CHAPTER

20

# Conic Sections (Parabola, Ellipse and Hyperbola)

**Chapter Highlights**

Conic Section, Important Terms, Section of a Right Circular Cone by Different Planes, Equation of Conic, General Equation, Centre of Conic, Parabola, Some Terms Related to Parabola, , Intersection of a Line and a Parabola, Equation of a Chord, Point of Intersection of Tangents, Position of a Point with Respect to a Parabola, Number of Tangents Drawn From a Point to a Parabola, Equation of the Pair of Tangents, Equations of Normal in Different Forms, Point of Intersection of Normals, Co-Normal Points, Chord of Contact, Chord with a Given Mid Point, Ellipse, Position of a Point with Respect to an Ellipse, Equation of Normal in Different Forms, Equation of the Pair of Tangents, Chord with a Given Mid Point, Optical Property of Parabola, Equation of a Hyperbola in Standard Form, Some Terms and Properties Related to a Hyperbola, Conjugate Hyperbola, Position of a Point with Respect to a Hyperbola, Equation of The Pair of Tangents, Chord with a Given Mid Point, Chord of Contact

## CONIC SECTION

A conic section or conic is the locus of a point which moves in a plane in such a way that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line.

The fixed point is called the focus and the fixed line is called the directrix of the conic. The constant ratio is called the eccentricity of the conic and is denoted by $e$. If

- $e = 1$, the conic is called **Parabola**.
- $e < 1$, the conic is called **Ellipse**.
- $e > 1$, the conic is called **Hyperbola**.
- $e = 0$, the conic is called **Circle**.
- $e = \infty$, the conic is called **pair of straight lines**.

## IMPORTANT TERMS

1. **Axis** The straight line passing through the focus and perpendicular to the directrix of the conic is known as its axis.
2. **Vertex** A point of intersection of a conic with its axis is known as a vertex of the conic.
3. **Centre** The point which bisects every chord of the conic passing through it, is called the centre of the conic.
4. **Focal Chord** A chord passing through the focus is known as focal chord of the conic.
5. **Latus Rectum** The focal chord which is perpendicular to the axis is known as latus rectum of the conic.
6. **Double Ordinate** A chord of the conic which is perpendicular to the axis is called the double ordinate of the conic.

**REMARK**

The curves defined above are called *conic sections*, because these are obtained when a right circular cone is cut by a plane in various ways.

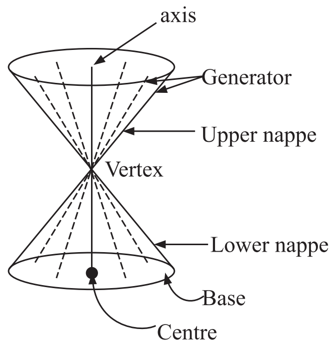



**Fig. 20.1**

## SECTION OF A RIGHT CIRCULAR CONE BY DIFFERENT PLANES

1. Section of a right circular cone by a plane passing through its vertex is a *pair of straight lines* passing through the vertex as shown in the figure.

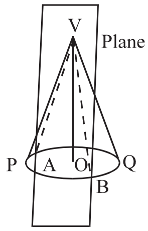



**Fig. 20.2**

2. Section of a right circular cone by a plane parallel to its base is a *circle* as shown in the figure.

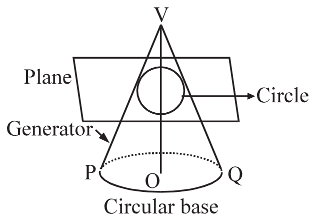



**Fig. 20.3**

3. Section of a right circular cone by a plane parallel to a generator of the cone is a *parabola* as shown in the figure.

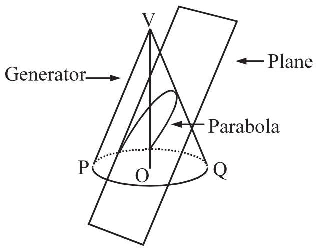



**Fig. 20.4**

4. Section of a right circular cone by a plane not parallel to any generator of the cone and not perpendicular or parallel to the axis of the cone is an *ellipse* as shown in the figure.

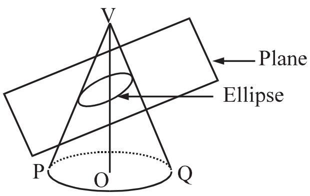



**Fig. 20.5**

5. Section of a right circular cone by a plane parallel to the axis of the cone is a *hyperbola* as shown in the figure.

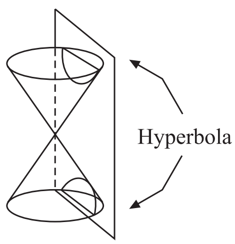



**Fig. 20.6**

## EQUATION OF CONIC

Let $S(h, k)$ be the focus and $QN$ be the directrix whose equation is $Ax + By + C = 0$

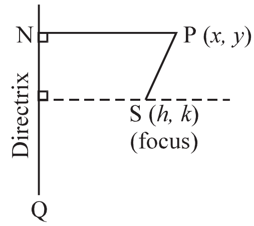



**Fig. 20.7**

Let $P(x, y)$ be any point on the conic. From $P$, draw $PN \perp QN$. If $e$ is the eccentricity of the conic, then by definition

$$\frac{PS}{PN} = e \quad \Rightarrow \quad PS^2 = e^2 PN^2$$

or $$(a-\alpha)^2 + (y-\beta)^2 = e^2\left(\frac{Ax+By+C}{\sqrt{A^2+B^2}}\right)^2$$

This is the cartesian equation of the conic which, on simplification, takes the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where $a$, $b$, $c$, $f$, $g$ and $h$ are constants.

Thus, the equation of a conic is an equation of second degree in $x$ and $y$.

## GENERAL EQUATION

The equation of conics is represented by the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (1) and discriminant of above equation is represented by $\Delta$

where $$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

***Case I:*** When $\Delta = 0$

In this case equation (1) represents the degenerate conic whose nature is given in the following table

**Table 9.1**

| S.No. | Condtion | Nature of conic |
|---|---|---|
| 1. | $\Delta = 0$ and $ab - h^2 = 0$ | A pair of coincident straight lines |
| 2. | $\Delta = 0$ and $ab - h^2 < 0$ | A pair of intersecting straight lines |
| 3. | $\Delta = 0$ and $ab - h^2 > 0$ | A point |

***Case II:*** When $\Delta \neq 0$

In this case equation (1) represents the non-degenerate conic whose nature is given in the following table:

**Table 9.2**

| S.No. | Condtion | Nature of conic |
|---|---|---|
| 1. | $\Delta \neq 0$, $h = 0$, $a = b$, $e = 0$ | A circle |
| 2. | $\Delta \neq 0$, $ab - h^2 = 0$, $e = 1$ | A parabola |
| 3. | $\Delta \neq 0$, $ab - h^2 > 0$, $e < 1$ | An ellipse |
| 4. | $\Delta \neq 0$, $ab - h^2 < 0$, $e > 1$ | A hyperbola |
| 5. | $\Delta \neq 0$, $ab - h^2 < 0$, $a + b = 0$, $e = \sqrt{2}$ | A rectangular hyperbola |

## CENTRE OF CONIC

Centre of the conic is the point which bisect every chord of the conic passing through it. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is the equation of the conic sectio and if $C$ is its centre, then the centre of the conic is

$$C\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

**SHORT-CUT METHOD**

Let $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be the given conic.

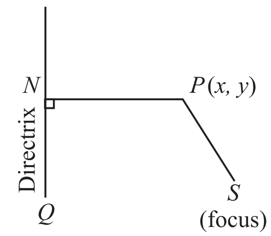



**Fig. 20.8**

- Differentiate $S$ partially with respect to $x$ treating $y$ as constant, we get

$$\frac{\partial S}{\partial x} = 2ax + 2hy + 2g.$$

- Differentiate $S$ partially with respect to $y$ treating $x$ as constant, we get

$$\frac{\partial S}{\partial y} = 2hx + 2by + 2f.$$

- Equating these two equations to zero and solving for $x$ and $y$, we get the coordinates of centre as

$$(x, y) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

## PARABOLA

A *parabola* is the locus of a point which moves in a plane such that its distance from a fixed point (called the focus) is equal to its distance from a fixed straight line (called the directrix).

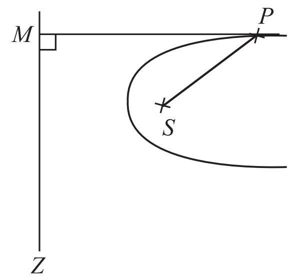



**Fig. 20.9**

Let $S$ be the focus, $QM$ be the directrix and $P$ be any point on the parabola. Then, by definition, $PS = PM$ where $PM$ is the length of the perpendicular from $P$ on the directrix $QM$.

## SOME TERMS RELATED TO PARABOLA

Let $y^2 = 4ax$ be the standard parabola, having focus $S$ at $(a, 0)$ and directrix represented by the line $MZ$ having the equation $x + a = 0$ (see figure). Then for such a parabola, the following important terms can be defined.

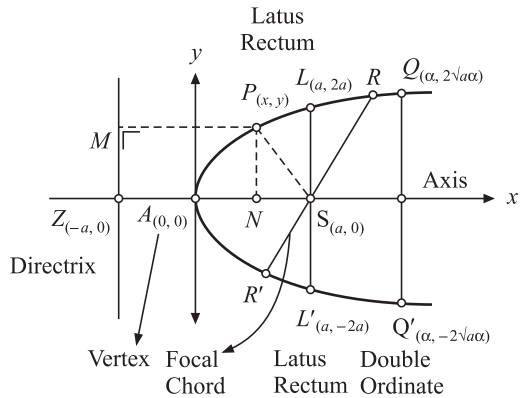



**Fig. 20.10**

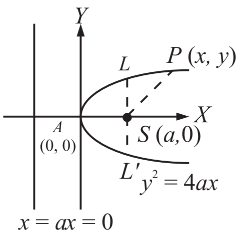
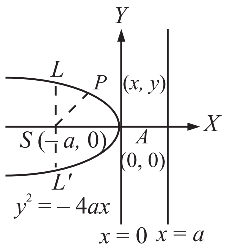
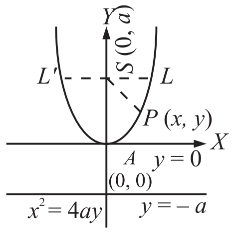
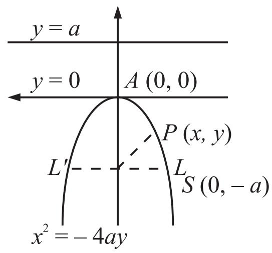
| | |
|---|---|
| **AxIs** | Axis of the conic section is the line passing through the focus and perpendicular to the directrix. For the Parabola, $y^2 = 4ax$, the line $y = 0$ (i.e. $x$-axis) is the AXIS. |
| **VERTEX** | The point $A(0, 0)$ is the point of intersection of the parabola and its axis, hence, $A$ is called the VERTEX of the parabola. |
| **Double Ordinate** | Through the point $Q(\alpha, 2\sqrt{a\alpha})$ on the parabola, a perpendicular drawn to the axis of the parabola such that it meets the other end of the curve at $Q'(\alpha, -2\sqrt{a\alpha})$. Then the line $QQ'$ is called Double Ordinate. Now, length of Double Ordinate $= QQ' = 2(2\sqrt{a\alpha}) =$ Double (Ordinate part) |
| **Latus** | The double ordinate passing through the |

| | |
|---|---|
| **Rectum** | focus is called Latus Rectum. Length of Latus Rectum = $LL' = 2(2a) = 4a$ |
| **Focal Chord** | Since, the chord of the parabola passing through the focus is called the focal chord of the parabola. In this case, the line $RR'$ is the focal chord of the parabola. |
| **Focal Distance** | Distance of any point $P$ on the parabola from the focus is called focal distance<br>$\therefore$ Focal Distance = $\lvert SP \rvert = \lvert PM \rvert$ i.e., Distance of $P$ from Directrix<br>$\Rightarrow \quad \lvert SP \rvert = x - a$ |

**NOTE**

- The vertex of the parabola bisects the join of focus and the point of intersection of directrix with the axis.
- Two parabolas are said to be equal when their latus recta are equal.
- Length of latus rectum $= 2\begin{pmatrix}\text{length of perpendicular} \\ \text{from focus to directrix}\end{pmatrix}$

## FOUR STANDARD FORMS OF THE PARABOLA

| | | | | |
|---|---|---|---|---|
| Standard Equation | $y^2 = 4ax$ (a > 0) | $y^2 = -4ax$ (a > 0) | $x^2 = 4ax$ ($a > 0$) | $x^2 = -4ax$ ($a > 0$) |
| Shape of the parabola |  |  |  |  |
| Vertex | $A(0, 0)$ | $A(0, 0)$ | $A(0, 0)$ | $A(0, 0)$ |
| Focus | $S(a, 0)$ | $S(-a, 0)$ | $S(0, a)$ | $S(0, -a)$ |
| Equation of directrix | $x = -a$ | $x = a$ | $y = -a$ | $y = a$ |
| Equation of axis | $y = 0$ | $y = 0$ | $x = 0$ | $x = 0$ |
| Length of latus rectum | $4a$ | $4a$ | $4a$ | $4a$ |
| Extermities of latus rectum | $(a, \pm 2a)$ | $(-a, \pm 2a)$ | $(\pm 2a, a)$ | $(\pm 2a, -a)$ |
| Equation of latus rectum | $x = a$ | $x = -a$ | $y = a$ | $y = -a$ |
| Equation of tangents at vertex | $x = 0$ | $x = 0$ | $y = 0$ | $y = 0$ |
| Focal distance of a point $P(x, y)$ | $x + a$ | $x - a$ | $y + a$ | $y - a$ |
| Parametric coordinates | $(at^2, 2at)$ | $(-at^2, 2at)$ | $(2at, at^2)$ | $(2at, -at^2)$ |
| Parametric Equations | $x = at^2$<br>$y = 2at$ | $x = -at^2$<br>$y = -2at$ | $x = 2at$<br>$y = at^2$ | $x = 2at$<br>$y = -at^2$ |
| Eccentricity ($e$) | 1 | 1 | 1 | 1 |

**TRICK(S) FOR PROBLEM SOLVING**

- $y^2 = 4a(x + a)$ is the equation of the parabola whose focus is the origin and the axis is $x$-axis.
- $y^2 = 4a(x - a)$ is the equation of parabola whose axis is $x$-axis and $y$-axis is directrix.
- $x^2 = 4a(y + a)$ is the equation of parabola whose focus is the origin and the axis is $y$-axis.
- $x^2 = 4a(y - a)$ is the equation of parabola whose axis is $y$-axis and the $x$-axis is directrix.
- The equation of the parabola whose vertex and focus are on $x$-axis at a distance $a$ and $a'$ respectively from the origin is $y^2 = 4(a' - a)(x - a)$.
- The equation of the parabola whose axis is parallel to $x$-axis is $x = Ay^2 + By + C$ and $y = Ax^2 + Bx + C$ is a parabola with its axis parallel to $y$-axis.

