CHAPTER

3

Complex Numbers

Chapter Highlights

Imaginary numbers, Integral powers of *i*, Complex numbers, Conjugate of a complex number, Modulus of a complex number, Square roots of a complex number, Argand plane and geometrical representation of complex numbers, Polar form of a complex number, Particular cases of polar form, Eulerian representation of a complex number, Logarithm of a complex number, Vectorial representation of a complex number, Roots of a complex number, Geometry of complex numbers.

IMAGINARY NUMBERS

Square root of a negative number is called an *imaginary number*.

Illustration 1

$\sqrt{-1}, \sqrt{-4}, \sqrt{-7}, \sqrt{-18}$, and so on are all imaginary numbers.

$\sqrt{-1}$ is denoted by the Greek letter i (pronounced as *iota*), where i is a number such that $i^2 = -1$. Thus,

$$\sqrt{-2} = \sqrt{2}i, \quad \sqrt{-3} = \sqrt{3}i, \quad \sqrt{-4} = 2i.$$

NOTE

If $a < 0$, then $\sqrt{a} = \sqrt{|a|}\, i$.

The symbol '*i*' was first introduced by the famous mathematician, Leonhard Euler (1707–1783) in 1748, possibly because '*i*' is the first letter of the Latin word 'imaginarius'.

INTEGRAL POWERS OF *I*

We have

$$i = \sqrt{-1},\ i^2 = -1.$$

Therefore,

$$i^3 = i^2 \times i = (-1) \times i = -i,$$

$$i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$$

TRICK(S) FOR PROBLEM SOLVING

- For any $n \in N$
  1. $i^{2n} = (i^2)^n = (-1)^n = \begin{bmatrix} 1, & \text{when } n \text{ is even} \\ -1, & \text{when } n \text{ is odd} \end{bmatrix}$
  2. $i^{2n+1} = (i^{2n})\, i = (-1)^n i = \begin{bmatrix} i, & \text{when } n \text{ is even} \\ -i, & \text{when } n \text{ is odd} \end{bmatrix}$.
- The sum of four consecutive powers of i is zero. For example,
  $$i^{10} + i^{11} + i^{13} + i^{14} = 0$$
- Also, for any $n \in N$, the value of i^{-n} is found out by writing this as $\frac{1}{i^n}$ and solving i^n.
- Thus, any integral power of i can be expressed in terms of ± 1 or $\pm i$.

For any two real numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either zero or positive. If both a and b are positive real numbers, then the calculation

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab} \text{ is wrong.}$$

The correct calculation is

$$\begin{aligned} \sqrt{-a} \times \sqrt{-b} &= (\sqrt{-1}\sqrt{a})(\sqrt{-1}\sqrt{b}) \\ &= (i\sqrt{a})(i\sqrt{b}) \\ &= i^2(\sqrt{a} \times \sqrt{b}) = (-1)(\sqrt{ab}) \\ &= -\sqrt{ab} \end{aligned}$$

Thus, the calculation $\sqrt{-2}\times\sqrt{-3} = \sqrt{(-2)\times(-3)}$

$= \sqrt{6}$ is wrong.

The correct result is

$$\sqrt{-2}\times\sqrt{-3} = (i\sqrt{2})\,(i\sqrt{3})$$
$$= i^2(\sqrt{2}\times\sqrt{3})$$
$$= -\sqrt{6}$$

SOLVED EXAMPLES

1. The value of $\left(\dfrac{1+i}{\sqrt{2}}\right)^8 + \left(\dfrac{1-i}{\sqrt{2}}\right)^8$ is equal to

(A) 4 (B) 6 (C) 8 (D) 2

Solution: (D)

We have, $\left(\dfrac{1+i}{\sqrt{2}}\right)^8 + \left(\dfrac{1-i}{\sqrt{2}}\right)^8$

$$= \left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^8 + \left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]^8$$

$$= \cos 2\pi + i\sin 2\pi + \cos 2\pi - i\sin 2\pi$$

$$= 2\cos 2\pi = 2\,(1) = 2 \quad \text{[By De-Moivre's theorem]}$$

2. $\sqrt{i} - \sqrt{-i}$ is equal to

(A) $i\sqrt{2}$ (B) $\dfrac{1}{i\sqrt{2}}$ (C) 0 (D) $-i\sqrt{2}$

Solution: (A, D)

We have, $i = 0 + i\cdot 1 = \dfrac{1}{2}\,(0 + 2i)$

$$= \frac{1}{2}\,(1 + i^2 + 2\cdot 1\cdot i) = \frac{1}{2}\,(1+i)^2$$

$$\therefore \quad \sqrt{i} = \pm\frac{1}{\sqrt{2}}\,(1+i)$$

$$\therefore \quad \sqrt{-i} = \pm\frac{1}{\sqrt{2}}\,(1-i)$$

Hence,

$$\sqrt{i} - \sqrt{-i} = \pm\frac{1}{\sqrt{2}}\,[(1+i)-(1-i)] = \pm\sqrt{2}\,i$$

3. The value of the sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals

(A) i (B) $i - 1$ (C) $-i$ (D) 0

Solution: (B)

$$\sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{n=1}^{13} i^n(1+i)$$

$$= (1+i)\left[\frac{i(1-i^{13})}{1-i}\right]$$

$$= (1+i)\left[\frac{i(1-i)}{1-i}\right]$$

$$= (1+i)\,i = -1 + i$$

4. The least positive integer n for which

$\left(\dfrac{1+i}{1-i}\right)^n = \dfrac{2}{\pi}\sin^{-1}\dfrac{1+x^2}{2x}$, where $x > 0$, is

(A) 1 (B) 2

(C) 4 (D) None of these

Solution: (C)

For $\sin^{-1}\dfrac{1+x^2}{2x}$ to be defined,

$$-1 \le \frac{1+x^2}{2x} \le 1$$

or $\quad \dfrac{1+x^2}{2x} \le 1$

or $\quad 1 + x^2 \le 2x$

or $\quad (1-x)^2 \le 0$ or $x = 1$

Now,

$$\left(\frac{1+i}{1-i}\right)^n = 1 \Rightarrow \left(\frac{(1+i)^2}{2}\right)^n = 1 \Rightarrow i^n = 1$$

COMPLEX NUMBERS

An expression of the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$, is called a *complex number*. It is usually denoted by z, i.e.,

$$z = x + iy$$

x is called the *real* part and y the *imaginary* part of z and may be denoted by Re (z) and Im (z) respectively.

If $y = 0$, z is called *purely real* and if $x = 0$, z is called *purely imaginary*.

The set of complex numbers is denoted by C.

If $x = 0$ and $y = 0$, the complex number reduces to $0 + i\cdot 0 = 0$, which is called the *zero complex number.*

IMPORTANT POINTS

- We observe that the system of complex numbers includes the system of real numbers, i.e., $R \subset C$.
- Every real number is a complex number.
- 0 is both purely real and purely imaginary number.
- A complex number is an imaginary number if and only if its imaginary part is non-zero. Here, real part may or may not be zero. $4 + 3i$ is an imaginary number but not purely imaginary.
- All purely imaginary numbers except zero are imaginary numbers but an imaginary number may or may not be purely imaginary.

EQUALITY OF COMPLEX NUMBERS

Two complex numbers are said to be *equal* if and only if their real parts and imaginary parts are separately equal.

i.e., $a + ib = c + id$

$\Leftrightarrow$ $a = c$ and $b = d$.

i.e., $z_1 = z_2$

$\Leftrightarrow$ $\text{Re}\,(z_1) = \text{Re}\,(z_2)$ and $\text{Im}\,(z_1) = \text{Im}\,(z_2)$

CAUTION

Inequality relation does not hold good in case of complex numbers having non-zero imaginary parts. For example, the statement $8 + 5i > 4 + 2i$ makes no sense.

Algebra of Complex Numbers

Addition

For two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, their sum is defined as

$$z = z_1 + z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$$

Subtraction

For two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, the subtraction of z_2 from z_1 is defined as

$$z_1 - z_2 = z_1 + (-z_2) = (a_1 - a_2) + i\,(b_1 - b_2)$$

Multiplication

Multiplication of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is defined as

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

TRICK(S) FOR PROBLEM SOLVING

The product of complex numbers can be easily computed if we actually carry out the multiplication as given below:

$$\begin{aligned}(a + ib)\,(c + id) &= ac + iad + ibc + i^2 bd \\ &= ac + i\,(ad + bc) - bd \qquad (\because i^2 = -1) \\ &= (ac - bd) + i\,(ad + bc)\end{aligned}$$

Division

Division of two complex numbers

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2,$$

where $x_2 + iy_2 \neq 0$, is defined as

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$

$$= \frac{x_1x_2 + y_1y_2 + i\,(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{i\,(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

Multiplicative Inverse of a Non-zero Complex Number

Multiplicative inverse of a non-zero complex number $z = a + ib$ is defined as

$$z^{-1} = \frac{1}{z} = \frac{1}{a + ib} = \frac{1}{a + ib} \times \frac{a - ib}{a - ib} = \frac{a - ib}{a^2 + b^2}$$

$$= \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$$

i.e., $$z^{-1} = \frac{\text{Re}\,(z)}{|z|^2} + i\frac{[-\text{Im}\,(z)]}{|z|^2}$$

SOLVED EXAMPLES

5. The number of integral solutions of the equation $(1 - i)^x = 2^x$ are

(A) 1 (B) 2

(C) 0 (D) None of these

Solution: (C)

Let k be an integral solution of the given equation.

Then, $(1 - i)^k = 2^k \Rightarrow |(1 - i)^k| = 2^k \Rightarrow \left(\sqrt{2}\right)^k = 2^k$,

which is possible only if $k = 0$.

6. Let z_1 and z_2 be two non real complex cube roots of unity and $|z - z_1|^2 + |z - z_2|^2 = \lambda$ be the equation of a circle with z_1, z_2 as ends of a diameter, then the value of λ is

(A) 4 (B) 3 (C) 2 (D) $\sqrt{2}$

Solution: (B)

We have,

$$|z - \omega|^2 + |z - \omega^2|^2 = \lambda$$

$$\Rightarrow \quad \lambda = |\omega - \omega^2|^2 = |\omega^2 + \omega^4 - 2\omega^3|$$

$$= |\omega^2 + \omega - 2| = |-1 - 2| = 3$$

CONJUGATE OF A COMPLEX NUMBER

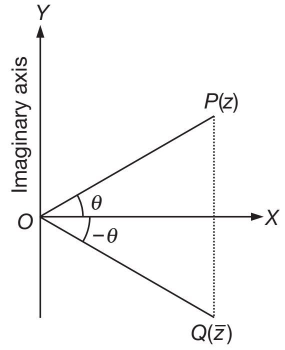
Conjugate of a complex number $z = a + ib$ is defined as

$$\bar{z} = a - ib.$$

For example, $\overline{4 + 5i} = 4 - 5i$ and $\overline{4 - 5i} = 4 + 5i$.

IMPORTANT POINTS

Geometrically, the conjugate of z is the reflection or point image of z in the real axis.



Properties of Conjugate

1. $\overline{(\overline{z})} = z$
2. $z = \overline{z}$ if and only if z is purely real
3. $z = -\overline{z}$ if and only if z is purely imaginary
4. $z + \overline{z} = 2\,\text{Re}\,(z)$ and $z - \overline{z} = 2i\,\text{Im}\,(z)$
5. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
6. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
7. $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
8. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$
9. If $z = f(z_1)$, then $\overline{z} = f(\overline{z_1})$
10. $\overline{(z^n)} = (\overline{z})^n$
11. $z_1 \overline{z_2} + \overline{z_1} z_2 = 2\,\text{Re}\,(\overline{z_1} z_2) = 2\,\text{Re}\,(z_1 \overline{z_2})$

Method of Writing the Complex Number $\dfrac{a+ib}{c+id}$ in the form A + iB

We have,

$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

[Multiplying the Nu. and the Dn. by the conjugate of the Dn.]

$$= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

$$= A + iB,$$

where

$$A = \frac{ac+bd}{c^2+d^2}$$

and

$$B = \frac{bc-ad}{c^2+d^2}.$$

TRICK(S) FOR PROBLEM SOLVING

To put the complex number $\dfrac{a+ib}{c+id}$ in the form $A + iB$ we should multiply the numerator and the denominator by the conjugate of the denominator.

MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = a + ib$, denoted as mod (z) or $|z|$, is defined as

$$|z| = \sqrt{a^2+b^2}, \text{ where } a = \text{Re}\,(z),\ b = \text{Im}\,(z).$$

Sometimes, $|z|$ is called absolute value of z. Note that $|z| \geq 0$.

For example, if $z = 3 + 2i$, then $|z| = \sqrt{3^2+2^2} = \sqrt{13}$.

