CHAPTER 16

# Definite Integral and Area

**Chapter Highlights**

Definite integral, Fundamental theorem of integral calculus, Differentiation under integral sign, Integration of mod functions, Area of plane regions, Curve tracing.

## DEFINITE INTEGRAL

Let $F(x)$ be any anti-derivative of $f(x)$, then for any two values of the independent variable $x$, say $a$ and $b$, the difference $F(b) - F(a)$ is called the *definite integral* of $f(x)$ from $a$ to $b$ and is denoted by $\int_a^b f(x)\,dx$. Thus

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is any anti-derivative of $f(x)$, The numbers $a$ and $b$ are called the *limits of integration*; $a$ is the lower limit and $b$ is the upper limit. Usually $F(b) - F(a)$ is abbreviated by writing $F(x)\big|_a^b$.

## FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If $f(x)$ is continuous on $[a, b]$ and if $F(x)$ be antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

**CAUTION**

The anti-derivative $F(x)$ must be continuous on the interval $[a, b]$.

### Why 'definete' Integral

The reason for using the term *definite integral* follows from the fact that the value of definite integral is independent of the particular choice of the antiderivative of $f(x)$. For if $F(x) + c$ is any other antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(x) + c\,\big|_a^b = [F(b) + c] - [F(a) + c]$$

$$= F(b) - F(a),$$

which is same as before.

**Working Rule to Evaluate $\int_a^b f(x)\,dx$**

**Step 1:** *Evaluate the indefinite integral $\int f(x)\,dx$ and omit the constant of integration. Let this be $F(x)$.*

**Step 2:** *Evaluate $F(b)$ and $F(a)$.*

**Step 3:** *Calculate $F(b) - F(a)$.*

Then, $\int_a^b f(x)\,dx = F(b) - F(a)$

### Evaluation of Definite Integrals by Substitution

In this method, we make a substitution $f(x) = t$ to reduce the given integral to a known form of integral.

**CAUTION**

When we make a substitution in a problem of definite integral, we have to change the limits according to the substitution.

For example, if we put $\phi(x) = t$ in the integral $\int_a^b f[\phi(x)]\phi'(x)\,dx$, then

$$\int_a^b f[\phi(x)]\phi'(x)dx = \int_{\phi(a)}^{\phi(b)} f(t)\,dt$$

**CAUTION**

The substitution must be continuous in the interval of integration.

## SOLVED EXAMPLES

**1.** $\int_0^a \frac{dx}{a+\sqrt{a^2-x^2}}$ is equal to

(A) $\frac{\pi}{2}+1$ (B) $\frac{\pi}{2}-1$

(C) $1-\frac{\pi}{2}$ (D) None of these

**Solution: (B)**

Let $I = \int_0^a \frac{dx}{a+\sqrt{a^2-x^2}} = \int_0^{\pi/2} \frac{a\cos\theta\, d\theta}{a+a\cos\theta}$

[Putting $x = a\sin\theta = \Rightarrow dx = a\cos\theta\, d\theta$]

$$= \int_0^{\pi/2} \frac{\cos\theta}{1+\cos\theta}\,d\theta = \int_0^{\pi/2}\left(1-\frac{1}{1+\cos\theta}\right)d\theta$$

$$= \int_0^{\pi/2} 1\,d\theta - \frac{1}{2}\int_0^{\pi/2}\sec^2\frac{\theta}{2}\,d\theta$$

$$= \left[\theta - \tan\frac{\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - \tan\frac{\pi}{4} = \left(\frac{\pi}{2}-1\right)$$

**2.** Given $\int_1^2 e^{x^2}dx = a$, the value of $\int_e^{e^4}\sqrt{\ln(x)}\,dx$ is

(A) $e^4 - e$ (B) $e^4 - a$

(C) $2e^4 - a$ (D) $2e^4 - e - a$

**Solution: (D)**

Given $\int_1^2 e^{x^2}dx = a$, let $I = \int_e^{e^4}\sqrt{\ln(x)}\,dx$

Put $\ln(x) = t^2 \Rightarrow \frac{1}{x}\,dx = 2tdt$

$$\therefore\quad I = \int_1^2 e^{t^2}\cdot 2t^2\,dt = \left(t\,e^{t^2}\right)_1^2 - \int_1^2 e^{t^2}dt$$

$$= 2e^4 - e - a$$

**3.** The value of the integral $\int_0^{\pi/2}\frac{1+2\cos x}{(2+\cos x)^2}\,dx$ is

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$

(C) $\frac{-1}{2}$ (D) $\frac{-1}{4}$

**Solution: (B)**

Let $P = \frac{\sin x}{2+\cos x}$

$$\Rightarrow \frac{dP}{dx} = \frac{(2+\cos x)\cdot\cos x - \sin x\cdot(-\sin x)}{(2+\cos x)^2}$$

$$= \frac{2\cos x + 1}{(2+\cos x)^2}$$

Integrating both sides with respect to $x$ between the limits 0 and $\frac{\pi}{2}$,

we get $(P)_0^{\pi/2} = \int_0^{\pi/2}\frac{2\cos x+1}{(2+\cos x)^2}\,dx$

i.e., $\int_0^{\pi/2}\frac{2\cos x+1}{(2+\cos x)^2}\,dx$

$$= \left(\frac{\sin x}{2+\cos x}\right)_0^{\pi/2} = \left(\frac{1}{2}-0\right)$$

$$= \frac{1}{2}$$

**4.** $\lim_{x\to 0}\frac{1}{x^3}\int_0^{x^2}\sin\sqrt{t}\,dt$ is equal to

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$

(C) $\frac{-1}{3}$ (D) $\frac{-2}{3}$

**Solution: (B)**

$$\lim_{x\to 0}\frac{1}{x^3}\int_0^{x^2}\sin\sqrt{t}\,dt = \lim_{x\to 0}\frac{\int_0^{x^2}\sin\sqrt{t}\,dt}{x^3}$$

$$\left(\frac{0}{0}\text{ form}\right)\left(\because \int_0^{x^2}\sin\sqrt{t}\,dt = 0 \text{ at } x=0\right)$$

$$= \lim_{x\to 0}\frac{\left[\sin\sqrt{t}\right]_{t=x^2}\cdot\frac{d}{dx}(x^2)}{3x^2}$$

(Using L' Hospital's Rule)

$$= \lim_{x\to 0}\frac{(\sin x)(2x)}{3x^2} = \frac{2}{3}$$

**5.** $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}\frac{dz}{z+\sqrt{z^2+1}}$ is equal to

(A) 0 (B) $\frac{1}{x+\sqrt{x^2+1}}$

(C) $\frac{1}{\sqrt{x^2+1}}$ (D) None of these

**Solution: (B)**

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}\frac{dz}{z+\sqrt{z^2+1}}$$

$$= \lim_{h\to 0}\frac{\int_x^{x+h}\frac{dz}{z+\sqrt{z^2+1}}}{h}\left(\frac{0}{0}\text{ form}\right)$$

$$\left(\because \int_x^{x+h}\frac{dz}{z+\sqrt{z^2+1}}\to 0 \text{ as } h\to 0\right)$$

$$= \lim_{h\to 0}\frac{\left(\frac{1}{z+\sqrt{z^2+1}}\right)_{z=x+h}\cdot\frac{d}{dh}(x+h)}{1}$$

(Using L' Hospital's Rule)

$$= \lim_{h\to 0}\frac{1}{x+h+\sqrt{(x+h)^2+1}} \quad .(1)$$

$$= \frac{1}{x+\sqrt{x^2+1}}$$

**6.** If $2f(x)+3f\left(\frac{1}{x}\right) = \frac{1}{x}-2$, $x\neq 0$, then $\int_1^2 f(x)\,dx$ is equal to

(A) $\frac{-2}{5}\log 2+\frac{1}{2}$ (B) $\frac{-2}{5}\log 2-\frac{1}{2}$

(C) $\frac{2}{5}\log 2+\frac{1}{2}$ (D) None of these

**Solution: (A)**

We have, $2f(x)+3f\left(\frac{1}{x}\right) = \frac{1}{x}-2$

$$\Rightarrow \quad 2f\left(\frac{1}{x}\right)+3f(x) = x-2$$

Solving the above two equations, we get

$$f(x) = \frac{-2}{5x}+\frac{3x}{5}-\frac{2}{5}$$

$$\therefore \quad \int_1^2 f(x)\,dx = \int_1^2\left(\frac{-2}{5x}+\frac{3x}{5}-\frac{2}{5}\right)dx$$

$$= \left(\frac{-2}{5}\log x+\frac{3x^2}{10}-\frac{2}{5}x\right)\Bigg|_1^2$$

$$= \left(\frac{-2}{5}\log 2+\frac{6}{5}-\frac{4}{5}\right)-\left(\frac{3}{10}-\frac{2}{5}\right)$$

$$= \left(\frac{-2}{5}\log 2+\frac{1}{2}\right)$$

**7.** Let $f(x)$ be a polynomial of degree 2 satisfying $f(0) = 1$, $f'(0) = -2$ and $f''(0) = 6$, then $\int_{-1}^2 f(x)\,dx$ is equal to

(A) 6 (B) 0

(C) 9 (D) None of these

**Solution: (C)**

Let $\quad f(x) = ax^2+bx+c$

$\Rightarrow \quad f'(x) = 2ax+b$ and $f''(x) = 2a$

We are given $f(0) = c = 1, f'(0) = b = -2$

and $\quad f''(0) = 2a = 6 \Rightarrow a = 3, b = -2$ and $c = 1$

$\therefore \quad f(x) = 3x^2-2x+1$

$$\therefore \int_{-1}^{2} f(x)\, dx = \int_{-1}^{2} (3x^2 - 2x + 1)\, dx$$

$$= [x^3 - x^2 + x]_{-1}^{2}$$

$$= (8 - 4 + 2) - (-1 - 1 - 1) = 6 + 3 = 9$$

**8.** If $f(x) = a + bx + cx^2$, then $\int_0^1 f(x)\, dx$ equals

(A) $\frac{1}{2}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$

(B) $\frac{1}{6}\left[f(0) + 2f\left(\frac{1}{2}\right) + f(1)\right]$

(C) $\frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$

(D) None of these

**Solution: (C)**

$f(0) = a, f(1) = a + b + c, f\left(\frac{1}{2}\right) = a + \frac{b}{2} + \frac{c}{4}$.

$$\therefore I = \int_0^1 f(x)\, dx = \int_0^1 (a + bx + cx^2)\, dx$$

$$= \left(ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right)_0^1 = \frac{1}{6}(6a + 3b + 2c)$$

$$= \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

**9.** $\int_0^1 \frac{x^a - 1}{\log x} dx$, where $a > 0$, is equal to

(A) $\log(a + 1)$ (B) $2\log(a + 1)$

(C) $3\log a$ (D) $2\log a$

**Solution: (A)**

Let $f(a) = \int_0^1 \frac{x^a - 1}{\log x} dx$

$$\Rightarrow f'(a) = \int_0^1 \frac{x^a \log x}{\log x} dx = \int_0^1 x^a\, dx = \left.\frac{x^{a+1}}{a+1}\right|_0^1$$

$$= \frac{1}{a+1}$$

$$\therefore f(a) = \int \frac{1}{a+1} da + C = \log(a + 1) + C$$

If $a = 0$, then $f(a) = 0$,

$\therefore C = 0$

Hence, $f(a) = \log(a + 1)$

**10.** The value of the integral $\int_3^4 \frac{[x^2]}{[x^2 - 14x + 49] + [x^2]}$ $dx$, where [.] denotes the greatest integer function, is

(A) 1 (B) $\frac{3}{2}$

(C) $\frac{1}{2}$ (D) None of these

**Solution: (C)**

Let $I = \int_3^4 \frac{[x^2]}{[x^2 - 14x + 49] + [x^2]} dx$ (1)

$$= \int_3^4 \frac{[x^2]}{[(7 - x)^2] + [x^2]} dx$$

$$= \int_3^4 \frac{[(7 - x)^2]}{[(7 - (7 - x))^2] + [(7 - x)^2]} dx$$

$$\left[\text{Using} \int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx\right]$$

$$= \int_3^4 \frac{[(7 - x)^2]}{(x^2) + [(7 - x)^2]} dx \quad (2)$$

Adding Eq. (1) and (2), we get

$$2I = \int_3^4 1\, dx = 1 \therefore I = \frac{1}{2}$$

## Properties of Definite Integrals

**1.** $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

**2.** $\int_a^b f(x)\, dx = \int_a^b f(y)\, dy$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

*Generalization:* The above property can be generalized into the form

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^{b} f(x)\,dx$$

where $a < c_1 < c_2 \ldots < c_{n-1} < c_n < b$.

4. $\int_o^a f(x)\,dx = \int_o^a f(a-x)\,dx$

5. $\int_o^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$

**NOTE**

This property is useful to evaluate a definite integral without first finding the corresponding indefinite integral which may be difficult or sometimes impossible to find.

6. $\int_o^{2a} f(x)\,dx = \int_o^a f(x)\,dx + \int_o^a f(2a-x)\,dx$

7. $\int_o^{2a} f(x)\,dx = \begin{cases} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_o^a f(x)\,dx & \text{if } f(2a-x) = f(x) \end{cases}$

8. $\int_{-a}^{a} f(x)\,dx$

$$= \begin{cases} 2\int_o^a f(x)\,dx \text{ if } f(-x) = f(x) \text{ i.e., } f(x) \text{ is even} \\ 0 \quad \text{if } f(-x) = -f(x) \text{ i.e., } f(x) \text{ is odd} \end{cases}$$

9. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

10. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} = \frac{b-a}{2}$

11. If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.

12. If $f(x) \leq g(x)$ on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

13. $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

14. If $f(x)$ is continuous on $[a, b]$, $m$ is the least and $M$ is the greatest value of $f(x)$ on $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

15. If $f(x)$ is a periodic function of period $a$, i.e., $f(a + x) = f(x)$, then

(A) $\int_o^{na} f(x)\,dx = n\int_o^a f(x)\,dx$

(B) $\int_a^{na} f(x)\,dx = (n-1)\int_o^a f(x)\,dx$

(C) $\int_{na}^{b+na} f(x)\,dx = \int_0^b f(x)\,dx$, where $b \in R^+$

(D) $\int_b^{b+a} f(x)\,dx$ is independent of $b$.

(E) $\int_b^{b+na} f(x)\,dx = n\int_0^a f(x)\,dx$, where $n \in I$

In particular,

(i) if $b = 0$, $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$

(ii) if $n = 1$, $\int_b^{b+a} f(x)\,dx = \int_0^a f(x)\,dx$

(F) $\int_{b+na}^{c+na} f(x)\,dx = \int_b^c f(x)\,dx$

16. For any two functions $f(x)$ and $g(x)$, integrable on the interval $[a, b]$, the Schwarz–Bunyakovsky inequality holds

$$\left|\int_a^b f(x)\cdot g(x)\,dx\right| \leq \sqrt{\int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx}$$

**17.** If a function $f(x)$ is continuous on the interval $[a, b]$, then there exists a point $c \in (a, b)$ such that

$$\int_a^b f(x)\ dx = f(c)\ (b-a), \text{ where } a < c < b$$

**18.** For a given function $f(x)$ continuous on $[a, b]$, if we are able to find two continuous function $f_1(x)$ and $f_2(x)$ on $[a, b]$ such that $f_1(x) \le f(x) \le f_2(x)$, $\forall\ x \in [a, b]$, then

$$\int_a^b f_1(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f_2(x)\,dx$$

**19.** If $f(t)$ is an odd function, then $F(x) = \int_a^x f(t)\,dt$ is an even function

**20.** If $f(t)$ is an even function, then $F(x) = \int_0^x f(t)\,dt$ is an odd function.

### CAUTION

If $f(t)$ is an even function, then $\int_a^x f(t)\,dt$; $a \neq 0$; is not necessarily an odd function. If will be odd if $\int_0^a f(t)\,dt = 0$

## USEFUL INTEGRAL

$$\int_0^{\pi/2} \log \sin x\ dx = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log\frac{1}{2}$$

## SOLVED EXAMPLES

**11.** The value of the integer $\int_0^{\pi} e^{\cos^2 x} \cdot \cos^3 (2n+1)\,x\,dx$, $n$ integer, is

(A) 0 (B) $\pi$
(C) $2\pi$ (D) None of these

**Solution: (A)**

Let $I = \int_0^{\pi} e^{\cos^2 x} \cdot \cos^3 (2n+1)\,x\,dx$

$$= \int_0^{\pi} e^{\cos^2 (\pi - x)} \cdot \cos^3 (2n+1)\,(\pi - x)\,dx$$

$$= \int_0^{\pi} e^{\cos^2 x} \cdot \cos^3 \left[(2n+1)\,\pi - (2n+1)\,x\right] dx$$

$$= -\int_0^{\pi} e^{\cos^2 x}\ \cos^3 (2n+1)\,x\,dx = -I$$

$\therefore\quad 2I = 0 \Rightarrow I = 0$

**12.** $\int_0^{\pi/2} (\tan x + \cot x)\ dx$ is equal to

(A) $\frac{\pi}{2}\log 2$ (B) $-\frac{\pi}{2}\log 2$
(C) $\pi \log 2$ (D) None of these

**Solution: (C)**

Let $I = \int_0^{\pi/2} \log(\tan x + \cot x)\ dx$

$$= \int_0^{\pi/2} \log\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2}{\sin 2x}\right) dx$$

$$= \log 2 \int_0^{\pi/2} 1\,dx - \int_0^{\pi/2} \log \sin 2x\ dx$$

$$= \frac{\pi}{2}\log 2 - \frac{1}{2}\cdot\int_0^{\pi} \log \sin z\ dz$$

$$\left(\text{Putting } 2x = z \Rightarrow dx = \frac{1}{2}\,dz\right)$$

$$= \frac{\pi}{2}\log 2 - \frac{1}{2}\cdot 2\int_0^{\pi/2} \log \sin z\ dz$$

$$= \frac{\pi}{2}\log 2 + \frac{\pi}{2}\log 2$$

$$= \pi \log 2$$

**13.** $\int_0^{\infty}\left[\frac{2}{e^x}\right] dx$ (where [ . ] denotes the greatest integer function) equals

(A) $\log_e 2$ (B) $e^2$
(C) 0 (D) $\frac{2}{e}$

**Solution: (A)**

$$\int_0^{\infty}\left[\frac{2}{e^x}\right]dx = \int_0^{\ln 2}\left[\frac{2}{e^x}\right]dx + \int_{\ln 2}^{\infty}\left[\frac{2}{e^x}\right]dx$$

$$= \int_0^{\ln 2} 1dx + 0 = \ln 2$$

**14.** $\int_{1/2}^{2}\frac{1}{x}\operatorname{cosec}^{101}\left(x-\frac{1}{x}\right)dx =$

(A) 1/4 (B) 1

(C) 0 (D) 101/2

**Solution: (C)**

$$I = \int_{1/2}^{2}\frac{1}{x}\operatorname{cosec}^{101}\left(x-\frac{1}{x}\right)dx$$

Let $\frac{1}{x} = t, -\frac{1}{x^2}dx = dt$

$\therefore$ $I = -\int_{1/2}^{2}\frac{1}{t}\operatorname{cosec}^{101}\left(t-\frac{1}{t}\right)dt$

$\Rightarrow I = -I$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

**15.** If $\int_{-1}^{-4} f(x)\,dx = 4$ and $\int_{2}^{-4}(3 - f(x))\,dx = 7$, then the value of $\int_{-2}^{1} f(-x)\,dx$ is

(A) 30 (B) 29

(C) 28 (D) None of these

**Solution: (B)**

$\int_{-1}^{-4} f(x)\,dx = 4$ and $\int_{2}^{-4}(3 - f(x))\,dx = 7,$

$\Rightarrow \int_{-4}^{2} f(x)\,dx = 7 + 18 = 25$

$I = \int_{-2}^{1} f(-x)\,dx = \int_{-1}^{2} f(x)\,dx = \int_{-1}^{-4} f(x)\,dx + \int_{-4}^{2} f(x)\,dx$

