\,dx = \int_1^{64} \frac{e^{\sin x}}{x}\,dx = F(k) - F(1)$

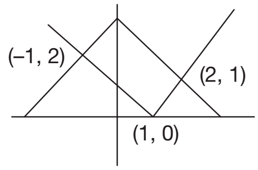
$$\Rightarrow \int_1^{64} F'(x)\,dx = F(k) - F(1)$$

$$\Rightarrow F(64) - F(1) = F(k) - F(1)$$

$$\Rightarrow k = 64\,\cdot$$

The correct option is (D)

189. Clearly area = $2\sqrt{2} \times \sqrt{2}$ sq. units



The correct option is (A)

190. $\int_0^1 f(x)[x^2 - f(x)]\,dx$

Solving this by substituting

$f(x) = e^x$

$\because f'(x) = f(x)$.

$\Rightarrow f(x) = e^x + c$

Also, $c = 0$ $(\because f(0) = 1)$

The correct option is (B)

191. The limit $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} e^{\frac{r}{n}}$ equals the integral

$$\int_0^1 e^x\,dx = (e - 1)\,.$$

The correct option is (B)

192. $\int_{-2}^{-1}(x^2-1)dx+\int_{-1}^{1}(1-x^2)dx+\int_{1}^{3}(x^2-1)dx$

$$=\left.\frac{x^3}{3}-x\right|_{-2}^{-1}+\left.x-\frac{x^3}{3}\right|_{-1}^{1}+\left.\frac{x^3}{3}-x\right|_{1}^{3}=\frac{28}{3}.$$

The correct option is (A)

193. $\int_0^{\frac{\pi}{2}}\frac{(\sin x+\cos x)^2}{\sqrt{(\sin x+\cos x)^2}}dx$

$$=\int_0^{\frac{\pi}{2}}(\sin x+\cos x)dx=\left|-\cos x+\sin x\right|_0^{\frac{\pi}{2}}=2.$$

The correct option is (C)

194. Let $I=\int_0^{\pi}xf(\sin x)dx=\int_0^{\pi}(\pi-x)f(\sin x)dx$

$$=\pi\int_0^{\pi}f(\sin x)dx-I\quad(\text{since }\sin(\pi-x)=\sin(x))$$

$$\Rightarrow I=\pi\int_0^{\pi/2}f(\sin x)\,dx\Rightarrow A=\pi.$$

The correct option is (B)

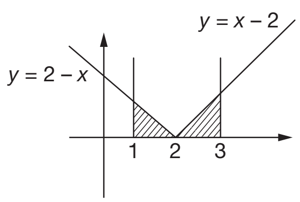
195. $f(-a)+f(a)=1$

$$I_1=\int_{f(-a)}^{f(a)}xg\{x(1-x)\}dx=\int_{f(-a)}^{f(a)}(1-x)g\{x(1-x)\}dx$$

$$\left(\because\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$$

The correct option is (A)

196. Required area $=\int_1^2(2-x)dx+\int_2^3(x-2)dx=1.$



The correct option is (A)

197. The limit

$$\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+\frac{3}{n^2}\sec^2\frac{9}{n^2}+....+\frac{1}{n}\sec^2 1\right]$$

is equal to

$$\lim_{n\to\infty}\frac{r}{n^2}\sec^2\frac{r^2}{n^2}=\lim_{n\to\infty}\frac{1}{n}\cdot\frac{r}{n}\sec^2\frac{r^2}{n^2}$$

So that given limit is equal to value of integral $\int_0^1 x\sec^2 x^2dx$

or $\frac{1}{2}\int_0^1 2x\sec x^2dx=\frac{1}{2}\int_0^1\sec^2 tdt$ [put $x^2=t$]

$$=\frac{1}{2}(\tan t)_0^1=\frac{1}{2}\tan 1$$

The correct option is (D)

198. Given that

$$I_1=\int_0^1 2^{x^2}dx,\ I_2=\int_0^1 2^{x^3}dx,\ I_3=\int_0^1 2^{x^2}dx,\ I_4=\int_0^1 2^{x^3}dx$$

$\forall\ 0<x<1,\ x^2>x^3$

$$\Rightarrow\int_0^1 2^{x^2}dx>\int_0^1 2^{x^3}dx$$

$\Rightarrow I_1>I_2$

The correct option is (B)