## INTERSECTION OF A LINE AND A PARABOLA

The line $y = mx + c$ does not intersect, touches or intersect a parabola $y^2 = 4ax$ according as $c \geq =, < \dfrac{a}{m}$.

## EQUATION OF A CHORD

1. The equation of chord joining the points $(x_1, y_1)$ and $(x_2, y_2)$ on the parabola $y^2 = 4ax$ is
$$y(y_1 + y_2) = 4ax + y_1y_2$$
2. The equation of chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is
$$y(t_1 + t_2) = 2(x + at_1t_2)$$
3. Length of the chord $y = mx + c$ to the parabola
$$y^2 = 4ax \text{ is given by } \frac{4}{m^2}\sqrt{1+m^2}\,\sqrt{a(a-mc)}$$

**Condition for the chord to be a focal chord** The chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ passes through focus provided $t_1t_2 = -1$.

**Length of focal chord** The length of a focal chord joining the points $(at_1^2, 2at_2)$ and $(at_2^2, 2at_2)$ is $(t_2 - t_1)^2$.

**TRICK(S) FOR PROBLEM SOLVING**

- The length of the focal chord through the point '$t$' on the parabola $y^2 = 4ax$ is $a(t + 1/t)^2$.
- The extremities of focal chord may be given by $(at^2, 2at)$ and $\left(\dfrac{a}{t^2}, -\dfrac{2a}{t}\right)$
- The slope of the chord joining the points '$t_1$' and '$t_2$' is $\dfrac{2}{t_1 + t_2}$

**Condition for tangency and point of contact** The line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \dfrac{a}{m}$ and the coordinates of the point of contact are $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

## POINT OF INTERSECTION OF TANGENTS

The point of intersection of tangents drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

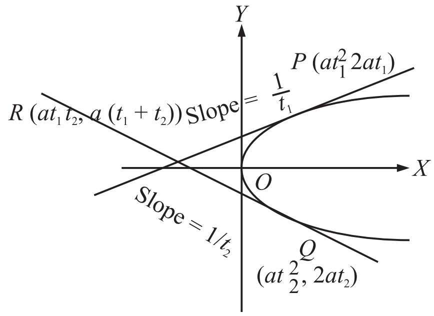



**Fig. 20.11**

$$R \equiv [at_1t_2, a(t_1 + t_2)].$$

**TRICK(S) FOR PROBLEM SOLVING**

- Angle between tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is:
$$\theta = \tan^{-1}\left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$$
- The G. M. of the $x$-coordinates of $P$ and $Q$ (i.e., $\sqrt{at_1^2 \times at_2^2} = at_1t_2$) is the $x$-coordinate of the point of intersection of tangents at $P$ and $Q$ on the parabola.
- The A.M. of the $y$-coordinates of $P$ and $Q$ $\left[\text{i.e., } \dfrac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right]$ is the $y$-coordinate of the point of intersection of tangents at $P$ and $Q$ on the parabola.
- The orthocentre of the triangle formed by three tangents to the parabola lies on the directrix.
- The locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle $\alpha$ is
$$(x + a)^2 \tan^2\alpha = y^2 - 4ax$$
- The tangents to the parabola $y^2 = 4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at $R$. Then the area of triangle $PQR$ is $\dfrac{1}{2}a^2(t_1 - t_2)^3$.
- If the straight line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$, then $ln = am^2$.

- If the line $\frac{x}{l} + \frac{y}{m} = 1$ touches the parabola $y^2 = 4a(x + b)$ then $m^2(l + b) + al^2 = 0$.
- If the two parabolas $y^2 = 4x$ and $x^2 = 4y$ intersect at point $P$, whose abscissa is not zero, then the tangent to each curve at $P$, make complementary angle with the $x$-axis.
- If the line $x \cos \alpha + y \sin a = p$ touches the parabola $y^2 = 4ax$, then $p \cos \alpha + a \sin^2\alpha = 0$ and the point of contact is $(a \tan^2\alpha, -2a \tan \alpha)$.
- Tangents at the extremities of any focul chord of a parabola meet at right angle on the directrix.
- Area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.
- If the tangents at the points $P$ and $Q$ on a parabola meet in $T$, then $ST$ is the geometric mean between $SP$ and $SQ$, i.e., $ST^2 = SP \cdot SQ$.

## POSITION OF A POINT WITH RESPECT TO A PARABOLA

The point $(x_1, y_1)$ lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, = \text{ or } < 0$, , respectively.

## NUMBER OF TANGENTS DRAWN FROM A POINT TO A PARABOLA

Two tangents can be drawn from a point to a parabola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

## EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $SS_1 = T^2$,

where $S \equiv y^2 - 4ax,\ S_1 \equiv y_1^2 - 4ax_1$

and $T \equiv yy_1 - 2a(x + x_1)$

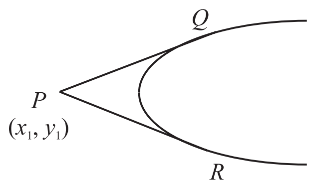



**Fig. 20.12**

## EQUATIONS OF NORMAL IN DIFFERENT FORMS

1. **Point Form** The equation of the normal to the parabola $y^2 = 4ax$ at a point $(x_1, y_1)$ is
$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$
2. **Parametric Form** The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is
$$y + tx = 2at + at^3$$
3. **Slope Form** The equation of normal to the parabola $y^2 = 4ax$ in terms of slope '$m$' is
$$y = mx - 2am - am^3$$

**NOTE**

The coordinates of the point of contact are $(am^2, -2am)$.

**Condition for Normality** The line $y = mx + c$ is a normal to the parabola

$$y^2 = 4ax \text{ if } c = -2am - am^3$$

## POINT OF INTERSECTION OF NORMALS

The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

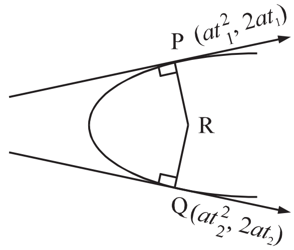



**Fig. 20.13**

$$R \equiv [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)].$$

**TRICK(S) FOR PROBLEM SOLVING**

1. If the normal at the point $P(at_1^2, 2at_2)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then
$$t_2 = -t_1 - \frac{2}{t_1}$$
Note that $PQ$ is normal to the parabola at $P$ and not at $Q$.

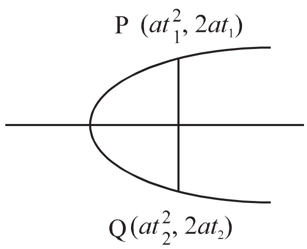



**Fig. 20.14**

2. If the normals at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet on the parabola $y^2 = 4ax$, then $t_1t_2 = 2$.

## CO-NORMAL POINTS

Any three points on a parabola normals at which pass through a common point are called co-normal points

### TRICK(S) FOR PROBLEM SOLVING

If three normals are drawn through a point $(h, k)$, then their slopes are the roots of the cubic:

$$k = mh - 2am - am^3$$

(i) The sum of the slopes of the normals at co-normal points is zero, i.e., $m_1 + m_2 + m_3 = 0$.

(ii) The sum of the ordinates of the co-normal points is zero (i.e. $-2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0$).

(iii) The centroid of the triangle formed by the co-normal points lies on the axis of the parabola [the vertices of the triangle formed by the co-normal points are $(am_1^2, -2am_1), (am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$. Thus, $y$-coordinate of the centroid becomes

$$\frac{-2a(m_1 + m_2 + m_3)}{3} = \frac{-2a}{3} \times 0 = 0$$

Hence, the centroid lies on the $x$-axis, i.e. axis of the parabola.]

(iv) If three normals drawn to any parabola $y^2 = 4ax$ from a given point $(h, k)$ be real, then $h > 2a$.

## CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $T = 0$ where $T \equiv yy_1 - 2a(x + x_1)$.

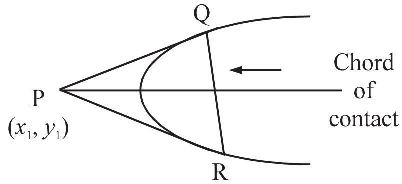



**Fig. 20.15**

## CHORD WITH A GIVEN MID POINT

The equation of the chord of the parabola $y^2 = 4ax$ with $P(x_1, y_1)$ as its middle point is given by

$$T = S_1$$

where $T \equiv yy_1 - 2a(x + x_1)$ and $S_1 \equiv y_1^2 - 4ax$.

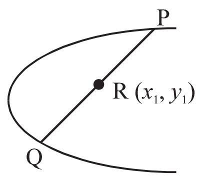



**Fig. 20.16**

## SOLVED EXAMPLES

**1.** If the tangent to the parabola $y^2 = 4ax$ meets the axis in $T$ and tangent at the vertex $A$ in $Y$ and the rectangle TAYG is completed, then the locus of $G$ is

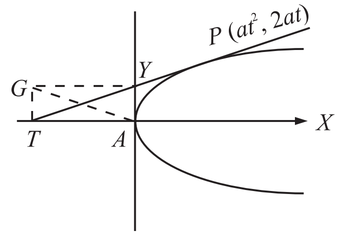
(A) $y^2 + 2ax = 0$ (B) $y^2 + ax = 0$
(C) $x^2 + ay = 0$ (D) none of these

**Solution: (B)**

Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$. The equation of tangent at $P$ is $ty = x + at^2$.

Since the tangent meets the axis of parabola in $T$ and tangent at the vertex $A$ in $Y$,

$\therefore$ coordinates of $T$ and $Y$ are $(-at^2, 0)$ and $(0, at)$ respectively.




Let the coordinates of $G$ be $(x_1, y_1)$.
Since TAYG is a rectangle,
$\therefore$ midpoint of diagonals $TY$ and $GA$ is same

$$\Rightarrow \frac{x_1 + 0}{2} = \frac{-at^2 + 0}{2} \text{ and } \frac{y_1 + 0}{2} = \frac{0 + at}{2}$$

$\Rightarrow$ $x_1 = -at^2$ (1)

and $y_1 = at$ (2)

Eliminating $t$ from (1) and (2), we get

$$x_1 = -a\left(\frac{y_1}{a}\right)^2 \Rightarrow y_1^2 + ax_1 = 0$$

$\therefore$ The locus of $G(x_1, y_1)$ is $y^2 + ax = 0$.

**2.** Equation of the parabola whose vertex is $(-3, -2)$, axis is horizontal and which passes through the point $(1, 2)$ is

(A) $y^2 + 4y + 4x - 8 = 0$ (B) $y^2 + 4y - 4x + 8 = 0$
(C) $y^2 + 4y - 4x - 8 = 0$ (D) none of these

**Solution: (C)**

Since the axis is horizontal and vertex is $(-3, -2)$, $\therefore$ the equation of the parabola must be of the form

$$(y + 2)^2 = 4a(x + 3)$$

It passes through $(1, 2)$, so $16 = 16a$ i.e. $a = 1$.
Hence, the equation of the required parabola is

$$(y + 2)^2 = 4(x + 3) \text{ or } y^2 + 4y - 4x - 8 = 0$$

**3.** Two tangents are drawn from the point $(-2, -1)$ to the parabola $y^2 = 4x$. If $\alpha$ is the angle between them, then $\tan\alpha =$

(A) 3 (B) $\frac{1}{3}$

(C) 2 (D) $\frac{1}{2}$

**Solution: (A)**

Given parabola is $y^2 = 4x$.

Here $4a = 4, \therefore a = 1$

Equation of pair of tangents drawn from the point (–2, –1) to the parabola is $SS_1 = T^2$

That is, $(y^2 - 4x)\,[(-1)^2 - 4(-2)] = [y(-1) - 2 \cdot 1(x - 2)]^2$

$\Rightarrow 9(y^2 - 4x) = (-2x - y + 4)^2$

$\Rightarrow 4x^2 - 8y^2 + 4xy + 20x - 8y + 16 = 0$

Since $\alpha$ is the angle between the tangents,

$$\therefore \tan\alpha = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{4-4(-8)}}{4-8}\right| = |-3| = 3$$

**4.** The portion of a tangent to a parabola $y^2 = 4ax$ cut off between the directrix and the curve subtends an angle $\theta$ at the focus, where $\theta =$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$ (D) none of these

**Solution: (C)**

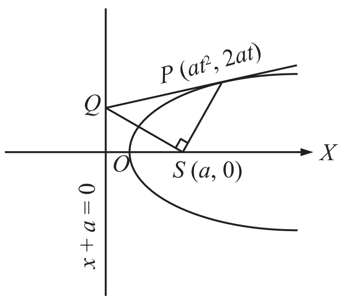
The equation of the tangent at $P(at^2, 2at)$ to

$$y^2 = 4ax \text{ is } ty = x + at^2 \quad (1)$$

It meets the directrix $x = -a$.

$$\therefore\ ty = -a + at^2 \Rightarrow y = \frac{a(t^2-1)}{t}$$

Thus, (1) meets the directrix at $Q\left(-a, \frac{a(t^2-1)}{t}\right)$.




Now, slope of $PS$ is $m_1 = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1}$ and slope of $QS$ is $m_2 = \frac{\frac{a(t^2-1)}{t-0}}{-a-a} = -\frac{(t^2-1)}{2t}$.

Since $m_1 m_2 = -1$, therefore $PQ$ subtends a right angle at the focus.

**5.** The length of the latus rectum of the parabola $25[(x-2)^2 + (y-4)^2] = (4x - 3y + 12)^2$ is

(A) $\frac{16}{5}$ (B) $\frac{8}{5}$

(C) $\frac{12}{5}$ (D) none of these

**Solution: (B)**

The given equation of the parabola can be written as

$$(x-2)^2 + (y-4)^2 = \left(\frac{4x - 3y + 12}{\sqrt{(4)^2 + (-3)^2}}\right)^2$$

$\therefore$ The coordinates of focus are (2, 4) and the equation of directrix is $4x - 3y + 12 = 0$.