$= 4 + 25 = 29$

**16.** The value of the integral $\int_0^{\pi}\frac{\sin 2kx}{\sin x}dx$, where $k \in I$, is

(A) $\frac{\pi}{2}$ (B) $\pi$

(C) 0 (D) None of these

**Solution: (C)**

Let $I = \int_0^{\pi}\frac{\sin 2kx}{\sin x}dx = \int_0^{\pi}\frac{\sin 2k(\pi - x)}{\sin(\pi - x)}dx$

$= \int_0^{\pi}\frac{\sin(2k\pi - 2kx)}{\sin x}dx = -\int_0^{\pi}\frac{\sin 2kx}{\sin x}dx = -I$

$\therefore$ $2I = 0 \Rightarrow I = 0$

**17.** The value of the integral $\int_0^{\pi}\frac{x^2\sin x}{(2x-\pi)(1+\cos^2 x)}dx$ is

(A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^2}{2}$

(C) $\frac{\pi^2}{6}$ (D) None of these

**Solution: (A)**

Let $I = \int_0^{\pi}\frac{x^2\sin x}{(2x-\pi)(1+\cos^2 x)}dx$

$= \int_0^{\pi}\frac{(\pi - x)^2\sin(\pi - x)}{(2\pi - 2x - \pi)(1+\cos^2(\pi - x))}dx$

$= \int_0^{\pi}\frac{(\pi^2 - 2\pi x + x^2)\sin x}{(\pi - 2x)(1+\cos^2 x)}dx$

$= \int_0^{\pi}\frac{(\pi^2 - 2\pi x)\sin x}{(\pi - 2x)(1+\cos^2 x)}dx - I$

$\Rightarrow$ $2I = \pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx = -\pi\int_1^{-1}\frac{dt}{1+t^2}$

(Putting $\cos x = t \Rightarrow \sin x\,dx = -dt$)

$= \pi\left[\tan^{-1} t\right]_{-1}^{1} = \pi\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi^2}{2}$

$\therefore I = \frac{\pi^2}{4}$

**18.** $\int_0^{1.5} [x^2]\, dx$, where [.] denotes the greatest integer function, is equal to

(A) $\sqrt{2} - 2$ (B) $2 - \sqrt{2}$

(C) $2 + \sqrt{2}$ (D) None of these

**Solution: (B)**

$$\int_0^{1.5} [x^2]\, dx = \int_0^1 [x^2]\, dx + \int_1^{\sqrt{2}} [x^2]\, dx + \int_{\sqrt{2}}^{1.5} [x^2]\, dx$$

$$\left[\because [x^2] = 0,\ 0 \le x < 1;\quad 1,\ 1 \le x < \sqrt{2};\quad 2,\ \sqrt{2} \le x < 1.5\right]$$

$$= \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1 \cdot dx + \int_{\sqrt{2}}^{1.5} 2\, dx$$

$$= (x)\Big|_1^{\sqrt{2}} + 2\,(x)\Big|_{\sqrt{2}}^{3/2}$$

$$= \sqrt{2} - 1 + 2\left(\frac{3}{2} - \sqrt{2}\right) = 2 - \sqrt{2}$$

**19.** $\int_0^3 \left[\sqrt{x}\right] dx$ is equal to

(A) 1 (B) 2

(C) – 1 (D) – 2

**Solution: (B)**

$$\int_0^3 \left[\sqrt{x}\right] dx = \int_0^1 \left[\sqrt{x}\right] dx + \int_1^3 \left[\sqrt{x}\right] dx$$

$$\left[\because \left[\sqrt{x}\right] = \begin{cases} 0 \text{ if } 0 \le x < 1 \\ 1 \text{ if } 1 \le x < 3 \end{cases}\right]$$

$$= \int_0^1 0\, dx + \int_1^3 1\, dx = x\Big|_1^3 = 2.$$

**20.** If $\{x\}$ represents the fraction part of $x$, then $\int_0^{100} \{\sqrt{x}\} dx =$

(A) $\frac{155}{3}$ (B) $\frac{235}{3}$

(C) $\frac{76}{3}$ (D) None of these

**Solution: (A)**

$$\int_0^{100} \left(\sqrt{x}\right) dx = \int_0^{100} \left[\sqrt{x} - \left(\sqrt{x}\right)\right] dx$$

$$= \int_0^{100} \sqrt{x}\, dx - \int_0^{100} \left(\sqrt{x}\right) dx = \frac{2000}{3} - \int_0^{100} \left(\sqrt{x}\right) dx$$

$$= \frac{2000}{3} - \left[\int_0^1 \left(\sqrt{x}\right) dx + \int_1^4 \left(\sqrt{x}\right) dx + \int_4^9 \left(\sqrt{x}\right) dx + \ldots + \int_{81}^{100} \left(\sqrt{x}\right) dx\right]$$

$$= \frac{2000}{3} - \left(\int_0^1 0\, dx + \int_1^4 1\, dx + \int_4^9 2\, dx + \ldots + \int_{81}^{100} 9\, dx\right)$$

$$= \frac{2000}{3} - (3 + 10 + 21 + 36 + 55 + 78 + 105 + 136 + 171)$$

$$= \frac{2000}{3} - 165 = \frac{155}{3}$$

**21.** $\int_{-2}^{2} [x^2]\, dx$ is equal to

(A) $10 - 2\sqrt{3} - 2\sqrt{2}$

(B) $10 + 2\sqrt{3} - 2\sqrt{2}$

(C) $10 - 2\sqrt{3} + 2\sqrt{2}$

(D) None of these

**Solution: (A)**

$$\int_{-2}^{2} (x^2)\, dx = 2\int_0^2 (x^2)\, dx \ (\because \text{ integrand is even})$$

$$= 2\left[\int_0^1 (x^2)\, dx + \int_1^{\sqrt{2}} (x^2)\, dx + \int_{\sqrt{2}}^{\sqrt{3}} (x^2)\, dx + \int_{\sqrt{3}}^{2} (x^2)\, dx\right]$$

$$\left[\because (x^2) = 0 \text{ if } 0 \le x < 1;\ 1 \text{ if } 1 \le x < \sqrt{2};\ 2 \text{ if } \sqrt{2} \le x < \sqrt{3};\ 3 \text{ if } \sqrt{3} \le x < 2\right]$$

$$= 2\left(\int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\, dx + \int_{\sqrt{3}}^{2} 3\, dx\right)$$

$$= 2\,(x)\Big|_1^{\sqrt{2}} + 4\,(x)\Big|_{\sqrt{2}}^{\sqrt{3}} + 6\,(x)\Big|_{\sqrt{3}}^{2}$$

$$= \left(10 - 2\sqrt{3} - 2\sqrt{2}\right)$$

**22.** $\int_{-2}^{1} [x+1]\,dx$ is equal to

(A) 0 (B) 1
(C) – 1 (D) None of these

**Solution: (A)**

$$\int_{-2}^{1} (x+1)\ dx$$

$$= \int_{-2}^{-1} (x+1)\,dx + \int_{-1}^{0} (x+1)\,dx + \int_{0}^{1} (x+1)\,dx$$

$$\left[\because (x+1) = -1 \text{ if } -2 \le x < -1;\ 0 \text{ if } -1 \le x < 0;\ 1 \text{ if } 0 \le x < 1\right]$$

$$= \int_{-2}^{-1} (-1)\ dx + \int_{-1}^{0} 0 \cdot dx + \int_{0}^{1} 1\ dx$$

$$= -\,(x)\Big|_{-2}^{-1} + (x)\Big|_{0}^{1} = 0.$$

**23.** $\int_{-1}^{2} [x-1]\ dx$ is equal to

(A) – 3 (B) 3
(C) –2 (D) 2

**Solution: (A)**

$$\int_{-1}^{2} [x-1]\,dx$$

$$= \int_{-1}^{0} (x-1)\ dx + \int_{0}^{1} (x-1)\ dx + \int_{1}^{2} (x-1)\ \ dx$$

$$\left[\because (x-1) = -2 \text{ if } -1 \le x < 0; -1 \text{ if } 0 \le x < 1;\ 0 \text{ if } 1 \le x < 2\right]$$

$$= \int_{-1}^{0} (-2)\ dx + \int_{0}^{1} (-1)\ dx + \int_{1}^{2} (0)\ dx$$

$$= -2\,(x)\Big|_{-1}^{0} - (x)\Big|_{0}^{1} = -3.$$

**24.** $\int_{0}^{100} \left(x - [x]\right)\ dx$ is equal to

(A) 50 (B) 100
(C) 200 (D) None of these

**Solution: (A)**

Since $x - [x]$ is a periodic function of period 1

$$\therefore\ \int_{0}^{100} [x - (x)]\ \ dx = 100\int_{0}^{1} [x - (x)]\ \ dx$$

$$= 100 \int_{0}^{1} x\ dx - 100 \int_{0}^{1} (x)\,dx$$

$$= 100 \left(\frac{x^2}{2}\right)\Bigg|_{0}^{1} - 100 \int_{0}^{1} 0\ dx\ = \mathbf{50}$$

**25.** If $f(x)$ is an odd function, then $\int_{a}^{x} f(t)\ dt$ is

(A) odd (B) even
(C) neither even nor odd (D) periodic

**Solution: (B)**

Let $\phi(x) = \int_{a}^{x} f(t)\ \ dt$, then

$$\phi(-x) = \int_{a}^{-x} f(t)\ \ dt = -\int_{-a}^{x} f(-z)\ \ dz$$

(Putting $t = -z \Rightarrow dt = -dz$)

$$= \int_{-a}^{x} f(z)\ \ dz\ (\because f \text{ is an odd function})$$

$$= \int_{-a}^{a} f(z)\,dz + \int_{a}^{x} f(z)\ \ dz = 0 + \int_{a}^{x} f(t)\ \ dt$$

$$\left[\because f(x) \text{ is odd i.e. } f(-x) = -f(x),\ \therefore \int_{-a}^{a} f(x)\,dx = 0\right]$$

$= \phi(x)\ \therefore\ \phi(x)$ is even.

**26.** If $f(x)$ is an even function, then $\int_{0}^{x} f(t)\ \ dt$ is

(A) odd (B) even
(C) neither even nor odd (D) periodic

**Solution: (A)**

Let $\phi(x) = \int_{0}^{x} f(t)\ \ dt$, then

$$\phi(-x) = \int_0^{-x} f(t)\ dt = -\int_0^{x} f(-z)\ dz$$

(Putting $t = -z \Rightarrow dt = -dz$)

$$= -\int_0^{x} f(z)\ dz \ (\because f \text{ is an even function})$$

$$= -\int_0^{x} f(t)\ dt = -\phi(x) \ \therefore\ \phi(x) \text{ is odd.}$$

**27.** $\displaystyle\int_{-100\pi}^{100\pi} \left(\sin^4 x + \cos^4 x\right) dx$ is equal to

(A) $100\pi$ (B) $150\pi$

(C) $200\pi$ (D) None of these

**Solution: (B)**

$$\int_{-100\pi}^{100\pi} (\sin^4 x + \cos^4 x)\, dx = 2\int_0^{100\pi} (\sin^4 x + \cos^4 x)\, dx$$

($\because$ integrand is even)

$$= 2\int_0^{\frac{\pi}{2}(200)} (\sin^4 x + \cos^4 x)\, dx$$

$$= 2 \cdot 200 \int_0^{\pi/2} (\sin^4 x + \cos^4 x)\ dx$$

$\left(\because \sin^4 x + \cos^4 x \text{ is a periodic function of period } \dfrac{\pi}{2}\right)$

$$= 400\left[\int_0^{\pi/2} \sin^4 x + \int_0^{\pi/2} \cos^4\left(\frac{\pi}{2} - x\right) dx\right]$$

$$= 800 \cdot \int_0^{\pi/2} \sin^4 x\ dx = 800 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 150\pi$$

**28.** If $f(x)$ and $\phi(x)$ are continuous functions on the interval [0, 4] satisfying $f(x) = f(4 - x)$, $\phi(x) + \phi(4 - x) = 3$ and

$\displaystyle\int_0^4 f(x)\ dx = 2$, then $\displaystyle\int_0^4 f(x)\ \phi(x)\, dx =$

(A) 3 (B) 6

(C) 2 (D) None of these

**Solution: (A)**

$$\int_0^4 f(x)\,\phi(x)\ dx = \int_0^4 f(4-x)\,\phi(4-x)\ dx$$

$$= \int_0^4 f(x)\cdot(3 - \phi(x))\ dx$$

$[\because f(x) = f(4 - x) \text{ and } \phi(x) + \phi(4 - x) = 3]$

$$= 3\int_0^4 f(x)\, dx - I$$

$$\Rightarrow 2I = 3 \cdot 2 \ \therefore\ I = 3$$

**29.** The value of the integral $\displaystyle\int_{\pi/2}^{3\pi/2} [\sin x]\ dx$, where $[\cdot]$ denotes the greatest integer function, is

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{-\pi}{2}$

(C) 0 (D) $\pi$

**Solution: (B)**

$$\int_{\pi/2}^{3\pi/2} (\sin x)\, dx = \int_{\pi/2}^{\pi} (\sin x)\, dx + \int_{\pi}^{3\pi/2} (\sin x)\ dx$$

$$= \int_{\pi/2}^{\pi} 0\, dx + \int_{\pi}^{3\pi/2} (-1)\, dx = -(x)_{\pi}^{3\pi/2}$$

$$= -\left(\frac{3\pi}{2} - \pi\right) = \frac{-\pi}{2}$$

**30.** The value of $\displaystyle\int_{-2}^{2} \max\,[(1 - x), (1 + x), 2]\ dx$ is

(A) 8 (B) $-8$

(C) 9 (D) $-9$

**Solution: (C)**

For $-2 \le x \le -1$, we have $1 - x \ge 2$

and $1 - x > 1 + x$

$\therefore \max\,[(1 - x), (1 + x), 2] = 1 - x$

For $-1 < x < 1$, we have $0 < 1 - x < 2$ and $0 < 1 + x < 2$

$\therefore \max[(1 - x), (1 + x), 2] = 2$

For $1 \le x \le 2$, we have $1 + x \ge 2$ and $1 + x > 1 - x$

$\therefore \max[(1-x), (1+x), 2] = 1 + x$

$$\therefore \int_{-2}^{2} \max[(1-x), (1+x), 2]\, dx$$

$$= \int_{-2}^{-1} (1-x)\, dx + \int_{-1}^{1} 2\, dx + \int_{1}^{2} (1+x)\, dx$$

$$= \left(x - \frac{x^2}{2}\right)_{-2}^{-1} + (2x)_{-1}^{1} + \left(x + \frac{x^2}{2}\right)_{1}^{2} = 9$$

**31.** If $f$ and $g$ are two continuous functions, then the value of the integral $\int_{-\pi/4}^{\pi/4} [f(x) + f(-x)] \cdot [g(x) - g(-x)]dx$ is

(A) $\frac{\pi}{4}$ (B) 0

(C) $\frac{-\pi}{4}$ (D) None of these

**Solution: (B)**

Since $f(x) + f(-x)$ is an even function and $g(x) - g(-x)$ is an odd function, therefore, $[f(x) + f(-x)] \times [g(x) - g(-x)]$ is an odd function, therefore

$$\int_{-\pi/4}^{\pi/4} [f(x) + f(-x)] \cdot [g(x) - g(-x)]\, dx = \mathbf{0}$$

**32.** If $f(a - x) = f(x)$ and $\int_0^{a/2} f(x)\ dx = p$, then $\int_0^a f(x)\ dx$ is equal to

(A) $2p$ (B) 0

(C) $p$ (D) None of these

**Solution: (A)**

$$\int_0^a f(x)\, dx = \int_0^{a/2} f(x)\, dx + \int_{a/2}^{a} f(x)\, dx$$

$$= p - \int_{a/2}^{0} f(a-z)\, dz$$

[Putting $x = a - z$ in the second integral so that $dx = -dz$]

$$= p + \int_0^{a/2} f(a-z)\, dz \quad \left(\because \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx\right)$$

$$= p + \int_0^{a/2} f(a-x)\, dx \quad \left(\because \int_a^b f(x)\, dx = \int_a^b f(y)\, dy\right)$$

$$= p + \int_0^{a/2} f(x)\, dx \quad [\because f(a-x) = f(x)]$$

$$= p + p = 2p$$

**33.** If $I = \int_{-a}^{a} (\alpha \sin^5 x + \beta \tan^3 x + \gamma \cos x)\ dx$, where $\alpha$, $\beta$, $\gamma$ are constants, then the value of $I$ depends on

(A) $\gamma, a$ (B) $\alpha, \beta, \gamma, a$

(C) $\alpha, \beta, a$ (D) None of these

**Solution: (A)**

We have, $I$

$$= \int_{-a}^{a} (\alpha \sin^5 x + \beta \tan^3 x + \gamma \cos x)\ dx$$

$$= \alpha \int_{-a}^{a} \sin^5 x\, dx + \beta \int_{-a}^{a} \tan^3 x\, dx + \gamma \int_{-a}^{a} \cos x\ dx$$

$$= \alpha \times 0 + \beta \times 0 + \gamma \times 2 \int_0^a \cos x\ dx$$

($\because$ $\sin^5 x$ and $\tan^3 x$ are odd functions)

$\therefore$ $I$ depends on $\gamma, a$

**34.** $\int_0^1 |\sin 2\pi x|\ dx$ is equal to

(A) 0 (B) $-1/\pi$

(C) $1/\pi$ (D) $2/\pi$

**Solution: (D)**

Since $\sin 2\pi x$ is positive for $0 < x \le \frac{1}{2}$ and negative for $\frac{1}{2} < x < 1$.

$$\therefore \quad |\sin 2\pi x| = \begin{bmatrix} \sin 2\pi x, \text{ for } 0 < x \le \frac{1}{2} \\ -\sin 2\pi x, \text{ for } \frac{1}{2} < x < 1 \end{bmatrix}$$

$$\therefore \int_0^1 |\sin 2\pi x|\ dx = \int_0^{1/2} \sin 2\pi x\ dx + \int_{1/2}^1 (-\sin 2\pi x)\ dx$$

$$= \left[\frac{-\cos 2\pi x}{2\pi}\right]_0^{1/2} + \left[\frac{\cos 2\pi x}{2\pi}\right]_{1/2}^1$$

$$= \frac{2}{\pi}$$

**35.** If $\int_0^{\pi} x f\left(\sin^3 x + \cos^2 x\right)\ dx =$

$k \int_0^{\pi/2} f\left(\sin^3 x + \cos^2 x\right) dx$, then $k =$

(A) $\frac{\pi}{2}$ (B) $\pi$

(C) $2\pi$ (D) None of these

**Solution: (B)**

$$\text{Let}\quad I = \int_0^{\pi} x f\ (\sin^3 x + \cos^2 x)\ dx$$

$$= \int_0^{\pi} (\pi - x)\ f\,[\sin^3(\pi - x) + \cos^2\ (\pi - x)]\,dx$$

$$= \pi \int_0^{\pi} f\ (\sin^3 x + \cos^2 x)\ dx - I$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} f\,(\sin^3 x + \cos^2 x)\ dx$$

$$= \frac{\pi}{2}\, 2 \int_0^{\pi/2} f(\sin^3 x + \cos^2 x)\ dx$$

$$= \pi \int_0^{\pi/2} f\ (\sin^3 x + \cos^2 x)\,dx$$

$$\therefore \quad k = \pi$$

**36.** If $x > 0$ and $\int_0^x [x]\,dx = [x]\left(\frac{1}{2}A + B\right)$, where $[\cdot]$ denotes the greatest integer function, then

(A) $A = [x] - 1$ (B) $B = x - [x]$

(C) $A = [x] + 1$ (D) $B = x + [x]$

**Solution: (A, B)**

Let $(x) = l$ then $x = l + k$, $0 \le k < 1$.