199. Required area (OAB) $=\int_{1-e}^{0}\ln(x+e)dx$

$$=\left[x\ln(x+e)-\int\frac{1}{x+e}xdx\right]_0^1=1.$$

The correct option is (A)

200. The parabolas $y^2=4x$ and $x^2=4y$ are symmetric about line $y=x$,

Now, area bounded between $y^2=4x$ and $y=x$ is $\int_0^4\left(2\sqrt{x}-x\right)dx=\frac{8}{3}$

$$\Rightarrow A_{s_2}=\frac{16}{3}\text{ and }A_{s_1}=A_{s_3}=\frac{16}{3}$$

$$\Rightarrow A_{s_1}:A_{s_2}:A_{s_3}::1:1:1$$

The correct option is (D)

201. $\lim_{x\to2}\int_0^{f(x)}\frac{4t^3}{x-2}dt$

Applying L Hopital rule

$$\lim_{x\to2}\left[4f(x)^2f'(x)\right]=4f(2)^3f'(2)$$

$$=4\times6^3\times\frac{1}{48}=18$$

The correct option is (D)

202. Given that $\int_{\pi/4}^{\beta}f(x)dx=\beta\sin\beta+\frac{\pi}{4}\cos\beta+\sqrt{2}\beta$

Differentiating w.r.t b

$$f(\beta)=\beta\cos\beta+\sin\beta-\frac{\pi}{4}\sin\beta+\sqrt{2}$$

The correct option is (D)

203. $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx = \int_0^{\pi} \cos^2 x dx = \frac{\pi}{2}$.

The correct option is (B)

204. Perpendicular distance of centre $\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$ from $x + 2y - 2 = 4$

$$= \frac{\left|\frac{1}{2} + \frac{1}{2} - 4\right|}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

$$\text{So, radius} = \sqrt{\frac{5}{2} - \frac{3}{2}} = 1.$$

The correct option is (B)

205. We have $I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x} + \sqrt{x}} dx$

$\Rightarrow I = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{x}} dx$ (applying property of definite integrals)

Adding both the above integrals, we find

$$2I = \int_3^6 dx = 3 \Rightarrow I = \frac{3}{2}$$

The correct option is (B)

206. Given integral

$$I = \int_0^{\pi} x f(\sin x) dx = \int_0^{\pi} (\pi - x)(\sin x) dx$$

$$= \pi \int_0^{\pi} f(\sin x) dx - I$$

$$\Rightarrow 2I = \pi \int_0^{\pi} f(\sin x) dx$$

This implies that

$$I = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx = \pi \int_0^{\pi/2} f(\sin x) dx$$

$$= \pi \int_0^{\pi/2} f(\cos x) dx$$

The correct option is (D)

207. Given integral

$$I = \int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2(x + 3\pi)] dx$$

Put $x + \pi = t$

$$I = \int_{-\pi/2}^{\pi/2} [t^3 + \cos^2 t] dt = 2 \int_0^{\pi/2} \cos^2 t dt$$

$$= \int_0^{\pi/2} (1 + \cos 2t) dt = \frac{\pi}{2} + 0$$

The correct option is (C)

208. Let $a = k + h$, where $[a] = k$ and $0 \le h < 1$

$$\therefore \int_1^a [x] f'(x) dx = \int_1^2 1 f'(x) dx + \int_2^3 2 f'(x) dx +$$

$$\ldots\ldots \int_{k-1}^{k} (k-1) dx + \int_k^{k+h} k f'(x) dx$$

$= \{f(2) - f(1)\} + 2\{f(3) - f(2)\} + 3\{f(4) - f(3)\} + \ldots\ldots + (k-1)\{f(k) - f(k-1)\} + k\{f(k+h) - f(k)\}$

$= - f(1) - f(2) - f(3) \ldots\ldots - f(k) + k\ f(k+h)$

$= [a]\ f(a) - \{f(1) + f(2) + f(3) + \ldots + f([a])\}$

The correct option is (B)

209. Given that $f(x) = \int_1^x \frac{\log t}{1+t} dt$

Now, $F(e) = f(e) + f\left(\frac{1}{e}\right)$ which implies that

$$F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{1+t} dt$$

$$= \int_1^e \frac{\log t}{1+t} + \int_1^e \frac{\log t}{t(1+t)} dt$$

$$= \int_1^e \frac{\log t}{t} dt = \frac{1}{2}$$

The correct option is (A)