The distance of the focus from the directrix

$$= \frac{|4(2) - 3(4) + 12|}{\sqrt{4^2 + (-3)^2}} = \frac{8}{5}$$

$\therefore$ The length of latus rectum $= 2 \times \frac{8}{5} = \frac{16}{5}$.

**6.** If the parabola $x^2 = ay$ makes an intercept of length $\sqrt{40}$ on the line $y - 2x = 1$, then $a$ is equal to

(A) 1 (B) –2

(C) –1 (D) 2

**Solution: (A, B)**

Solving the two equation $x^2 = ay$ and $y - 2x = 1$, we get

$$x^2 = a(2x + 1) \text{ or } x^2 - 2ax - a = 0$$

$\therefore\ x_1 + x_2 = 2a$ and $x_1 x_2 = -a$

So, the given line cuts the parabola at two points $(x_1, y_1)$ and $(x_2, y_2)$.

Now, $(\sqrt{40})^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ [Given]

$$\Rightarrow 40 = (x_1 - x_2)^2 + \left(\frac{x_1^2}{a} - \frac{x_2^2}{a}\right)^2$$

$$= (x_1 - x_2)^2 \left[1 + \frac{(x_1 + x_2)^2}{a^2}\right]$$

$$= [(x_1 + x_2)^2 - 4x_1x_2]\left(\frac{4a^2}{a^2} + 1\right) = 5(4a^2 + 4a)$$

$\therefore\ a^2 + a - 2 = 0$ or $(a + 2)(a - 1) = 0$

$\therefore\ a = 1, -2$

**7.** A ray of light is coming along the line which is parallel to $y$-axis and strikes a concave mirror whose intersection with the $xy$-plane is a parabola $(x - 4)^2 = 4(y + 2)$. After reflection, the ray must pass through the point

(A) (4, –1) (B) (0, 1)

(C) (–4, 1) (D) none of these

**Solution: (A)**

The equation of axis of the parabola is $x - 4 = 0$ which is parallel to $y$-axis. So the ray of light is parallel to the axis of the parabola. We know that any ray parallel to the axis of a parabola passes through the focus after reflection.

$\therefore$ The ray must pass through the point (4, –1).

**8.** With respect to the parabola $y^2 = 2x$, the points $P(4, 2)$ and $Q(1, 4)$ are such that

(A) $P$ and $Q$ both lie inside the parabola
(B) $P$ lies inside whereas $Q$ lies outside the parabola
(C) $P$ lies outside whereas $Q$ lies inside the parabola
(D) $P$ and $Q$ both lie outside the parabola

**Solution: (B)**

Let $S \equiv y^2 - 2x$.
Then $S\,]_{P(4,\,2)} = (2)^2 - 2(4) = 4 - 8 < 0$
$\therefore$ Point $P$ lies inside the parabola.
Also, $S]_{Q\,(1,\,4)} = (4)^2 - 2(1) = 16 - 2 = 14 > 0$
$\therefore$ Point $Q$ lies outside the parabola.

**9.** If $y + 3 = m_1(x + 2)$ and $y + 3 = m_2(x + 2)$ are two tangents to the parabola $y^2 = 8x$, then

(A) $m_1 + m_2 = 0$ (B) $m_1m_2 = -1$
(C) $m_1m_2 = 1$ (D) none of these

**Solution: (B)**

Clearly the two tangents, having slopes $m_1$ and $m_2$, meet on the line $x = -2$, which is the directrix of the parabola $y^2 = 8x$, therefore the two tangents must be at right angles, i.e. $m_1m_2 = -1$.

**10.** The parametric representation $(3 + t^2, 3t - 2)$ represents a parabola with

(A) focus at (–3, –2) (B) vertex at (3, –2)
(B) directrix $x = -5$ (D) all of these

**Solution: (B)**

We have, $x = 3 + t^2$ and $y = 3t - 2$
$\Rightarrow$ $x - 3 = t^2$ and $y + 2 = 3t$
$\Rightarrow$ $(y + 2)^2 = 9(x - 3)$
which is a parabola with vertex at (3, –2), focus at $\left(\dfrac{21}{4}, -2\right)$ and directrix $x = \dfrac{3}{4}$.

**11.** If $PSQ$ is the focal chord of the parabola $y^2 = 8x$ such that $SP = 6$, then the length $SQ$ is

(A) 6 (B) 4
(C) 3 (D) none of these

**Solution: (C)**

Since the semi latus rectum of a parabola is the harmonic mean between the segment of any focal chord of a parabola, therefore $SP$, 4, $SQ$ are in H.P.

$$\Rightarrow \quad 4 = 2\frac{SP \cdot SQ}{SP + SQ} \quad \Rightarrow \quad 4 = 2\frac{6 \cdot SQ}{6 + SQ} \quad \Rightarrow \quad SQ = 3$$

**12.** The circle on focal radii of a parabola as diameter touches the

(A) axis (B) directrix
(C) tangent at the vertex (D) none of these

**Solution: (C)**

Let the parabola be $y^2 = 4ax$.
Let $P(at^2, 2at)$ be any point on the parabola.
Then $SP$ is focal radii of the parabola.
The equation of a circle with $SP$ as diameter is

$$(x - a)\,(x - at^2) + (y - 0)\,(y - 2at) = 0$$

It meets $y$-axis at $x = 0$
$\therefore$ $y^2 - 2aty + a^2t^2 = 0$ i.e., $(y - at)^2 = 0$
$\Rightarrow$ $y$-axis meets the circle only at one point.
Therefore, the circle touches the tangent at the vertex.

**13.** If the two parabolas $y^2 = 4a(x - 2)$ and $x^2 = 4a(y - 3)$ touch each other, then their point of contact lies on a

(A) circle (B) parabola
(C) ellipse (D) hyperbola

**Solution: (D)**

Let $P(x_1, y_1)$ be the point of contact of the two given parabolas

$$y^2 = 4a(x - 2) \qquad (1)$$

and

$$x^2 = 4a\,(y - 3) \qquad (2)$$

Equation of tangent at $P$ to (1) is

$$yy_1 = 2a(x + x_1) - 8a$$

or

$$2ax - y_1y + (2ax_1 - 8a) = 0 \qquad (3)$$

Equation of tangent at $P$ to (2) is

$$xx_1 = 2a(y + y_1) - 12a$$

or

$$x_1x - 2ay - (2ay_1 - 12a) = 0 \qquad (4)$$

Since (3) and (4) represent the same line,

$$\therefore \quad \frac{2a}{x_1} = \frac{-y_1}{-2a} \quad \Rightarrow \quad x_1y_1 = 4a^2$$

$\therefore$ point of contact $(x_1, y_1)$ lies on the curve $xy = 4a^2$. which is a hyperbola.

**14.** If $ASB$ is a focal chord of a parabola such that $AS = 2$ and $SB = 4$, then the latus rectum of the parabola is

(A) $\dfrac{8}{3}$ (B) $\dfrac{16}{3}$

(C) $\dfrac{25}{3}$ (D) none of these

**Solution: (B)**

We know that semi-latus rectum of a parabola is the harmonic mean of segments of a focal chord.

$\therefore$ Semi latus rectum $= 2\dfrac{AS \cdot SB}{AS + SB} = \dfrac{16}{6}$. $\therefore$ Latus rectum $= \dfrac{32}{6}$ i.e., $\dfrac{16}{3}$.

**15.** If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of $k$ is

(A) $\dfrac{1}{8}$ (B) 8
(C) 4 (D) $\dfrac{1}{4}$

**Solution: (C)**

The given parabola is

$$y^2 = kx - 8 = k\left(x - \frac{8}{k}\right)$$

Shifting the origin to $(8/k, 0)$, the parabola becomes $Y^2 = kX$ where $X = x - 8/k$ and $Y = y$.

Directrix of this parabola is $X = \dfrac{-k}{4}$

or $$x - \frac{8}{k} = \frac{-k}{4}$$

This will coincide with $x = 1$ if $\dfrac{8}{k} - \dfrac{k}{4} = 1$

$\Rightarrow\ 32 - k^2 = 4k \ \Rightarrow\ k^2 + 4k - 32 = 0$

or $(k + 8)(k - 4) = 0 \ \Rightarrow\ k = -8, 4$

Thus, $k = 4$

**16.** A line bisecting the ordinate $PN$ of a point $P(at^2, 2at)$, $t > 0$, on the parabola $y^2 = 4ax$ is drawn parallel to the axis to meet the curve at $Q$. If $NQ$ meets the tangent at the vertex at the point $T$, then the coordinates of $T$ are

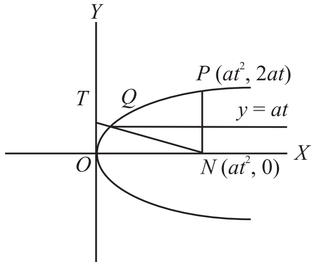
(A) $\left(0, \dfrac{4}{3}at\right)$ (B) $(0, 2at)$
(C) $\left(\dfrac{1}{4}at^2, at\right)$ (D) $(0, at)$

**Solution: (A)**

Equation of the line parallel to the axis and bisecting the ordinate $PN$ of the point $P(at^2, 2at)$ is $y = at$ which meets the parabola $y^2 = 4ax$ at the point $Q\left(\dfrac{1}{4}at^2, at\right)$.

Coordinates of $N$ are $(at^2, 0)$.

Equation of $NQ$ is $y = \dfrac{0 - at}{at^2 - \frac{1}{4}at^2}(x - at^2)$,




which meets the tangent at the vertex, $x = 0$, at the point

$$y = \frac{4}{3}at$$

**17.** The difference of the squares of the perpendiculars drawn from the points $(a \pm k, 0)$ on any tangent to a parabola $y^2 = 4ax$ is

(A) 4 (B) $4a$
(C) $4k$ (D) $4ak$

**Solution: (B)**

Equation of any tangent to the parabola

$$y^2 = 4ax \text{ is}$$
$$ty = x + at^2 \quad (1)$$

The difference of the squares of the perpendiculars from $(a \pm k, 0)$ to (1) is

$$\frac{(a + k + at^2)^2 (a - k + at^2)^2}{1 + t^2} = \frac{4a(1 + t^2)k}{1 + t^2} = 4ak$$

**18.** The locus of a point whose sum of the distances from the origin and the line $x = 2$ is 4 units, is

(A) $y^2 = -12(x - 3)$ (B) $y^2 = 12(x - 3)$
(C) $x^2 = 12(y - 3)$ (D) $x^2 = -12(y - 3)$

**Solution: (A)**

Let the coordinates of the point be $(h, k)$.

Distance of the point from origin

$$= \sqrt{(h - 0)^2 + (k - 0)^2} = \sqrt{h^2 + k^2}.$$

Distance of the point from the line $x - 2 = 0$ is $h - 2$

According to the given condition,

$$\sqrt{h^2 + k^2} + h - 2 = 4 \ \Rightarrow\ \sqrt{h^2 + k^2} = 6 - h$$

Squaring both sides, we have,

$$h^2 + k^2 = 36 + h^2 - 12h \text{ or } k^2 = -12(h - 3)$$

$\therefore$ The path of the point is $y^2 = -12(x - 3)$.

**19.** If three points $E$, $F$, $G$ are taken on the parabola $y^2 = 4ax$ so that their ordinates are in G.P. , then the tangents at $E$ and $G$ intersect on the

(A) directrix (B) axis
(C) ordinate of $F$ (D) none of these

**Solution: (B)**

Given parabola is $y^2 = 4ax$ (1)

Let the coordinates of $E$, $F$ and $G$ be respectively

$$(at_1^2, 2at_1), (at_2^2, 2at_2) \text{ and } (at_2^3, 2at_3)$$

Since ordinates of $E$, $F$ and $G$ are in G.P.

$$\therefore\ (2at_2)^2 = (2at_1)(2at_3) \text{ or } t_2^2 = t_1 t_3 \quad (2)$$

The tangents at $E$ and $G$ are

$$t_1 y = x + at_1^2 \quad (3)$$

and $$t_3 y = x + at_3^2 \quad (4)$$

Solving (3) and (4), we get $x = at_1t_3 = at_2^2$ [from (2)].

Since the $x$-coordinate of the point of intersection is $at_2^2$, the point lies on the line $x = at_2^2$ i.e., on the ordinate of $F(at_2^2, 2at_2)$.

**20.** If the two parabolas $y^2 = 4a(x - k_1)$ and $x^2 = 4a(y - k_2)$ always touch each other, $k_1$ and $k_2$ being variable parameters, then their point of contact lies on the curve

(A) $xy = a^2$ (B) $xy = 2a^2$
(C) $xy = 4a^2$ (D) none of these

**Solution: (C)**

Given parabolas are $y^2 = 4a(x - k_1)$ (1)

and $x^2 = 4a(y - k_2)$ (2)

Let $(\alpha, \beta)$ be their point of contact.

Equation of tangent to (1) at $(\alpha, \beta)$ is $\beta y = 2a(x - k_1 + \alpha)$

or $2ax - \beta y = 2a(k_1 - \alpha)$ (3)

Equation of tangent to (2) at $(\alpha, \beta)$ is

$$\alpha x = 2a(y - k_2 + \beta)$$

or $\alpha x - 2ay = 2a(\beta - k_2)$ (4)

Since (3) and (4) are identical, comparing coefficients of $x$ and $y$ in (3) and (4), we get

$$\frac{2a}{\alpha} = \frac{\beta}{2a}$$

$\Rightarrow \alpha\beta = 4a^2$. i.e., the point of contact $(\alpha, \beta)$ lies on the curve $xy = 4a^2$.

**21.** Coordinates of any point on the parabola, whose focus is $\left(\frac{-3}{2}, -3\right)$ and the directrix is $2x + 5 = 0$ is given by

(A) $(2t^2 + 2, 2t - 3)$ (B) $(2t^2 - 2, 2t - 3)$
(C) $(2t^2 - 2, 2t + 3)$ (D) none of these

**Solution: (B)**

The equation of the parabola is

$$\left(x + \frac{3}{2}\right)^2 + (y + 3)^2 = \left(\frac{2x + 5}{2}\right)^2$$

$$\Rightarrow 4\left[x^2 + \frac{9}{4} + 3x\right] + 4[y^2 + 9 + 6y] = (4x^2 + 25 + 20x)$$

$\Rightarrow$ $(y^2 + 6y + 9) - 2(x + 2) = 0$

or $(y + 3)^2 = 2(x + 2)$

Clearly, $x = 2t^2 - 2$ and $y = 2t - 3$ satisfy it for all $t$.