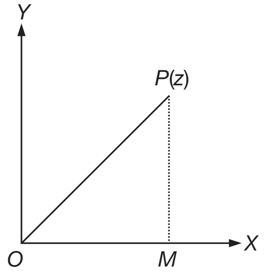
Properties of Modulus

1. $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$, i.e., $x = 0$, $y = 0$
2. $|z| = |\overline{z}| = |-z| = |-\overline{z}|$.
3. $z\overline{z} = |z|^2$
4. $-|z| \leq \text{Re}\,(z) \leq |z|$ and $-|z| \leq \text{Im}\,(z) \leq |z|$
5. $|z^n| = |z|^n$
6. $|z_1 z_2| = |z_1|\,|z_2|$
7. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
8. $|z_1 \pm z_2| \leq |z_1| + |z_2|$
9. $|z_1 - z_2| \geq |z_1| - |z_2|$
10. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2)$
11. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}\,(z_1 \overline{z_2})$
12. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}\,(z_1 \overline{z_2})$
13. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$$\Rightarrow \frac{z_1}{z_2} \text{ is purely imaginary or Re}\left(\frac{z_1}{z_2}\right) = 0$$

IMPORTANT POINTS

Geometrically $|z|$ represents the distance of point P from the origin. i.e., $|z| = OP$



TRICK(S) FOR PROBLEM SOLVING

Most of the complex equations are solved using the property $z\bar{z} = |z|^2$.

SOLVED EXAMPLES

7. The solution of the equation $|z| - z = 1 + 2i$ is

(A) $\frac{3}{2} - 2i$ (B) $\frac{3}{2} + 2i$

(C) $2 - \frac{3}{2}i$ (D) None of these

Solution: (A)

We have, $|z| - z = 1 + 2i$

$$\Rightarrow \sqrt{x^2 + y^2} - (x + iy) = 1 + 2i,$$

where $z = x + iy$

$$\Rightarrow \sqrt{x^2 + y^2} - x = 1 \text{ and } y = -2$$

[Comparing real and imaginary parts]

$$\Rightarrow x = \frac{3}{2} \text{ and } y = -2.$$

$\therefore$ The solution of the given equation is $\frac{3}{2} - 2i$.

8. If $\frac{z-1}{z+1}$ is purely imaginary, then

(A) $|z| > 1$ (B) $|z| < 1$

(C) $|z| = 1$ (D) None of these

Solution: (C)

Let $\frac{z-1}{z+1} = iy$, where y is real

$$\Rightarrow \frac{z+1}{z-1} = \frac{1}{iy}$$

$$\Rightarrow \frac{2z}{2} = \frac{1+iy}{1-iy} \quad \text{(by componendo and dividendo)}$$

$$\Rightarrow z = \frac{1+iy}{1-iy} \Rightarrow |z| = \frac{\sqrt{1+y^2}}{\sqrt{1+y^2}} = 1$$

9. If $|z - i| < 1$, then $|z + 12 - 6i|$

(A) <14 (B) <16

(C) >14 (D) =14

Solution: (A)

Given, $|z - i| < 1$

Now, $|z + 12 - 6i| = |(z - i) + (12 - 5i)|$

$\leq |z - i| + |12 - 5i|$

$(\because |z_1 + z_2| \leq |z_1| + |z_2|)$

$< 1 + 13 = 14$

Hence $|z + 12 - 6i| < 14$.

10. The maximum value of $|z|$ when z satisfies the condition $\left|z + \frac{2}{z}\right| = 2$ is

(A) $\sqrt{3} - 1$ (B) $\sqrt{3} + 1$

(C) $\sqrt{3}$ (D) $\sqrt{2} + \sqrt{3}$

Solution: (B)

We have, $|z| = \left|z + \frac{2}{z} - \frac{2}{z}\right| \leq \left|z + \frac{2}{z}\right| + \frac{2}{|z|}$.

$$\Rightarrow |z| \leq 2 + \frac{2}{|z|} \Rightarrow |z|^2 \leq 2|z| + 2$$

$$\Rightarrow |z|^2 - 2|z| + 1 \leq 1 + 2 \Rightarrow (|z| - 1)^2 \leq 3$$

$$\Rightarrow -\sqrt{3} \leq |z| - 1 \leq \sqrt{3} \Rightarrow 1 - \sqrt{3} \leq |z| \leq 1 + \sqrt{3}$$

That is, the maximum value of $|z|$ is $1 + \sqrt{3}$.

11. If $|z| = \text{Max.}\{|z - 1|, |z + 1|\}$, then

(A) $|z + \bar{z}| = \frac{1}{2}$ (B) $z + \bar{z} = 1$

(C) $|z + \bar{z}| = 1$ (D) None of these

Solution: (C)

We have, $|z| = |z - 1|$

$$\Rightarrow |z|^2 = |z - 1|^2 \Rightarrow z\bar{z} = (z - 1)(\bar{z} - 1)$$

$$\Rightarrow z\bar{z} = z\bar{z} - z - \bar{z} + 1 \Rightarrow z + \bar{z} = 1$$

Also, $|z| = |z+1| \Rightarrow |z|^2 = |z+1|^2$

$\Rightarrow \quad z\,\bar{z} = (z+1)(\bar{z}+1) = z\bar{z} + z + \bar{z} + 1$

$\Rightarrow \quad z + \bar{z} = -1,$

$\therefore \quad |z + \bar{z}| = 1$

12. If $\dfrac{z-2}{z+2}$ $(z \neq -2)$ is purely imaginary then $|z|$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Solution: (B)

Let $\quad z = x + iy$

Then, $\dfrac{z-2}{z+2} = \dfrac{x+iy-2}{x+iy+2} = \dfrac{(x-2)+iy}{(x+2)+iy}$

$$= \frac{[(x-2)+iy][(x+2)-iy]}{(x+2)^2+y^2}$$

$$= \frac{(x^2+y^2-4)+i(4y)}{(x+2)^2+y^2}$$

Since $\dfrac{z-2}{z+2}$ is purely imaginary,

$\therefore \quad x^2 + y^2 - 4 = 0$

$\Rightarrow \quad x^2 + y^2 = 4 \Rightarrow |z|^2 = 4 \Rightarrow |z| = 2.$

SQUARE ROOTS OF A COMPLEX NUMBER

Let $z = a + ib$ and let the square root of z be the complex number $x + iy$. Then

$$\sqrt{a+ib} = x + iy$$

or $\quad (a + ib) = (x + iy)^2 = (x^2 - y^2) + (2xy)\,i$

Equating real and imaginary part, we get

$$a = x^2 - y^2 \tag{1}$$

and
$$b = 2xy \tag{2}$$

Now, $\quad x^2 + y^2 = \sqrt{(x^2-y^2)^2 + 4x^2y^2}$

$$= \sqrt{a^2+b^2} \tag{3}$$

Solving the equations (1) and (3), we get

$$x = \pm\sqrt{\left(\frac{\sqrt{a^2+b^2}+a}{2}\right)}$$

and
$$y = \pm\sqrt{\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)}$$

From Eq. (2), we can determine the sign of xy. If $xy > 0$, then x and y will have same sign. Thus,

$$\sqrt{a+ib} = \pm\left[\sqrt{\left(\frac{\sqrt{a^2+b^2}+a}{2}\right)} + i\sqrt{\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)}\right]$$

If $xy < 0$, then

$$\sqrt{a+ib} = \pm\left[\sqrt{\left(\frac{\sqrt{a^2+b^2}+a}{2}\right)} - i\sqrt{\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)}\right]$$

TRICK(S) FOR PROBLEM SOLVING

Square roots of $z = a + ib$ are:

$\pm\left[\sqrt{\dfrac{|z|+a}{2}} + i\sqrt{\dfrac{|z|-a}{2}}\right]$ for $b > 0$ and

$\pm\left[\sqrt{\dfrac{|z|+a}{2}} - i\sqrt{\dfrac{|z|-a}{2}}\right]$ for $b < 0$

$\sqrt{a+tb} + \sqrt{a-tb} = \pm\sqrt{2}\left\{\sqrt{\sqrt{a^2+b^2}+a}\right\}$

where $b > 0$

$\Rightarrow \quad \sqrt{z} + \sqrt{\bar{z}} = \pm\sqrt{2}\left\{\sqrt{|z|+a}\right\}$

where $\operatorname{Im}(z) > 0$

Also, $\sqrt{a+tb} - \sqrt{a-tb} = \pm\sqrt{2}\left\{\sqrt{\sqrt{a^2+b^2}-a}\right\}t$

where $b > 0$

$\Rightarrow \quad \sqrt{z} - \sqrt{\bar{z}} = \pm\sqrt{2}\left\{\sqrt{|z|-a}\right\}t$

where $b > 0$

SOLVED EXAMPLES

13. If $\sqrt[3]{a-ib} = x - iy$, then $\sqrt[3]{a+ib} =$

(A) $x + iy$ (B) $x - iy$
(C) $y + ix$ (D) $y - ix$

Solution: (A)

We have, $\quad \sqrt[3]{a-ib} = x - iy$

$\Rightarrow \quad a - ib = (x - iy)^3 = x^3 - 3x^2 \cdot iy + 3x\,(iy)^2 - (iy)^3$

$= (x^3 - 3xy^2) - i\,(3x^2y - y^3)$

$\therefore \quad a + ib = (x^3 - 3xy^2) + i\,(3x^2y - y^3)$

$= x^3 + 3x^2 \cdot (iy) + 3x\,(iy)^2 + (iy)^3$

$= (x + iy)^3$

$\therefore \quad \sqrt[3]{a+ib} = x + iy.$

14. The complex number z satisfying the equations $|z - i| = |z + 1| = 1$ is

(A) 0 (B) $1 + i$
(C) $-1 + i$ (D) $1 - i$

Solution: (A, C)

Let $z = x + iy$. Then,

$$|(x + iy) - i| = |(x + iy) + 1| = 1$$

or $\sqrt{x^2 + (y-1)^2} = \sqrt{(x+1) + y^2} = 1$

$\therefore$ $x^2 + y^2 - 2y + 1 = x^2 + y^2 + 2x + 1$

i.e., $x = -y$ (1)

and $x^2 + y^2 - 2y + 1 = 1$ (2)

From Eq. (1) and (2), $x^2 + x^2 + 2x = 0$; or $x\,(x + 1) = 0$

$\therefore$ $x = 0, -1;$

$\therefore$ $y = 0, 1$

$\therefore$ $z = x + iy = 0, -1 + i.$

15. The complex number z satisfying the equations $|z| - 4 = |z - i| - |z + 5i| = 0$, is

(A) $\sqrt{3} - i$ (B) $2\sqrt{3} - 2i$
(C) $-2\sqrt{3} - 2i$ (D) 0

Solution: (B, C)

We have two equations

$$|z| - 4 = 0 \text{ and } |z - i| - |z + 5i| = 0$$

Putting $z = x + iy$, these equations become

$|x + iy| = 4$ i.e., $x^2 + y^2 = 16$ (1)

and $|x + iy - i| = |x + iy + 5i|$

or $x^2 + (y - 1)^2 = x^2 + (y + 5)^2$

i.e. $y = -2$ (2)

Putting $y = -2$ in (1), $x^2 + 4 = 16$ or $x = \pm 2$.
Hence, the complex numbers z satisfying the given equations are

$$z_1 = 2 - 2i, \text{ and } z_2 = -2 - 2i.$$

16. If $i z^3 + z^2 - z + i = 0$, then

(A) $|z| < 1$ (B) $|z| > 1$
(C) $|z| = 1$ (D) $|z| = 0$

Solution: (C)

Given, $iz^3 + z^2 - z + i = 0$

$\Rightarrow$ $i z^2 (z - i) - (z - i) = 0$

$\Rightarrow$ $(z - i)(i z^2 - 1) = 0 \Rightarrow z = i$

or $z^2 = \frac{1}{i} = -i$

Now, $z = i \Rightarrow |z| = |i| = 1$

and $z^2 = -i \Rightarrow |z^2| = |-i|$

$\Rightarrow$ $|z|^2 = 1 \Rightarrow |z| = 1$

Thus, in both cases $|z| = 1$.

17. The greatest value of $|z + 1|$ if $|z + 4| \le 3$ is

(A) 4 (B) 5
(C) 6 (D) None of these

Solution: (C)

We have,

$$|z + 1| = |z + 4 - 3| = |(z + 4) + (-3)|$$
$$\le |z + 4| + |-3| = |z + 4| + 3$$
$$\le 3 + 3 = 6 \qquad (\because |z + 4| \le 3)$$

Hence, the greatest value of $|z + 1|$ is 6.

ARGAND PLANE AND GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

Let O be the origin and OX and OY be the x-axis and y-axis respectively. Then, any complex number $z = x + iy = (x, y)$ may be represented by a unique point P whose coordinates are (x, y).

The representation of complex numbers as points in a plane forms an **Argand diagram**.

The plane on which complex numbers are represented is known as the **complex plane** or **Argand's plane** or **Gaussian plane**. The x-axis is called the **real axis** and y-axis the **imaginary axis**.

The complex number $z = x + iy$ is known as the **affix** of the point (x, y) which it represents.

POLAR FORM OF A COMPLEX NUMBER

Let O be the origin and OX and OY be the x-axis and y-axis respectively. Let $z = x + iy$ be a complex number represented by the point $P(x, y)$.

Draw $PM \perp OX$. Then,

$$OM = x \text{ and } PM = y. \text{ Join } OP$$

Let $OP = r$ and $\angle XOP = \theta$.

Then

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

This form of z is called **polar or trigonometric form.**

Comparing real and imaginary parts, we get

$$x = r\cos\theta \qquad (1)$$

and $$y = r\sin\theta \qquad (2)$$

Squaring Eq. (1) and (2) and adding, we get

$$r^2 = x^2 + y^2 \quad \text{or} \quad r = \sqrt{x^2 + y^2} = |z|$$

Thus, r is known and is equal to the modulus of the complex number z.