210. $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2 - 1}} = \frac{\pi}{2}$

$$\Rightarrow \left[\sec^{-1} t\right]_{\sqrt{2}}^{x} = \frac{\pi}{2}$$

$$\Rightarrow \sec^{-1} x - \frac{\pi}{4} = \frac{\pi}{2}$$

$$\Rightarrow \sec^{-1} x = \frac{3\pi}{4}$$

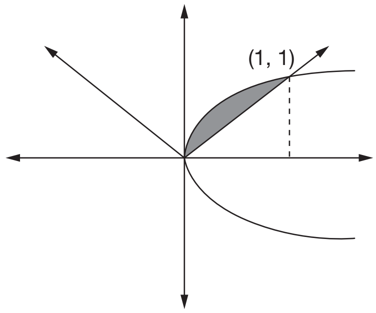
$$\Rightarrow x = -\sqrt{2}$$

The correction option is (D)

211. Area, $A = \int_0^1 (\sqrt{x} - x) dx$

$$= \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$



The correct option is (C)

212. The integral

$$I=\int_0^1 \frac{\sin x}{\sqrt{x}}dx<\int_0^1 \frac{x}{\sqrt{x}}dx=\int_0^1 \sqrt{x}dx=\frac{2}{3}x^{3/2}\Big|_0^1=\frac{2}{3}$$

$$\Rightarrow I<\frac{2}{3}$$

And the integral $J=\int_0^1 \frac{\cos x}{\sqrt{x}}dx<\int_0^1 \frac{1}{\sqrt{x}}dx=2\sqrt{x}\,\Big|_0^1=2$

$\therefore$ J ≤ 2.

The correct option is (B)

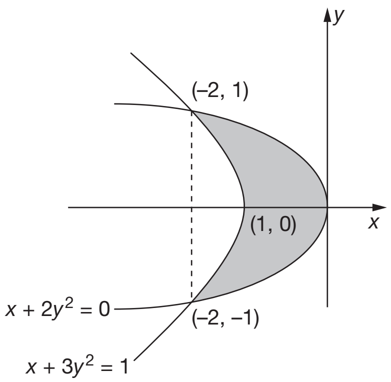
213. Solving the equations we get the points of intersection (–2, 1) and (–2, -1)

The bounded region is shown as shaded region.

The required area of the shaded portion

$$=2\int_0^1 (1-3y^2)-(-2y^2)$$

$$=2\int_0^1 (1-y^2)dy=2\left[y-\frac{y^3}{3}\right]_0^1=2\times\frac{2}{3}=\frac{4}{3}.$$



The correct option is (D)

214. If the given integral is

$$I=\int_0^\pi [\cot x]dx, \qquad (1)$$

then, $I=\int_0^\pi [\cot(\pi-x)]dx=\int_0^\pi [-\cot x]dx \qquad (2)$

Adding (1) and (2)

$$2I=\int_0^\pi [\cot x]dx+\int_0^\pi [-\cot x]dx==\int_0^\pi (-1)dx$$

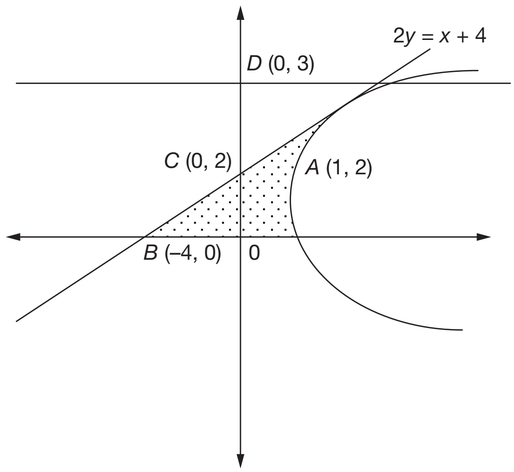
$$\left[\because [x]+[-x]=-1 \text{ if } x\notin Z \atop \qquad =0 \text{ if } x\in Z\right]$$

$$=[-x]_0^x=-\pi$$

$$\therefore \quad I=-\frac{\pi}{2}$$

The correct option is (D)