**22.** The conic represented by the equation $\sqrt{ax} + \sqrt{by} = 1$ is

(A) ellipse (B) hyperbola
(C) parabola (D) none of these

**Solution: (C)**

The given conic is $\sqrt{ax} + \sqrt{by} = 1$

Squaring both sides,

$$ax + by + 2\sqrt{abxy} = 1$$

or $$ax + by - 1 = -2\sqrt{abxy}$$

Squaring again, $(ax + by - 1)^2 = 4abxy$

or $a^2x^2 - 2abxy + b^2y^2 - 2ax - 2by + 1 = 0$ (1)

Comparing the equation (1) with the equation

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

$\therefore$ $A = a^2, H = -ab, B = b^2, G = -a, F = -b, C = 1$

Then, $\Delta = ABC + 2FGH - AF^2 - BG^2 - CH^2$
$= a^2b^2 - 2a^2b^2 - a^2b^2 - a^2b^2 - a^2b^2$
$= -4a^2b^2 \neq 0$

and $H^2 = a^2b^2 = AB$

So we have $\Delta \neq 0$ and $H^2 - AB = 0$. Hence the given equation represents a parabola.

**23.** If the tangents to the parabola $y^2 = 4ax$ at $(x_1, y_1)$ and $(x_2, y_2)$ intersect at $(x_3, y_3)$, then

(A) $x_1, x_2, x_3$ are in G. P (B) $x_1, x_2, x_3$ are in A. P
(C) $y_1, y_2, y_3$ are in G.P (D) $y_1, y_2, y_3$ are in A. P

**Solution: (A, B)**

Let $(x_1, y_1) \equiv (at_1^2, 2at_1)$

and $(x_2, y_2) \equiv (at_2^2, 2at_2)$.

Then, $(x_3, y_3) = [at_1t_2, a(t_1 + t_2)]$

$\therefore$ $x_1x_2 = at_1^2 \cdot at_2^2 = (at_1t_2)^2 = x_3^2$

and $$y_3 = a(t_1 + t_2) = \frac{1}{2}(y_1 + y_2)$$

$\therefore$ $x_1, x_2, x_3$ are in G. P and $y_1, y_2, y_3$ are in A.P.

**24.** A circle has its centre at the vertex of the parabola $x^2 = 4y$ and the circle cuts the parabola at the ends of its latus rectum. The equation of the circle is

(A) $x^2 + y^2 = 5$ (B) $x^2 + y^2 = 4$
(C) $x^2 + y^2 = 1$ (D) none of these

**Solution: (A)**

Coordinates of the vertex of the parabola $x^2 = 4y$ are (0, 0) and the ends of latus rectum are (2, 1) and (–2, 1).

$\therefore$ Centre of the circle is (0, 0) and radius of the circle is

$$= \sqrt{(2)^2 + (1)^2} = \sqrt{5}$$

$\therefore$ Equation of circle is

$$x^2 + y^2 = 5$$

**25.** If $b$ and $c$ are the lengths of the segments of any focal chord of a parabola $y^2 = 4ax$, then the length of the semi-latus rectum is

(A) $\frac{bc}{b+c}$ (B) $\sqrt{bc}$

(C) $\frac{b+c}{2}$ (D) $\frac{2bc}{b+c}$

**Solution: (D)**

Since the semi latus rectum of a parabola is the harmonic mean between the segments of any focal chord of the parabola.

$\therefore$ $l$ is the harmonic mean between $b$ and $c$.

Hence, $$l = \frac{2bc}{b+c}$$

**26.** The mirror image of the directrix of the parabola $y^2 = 4(x + 1)$ in the line mirror $x + 2y = 3$ is

(A) $x = -2$ (B) $4y - 3x = 16$

(C) $3x - 4y + 16 = 0$ (D) none of these

**Solution: (C)**

Directirx of $y^2 = 4(x + 1)$ is $x = -2$

Any point on $x = -2$ is $(-2, k)$

Now, mirror image $(x, y)$ of $(-2, k)$ in the line $x + 2y = 3$ is given by

$$\frac{x+2}{1} = \frac{y-k}{2} = -2\left(\frac{-2+2k-3}{5}\right)$$

$$\Rightarrow \quad x = \frac{10-4k}{5} - 2 \Rightarrow x = -\frac{4k}{5} \quad (1)$$

Also, $$y = \frac{20-8k}{5} \quad (2)$$

From (1) and (2)

$$y = 4 + \frac{3}{5}\left(\frac{5x}{4}\right)$$

or $4y = 16 + 3x$ is the equation of the mirror image of the directrix.

**27.** Through the vertex $O$ of a parabola $y^2 = 4x$, chords $OP$ and $OQ$ are drawn at right angles to one another. The locus of the middle point of $PQ$ is

(A) $y^2 = 2x + 8$ (B) $y^2 = x + 8$

(C) $y^2 = 2x - 8$ (D) none of these

**Solution: (C)**

Given parabola is $y^2 = 4x$ (1)

Here $4a = 4$, $\therefore a = 1$

Let $P \equiv (t_1^2, 2t_1)$ and $Q \equiv (t_2^2, 2t_2)$

Slope of $OP = \frac{2t_1}{t_1^2} = \frac{2}{t_1}$ and slope of $OQ = \frac{2}{t_2}$.

Since $OP \perp OQ$, $\therefore \frac{4}{t_1 t_2} = -1$ or $t_1 t_2 = -4$ (2)

Let $R(\alpha, \beta)$ be the middle point of $PQ$, then

$$\alpha = \frac{t_1^2 + t_2^2}{2} \quad (3) \text{ and } \beta = t_1 + t_2 \quad (4)$$

From (4), $\beta^2 = t_1^2 + t_2^2 + 2t_1 t_2 = 2\alpha - 8$

[From (2) and (3)]

Hence locus of $R(\alpha, \beta)$ is $y^2 = 2x - 8$.

**28.** If the parabolas $y^2 = 4a(x - c_1)$ and $x^2 = 4a(y - c_2)$ touch each other, then the locus of their point of contact is

(A) $xy = 4a^2$ (B) $xy = 2a^2$

(C) $xy = a^2$ (D) none of these

**Solution: (A)**

Let $P(x, y)$ be the point of contact.

$$\therefore \quad 2y\frac{dy}{dx} = 4a \text{ and } 2x = 4a\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{4a}{2y} = \frac{2x}{4a} \quad \Rightarrow \quad xy = 4a^2,$$

which is the required locus.

**29.** Maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$ may be equal to

(A) 3 (B) 5

(C) 4 (D) none of these

**Solution: (B)**

Equations of normals to $y^2 = 4ax$ and $x^2 = 4by$ are given by

$$y = mx - 2am - am^3 \text{ and } y = mx + 2b + \frac{b}{m^2}$$

For common normals, $2b + \frac{b}{m^2} + 2am + am^3 = 0$

$\Rightarrow \quad am^5 + 2am^3 + 2bm^2 + b = 0$

So, a maximum of 5 normals are possible.

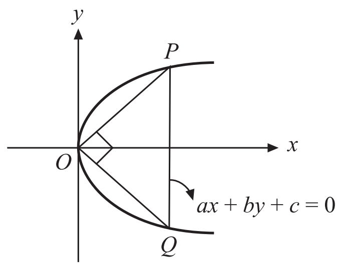
**30.** If the segment intercepted by the parabola $y^2 = 4\alpha x$ on the line $ax + by + c = 0$ subtends a right angle at the vertex, then

(A) $4a\alpha + c = 0$ (B) $4b\alpha + c = 0$

(C) $4a\alpha + b = 0$ (D) none of these

**Solution: (A)**

Making the equation of parabola $y^2 = 4\alpha x$ homogeneous using the equation of line $ax + by + c = 0$, we get



$$y^2 = 4\alpha x\left(\frac{ax+by}{-c}\right)$$

$\Rightarrow$ $4a\alpha x^2 + 4b\alpha xy + cy^2 = 0$,
which represents the combined equation of $OP$ and $OQ$.
Since $\angle POQ = 90°$, coefficient of $x^2$ + coefficient of $y^2 = 0$

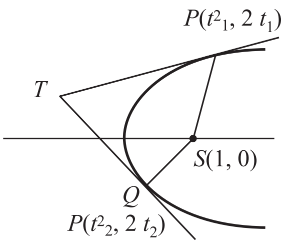
$\Rightarrow$ $$4a\alpha + c = 0$$

**31.** The tangents at two points $P$ and $Q$ on the parabola $y^2 = 4x$ intersect at $T$. If $SP$, $ST$ and $SQ$ are equal to $a$, $b$ an $c$ respectively, where $S$ is the focus, then the roots of the equation $ax^2 + 2bx + c = 0$ are

(A) real and equal (B) real and unequal
(C) complex numbers (D) irrational

**Solution: (A)**

The tangents at the points $P(t_1^2, 2t_1)$ and $Q(t_2^2, 2t_2)$ intersect at the point $T(t_1t_2, t_1 + t_2)$.




Now, $a = SP = 1 + t_1^2$ and $c = SQ = 1 + t_2^2$

$\therefore$ $b^2 = ST^2 = (t_1t_2 - 1)^2 + (t_1 + t_2)^2$

$$= t_1^2 + t_2^2 + 1 + t_1^2t_2^2$$

$$= (1 + t_1^2)(1 + t_2^2) = ac$$

$\therefore$ Roots of the equation $ax^2 + 2bx + c = 0$ are real and equal.

**32.** The centroid of the triangle formed by the feet of the normals from the point $(h, k)$ to the parabola $y^2 + 4ax = 0$, $(a > 0)$ lies on

(A) $x$-axis (B) $y$-axis
(C) $x = h$ (B) $y = k$

**Solution: (A)**

Co-ordinates of any point on the parabola $y^2 = -4ax$ are $(-at^2, 2at)$.
Equation of the normal at $(-at^2, 2at)$ is

$$y - xt = 2at + at^3$$

If the normal passes through the point $(h, k)$, then

$$k - th = 2at + at^3$$

or $$at^3 + (2a + h)t - k = 0,$$

which is a cubic equation whose three roots $t_1$, $t_2$, $t_3$ are the parameters of the feet of the three normals.

$\therefore$ Sum of the roots $= t_1 + t_2 + t_3 = -\dfrac{\text{Coefficient of } t^2}{\text{Coefficient of } t^3} = 0$

$\therefore$ Centroid of the triangle formed by the feet of the normals

$$= \left(-\frac{a}{3}(t_1^2 + t_2^2 + t_3^2), \frac{2a}{3}(t_1 + t_2 + t_3)\right)$$

$$= \left(-\frac{a}{3}(t_1^2 + t_2^2 + t_3^2), 0\right)$$

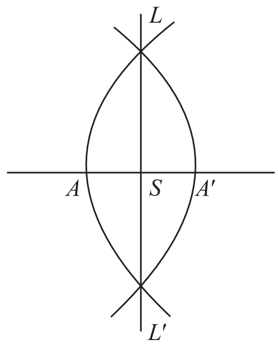
which, clearly, lies on the $x$-axis.

**33.** Given the two ends of the latus rectum, the maximum number of parabolas that can be drawn is

(A) 1 (B) 2
(C) 3 (D) none of these

**Solution: (B)**

$L$ and $L'$ are the ends of latus rectum. $S$ bisects $LL'$. As $A'$ is perpendicular bisector of $LL'$, where $AS = \frac{1}{4}LL' = A'S$.




Clearly, two parabolas are possible.

**34.** A line $L$ passing through the focus of the parabola $y^2 = 4(x - 1)$ intersects the parabola in two distinct points. If $m$ be the slope of the line $L$, then

(A) $m \in R - \{0\}$ (B) $-1 < m < 1$
(C) $m < -1$ or $m > 1$ (D) none of these

**Solution: (A)**

The focus of the parabola $y^2 = 4(x - 1)$ is $(2, 0)$. Any line through the focus is

$$(y - 0) = m(x - 2), \text{ i.e., } y = m(x - 2)$$

It will meet the given parabola if

$$m^2(x - 2)^2 = 4(x - 1)$$

or $$m^2x^2 - 4(m^2 + 1)x + 4(m^2 + 1) = 0$$

If $m \neq 0$, discriminant $= 16(m^2 + 1)^2 - 16m^2(m^2 + 1) = 0$

$$= 16(m^2 + 1) > 0 \text{ for all } m$$

But if $m = 0$, then $x$ does not have two real distinct values

$\therefore$ $$m \in R - \{0\}$$

**35.** If the length of a focal chord of the parabola $y^2 = 4ax$ at a distance $b$ from the vertex is $c$, then

(A) $a^2c = 4b^3$ (B) $b^2c = 4a^3$
(C) $c^2b = 4a^3$ (D) none of these

**Solution: (B)**

Let the ends of the focal chord be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$. Then $t_1t_2 = -1$.
Equation of the focal chord is

$$(t_1 + t_2)y = 2x + 2at_1t_2$$

Given: $$b = \frac{2at_1t_2}{\sqrt{(t_1+t_2)^2+4}} = \frac{-2a}{\sqrt{2+t_1^2+t_2^2}}$$

Also, $$c^2 = a^2(t_1^2 - t_2^2)^2 + 4a^2(t_1 - t_2)^2$$
$$= a^2(t_1 - t_2)^2[(t_1+t_2)^2 + 4]$$
$$= a^2(t_1^2 + t_2^2 + 2)^2 \quad (\because \quad t_1t_2 = -1)$$

$\therefore$ $$c = a(t_1^2 + t_2^2 + 2)$$

Now, $$b^2 = \frac{4a^2}{t_1^2+t_2^2+2} = \frac{4a^2}{c/a} = \frac{4a^3}{c}$$

$\therefore$ $b^2c = 4a^3$

**36.** If from a point, the two tangents drawn to the parabola $y^2 = 4ax$ are normals to the parabola $x^2 = 4by$, then

(A) $a^2 > 8b^2$ (B) $b^2 > 8a^2$
(C) $a^2 < 8b^2$ (D) none of these

**Solution: (A)**

The coordinates of any point on the parabola $x^2 = 4by$ are $(2bt, bt^2)$.