Substituting the value of r in Eq. (1) and (2), we get

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}} \qquad (3)$$

Dividing Eq. (2) by (1), we get $\tan\theta = \dfrac{y}{x}$.

The form $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ of the complex number z is called exponential form.

Any value of θ satisfying (3) is known as **amplitude or argument** of z and written as $\theta = \arg(z)$ or $\theta = \text{amp } z$.

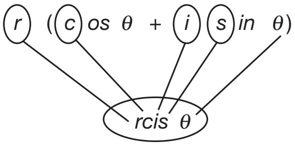
IMPORTANT POINTS

The unique value of θ such that $-\pi < \theta \le \pi$ for which $x = r\cos\theta$ and $y = r\sin\theta$, is known as the **principal value of the argument.**

The general value of the argument is $(2n\pi + \theta)$, where n is an integer and θ is the principal value of $\arg(z)$.

While reducing a complex number to polar form, we always take the principal value.

The complex number $z = r(\cos\theta + i\sin\theta)$ can also be written as $r\text{cis}\theta$.



TRICK(S) FOR PROBLEM SOLVING

- If $x > 0, y > 0$ (i.e., z is in first quadrant), then $\arg z = \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$.
- If $x < 0, y > 0$ (i.e., z is in second quadrant), then $\arg z = \theta = \pi - \tan^{-1}\left(\dfrac{y}{|x|}\right)$.
- If $x < 0, y < 0$ (i.e., z is in third quadrant), then $\arg z = \theta = -\pi + \tan^{-1}\left(\dfrac{y}{x}\right)$.
- If $x > 0, y < 0$ (i.e., z is in fourth quadrant), then $\arg z = \theta = -\tan^{-1}\left(\dfrac{|y|}{x}\right)$.
- Argument of the complex number 0 is not defined.
- $\arg(x + i0) = \begin{cases} 0, & \text{if } x > 0 \\ \pi, & \text{if } x < 0 \end{cases}$
- $\arg(0 + iy) = \begin{cases} \pi/2, & \text{if } y > 0 \\ 3\pi/2, & \text{if } y < 0 \end{cases}$.

Properties of Argument

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
2. $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
3. $\arg\left(\dfrac{z}{\bar{z}}\right) = 2\arg z$
4. $\arg(z^n) = n\arg z$
5. If $\arg\left(\dfrac{z_2}{z_1}\right) = \theta$, then $\arg\left(\dfrac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k \in I$
6. $\arg\bar{z} = -\arg z$
7. $\arg(z\bar{z}) = \arg\left(|z|^2\right) = \arg(\text{positive real number}) = 0$

SOLVED EXAMPLES

18. The inequality $|z - 4| < |z - 2|$ represents the region given by,

(A) $\text{Re}(z) > 0$
(B) $\text{Re}(z) < 0$
(C) $\text{Re}(z) > 3$
(D) None of these

Solution: (C)

Given $\quad |z - 4|^2 < |z - 2|^2$

$$\Rightarrow \quad |(x-4) + iy|^2 < |(x-2) + iy|^2$$

$$\Rightarrow \quad (x-4)^2 + y^2 < (x-2)^2 + y^2$$

$$\Rightarrow \quad -4x < -12 \Rightarrow 4x > 12;\ x > 3$$

$$\Rightarrow \quad \text{Re}(z) > 3.$$

19. If $z_r = \cos\left(\dfrac{\pi}{3^r}\right) + i\sin\left(\dfrac{\pi}{3^r}\right)$, $r = 1, 2, 3, \ldots$, then $z_1 z_2 z_3 \ldots \infty =$

(A) i (B) $-i$
(C) 1 (D) -1

Solution: (A)

Since $z_r = \cos\left(\dfrac{\pi}{3^r}\right) + i\sin\left(\dfrac{\pi}{3^r}\right),$

$r = 1, 2, 3, \ldots$

we have, $z_1 \cdot z_2 \cdot z_3 \ldots \infty$

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}\right)\ldots\infty$$

$$= \cos\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots\right)$$

$$= \cos\left(\frac{\frac{\pi}{3}}{1-\frac{1}{3}}\right) + i\sin\left(\frac{\frac{\pi}{3}}{1-\frac{1}{3}}\right)$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i \cdot 1 = i$$

20. The value of $\displaystyle\sum_{k=1}^{10}\left(\sin\frac{2\pi k}{11} - i\cos\frac{2\pi k}{11}\right)$ is

(A) 1 (B) -1 (C) i (D) $-i$

Solution: (C)

We have,

$$\sum_{k=1}^{10}\left(\sin\frac{2\pi k}{11} - i\cos\frac{2\pi k}{11}\right)$$

$$= \sum_{k=1}^{10}\left(-i^2\sin\frac{2\pi k}{11} - i\cos\frac{2\pi k}{11}\right)$$

$$= -i\sum_{k=1}^{10}\left(\cos\frac{2\pi k}{11} + i\sin\frac{2\pi k}{11}\right) = -i\sum_{k=1}^{10} e^{i\frac{2\pi k}{11}}$$

$$= -i\left[\sum_{k=0}^{10} e^{i\frac{2\pi k}{11}} - 1\right]$$

$= -i$ (sum of 11th roots of unity $- 1$)

$= -i(0 - 1) = i.$

21. The inequality $|z - 4| < |z - 2|$ represents the region given by

(A) $\text{Re}(z) > 0$ (B) $\text{Re}(z) < 0$
(C) $\text{Re}(z) > 2$ (D) None of these

Solution: (D)

We have,

$$|z - 4| < |z - 2| \Rightarrow |z - 4|^2 < |z - 2|^2$$

$$\Rightarrow \quad |x + iy - 4|^2 < |x + iy - 2|^2$$

(Putting $z = x + iy$)

$$\Rightarrow \quad (x - 4)^2 + y^2 < (x - 2)^2 + y^2$$

$$\Rightarrow \quad x^2 - 8x + 16 + y^2 < x^2 - 4x + 4 + y^2$$

$$\Rightarrow \quad -4x < -12 \Rightarrow x > 3 \Rightarrow \text{Re}(z) > 3$$

22. The argument of $\dfrac{1 - i\sqrt{3}}{1 + i\sqrt{3}}$ is

(A) $\dfrac{\pi}{3}$ (B) $\dfrac{2\pi}{3}$ (C) $\dfrac{4\pi}{3}$ (D) $-\dfrac{2\pi}{3}$

Solution: (D)

$$\frac{1 - i\sqrt{3}}{1 + i\sqrt{3}} = \frac{(1 - i\sqrt{3})^2}{4} = \frac{-2 - 2\sqrt{3}\,i}{4}$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i.$$

$$\therefore \quad \arg\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right) = -(\pi - \tan^{-1}\sqrt{3}) = -\frac{2\pi}{3}$$

23. $\arg bi\ (b > 0)$ is

(A) π (B) $\dfrac{\pi}{2}$ (C) $-\dfrac{\pi}{2}$ (D) 0

Solution: (B)

Since $b > 0$, bi represents a point on the positive side of the imaginary axis on which the argument of every point is $\dfrac{\pi}{2}$.

24. Let z_k $(k = 0, 1, 2, \ldots, 6)$ be the roots of the equation $(z + 1)^7 + z^7 = 0$, then $\displaystyle\sum_{k=0}^{6}\text{Re}(z_k)$ is equal to

(A) 0 (B) $\dfrac{3}{2}$ (C) $-\dfrac{7}{2}$ (D) $\dfrac{7}{2}$

Solution: (C)

Let $z_k = x_k + iy_k,$

we have $(z_k + 1)^7 + z_k^7 = 0$

$$\Rightarrow \quad (z_k + 1)^7 = -z_k^7 \Rightarrow |z_k + 1|^7 = |z_k|^7$$

$$\Rightarrow \quad |z_k + 1| = |z_k| \Rightarrow |x_k + iy_k + 1|^2 = |x_k + iy_k|^2$$

$$\Rightarrow \quad (x_k + 1)^2 + y_k^2 = x_k^2 + y_k^2$$

$$\Rightarrow \quad 2x_k + 1 = 0 \quad \text{or} \quad x_k = -\frac{1}{2}$$

Thus, $\displaystyle\sum_{k=0}^{6}\text{Re}(z_k) = \sum_{k=0}^{6} x_k = -\frac{7}{2}.$

25. If arg $(z) < 0$, then arg $(-z)$ – arg (z) =

(A) π (B) $-\pi$

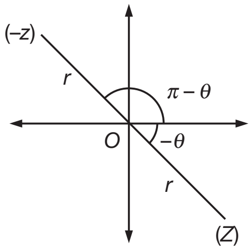
(C) $-\dfrac{\pi}{2}$ (D) $\dfrac{\pi}{2}$

Solution: (A)

As $-\theta = \arg(z) < 0$,

we take $z = r\,[\cos(-\theta) + i\sin(-\theta)]$

$= r\,(\cos\theta - i\sin\theta)$



$$\Rightarrow \quad -z = r\,(-\cos\theta + i\sin\theta)$$

$$= r\,[\cos(\pi - \theta) + i\sin(\pi - \theta)]$$

$\therefore$ $\arg(-z) = \pi - \theta$

Thus,

$$\arg(-z) - \arg(z) = \pi - \theta + (\theta) = \pi$$

PARTICULAR CASES OF POLAR FORM

1. $1 = 1 + i0 = \cos 0 + i\sin 0$
2. $-1 = -1 + i0 = \cos\pi + i\sin\pi$
3. $i = 0 + i1 = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$
4. $-i = 0 + i(-1) = \cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)$
5. $1 - i = \sqrt{2}\left[\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right]$
6. $-1 - i = \sqrt{2}\left[\cos\left(-\dfrac{3\pi}{4}\right) + i\sin\left(-\dfrac{3\pi}{4}\right)\right]$

EULERIAN REPRESENTATION OF A COMPLEX NUMBER

Since $e^{i\theta} = \cos\theta + i\sin\theta$, thus any non zero complex number $z = x + iy = r(\cos\theta + i\sin\theta)$ can be represented in Eulerian form as

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

where $|z| = r$ and $\theta = \arg(z)$.

NOTE

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

SOLVED EXAMPLE

26. For any integer n, the argument of $z = \dfrac{(\sqrt{3} + i)^{4n+1}}{(1 - i\sqrt{3})^{4n}}$ is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{3}$

(C) $\dfrac{\pi}{2}$ (D) $\dfrac{2\pi}{3}$

(E) All of the above

Solution: (A)

We have,

$$z = \frac{(\sqrt{3} + i)^{4n+1}}{(1 - i\sqrt{3})^{4n}}$$

$$= \frac{\left(2e^{i\frac{\pi}{6}}\right)^{4n+1}}{\left(2e^{-i\frac{\pi}{3}}\right)^{4n}} = \frac{2^{4n+1}e^{i(4n+1)\frac{\pi}{6}}}{2^{4n}e^{-i4n\frac{\pi}{3}}}$$

$$= 2 \cdot e^{i(12n+1)\frac{\pi}{6}} = 2 \cdot e^{2n\pi i} \cdot e^{\frac{\pi i}{6}}$$

$$= 2 \cdot e\pi^{i/6} \qquad (\because e^{2n\pi i} = 1)$$

$$\therefore \quad \arg z = \frac{\pi}{6}.$$

LOGARITHM OF A COMPLEX NUMBER

$$\log(x + iy) = \log_e(re^{i\theta}) = \log_e e^{i\theta} = \log_e r + i\theta$$

$$= \log_e \sqrt{(x^2 + y^2)} + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$\log_e(z) = \log_e |z| + i\arg(z)$$

CAUTION

Since the argument of a complex number is not unique, the log of a complex number cannot be unique. In general,

$$\log_e(z) = \log_e |z| + i\,[2k\pi + \arg(z)],\ k \in I$$

NOTE

$$\log i = \log e^{\frac{i\pi}{2}} = \frac{i\pi}{2},\ \log(\log i) = \log\left(\frac{i\pi}{2}\right)$$

$$= \log i + \log\left(\frac{\pi}{2}\right) = \frac{i\pi}{2} + \log(\pi/2).$$

VECTORIAL REPRESENTATION OF A COMPLEX NUMBER

If P is the point (a, b) on the argand plane corresponding to the complex number $z = a + ib$.