215. Equation of tangent at (2, 3) to the parabola $(y-2)^2=x-1$ is $x-2y+4=0$



Required Area = Area of ΔOCB + Area of OAPD – Area of ΔPCD

$$=\frac{1}{2}(4\times 2)+\int_0^3 (y^2-4y+5)dy-\frac{1}{2}(1\times 2)$$

$$=4+\left[\frac{y^3}{3}-2y^2+5y\right]_0^3-1=4-9-18+15-1$$

= 28 – 19 = 9 sq. units

(or)

$$\text{Area}=\int_0^3 (2y-4-y^2+4y-5)dy$$

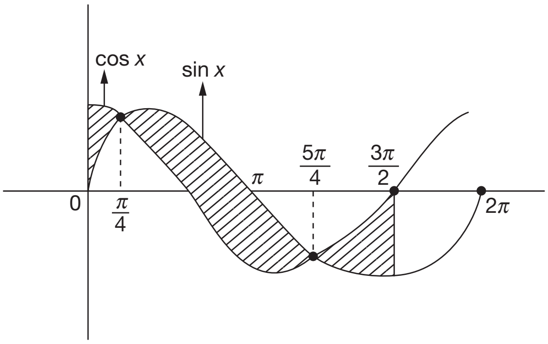
$$=\int_0^3 (-y^2+6y-5)dy=-\int_0^3 (3-y)^2dy$$

$$=\left[\frac{(y-3)^3}{3}\right]_0^3=\frac{27}{3}=9 \text{ sq.units}$$

The correct option is (C)

216. $\int_0^{\frac{\pi}{4}}(\cos x-\sin x)dx+\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}(\sin x-\cos x)dx+$

$\int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}}(\cos x-\sin x)=4\sqrt{2}-2$



The correct option is (D)

217. $p'(x) = p'(1-x)$

$\Rightarrow p(x) = -p(1-x) + c$

at $x = 0$

$p(0) = -p(1) + c = \Rightarrow 42 = c$

now $p(x) = -p(1-x) + 42$

$\Rightarrow p(x) + p(1-x) = 42$

$I=\int_0^1 p(x)dx=\int_0^1 p(1-x)dx$

$2I=\int_0^1(42)dx \Rightarrow I=21$

The correct option is (A)

218. $I=8\int_0^1\frac{\log(1+x)}{1+x^2}dx$

$=8\int_0^{\frac{\pi}{4}}\frac{\log(1+\tan\theta)}{1+\tan 2\theta}\sec^2\theta d\theta\ \ (\text{let} x=\tan\theta)$

$=8\int_0^{\frac{\pi}{4}}\log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$

$=8\int_0^{\frac{\pi}{4}}\log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$

$=8\int_0^{\frac{\pi}{4}}\log 2d\theta-8\int_0^{\frac{\pi}{4}}\log(1+\tan\theta)d\theta$

$=8\log 2\frac{\pi}{4}-1$

$2I = 2\pi\log 2$

$I = \pi\log 2$

The correct option is (D)

219. Area $=\int_0^1 xdx+\int_1^e\frac{1}{x}dx=\frac{1}{2}+1=\frac{3}{2}$

The correct option is (B)

220. $f'(x)=\sqrt{x}\ \sin x$

Given $x\in\left(0,\frac{5\pi}{2}\right)$

$f'(x)$ changes sign from +ve to –ve at π

$f'(x)$ changes sign from –ve to +ve at 2π

f has local max at π, local min at 2π

The correct option is (C)

221. Required area

$A=2\left[\int_0^2\left(3\sqrt{y}-\frac{\sqrt{y}}{2}\right)dy\right]=2\int_0^2\frac{5\sqrt{y}}{2}dy$

$=5\left[\frac{y^{3/2}}{3/2}\right]_0^2=\frac{10}{3}\left[2^{3/2}-0\right]=\frac{20\sqrt{2}}{3}$

The correct option is (C)

222. $g(x)=\int_0^x\cos 4t dt$

$\Rightarrow g'(x)=\cos 4x$

$\Rightarrow g(x)=\frac{\sin 4x}{4}+k$

$\Rightarrow g(x)=\frac{\sin 4x}{4}[\because g(0)=0]$

$g(x+\pi)=g(x)+g(\pi)=g(x)-g(\pi)(\because g(\pi)=0)$

The correct option is (B) or (C)

223. The derivative $\frac{dy}{dx}=|x|=2\Rightarrow x=\pm 2\Rightarrow y=\int_0^2|t|dt=2$ for $x=2$.

And, $y=\int_0^{-2}|t|dt=-2$ for $x=-2$.