For the parabola $x^2 = 4by, \dfrac{dy}{dx} = \dfrac{x}{2b}$.

Slope of the normal at $(2bt, bt^2) = -\dfrac{2b}{2bt} = -\dfrac{1}{t}$

$\therefore$ Equation of normal is $y - bt^2 = -\dfrac{1}{t}(x - 2bt)$

or $$y = -\frac{x}{t} + 2b + bt^2$$

It will touch the parabola $y^2 = 4ax$ if

$$2b + bt^2 = \frac{a}{-1/t} \quad \left(\because \quad c = \frac{a}{m}\right)$$

$\Rightarrow$ $bt^2 + at + 2b = 0$
For distinct real roots, discriminant > 0
$\Rightarrow$ $a^2 - 8b^2 = 0$ or $a^2 > 8b^2$

**37.** If two tangents drawn from the point $(x_1, y_1)$ to the parabola $y^2 = 4x$ be such that the slope of one tangent is double of the other, then

(A) $2y_1^2 = 9x_1$ (B) $2x_1^2 = 9y_1$
(C) $4y_1^2 = 9x_1$ (D) none of these

**Solution: (A)**

The equation of any tangent to the parabola $y^2 = 4x$ is

$$y = mx + \frac{1}{m}$$

If it passes through the point $(x_1, y_1)$, then

$$y_1 = mx_1 + \frac{1}{m} \quad \text{or} \quad x_1m^2 - y_1m + 1 = 0$$

Its roots are given to be $m_1$ and $2m_1$

$$\therefore \quad m_1 + 2m_1 = \frac{y_1}{x_1} \quad \Rightarrow \quad 3m_1 = \frac{y_1}{x_1}$$

and $$m_1 \cdot 2m_1 = \frac{1}{x_1} \quad \Rightarrow \quad 2m_1^2 = \frac{1}{x_1}$$

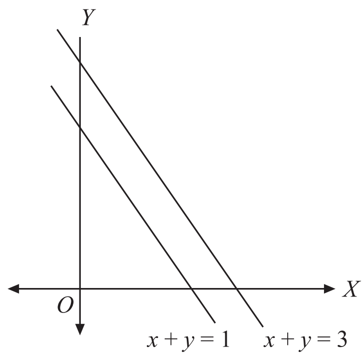
$$\therefore \quad 2\left(\frac{y_1}{3x_1}\right)^2 = \frac{1}{x_1} \quad \text{or} \quad 2y_1^2 = 9x_1$$

**38.** If the focus of the parabola $(y - \beta)^2 = 4(x - \alpha)$ always lies between the lines $x + y = 1$ and $x + y = 3$, then

(A) $1 < \alpha + \beta < 2$ (B) $0 < \alpha + \beta < 1$
(C) $0 < \alpha + \beta < 2$ (D) none of these

**Solution: (C)**

The coordinates of the focus of the given parabola are $(\alpha + 1, \beta)$.




Clearly, focus must lie to the opposite side of the origin w.r.t. the line $x + y - 1 = 0$ and same side as origin with respect to the line $x + y - 3 = 0$. Hence, $\alpha + \beta > 0$ and $\alpha + \beta < 2$.

## ELLIPSE

An ellipse is the locus of a point which moves in a plane so that the ratio of its distance from a fixed point (called focus) and a fixed line (called directrix) is a constant which is less than one. This ratio is called eccentricity and is denoted by $e$. For an ellipse, $e < 1$.

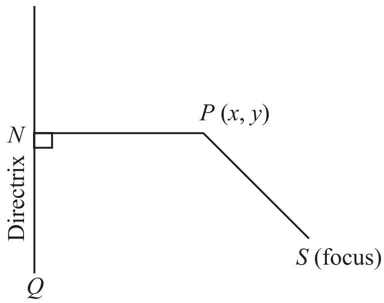



**Fig. 20.17**

Let $S$ be the focus, $QN$ be the directrix and $P$ be any point on the ellipse. Then, by definition, $\frac{PS}{PN} = e$ or $PS = e\ PN$, $e < 1$, where $PN$ is the length of the perpendicular from $P$ on the directrix $QN$.

**An alternate definition** An ellipse is the locus of a point that moves in such a way that the sum of its distances from two fixed points (called foci) is constant.

## Equation of An Ellipse in Standard Form

The standard form of the equation of an ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b),$$

where $a$ and $b$ are constants.

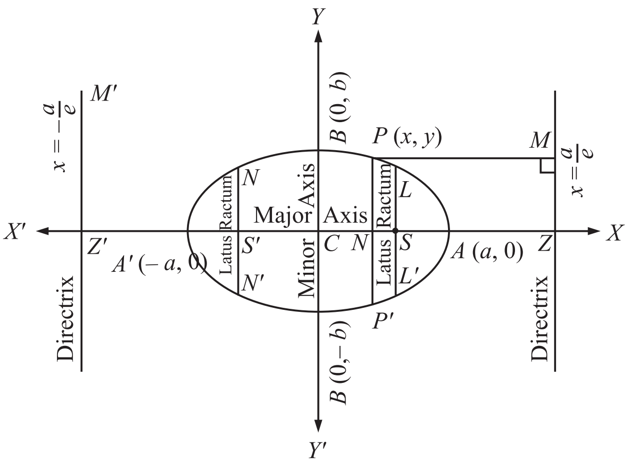



**Fig. 20.18**

## Some Terms and Properties Related to an Ellipse

A sketch of the locus of a moving point satisfying the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$, has been shown in the figure given above.

1. **Symmetry**
   (a) On replacing $y$ by $-y$, the above equation remains unchanged. So, the curve is symmetrical about $x$-axis
   (b) On replacing $x$ by $-x$, the above equation remains unchanged. So, the curve is symmetrical about $y$-axis
2. **Foci** If $S$ and $S'$ are the two foci of the ellipse and their coordinates are ($ae$, o) and ($-ae$, 0) respectively, then distance between foci is given by

$$SS' = 2ae$$

3. **Directrices** If $ZM$ and $Z'M'$ are the two directrices of the ellipse and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively, then the distance between directrices is given by

$$ZZ' = -\frac{2a}{e}$$

4. **Axes** The lines $AA'$ and $BB'$ are called the major axis and minor axis respectively of the ellipse.
   The length of major axis = $AA' = 2a$
   The length of minor axis = $BB' = 2b$
5. **Centre** The point of intersection $C$ of the axes of the ellipse is called the centre of the ellipse. All chords, passing through $C$ are bisected at $C$.
6. **Vertices** The end points $A$ and $A'$ of the major axis are known as the vertices of the ellipse

$$A \equiv (a, 0) \text{ and } A' \equiv (-a, 0)$$

   **Remember:** The vertex divides the join of focus and the point of interesection of directrix with axis internally and enternally the ratio $e : 1$.
7. **Focal chord** A chord of the ellipse passing through its focus is called a focal chord.
8. **Ordinate and double ordinate** Let $P$ be a point on the ellipse. From $P$, draw $PN \perp AA'$ (major axis of the ellipse) and produce $PN$ to meet the ellipse at $P'$. Then $PN$ is called an *ordinate* and $PNP'$ is called the *double ordinate* of the point $P$.
9. **Latus rectum** If $LL'$ and $NN'$ are the latus rectum of the ellipse, then these lines are $\perp$ to the major axis $AA'$, passing through the foci $S$ and $S'$ respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right), L' \equiv \left(ae, \frac{-b^2}{a}\right),$$

$$N \equiv \left(-ae, \frac{b^2}{a}\right), N' \equiv \left(-ae, \frac{-b^2}{a}\right)$$

   Length of latus rectum $= LL' = \frac{2b^2}{a} = NN'$.

10. By definition, $SP = ePM = e\left(\frac{a}{e} - x\right) = a - ex$

    and $S'P = e\left(\frac{a}{e} + x\right) = a + ex$

    This implies that distances of any point $P(x, y)$ lying on the ellipse from foci are: $(a - ex)$ and $(a + ex)$. In other words

$$SP + S'P = 2a$$

i.e., sum of distances of any point $P(x, y)$ lying on the ellipse from foci is constant.

**11. Eccentricity of the ellipse** Since, $SP = ePM$, therefore,

$$SP^2 = e^2PM^2$$

or $$(x-ae)^2+(y-0)^2 = e^2\left(\frac{a}{e}-x\right)^2$$

$$(x - ae)^2 + y^2 = (a - ex)^2$$

$$x^2 + a^2e^2 - 2aex + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$

On comparing with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$b^2 = a^2(1-e^2) \text{ or } e=\sqrt{1-\frac{b^2}{a^2}}$$

**12. Auxiliary circle** The circle drawn on major axis $AA'$ as diameter is known as the Auxiliary circle.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the equation of its auxiliary circle is

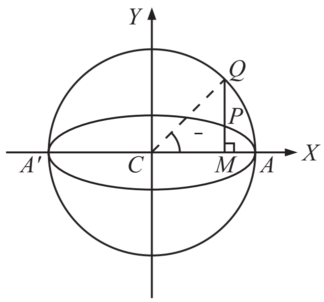



**Fig. 20.19**

$$x^2 + y^2 = a^2$$

Let $Q$ be a point on auxiliary circle so that $QM$, perpendicular to major axis meets the ellipse at $P$. The points $P$ and $Q$ are called as corresponding points on the ellipse and auxiliary circle respectively.

The angle $\theta$ is known as *eccentric angle* of the point $P$ on the ellipse.

It may be noted that the $CQ$ and not $CP$ is inclined at $\theta$ with $x$-axis.

**13. Parametric equation of the ellipse** The coordinates $x = a\cos\theta$ and $y = b\sin\theta$ satisfy the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

for all real values of $\theta$. Thus, $x = a\cos\theta$, $y = b\sin\theta$ are the parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameter $0 \le \theta < 2\pi$.

Hence the coordinates of any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

may be taken as $(a\cos\theta, b\sin\theta)$. This point is also called the point '$\theta$'.

The angle $\theta$ is called the eccentric angle of the point $(a\cos\theta, b\sin\theta)$ on the ellipse.

**14. Equation of Chord** The equation of the chord joining the points $P \equiv (a\cos\theta_1, b\sin\theta_1)$ and $Q \equiv (a\cos\theta_2, b\sin\theta_2)$ is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

**NOTE**

If the centre of the ellipse lies at $(h, k)$ and the axes are parallel to the coordintate axes, then the equation of the ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

## POSITION OF A POINT WITH RESPECT TO AN ELLIPSE

The point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0, =0$ or $0$.

Intersection of line and an Ellipse

The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two distinct points if $a^2m^2 + b^2 > c^2$, in one point if $c^2 = a^2m^2 + b^2$ and does not intersect if $a^2m^2 + b^2 < c^2$.

## Condition for Tangency and Points of Contact

The condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is that $c^2 = a^2m^2 + b^2$ and the coordinates of the points of contact are

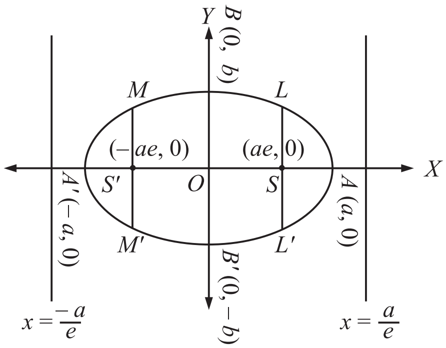
$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

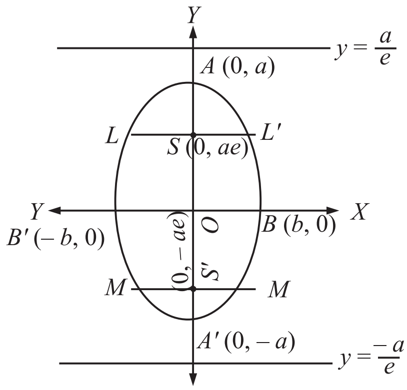
Two standard forms of the ellipse are shown below along with their properties:

## TWO STANDARD FORMS OF THE ELLIPSE

| | | |
|---|---|---|
| Standard Equation | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b)$ (Horizontal Form of an Ellipse) | $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1(a, b)$ (Vertical Form of an Ellipse |

Shape of the Ellipse




| | | |
|---|---|---|
| Centre | (0, 0) | (0, 0) |
| Equation of major axis | $y=0$ | $x=0$ |
| Equation of minor axis | $x=0$ | $y=0$ |
| Length of major axis | $2a$ | $2a$ |
| Length of minor axis | $2b$ | $2b$ |
| Foci | $(\pm ae, 0)$ | $(0, \pm ae)$ |
| Vertices | $(\pm a, 0)$ | $(0, \pm a)$ |
| Equation of directrices | $x=\pm\frac{a}{e}$ | $y=\pm\frac{a}{e}$ |
| Eccentricity | $e=\sqrt{\frac{a^2-b^2}{a^2}}$ | $e=\sqrt{\frac{a^2-b^2}{a^2}}$ |
| Length of latus rectum | $\frac{2b^2}{a}$ | $\frac{2b^2}{a}$ |
| Ends of latra-recta | $\left(\pm ae, \pm\frac{b^2}{a}\right)$ | $\left(\pm\frac{b^2}{a}, \pm ae\right)$ |
| Parametric coordinates | $(a\cos\theta, b\sin\theta)$ | $(a\cos\theta, b\sin\theta)$ |
| Focal radii | $SP=a-ex_1$ and $S'P=a+ex_1$ | $SP=a-ey_1$ and $S'P=a+ey_1$ |
| Sum of focal radii $SP+S'P=$ | $2a$ | $2a$ |
| Distance between foci | $2ae$ | $2ae$ |
| Distance between directrices | $\frac{2a}{e}$ | $\frac{2a}{e}$ |
| Tangents at the vertices | $x=\pm a$ | $y=\pm a$ |

## Equation of Tangent in Different Forms

1. **Point form** The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
   at the point $(x_1, y_1)$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

**TRICK(S) FOR PROBLEM SOLVING**

The equation of tangent at $(x_1, y_1)$ can also be obtained by replacing $x^2$ by $xx_1$, $y^2$ by $yy_1$, $x$ by $\frac{x+x_1}{2}$, $y$ by $\frac{y+y_1}{2}$, and $xy$ by $\frac{xy_1+x_1y}{2}$. This method is used only when the equation of ellipse is a polynomial of second degree in $x$ and $y$.