Then

$$\overrightarrow{OP} = a\hat{i} + b\hat{j},$$

$$\therefore \qquad \left|\overrightarrow{OP}\right| = \sqrt{a^2 + b^2} = |z|$$

and

$$arg(z) = \text{direction of the vector } \overrightarrow{OP} = \tan^{-1}\left(\frac{b}{a}\right).$$

De'Moivre's Theorem

If n is any integer, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

TRICK(S) FOR PROBLEM SOLVING

- If n is any rational number, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.
- $(\cos\theta + i\sin\theta)^{-n} = \cos(-n)\theta + i\sin(-n)\theta$
  $= \cos n\theta - i\sin n\,\theta$
- $(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$
  $= \cos(-n\theta) + i\sin(-n\theta)$
  $= \cos n\theta - i\sin n\theta$
- $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$
- The theorem cannot be applied to $(\cos\theta + i\sin\phi)^n$ i.e., θ must be same with cos and sin both.
- The theorem is not directly applicable to $(\sin\theta + i\cos\theta)^n$, rather
  $$(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n$$
  $$= \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)$$
- $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ \ldots\ (\cos\theta_n + i\sin\theta_n)$
  $= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$

SOLVED EXAMPLES

27. If $z = \cos\theta + i\sin\theta$, then

(A) $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

(B) $z^n + \dfrac{1}{z^n} = 2^n\cos n\theta$

(C) $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$

(D) $z^n - \dfrac{1}{z^n} = (2i)^n\sin n\theta$

Solution: (A, C)

We have,

$$\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta.$$

$$\therefore \qquad z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

and $$\frac{1}{z^n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$$

Hence, $$z^n + \frac{1}{z^n} = 2\cos n\theta$$

and $$z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

28. If $z = \cos\theta + i\sin\theta$, then $\dfrac{z^{2n} - 1}{z^{2n} + 1} =$

(A) $i\cot n\theta$ (B) $i\tan n\theta$
(C) $\tan n\theta$ (D) $\cot n\theta$
(n is an integer)

Solution: (B)

We have,

$$\frac{z^{2n}-1}{z^{2n}+1} = \frac{(\cos\theta + i\sin\theta)^{2n} - 1}{(\cos\theta + i\sin\theta)^{2n} + 1}$$

$$= \frac{\cos 2n\theta + i\sin 2n\theta - 1}{\cos 2n\theta + i\sin 2n\theta + 1}$$

(Using De Moivre's Theorem)

$$= \frac{(1 - 2\sin^2 n\theta) + 2i\sin n\theta\cos n\theta - 1}{(2\cos^2 n\theta - 1) + 2i\sin n\theta\cos n\theta + 1}$$

$$= \frac{i\sin n\theta\cos n\theta + i^2\sin^2 n\theta}{\cos^2 n\theta + i\sin n\theta\cos n\theta} \qquad (\because i^2 = -1)$$

$$= \frac{i\sin n\theta(\cos n\theta + i\sin n\theta)}{\cos n\theta(\cos n\theta + i\sin n\theta)} = i\tan n\theta.$$

29. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$ and $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} = -1$, then

$\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) =$

(A) 0 (B) 1
(C) –1 (D) None of these

Solution: (C)

We have,

$$\frac{1}{a} = \cos\alpha - i\sin\alpha,\ \frac{1}{b} = \cos\beta - i\sin\beta$$

Now $\quad \dfrac{a}{b} = (\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)$

or $\quad \dfrac{a}{b} = \cos(\alpha - \beta) + i\sin(\alpha - \beta)$

Similarly, $\quad \dfrac{b}{c} = \cos(\beta - \gamma) + i\sin(\beta - \gamma)$

and $\quad \dfrac{c}{a} = \cos(\gamma - \alpha) + i\sin(\gamma - \alpha)$

Putting these values in $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} = -1$,
we get

$$\begin{aligned}&[\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha)]\\ &+ i\,[\sin(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha)]\\ &= -1 = -1 + 0\,i\end{aligned}$$

Comparing real part on both sides, we get

$$\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -1$$

30. If n is a positive integer, then $(\sqrt{3} + i)^n + (\sqrt{3} - i)^n$ is equal to

(A) $2^n \cos\dfrac{n\pi}{6}$ (B) $2^{n+1} \cos\dfrac{n\pi}{6}$

(C) $2^{n-1} \cos\dfrac{n\pi}{6}$ (D) None of these

Solution: (B)

Let $\sqrt{3} = r\cos\theta$ and $1 = r\sin\theta$

so that

$$r^2 = 4 \text{ and } \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow r = 2,\ \theta = \frac{\pi}{6}$$

$$\therefore \quad (\sqrt{3} + i)^n = 2^n\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n$$

or $\quad (\sqrt{3} + i)^n = 2^n\left\{\cos\left(\dfrac{n\pi}{6}\right) + i\sin\left(\dfrac{n\pi}{6}\right)\right\} \qquad (1)$

Similarly,

$$(\sqrt{3} - i)^n = 2^n\left\{\cos\left(\frac{n\pi}{6}\right) - i\sin\left(\frac{n\pi}{6}\right)\right\} \qquad (2)$$

Adding Eq. (1) and (2), we obtain

$$\begin{aligned}(\sqrt{3} + i)^n + (\sqrt{3} - i)^n &= 2\cdot 2^n\cos\left(\frac{n\pi}{6}\right)\\ &= 2^{n+1}\cos\left(\frac{n\pi}{6}\right)\end{aligned}$$

31. If $(\sin\theta_1 + i\cos\theta_1)(\sin\theta_2 + i\cos\theta_2)\ldots(\sin\theta_n + i\cos\theta_n) = a + ib$, then $a^2 + b^2 =$

(A) 4 (B) 2
(C) 1 (D) None of these

Solution: (C)

Given expression

$$\begin{aligned}&= \prod_{r=1}^{n}\left(\cos\left(\frac{\pi}{2} - \theta_r\right) + i\sin\left(\frac{\pi}{2} - \theta_r\right)\right)\\ &= \cos\sum_{r=1}^{n}\left(\frac{\pi}{2} - \theta_r\right) + i\sin\sum_{r=1}^{n}\left(\frac{\pi}{2} - \theta_r\right)\\ &= \cos\alpha + i\sin\alpha,\end{aligned}$$

where $\quad \alpha = \displaystyle\sum_{r=1}^{n}\left(\frac{\pi}{2} - \theta_r\right)$

$$= a + ib$$

$$\therefore \quad a^2 + b^2 = \cos^2\alpha + \sin^2\alpha = 1.$$

32. If $z^2 - 2z\cos\theta + 1 = 0$, then $z^2 + z^{-2}$ is equal to

(A) $2\cos 2\theta$ (B) $2\sin 2\theta$ (C) $2\cos\theta$ (D) $2\sin\theta$

Solution: (A)

We have,

$$z^2 - 2z\cos\theta + 1 = 0$$

$$\Rightarrow \quad z = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$\begin{aligned}&= \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm \sqrt{i^2\sin^2\theta}\\ &= \cos\theta \pm i\sin\theta.\end{aligned}$$

When $\quad z = \cos\theta + i\sin\theta$

$$\begin{aligned}z^2 + z^{-2} &= \cos 2\theta + i\sin 2\theta + (\cos 2\theta - i\sin 2\theta)\\ &= 2\cos 2\theta\end{aligned}$$

and when $\quad z = \cos\theta - i\sin\theta$,

$$\begin{aligned}z^2 + z^{-2} &= \cos 2\theta - i\sin 2\theta + \cos 2\theta + i\sin 2\theta\\ &= 2\cos 2\theta\end{aligned}$$

ROOTS OF A COMPLEX NUMBER

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right],$$

where $k = 0, 1, 2, 3, \ldots (n - 1)$.

Cube Roots of Unity

Let $\quad z = 1^{1/3}$ or $z^3 - 1 = 0$

$\Rightarrow \quad (z - 1)(z^2 + z + 1) = 0$

i.e., $z = 1, \dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$

Put $\omega = \dfrac{-1+i\sqrt{3}}{2}$,

then $\omega^2 = \dfrac{-1-i\sqrt{3}}{2}$.

Thus cube roots of unity are 1, ω, ω^2.

Properties of Cube Roots of Unity

1. $1 + \omega + \omega^2 = 0$
2. $\omega^3 = 1$
3. $\omega^{3n} = 1$, $\omega^{3n+1} = \omega$, $\omega^{3n+2} = \omega^2$
4. $\bar{\omega} = \omega^2$ and $(\bar{\omega})^2 = \omega$, $\omega\,\bar{\omega} = \omega^3$, $\omega = e^{\frac{2\pi i}{3}}$, $\omega^2 = e^{-\frac{2\pi i}{3}}$
5. If $a + b\omega + c\omega^2 = 0$, then $a = b = c$ provided a, b, c are real.
6. If these roots are marked on the argand plane, then these are vertices of an equilateral triangle with circumcentre at origin, as shown in the Fig. 3.1.

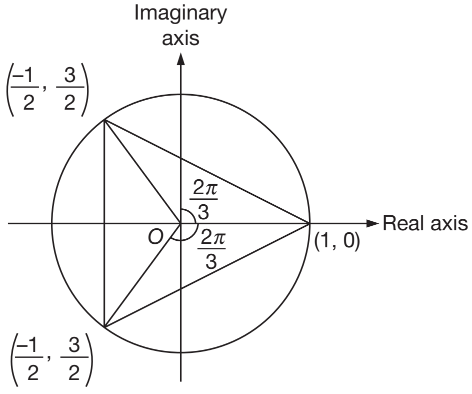


Fig. 3.1

Fourth Roots of Unity

The four, fourth roots of unity are given by the solution set of the equation $x^4 - 1 = 0$

$\Rightarrow \quad (x^2 - 1)(x^2 + 1) = 0 \Rightarrow x = \pm 1, \pm i$

Fourth roots of unity are vertices of a square which lies on coordinate axes.

Some Useful Relations

1. $x^2 + y^2 = (x + iy)(x - iy)$
2. $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$
3. $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$
4. $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$, in particular, $x^2 + x + 1 = (x - \omega)(x - \omega^2)$
5. $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$, in particular, $x^2 - x + 1 = (x + \omega)(x + \omega^2)$
6. $x^2 + y^2 + z^2 - xy - xz - yz = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$
7. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

SOLVED EXAMPLES

33. If 1, ω, ω^2 be the three cube roots of unity, then $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots$ to $2n$ factors =

(A) 1	(B) –1
(C) 0	(D) None of these

Solution: (A)

We have,

$(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots$ to $2n$ factors

$= (1 + \omega)(1 + \omega^2)(1 + \omega^3 \cdot \omega)(1 + \omega^6 \cdot \omega^2) \ldots$ to $2n$ factors

$= (1 + \omega)(1 + \omega^2)(1 + \omega)(1 + \omega^2) \ldots$ to $2n$ factors

$(\because \omega^3 = \omega^6 = \ldots = 1)$

$= [(1 + \omega)(1 + \omega) \ldots$ to n factors$]$

$[(1 + \omega^2)(1 + \omega^2) \ldots$ to n factors$]$

$= (1 + \omega)^n (1 + \omega^2)^n = [(1 + \omega)(1 + \omega^2)]^n$

$= (1 + \omega + \omega^2 + \omega^3)^n = (0 + 1)^n = 1$

$(\because 1 + \omega + \omega^2 = 0, \omega^3 = 1)$.

34. If 1, ω, ω^2 are the three cube roots of unity, then $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \ldots$ to $2n$ factors =

(A) 2^n	(B) 2^{2n}
(C) 2^{4n}	(D) None of these

Solution: (B)

We have,

$(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$

$(1 - \omega^8 + \omega^{16}) \ldots$ to $2n$ factors

$= (1 - \omega + \omega^2)(1 - \omega^2 + \omega)(1 - \omega + \omega^2)$

$(1 - \omega^2 + \omega) \ldots$ to $2n$ factors.

$[\because \omega^4 = \omega, \omega^8 = \omega^2, \omega^{16} = \omega$ and so on$]$

$= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2) \ldots$ to $2n$ factors

$= (2^2\,\omega^3)(2^2\,\omega^3) \ldots$ to n factors

$[\because (-2\omega)(-2\omega^2) = 2^2\,\omega^3 = 2^2]$

$= (2^2)^n = 2^{2n}$.

35. $\sqrt{-1-\sqrt{-1-\sqrt{-1-\ldots \text{ to } \infty}}} =$

(A) 1 (B) -1 (C) ω (D) ω^2

Solution: (C, D)

Let $x = \sqrt{-1-\sqrt{-1-\sqrt{-1-\ldots \text{ to } \infty}}}$

Then $x = \sqrt{-1-x}$ or $x^2 = -1-x$

or $x^2 + x + 1 = 0$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1-4\cdot 1\cdot 1}}{2\cdot 1} = \frac{-1\pm\sqrt{-3}}{2}$$

$$= \frac{-1\pm\sqrt{3}i}{2} = \omega \text{ or } \omega^2.$$

36. $\left(\frac{\sqrt{3}+i}{2}\right)^6 + \left(\frac{i-\sqrt{3}}{2}\right)^6 =$

(A) -2 (B) 2 (C) -1 (D) 1

Solution: (A)

We have,

$$\frac{\sqrt{3}+i}{2} = \frac{i\sqrt{3}+i^2}{2i} = -i\left(\frac{-1+\sqrt{3}i}{2}\right) = -i\omega$$

and
$$\frac{i-\sqrt{3}}{2} = \frac{i^2-i\sqrt{3}}{2i} = -i\left(\frac{-1-\sqrt{3}i}{2}\right) = -i\omega^2$$

Hence,
$$\left(\frac{\sqrt{3}+i}{2}\right)^6 + \left(\frac{i-\sqrt{3}}{2}\right)^6 = (-i\omega)^6 + (-i\omega^2)^6$$

$$= i^6(\omega^6 + \omega^{12})$$

$$= -1\,(1+1) = -2.$$

37. The common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ are

(A) $-1, \omega$
(B) $-1, \omega^2$
(C) ω, ω^2
(D) None of these

Solution: (C)

We have, $z^3 + 2z^2 + 2z + 1 = 0$

$\Rightarrow \quad (z+1)(z^2+z+1) = 0$

Its roots are -1, ω and ω^2. The root $z = -1$ does not satisfy the equation $z^{1985} + z^{100} + 1 = 0$ but $z = \omega$ and $z = \omega^2$ satisfy it. Hence, ω and ω^2 are the common roots.