$$\therefore \quad \int_0^x (x)\ dx = \int_0^{l+k} (x)\ dx$$

$$= \int_0^1 (x)\ dx + \int_1^2 (x)\ dx + \cdots + \int_{l-1}^{l} (x)\ dx + \int_l^{l+k} (x)\ dx$$

$$= 0 + 1 + 2 + \cdot + (l - 1) + l \cdot k$$

$$= \frac{1}{2}\ (l - 1)\ l + lk = \frac{1}{2}(x)\ [(x) - 1] + (x)\,[x - (x)]$$

$$= (x)\ \left\{\frac{1}{2}[(x) - 1] + [x - (x)]\right\}$$

$$\therefore \quad A = (x) - 1 \text{ and } B = x - (x)$$

**37.** The value of the integral $\int_0^1 e^{x^2}\ dx$ is

(A) greater than $e$ (B) less than $e$

(C) greater than 1 (D) less than 1

**Solution: (B, C)**

We have, $\quad 0 < x^2 < 1, \Rightarrow e^0 < e^{x^2} < e^1$

$$\Rightarrow \int_0^1 1\ dx < \int_0^1 e^{x^2}\ dx < \int_0^1 e\ dx \Rightarrow 1 < \int_0^1 e^{x^2}\,dx < e\ .$$

**38.** The value of the integral $\int_0^{100} 2^{x-[x]}\ dx$, where $[\cdot]$ denotes the greatest integer function, is

(A) log 2 (B) 50 log 2

(C) 100 log 2 (D) None of these

**Solution: (C)**

$$\int_0^{100} 2^{x-[x]}dx = \sum_{k=1}^{100} \int_{k-1}^{k} 2^{x-[x]}\ dx$$

$$= \sum_{k=1}^{100} \int_{k-1}^{k} 2^{x-(k-1)}\ dx$$

$$= \sum_{k=1}^{100} 2^{-(k-1)} \int_{k-1}^{k} 2^x\ dx$$

$$= \sum_{k-1}^{100} 2^{(1-k)} \cdot (2^k - 2^{k-1}) \log 2$$

$$= \log 2 \sum_{k=1}^{100} (2-1) = 100 \log 2$$

**39.** Let $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x\, f[x(3-x)]\, dx$

and $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f[x(3-x)]\, dx$,

where $f$ is a continuous function and $z$ is any real number, then $I_1/I_2 =$

(A) 3/2 (B) 1/2
(C) 1 (D) None of these

**Solution: (A)**

We have, $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x\, f[x(3-x)]\, dx$

$$= \int_{\sec^2 z}^{2-\tan^2 z} (3-x)\, f\{(3-x)[3-(3-x)]\}\, dx$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right]$$

$$= \int_{\sec^2 z}^{2-\tan^2 z} (3-x)\, f[x(3-x)]$$

$$= 3\int_{\sec^2 z}^{2-\tan^2 z} f[x(3-x)]\, dx - \int_{\sec^2 z}^{2-\tan^2 z} x\, f[x(3-x)]\, dx$$

$= 3\, I_2 - I_1$

$\therefore\ 2\, I_1 = 3\, I_2$ and so $I_1/I_2 = \frac{3}{2}$

**40.** If $f(x)$ is a periodic function with period $T$, then $\int_{a+2T}^{b+2T} f(x)\, dx$ is equal to

(A) $\int_{a+T}^{b+T} f(x)\, dx$ (B) $\int_{a+T}^{b} f(x)\, dx$

(C) $\int_{a}^{b+T} f(x)\, dx$ (D) $\int_{a}^{b} f(x)\, dx$

**Solution: (A, D)**

$$\int_{a+2T}^{b+2T} f(x)\, dx = \int_{a+T}^{b+T} f(y+T)\, dy$$

(Putting $x = y + T$ so that $dx = dy$)

$$= \int_a^b f(z)\, dz$$

(Again, putting $z = y + T$ so that $dz = dy$)

$$= \int_a^b f(x)\, dx.$$

**41.** The value of the integral $\int_1^2 \sqrt{(2x+3)(3x^2+4)}\, dx$ cannot exceed

(A) $\sqrt{48}$ (B) $\sqrt{66}$

(C) $\sqrt{73}$ (D) None of these

**Solution: (B)**

$$\int_1^2 \sqrt{(2x+3)(3x^2+4)}\, dx$$

$$\le \sqrt{\int_1^2 (2x+3)\, dx \cdot \int_1^2 (3x^2+4)\, dx}$$

$$\left[\because \left|\int_a^b f(x)\cdot g(x)\, dx\right| \le \sqrt{\int_a^b f^2(x)\, dx \cdot \int_a^b g^2(x)\, dx}\right]$$

$$= \sqrt{(x^2+3x)_1^2 \cdot (x^3+4x)_1^2} = \sqrt{6\times 11} = \sqrt{66}$$

**42.** If $I = \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$, then

(A) $\frac{1}{2} < I < \frac{1}{3}$ (B) $\frac{1}{4} < I < \frac{1}{3}$

(C) $\frac{1}{4} < I < 1$ (D) None of these

**Solution: (C)**

Let $f(x) = 2x^3 - 9x^2 + 12x + 4$, then $f(x)$ is a decreasing function on the interval [1, 2].

$\therefore\quad 8 = f(2) < f(x) < f(1) = 9$

$$\therefore\quad \frac{1}{3} < \frac{1}{\sqrt{2x^3 - 9x^2 + 12x + 4}} < \frac{1}{\sqrt{8}}$$

$$\Rightarrow\quad \frac{1}{3}\int_1^2 dx < \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}} < \frac{1}{\sqrt{8}}\int_1^2 dx$$

$$\Rightarrow \qquad \frac{1}{4} < \frac{1}{3} < I < \frac{1}{\sqrt{8}} < 1$$

Hence $\qquad \dfrac{1}{4} < I < 1$

**43.** $\int_0^3 \{\sqrt{x}\}\, dx$, where $\{x\}$ denotes the fractional part of $x$, is equal to

(A) $\frac{5}{2}$ (B) $\frac{7}{2}$

(C) $\frac{3}{2}$ (D) None of these

**Solution: (A)**

$$\int_0^3 \{\sqrt{x}\}\, dx = \int_0^3 \left(x - [\sqrt{x}]\right) dx = \int_0^3 x\, dx - \int_0^3 [\sqrt{x}]\, dx$$

$$= \left|\frac{x^2}{2}\right|_0^3 - \left(\int_0^1 [\sqrt{x}]\, dx + \int_1^3 [\sqrt{x}]\, dx\right)$$

$$= \frac{9}{2} - \left(\int_0^1 0\, dx + \int_1^3 1\, dx\right) = \frac{9}{2} - 2 = \frac{5}{2}$$

**44.** The value of the integral $\int_0^{2[x]} \left(x - [x]\right) dx$ is

(A) $[x]$ (B) $\frac{1}{2}[x]$

(C) $3[x]$ (D) $2[x]$

**Solution: (A)**

$$\int_0^{2(x)} [x - (x)]\, dx = \int_0^{2(x)\cdot 1} [x - (x)]\, dx$$

$$= 2(x) dx \int_0^1 [x - (x)]$$

$[\because\ x - (x)$ is a periodic function of period 1]

$$= 2(x)\left[\left|\frac{x^2}{2}\right|_0^1 - \int_0^1 (x)\, dx\right]$$

$$= 2(x)\left(\frac{1}{2} - 0\right) = (x)$$

**45.** The value of the integral $\int_0^a \frac{e^x}{e^{[x]}}\, dx,\ a \in Z_+$, is

(A) $ae$ (B) $a(e+1)$

(C) $a(1-e)$ (D) $a(e-1)$

**Solution: (D)**

$$\int_0^a \frac{e^x}{e^{(x)}}\, dx = \int_0^{a\cdot 1} e^{x-(x)}\, dx = a\int_0^1 e^{x-(x)}\, dx$$

$[\because\ x - (x)$ is a periodic function of period 1]

$$= a\int_0^1 e^x\, dx = a(e-1)$$

**46.** If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all $x\ (x \neq 0)$, then the value of the integral $\int_{\tan\theta}^{\cot\theta} f(x)\, dx$ is

(A) $\tan^2\theta$ (B) $2\tan\theta$

(C) 0 (D) None of these

**Solution: (C)**

$$\text{Let } I = \int_{\tan\theta}^{\cot\theta} f(x)\, dx = \int_{\tan\theta}^{\cot\theta} \frac{-1}{x^2} f\left(\frac{1}{x}\right) dx$$

$$\left[\because f(x) = \frac{-1}{x^2} f\left(\frac{1}{x}\right)\right]$$

$$= \int_{\cot\theta}^{\tan\theta} f(z)\, dz \quad \left(\text{Putting } \frac{1}{x} = z \text{ so that } \frac{-1}{x^2}dx = dz\right)$$

$$= -\int_{\tan\theta}^{\cot\theta} f(x)\, dx = -I \Rightarrow 2I = 0,\ \therefore\ I = 0$$

## DIFFERENTIATION UNDER INTEGRAL SIGN

If the functions $g(x)$ and $h(x)$ are defined on $[a, b]$ and differentiable at all points $x \in (a, b)$ and $f(t)$ is continuous on $[a, b]$, then

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\, dt\right] = \frac{d}{dx}[h(x)]\cdot f[h(x)] - \frac{d}{dx}[g(x)]\cdot f[g(x)]$$

## INTEGRATION OF MOD FUNCTIONS

$$\int_a^b |f(x)|\, dx$$

### WORKING RULE

(i) Solve the equation $f(x) = 0$.
Let the roots be $\alpha$ and $\beta$ such that $a < \alpha < \beta < b$.

(ii) Divide the interval $(a, b)$ into subintervals $(a, \alpha)$, $(\alpha, \beta)$ and $(\beta, b)$.

(iii) Check the sign of $f(x)$ by taking any point in each of these subintervals.

(iv) Use the property $\int_a^b f(x)\, dx$

$$= \int_a^\alpha f(x)\, dx + \int_\alpha^\beta f(x)\, dx + \int_\beta^b f(x)\, dx$$

with proper sign of $f(x)$ in each $(a, \alpha)$, $(\alpha, \beta)$, $(\beta, b)$.

### SOLVED EXAMPLES

**47.** $\int_0^{2\pi} \sqrt{\frac{1-\cos 2x}{2}}\, dx$ is equal to

(A) 2 (B) – 2
(C) 4 (D) – 4

**Solution: (C)**

$$\int_0^{2\pi} \sqrt{\frac{1-\cos 2x}{2}}\, dx = \int_0^{2\pi} |\sin x|\, dx$$

$$= \int_0^{\pi} |\sin x|\, dx + \int_\pi^{2\pi} |\sin x|\, dx$$

$\left(\because \sin x = 0 \text{ gives } x = n\pi, n = 0, \pm 1, \pm 2, \ldots; \text{ out of which only } x = \pi \in (0, 2\pi)\cdot\right)$

$$= \int_0^{\pi} \sin x\, dx - \int_\pi^{2\pi} \sin x\, dx$$

$$= -[\cos x]_0^\pi + [\cos x]_\pi^{2\pi}$$

$$= -(-1-1) + (1+1) = \mathbf{4}$$

**48.** $\int_0^{\pi} |\sin x + \cos x|\, dx$ is equal to

(A) $\sqrt{2}$ (B) $2\sqrt{2}$
(C) $3\sqrt{2}$ (D) None of these

**Solution: (B)**

$$\int_0^{\pi} |\sin x + \cos x|\, dx$$

$$dx = \int_0^{3\pi/4} |\sin x + \cos x|\, dx + \int_{3\pi/4}^{\pi} |\sin x + \cos x|\, dx$$

$$\left[\begin{array}{l}\because \sin x + \cos x = 0 \Rightarrow \cos x = -\sin x \\ \Rightarrow \tan x - 1 \Rightarrow x = \frac{3\pi}{4} \in (0, \pi)\end{array}\right]$$

$$= \int_0^{3\pi/4} (\sin x + \cos x)\, dx - \int_{3\pi/4}^{\pi} (\sin x + \cos x)\, dx$$

$$= (-\cos x + \sin x)\Big|_0^{3\pi/4} - (-\cos x + \sin x)\Big|_{3\pi/4}^{\pi}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 1\right] - \left[1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right]$$

$$= 2\sqrt{2}.$$

**49.** $\int_0^{100\pi} \sqrt{1-\cos 2x}\, dx$ is equal to

(A) $100\sqrt{2}$ (B) $50\sqrt{2}$
(C) $200\sqrt{2}$ (D) $400\sqrt{2}$

**Solution: (C)**

Let $I = \int_0^{100\pi} \sqrt{1-\cos 2x}\, dx$

$$= \sqrt{2} \int_0^{100\pi} |\sin x|\, dx$$

Since $|\sin x|$ is a periodic function of the period $\pi$

$$\therefore \quad I = 100\sqrt{2} \int_0^{\pi} |\sin x|\, dx = 100\sqrt{2} \int_0^{\pi} \sin x\, dx$$

$[\because \sin x \geq 0 \text{ in } (0, \pi)]$

$$= 100\sqrt{2}\, [-\cos x]_0^\pi = 200\sqrt{2}$$

**50.** $\int_\pi^{10\pi} |\sin x|\, dx$ is equal to

(A) 18 (B) 20
(C) 40 (D) None of these

**Solution: (A)**

Since | sin $x$ | is a periodic function of period $\pi$,

$$\therefore \int_{\pi}^{10\pi} |\sin x|\, dx = 9\int_0^{\pi} |\sin x|\, dx = 9\int_0^{\pi} \sin x\, dx$$

$[\because \sin x > 0 \text{ in } (0, \pi)]$

$$= -9\,[\cos x]_0^{\pi} = -9\,(-1-1) = 18$$

**51.** $\int_0^{10\pi} (|\sin x| + |\cos x|)\, dx$ is equal to

(A) 20 (B) 40
(C) 10 (D) None of these

**Solution: (B)**

$$\int_0^{10\pi} (|\sin x| + |\cos x|)\, dx = \int_0^{20\left(\frac{\pi}{2}\right)} (|\sin x| + |\cos x|)\, dx$$

$$= 20\int_0^{\pi/2} (|\sin x| + |\cos x|)\, dx$$

$\left(\because |\sin x| + |\cos x| \text{ is a periodic function of period } \frac{\pi}{2}\right)$

$$= 20\int_0^{\pi/2} (\sin x + \cos x)\, dx$$

$\left[\because \text{ in } \left(0, \frac{\pi}{2}\right) \text{ both } \sin x \text{ and } \cos x \text{ are positive}\right]$

$$= 20\,(-\cos x + \sin x)_0^{\pi/2}$$

$$= 20\,(1+1) = 40.$$

**52.** The value of $\int_0^{2\pi} \sqrt{1+\sin\frac{x}{2}}\, dx$ is

(A) 0 (B) 2
(C) 8 (D) 4

**Solution: (C)**

$$\int_0^{2\pi} \sqrt{1+\sin\frac{x}{2}}\, dx$$

$$= \int_0^{2\pi} \sqrt{\sin^2\frac{x}{4} + \cos^2\frac{x}{4} + 2\sin\frac{x}{4}\cos\frac{x}{4}}\, dx$$

$$= \int_0^{2\pi} \left(\sin\frac{x}{4} + \cos\frac{x}{4}\right) dx$$

$$= 4\left(-\cos\frac{x}{4} + \sin\frac{x}{4}\right)\Bigg|_0^{2\pi} = 4\,[(1+1)] = 8$$

**53.** The value of the integral $\int_{1/2e}^{e/2} |\log 2x|\, dx$ is

(A) $1 + e^{-1}$ (B) $1 - e^{-1}$
(C) $e^{-1} - 1$ (D) None of these

**Solution: (B)**

Put $2x = t \Rightarrow dx = \frac{1}{2}\, dt$

$$\therefore \int_{1/2e}^{e/2} |\log 2x|\, dx = \frac{1}{2}\int_{1/e}^{e} |\log t|\, dt$$

$$= \frac{1}{2}\left[\int_{1/e}^{1} |\log t|\, dt + \int_1^{e} |\log t|\, dt\right]$$

$$= \frac{1}{2}\left(-\int_{1/e}^{1} \log t\, dt + \int_1^{e} \log t\, dt\right)$$

$$= \frac{1}{2}\left[(-t\log t)\Big|_{1/e}^{1} + t\Big|_{1/e}^{1} + (t\log t)\Big|_1^{e} - t\Big|_1^{e}\right]$$

$$= \frac{1}{2}\left(\frac{1}{e}\log\frac{1}{e} + 1 - \frac{1}{e} + e - e + 1\right)$$

$$= \frac{1}{2}\left(\frac{-2}{e} + 2\right) = \left(1 - \frac{1}{e}\right)$$

**54.** The value of the integral $\int_a^{a+\frac{\pi}{2}} (|\sin x| + |\cos x|)\, dx$ is

(A) $a\pi$ (B) $2a\,\pi$
(C) $\frac{a\pi}{2}$ (D) Independent of $a$

**Solution: (D)**

Since ( | sin $x$ | + | cos $x$ | is a periodic function with period $\frac{\pi}{2}$, therefore, $\int_a^{a+\frac{\pi}{2}} (|\sin x| + |\cos x|)\quad dx$ is independent of $a$.

## SUMMATION OF SERIES WITH THE HELP OF DEFINITE INTEGRAL AS THE LIMIT OF A SUM

If $f(x)$ is a continuous and single valued function defined on the interval $[a, b]$, then the definite integral $\int_a^b f(x)\, dx$ is defined as follows:

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h\,[f(a+h) + f(a+2h) + \ldots + f(a+nh)]$$

where $nh = b - a$.

or $$\lim_{\substack{n\to\infty \\ h \to 0}} h \sum_{r=1}^{n} f(a + rh) = \int_a^b f(x)\, dx \qquad (1)$$

Put $a = 0$ and $b = 1 \Rightarrow nh = 1 \Rightarrow h = \frac{1}{n}$

Substitute $a = 0$, $b = 1$ and $h = \frac{1}{n}$ in (1), we get

$$\lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\, dx$$

### WORKING RULE

(i) Find the $r$th term of the given series and express it in the form $\frac{1}{n} f\left(\frac{r}{n}\right)$. Then the given series can be written as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

(ii) Write the above series equal to a definite integral by replacing $\frac{1}{n}$ by $dx$, $\frac{r}{n}$ by $x$ and

$\lim_{n\to\infty} \sum$ by $\int$

Also, lower limit of integration $= \lim_{n\to\infty} \frac{r_1}{n}$

and upper limit of integration $= \lim_{n\to\infty} \frac{r_2}{n}$,

where $r_1$ and $r_2$ are the least and greatest values of $r$ respectively.

*Note:* Here each term tends to zero when $n \to \infty$, so addition or omission of finite number of terms does not affect the required limit.

## SOLVED EXAMPLES

**55.** $\lim_{n\to\infty}\left(\frac{1^2}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \cdots + \frac{1}{2n}\right)$ is equal to

(A) $\frac{1}{3}\log 2$ (B) $\frac{1}{2}\log 2$

(C) $\log 2$ (D) None of these

**Solution: (A)**

$$\lim_{n\to\infty}\left(\frac{1^2}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \cdots + \frac{1}{2n}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{r^2}{r^3+n^3}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{\left(\frac{r}{n}\right)^2}{\left(\frac{r}{n}\right)^3 + 1}$$

$$= \int_0^1 \frac{x^2}{x^3+1}\, dx = \frac{1}{3}\left[\log(x^3+1)\right]_0^1$$

$$= \frac{1}{3}(\log 2 - \log 1) = \frac{1}{3}\log 2$$

**56.** The value of the $\lim_{n\to\infty}$

$$\left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2-1}} + \frac{1}{\sqrt{n^2-2^2}} + \cdots + \frac{n^2}{\sqrt{n^2-(n-1)^2}}\right]$$

is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$ (D) None of these

**Solution: (C)**

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2-1}} + \frac{1}{\sqrt{n^2-2^2}} + \cdots + \frac{n^2}{\sqrt{n^2-(n-1)^2}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-0^2}} + \frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{\sqrt{n^2 - r^2}} = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} \cdot \frac{1}{\sqrt{1 - r^2/n^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} = (\sin^{-1} x)_0^1 = \frac{\pi}{2}$$

## SOME USEFUL REDUCTION FORMULAE

1. If $n$ is a positive integer, then

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$$

$$= \begin{bmatrix} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}, \text{ when } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, \text{ when } n \text{ is even} \end{bmatrix}$$

2. $\int_0^{\pi/2} \sin^n x \cos^m x\, dx$

$$= \begin{bmatrix} \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{2}{3+n} \cdot \frac{1}{1+n}; \text{ if } m \text{ is odd} \\ \text{and } n \text{ may be even or odd} \\ \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{1}{2+n} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \\ \text{if } m \text{ is even and } n \text{ is odd} \\ \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{1}{2+n} \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, \\ \text{if } m \text{ is even and } n \text{ is even} \end{bmatrix}$$

These formulae can be expressed as a single formula :

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x\, dx$$

$$= \frac{[(m-1)(m-3)\ldots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)\cdots}$$

to be multiplied by $\frac{\pi}{2}$ when $m$ and $n$ are both even integers.