2. **Parametric form** The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

3. **Slope form** The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of slope '$m$' is

$$y=mx\pm\sqrt{a^2m^2+b^2}$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

**TRICK(S) FOR PROBLEM SOLVING**

- **Number of tangents drawn from a point** Two tangents can be drawn from a point to an ellipse. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.
- **Director circle** It is the locus of points from which perpendicular tangents are drawn to the ellipse.
  The equation of Director Circle of the ellipse

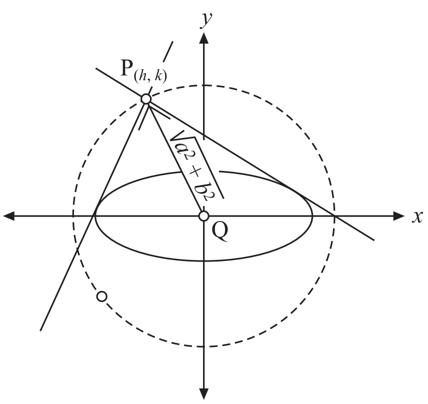



Fig. 20.20

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2+b^2$.

- The product of perpendiculars from the foci on any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to $b^2$.

## EQUATION OF NORMAL IN DIFFERENT FORMS

1. **Point form** The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(x_1, y_1)$ is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

2. **Parametric form** The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta, b\sin\theta)$ is

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$$

or $$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$$

3. **Slope form** The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of slope '$m$' is

$$y=mx\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$$

**TRICK(S) FOR PROBLEM SOLVING**

- The coordinates of the points of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2+b^2m^2}}\right)$$

- **Condition for normality** The line $y = mx + c$ is a normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ if } c^2=\frac{m^2(a^2-b^2)^2}{(a^2+b^2m^2)}.$$

## EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point

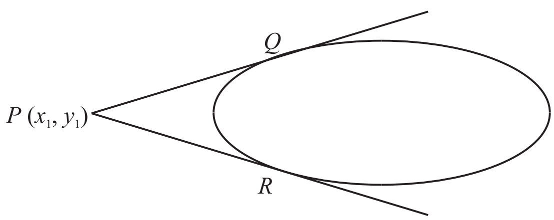



Fig. 20.21

$P(x_1, y_1)$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

$$SS_1 = T^2$$

where $$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1,\ S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

and $$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

## CHORD WITH A GIVEN MID POINT

The equation of the chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with $P(x_1, y_1)$ as its middle point is given by

$$T = S_1$$

where $$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1 \text{ and } S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

## Chord of Contact

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $T = 0$, where

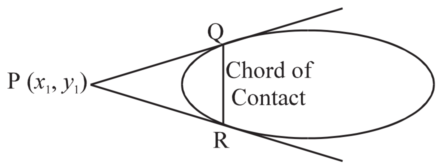


**Fig. 20.22**

$$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

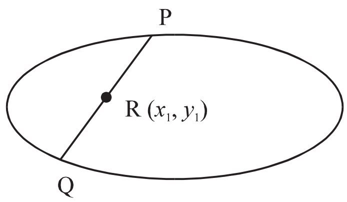


**Fig. 20.23**

## SOLVED EXAMPLES

**39.** The equation of the normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the end of the latus rectum in the first quadrant, is

(A) $x + ey - ae^3 = 0$ (B) $x - ey + ae^3 = 0$
(C) $x - ey - ae^3 = 0$ (D) none of these

**Solution: (C)**
The end of the latus rectum in the first quadrant is $\left(ae, \dfrac{b^2}{a}\right)$.

Equation of normal at $\left(ae, \dfrac{b^2}{a}\right)$ is

$$\frac{a^2x}{ae}-\frac{b^2y}{b^2/a}=a^2-b^2 \quad \left(\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2\right)$$

or $$\frac{a}{e}x - ay = a^2e^2 \quad \left(\because\ e^2=\frac{a^2-b^2}{a^2}\right)$$

or $x - ey - ae^3 = 0$

**40.** If the normal at the end of a latus rectum of an ellipse passes through one extremity of the minor axis, then

(A) $e^4 + e^2 - 1 = 0$ (B) $e^4 - e^2 + 1 = 0$
(C) $e^4 - e^2 - 1 = 0$ (D) none of these

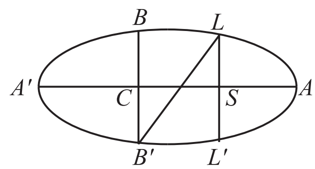
**Solution: (A)**
Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Let the normal at the extremity $L$ of the latus rectum passes through the extremity $B'$ of the minor axis.

Coordinates of $L$ are $\left(ae, \dfrac{b^2}{a}\right)$ and coordinates of $B'$ are $(0, -b)$.

Equation of the normal at $L$ is




$$\frac{a^2\cdot x}{ae}-\frac{b^2\cdot y}{b^2/a}=a^2-b^2\left(\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2\right)$$

or $$\frac{ax}{e}-ay=a^2-b^2$$

If it passes through $B'(0, -b)$, then $0 + ab = a^2 - b^2$

$\Rightarrow$ $a^2b^2 = (a^2 - b^2)^2$
But $b^2 = a^2(1 - e^2)$.
$\therefore$ $a^2 \times a^2(1 - e^2) = [a^2 - a^2(1 - e^2)]^2$
$\Rightarrow$ $a^4(1 - e^2) = a^4(1 - 1 + e^2)^2$
$\Rightarrow$ $1 - e^2 = e^4$ or $e^4 + e^2 - 1 = 0$

**41.** Eccentric angle of a point on the ellipse $x^2 + 3y^2 = 6$ at a distance 2 units from the centre of the ellipse is

(A) $\dfrac{\pi}{4}$ (B) $\dfrac{\pi}{3}$
(C) $\dfrac{3\pi}{4}$ (D) $\dfrac{2\pi}{3}$

**Solution: (A, C)**
The equation of ellipse can be written in the form

$$\frac{x^2}{(\sqrt{6})^2}+\frac{y^2}{(\sqrt{2})^2}=1$$

Let the eccentric angle of the point be $\theta$, then its coordinates are $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$.

Since the distance of the point from the centre is 2 units

$\therefore\ (\sqrt{6}\cos\theta - 0)^2 + (\sqrt{2}\sin\theta - 0)^2 = 4$

$\Rightarrow\ 6\cos^2\theta + 2(1 - \cos^2\theta) = 4 \ \Rightarrow\ 4\cos^2\theta = 2$

$\Rightarrow\ \cos\theta = \pm\dfrac{1}{\sqrt{2}}.\ \therefore\ \theta = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$

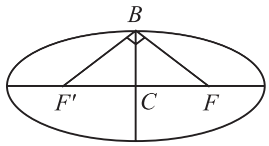
**42.** An ellipse has $CB$ as a semi minor axis, $F$, $F'$ are its foci and the angle $FBF'$ is a right angle. Then the eccentricity of the ellipse is

(A) $\dfrac{1}{\sqrt{2}}$ (B) $\dfrac{1}{2}$

(C) $\sqrt{\dfrac{3}{2}}$ (D) none of these

**Solution: (A)**

Since $\angle FBF' = \dfrac{\pi}{2}$




$\therefore\ \angle FBC = \angle F'BC = \dfrac{\pi}{4}$

$\therefore\ CB = CF \ \Rightarrow\ b = ae$

$\Rightarrow\ b^2 = a^2e^2 \ \Rightarrow\ a^2(1 - e^2) = a^2e^2$

$\Rightarrow\ 2e^2 = 1 \ \Rightarrow\ e = \dfrac{1}{\sqrt{2}}$

**43.** Let $P$ be a variable point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci $F_1$ and $F_2$. If $A$ is the area of the triangle $PF_1F_2$, then the maximum value of $A$ is

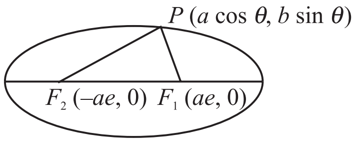
(A) $2abe$ (B) $abe$

(C) $\dfrac{1}{2}abe$ (D) none of these

**Solution: (B)**

Let $P \equiv (a\cos\theta, b\sin\theta)$

Then, $A$ = area of $\Delta PF_1F_2$




$$= \frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix}$$

$$= \left|\frac{1}{2}\cdot 2ae\cdot b\sin\theta\right| = abe|\sin\theta|.$$

Clearly, $A$ is maximum when $|\sin\theta| = 1$.

$\therefore$ Maximum value of $A = abe$.

**44.** The number of real tangents that can be drawn to the ellipse $3x^2 + 5y^2 = 32$ and $25x^2 + 9y^2 = 450$ passing through (3, 5) is

(A) 0 (B) 2

(C) 3 (D) 4

**Solution: (C)**

Since $3 \times 3^2 + 5 \times 5^2 - 32 > 0$, the point (3, 5) lies outside the ellipses $3x^2 + 5y^2 = 32$.

Also, $25 \times 3^2 + 9 \times 5^2 - 450 = 0$, $\therefore$ the point (3, 5) lies on the ellipse $25x^2 + 9y^2 = 450$. So the required number of tangents is 3.

**45.** The locus of mid-points of focal chords of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

(A) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{ex}{a}$ (B) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{ex}{a}$

(C) $x^2 + y^2 = a^2 + b^2$ (D) none of these

**Solution: (A)**

Let $(h, k)$ be the mid point of a focal chord. Then its equation is $T = S_1$

i.e., $$\frac{xh}{a^2} + \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$$

Since it passes through $(ae, 0)$

$\therefore\ \dfrac{hae}{a^2} = \dfrac{h^2}{a^2} + \dfrac{k^2}{b^2}$

$\therefore$ Locus of $(h, k)$ is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{xe}{a}$

**46.** If $\dfrac{x}{a} + \dfrac{y}{b} = \sqrt{2}$ touches the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then its eccentric angle $\theta$ is equal to

(A) 0 (B) 90°

(C) 45° (D) 60°

**Solution: (C)**

Equation of any tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is}$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad (1)$$

Also, $\dfrac{x}{a} + \dfrac{y}{b} = \sqrt{2}$ touches the given ellipse.

Comparing coefficients in (1) and (2), we get

$$\frac{\cos\theta/a}{1/a} = \frac{\sin\theta/b}{1/b} = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}} = \sin\theta$

$\therefore \quad \theta = 45°$

**47.** If the angle between the straight lines joining foci and the ends of minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is 90°, then the eccentricity is

(A) $\frac{1}{2}$ (B) $\sqrt{\frac{3}{2}}$

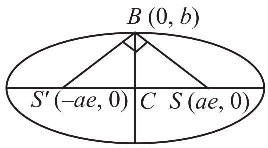
(C) $\frac{1}{\sqrt{2}}$ (D) none of these

**Solution: (C)**

Slope of $BS$ is $m_1 = \frac{b-0}{0-ae} = -\frac{b}{ae}$

and slope of $BS'$ is

$$m_2 = \frac{b-0}{0+ae} = \frac{b}{ae}$$




Since $\angle SBS' = 90°$,

$\therefore \quad m_1 m_2 = -1 \quad \Rightarrow \quad \frac{-b}{ae}\times\frac{b}{ae} = -1 \quad \Rightarrow \quad b^2 = a^2e^2$

$\Rightarrow \quad a^2(1-e^2) = a^2e^2 \quad \Rightarrow \quad 1-e^2 = e^2 \quad \Rightarrow \quad 2e^2 = 1$

$\therefore \quad e = \frac{1}{\sqrt{2}}$

**48.** The line $y = 2t^2$ meets the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ in real points if

(A) $|t| \le 1$ (B) $|t| > 1$

(C) $|t| < 3$ (D) none of these

**Solution: (A)**

Putting $y = 2t^2$ in the equation of the given ellipse

$$\frac{x^2}{9}+\frac{y^2}{4}=1, \text{ we get}$$

$\frac{x^2}{9}+\frac{4t^4}{4}=1 \quad \Rightarrow \quad x^2 = 9(1-t^4) = 9(1-t^2)(1+t^2)$

This will give real values of $x$ if $1 - t^2 \ge 0$ i.e., $|t| \le 1$.

**49.** The eccentricity of the ellipse which meets the straight line $\frac{x}{7}+\frac{y}{2}=1$ on the axis of $x$ and the straight line $\frac{x}{3}-\frac{y}{5}=1$ on the axis of $y$ and whose axes lie along the axes of coordinates is

(A) $\frac{2\sqrt{6}}{7}$ (B) $\frac{3\sqrt{2}}{7}$

(C) $\frac{\sqrt{6}}{7}$ (D) none of these

**Solution: (A)**

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

It is given that it passes through (7, 0) and (0, – 5)

$\therefore \quad a^2 = 49$ and $b^2 = 25$

Since $\quad b^2 = a^2(1-e^2), \therefore \quad 25 = 49(1-e^2)$

$\Rightarrow \quad 1-e^2 = \frac{25}{49} \quad \Rightarrow \quad e^2 = 1-\frac{25}{49} = \frac{24}{49} \quad \Rightarrow \quad e = \frac{2\sqrt{6}}{7}$

**50.** A point on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at a distance equal to the mean of the lengths of the semi-major axis and semi-minor axis from the centre is

(A) $\left(\frac{2\sqrt{91}}{7}, \frac{3\sqrt{105}}{14}\right)$

(B) $\left(\frac{2\sqrt{91}}{7}, \frac{-3\sqrt{105}}{14}\right)$

(C) $\left(-\frac{2\sqrt{105}}{7}, \frac{-3\sqrt{91}}{14}\right)$

(D) $\left(-\frac{2\sqrt{105}}{7}, \frac{3\sqrt{91}}{14}\right)$

**Solution: (A)**

Given ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$ i.e., $\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$

$\therefore$ Lengths of semi-major and semi-minor axes are 4 and 3 respectively. So, the mean of these lengths is 7/2.

Let $P(4\cos\theta, 3\sin\theta)$ be a point on the ellipse at a distance 7/2 from the centre (0, 0).