38. If the cube roots of unity are $1, \omega, \omega^2$, then the roots of the equation $(x-1)^3 + 8 = 0$ are

(A) $-1, 1+2\omega, 1+2\omega^2$
(B) $-1, 1-2\omega, 1-2\omega^2$
(C) $-1, -1, -1$
(D) None of these

Solution: (B)

We have, $(x-1)^3 + 8 = 0$

$\Rightarrow \quad (x-1)^3 = -8$

$\therefore \quad x - 1 = (-8)^{1/3} = -2, -2\omega, -2\omega^2$

Hence, $x = -1, 1-2\omega, 1-2\omega^2$

39. $(i+\sqrt{3})^{100} + (i-\sqrt{3})^{100} + 2^{100} =$

(A) 1
(B) -1
(C) 0
(D) None of these

Solution: (C)

We have,
$$i+\sqrt{3} = \frac{-1+\sqrt{3}i}{2}\cdot\frac{2}{i} = \frac{2\omega}{i}$$

and
$$i-\sqrt{3} = \frac{-1-\sqrt{3}i}{2}\cdot\frac{2}{i} = \frac{2\omega^2}{i}$$

$$\therefore \quad (i+\sqrt{3})^{100} + (i-\sqrt{3})^{100} + 2^{100}$$

$$= \left(\frac{2\omega}{i}\right)^{100} + \left(\frac{2\omega^2}{i}\right)^{100} + 2^{100}$$

$$= \frac{2^{100}}{i^{100}}(\omega^{100} + \omega^{200}) + 2^{100}$$

$$= 2^{100}(\omega + \omega^2) + 2^{100}$$

$$= -2^{100} + 2^{100} = 0.$$

*n*th Roots of Unity

Since $1 = \cos 0 + i \sin 0$, therefore,

$$(1)^{1/n} = (\cos 0 + i\sin 0)^{1/n}$$

$$= \cos\frac{2\pi r + 0}{n} + i\sin\frac{2\pi r+0}{n};\ r = 0, 1, 2, \ldots, (n-1)$$

$$= \cos\frac{2\pi r}{n} + i\sin\frac{2\pi r}{n};\ r = 0, 1, 2, \ldots, (n-1)$$

$$= e^{i\frac{2r\pi}{n}};\ r = 0, 1, 2, \ldots, (n-1)$$

$$= 1, e^{(i2\pi/n)}, e^{(i4\pi/n)}, \ldots, e^{[i2(n-1)\pi/n]}$$

$$= 1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1},$$

where $\alpha = e^{(i2\pi/n)}$

Properties of nth Roots of Unity

1. $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$
2. $1\cdot\alpha\cdot\alpha^2\cdot\ldots\,\alpha^{n-1} = (-1)^{n-1}$
3. The n, nth roots of unity lie on the unit circle $|z| = 1$ and form the vertices of a regular polygon of n sides.
4. nth roots of unity form a G.P. with common ratio $e^{(i2\pi/n)}$.

SOLVED EXAMPLES

40. If r is non-real and $r = \sqrt[5]{1}$, then the value of $\left(1+r+r^2+r^{-2}-r^{-1}\right)$ is equal to

(A) 2 (B) 4
(C) 8 (D) None of these

Solution: (B)

$$|1+r+r^2+r^{-2}-r^{-1}| = |1+r+r^2+r^3-r^4|$$

$$[\because r^5 = 1 \Rightarrow r^3 \cdot r^2 = 1 \text{ or } r^{-2} = r^3$$

and $\quad r^4 \cdot r = 1 \quad \text{or} \quad r^{-1} = r^4]$

$$= |1+r+r^2+r^3+r^4-2r^4|$$

$$\left|\frac{1-r^5}{1-r} - 2r^4\right| = \left|0-2r^4\right| \qquad (\because r^5 = 1)$$

$$= 2|r|^4 = 2(1) = 2 \qquad (\because |r| = 1 \text{ as } r^5 = 1)$$

$$\therefore \qquad 2^{|1+r+r^2+r^{-2}-r^{-1}|} = 2^2 = 4$$

41. The values of $(16)^{1/4}$ are

(A) $\pm 2, \pm 2\,i$ (B) $\pm 4, \pm 4\,i$
(C) $\pm 1, \pm i$ (D) None of these

Solution: (A)

We have

$$(16)^{1/4} = (2^4)^{1/4} = 2\,(1)^{1/4}$$

$$= 2\,(\cos 0 + i \sin 0)^{1/4}$$

$$= 2\left\{\cos\frac{1}{4}(2k\pi+0) + i\,\sin\frac{1}{4}(2k\pi+0)\right\},$$

$$k = 0, 1, 2, 3$$

$$= 2\times 1, 2\times i, 2\times -1, 2\times -i = \pm 2, \pm 2i$$

GEOMETRY OF COMPLEX NUMBERS

1. **Distance Formula:** The distance between two points $P(z_1)$ and $Q(z_2)$ is given by $PQ = |z_2 - z_1| = |$*affix of Q – affix of P*$|$ (see Fig. 3.2)

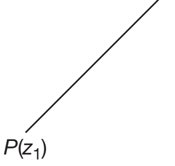


Fig. 3.2

2. **Section Formula:** If $R(z)$ divides the line segment joining $P(z_1)$ and $Q(z_2)$ in the ratio $m_1 : m_2 (m_1, m_2 > 0)$ then
   - **(i)** For internal division, $z = \dfrac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$
   - **(ii)** For external division, $z = \dfrac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$
3. **Equation of the Perpendicular Bisector:** If $P(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is moving point (see Fig. 3.3) such that it is always at equal distance from $P(z_1)$ and $Q(z_2)$ then locus of $R(z)$ is perpendicular bisector of PQ

i.e., $\qquad PR = QR$ or $|z - z_1| = |z - z_2|$

$\Rightarrow \qquad |z - z_1|^2 = |z - z_2|^2$

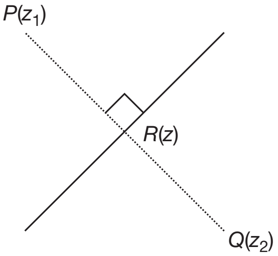


Fig. 3.3

After solving,

$$z\left(\bar{z}_1 - \bar{z}_2\right) + \bar{z}\left(z_1 - z_2\right) = |z_1|^2 - |z_2|^2$$

4. **Equation of a Straight Line**
   - **(i) Parametric form:** Equation of a straight line joining the points having affixes z_1 and z_2 is $z = t\,z_1 + (1-t)z_2$, where $t \in R$
   - **(ii) Non-parametric form:** Equation of a straight line joining the points having affixes z_1 and z_2 is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad z\left(\bar{z}_1 - \bar{z}_2\right) - \bar{z}\left(z_1 - z_2\right) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

TRICK(S) FOR PROBLEM SOLVING

- Three points z_1, z_2 and z_3 are collinear if,

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

- If three points $A(z_1)$, $B(z_2)$, $C(z_3)$ are collinear then slope of AB = slope of BC = slope of AC

$$\Rightarrow \qquad \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$$

(iii) **General equation of a straight line:** The general equation of a straight line is of the form $\bar{a}z + a\bar{z} + b$, where a is complex number and b is real number.

(iv) **Slope of a line:** The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is $-\dfrac{a}{\bar{a}} = -\dfrac{\text{coeff. of } \bar{z}}{\text{coeff. of } z}$ and real slope of the line $\bar{a}z + a\bar{z} + b$ is $-\dfrac{\text{Re}(a)}{\text{Im}(a)} = -i\dfrac{(a+\bar{a})}{(a-\bar{a})}$.

(v) **Length of perpendicular:** The length of perpendicular from a point z_1 to the line $\bar{a}z + a\bar{z} + b = 0$ is given by $\dfrac{|\bar{a}z_1 + a\bar{z}_1 + b|}{|a|+|\bar{a}|}$ or $\dfrac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$.

5. **Equation of a circle**

(i) The equation of a circle whose centre is at point having affix z_0 and radius r is $|z - z_0| = r$.

(ii) If the centre of the circle is at origin and radius r, then its equation is $|z| = r$ (see Fig. 3.4).

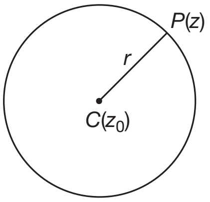


Fig. 3.4

(iii) $|z - z_0| < r$ represents interior of a circle $|z - z_0| = r$ whereas $|z - z_0| > r$ represents exterior of the circle $|z - z_0| = r$.

$$\overrightarrow{AC} = \overrightarrow{AB}e^{i\theta} \quad \text{or} \quad (z_3 - z_1) = (z_2 - z_1)e^{i\theta}$$

$$\text{or} \quad \frac{z_3 - z_1}{z_2 - z_1} = e^{i\theta}$$

(iv) If A, B and C are three points in argand plane such that $AC = AB$ and $\angle CAB = \theta$ then use the rotation about A to find $e^{i\theta}$, but if $AC \neq AB$ use coni method.

(v) If four points z_1, z_2, z_3 and z_4 are con-cyclic then

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_2)(z_1 - z_3)} = \text{real}$$

$$\text{or} \quad \arg\left(\frac{(z_2 - z_3)(z_4 - z_1)}{(z_1 - z_3)(z_4 - z_2)}\right) = \pm\pi, 0$$

TRICK(S) FOR PROBLEM SOLVING

If z is a variable point and z_1, z_2 are two fixed points in the argand plane, then

1. $|z - z_1| = |z - z_2| \Rightarrow$ Locus of z is the perpendicular bisector of the line segment joining z_1 and z_2.
2. $|z - z_1| + |z - z_2| = \text{constant } (\neq |z_1 - z_2|)$ $\Rightarrow$ Locus of z is an ellipse
3. $|z - z_1| + |z - z_2| = |z_1 - z_2|$ $\Rightarrow$ Locus of z is the line segment joining z_1 and z_2
4. $|z - z_1| - |z - z_2| = |z - z_2|$ $\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2.
5. $|z - z_1| - |z - z_2| = \text{constant } (\neq |z_1 - z_2|)$ $\Rightarrow$ Locus of z is a hyperbola.
6. $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2 \Rightarrow$ Locus of z is a circle with z_1 and z_2 as the extremities of diameter.
7. $|z - z_1| = k\,|z - z_2|$, $(k \neq 1) \Rightarrow$ Locus of z is a circle.
8. $arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$ (fixed) $\Rightarrow$ Locus of z is a segment of circle.
9. $arg\left(\dfrac{z - z_1}{z - z_2}\right) = \pm\dfrac{\pi}{2}$ $\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the vertices of diameter.
10. $arg\left(\dfrac{z - z_1}{z - z_2}\right) = 0$ or $\pi \Rightarrow$ Locus of z is a straight line passing through z_1 and z_2.

TIME SAVING TIPS

- $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
  Thus $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 + z_2|$.
- If $\left|z + \dfrac{1}{z}\right| = a$, the greatest and least values of $|z|$ are respectively $\dfrac{a + \sqrt{a^2+4}}{2}$ and $\dfrac{-a + \sqrt{a^2+4}}{2}$.
- The area of the triangle whose vertices are z, iz and $z + iz$ is $\dfrac{1}{2}|z|^2$.
- The area of the triangle with vertices z, wz and $z + wz$ is $\dfrac{\sqrt{3}}{4}|z|^2$.
- If z_1, z_2, z_3 be the vertices of an equilateral triangle and z_0 be the circumcentre, then $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.
- If z_1, z_2, z_3 be the vertices of a triangle, then the triangle is equilateral iff $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$

or $\quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

or $\quad \dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$

- The equation $|z - z|^2 + |z - z_2|^2 = k$ (where k is a real number) will represent a circle with centre at $\frac{1}{2}\,(z_1 + z_2)$ and radius $\frac{1}{2}\,\sqrt{2k - |z_1 - z_2|^2}$ provided $k \geq \frac{1}{2}\,|z_1 - z_2|^2$.
- The one and only one case in which $|z_1| + |z_2| + \ldots + |z_n| = |z_1 + z_2 + \ldots + z_n|$ is that the numbers $z_1, z_2, \ldots z_n$ have the same amplitude.
- If three points z_1, z_2, z_3 are connected by relation $az_1 + bz_2 + cz_3 = 0$ where $a + b + c = 0$, then the three points are collinear.
- If z is a complex number, then e^z is periodic.
- If three complex numbers are in A.P., then they lie on a straight line in the complex plane.