## SOLVED EXAMPLES

**57.** $\int_0^2 x^3 \sqrt{2x - x^2}\, dx$ is equal to

(A) $\frac{7\pi}{2}$ (B) $\frac{7\pi}{4}$ (C) $\frac{7\pi}{8}$ (D) $\frac{7\pi}{16}$

**Solution: (C)**

$$\int_0^2 x^3 \sqrt{2x - x^2}\, dx$$

$$= \int_0^{\pi/2} 8 \sin^6\theta \sqrt{4\sin^2\theta - 4\sin^4\theta}\, (4 \sin\theta \cos\theta)\, d\theta$$

[Putting $x = 2\sin^2\theta \Rightarrow dx = 4\sin\theta\cos\theta\, d\theta$]

$$= 64 \int_0^{\pi/2} \sin^8\theta \cdot \cos^2\theta\, d\theta = 64 \cdot \frac{7\cdot5\cdot3\cdot1\cdot1}{10\cdot8\cdot6\cdot4\cdot2} \cdot \frac{\pi}{2}$$

$$= \frac{7\pi}{8}$$

**58.** The value of $\int_a^b (x-a)^3 (b-x)^4\, dx$ is

(A) $\frac{(b-a)^6}{82}$ (B) $\frac{(b-a)^7}{136}$

(C) $\frac{(b-a)^8}{280}$ (D) None of these

**Solution: (C)**

Put $x = a\cos^2\theta + b\sin^2\theta$

$\Rightarrow dx = 2(b-a)\sin\theta\cos\theta\, d\theta$

$$\therefore \int_a^b (x-a)^3 (b-x)^4\, dx$$

$$= 2(b-a) \int_0^{\pi/2} (a\cos^2\theta + b\sin^2\theta - a)^3 (b - a\cos^2\theta - b\sin^2\theta)^4 \sin\theta\cos\theta\, d\theta$$

$$= 2(b-a)^8 \int_0^{\pi/2} \sin^7\theta \cos^9\theta\, d\theta$$

$$= 2(b-a)^8 \frac{6\cdot4\cdot2\cdot8\cdot6\cdot4\cdot2}{16\cdot14\cdot12\cdot10\cdot8\cdot6\cdot4\cdot2}$$

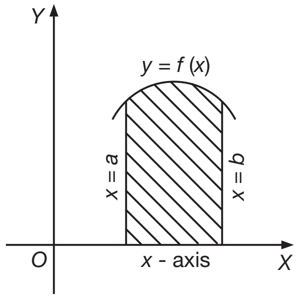
$$= \frac{(b-a)^8}{280}$$

## AREA OF PLANE REGIONS

1. The are a bounded by the curve $y = f(x)$, $x$-axis and the ordinates $x = a$. and $x = b$ (where $b > a$) is given by

$$A = \int_a^b y\, dx$$

$$= \int_a^b f(x)\, dx$$




2. The area bounded by the curve $x = g(y)$, $y$ – axis and the abscissae $y = c$ and $y = d$ (where $d > c$) is given by

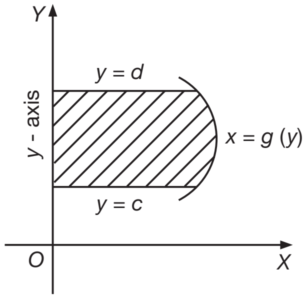



**Fig. 16.2**

$$A = \int_c^d x\,dy = \int_c^d g(y)\ dy$$

3. The area bounded by the curve $y = h(x)$, $x$-axis and the two ordinates $x = a$ and $x = b$ is given by

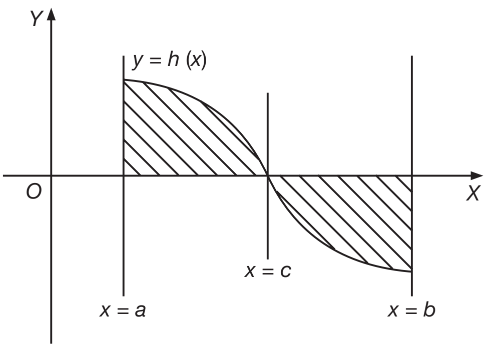


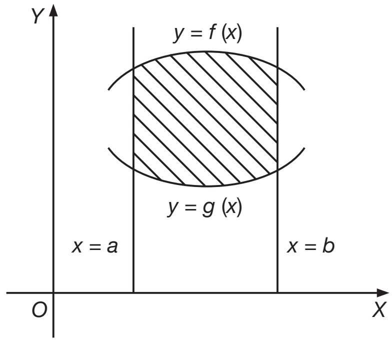

**Fig. 16.3**

$$A = \left|\int_a^c y\,dx\right| + \left|\int_c^b y\,dx\right|$$

where $c$ is a point in between $a$ and $b$.

4. If we have two curves $y = f(x)$ and $y = g(x)$, such that $y = f(x)$ lies above the curve $y = g(x)$ and both are above the axis of $x$ then the area bounded between them and the ordinates $x = a$ and $x = b$ $(b > a)$, is given by

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\ dx$$




i.e., upper curve area – lower curve area.

5. The area bounded by the curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = a$ and $x = b$ is given by

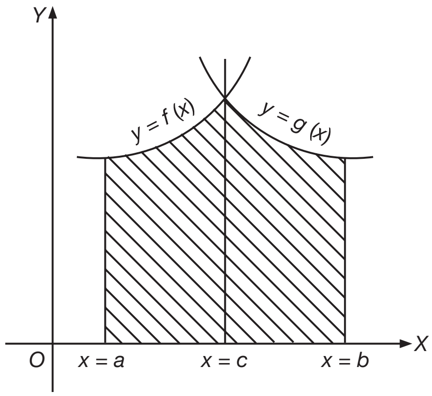



**Fig. 16.5**

$$A = \int_a^c f(x)\,dx + \int_c^b g(x)\ dx,$$

where $x = c$ is the point of intersection of the two curves.

## CURVE TRACING

In order to find the area bounded by several curves, it is important to have rough sketch of the required portion. The following steps are very useful in tracing a cartesian curve $f(x, y) = 0$.

**Step 1: Symmetry**

**(i)** The curve is symmetrical about $x$-axis if all powers of $y$ in the equation of the given curve are even.

**(ii)** The curve is symmetrical about $y$-axis if all powers of $x$ in the equation of the given curve are even.

**(iii)** The curve is symmetrical about the line $y = x$, if the equation of the given curve remains unchanged on interchanging $x$ and $y$.

**(iv)** The curve is symmetrical in opposite quadrants, if the equation of the given curve remains unchanged when $x$ and $y$ are replaced by $-x$ and $-y$ respectively.

**Step 2: Origin**

If there is no constant term in the equation of the given curve, then the curve passes through the origin.

In that case, the tangents at the origin are given by equating to zero the lowest degree terms in the equation of the given curve.

For example, the curve $y^3 = x^3 + axy$ passes through the origin and the tangents at the origin are given by $axy = 0$ i.e., $x = 0$ and $y = 0$.

**Step 3: Intersection with the Coordinate Axes**

**(i)** To find the points of intersection of the curve with X-axis, put $y = 0$ in the equation of the given curve and get the corresponding values of $x$.

**(ii)** To find the points of intersection of the curve with Y-axis, put $x = 0$ in the equation of the given curve and get the corresponding values of $y$.

**Step 4: Asymptotes**

Find out the asymptotes of the curve.

**(i)** The vertical asymptotes or the asymptotes parallel to $y$-axis of the given curve are obtained by equating to zero the coefficient of the highest power of $y$ in the equation of the given curve.

**(ii)** The horizontal asymptotes or the asymptotes parallel to $x$-axis of the given curve are obtained by equating to zero the coefficient of the highest power of $x$ in the equation of the given curve.

**Step 5: Regions where the curve does not exist**

Find out the regions of the plane in which no part of the curve lies. To determine such regions we solve the given equation for $y$ in terms of $x$ or vice-versa. Suppose that $y$ becomes imaginary for $x > a$, the curve does not lie in the region $x > a$.

**Step 6: Critical points**

Find out the values of $x$ at which $\dfrac{dy}{dx} = 0$.

At such points $y$ generally changes its character from an increasing function of $x$ to a decreasing function of $x$ or vice-versa.

**Step 7:** Trace a rough sketch of the curve, keeping in mid the above facts and plotting some points on the curve.

## Some Well-known Curves

**1. Circle**

(A) $x^2 + y^2 = a^2$

This represents a circle; the centre at (0, 0) and the radius $a$. It is symmetric about both the axes.

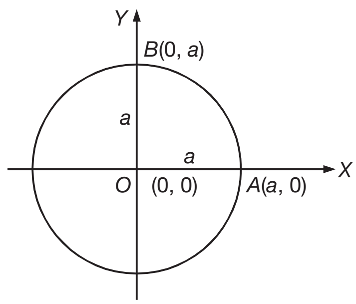



**Fig. 16.6**

(B) $(x - \alpha)^2 + (y - \beta)^2 = a^2$

This is a circle; the centre at $(\alpha, \beta)$ and the radius $a$,

(C) $x^2 + y^2 + 2gx + 2fy + c = 0$ (general equation).

This is a circle; centre at $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

**DO NOT FORGET**

- $(x - \alpha)^2 + (y - \beta)^2 < a^2$ represents the interior of the circle.
- $(x - \alpha)^2 + (y - \beta)^2 > a^2$ represents the exterior (i.e., region lying outside) of the circle.

**2. Ellipse**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > 0, b > 0.$$

It is symmetric about both the axes, meeting $x$-axis at $(\pm a, 0)$ and $y$-axis at $(0, \pm b)$, where $b < a$. Here $AA_1 = 2a$ = length of major axis, $BB_1 = 2b$ = length of minor axis.

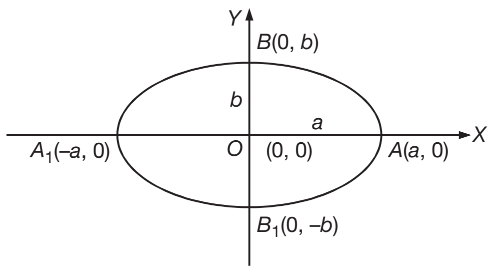
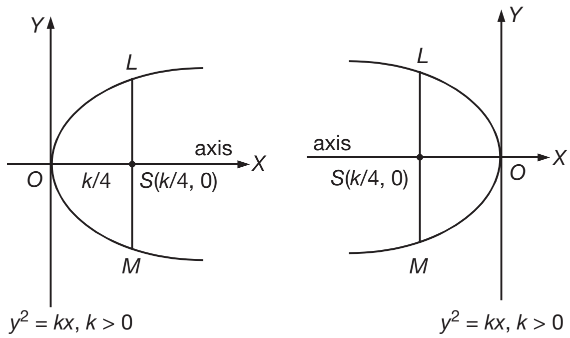



**Fig. 16.7**

**3. Parabola**

$y^2 = kx$ or $x^2 = ky$ where $k > 0$ or $k < 0$. (standard equation)

(A)

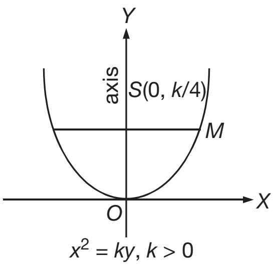



(A) (B)

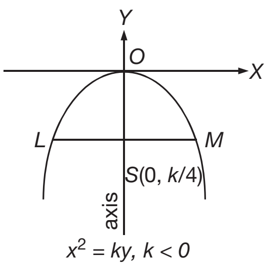
**Fig. 16.8**

It is symmetric about $x$-axis, where $S$ is the focus, $O$ is the vertex, $LM$ is the latus rectum; $LM \perp OX$; $O$ being the axis of the parabola.

(B)




(A)

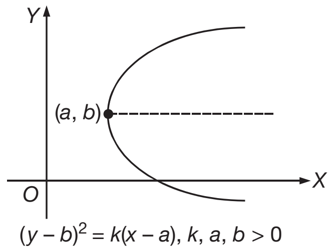



(B)

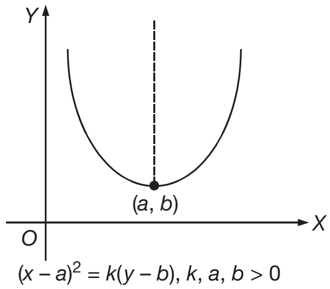
**Fig. 16.9**

It is symmetric about $y$-axis, where $S$ is the focus, $O$ is the vertex, $LM$ is the latus-rectum, $LM \perp OY$.

(C)




**(A)**

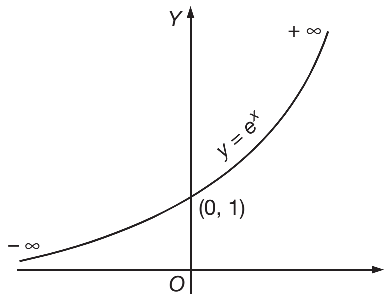



**(B) Fig. 16. 10**

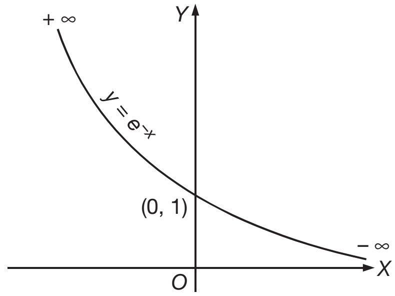
(D) $x = ay^2 + by + c$ or $y = ax^2 + bx + c$ (general equation)

Change it in the form either (i) or (ii).

**4.** Exponential curves




**(A)**




**(B) Fig. 16.11**

**5.** Logarithmic curves

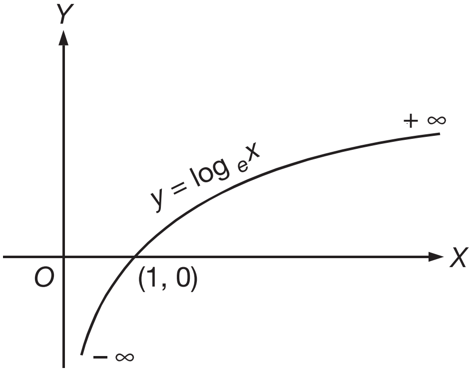



**Fig. 16.12**

**6.** Modulus function

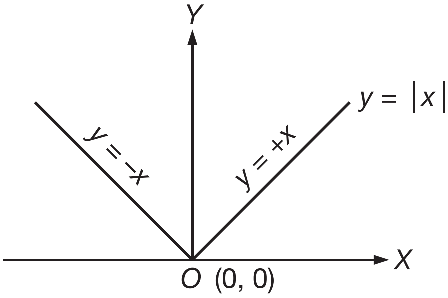
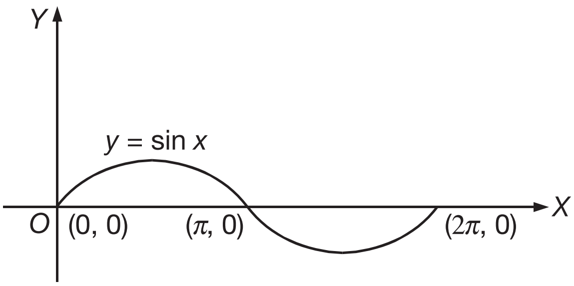
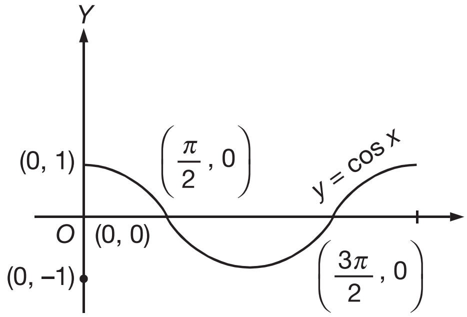



**Fig. 16.13**

**7.** Trigonometric curves




**(A)**




**(B) Fig. 16.14**

**8.**

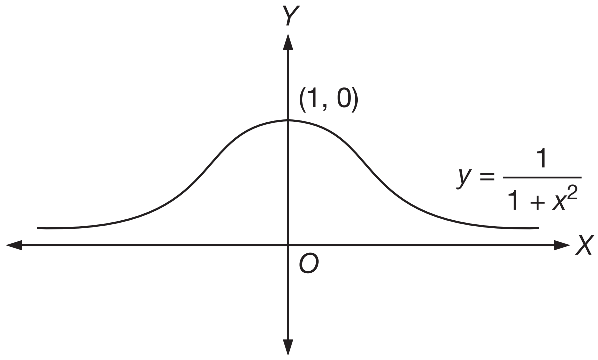



**Fig. 16.15**

# EXERCISES

## Single Option Correct Type

**1.** If $\int_0^1 e^{x^2}(x-\alpha)\,dx = 0$, then

(A) $1<\alpha<2$ (B) $\alpha<0$
(C) $0<\alpha<1$ (D) $\alpha=0$

**2.** If $x=\int_2^{\sin t}\sin^{-1}z\,dz$ and $y=\int_n^{\sqrt{t}}\frac{\sin z^2}{z}dz$, then $\frac{dy}{dx}$ is equal to

(A) $\frac{\tan t}{2t^2}$ (B) $\frac{2t^2}{\tan t}$

(C) $\frac{\tan t}{t^2}$ (D) None of these

**3.** Let $g(x)=\int_0^x f(t)\,dt$, where $f$ is such that $\frac{1}{2}\le f(t)\le 1$ for $t\in[0,1]$ and $0\le f(t)\le\frac{1}{2}$ for $t\in[1,2]$. Then $g(2)$ satisfies the inequality:

(A) $-\frac{3}{2}\le g(2)<\frac{1}{2}$ (B) $0<g(2)<2$

(C) $\frac{3}{2}<g(2)\le\frac{5}{2}$ (D) $2<g(2)<4$

**4.** $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{n^2}\cdot\sec^2\frac{r^2}{n^2}$ is equal to

(A) tan 1 (B) $\frac{1}{3}$ tan 1

(C) $\frac{1}{2}$ tan 1 (D) None of these

**5.** $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sin^{2k}\frac{r\pi}{2n}$ is equal to

(A) $\frac{2k!}{2^{2k}(k!)^2}$ (B) $\frac{2k!}{2^k(k!)}$

(C) $\frac{2k!}{2^k(k!)^2}$ (D) None of these

**6.** The value of $\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}dx$ is

(A) $\frac{\pi}{2}$ (B) 0 (C) $\pi$ (D) $2\pi$

**7.** $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$ is equal to

(A) $\frac{2}{e}$ (B) $\frac{e}{2}$ (C) $\frac{e}{4}$ (D) $\frac{4}{e}$

**8.** Let $T>0$ be a fixed real number. Suppose $f$ is a continuous function such that for all $x\in R$, $f(x+T)=f(x)$. If $I=\int_0^T f(x)\,dx$ then the value of $\int_3^{3+3T} f(2x)\,dx$ is