$\therefore \quad 16\cos^2\theta + 9\sin^2\theta = \frac{49}{4}$

$16\cos^2\theta + 9(1-\cos^2\theta) = \frac{49}{4} \quad \Rightarrow \quad 28\cos^2\theta = 13$

$\Rightarrow \quad \cos\theta = \pm\sqrt{\frac{13}{28}} = \pm\sqrt{\frac{91}{14}}$ and $\sin\theta = \pm\sqrt{\frac{105}{14}}$

So, the coordinates of the required point are

$\left(\pm\frac{4\sqrt{91}}{14}, \frac{3\sqrt{105}}{14}\right)$ i.e., $\left(\pm\frac{2\sqrt{91}}{7}, \frac{3\sqrt{105}}{14}\right)$

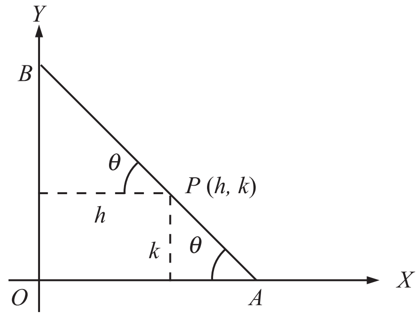
**51.** If the extremities of a line segment of length $l$ move in two fixed perpendicular straight lines, then the locus of that point which divides this line segment in the ratio 1 : 2, is

(A) a parabola (B) an ellipse
(C) a hyperbola (D) none of these

**Solution: (B)**

Let the two fixed $\perp$ straight lines be the coordinates axes.

Let $P(h, k)$ be the point whose locus is required.




Let $PA : PB = 1 : 2$

Then $PA = \frac{l}{3}$ and $PB = \frac{2l}{3}$

$$k = \frac{l}{3}\sin\theta \quad \text{or} \quad 3k = l\sin\theta \qquad (1)$$

and $$h = \frac{2l}{3}\cos\theta \quad \text{or} \quad \frac{3h}{2} = l\cos\theta \qquad (2)$$

Squaring and adding (1) and (2), we get

$$9k^2 + \frac{9h^2}{4} = l^2 \text{ or } 9h^2 + 36k^2 = 4l^2$$

$\therefore$ locus of $P(h, k)$ is $9x^2 + 36y^2 = 4l^2$, which is an ellipse.

**52.** If the distance of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ from the centre is 2, then the eccentric angle is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$
(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{2}$

**Solution: (B)**

Equation of the ellipse is $\frac{x^2}{6} + \frac{y^2}{2} = 1$.

Here $a^2 = 6$ and $b^2 = 2 \Rightarrow a = \sqrt{6}$ and $b = \sqrt{2}$.

Let '$\theta$' be the eccentric angle of the point so that the coordinates of the point are $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$.

Since distance of this point from the centre $C(0, 0)$ is 2.

$$\therefore \sqrt{6\cos^2\theta + 2\sin^2\theta} = 2 \Rightarrow 6\cos^2\theta + 2\sin^2\theta = 4$$

$$\Rightarrow 6\cos^2\theta + 2(1 - \cos^2\theta) = 4 \text{ or } 4\cos^2\theta = 2$$

$$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$$

$$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \quad (\because 0 \le \theta < 2\pi)$$

**53.** If the focal distance of an end of the minor axis of any ellipse (referred to its axes as the axes of $x$ and $y$ respectively) is $k$ and the distance between the foci is $2h$, then its equation is

(A) $\frac{x^2}{k^2} + \frac{y^2}{k^2 + h^2} = 1$ (B) $\frac{x^2}{k^2} + \frac{y^2}{h^2 - k^2} = 1$
(C) $\frac{x^2}{k^2} + \frac{y^2}{k^2 - h^2} = 1$ (D) $\frac{x^2}{k^2} + \frac{y^2}{h^2} = 1$

**Solution: (C)**

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let $e$ be the eccentricity of the ellipse.

Since distance between foci = $2h$

$$\therefore \quad 2ae = 2h \Rightarrow ae = h \qquad (1)$$

Focal distance of one end of minor axis say $(0, b)$ is $k$.

$$\therefore \quad a + e(0) = k \Rightarrow a = k \qquad (2)$$

From (1) and (2)

$$b^2 = a^2(1 - e^2) = k^2 - h^2$$

$\therefore$ The equation of the ellipse is

$$\frac{x^2}{k^2} + \frac{y^2}{k^2 - h^2} = 1$$

**54.** Let $E$ be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and $C$ be the circle $x^2 + y^2 = 9$. Let $P$ and $Q$ be the points (1, 2) and (2, 1) respectively. Then

(A) $Q$ lies inside $C$ but outside $E$
(B) $Q$ lies outside both $C$ and $E$
(C) $P$ lies inside both $C$ and $E$
(D) $P$ lies inside $C$ but outside $E$

**Solution: (D)**

Since $1^2 + 2^2 = 5 < 9$ and $2^2 + 1^2 = 5 < 9$,

$\therefore$ both $P$ and $Q$ lie inside $C$.

Also, $$\frac{1^2}{9} + \frac{2^2}{4} = \frac{1}{9} + 1 > 1$$

and $$\frac{2^2}{9} + \frac{1}{4} = \frac{16 + 9}{36} = \frac{25}{36} < 1$$

$\therefore$ $P$ lies outside $E$ and $Q$ lies inside $E$.

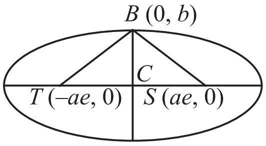
Thus, $P$ lies inside $C$ but outside $E$.

**55.** $S$ and $T$ are the foci of an ellipse and $B$ is an end of the minor axis. If $STB$ is an equilateral triangle, the eccentricity of the ellipse is

(A) 1/4 (B) 1/3
(C) 1/2 (D) 2/3

**Solution: (C)**

We have, $S \equiv (ae, 0)$, $T \equiv (-ae, 0)$ and $B \equiv (0, b)$.




Since $STB$ is an equilateral triangle

$\therefore \quad ST^2 = TB^2 \quad \Rightarrow \quad 4a^2e^2 = a^2e^2 + b^2$

$\Rightarrow \quad 3a^2e^2 = b^2 = a^2(1 - e^2)$

$$\Rightarrow \quad 3e^2 = 1 - e^2 \quad \Rightarrow \quad 4e^2 = 1 \quad \Rightarrow \quad e = \frac{1}{2}$$

**56.** An ellipse slides between two lines at right angles to one another. The locus of its centre is

(A) a parabola (B) an ellipse
(C) a circle (D) None of these

**Solution: (C)**

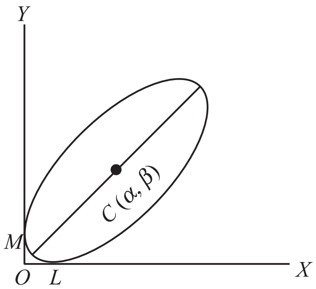
Let the two given lines be taken as the coordinate axes.

Let $C(\alpha, \beta)$ be the centre of the ellipse in any position. Here the position of centre $C$ changes as the ellipse slides.

Let $a$ and $b$ be the semi major and semiminor axes of the ellipse.

Equation of the director circle of the ellipse is

$$(x - \alpha)^2 + (y - \beta)^2 = a^2 + b^2 \qquad (1)$$




Since $OX$ and $OY$ are mutually perpendicular tangents to sliding ellipse for all its positions, therefore, $O(0, 0)$ will lie on circle (1)

$$\therefore \qquad \alpha^2 + \beta^2 = a^2 + b^2$$

Hence, locus of $C(\alpha, \beta)$ is $x^2 + y^2 = a^2 + b^2$.

**57.** If any tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ intercepts equal lengths $l$ on the axes, then $l =$

(A) 3 (B) 5
(C) $\sqrt{5}$ (D) none of these

**Solution: (B)**

The equation of any tangent to the given ellipse is

$$\frac{x}{4}\cos\theta + \frac{y}{3}\sin\theta = 1.$$

The tangent meets $x$-axis at $A\left(\frac{4}{\cos\theta}, 0\right)$ and $y$-axis at $\left(0, \frac{3}{\sin\theta}\right)$.

Given : $$\frac{4}{\cos\theta} = l = \frac{3}{\sin\theta}$$

$$\Rightarrow \quad \cos\theta = \frac{4}{l} \text{ and } \sin\theta = \frac{3}{l}$$

$$\Rightarrow \quad \cos^2\theta + \sin^2\theta = \frac{16}{l^2} + \frac{9}{l^2} \quad \Rightarrow \quad l^2 = 25. \quad \therefore \quad l = 5$$

**58.** If $P(a\cos\theta, b\sin\theta)$ is a point on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then '$\theta$' is

(A) angle of $OP$ line from positive direction of $x$-axis ($O$ is origin)
(B) angle of $OQ$ line from positive direction of $x$-axis [when $Q$ is $(a\cos\theta, a\sin\theta)$]
(C) it depends on the pont $P$
(D) none of the above

**Solution: (B)**

$P(a\cos\theta, b\sin\theta)$, then $\theta$ is angle of a corresponding pont on auxilliary circle $x^2 + y^2 = a^2$ i.e., $(a\cos\theta, a\sin\theta)$.

**59.** If (5, 12) and (24, 7) are the foci of an ellipse passing through the origin, then the eccentricity of the conic is

(A) $\frac{\sqrt{386}}{12}$ (B) $\frac{\sqrt{386}}{13}$
(C) $\frac{\sqrt{386}}{25}$ (D) $\frac{\sqrt{386}}{38}$

**Solution: (D)**

Let $S(5, 12)$, $S'(24, 7)$ be the two foci.

$P(0, 0)$ is a point on the conic.

$$SP = \sqrt{25 + 144} = 13$$

$$S'P = \sqrt{576 + 46} = \sqrt{625} = 25$$

$$SS' = \sqrt{(24-5)^2 + (7-12)^2} = \sqrt{19^2 + 5^2} = \sqrt{386}$$

If the conic is an ellipse,

then $\qquad SP + S'P = 2a$ and $SS' = 2ae$

$$\therefore \quad e = \frac{SS'}{SP + S'P} = \frac{\sqrt{386}}{13 + 25} = \frac{\sqrt{386}}{38}$$

**60.** A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at $P$ and $Q$. The angle between the tangents at $P$ and $Q$ of the ellipse $x^2 + 2y^2 = 6$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

**Solution: (D)**

Given ellipses are $x^2 + 4y^2 = 4$

i.e., $$\frac{x^2}{2^2}+\frac{y^2}{1^2}=1 \quad (1)$$

and $x^2 + 2y^2 = 6$ i.e., $$\frac{x^2}{(\sqrt{6})^2}+\frac{y^2}{(\sqrt{3})^2}=1 \quad (2)$$

Let $R(\alpha, \beta)$ be the point of intersection of the tangents to ellipse (2) at $P$ and $Q$. then $PQ$ will be chord of contact of $R$.

∴ its equation is

$$\frac{\alpha x}{6}+\frac{\beta y}{3}=1$$

i.e., $$\alpha x + 2y\beta = 6$$

or $$y=-\frac{\alpha}{2\beta}x+\frac{3}{\beta} \quad (3)$$

Since (3) touches (1)

∴ $$\left(\frac{3}{\beta}\right)^2 = 2^2\cdot\frac{\alpha^2}{4\beta^2}+1^2 \quad (c^2=a^2m^2+b^2)$$

⇒ $$\frac{9}{\beta^2}=\frac{\alpha^2}{\beta^2}+1=\frac{\alpha^2+\beta^2}{\beta^2}$$

⇒ $$\alpha^2+\beta^2=9$$

∴ locus of $(\alpha, \beta)$ is

$$x^2+y^2=9=(\sqrt{6})^2+(\sqrt{3})^2$$

i.e., director circle.

∴ tangent at $P$, $Q$ meet at right angles.

**61.** If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ intersects it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to:

(A) 2/3 (B) –2/3

(C) $\frac{1}{3}$ (B) $-\frac{1}{3}$

**Solution: (B)**

Equation of normal at $(a \cos 2\theta, b \sin 2\theta)$ is

$$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2$$

⇒ $$\sqrt{14}x\sec\theta-\sqrt{5}y\,\mathrm{cosec}\,\theta=9$$

It passes through $(a \cos 2\theta, b \sin 2\theta)$

⇒ $$\sqrt{14}\sqrt{14}\cos 2\theta\cdot\sec\theta-5\sin 2\theta\,\mathrm{cosec}\,\theta=9$$

⇒ $$14\frac{\cos 2\theta}{\cos\theta}-5\frac{\sin 2\theta}{\sin\theta}=9$$

⇒ $$14(2\cos^2\theta-1)-10\cos^2\theta=9(0)\theta$$

⇒ $$18\cos^2\theta-9\cos\theta-14=0$$

⇒ $$\cos\theta=-\frac{2}{3}$$

**62.** If the tangent line to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts intercepts $h$ and $k$ from axes, then $\frac{a^2}{h^2}+\frac{b^2}{k^2}=$

(A) 0 (B) 1

(C) –1 (D) 2

**Solution: (B)**

Let the equation of tangent at the point $(x_1, y_1)$ be

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \Rightarrow \frac{x}{\frac{a^2}{x_1}}+\frac{y}{\frac{b^2}{y_1}}=1$$

which meets axes at $\left(\frac{a^2}{x_1},0\right)$ and $\left(0,\frac{b^2}{y_1}\right)$

but $\frac{a^2}{x_1}=h$ and $\frac{b^2}{y_1}=k$ (given)

⇒ $$x_1=\frac{a^2}{h}, y_1=\frac{b^2}{k} \Rightarrow \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

∴ $$\frac{a^2}{h^2}+\frac{b^2}{k^2}=1$$

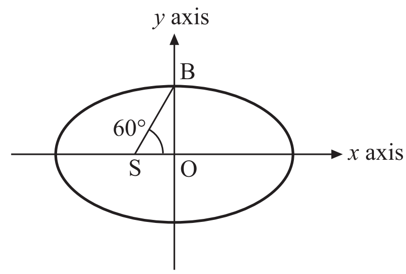
**63.** $S$ and $T$ are the foci of an ellipse and $B$ is an end of the minor axis. if $\Delta STB$ is equilateral, then $e$ is

(A) 1/4 (B) 1/3

(C) 1/2 (D) none of these

**Solution: (C)**

$$\tan 60^\circ=\frac{OB}{OS}$$




⇒ $$\sqrt{3}=-\frac{b}{ae} \Rightarrow \frac{b}{a}=e\sqrt{3} \quad (1)$$

Now $$e^2=1-\frac{b^2}{a^2} \Rightarrow e^2=1-3e^2$$

⇒ $$4e^2=1 \Rightarrow e=\frac{1}{2}$$

**64.** The centre of the ellipse $\frac{(x-y-2)^2}{16}+\frac{(x-y)^2}{9}=1$ is

(A) (0, 0) (B) (1, 1)
(C) (1, 0) (D) (0, 1)

**Solution: (B)**

The given equation can be written as

$$9(x+y-2)^2+16(x-y)^2=144$$

Let $t = 9(x+y-2)^2+16(x-y)^2-144$

∴ Centre of the conic (ellipse in this case) is the point of intersection of lines

$$\frac{\partial f}{\partial x}=0 \text{ and } \frac{\partial f}{\partial y}=0$$

Now, $\frac{\partial f}{\partial x}=0$

$\Rightarrow$ $18(x+y-2)+16\cdot 2(x-y)=0$
$\Rightarrow$ $18(x+y-2)+32(x-y)=0$
$\Rightarrow$ $50x-14y-36=0$ (1)

Also, $\frac{\partial f}{\partial y}=0$

$\Rightarrow$ $18(x+y-2)+16\cdot 2(x-y)(-1)=0$
$\Rightarrow$ $9(x+y-2)-16(x-y)=0$
$\Rightarrow$ $-7x+25y-18=0$
$\Rightarrow$ $7x-25y+18=0$ (2)

Solving (1) and (2), we have centre of conic is $C(1, 1)$.