SOLVED EXAMPLES

42. The centre of a square $ABCD$ is at $z = 0$. If A is z_1, then the centroid of triangle ABC is

(A) $\dfrac{z_1}{3}\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$

(B) $\dfrac{z_1}{3}\,(\cos\pi + i\sin\pi)$

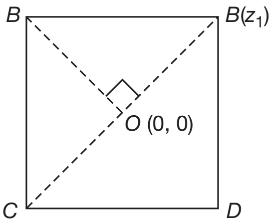
(C) $z_1\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$

(D) None of these

Solution: (A)

Since A is z_1 and $\angle AOB = \dfrac{\pi}{2}$

$\therefore$ B is $z_1\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$



Also, c is $z_1(\cos\pi + i\sin\pi)$

$\therefore$ Centroid of ΔABC is

$$\frac{z_1}{3}\left(1 + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} + \cos\pi + i\sin\pi\right)$$

$$= \frac{z_1}{3}\,(1 + 0 + i - 1 + 0) = \frac{z_1}{3}\,i$$

$$= \frac{z_1}{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

43. If z_1 and z_2 ($\neq 0$) are two complex numbers such that $\left|\dfrac{z_1 - z_2}{z_1 + z_2}\right| = 1$, then

(A) $z_2 = ikz_1, k \in R$ (B) $z_2 = kz_1, k \in R$

(C) $z_2 = z_1$ (D) None of these

Solution: (A)

We have,

$$\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1 \Rightarrow \left|\frac{z_1/z_2 - 1}{z_1/z_2 + 1}\right| = 1$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2} - 1\right| = \left|\frac{z_1}{z_2} + 1\right|$$

$\Rightarrow$ $\dfrac{z_1}{z_2}$ lies on the perpendicular bisector of the segment joining A ($-1 + 0i$) and B ($1 + 0i$).

$\therefore \quad \dfrac{z_1}{z_2} = ai$ for some $a \in R$

$\Rightarrow \quad \dfrac{z_2}{z_1} = \dfrac{1}{ai} = \dfrac{-i}{a}$

$\therefore \quad z_2 = i\,kz_1$ for some $k \in R$

44. If $z = x + iy$ and 'a' is a real number such that $|z - ai| = |z + ai|$, then locus of z is

(A) x-axis (B) y-axis

(C) $x = y$ (B) $x^2 + y^2 = 1$

Solution: (A)

We have, $\quad |z - ai| = |z + ai|$

$\Rightarrow \quad |x + i\,(y - a)|^2 = |x + i\,(y + a)|^2$

$\Rightarrow \quad x^2 + (y - a)^2 = x^2 + (y + a)^2$

$\Rightarrow \quad 4ay = 0$; $y = 0$, which is x-axis.

45. The locus represented by $|z - 1| = |z + i|$ is

(A) a circle of radius 1

(B) an ellipse with foci at 1 and $-i$

(C) a line through the origin

(D) a circle on the join of 1 and $-i$ as diameter

Solution: (C)

We have, $\quad |z - 1| = |z + i|$

$\Rightarrow \quad |(x - 1) + iy| = |x + i\,(y + 1)|$

$\Rightarrow \quad (x - 1)^2 + y^2 = x^2 + (y + 1)^2$

$\Rightarrow \quad x + y = 0$, which is a line through the origin.

46. The centre of a regular polygon of n sides is located at the point $z = 0$, and one of its vertex z_1 is known. If z_2 be the vertex adjacent to z_1, then z_2 is equal to

(A) $z_1\left(\cos\frac{2\pi}{n} \pm i\sin\frac{2\pi}{n}\right)$

(B) $z_1\left(\cos\frac{\pi}{n} \pm i\sin\frac{\pi}{n}\right)$

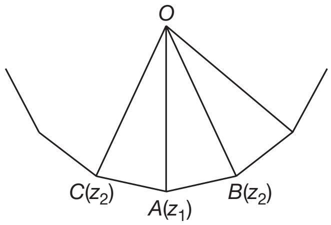
(C) $z_1\left(\cos\frac{\pi}{2n} \pm i\sin\frac{\pi}{2n}\right)$

(D) None of these

Solution: (A)

Let A be the vertex with affix z_1. There are two possibilities and can be obtained by rotating z_1 through $\frac{2\pi}{n}$ either in clockwise or in anti-clockwise direction.

$$z_2 = z_1 e^{\pm\frac{12\pi}{n}} \qquad (\because |z_2| = |z_1|)$$



47. The locus of the complex number z in the Argand plane if $\left|\frac{1-iz}{z-i}\right| = 1$, is

(A) a circle (B) x-axis
(C) y-axis (D) None of these

Solution: (B)

Let $z = x + iy$

Given, $\left|\frac{1-iz}{z-i}\right| = 1$

$$\Rightarrow \left|\frac{1-i(x+iy)}{x+iy-i}\right| = 1$$

$$\Rightarrow \left|\frac{1+y-ix}{x+i(y-1)}\right| = 1$$

$$\Rightarrow \frac{\sqrt{(1+y)^2+x^2}}{\sqrt{x^2+(y-1)^2}} = 1$$

$$\Rightarrow (1+y)^2 + x^2 = x^2 + (y-1)^2$$

$$\Rightarrow 1 + y^2 + 2y + x^2 = x^2 + y^2 - 2y + 1$$

$$\Rightarrow 4y = 0$$

$\Rightarrow$ $y = 0$, which is the equation of x-axis.

48. The equation $|z-1|^2 + |z+1|^2 = 4$ represents on the Argand plane

(A) a straight line
(B) an ellipse
(C) a circle with centre origin and radius 2
(D) a circle with centre origin and radius unity

Solution: (D)

We have, $|z-1|^2 + |z+1|^2 = 4$ (1)

$$\Rightarrow (x-1)^2 + y^2 + (x+1)^2 + y^2 = 4$$

(Putting $z = x + iy$)

$$\Rightarrow 2(x^2 + y^2 + 1) = 4$$

$$\therefore x^2 + y^2 = 1 \quad \text{or} \quad |z|^2 = 1$$

$$\Rightarrow |z| = 1 \qquad \text{(since } |z| \text{ cannot be –ve)}$$

Thus, the Eq. (1) represents all points z on the circle with centre origin and radius unity.

49. The locus of the point z satisfying the condition

$\arg\frac{z-1}{z+1} = \frac{\pi}{3}$ is

(A) a straight line (B) circle
(C) a parabola (D) None of these

Solution: (B)

We have, $\arg\frac{z-1}{z+1} = \frac{\pi}{3}$

$$\Rightarrow \arg\frac{x+iy-1}{x+iy+1} = \frac{\pi}{3} \qquad \text{(Putting } z = x+iy\text{)}$$

$$\Rightarrow \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = \frac{\pi}{3}$$

$$\left(\because \operatorname{Arg}\frac{z_1}{z_2} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2\right)$$

$$\Rightarrow \tan^{-1}\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1+\frac{y^2}{x^2-1}} = \frac{\pi}{3}$$

$$\Rightarrow \frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\Rightarrow x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 = 0, \text{ which is a circle.}$$

50. If $\omega = \left(\frac{z-i}{1+iz}\right)^n$, n integral, then ω lies on the unit circle for

(A) only even n (B) only odd n
(C) only positive n (D) all n

Solution: (D)

We have, $\omega = \left(\dfrac{z-i}{1+iz}\right)^n = \left(\dfrac{z-i}{i(z-i)}\right)^n$

$$= \left(\frac{1}{i}\right)^n = (-i)^n$$

$\therefore$ $|\omega| = |(-i)^n| = |-i|^n = 1$ for all n.

$\therefore$ ω lies on unit circle for all n.

51. The equation $z\,\bar{z} + a\,\bar{z} + \bar{a}\,z + b = 0, b \in R$ represents a circle (not point circle) if

(A) $|a|^2 > b$ (B) $|a|^2 < b$
(C) $|a| > b$ (D) $|a| < b$

Solution: (A)

We have, $z\,\bar{z} + a\bar{z} + \bar{a}\,z + b = 0$

$\Rightarrow$ $z\bar{z} + a\bar{z} + \bar{a}z + a\,\bar{a} = a\bar{a} - b$

$\Rightarrow$ $(z + a)\,(\bar{z} + \bar{a}) = a\bar{a} - b$

$\Rightarrow$ $|z + a|^2 = |a|^2 - b$

This represents a circle (not point circle) if $|a|^2 > b$.

52. If $z^4 = (z - 1)^4$, then the roots are represented in the argand plane by the points that are

(A) collinear
(B) concyclic
(C) vertices of a parallelogram
(D) None of these

Solution: (A)

We have, $z^4 = (z - 1)^4$

$\Rightarrow$ $\left(\dfrac{z-1}{z}\right) = 1^{1/4} = e^{\frac{2n\pi i}{4}}$, $n = 0, 1, 2, 3$

Since for all these values of z,

$\left|\dfrac{z-1}{z}\right| = 1$ so they lie on the line bisecting perpendicularly the join of $z = 1$ and $z = 0$.

53. The equation $z^2 + \bar{z}^2 - 2|z|^2 + z + \bar{z} = 0$ represents a

(A) straight line (B) circle
(C) hyperbola (D) parabola

Solution: (D)

We have, $z^2 + \bar{z}^2 - 2|z|^2 + z + \bar{z} = 0$

$\Rightarrow (x + iy)^2 + (x - iy)^2 - 2(x^2 + y^2) + x + iy + x - iy = 0$
(Putting $z = x + iy$)

$\Rightarrow$ $2x^2 + 2\,(iy)^2 - 2x^2 - 2y^2 + 2x = 0$

$\Rightarrow$ $-4y^2 + 2x = 0$ or $y^2 = \dfrac{1}{2}x$,

which is a parabola.

54. Let z_1 and z_2 be two non real complex cube roots of unity and $|z - z_1|^2 + |z - z^2|^2 = \lambda$ be the equation of a circle with z_1, z_2 as ends of a diameter, then the value of λ is

(A) 4 (B) 3 (C) 2 (D) $\sqrt{2}$

Solution: (B)

We have,

$$|z - \omega|^2 + |z - \omega^2|^2 = \lambda$$

$\Rightarrow$ $\lambda = |\omega - \omega^2|^2 = |\omega^2 + \omega^4 - 2\omega^3|$

$= |\omega^2 + \omega - 2| = |-1 - 2| = 3$

55. The region in the Argand diagram defined by $|z - 3| + |z + 3| < 6$ is the interior of the ellipse with major axis along

(A) real axis (B) imaginary axis
(C) $y = x$ (D) $y = -x$

Solution: (A)

The equation $|z - (3 + 0i)| + |z - (-3 + 0i)| < 6$ represents the interior of ellipse with foci at (3, 0) and (–3, 0). So, major axis is along real axis.

56. If the area of the triangle on the argand plane formed by the complex numbers $-z$, iz, $z - iz$ is 600 square units, then $|z|$ is equal to

(A) 10 (B) 20
(C) 30 (D) None of these

Solution: (B)

Area of the triangle on the argand plane formed by the complex numbers $-z$, iz, $z - iz$ is $\dfrac{3}{2}|z|^2$.

$\therefore$ $\dfrac{3}{2}|z|^2 = 600 \Rightarrow |z| = 20$

57. If $|z + \bar{z}| + |z - \bar{z}| = 8$, then z lies on

(A) a circle
(B) a straight line
(C) a square
(D) None of these

Solution: (C)

We have, $|z + \bar{z}| + |z - \bar{z}| = 8$

$\Rightarrow$ $2|x| + 2|y| = 8$ or $|x| + |y| = 4$

58. If $\text{Im}\left(\dfrac{z+2i}{z+2}\right) = 0$, then z lies on the curve

(A) $x^2 + y^2 + 2x + 2y = 0$
(B) $x^2 + y^2 - 2x = 0$
(C) $x + y + 2 = 0$
(D) None of these

Solution: (C)

Let $z = x + iy$

Then, $\dfrac{z+2i}{z+2} = \dfrac{x+iy+2i}{x+iy+2} = \dfrac{x+(y+2)i}{(x+2)+iy}$

$$= \frac{[x+(y+2)i]\,[(x+2)-iy]}{(x+2)^2+y^2}$$

$$= \frac{(x^2+y^2+2x+2y)+i(2x+2y+4)}{(x+2)^2+y^2}$$

Since $\text{Im}\left(\dfrac{z+2i}{z+2}\right) = 0 \Rightarrow x+y+2=0$

which represents a straight line.

59. The cube roots of unity

(A) lie on the circle $|z| = 1$

(B) are collinear

(C) form an equilateral triangle

(D) None of these

Solution: (A, C)

Clearly, cube roots of unity 1, ω, ω^2 satisfy $|z| = 1$.

Also, $\quad |1-\omega|^2 = \left(\dfrac{3}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2 = 3$

$\Rightarrow \quad |1-\omega| = \sqrt{3}$

$\quad |\omega-\omega^2| = |\sqrt{3}\,i| = \sqrt{3}$

and $\quad |1-\omega^2| = \left|1-\left(-\dfrac{1}{2}-\dfrac{i\sqrt{3}}{2}\right)\right|$

$$= \left|\frac{3}{2}+\frac{i\sqrt{3}}{2}\right| = \sqrt{3}.$$

Therefore, 1, ω, ω^2 form an equilateral triangle.