(A) $\frac{3}{2}I$ (B) $2I$ (C) $3I$ (D) $6I$

**9.** The value of the integral $\int_0^{\pi/2}\frac{\sin 8x\log(\cot x)}{\cos 2x}dx$ is

(A) $\frac{1}{2}$ (B) $\frac{-1}{2}$ (C) 1 (D) 0

**10.** If $I_m=\int_1^x(\log x)^m\,dx$ satisfies the relation $I_m=k-l\,I_{m-1}$, then

(A) $k=e$ (B) $l=m$
(C) $k=\frac{1}{e}$ (D) None of these

**11.** The value of $\int_{-1}^{1}[x[1+\sin\pi x]+1]\,dx$ is, ([·] denotes the greatest integer)

(A) 2 (B) 0
(C) 1 (D) None of these

**12.** The value of the integral $\int_0^{41\pi/4}|\cos x|\,dx$ is

(A) $20-\frac{1}{\sqrt{2}}$ (B) $20+\frac{1}{\sqrt{2}}$

(C) $19+\frac{1}{\sqrt{2}}$ (D) $19-\frac{1}{\sqrt{2}}$

**13.** If $\int_0^y e^{-t^2}\,dt + \int_0^{x^2} \sin^2 t\,dt = 0$, then $\frac{dy}{dx}$ is equal to

(A) $2x \sin^2 x^2\, e^{y^2}$ (B) $-2x \sin^2 x^2\, e^{y^2}$

(C) $x \sin^2 x^2\, e^{y^2}$ (D) $-x \sin^2 x^2\, e^{y^2}$

**14.** Given that $h(a) = h(b)$. Then the value of

$$\int_a^b [f\{g[h(x)]\}]^{-1} f'\{g[h(x)]\} g'[h(x)]h'(x)\,dx \text{ is}$$

(A) 0

(B) $\frac{h(a)+h(b)}{2}$

(C) $g(h(b)) - g(h(a))$

(D) $\log[f\{g[h(b)]\}] - \log[f\{g[h(a)]\}]$

**15.** $\int_0^\infty x^n e^{-x}\,dx$ ($n$ is a + *ve* integer) is equal to

(A) $n!$ (B) $(n-1)!$

(C) $(n-2)!$ (D) None of these

**16.** The area of the smaller part of the circle $x^2 + y^2 = a^2$, cut off by the line $x = \frac{a}{\sqrt{2}}$, is given by

(A) $\frac{a^2}{2}\left(\frac{\pi}{2}+1\right)$ (B) $\frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$

(C) $a^2\left(\frac{\pi}{2}-1\right)$ (D) None of these

**17.** If the ordinate $x = a$ divides the area bounded by $x$-axis, part of the curve $y = 1 + \frac{8}{x^2}$ and the ordinates $x = 2$, $x = 4$, into two equal parts, then $a$ is equal to

(A) $\sqrt{2}$ (B) $2\sqrt{2}$

(C) $3\sqrt{2}$ (D) None of these

**18.** If $[x]$ denotes the greatest integer $\le x$, then the value of

$$\int_4^{10} \frac{[x^2]}{[x^2 - 28x + 196] + [x^2]}\,dx \text{ is}$$

(A) 3 (B) 2 (C) 1 (D) 0

**19.** The area bounded by the $y = |\sin x|$, $x$-axis and the lines $|x| = \pi$ is

(A) 2 (B) 1

(C) 4 (D) None of these

**20.** If the area bounded by the curve $y = f(x)$, $x$-axis and the ordinates $x = 1$ and $x = b$ is $(b - 1)\sin(3b + 4)$, then

(A) $f(x) = \cos(3x + 4) + 3(x - 1)\sin(3x + 4)$

(B) $f(x) = \sin(3x + 4) + 3(x - 1)\cos(3x + 4)$

(C) $f(x) = \sin(3x + 4) - 3(x - 1)\cos(3x + 4)$

(D) None of these

**21.** The area bounded by the curve $y = \sin^{-1}x$ and the lines $x = 0$, $|y| = \frac{\pi}{2}$ is

(A) 2 (B) 4 (C) 8 (D) 16

**22.** For any $t \in \mathbb{R}$ and $f$ a continuous function, let

$$I_1 = \int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))\,dx \text{ and } I_2 = \int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))\,dx,$$

then $I_1/I_2$ is equal to

(A) 2 (B) 1

(C) 4 (D) None of these

**23.** If $\int_0^{100} f(x)\,dx = a$, then $\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)\,dx\right) =$

(A) $100a$ (B) $a$

(C) 0 (D) $100a$

**24.** The total area enclosed by the lines $y = |x|$, $y = 0$ and $|x| = 1$ is

(A) 2 (B) 4

(C) 1 (D) None of these

**25.** The area bounded by $y = \tan x$, $y = \cot x$, $x$-axis in $0 \le x \le \frac{\pi}{2}$ is

(A) $3 \log 2$ (B) $\log 2$

(C) $2 \log 2$ (D) None of these

**26.** The value of

$$\int_{-1/2}^{1/2} \left[\alpha \log\left(\frac{1+x}{1-x}\right) + \beta \log\left(\frac{1-x}{1+x}\right)^{-2} + \gamma\right] dx \text{ depends}$$

on the value of

(A) $\gamma$ (B) $\beta$ (C) $\alpha$ (D) $\alpha$ and $\beta$

**27.** The area of the smaller part bounded by the semi-circle $y = \sqrt{4 - x^2}$, $y = x\sqrt{3}$ and $x$-axis is

(A) $\frac{\pi}{3}$  (B) $\frac{2\pi}{3}$
(C) $\frac{4\pi}{3}$  (D) None of these

**28.** The number of possible solutions of the equation

$\int_0^x (t^2 - 8t + 13)\,dt = x\sin\left(\frac{a}{x}\right)$ is

(A) 2  (B) 1
(C) no solution  (D) infinite

**29.** The area bounded by the lines $y = 2$, $x = 1$, $x = a$ and the curve $y = f(x)$, which cuts the last two lines above the first line for all $a \ge 1$, is equal to

$\cdot\frac{2}{3}\left[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}\right]$ Then $f(x) =$

(A) $2\sqrt{2x}\; x \ge 1$  (B) $\sqrt{2x}$, $x \ge 1$
(C) $2\sqrt{x}$, $x \ge 1$  (D) None of these

**30.** The area above $x$-axis, bounded by the line $x = 4$ and the curve $y = f(x)$, where $f(x) = x^2$, $0 \le x \le 1$and $f(x) = \sqrt{x}$, $x \ge 1$, is

(A) 1  (B) 2  (C) 4  (D) 5

**31.** The area of the portion of the circle $x^2 + y^2 = 1$, which lies inside the parabola $y^2 = 1 - x$, is

(A) $\frac{\pi}{2} - \frac{2}{3}$  (B) $\frac{\pi}{2} + \frac{2}{3}$  (C) $\frac{\pi}{2} + \frac{4}{3}$  (D) $\frac{\pi}{2} - \frac{4}{3}$

**32.** If $f(x) = \int_0^1 \frac{dt}{1 + |x - t|}$, then $f'\left(\frac{1}{2}\right)$ is equal two

(A) 1  (B) –1  (C) $\frac{1}{2}$  (D) 0

**33.** The area bounded by the parabolas $y^2 = 4a(x + a)$ and $y^2 = -4a(x - a)$ is

(A) $\frac{16}{3}a^2$  (B) $\frac{8}{3}a^2$
(C) $\frac{4}{3}a^2$  (D) None of these

**34.** Let $f$ beintegrable over $[0, a]$ for any real $a$. If we define

$I_1 = \int_0^{\pi/2} \cos\theta\, f(\sin\theta + \cos^2\theta)\,d\theta$ and

$I_2 = \int_0^{\pi/2} \sin 2\theta\, f(\sin\theta + \cos^2\theta)\,d\theta$, then

(A) $I_1 = I_2$  (B) $I_1 = -I_2$
(C) $I_1 = 2I_2$  (D) $I_1 = -2I_2$

**35.** The area bounded by the circle $x^2 + y^2 = 8$, the parabola $x^2 = 2y$ and the line $y = x$ in $y \ge 0$ is

(A) $\frac{2}{3} + 2\pi$  (B) $\frac{2}{3} - 2\pi$
(C) $\frac{2}{3} + \pi$  (D) $\frac{2}{3} - \pi$

**36.** The area lying in the first quadrant inside the circle $x^2 + y^2 = 12$ and bounded by the parabolas $y^2 = 4x$, $x^2 = 4y$ is

(A) $2\left(\frac{\sqrt{2}}{3} + \frac{3}{2}\sin^{-1}\frac{1}{3}\right)$

(B) $4\left(\frac{\sqrt{2}}{3} + \frac{3}{2}\sin^{-1}\frac{1}{3}\right)$

(C) $\left(\frac{\sqrt{2}}{3} + \frac{3}{2}\sin^{-1}\frac{1}{3}\right)$

(D) None of these

**37.** If $f(y) = e^y$, $g(y) = y$; $y > 0$ and $\phi(t) = \int_0^t f(t - y)\,g(y)\,dy$, then $\phi(t) =$

(A) $e^t - (1 + t)$  (B) $1 - e^{-t}(1 + t)$
(C) $te^t$  (D) None of these

**38.** The value of $\int_{-1}^{1} \frac{\sin^2 x}{\left[\frac{x}{\sqrt{2}}\right] + \frac{1}{2}}dx$, where $[x]$ = greatest integer less than or equal to $x$, is

(A) 1  (B) 0
(C) $4 - \sin 4$  (D) None of these

**39.** Suppose that $f''(x)$ is continuous for all $x$ and $f(0) = f'(1) = 1$. If $\int_0^1 t\, f''(t)\,dt = 0$, then the value of $f(1)$ is

(A) 3  (B) 2
(C) $4\frac{1}{2}$  (D) None of these

**40.** Let $\frac{d}{dx}\phi(x)=\left(\frac{e^{\sin x}}{x}\right), x>0$. If $\int_1^4 \frac{3}{x}e^{\sin x^3}\,dx = \phi(k) - \phi(1)$, then one of the possible values of $k$ is

(A) 48 (B) 32
(C) 64 (D) None of these

**41.** If $I_1 = \int_0^{3\pi} f\left(\cos^2 x\right) dx$ and $I_2 = \int_0^{\pi} f\left(\cos^2 x\right) dx$, then

(A) $I_1 = 5I_2$ (B) $I_1 = I_2$
(C) $I_1 = 3I_2$ (D) None of these

**42.** If $u_{10} = \int_0^{\pi/2} x^{10} \sin x\, dx$, then the value of $u_{10} + 90u_8$ is

(A) $9\left(\frac{\pi}{2}\right)^9$ (B) $10\left(\frac{\pi}{2}\right)^9$
(C) $\left(\frac{\pi}{2}\right)^9$ (D) $9\left(\frac{\pi}{2}\right)^8$

**43.** If $a$, $b$ ($a < b$) be the points of discontinuity of function $fofof(x)$, where $f(x) = \frac{1}{1-x}$, $x \neq 1$, then $\int_a^b \frac{f(x)}{f(x)+f(1-x)}dx =$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2

**44.** One value of $k$ for which the area of the figure bounded by the curve $y = 8x^2 - x^5$, the straight lines $x = 1$ and $x = k$ and the $x$-axis is equal to $\frac{16}{3}$, is

(A) – 1 (B) 3
(C) 2 (D) $\sqrt[3]{8-\sqrt{17}}$

**45.** If $\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt$, then the value of $f(1)$ is

(A) $\frac{1}{2}$ (B) 0 (C) 1 (D) $\frac{-1}{2}$

**46.** If $\int_0^1 \frac{\sin t}{1+t}dt = \alpha$, then the value of the integral $\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi+2-t}dt$ in terms of $\alpha$ is given by

(A) $2\,\alpha$ (B) $-2\,\alpha$ (C) $\alpha$ (D) $-\alpha$

**47.** Given $\int_1^2 e^{x^2}dx = a$, the value of $\int_e^{e^4}\sqrt{\log_e x}\;dx$ is

(A) $e^4 - e$ (B) $e^4 - a$
(C) $2e^4 - a$ (D) $2e^4 - e - a$

**48.** The value of $\int_{-\pi/2}^{\pi/2}\left(\left[\frac{x}{\pi}\right]+0.5\right)dx$ is

[where [.] denotes the greates integer function)

(A) $\pi$ (B) $\pi/2$ (C) 0 (D) $-\pi/2$

**49.** If $\frac{1}{\sqrt{a}}\int_1^a\left(\frac{3}{2}\sqrt{x}+1-\frac{1}{\sqrt{x}}\right)dx < 4$, then '$a$' may take the value

(A) 0 (B) 4
(C) 9 (D) None of these

**50.** If $I_1 = \int_0^{\pi/2}\cos(\sin x)\;dx$; $I_2 = \int_0^{\pi/2}\sin(\cos x)\,dx$ and $I_3 = \int_0^{\pi/2}\cos x\,dx$, then

(A) $I_1 > I_3 > I_2$ (B) $I_3 > I_1 > I_2$
(C) $I_1 > I_2 > I_3$ (D) $I_3 > I_2 > I_1$.

**51.** The value of $c$ for which the area of the figure bounded by the curve $y = 8x^2 - x^5$, the straight lines $x = 1$ and $x = c$ and the $x$-axis is equal to 16/3, is

(A) 2 (B) $\sqrt{8-\sqrt{17}}$
(C) 3 (D) –1

**52.** Let $f(x)$ be a continuous function such that the area bounded by the curve $y = f(x)$, $x$-axis and the lines $x = 0$ and $x = a$ is $\frac{a^2}{2}+\frac{a}{2}\sin a+\frac{\pi}{2}\cos a$, then $f\left(\frac{\pi}{2}\right) =$

(A) 1 (B) $\frac{1}{2}$
(C) $\frac{1}{3}$ (D) None of these

**53.** The value of the integral $\int_1^{e^6}\left[\frac{\log x}{3}\right]dx$, where [·] denotes the greatest integer function, is

(A) 0 (B) $e^6 - e^3$
(C) $e^6 + e^3$ (D) $e^3 - e^6$

**54.** If $\int_a^b |\sin x|dx = 8$ and $\int_a^{a+b}|\cos x|\,dx = \frac{9}{2}$, then $a$ is equal to:

(A) $\frac{\pi}{2}$ (B) $\pi$
(C) $\frac{\pi}{4}$ (D) None of these

**55.** $\int_0^{\pi/6} \sec^2 x \, d(x - [x])$ is equal to

(A) $\sqrt{3}$ (B) $\frac{1}{\sqrt{3}}$
(C) 1 (D) None of these

**56.** Let $f(x) = \max\{x + |x|, x - [x]\}$, where $[x]$ denotes the greatest integer less than or equal to $x$, then $\int_{-2}^{2} f(x)\,dx$ is equal to

(A) 1 (B) 3
(C) 5 (D) None of these

**57.** $\left|\int_{10}^{19} \frac{\sin x \, dx}{1 + x^8}\right|$ is less than

(A) $10^{-10}$ (B) $10^{-11}$
(C) $10^{-7}$ (D) $10^{-9}$

**58.** If $\int_0^1 \frac{dx}{2e^x - 1} = p \log(qe - 1) - r$, then

(A) $p = 1, q = 1, r = -1$ (B) $p = 1, q = 2, r = 1$
(C) $p = 1, q = 2, r = -1$ (D) None of these

**59.** $\int_0^{\pi} [\tan^{-1} x]\,dx$ is equal to (where $[\cdot]$ denotes greatest integer function)

(A) $\pi$ (B) tan 1
(C) $\pi$+ tan 1 (D) $\pi$ – tan 1

**60.** For $y = f(x) = \int_0^x 2|t|\,dt$, the tangent lines parallel to the bisector of the first quadrant angle are

(A) $y = x \pm \frac{1}{4}$ (B) $y = x \pm \frac{3}{2}$
(C) $y = x \pm \frac{1}{2}$ (D) None of these

**61.** Let $f(x)$ be a continuous function in $\mathbb{R}$ such that $f(x) + f(y) = f(x + y)$, then $\int_{-2}^{2} f(x)\,dx =$

(A) $2\int_0^2 f(x)\,dx$ (B) 0
(C) $2f(2)$ (D) None of these

**62.** $\int_0^{\pi} |1 + 2\cos x|\,dx$ is equal to

(A) $\frac{\pi}{3} - 2\sqrt{3}$ (B) $\frac{\pi}{3} - \sqrt{3}$
(C) $\frac{\pi}{3} + \sqrt{3}$ (D) $\frac{\pi}{3} + 2\sqrt{3}$

**63.** If $\int_0^{\infty} e^{-ax}\,dx = \frac{1}{a}$, then $\int_0^{\infty} x^n e^{-ax}\,dx$ is

(A) $\frac{(-1)^n n!}{a^{n+1}}$ (B) $\frac{(-1)^n (n-1)!}{a^n}$
(C) $\frac{n!}{a^{n+1}}$ (D) None of these

**64.** The tangent to the curve $y = f(x)$ at the point with abscissa $x = 1$ form an angle of $\pi/6$ and at the point $x = 2$ an angle of $\pi/3$ and at the point $x = 3$ an angle of $\pi/4$. If $f''(x)$ is continuous, then the value of $\int_1^3 f''(x) f'(x)\,dx + \int_2^3 f''(x)\,dx$ is

(A) $\frac{4\sqrt{3} - 1}{3\sqrt{3}}$ (B) $\frac{3\sqrt{3} - 1}{2}$
(C) $\frac{4 - 3\sqrt{3}}{3}$ (D) None of these

**65.** If $f(x) = \frac{e^x}{1 + e^x}$, $I_1 = \int_{f(-a)}^{f(a)} x\, g[x(1 - x)]\,dx$ and $I_2 = \int_{f(-a)}^{f(a)} g[x(1 - x)]\,dx$, then the value of $\frac{I_2}{I_1}$ is

(A) 1 (B) –3 (C) –1 (D) 2

**66.** The value of $\int_0^1 \frac{8\log(1 + x)}{1 + x^2}\,dx$ is

(A) log 2 (B) $\pi \log 2$
(C) $\frac{\pi}{8}\log 2$ (D) $\frac{\pi}{2}\log 2$

**67.** For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$

Then $f$ has

(A) Local maximum at $\pi$ and local $2\pi$
(B) Local maximum at $\pi$ and $2\pi$
(C) Local minimum at $\pi$ and $2\pi$
(D) Local minimum at $\pi$ and local maximum at $2\pi$

**68.** The shortest distance between line $y - x = 1$ and curve $x = y^2$ is

(A) $\frac{4}{\sqrt{3}}$ (B) $\frac{\sqrt{3}}{4}$

(C) $\frac{3\sqrt{2}}{8}$ (D) $\frac{8}{3\sqrt{2}}$

**69.** The area of the region enclosed by the curves $y = x$, $x = e$, $y = 1/x$ and the positive $x$-axis is

(A) 5/2 square units (B) 1/2 square units
(C) 1 square units (D) 3/2 square units

**70.** The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line $y = 2$ is

(A) $20\sqrt{2}$ (B) $\frac{10\sqrt{2}}{3}$

(C) $\frac{20\sqrt{2}}{3}$ (D) $10\sqrt{2}$

**71.** Using the fact that $0 \le f(x) \le g(x)$, $c < x < d \Rightarrow \int_c^d f(x)\,dx \le \int_c^d g(x)\,dx$, we can conclude that $\int_1^3 \sqrt{3 + x^3}\, dx$ lies in the interval

(A) $\left(\frac{1}{2}, 3\right)$ (B) $(2, \sqrt{30})$

(C) $\left(\frac{3}{2}, 5\right)$ (D) $(4, 2\sqrt{30})$

**72.** $\lim_{n \to \infty} \frac{1}{n}\left[1 + \frac{n^2}{n^2 + 1^2} + \frac{n^2}{n^2 + 2^2} + \ldots + \frac{n^2}{n^2 + (n-1)^2}\right]$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

**73.** Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx$. Then which one of the following is true?

(A) $I > \frac{2}{3}$ and $J > 2$ (B) $I < \frac{2}{3}$ and $J \le 2$

(C) $I < \frac{2}{3}$ and $J > 2$ (D) $I > \frac{2}{3}$ and $J < 2$

**74.** The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to

(A) 5/3 (B) 1/3 (C) 2/3 (D) 4/3

**75.** $\int_0^{\pi} [\cot x] dx$, $[\cdot]$ denotes the greatest integer function, is equal to

(A) $\frac{\pi}{2}$ (B) 1

(C) –1 (D) $-\frac{\pi}{2}$

**76.** Given $\int_1^2 e^{x^2} dx = a$, the value of $\int_e^{e^4} \sqrt{\ln(x)}\, dx$ is

(A) $e^4 - e$ (B) $e^4 - a$
(C) $2e^4 - a$ (D) $2e^4 - e - a$

**77.** $\int_{-2}^{2} [x^2]\, dx$ is equal to

(A) $10 - 2\sqrt{3} - 2\sqrt{2}$
(B) $10 + 2\sqrt{3} - 2\sqrt{2}$
(C) $10 - 2\sqrt{3} + 2\sqrt{2}$
(D) None of these