**65.** On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are:

(A) $\left(\frac{2}{5},\frac{1}{5}\right)$ (B) $\left(-\frac{2}{5},\frac{1}{5}\right)$

(C) $\left(-\frac{2}{5},-\frac{1}{5}\right)$ (D) none of these

**Solution: (B)**

We have $PF_1 + PF_2 = 2a = 10$ for every point $P$ on the ellipse. Differentiating w.r.t. $x$, we get

$$8x+18y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{8x}{18y}=-\frac{4x}{9y}$$

The tangent at point $(x, y)$ will be parallel to $8x = 9y$ if

$$\frac{-4x}{9y}=\frac{8}{9} \Rightarrow x=-2y$$

Substituting $x = -2y$ in $4x^2 + 9y^2 = 1$, we get

$$4(-2y)^2+9y^2=1 \text{ or } 25y^2=1 \Rightarrow y=\pm\frac{1}{5}.$$

Thus, the points where the tangents are parallel to $8x = 9y$ are $\left(-\frac{2}{5},\frac{1}{5}\right)$ and $\left(\frac{2}{5},-\frac{1}{5}\right)$.

**66.** If $y = x$ and $3y + 2x = 0$ are the equations of a pair of conjugate diameters of an ellipse, then the eccentricity of the ellipse is

(A) $\sqrt{\frac{2}{3}}$ (B) $\frac{1}{\sqrt{3}}$

(C) $\frac{1}{\sqrt{2}}$ (C) $\frac{2}{\sqrt{5}}$

**Solution: (B)**

Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$

Slope of the given diameter are $m_1 = 1$, $m_2 = -2/3$.

$\Rightarrow$ $m_1m_2 = -2/3 = -b^2/a^2$

(using the condition of conjugacy of two diameters)

$\Rightarrow$ $3b^2 = 2a^2$ $\Rightarrow$ $3a^2(1-e^2) = 2a^2$

$\Rightarrow$ $1-e^2 = 2/3$ $\Rightarrow$ $e^2 = 1/3$ $\Rightarrow$ $e =$

**67.** The product of length of perpendiculars drawn from the two foci to the tangent at any point on the ellipse $25x^2 + 4y^2 = 100$ is

(A) $30\sqrt{5}$ (B) $25\sqrt{2}$
(C) 152 (D) none of these

**Solution: (D)**

Ellipse $\frac{x^2}{4}+\frac{y^2}{25}=1$ has length of semi-minor axis as 2, hence $b^2 = 4$. Product of length of perpendiculars $= b^2 = 4$.

**68.** Equation $\frac{x^2}{k}+\frac{y^2}{(k-1)}=1$ represents an ellipse if:

(A) $0 < k < 1$ (B) $k > 1$
(C) $k < 0$ or $k > 1$ (D) none of these

**Solution: (B)**

$k > 0, k-1 > 0 \Rightarrow k > 1$.

## OPTICAL PROPERTY OF PARABOLA

**(a)** A ray parallel to the axis of the parabola after reflection from its internal surface passes through the focus.
**(b)** If a point is at a minimum distance from a parabola, then this point must lie on a normal to the parabola through this point.

## Hyperbola

A hyperbola is the locus of a point which moves in a plane so that the ratio of its distances from a fixed point (called focus) and a fixed line (called directrix) is a constant which is greater then one. This ratio is called eccentricity and is denoted by $e$. For a hyperbola, $e > 1$.

Let $S$ be the focus, $QN$ be the directrix and $P$ be any point on the hyperbola. Then, by definition,

$$\frac{PS}{PN}=e \quad\text{or}\quad PS=ePN,\ e>1,$$

where $PN$ is the length of the perpendicular from $P$ on the directrix $QN$.

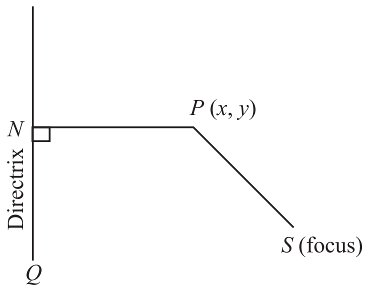



**Fig. 20.24**

**An alternate definition** A hyperbola is the locus of a point which moves in such a way that the difference of its distances from two fixed points (called foci) is constant.

## EQUATION OF A HYPERBOLA IN STANDARD FORM

The general form of standard hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where $a$ and $b$ are constants.

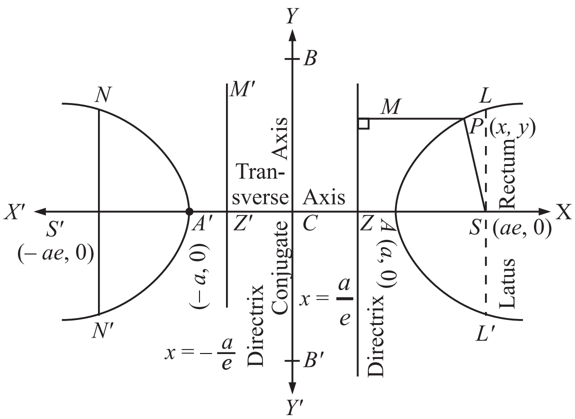



**Fig. 20.25**

## SOME TERMS AND PROPERTIES RELATED TO A HYPERBOLA

A sketch of the locus of a moving point satisfying the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, has been shown in the figure above.

1. **Symmetry** Since only even powers of $x$ and $y$ occur in the above equation, so the curve is symmetrical about both the axes.
2. **Foci** If $S$ and $S'$ are the two foci of the hyperbola and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively, then distance between foci is given by $SS' = 2ae$.
3. **Directrices** $ZM$ and $Z'M'$ are the two directrices of the hyperbola and their equations are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$ respectively, then the distance between directrices is given by $ZZ'=\frac{2a}{e}$.
4. **Axes** The lines $AA'$ and $BB'$ are called the transverse axis and conjugate axis respectively of the hyperbola.
   The length of transverse axis $= AA' = 2a$
   The length of conjugate axis $= BB' = 2b$
5. **Centre** The point of intersection $C$ of the axes of the hyperbola is called the centre of the hyperbola. All chords, passing through $C$, are bisected at $C$.
6. **Vertices** The points $A \equiv (a, 0)$ and $A' \equiv (-a, 0)$, where the curve meets the line joining the foci $S$ and $S'$, are called the vertices of the hyperbola.

**NOTE**

The vertex divides the join of focuss and the point of intersection of directrix with axis internally and externally in the ratio $e : 1$.

7. **Focal chord** A chord of the hyperbola passing through its focus is called a focal chord.
8. **Focal distances of a point** The difference of the focal distances of any point on the hyperbola is constant and equal to the length of the transverse axis of the hyperbola. If $P$ is any point on the hyperbola, then
   $$S'P - SP = 2a = \text{Transverse axis}.$$
9. **Latus rectum** If $LL'$ and $NN'$ are the latus rectum of the hyperbola then these lines are perpendicular to the transverse axis $AA'$, passing through the foci $S$ and $S'$ respectively.
   $$L \equiv \left(ae, \frac{b^2}{a}\right), \quad L' \equiv \left(ae, \frac{-b^2}{a}\right),$$
   $$N \equiv \left(-ae, \frac{b^2}{a}\right), \quad N' \equiv \left(-ae, \frac{-b^2}{a}\right)$$
   Length of latus rectum $= LL' = \frac{2b^2}{a} = NN'$.
10. **Eccentricity of the hyperbola** We know that
    $$SP = e\,PM \text{ or } SP^2 = e^2\,PM^2$$
    or
    $$(x-ae)^2+(y-0)^2 = e^2\left(x-\frac{a}{e}\right)^2$$
    $$(x-ae)^2+y^2=(ex-a)^2$$
    $$x^2+a^2e^2-2aex+y^2=e^2x^2-2aex+a^2$$
    $$x^2(e^2-1)-y^2=a^2(e^2-1)$$
    $$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1.$$
    On comparing with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$$b^2 = a^2(e^2 - 1) \quad \text{or} \quad e = \sqrt{1 + \frac{b^2}{a^2}}$$

**11. Parametric equations of the hyperbola** Since the coordinates $x = a \sec\theta$ and $y = b \tan\theta$ satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for all real values of $\theta$, therefore, $x = a \sec\theta$, $y = b \tan\theta$ are the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the parameter $0 \le \theta < 2\pi$.

Hence, the coordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be taken as $(a \sec\theta, b \tan\theta)$. This point is also called the point '$\theta$'.

The angle $\theta$ is called the eccentric angle of the point $(a \sec\theta, b \tan\theta)$ on the hyperbola.

**12. Equation of chord** The equation of the chord joining the points $P \equiv (a \sec\theta_1, b \tan\theta_1)$ and $Q \equiv (a \sec\theta_2, b \tan\theta_2)$ is

$$\frac{x}{a}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 + \theta_2}{2}\right).$$

or

$$\begin{vmatrix} x & y & 1 \\ a\sec\theta_1 & b\tan\theta_1 & 1 \\ a\sec\theta_2 & b\tan\theta_2 & 1 \end{vmatrix} = 0$$

## CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \left(\text{i.e., } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1\right)$

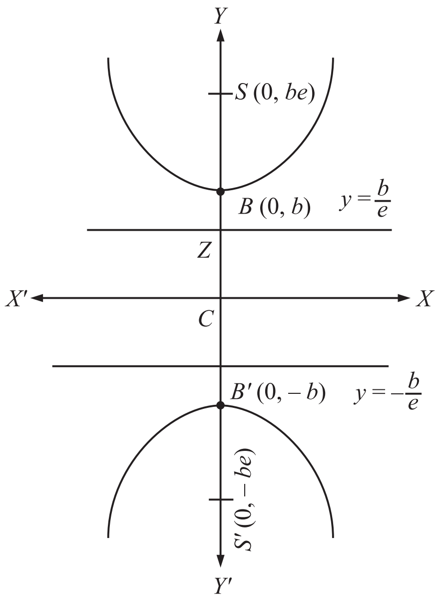



**Fig. 20.26**

Properties of hyperbola and conjugate are given below in the table:

## PROPERTIES OF HYPERBOLA AND ITS CONJUGATE

| | Hyperbola | Conjugate Hyperbola |
|---|---|---|
| Standard equation | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ | $\frac{-x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ |
| centre | (0, 0) | (0, 0) |
| Equation of transverse axis | $y = 0$ | $x = 0$ |
| Equation of conjugate axis | $x = 0$ | $y = 0$ |
| Length of transverse axis | $2a$ | $2b$ |
| Length of conjugate axis | $2b$ | $2a$ |
| Foci | | $(\pm ae, 0)$ $(0, \pm be)$ |
| Equation of directrices | $x = \pm\frac{a}{e}$ | $y = \pm\frac{b}{e}$ |
| Vertices | $(\pm a, 0)$ | $(0, \pm b)$ |

| | | |
|---|---|---|
| Eccentricity | $e=\sqrt{\frac{a^2+b^2}{a^2}}$ | $e=\sqrt{\frac{a^2+b^2}{b^2}}$ |
| Length of latus rectum | $\frac{2b^2}{a}$ | $\frac{2a^2}{b}$ |
| Parametrc coordinates | $(a\sec\theta, b\tan\theta)$ | $(b\sec\theta, a\tan\theta)$ |
| Focal radii | $SP = ex_1 - a$ and $S'P = ex_1 + a$ | $SP = ey_1 - b$ and $S'P = ey_1 + b$ |
| Difference of focal radii ($S'P - SP$) | $2a$ | $2b$ |
| Tangents at the vertices | $x = \pm a$ | $y = \pm b$ |

**NOTE**

If the centre of the hyperbola lies at a point ($h$, $k$) and the axes are paralllel to the coordinate axes, then the equation of the hyperbola is

$$\frac{(x-a)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

## POSITION OF A POINT WITH RESPECT TO A HYPERBOLA

The point $P(x_1, y_1)$ lies outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1>0, =0$ or $<0$.

### Intersection of a line and a Hyperbola

The straight line $y = mx + c$ will cut the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in two points which may be real, concident or imaginary according as $C^2 > 1 = 1 < a^2m^2 - b^2$.

### Condition for Tangency and Points of Contact

The condition for the line $y = mx + c$ to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is that $c^2 = a^2m^2 - b^2$ and the coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

### Equation of Tangent in Different Forms

1. **Point form** The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(x_1, y_1)$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

**TRICK(S) FOR PROBLEM SOLVING**

The equation of tangent at $(x_1, y_1)$ can also be obtained by replacing $x^2$ by $xx_1$, $y^2$ by $yy_1$, $x$ by $\frac{x+x_1}{2}$, $y$ by $\frac{y+y_1}{2}$ and $xy$ by $\frac{xy_1+x_1y}{2}$. This method is used only when the equation of hyperbola is a polynomial of second degree in $x$ and $y$.

2. **Parametric form** The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$ is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$$

3. **Slope form** The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of slope '$m$' is

$$y = mx \pm \sqrt{a^2m^2-b^2}$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

**TRICK(S) FOR PROBLEM SOLVING**

- **Number of tangents from a point** Two tangents can be drawn from a point to a hyperbola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the hyperbola.
- **Director circle** It is the locus of points from which ⊥ tangents are drawn to the hyperbola. The equation of director circle of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is } x^2+y^2=a^2-b^2$$