60. If $|z-1| + |z+3| \le 8$, then the range of values of $|z-4|$ is

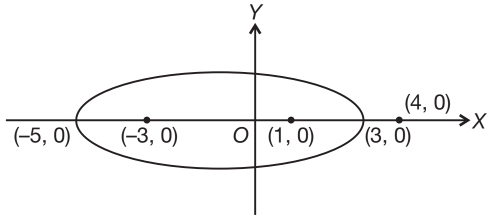
(A) (0, 8) (B) [0, 8]

(C) [1, 9] (D) [5, 9]

Solution: (C)

Given $\quad |z-1| + |z+3| \le 8$

$\therefore$ z lies inside or on the ellipse whose foci are (1, 0) and (– 3, 0) and vertices are (– 5, 0) and (3, 0).



Now, $|z-4|$ is distance of z from (4, 0). Minimum distance is 1 and maximum is 9.

EXERCISES

Single Option Correct Type

1. If a, b, c, p, q, r are three complex numbers such that $\dfrac{p}{a}+\dfrac{q}{b}+\dfrac{r}{c} = 1+i$ and $\dfrac{a}{p}+\dfrac{b}{q}+\dfrac{c}{r} = 0$, then the value of $\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}+\dfrac{r^2}{c^2}$ is

(A) $2i$ (B) i

(C) $-2i$ (D) None of these

2. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other, for

(A) $x = n\pi$ (B) $x = 0$

(C) $x = \left(n+\dfrac{1}{2}\right)\pi$ (D) no value of x

3. If z_1 and z_2 are two non-zero complex numbers such that $|z_1+z_2| = |z_1| + |z_2|$, then $\arg z_1 - \arg z_2$ is equal to

(A) $-\pi$ (B) $-\dfrac{\pi}{2}$ (C) π (D) $\dfrac{\pi}{2}$

4. The number of solutions of the equation $z^2 + |z|^2 = 0$, where $z \in C$ is

(A) one (B) two

(C) three (D) infinitely many

5. If ω is the nth root of unity, then

$$(1+\omega+\omega^2+\ldots+\omega^{n-1}) \text{ is}$$

(A) 2 (B) 0 (C) 1 (D) –1

6. The complex number which satisfies the equation

$$z + \sqrt{2}\,|z+1| + i = 0 \text{ is}$$

(A) $2-i$ (B) $-2-i$

(C) $2+i$ (D) $-2+i$

7. z_1, z_2 are two non-real complex numbers such that $\dfrac{z_1}{z_2}+\dfrac{z_2}{z_1} = 1$. Then z_1, z_2 and the origin

(A) are collinear
(B) form right angled triangle
(C) form right angle isosceles triangle
(D) form an equilateral triangle

8. $\tan\left[i \log \dfrac{a-ib}{a+ib}\right]$ is equal to

(A) $\dfrac{2ab}{a^2+b^2}$ (B) $\dfrac{a^2-b^2}{2ab}$
(C) $\dfrac{2ab}{a^2-b^2}$ (D) ab

9. If $(\sqrt{3}+i)^{100} = 2^{99}(a+ib)$, then $b =$
(A) $\sqrt{3}$ (B) $\sqrt{2}$
(C) 1 (D) None of these

10. The real value of α for which the expression $\dfrac{1-i\sin\alpha}{1+2i\sin\alpha}$ is purely real is

(A) $(2n+1)\dfrac{\pi}{2}$ (B) $(n+1)\dfrac{\pi}{2}$
(C) $n\pi$ (D) None of these

11. The locus of z which satisfies the inequality

$$\log_{0.3}|z-1| > \log_{0.3}|z-i| \text{ is given by,}$$

(A) $x+y>0$ (B) $x-y<0$
(C) $x+y<0$ (D) $x-y>0$

12. If centre of a regular hexagon is at origin and one of the vertices on argand diagram is $1+2i$ then its perimeter is

(A) $2\sqrt{5}$ (B) $6\sqrt{2}$ (C) $4\sqrt{5}$ (D) $6\sqrt{5}$

13. If z_1, z_2, z_3 are three complex numbers, then $z_1 \operatorname{Im}(\bar{z}_2 z_3) + z_2 \operatorname{Im}(\bar{z}_3 z_1) + z_3 \operatorname{Im}(\bar{z}_1 z_2)$ is equal to

(A) 1 (B) –1
(C) 0 (D) None of these

14. If $\dfrac{2z_1}{3z_2}$ is purely imaginary number, then $\left|\dfrac{z_1-z_2}{z_1+z_2}\right|^4$ is equal to

(A) $\dfrac{3}{2}$ (B) 1 (C) $\dfrac{2}{3}$ (D) $\dfrac{4}{9}$

15. If $x^6 = (4-3i)^5$, then the product of all of its roots is (where $\theta = -\tan^{-1}(3/4)$)

(A) $5^5(\cos 5\theta + i\sin 5\theta)$
(B) $-5^5(\cos 5\theta + i\sin 5\theta)$
(C) $5^5(\cos 5\theta - i\sin 5\theta)$
(D) $-5^5(\cos 5\theta - i\sin 5\theta)$

16. $|z_1+z_2| = |z_1| + |z_2|$ is possible if

(A) $z_2 = \bar{z}_1$ (B) $z_2 = \dfrac{1}{z_1}$
(C) $\arg z_1 = \arg z_2$ (D) $|z_1| = |z_2|$

17. If $z = x+iy$, x, y real, then $|x| + |y| \le k|z|$, where k is equal to

(A) 1 (B) $\sqrt{2}$
(C) $\sqrt{3}$ (D) None of these

18. If $(1+i)(1+2i)(1+3i) \ldots (1+ni) = \alpha + i\beta$ then $2 \times 5 \times 10 \ldots (1+n^2) =$

(A) $\alpha - i\beta$ (B) $\alpha^2 - \beta^2$
(C) $\alpha^2 + \beta^2$ (D) None of these

19. Let $z_1 = a+ib$, $z_2 = p+iq$ be two unimodular complex numbers such that $\operatorname{Im}(z_1\bar{z}_2) = 1$.
If $\omega_1 = a+ip$, $\omega_2 = b+iq$, then

(A) $\operatorname{Re}(\omega_1\omega_2) = 1$ (B) $\operatorname{Im}(\omega_1\omega_2) = 1$
(C) $\operatorname{Re}(\omega_1\omega_2) = 0$ (D) $\operatorname{Im}(\omega_1\overline{\omega_2}) = 1$

20. If $\sqrt[3]{a+ib} = x+iy$, then $\dfrac{a}{x} + \dfrac{b}{y} =$

(A) $4(x^2+y^2)$ (B) $4(x^2-y^2)$
(C) $2(x^2-y^2)$ (D) None of these

21. If $z = a+ib$ where $a>0$, $b>0$, then

(A) $|z| \ge \dfrac{1}{\sqrt{2}}(a-b)$ (B) $|z| \ge \dfrac{1}{\sqrt{2}}(a+b)$
(C) $|z| < \dfrac{1}{\sqrt{2}}(a+b)$ (D) None of these

22. The complex numbers z_1, z_2 and z_3 satisfying $\dfrac{z_1-z_3}{z_2-z_3} = \dfrac{1-\sqrt{3}i}{2}$ are the vertices of a triangle which is

(A) of area zero (B) right angled isosceles
(C) equilateral (D) obtuse angled isosceles

23. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$, then $a_0 + a_3 + a_6 + \ldots =$

(A) 3^n (B) 3^{n-1}
(C) 3^{n-2} (D) None of these

24. If $1, a_1, a_2, \ldots, a_{n-1}$ are the n nth roots of unity, then $(1-a_1)(1-a_2)(1-a_3) \ldots (1-a_{n-1}) =$

(A) $n+1$ (B) n
(C) $n-1$ (D) None of these

25. The closest distance of the origin from a curve given as = 0 (a is a complex number) is

(A) 1 (B) $\dfrac{|a|}{2}$ (C) $\dfrac{\operatorname{Re} a}{|a|}$ (D) $\dfrac{\operatorname{Im} a}{|a|}$

26. $|z-1|+|z+3|\le 8$, then the range of values of $|z-4|$ is
(A) (0, 8) (B) [0, 8] (C) [1, 9] (D) [5, 9]

27. The roots of the equation $z^4+1=0$ are
(A) $(\pm 1 \pm i)$ (B) $(\pm 2 \pm 2i)$
(C) $\frac{1}{\sqrt{2}}(\pm 1 \pm i)$ (D) None of these

28. The integral solution of the equation $(1-i)^n = 2^n$ is
(A) $n=0$ (B) $n=1$
(C) $n=-1$ (D) None of these

29. The greatest value of the moduli of complex numbres z satisfying the equation $\left|z-\frac{4}{z}\right|=2$ is
(A) $\sqrt{5}$ (B) $\sqrt{5}-1$
(C) $\sqrt{5}+1$ (D) None of these

30. The locus of the complex number z in an argand plane satisfying the equation

$$\text{Arg}(z+i)-\text{Arg}(z-i)=\frac{\pi}{2} \text{ is}$$

(A) boundary of a circle (B) interior of a circle
(C) exterior of a circle (D) None of these

31. $\frac{z^2}{z-1}$ is always real, then
(A) z lies only on a circle
(B) z lies only on the real axis
(C) z lies either on the real axis or on a circle
(D) None of these

32. z_1 and z_2 are two complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular whereas z_2 is not unimodular. Then $|z_1|=$
(A) 1 (B) 2 (C) 3 (D) 4

33. If for the complex numbers z_1 and z_2, $|z_1+z_2|=|z_1-z_2|$, then $amp\ z_1 \sim amp\ z_2 =$
(A) π (B) $\frac{\pi}{2}$
(C) $\frac{\pi}{4}$ (D) None of these

34. The locus of the complex number z in an argand plane satisfying the inequality

$$\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right)>1 \left(\text{where } |z-1|\ne\frac{2}{3}\right) \text{ is}$$

(A) a circle
(B) interior of a circle
(C) exterior of a circle
(D) None of these

35. The equation $z^3+iz-1=0$ has
(A) three real roots
(B) one real root
(C) no real roots
(D) no real or complex roots

36. If all the roots of $z^3+az^2+bz+c=0$ are of unit modulus, then
(A) $|a|\le 3$ (B) $|b|>3$
(C) $|c|\le 3$ (D) None of these

37. Let z_1 and z_2 be two complex numbers such that $\frac{z_1}{z_2}+\frac{z_2}{z_1}=1$, then
(A) z_1, z_2 are collinear
(B) z_1, z_2 and the origin from a right angled triangle
(C) z_1, z_2 and the origin form an equilateral triangle
(D) None of these

38. If $S(n)=i^n+i^{-n}$, where $i=\sqrt{-1}$ and n is a positive integer, then the total number of distinct values of $S(n)$ is
(A) 1 (B) 2 (C) 3 (D) 4

39. If $z_1 \ne -z_2$ and $|z_1+z_2|=\left|\frac{1}{z_1}+\frac{1}{z_2}\right|$, then
(A) at least one of z_1, z_2 is unimodular
(B) $z_1 \times z_2$ is unimodular
(C) both z_1, z_2 are unimodular
(D) None of these

40. If $z=x+iy$ satisfies amp $(z-1)=$ amp $(z+3i)$ then the value of $(x-1):y$ is equal to
(A) 2 : 1 (B) – 1 : 3
(C) 1 : 3 (D) None of these

41. If z_1, z_2, z_3, z_4 are the four complex numbers represented by the vertices of a quadrilateral taken in order such that $z_1-z_4=z_2-z_3$ and amp $\frac{z_4-z_1}{z_2-z_1}=\frac{\pi}{2}$ then the quadrilateral is a
(A) square
(B) rhombus
(C) rectangle
(D) a cyclic quadrilateral

42. Let z be a complex number with modulus 2 and argument $\frac{2\pi}{3}$, then z is equal to
(A) $-1+i\sqrt{3}$ (B) $1-i\sqrt{3}$
(C) $-\frac{1}{2}+\frac{i\sqrt{3}}{2}$ (D) None of these

43. If $\log_{\sqrt{3}}\left(\frac{|z|^2 - |z| + 1}{2 + |z|}\right) < 2$, then the locus of z is

(A) $|z| < 5$ (B) $|z| = 5$
(C) $|z| > 5$ (D) None of these

44. If $|z| = 1$, then the value of $\left(\frac{z-1}{z+1}\right)$ is

(A) 0 (B) purely real
(C) purely imaginary (D) complex number

45. If z_1 and z_2 are complex numbers, such that $z_1 + z_2$ is a real number, then

(A) $z_1 = -\bar{z}_2$
(B) $z_2 = \bar{z}_1$
(C) z_1 and z_2 are any two complex numbers
(D) $z_1 = \bar{z}_1, z_2 = \bar{z}_2$

46. The locus of the points representing the complex numbers which satisfy $|z| - 2 = 0$, $|z - i| - |z + 5i| = 0$ is:

(A) a circle with centre at origin
(B) a straight line passing through origin
(C) the single point (0, –2)
(D) None of these

47. Let the affix of $2 - 4i$ be P. Then OP is rotated about O through an angle of 180° and is stretched 5/2 times. The complex number corresponding to the new position of P is

(A) $5 - 10i$ (B) $5 + 10i$
(C) $-5 + 10i$ (D) None of these

48. If P, P' represent the complex number z_1 and its additive inverse respectively then the complex equation of the circle with PP' as a diameter is

(A) $\frac{z}{z_1} = \overline{\left(\frac{z_1}{z}\right)}$ (B) $z\bar{z} + z_1\bar{z}_1 = 0$
(C) $z\bar{z}_1 + \bar{z}z_1$ (D) None of these