**78.** If $f: R \to R$ is continuous and differentiable function such that $\int_{-1}^{x} f(t)\, dt + f'''(3)\int_x^0 dt = \int_1^x t^3\, dt - f'(1)\int_0^x t^2\, dt + f''(2)\int_x^3 t\, dt$, then the value of $f'(4)$ is

(A) $48 - 8f'(1) - f''(2)$
(B) $48 + 8f'(1) - f''(2)$
(C) $48 - 8f'(1) + f''(2)$
(D) $48 + 8f'(1) + f''(2)$

**79.** The value of $\int_{-2}^{2} \max\{(1 - x), (1 + x), 2\}\, dx$ is

(A) 8 (B) – 8 (C) 9 (D) – 9

**80.** Let $I_1 = \left(-4\sqrt{2} + \frac{20}{3}\right) dx$ and, $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x))$ $dx$, where $f$ is a continuous function and $z$ is any real number, then $I_1/I_2 =$

(A) $\frac{3}{2}$ (B) $\frac{1}{2}$

(C) 1 (D) None of these

**81.** The value of the integral

$\int_1^2 \sqrt{(2x+3)(3x^2+4)}\, dx$ cannot exceed

(A) $\sqrt{48}$ (B) $\sqrt{66}$

(C) $\sqrt{73}$ (D) None of these

**82.** If $I = \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$, then

(A) $\frac{1}{2} < I < \frac{1}{3}$ (B) $\frac{1}{4} < I < \frac{1}{3}$

(C) $\frac{1}{4} < I < 1$ (D) None of these

**83.** If $I_{1,\,n} = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx$ and $I_{2,\,n} = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$, $n \in N$, then

(A) $I_{2,\,n+1} - I_{2,\,n} = I_{1,n}$

(B) $I_{2,\,n+1} - I_{2,\,n} = I_{1,n+1}$

(C) $I_{2,\,n+1} + I_{1,\,n} = I_{2,n}$

(D) $I_{2,\,n+1} + I_{1,\,n+1} = I_{2,n}$

**84.** The value of the integral $\int_0^{2[x]} (x - [x])\, dx$ is

(A) $[x]$ (B) $\frac{1}{2}[x]$ (C) $3[x]$ (D) $2[x]$

**85.** $f(x)$ is a continuous function for all real values of $x$ and satisfies $\int_0^x f(t)\, dt = \int_x^1 t^2 f(t)\, dt + \frac{x^{16}}{8} + \frac{x^6}{3} + k$. The value of $k$ is

(A) $\frac{167}{840}$ (B) $-\frac{167}{840}$

(C) $\frac{17}{38}$ (D) None of these

**86.** If $f(x) = \frac{x-1}{x+1}$, $f^2(x) = f(f(x))$, . . . $f_{(x)}^{k+1} = f(f^k(x))$, $k = 1, 2, 3, \ldots$ and $\phi(x) = f^{1998}(x)$, then $\int_{1/e}^{1} \phi(x)\, dx =$

(A) 1 (B) –1

(C) 0 (D) None of these

**87.** Let $g(x) = \int_0^x f(t)\, dt$, where $f$ is such that $\frac{1}{2} \le f(t) \le 1$ for $t \in [0, 1]$ and $0 \le f(t) \le \frac{1}{2}$ for $t \in [1, 2]$. Then,

(A) $-\frac{3}{2} \le g(2) \le \frac{1}{2}$ (B) $\frac{3}{2} \le g(2) \le \frac{5}{2}$

(C) $\frac{1}{2} \le g(2) \le \frac{3}{2}$ (D) None of these

**88.** $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \sin^{2k} \frac{r\pi}{2n}$ is equal to

(A) $\frac{2k!}{2^{2k}(k!)^2}$ (B) $\frac{2k!}{2^k(k!)}$

(C) $\frac{2k!}{2^k(k!)^2}$ (D) None of these

**89.** The smaller area enclosed by $y = f(x)$, where $f(x)$ is a polynomial of least degree satisfying $\lim_{x\to 0}\left\{1 + \frac{f(x)}{x^3}\right\}^{1/x} = e$ and the circle $x^2 + y^2 = 2$ above the $x$-axis, is

(A) $\left(\frac{\pi}{3} + \frac{2}{5}\right)$ square units

(B) $\left(\frac{\pi}{2} + \frac{3}{5}\right)$ square units

(C) $\left(\frac{\pi}{6} + \frac{3}{5}\right)$ square units

(D) $\left(\frac{\pi}{3} + \frac{6}{5}\right)$ square units

**90.** The value of the integral $\int_0^{41\pi/4} |\cos x|\, dx$ is

(A) $20 - \frac{1}{\sqrt{2}}$ (B) $20 + \frac{1}{\sqrt{2}}$

(C) $19 + \frac{1}{\sqrt{2}}$ (D) $19 - \frac{1}{\sqrt{2}}$

**91.** If the ordinate $x = a$ divides the area bounded by $x$-axis, part of the curve $y = 1 + \frac{8}{x^2}$ and the ordinates $x = 2$, $x = 4$, into two equal parts, then $a$ is equal to

(A) $\sqrt{2}$ (B) $2\sqrt{2}$
(C) 3 (D) None of these

**92.** If $\int_0^{100} f(x)\,dx = a$, then $\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)\,dx\right)$

(A) $100a$ (B) $a$ (C) 0 (D) $100a$

**93.** The area bounded by the lines $y = 2$, $x = 1$, $x = a$ and the curve $y = f(x)$, which cuts the last two lines above the first line for all $a \geq 1$, is equal to $\frac{2}{3}\left[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}\right]$. Then, $f(x) =$

(A) $2\sqrt{2x}\ x \geq 1$ (B) $\sqrt{2x}\ , x \geq 1$
(C) $2\sqrt{x}\ , x \geq 1$ (D) None of these

**94.** The area bounded by the parabolas $y^2 = 4a(x + a)$ and $y^2 = -4a(x - a)$ is

(A) $\frac{16}{3}a^2$ (B) $\frac{8}{3}a^2$
(C) $\frac{4}{3}a^2$ (D) None of these

**95.** If $\int_0^1 \frac{\sin t}{1+t}\,dt = \alpha$, then the value of the integral $\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi + 2 - t}\,dt$ in terms of $\alpha$ is given by

(A) $2\alpha$ (B) $-2\alpha$
(C) $\alpha$ (D) $-\alpha$

**96.** If $I_1 = \int_0^{\pi/2} \cos(\sin x)\,dx$; $I_2 = \int_0^{\pi/2} \sin(\cos x)\,dx$ and $I_3 = \int_0^{\pi/2} \cos x\,dx$, then

(A) $I_1 > I_3 > I_2$ (B) $I_3 > I_1 > I_2$
(C) $I_1 > I_2 > I_3$ (D) $I_3 > I_2 > I_1$

**97.** If $a_n = \int_0^{\pi/4} \cot^n x\,dx$, then $a_2 + a_4$, $a_3 + a_5$, $a_4 + a_6$ are in

(A) GP (B) AP
(C) HP (D) None of these

**98.** Let $f(x)$ be a continuous function in [–2, 2] such that $f(x) + f(y) = f(x + y)$, then $\int_{-2}^{2} f(x)\,dx =$

(A) $2\int_0^2 f(x)\,dx$ (B) 0
(C) $2f(2)$ (D) None of these

**99.** If $\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$, then $\int_0^\infty x^n e^{-ax}\,dx$ is

(A) $\frac{(-1)^n n!}{a^{n+1}}$ (B) $\frac{(-1)^n (n-1)!}{a^n}$
(C) $\frac{n!}{a^{n+1}}$ (D) None of these

**100.** Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f(x)$, $x$-axis and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$ is $\left(\beta\sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta\right)$. Then, $\left(\frac{\pi}{2}\right)f$ is

(A) $\left(1 - \frac{\pi}{4} - \sqrt{2}\right)$ (B) $\left(1 - \frac{\pi}{4} + \sqrt{2}\right)$
(C) $\left(\frac{\pi}{4} + \sqrt{2} - 1\right)$ (D) $\left(\frac{\pi}{4} - \sqrt{2} + 1\right)$

**101.** The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4$, $y = 4$ and the coordinate axes. If $S_1$, $S_2$, $S_3$ are respectively the areas of these parts numbered from top to bottom; then $S_1 : S_2 : S_3$ is

(A) 2 : 1 : 2 (B) 1 : 1 : 1
(C) 1 : 2 : 1 (D) 1 : 2 : 3

**102.** The value of $\int_1^a [x]f'(x)dx, a > 1$, where $[x]$ denotes the greatest integer not exceeding $x$ is

(A) $a\,f([a]) - \{f(1) + f(2) + ... + f(a)\}$
(B) $a\,f(a) - \{f(1) + f(2) + ... + f([a])\}$
(C) $[a]\,f(a) - \{f(1) + f(2) + ... + f([a])\}$
(D) $[a]\,f([a]) - \{f(1) + f(2) + ... + f(a)\}$

**103.** If $I_1 = \int_0^a [x]\,dx$ and $I_2 = \int_0^a \{x\}\,dx$, where $[x]$ and $\{x\}$ denote, respectively, the integral and fractional parts of $x$ and $a$ is a positive integer, then

(A) $I_2 = (a-1) I_1$ (B) $I_1 = (a-1) I_2$
(C) $I_1 = a I_2$ (D) $I_2 = aI_1$

**104.** $\int_0^1 \frac{dx}{1+x^2+2x^5}$ lies between

(A) $\frac{1}{4}$ and 1 (B) $\frac{1}{4}$ and $\frac{1}{2}$
(C) $\frac{1}{2}$ and 1 (D) None of these

**105.** $\int_0^1 \frac{dx}{1+x^2+2x^5}$ lies between

(A) $\frac{\pi}{6\sqrt{3}}$ and $\frac{\pi}{4}$ (B) $\frac{\pi}{3\sqrt{3}}$ and $\frac{\pi}{2}$
(C) $\frac{\pi}{3\sqrt{3}}$ and $\frac{\pi}{4}$ (D) None of these

**106.** $\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})\, dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})\, dt$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$
(C) 0 (D) None of these

**107.** If $f(x)$ satisfies the relation $\int_{-2}^{x} f(t)\, dt + xf'''(3)$

$= \int_1^x x^3 dx + f'(1)\int_2^x x^2 dx + f''(2)\int_3^x x\, dx$, then

(A) $f(x) = x^3 + 5x^2 + 2x - 6$
(B) $f(x) = x^3 - 5x^2 + 2x + 6$
(C) $f(x) = x^3 + 5x^2 + 2x - 6$
(D) $f(x) = x^3 - 5x^2 + 2x - 6$

**108.** $\lim_{x\to 0} \frac{1}{x}\left[\int_0^{x+y} e^{\sin^2 t} dt - \int_0^{y} e^{\sin^2 t} dt\right]$, where $y$ is a constant independent of $x$, is equal to

(A) $e^{\sin^2 y}$ (B) $2\, e^{\sin^2 y}$
(C) $-e^{\sin^2 y}$ (D) None of these

**109.** $\int_0^5 \frac{\tan^{-1}(x-[x])}{1+(x-[x])^2}\, dx$, where $[\,\cdot\,]$ denotes the greatest integer function, is equal to

(A) $\frac{\pi^2}{32}$ (B) $\frac{3\pi^2}{32}$
(C) $\frac{5\pi^2}{32}$ (D) None of these

**110.** If $[x]$ and $\{x\}$ denote the integral and fractional parts of $x$, respectively, then $\int_0^x \left(x - [x] - \frac{1}{2}\right) dx$ is equal to

(A) $\frac{1}{2}\{x\}(\{x\}-1)$ (B) $\frac{1}{2}\{x\}(\{x\}+1)$
(C) $\{x\}(\{x\}-1)$ (D) None of these

**111.** $\int_0^{k\pi} \sin\left[\frac{2x}{\pi}\right] dx = A.\ \frac{\sin k \sin\left(k+\frac{1}{2}\right)}{\sin\frac{1}{2}}$, where $A$ is equal to

(A) $\pi$ (B) $\frac{\pi}{4}$
(C) $\frac{\pi}{2}$ (D) None of these

**112.** If $f(x) = \int_0^x (1+t^3)^{-1/2} dt$ and $g$ is the inverse of $f$, then the value of $\frac{g''}{g^2}$ is

(A) $\frac{1}{2}$ (B) $\frac{3}{2}$
(C) 1 (D) Cannot be determined

**113.** If $I_n = \int_0^{\pi/2} \cos^n x \cos nx\, dx$, then $I_1, I_2, I_3$ are in

(A) AP (B) GP
(C) HP (D) None of these

**114.** If $I_n = \int_0^1 (1-x^a)^n\, dx$, then $\frac{I_n}{I_{n+1}} = 1 + \frac{1}{k}$, where $k =$

(A) $(n+1)a$ (B) $na$
(C) $(n-1)a$ (D) None of these

**115.** $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$ is equal to

(A) $-1$ (B) $e^{-1}$ (C) 1 (D) $e$

**116.** $\lim_{n\to\infty} \dfrac{1+\sqrt[3]{2}+\sqrt[3]{3}+...+\sqrt[3]{n-1}}{\sqrt[3]{n^4}}$

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1

**117.** Let $\phi(x) = \int_0^x g(t)\,dt$, where the function $g$ is such that $-\frac{1}{2} \le g(t) \le 0,\ \forall t \in [0, 1]$ $\frac{1}{2} \le g(t) \le 1,\ \forall t \in [1, 3]$ $g(t) \le 1,\ \forall t \in [3, 4]$ Then, $\phi(4)$ satisfies the inequality

(A) $\frac{1}{2} \le \phi(4) \le 3$ (B) $0 \le \phi(4) \le 2$

(C) $\phi(4) \le 3$ (D) None of these

**118.** $\int_1^4 (\{x\})^{[x]}\,dx$, where $\{\cdot\}$ and $[\cdot]$ denote the fractional part and greatest integer function, respectively, is equal to

(A) 1 (B) $\frac{12}{13}$ (C) $\frac{13}{12}$ (D) $\frac{6}{7}$

**119.** If $[\cdot]$ denotes the greatest integer function, then $\int_0^2 [x+[x+[x]]]\,dx =$

(A) 1 (B) 2 (C) 3 (D) 0

**120.** $\int_{\sqrt{(3a^2+b^2)/4}}^{\sqrt{(a^2+b^2)/2}} \dfrac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,dx =$

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

**121.** $\lim_{n\to\infty}\left(\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdot\sin\frac{3\pi}{2n}\cdots\sin\frac{(n-1)\pi}{n}\right)^{1/n}$

(A) $\frac{1}{4}$ (B) 4

(C) 1 (D) None of these

**122.** $\int_{-2\pi}^{5\pi} \cot^{-1}(\tan x)\,dx$

(A) $7\pi^2$ (B) $\frac{7\pi^2}{2}$ (C) 0 (D) $\frac{3\pi^2}{2}$

**123.** $\int_0^{\sqrt{3}} \dfrac{1}{1+x^2}\cdot\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)dx$

(A) $\frac{7}{72}\pi^2$ (B) $\frac{3}{42}\pi^2$

(C) $\frac{17}{72}\pi^2$ (D) None of these

**124.** The value of the definite integral $\int_0^1 \dfrac{x}{x^2+16}\,dx$ lies in the interval $[a, b]$. The smallest such interval is

(A) $[0, 1]$ (B) $\left[0, \frac{1}{7}\right]$

(C) $\left[0, \frac{1}{17}\right]$ (D) None of these

**125.** $\int_0^1 \dfrac{dx}{\sqrt{4-x^2-x^3}}$ belongs to the interval

(A) $\left[0, \frac{\pi}{6}\right]$ (B) $\left[\frac{\pi}{6}, \frac{\pi}{4\sqrt{2}}\right]$

(C) $\left[\frac{\pi}{4\sqrt{2}}, \frac{\pi}{2}\right]$ (D) None of these

**126.** The value of a positive integer $n \le 5$ such that $\int_0^1 e^x(x-1)^n\,dx = 16-6e$ is

(A) 1 (B) 2 (C) 3 (D) 4

**127.** Let $f(x)$ be a function defined by $f(x) = \int_1^x x(x^2-3x+2)\,dx$, $1 \le x \le 3$, then the range of $f(x)$ is

(A) $\left[-\frac{1}{4}, 2\right]$ (B) $\left[-\frac{1}{4}, 4\right]$

(C) $[0, 2]$ (D) None of these

**128.** The value of the integral $\int_2^3 \left(\sqrt{2x-\sqrt{5(4x-5)}} + \sqrt{2x+\sqrt{5(4x-5)}}\right)dx$ is equal to

(A) $\dfrac{7\sqrt{7}+2\sqrt{5}}{3\sqrt{2}}$ (B) $\dfrac{7\sqrt{7}-2\sqrt{5}}{3\sqrt{2}}$

(C) $\dfrac{2\sqrt{5}-7\sqrt{7}}{3\sqrt{2}}$ (D) None of these

**129.** If $f$ and $g$ are two continuous functions being even and odd, respectively, then $\int_{-a}^{a} \frac{f(x)}{b^{g(x)}+1}\,dx$ is equal to ($a$ being any non-zero number and $b$ is positive real number, $b \neq 1$)

(A) Independent of $f$
(B) Independent of $g$
(C) Independent of both $f$ and $g$
(D) None of these

**130.** For $x > 0$, let $f(x) = \int_{1}^{x} \frac{ln\, t}{1+t}\,dt$. Then, the value of $f(e) + f\left(\frac{1}{e}\right)$ is

(A) 1 (B) 2
(C) $\frac{1}{2}$ (D) None of these

**131.** The value of the integral $\int_{0}^{2\pi} e^{\cos\theta} \cos(\sin\theta)\,d\theta$ is

(A) 0 (B) $\pi$
(C) $2\pi$ (D) cannot be determined

**132.** Let $f$ be a real valued function satisfying $f(x) + f(x+6) = f(x+3) + f(x+9)$. Then, $\int_{x}^{x+12} f(t)\,dt$ is

(A) A linear function
(B) An exponential function
(C) A constant function
(D) None of these

**133.** If $f(x) = x + \int_{0}^{1} (xy^2 - x^2 y)\, f(y)\,dy$, then $f(x)$ attains a minimum at

(A) $x = \frac{8}{9}$ (B) $x = -\frac{8}{9}$
(C) $\frac{9}{8}$ (D) $-\frac{8}{9}$

**134.** $\lim_{n\to\infty} \frac{(1^2+2^2+3^2+...+n^2)(1^3+2^3+3^3+...+n^3)}{(1^6+2^6+3^6+...+n^6)}$ is equal to

(A) $\frac{7}{12}$ (B) $\frac{12}{7}$
(C) $\frac{5}{12}$ (D) None of these

**135.** The smaller area enclosed by $y = f(x)$, where $f(x)$ is a polynomial of least degree satisfying $\lim_{x\to 0}\left[1+\frac{f(x)}{x^3}\right]^{1/x} = e$ and the circle $x^2 + y^2 = 2$ above the $x$-axis is

(A) $\frac{\pi}{2}+\frac{3}{5}$ (B) $\frac{\pi}{2}-\frac{3}{5}$
(C) $\frac{\pi}{2}-\frac{6}{5}$ (D) None of these

**136.** If $I$ is the greatest of the definite integrals

$$I_1 = \int_0^1 e^{-x}\cos^2 x\,dx,\ I_2 = \int_0^1 e^{-x^2}\cos^2 x\,dx,$$

$$I_3 = \int_0^1 e^{-x^2}\,dx,\ \text{and } I_4 = \int_0^1 e^{-x^2/2}\,dx,\ \text{then,}$$