$\therefore$ Equation of tangents are

$y-2=2(x-2)\Rightarrow y=2x-2$,

and $y+2=2(x+2)\Rightarrow y=2x+2$

Putting $y = 0$, we get $x = 1$ and -1.

The correct option is (D)

224. $2\sqrt{x}=x-3$

$4x=x^2-6x+9$

$x^2=10x+9$

$x = 9, x = 1$

And, so

$$\int_0^3 \left[(2y+3) - y^2\right] dy$$

$$\left[y^2 + 3y - \frac{y^3}{3}\right]_0^3 = 9 + 9 - 9 = 9$$

The correct option is (D)

225. The integral $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} dx$$

$$2I = \frac{\pi}{6}$$

$$I = \frac{\pi}{12}.$$

The correct option is (C)

226. $I = \int_0^{\pi} \sqrt{1 + 4\sin^2 \frac{x}{2} - 4\sin\frac{x}{2}} dx$

$$= \int_0^{\pi} \left|1 - 2\sin\frac{x}{2}\right| dx$$

$$= \int_0^{\pi/3} \left(1 - 2\sin\frac{x}{2}\right) dx + \int_{\pi/3}^{\pi} \left(2\sin\frac{x}{2} - 1\right) dx$$

$$= \left(x + 4\cos\frac{x}{2}\right)\Bigg|_0^{\pi/3} + \left(-4\cos\frac{x}{2} - x\right)\Bigg|_{\pi/3}^{\pi}$$

$$= -\frac{\pi}{3} + 8 \cdot \frac{\sqrt{3}}{2} - 4$$

$$= 4\sqrt{3} - 4 - \frac{\pi}{3}$$

The correct option is (D)

227. $A = \frac{1}{2} \times \pi + 2\int_0^1 \sqrt{1-x}\, dx$

$$= \frac{\pi}{2} + \frac{4}{3}.$$

The correct option is (A)

228. The required area

$$= \int_{\frac{1}{2}}^{1} \left(\frac{y+1}{4} - \frac{y^2}{2}\right) dy$$

$$= \frac{1}{4}\left(\frac{y2}{2} + y\right)_{\frac{1}{2}}^{1} - \frac{1}{2}\left(\frac{y^3}{3}\right)_{\frac{1}{2}}^{1}$$

$$= \frac{1}{4}\left(\frac{1}{2} + 1 - \left(\frac{1}{8} - \frac{1}{2}\right)\right) - \frac{1}{2}\frac{1}{3}\left(1 + \frac{1}{8}\right)$$

$$= \frac{1}{4} \times \frac{15}{8} - \frac{3}{16} = \frac{9}{32}.$$

The correct option is (C)

229. Applying the property

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

And then add

$$2I = \int_2^4 1 dx$$

$$\therefore 2I = 2$$

$$\therefore I = 1.$$

The correct option is (B)

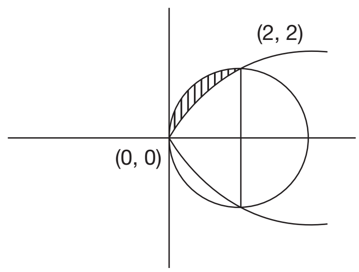
230. Given expression

$$= e^{\lim\limits_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} n\left(1 + \frac{r}{n}\right)} = e^{\int_0^2 \ln(1+x)\, dx}$$

$$= e^{\left((x+1)(ln(x+1)-1)\right)_0^2} = e^{3\ln 3 - 2} = \frac{27}{e^2}$$

The correct option is (C)

231.



$$\text{Required Area } = \frac{\pi(2)^2}{4} - \sqrt{2}\int_0^2 \sqrt{x}\, dx$$

$$= \pi - \sqrt{2} \cdot \frac{2}{3}\ 2\sqrt{2} = \pi - 8/3$$

The correct option is (C)