(A) $I = I_1$ (B) $I = I_2$
(C) $I = I_3$ (D) $I = I_4$

**137.** The values of $a$ for which the equation $\int_0^x \sin^2\frac{t}{2}\,dt = a^2x^2 - \frac{1}{2}(3x-1) + \frac{1}{a^2}$ possesses a solution are

(A) $\pm\frac{1}{\sqrt{n\pi+\frac{\pi}{2}}}, n \in N$
(B) $\pm\frac{1}{\sqrt{2n\pi+\pi}}, n \in N$
(C) $\pm\frac{1}{\sqrt{2n\pi-\frac{\pi}{2}}}, n \in N$
(D) None of these

**138.** The value of the integral $\int_0^{\infty}\frac{dx}{\left(x+\sqrt{x^2+1}\right)^n}$, where $n > 1$, is

(A) $\frac{n^2}{n^2-1}$ (B) $\frac{n}{n^2-1}$
(C) $\frac{n^2}{n^2+1}$ (D) $\frac{n}{n^2+1}$

**139.** If $\alpha$ is a parameter independent of $x$ and $\alpha \neq (2n+1)\pi$, $n \in Z$, then the value of the integral $\int_0^1 \frac{x^{\cos\alpha}-1}{\ln x}\,dx,\ x > 0, x \neq 1$ is

(A) $\ln(1+\cos\alpha)$
(B) $\ln(1-\cos\alpha)$
(C) $\ln|\cos\alpha|$
(D) None of these

**140.** Let $f(x)$ be positive, continuous and differentiable on the interval $(a, b)$ and $\lim_{x\to a^+} f(x) = 1$, $\lim_{x\to b^-} f(x) = 3^{1/4}$. If $f'(x) \geq f^3(x) + \frac{1}{f(x)}$, then the greatest value of $b-a$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{12}$

**141.** The value of $\int_0^1 (\{2x\}-1)(\{3x\}-1)\,dx$, where $\{\cdot\}$ denotes the fractional part is,

(A) $\frac{19}{72}$ (B) $\frac{31}{9}$ (C) $\frac{1}{8}$ (D) $\frac{72}{19}$

**142.** The area included between the curves $x^2+y^2=a^2$ and $\sqrt{|x|}+\sqrt{|y|}=\sqrt{a}\ (a>0)$ is

(A) $\left(\pi+\frac{2}{3}\right)a^2$ (B) $\left(\pi-\frac{2}{3}\right)a^2$
(C) $\frac{2}{3}a^2$ (D) $\frac{2\pi}{3}a^2$

## More than One Option Correct Type

**143.** If $I_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\,dx$, then

(A) $I_n = \frac{n\pi}{2}$
(B) $I_n = 2\int_0^{\pi/2} \frac{\sin nx \cos 2nx}{\sin x}\,dx$
(C) $I_1, I_2, I_3, \ldots I_n, \ldots$ is an A.P.
(D) $\sin(I_{16}) = 0$

**144.** If $A_n = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x}\,dx$;

$B_n = \int_0^{\pi/2} \left(\frac{\sin nx}{\sin x}\right)^2 dx$; for $n \in N$, then

(A) $A_{n+1} = A_n$ (B) $B_{n+1} = B_n$
(C) $A_{n+1} - A_n = B_{n+1}$ (D) $B_{n+1} - B_n = A_{n+1}$

**145.** If $g(x) = \int_0^x \cos 4t\,dt$, then $g(x+\pi)$ equals

(A) $\frac{g(x)}{g(\pi)}$ (B) $g(x)+g(\pi)$
(C) $g(x)-g(\pi)$ (D) $g(x)\cdot g(\pi)$

**146.** If $x > 0$ and $\int_0^x [x]\,dx = [x]\left(\frac{1}{2}A+B\right)$, where [.] denotes the greatest integer function, then

(A) $A = [x]-1$
(B) $B = x-[x]$
(C) $A = [x]+1$
(D) $B = x+[x]$

**147.** If $\int_1^4 |x-3|\,dx = 2A+B$, Then

(A) $A = 3/2, B = 4$
(B) $A = 1, B = 1/2$
(C) $A = 2, B = -3/2$
(D) $A = 1/2, B = 3/2$

**148.** If $\int_a^b |\sin x|\,dx = 8$ and $\int_a^{a+b} |\cos x|\,dx = \frac{9}{2}$, then

(A) $a = \frac{\pi}{2}$ (B) $b = \frac{11\pi}{4}$
(C) $a = \frac{\pi}{4}$ (D) $b = \frac{17\pi}{4}$

**149.** If $\int_0^1 \frac{dx}{2e^x-1} = p\log(qe-1)-r$, then

(A) $p = 1$ (B) $q = 2$
(C) $r = 1$ (D) $r = 2$

**150.** If $f(x)$ satisfies the relation

$$f(x) = e^x + \int_0^1 (x+ye^x)\,f(y)\,dy,$$

then $f(x) = ae^x + bx$, where

(A) $a = -\dfrac{3}{2(e-1)}$ (B) $b = -3$

(C) $a = \dfrac{3}{2(e-1)}$ (D) $b = 3$

**151.** $\int_0^{\pi/2} f(\sin 2x)\sin x\, dx$ is equal to

(A) $\int_0^{\pi/2} f(\sin 2x)\cos x\, dx$

(B) $\sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx$

(C) $\sqrt{2}\int_0^{\pi/4} f(\cos 2x)\sin x\, dx$

(D) None of these

**152.** The absolute value of $\int_{10}^{19} \dfrac{\sin x}{1+x^8}dx$ is

(A) less than $10^{-7}$ (B) more than $10^{-7}$
(C) less than $10^{-6}$ (D) more than $10^{-6}$

**153.** $\int_{-1/2}^{1/2} \sqrt{\left(\dfrac{x+1}{x-1}\right)^2 + \left(\dfrac{x-1}{x+1}\right)^2 - 2}\, dx$ is equal to

(A) $4\log\dfrac{3}{4}$ (B) $4\log\dfrac{4}{3}$

(C) $2\log\dfrac{16}{9}$ (D) $-\log\dfrac{81}{256}$

**154.** If $I_n = \int_0^1 \dfrac{dx}{(1+x^2)^n}$ ; $n \in N$, then

(A) $2n\, I_{n+1} = 2^{-n} - (2n-1)\, I_n$
(B) $2n\, I_{n+1} = 2^{-n} + (2n-1)\, I_n$

(C) $I_2 = \dfrac{\pi}{8} + \dfrac{1}{4}$

(D) $I_2 = \dfrac{\pi}{8} - \dfrac{1}{4}$

**155.** Given $f$ is an odd function and periodic with period 2. If $f(x)$ is continuous $\forall\, x$ and $g(x) = \int_0^x f(t)\,dt$, then

(A) $g$ is an odd function
(B) $g$ is periodic with period 2
(C) $g(2n) = 0$
(D) $g(2n) = 1$

**156.** If $e$ be the eccentricity of a hyperbola and $f(e)$ be the eccentricity of its conjugate hyperbola, then the value of the integral $\int_1^3 \underbrace{f f f \ldots f(e)}_{n \text{ times}}\, de$ is

(A) $2\sqrt{2}$ if $n$ is even (B) $2\sqrt{2}$ if $n$ is odd

(C) 4 if $n$ is even (D) 4 if $n$ is odd

**157.** If $A_n$ be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \dfrac{\pi}{4}$, then for $n > 2$,

(A) $A_n + A_{n-2} = \dfrac{1}{n-1}$

(B) $A_n + A_{n+2} = \dfrac{1}{n+1}$

(C) $\dfrac{1}{2n+1} < A_n < \dfrac{1}{2n-2}$

(D) $\dfrac{1}{2n-1} < A_n < \dfrac{1}{2n}$

## Passage Based Questions

### *Passage 1*

A function is said to be bounded if its range is bounded, otherwise, it is unbounded. Thus, a function $f(x)$ is bounded in the domain $D$, if there exist two real numbers $k$ and $K$ such that

$$k \le f(x) \le K \text{ for all } x \in D.$$

Again, the bounds of the range of a bounded function are called the bounds of the function.

Let $f: X \to Y$ (i.e., $f$ is a function whose domain is $X$ and range $f(X) \subseteq Y$ the co-domain).

$f$ is called a monotonically increasing function if $x_1, x_2 \in X$ with $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$.

$f$ is called a monotonically decreasing function if $x_1, x_2 \in X$ with $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$.

1. If a bounded function $f: [a, b] \to R$ is continuous on $[a, b]$, then $f$ is integrable on $[a, b]$.
2. If a bounded function $f: [a, b] \to R$ is monotinic on $[a, b]$, then $f$ is integrable on $[a, b]$.
3. If the set of points of discontinuity of a bounded function $f: [a, b] \to R$ is finite, then $f$ is integrable on $[a, b]$.

4. If the set of points of discontinuity of a bounded function $f$: $[a, b] \to R$ has a finite number of limit points, then $f$ is integrable on $[a, b]$.

**158.** If $f(x) = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ 1, & \text{otherwise} \end{cases}$, then $\int_0^m f(x)\,dx =$

(A) 0 (B) $m$
(C) $m - 1$ (D) does not exist

**159.** If $f(x) = \begin{cases} \frac{1}{2^n}, & \text{when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, (n = 0, 1, 2, ...) \\ 0, & \text{when } x = 0 \end{cases}$

then $\int_0^1 f(x)\,dx =$

(A) $\frac{1}{2}$ (B) 0
(C) $\frac{2}{3}$ (D) does not exist

**160.** If $f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n}, (n = 1, 2, ....) \\ 0, & x = 0 \end{cases}$

then $\int_0^1 f(x)\,dx =$

(A) $\frac{\pi^2}{6} - 1$ (B) $\frac{\pi^2}{6}$
(C) 0 (D) does not exist

### Passage 2

A function $f$ defined on $[a, b]$ is bounded above if there exists a real number $K$ such that $|f(x)| \le K$, $\forall x \in [a, b]$. The real number $K$ is called an upper bound of $f$. $f$ is called bounded below if there exists a real number $k$ such that $|f(x)| \ge k$, $\forall x \in [a, b]$. The real number $k$ is called a lower bound of $f$.

It may be observed that if $K$ is an upper bound of $f$, then every real number greater than or equal to $K$ is also an upper bound of $f$. The smallest of all the upper bounds of $f$ is called the least upper bound or the supremum (sup.) of $f$. Similarly, the greatest of all the lower bounds of $f$ is called the greatest lower bound or the infimum (Inf.) of $f$. If the functions $f$ and $g$ are bounded and continuous on $[a, b]$ and $g$ keeps the same sign on $[a, b]$, then there exists a number $\mu$ between the infimum and supremum of $f$ on $[a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = \mu \int_a^b g(x)\,dx$$

**161.** $\int_0^1 \frac{\sin \pi x}{1 + x^2}\,dx$ lies between

(A) $\frac{1}{\pi}$ and $\frac{2}{\pi}$ (B) $\frac{\pi}{2}$ and $\pi$
(C) $\pi$ and $\frac{3\pi}{2}$ (D) None of these

**162.** $\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx$ lies between

(A) $\frac{\pi^2}{3}$ and $\frac{2\pi^2}{3}$ (B) $\frac{\pi^2}{9}$ and $\frac{2\pi^2}{9}$
(C) $\frac{2\pi^2}{9}$ and $\frac{4\pi^2}{9}$ (D) None of these

**163.** $\int_0^1 \frac{x^2}{\sqrt{1 + x^2}}$ lies between

(A) $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{\sqrt{2}}$ and 1
(C) $\frac{1}{3\sqrt{2}}$ and $\frac{1}{3}$ (D) None of these

### Passage 3

We know that for a continuous function $f$ in $[a, b]$

$$\lim_{n\to\infty} \sum_{r=1}^{n} hf(a + rh) = \int_a^b f(x)\,dx,\ h = \frac{b-a}{n} \qquad (1)$$

On putting $a = 0$, $b = 1$; (1) becomes

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$$

This formula enables us to evaluate limits of the form:

$$\lim_{n\to\infty} \left[\phi_1(n) + \phi_2(n) + ... + \phi_n(n)\right]$$

To evaluate this limit we express the $r$th term as

$$\phi_r(n) = \frac{1}{n} f\left(\frac{r}{n}\right)$$

and then replace $\frac{r}{n}$ by $x$, $\frac{1}{n}$ by $dx$ and $\lim_{n\to\infty} \sum_{r=1}^{n}$ by $\int_0^1$

Also, $\lim_{n\to\infty} \sum_{r=1}^{kn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^k f(x)\,dx$

**164.** $\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$ is equal to

(A) $\frac{2}{e}$ (B) $\frac{e}{2}$ (C) $\frac{e}{4}$ (D) $\frac{4}{e}$

***Passage 4***

We know that the definite integral $\int_a^b y\,dx$ gives the area of the region which is bounded by the curve $y = f(x)$, the axis of $x$, and the two ordinates $x = a$, $x = b$.

We now consider a closed curve represented by the parametric equations $x = f(t)$, $y = \phi(t)$, '$t$' being the parameter. We suppose that the curve does not intersect itself. Also suppose that as the parameter '$t$' increases from a value $t_1$ to the value $t_2$, the point $P(x, y)$ describes the curve completely in the counter-clockwise since. The curve being closed, the point on it corresponding to the value $t_2$, of the parameter is the same as the point corresponding to the value $t_1$ of the parameter. The area of the region bounded by such a curve is

$$\frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

The above formula gives the area enclosed by any closed curve whatsoever, provided only, that it does not intersect itself; there being no restriction as to the manner in which the curve is situated to the coordinate axes.

**165.** The area enclosed by the curve $x = a\cos^3 t$, $y = b\sin^3 t$, $0 \le t \le 2\pi$ is

(A) $\frac{3\pi ab}{8}$ (B) $\frac{3\pi ab}{4}$

(C) $\frac{3ab}{8}$ (D) None of these

## Match the Column Type

**166.** If [·] denotes the greatest integer function, then

| Column-I | Column-II |
|---|---|
| I. $\int_0^\infty \left[\frac{2}{e^x}\right] dx =$ | (A) 2 |
| II. $\int_0^{1.5} [x^2]\,dx =$ | (B) ln 2 |
| III. $\int_0^{\pi/2} \frac{1 + 2\cos x}{(2 + \cos x)^2}\,dx =$ | (C) $2 - \sqrt{2}$ |
| IV. $\int_3^4 \frac{[x^2]}{[x^2 - 14x + 49] + [x^2]}\,dx =$ ([·] denotes the greatest integer function) | (D) $\frac{1}{2}$ |

**167.**

| Column-I | Column-II |
|---|---|
| I. $\int_0^2 x^3\sqrt{2x - x^2}\,dx =$ | (A) $\pi$ |
| II. $\int_{\pi/2}^{3\pi/2} [\sin x]\,dx =$ (where [·] denotes the greatest integer function) | (B) $\frac{7\pi}{8}$ |
| III. $\int_0^1 \lvert\sin 2\pi x\rvert\,dx =$ | (C) $-\frac{\pi}{2}$ |
| IV. $\int_0^\pi \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}\,dx =$ | (D) $\frac{2}{\pi}$ |

**168.**

| Column-I | Column-II |
|---|---|
| I. $\int_{-1}^{3} (\lvert x-2\rvert + [x])\,dx =$ ([x] stands for greatest integer to x) less than or equal | (A) 2 |
| II. $\int_{-1}^{1} \frac{\sin^2 x}{\left[\frac{x}{\sqrt{2}}\right] + \frac{1}{2}}\,dx =$ ([x] stands for greatest integer less than or equal to x) | (B) 3 |
| III. If $\int_1^b (b - 4x)\,dx \ge 6 - 5b$, $b > 1$, then $b =$ | (C) 7 |
| IV. If $I_1 = \int_0^{3\pi} f(\cos^2 x)\,dx$ and $I_2 = \int_0^{\pi} f(\cos^2 x)\,dx$ then $I_1 = kI_2$, where $k =$ | (D) 0 |

## Assertion-Reason Type

**Instructions:** *In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:*

(A) Assertion (A) is True and Reason (R) is True; Reason (R) is a correct explanation for Assertion (A)
(B) Assertion (A) is True, Reason (R) is True; Reason (R) is not a correct explanation for Assertion (A)
(C) Assertion (A) is True, Reason (R) is False
(D) Assertion (A) is False, Reason (R) is True

**169. Assertion:** If $2f(x) + 3f\left(\frac{1}{x}\right) = \frac{1}{x} - 2, x \neq 0$, then

**Reason:** $f(x) = -\frac{2}{5x} + \frac{3x}{5} - \frac{2}{5}\int_1^2 f(x)\,dx = -\frac{2}{5}\log 2 + \frac{1}{2}$

**170. Assertion:** $I_n = \int_0^\infty x^n e^{-x} dx$ ($n$ is a positive integer) $= n!$

**Reason:** $I_n = nI_{n-1}$

**171. Assertion:** If $a$, $b$ ($a < b$) be the points of discontinuity of function $fofof(x)$, where $f(x) = \frac{1}{1-x}, x \neq 1$, then

$$\int_a^b \frac{f(x)}{f(x) + f(1-x)} dx = \frac{1}{2}$$

**Reason:** $a = 0, b = 1$

**172. Assertion:** $\int_0^{\pi/6} \frac{\sqrt{3\cos 2x - 1}}{\cos x} dx = \sqrt{\frac{2}{3}}\,\pi - 2\tan^{-1}\sqrt{2}$

**Reason:** $\int_0^{\pi/3} \frac{dx}{5 + \cos 2x} = \frac{1}{2\sqrt{6}} \tan^{-1}\sqrt{2}$

**173. Assertion:** $\int_{1/3}^{11/2} \{x\}\,dx = \frac{185}{72}$

(where $\{x\}$ denotes the fractional part of $x$)

**Reason:** If $f(x)$ is a periodic function having period, $T$, then

$$\int_a^{b+nT} f(x)\,dx = n\int_0^T f(x)\,dx + \int_a^b f(x)\,dx$$

**174. Assertion:** If $f(x)$ is a non-negative continuous function such that $f(x) + f\left(x + \frac{1}{2}\right) = 1$, then $\int_0^2 f(x)\,dx = 1$

**Reason:** $f(x)$ is a periodic function having period 1.

**175. Assertion:** $\int_0^{\pi/2} \sin 2kx \cot x\,dx = \frac{\pi}{2}$

**Reason:** $\frac{\sin 2kx}{\sin x} = 2$ (cos $x$ + cos $3x$ + ... + cos $(2k-1)x$)

**176. Assertion:** If $f(x) = \int_0^x g(t)\,dt$, where $g$ is an even function and $f(x + 5) = g(x)$, then $g(0) - g(x) = \int_0^x f(t)\,dt$.

**Reason:** $f$ is an odd function.

**177. Assertion:** If $f$, $g$ and $h$ be continuous functions on $[0, a]$ such that $f(x) = f(a-x)$, $g(x) = -g(a-x)$ and $3h(x) - 4h(a-x) = 5$, then $\int_0^a f(x)g(x)h(x)\,dx = 0$

**Reason:** $\int_0^a f(x)g(x)\,dx = 0$

## Previous Year's Questions

**178.** $\int_0^{10\pi} |\sin x|\,dx$ is **[2002]**

(A) 20 (B) 8 (C) 10 (D) 18

**179.** $I_n = \int_0^{\pi/4} \tan^n x\,dx$, then $\lim_{n\to\infty} n\,[I_n + I_{n+2}]$ equals **[2002]**

(A) $\frac{1}{2}$ (B) 1 (C) $\infty$ (D) zero

**180.** $\int_0^2 [x^2]\,dx$ is **[2002]**

(A) $2 - \sqrt{2}$ (B) $2 + \sqrt{2}$
(C) $\sqrt{2} - 1$ (D) $-\sqrt{2} - \sqrt{3} + 5$

**181.** $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$ is **[2002]**

(A) $\frac{\pi^2}{4}$ (B) $\pi^2$

(C) zero (D) $\frac{\pi}{2}$

**182.** Evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ **[2002]**

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) zero (D) 1

**183.** The area bounded by the curve $y = 2x\text{-}x^2$ and the straight line $y = \text{-}x$ is given by **[2002]**

(A) $\frac{9}{2}$ sq unit (B) $\frac{43}{6}$ sq unit

(C) $\frac{35}{6}$ sq unit (D) None of these

**184.** If $f(y) = e^y$, $g(y) = y$; $y > 0$ and $F(t) = \int_0^t f(t-y)g(y)dy$, then **[2003]**

(A) $F(t) = 1 - e^{-t}(1+t)$ (B) $F(t) = e^t - (1+t)$

(C) $F(t) = te^t$ (D) $F(t) = te^{-t}$

**185.** If $f(a + b\text{-}x) = f(x)$, then $\int_a^b x\, f(x)\, dx$ is equal to **[2003]**

(A) $\frac{a+b}{2}\int_a^b f(b-x)dx$

(B) $\frac{a+b}{2}\int_a^b f(x)dx$

(C) $\frac{b-a}{2}\int_a^b f(x)dx$

(D) $\frac{a+b}{2}\int_a^b f(a+b-x)dx$

**186.** The value of $\lim_{x\to 0} \frac{\int_0^{x^2} \sec^2 t\, dt}{x \sin x}$ is **[2003]**

(A) 3 (B) 2 (C) 1 (D) 0

**187.** The value of the integral $I = \int_0^1 x(1-x)^n dx$ is **[2003]**

(A) $\frac{1}{n+1}$ (B) $\frac{1}{n+2}$

(C) $\frac{1}{n+1} - \frac{1}{n+2}$ (D) $\frac{1}{n+1} + \frac{1}{n+2}$

**188.** Let $\frac{d}{dx}F(x) = \left(\frac{e^{\sin x}}{x}\right), x > 0$.

If $\int_1^4 \frac{3}{x} e^{\sin x^3} dx = F(k) - F(1)$, then one of the possible values of $k$, is **[2003]**

(A) 15 (B) 16 (C) 63 (D) 64

**189.** The area of the region bounded by the curves $y = |x-1|$ and $y = 3-|x|$ is **[2003]**

(A) 2 sq. units (B) 3 sq. units

(C) 4 sq. units (D) 6 sq. units

**190.** Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and $g(x)$ be a function that satisfies $f(x) + g(x) = x^2$. Then the value of the integral $\int_0^1 f(x)g(x)\, dx$, is **[2003]**

(A) $e - \frac{e^2}{2} - \frac{5}{2}$ (B) $e + \frac{e^2}{2} - \frac{3}{2}$

(C) $e - \frac{e^2}{2} - \frac{3}{2}$ (D) $e + \frac{e^2}{2} + \frac{5}{2}$

**191.** $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} e^{\frac{r}{n}}$ **[2004]**

(A) $e$ (B) $e - 1$

(C) $1 - e$ (D) $e + 1$

**192.** The value of $\int_{-2}^{3} |1 - x^2|\, dx$ is **[2004]**

(A) $\frac{28}{3}$ (B) $\frac{14}{3}$

(C) $\frac{7}{3}$ (D) $\frac{1}{3}$

**193.** The value of $I = \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{1+\sin 2x}} dx$ is **[2004]**

(A) 0 (B) 1

(C) 2 (D) 3

**194.** If $\int_0^{\pi} x f(\sin x)dx = A\int_0^{\pi/2} f(\sin x)dx$, then $A$ is **[2004]**

(A) 0 (B) $\pi$

(C) $\frac{\pi}{4}$ (D) $2\pi$

**195.** If $f(x)=\frac{e^x}{1+e^x}$, $I_1=\int_{f(-a)}^{f(a)} xg\{x(1-x)\}dx$ and $I_2=\int_{f(-a)}^{f(a)} g\{x(1-x)\}dx$ then the value of $\frac{I_2}{I_1}$ is **[2004]**

(A) 2 (B) –3 (C) –1 (D) 1

**196.** The area of the region bounded by the curves $y=|x-2|$, $x=1$, $x=3$ and the $x$-axis is **[2004]**

(A) 1 (B) 2 (C) 3 (D) 4

**197.** $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+....+\frac{1}{n^2}\sec^2 1\right]$ equals **[2005]**

(A) $\frac{1}{2}\sec 1$ (B) $\frac{1}{2}\text{cosec}1$

(C) $\tan 1$ (D) $\frac{1}{2}\tan 1$

**198.** If

$$l_1=\int_0^1 2^{x^2}dx, l_2=\int_0^1 2^{x^3}dx, l_3=\int_1^2 2^{x^2}dx, \text{and } l_4=\int_1^2 2^{x^3}dx$$

then **[2005]**

(A) $l_2>l_1$ (B) $l_1>l_2$

(C) $l_3=l_4$ (D) $l_3>l_4$

**199.** The area enclosed between the curve $y=\log_e(x+e)$ and the coordinate axes is **[2005]**

(A) 1 (B) 2 (C) 3 (D) 4

**200.** The parabolas $y^2=4x$ and $x^2=4y$ divide the square region bounded by the lines $x=4$, $y=4$ and the coordinate axes. If $S_1$, $S_2$, $S_3$ are respectively the areas of these parts numbered from top to bottom; then $S_1:S_2:S_3$ is **[2005]**

(A) 1 : 2 : 1 (B) 1 : 2 : 3

(C) 2 : 1 : 2 (D) 1 : 1 : 1

**201.** Let $f: R\to R$ be a differentiable function having $f(2)=6, f'(2)=\left(\frac{1}{48}\right)$. Then $\lim_{x\to 2}\int_6^{f(x)}\frac{4t^3}{x-2}dt$ equals **[2005]**

(A) 24 (B) 36 (C) 12 (D) 18

**202.** Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y=f(x)$, $x$-axis and the ordinates $x=\frac{\pi}{4}$ and $x=\beta>\frac{\pi}{4}$ is $\left(\beta\sin\beta+\frac{\pi}{4}\cos\beta+\sqrt{2}\beta\right)$. Then $f\left(\frac{\pi}{2}\right)$ is **[2005]**

(A) $\left(\frac{\pi}{4}+\sqrt{2}-1\right)$ (B) $\left(\frac{\pi}{4}-\sqrt{2}+1\right)$

(C) $\left(1-\frac{\pi}{4}-\sqrt{2}\right)$ (D) $\left(1-\frac{\pi}{4}+\sqrt{2}\right)$

**203.** The value of $\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx$, $a>0$, is **[2005]**

(A) $a\pi$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{a}$ (D) $2\pi$

**204.** The plane $x+2y-z=4$ cuts the sphere $x^2+y^2+z^2-x+z-2=0$ in a circle of radius **[2005]**

(A) 3 (B) 1 (C) 2 (D) $-\sqrt{2}$

**205.** The value of the integral, $\int_3^6\frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}dx$ is **[2006]**

(A) 1/2 (B) 3/2

(C) 2 (D) 1

**206.** $\int_0^{\pi} xf(\sin x)dx$ equal to **[2006]**

(A) $\pi\int_0^{\pi} f(\cos x)dx$

(B) $\pi\int_0^{\pi}(\pi-x)f(\sin x)dx$

(C) $\frac{\pi}{2}\int_0^{\pi/2} f(\sin x)dx\, fgp$

(D) $\pi\int_0^{\pi/2} f(\cos x)dx$

**207.** $\int_{-3\pi/2}^{-\pi/2}[(x+\pi)^3+\cos^2(x+3\pi)]dx$ is equal to **[2006]**

(A) $\frac{\pi^4}{32}$ (B) $\frac{\pi^4}{32}+\frac{\pi}{2}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}-1$

**208.** The value of $\int_1^a [x]f'(x)dx, a>1$ where $[x]$ denotes the greatest integer not exceeding $x$ is **[2006]**

(A) $af(A)-\{f(1)+f(2)+\ldots+f([a])\}$

(B) $[a]f(A)-\{f(1)+f(2)+\ldots+f([a])\}$

(C) $[a]f([a]) - \{f(1)+f(2)+\ldots+f(a)\}$
(D) $af([a]) - \{f(1)+f(2)+\ldots+f(a)\}$

**209.** Let $F(x) = f(x) + f\left(\frac{1}{x}\right)$, where $f(x)=\int_1^x \frac{\log t}{1+t}\,dt$. Then $F(e)$ equals **[2007]**

(A) $\frac{1}{2}$ (B) 0 (C) 1 (D) 2

**210.** The solution for $x$ of the equation **[2007]**

$$\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2}$$

(A) 2 (B) $\pi$ (C) $\frac{\sqrt{3}}{2}$ (D) $-\sqrt{2}$

**211.** The area enclosed between the curves $y^2 = x$ and $y=|x|$ is **[2007]**

(A) 2/3 (B) 1 (C) 1/6 (D) 1/3

**212.** Let $I=\int_0^1 \frac{\sin x}{\sqrt{x}}dx$ and $J=\int_0^1 \frac{\cos x}{\sqrt{x}}dx$. Then which one of the following is true? **[2008]**

(A) $I>\frac{2}{3}$ and $J>2$ (B) $I<\frac{2}{3}$ and $J<2$

(C) $I<\frac{2}{3}$ and $J>2$ (D) $I>\frac{2}{3}$ and $J<2$

**213.** The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to **[2008]**

(A) $\frac{5}{3}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$

**214.** $\int_0^{\pi}[\cot x]dx$, $[\cdot]$ denotes the greatest integer function, is equal to **[2009]**

(A) $\frac{\pi}{2}$ (B) 1 (C) $-1$ (D) $-\frac{\pi}{2}$

**215.** The area of the region bounded by the parabola $(y-2)^2 = x-1$, the tangent to the parabola at the point (2, 3) and the $x$-axis is **[2009]**

(A) 3 (B) 6 (C) 9 (D) 12

**216.** The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the lines $x = 0$ and $x=\frac{3\pi}{2}$ is **[2010]**

(A) $4\sqrt{2}+2$ (B) $4\sqrt{2}-1$

(C) $4\sqrt{2}+1$ (D) $4\sqrt{2}-2$

**217.** Let $p(x)$ be a function defined on $R$ such that $p'(x) = p'(1-x)$, for all $x \in [0, 1]$, $p(0) = 1$ and $p(1) = 41$. Then $\int_0^1 p(x)dx$ equals **[2010]**

(A) 21 (B) 41 (C) 42 (D) 20

**218.** The value of the integral $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$ is **[2011]**

(A) $\frac{\pi}{8}\log 2$ (B) $\frac{\pi}{2}\log 2$

(C) log2 (D) $\pi\log 2$

**219.** The area of the region enclosed by the lines $y = x, x = e$, the curve $y=\frac{1}{x}$ and the positive $x$-axis is **[2011]**

(A) 1 sq. units (B) $\frac{3}{2}$ sq. units

(C) $\frac{5}{2}$ sq. units (D) $\frac{1}{2}$ sq. units

**220.** Let $f(x)=\int_0^x \sqrt{t}\sin t\,dt$, for $x \in \left(0, \frac{5\pi}{2}\right)$. Then, $f$ has **[2011]**

(A) local minimum at $\pi$ and $2\pi$
(B) local minimum at $\pi$ and local maximum at $2\pi$
(C) local maximum at $\pi$ and local minimum at $2\pi$
(D) local maximum at $\pi$ and $2\pi$

**221.** The area bounded between the parabolas $x^2=\frac{y}{4}$ and $x^2 = 9y$, and the straight line $y = 2$ is **[2012]**

(A) $20\sqrt{2}$ (B) $\frac{10\sqrt{2}}{3}$

(C) $\frac{20\sqrt{2}}{3}$ (D) $10\sqrt{2}$

**222.** If $g(x)=\int_0^x \cos 4t\,dt$, then $g(x+\pi)$ equals **[2012]**

(A) $\frac{g(x)}{g(\pi)}$ (B) $g(x)+g(\pi)$

(C) $g(x)-g(\pi)$ (D) $g(x).g(\pi)$

**223.** The intercepts on $x$-axis made by tangents to the curve, $y=\int_0^x |t|\,dt, x\in R$, which are parallel to the line $y = 2x$, are equal to **[2013]**

(A) $\pm 2$ (B) $\pm 3$ (C) $\pm 4$ (D) $\pm 1$

**224.** The area (in sq. units) bounded by the curves $y=\sqrt{x}, 2y-x+3=0$, $x$-axis, and lying in the first quadrant is **[2013]**

(A) 36 (B) 18 (C) $\frac{27}{4}$ (D) 9

**225. Statement-I:** The value of the integral $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$ is equal to $\frac{\pi}{6}$.

**Statement-II:** $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx.$ **[2013]**

(A) Statement-I is True; Statement-II is true; Statement-II is **not** a correct explanation for Statement-I
(B) Statement-I is True; Statement-II is False.
(C) Statement-I is False; Statement-II is True
(D) Statement-I is True; Statement-II is True; Statement-II is a **correct** explanation for Statement-I

**226.** The value of the integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals **[2014]**

(A) $\pi-4$ (B) $\frac{2\pi}{3}-4-4\sqrt{3}$

(C) $4\sqrt{3}-4$ (D) $4\sqrt{3}-4-\frac{\pi}{3}$

**227.** The area of the region described by the set $A=\{(x,y): x^2+y^2\le 1, y^2\le 1-x\}$ is **[2014]**

(A) $\frac{\pi}{2}+\frac{4}{3}$ (B) $\frac{\pi}{2}-\frac{4}{3}$

(C) $\frac{\pi}{2}-\frac{2}{3}$ (D) $\frac{\pi}{2}+\frac{2}{3}$

**228.** The area (in sq. units) of the region described by $\{(x, y) : y^2 \le 2x \text{ and } y \ge 4x - 1\}$ is : **[2015]**

(A) $\frac{5}{64}$ (B) $\frac{15}{64}$

(C) $\frac{9}{32}$ (D) $\frac{7}{32}$

**229.** The integral $\int_2^4\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$ is equal to: **[2015]**

(A) 4 (B) 1 (C) 6 (D) 2

**230.** $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)\ldots 3n}{n^{2n}}\right)^{1/n}$ is equal to **[2016]**

(A) 3 log 3 –2 (B) $\frac{18}{e^4}$

(C) $\frac{27}{e^2}$ (D) $\frac{9}{e^2}$

**231.** The area (in sq. units) of the region $\{(x,y): y^2\ge 2x \text{ and } x^2+y^2\le 4x, x\ge 0, y\ge 0\}$ is **[2016]**

(A) $\frac{\pi}{2}-\frac{2\sqrt{2}}{3}$ (B) $\pi-\frac{4}{3}$

(C) $\pi-\frac{8}{3}$ (D) $\pi-\frac{4\sqrt{2}}{3}$

## ANSWER KEYS

### Single Option Correct Type

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** (C) | 2. (A) | 3. (B) | 4. (C) | 5. (A) | **6.** (C) | 7. (D) | 8. (C) | 9. (D) | **10.** (B) |
| 11. (A) | **12.** (B) | **13.** (B) | **14.** (A) | **15.** (A) | **16.** (B) | **17.** (B) | 18. (A) | **19.** (C) | **20.** (B) |
| **21.** (A) | **22.** (B) | **23.** (B) | **24.** (C) | **25.** (B) | **26.** (A) | **27.** (B) | **28.** (B) | **29.** (A) | **30.** (D) |
| **31.** (C) | **32.** (D) | **33.** (A) | **34.** (A) | **35.** (A) | **36.** (B) | **37.** (A) | **38.** (B) | **39.** (B) | **40.** (C) |
| **41.** (C) | **42.** (B) | **43.** (B) | **44.** (D) | **45.** (A) | **46.** (D) | **47.** (D) | **48.** (C) | **49.** (A) | **50.** (A) |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **51.** (D) | **52.** (B) | **53.** (B) | **54.** (C) | **55.** (B) | **56.** (C) | **57.** (C) | **58.** (B) | **59.** (D) | **60.** (A) |
| **61.** (B) | **62.** (D) | **63.** (C) | **64.** (C) | **65.** (D) | **66.** (B) | **67.** (A) | **68.** (C) | **69.** (D) | **70.** (C) |
| **71.** (D) | **72.** (C) | **73.** (B) | **74.** (D) | **75.** (D) | **76.** (D) | **77.** (A) | **78.** (A) | **79.** (C) | **80.** (A) |
| **81.** (B) | **82.** (C) | **83.** (B) | **84.** (A) | **85.** (B) | **86.** (B) | **87.** (C) | **88.** (A) | **89.** (B) | **90.** (B) |
| **91.** (B) | **92.** (B) | **93.** (A) | **94.** (A) | **95.** (D) | **96.** (A) | **97.** (C) | **98.** (B) | **99.** (C) | **100.** (B) |
| **101.** (B) | **102.** (C) | **103.** (B) | **104.** (A) | **105.** (C) | **106.** (A) | **107.** (D) | **108.** (A) | **109.** (C) | **110.** (A) |
| **111.** (C) | **112.** (B) | **113.** (B) | **114.** (A) | **115.** (B) | **116.** (C) | **117.** (C) | **118.** (C) | **119.** (C) | **120.** (D) |
| **121.** (A) | **122.** (B) | **123.** (A) | **124.** (C) | **125.** (B) | **126.** (C) | **127.** (A) | **128.** (B) | **129.** (B) | **130.** (C) |
| **131.** (C) | **132.** (C) | **133.** (D) | **134.** (A) | **135.** (B) | **136.** (D) | **137.** (C) | **138.** (B) | **139.** (A) | **140.** (C) |
| **141.** (A) | **142.** (B) | | | | | | | | |

## More than One Option Correct Type

| | | | |
|---|---|---|---|
| **143.** (A), (C) and (D) | **144.** (A) and (D) | **145.** (B) and (C) | **146.** (A) and (B) |
| **147.** (B), (C) and (D) | **148.** (C) and (D) | **149.** (A), (B) and (C) | **150.** (A) and (B) |
| **151.** (A) and (B) | **152.** (A) and (C) | **153.** (B), (C) and (D) | **154.** (B) and (C) |
| **155.** (B) and (C) | **156.** (B) and (C) | **157.** (A) and (B) | |

## Passage Based Questions

***Passage 1***

**158.** (B) **159.** (C) **160.** (A)

***Passage 2***

**161.** (A) **162.** (B) **163.** (C)

***Passage 3***

**164.** (D)

***Passage 4***

**165.** (A)

## Match the Column Type

**166.** I $\rightarrow$ (B); II $\rightarrow$ (C); III $\rightarrow$ (D); IV $\rightarrow$ (D)
**167.** I $\rightarrow$ (B); II $\rightarrow$ (C); III $\rightarrow$ (D); IV $\rightarrow$ (A)
**168.** I $\rightarrow$ (C); II $\rightarrow$ (D); III $\rightarrow$ (A); IV $\rightarrow$ (B)

## Assertion-Reason Type

**169.** (A) **170.** (A) **171.** (A) **172.** (A) **173.** (A) **174.** (A) **175.** (A) **176.** (A) **177.** (A)

## Previous Year's Questions

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **178.** (A) | **179.** (B) | **180.** (D) | **181.** (B) | **182.** (A) | **183.** (A) | **184.** (B) | **185.** (B) | **186.** (C) | **187.** (C) |
| **188.** (D) | **189.** (A) | **190.** (B) | **191.** (B) | **192.** (A) | **193.** (C) | **194.** (B) | **195.** (A) | **196.** (A) | **197.** (D) |
| **198.** (B) | **199.** (A) | **200.** (D) | **201.** (D) | **202.** (D) | **203.** (B) | **204.** (B) | **205.** (B) | **206.** (D) | **207.** (C) |
| **208.** (B) | **209.** (A) | **210.** (D) | **211.** (C) | **212.** (B) | **213.** (D) | **214.** (D) | **215.** (C) | **216.** (D) | **217.** (A) |
| **218.** (D) | **219.** (B) | **220.** (C) | **221.** (C) | **222.** (B) or (C) | | **223.** (D) | **224.** (D) | **225.** (C) | **226.** (D) |
| **227.** (A) | **228.** (C) | **229.** (B) | **230.** (C) | **231.** (C) | | | | | |