49. If a, b, c, p, q, r are three non-zero complex numbers such that $\frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1 + i$ and $\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0$, then value of $\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2}$ is

(A) 0 (B) -1
(C) $2i$ (D) $-2i$

50. If z_1, z_2 are two complex numbers such that $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$ and $tz_1 = kz_2$ where $k \in \mathbb{R}$, then the angle between $(z_1 - z_2)$ and $(z_1 + z_2)$ is

(A) $\tan^{-1}\left(\frac{2k}{k^2+1}\right)$ (B) $\tan^{-1}\left(\frac{2k}{1-k^2}\right)$
(C) $-2\tan^{-1}(k)$ (D) $2\tan^{-1}(k)$

51. $1 + x^2 = \sqrt{3}x$, then $\sum_{n=1}^{24}\left(x^n - \frac{1}{x^n}\right)^2$ is equal to

(A) 48 (B) -48
(C) $\pm 48(\omega - \omega^2)$ (D) $1 \pm 48\omega$

52. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is

(A) 0 (B) 2 (C) 7 (D) 17

53. For any two complex numbers z_1 and z_2 with $|z_1| \neq |z_2|$
$\left|\sqrt{2}z_1 + i\sqrt{3}\bar{z}_2\right|^2 + \left|\sqrt{3}\bar{z}_1 + i\sqrt{2}\,z_2\right|^2$ is

(A) less than $5\,|z_1|^2 + |z_2|^2$
(B) greater than $10\,|z_1z_2|$
(C) equal to $2|z_1|^2 + 3\,|z_2|^2$
(D) zero

54. If the complex numbers z_1, z_2, z_3 are in AP, then they lie on a

(A) circle (B) parabola
(C) line (D) ellipse

55. If the roots of $(z - 1)^{25} = 2\omega^2(z + 1)^{25}$ where ω is a complex cube root of unity are plotted in the argand plane, they lie on

(A) a straight line (B) a circle
(C) an ellipse (D) None of these

56. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is

(A) $\frac{3}{4}$ (B) $3\sqrt{3}$ (C) 3 (D) $\frac{3\sqrt{3}}{2}$

57. If z_1 and z_2 are the two complex roots of equal magnitude and their arguments differ by $\frac{\pi}{2}$, of the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ then a (in terms of b and c) is

(A) $\frac{b^2}{2c}$ (B) $\frac{b^2}{c}$
(C) $\frac{b}{2c}$ (D) None of these

58. Common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ are

(A) ω, ω^2 (B) $1, \omega, \omega^2$
(C) $-1, \omega, \omega^2$ (D) $-\omega, -\omega^2$

59. $\sin^{-1}\left[\frac{1}{i}(z-1)\right]$, where z is non-real, can be the angle of a triangle if

(A) Re $(z) = 1$, Im $(z) = 2$
(B) Re $(z) = 1$, $-1 \le$ Im $(z) \le 1$
(C) Re (z) + Im $(z) = 0$
(D) Re (z) = Im (z)

60. If $x^2 - x + 1 = 0$ then the value of $\sum_{n=1}^{5}\left(x^n + \frac{1}{x^n}\right)^2$ is

(A) 8
(B) 10
(C) 12
(D) None of these

61. The triangle formed by the points 1, $\frac{1+i}{\sqrt{2}}$ and i as vertices in the Argand diagram is

(A) scalene
(B) equilateral
(C) isosceles
(D) right-angled

62. If the quadratic equation $z^2 + (a + ib)z + c + id = 0$, where a, b, c, d are non-zero real numbers, has a real root, then

(A) $d^2 - abd - c^2 = 0$
(B) $d^2 - abd + b^2c = 0$
(C) $d^2 + abd + c^2 = 0$
(D) None of these

63. If $|z - i| \le 2$ and $z_0 = 5 + 3i$, the maximum value of $|iz + z_0|$ is

(A) 7
(B) 9
(C) 13
(D) None of these

64. The solutions of the equation $z\left(\overline{z-2i}\right) = 2\,(2 + i)$ are

(A) $3 + i$, $3 - i$
(B) $1 + 3i$, $1 - 3i$
(C) $1 + 3i$, $1 - i$
(D) $1 - 3i$, $1 + i$

65. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re $z = 1$, then it is necessary that

(A) $\beta \in (1, \infty)$
(B) $\beta \in (0, 1)$
(C) $\beta \in (-1, 0)$
(D) $|\beta| = 1$

66. If $\omega(\ne 1)$ is a cube root of unity, and $(1 + \omega)^7 = A + B\omega$. Then (A, B) equals

(A) $(-1, 1)$ (B) $(0, 1)$ (C) $(1, 1)$ (D) $(1, 0)$

67. If $z \ne 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies

(A) either on the real axis or on a circle passing through the origin.
(B) on a circle with centre at the origin.
(C) either on the real axis or on a circle not passing through the origin.
(D) on the imaginary axis.

68. Two circles in complex plane are
C_1: $|z - i| = 2$
C_2: $|z - 1 - 2i| = 4$. Then

(A) C_1 and C_2 touch each other.
(B) C_1 and C_2 intersect at two distinct points.
(C) C_1 lies within C_2.
(D) C_2 lies within C_1.

69. The conjugate of a complex number is $\frac{1}{i-1}$. Then the complex number is

(A) $\frac{-1}{i-1}$ (B) $\frac{1}{i+1}$ (C) $\frac{-1}{i+1}$ (D) $\frac{1}{i-1}$

70. If $\left|z - \frac{4}{z}\right| = 2$, then the maximum value of $|z|$ is equal to

(A) $\sqrt{3} + 1$
(B) $\sqrt{5} + 1$
(C) 2
(D) $2 + \sqrt{2}$

71. If $z_1z_2 \in C$, $z^2_1 + z^2_2 \in R$, $z_1(z^2_1 - 3z^2_2) = 2$ and $z_2(3z^2_1 - z^2_2) = 11$, then the value of $z^2_1 + z^2_2$ is

(A) 2 (B) 3 (C) 4 (D) 5

72. If $\sqrt{1 - C^2} = nc - 1$ and $z = e^{i\theta}$,

then $\frac{c}{2n}(1 + nz)\left(1 + \frac{n}{z}\right) =$

(A) $1 + c\cos\theta$
(B) $1 - c\cos\theta$
(C) $1 + 2c\cos\theta$
(D) $1 - 2c\cos\theta$

73. Let 'a' be a complex number such that $|\,a\,| < 1$ and z_1, z_2,..., z_n be the vetices of a polygon such that $z_k = 1 + a + a^2 + \ldots + a^k$, then the vertices of the polygon lie within the circle

(A) $|\,z\,| = \frac{1}{|1-a|}$
(B) $|z - a\,| = \frac{1}{|1-a|}$
(C) $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$
(D) None of these

74. All the roots of the equation $a_1z^3 + a_2z^2 + a_3z + a_4 = 3$, where $|a_i| \le 1$, $i = 1, 2, 3, 4$ lie outside the circle with centre origin and radius

(A) $\frac{1}{3}$
(B) $\frac{2}{3}$
(C) 1
(D) None of these

75. If $z^4 = (z - 1)^4$, then the roots are represented in the argand plane by the points that are

(A) collinear
(B) concyclic
(C) vertices of a parallelogram
(D) None of these

76. The maximum value of $|z|$ when z satisfies the condition $\left|z+\frac{2}{z}\right|=2$ is

(A) $\sqrt{3}-1$ (B) $\sqrt{3}+1$

(C) $\sqrt{3}$ (D) $\sqrt{2}+\sqrt{3}$

77. If $|z+\bar{z}|+|z-\bar{z}|=8$, then z lies on

(A) a circle (B) a straight line

(C) a square (D) None of these

78. The complex number which satisfies the equation

$$z+\sqrt{2}\,|z+1|+i=0 \text{ is}$$

(A) $2-i$ (B) $-2-i$

(C) $2+i$ (D) $-2+i$

79. $\tan\left[i\log\frac{a-ib}{a+ib}\right]$ is equal to

(A) $\frac{2ab}{a^2+b^2}$ (B) $\frac{a^2-b^2}{2ab}$

(C) $\frac{2ab}{a^2-b^2}$ (D) ab

80. $|z_1+z_2|=|z_1|+|z_2|$ is possible if

(A) $z_2=\bar{z}_1$ (B) $z_2=\frac{1}{z_1}$

(C) $\arg z_1=\arg z_2$ (D) $|z_1|=|z_2|$

81. If $z=x+iy$, x, y real, then $|x|+|y|\le k\,|z|$, where k is equal to

(A) 1 (B) $\sqrt{2}$

(C) $\sqrt{3}$ (D) None of these

82. If $z=a+ib$ where $a>0$, $b>0$, then

(A) $|z|\ge\frac{1}{\sqrt{2}}(a-b)$ (B) $|z|\ge\frac{1}{\sqrt{2}}(a+b)$

(C) $|z|<\frac{1}{\sqrt{2}}(a+b)$ (D) None of these

83. If $(1+x+x^2)^n=a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n}$, then $a_0+a_3+a_6+\ldots=$

(A) 3^n (B) 3^{n-1}

(C) 3^{n-2} (D) None of these

84. The closest distance of the origin from a curve given as $a\bar{z}+\bar{a}z+a\bar{a}=0$ (a is a complex number) is

(A) 1 (B) $\frac{|a|}{2}$ (B) $\frac{\operatorname{Re} a}{|a|}$ (D) $\frac{\operatorname{Im} a}{|a|}$

85. The integral solution of the equation $(1-i)^n=2^n$ is

(A) $n=0$ (B) $n=1$

(C) $n=-1$ (D) None of these

86. The locus of the complex number z in an argand plane satisfying the equation

$$\operatorname{Arg}(z+i)-\operatorname{Arg}(z-i)=\frac{\pi}{2} \text{ is}$$

(A) boundary of a circle (B) interior of a circle

(C) exterior of a circle (D) None of these

87. If for the complex numbers z_1 and z_2, $|z_1+z_2|=|z_1-z_2|$, then $amp\ z_1 \sim amp\ z_2=$

(A) π (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{4}$ (D) None of these

88. The locus of the complex number z in an argand plane satisfying the inequality

$$\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right)>1\left(\text{where } |z-1|\ne\frac{2}{3}\right) \text{ is}$$

(A) a circle (B) interior of a circle

(C) exterior of a circle (D) None of these

89. Let z_1 and z_2 be two complex numbers such that $\frac{z_1}{z_2}+\frac{z_2}{z_1}=1$, then

(A) z_1, z_2 are collinear

(B) z_1, z_2 and the origin from a right angled triangle

(C) z_1, z_2 and the origin form an equilateral triangle

(D) None of these

90. If P, P' represent the complex number z_1 and its additive inverse respectively, then the complex equation of the circle with PP' as a diameter is

(A) $\frac{z}{z_1}=\overline{\left(\frac{z_1}{z}\right)}$ (B) $z\bar{z}+z_1\bar{z}_1=0$

(C) $z\bar{z}_1+\bar{z}z_1$ (D) None of these

91. If a, b, c, p, q, r are three non-zero complex numbers such that $\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1+i$ and $\frac{a}{p}+\frac{b}{q}+\frac{c}{r}=0$, then value of $\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}$ is

(A) 0 (B) -1 (C) $2i$ (D) $-2i$

92. If z_1, z_2 are two complex numbers such that $\left|\frac{z_1-z_2}{z_1+z_2}\right|=1$ and $iz_1=kz_2$ where $k\in R$, then the angle between (z_1-z_2) and (z_1+z_2) is

(A) $\tan^{-1}\left(\frac{2k}{k^2+1}\right)$ (B) $\tan^{-1}\left(\frac{2k}{1-k^2}\right)$

(C) $-2\tan^{-1}(k)$ (D) $2\tan^{-1}(k)$

93. $1 + x^2 = \sqrt{3}x$, then $\sum_{n=1}^{24}\left(x^n - \frac{1}{x^n}\right)^2$ is equal to

(A) 48 (B) –48
(C) $\pm 48(\omega - \omega^2)$ (D) $1 \pm 48\omega$

94. For any two complex numbers z_1 and z_2 with $|z_1| \neq |z_2|$

$$\left|\sqrt{2}z_1 + i\sqrt{3}\bar{z}_2\right|^2 + \left|\sqrt{3}\bar{z}_1 + i\sqrt{2}\,z_2\right|^2 \text{ is}$$

(A) less than $5\,(|z_1|^2 + |z_2|^2)$
(B) greater than $10\,|z_1 z_2|$
(C) equal to $2|z_1|^2 + 3\,|z_2|^2$
(D) zero

95. If the roots of $(z - 1)^{25} = 2\omega^2(z + 1)^{25}$ (where ω is a complex cube root of unity) are plotted in the argand plane, they lie on

(A) a straight line (B) a circle
(C) an ellipse (D) None of these

96. If z_1 and z_2 are the two complex roots of equal magnitude and their arguments differ by $\frac{\pi}{2}$, of the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ then a (in terms of b and c) is

(A) $\frac{b^2}{2c}$ (B) $\frac{b^2}{c}$
(C) $\frac{b}{2c}$ (D) None of these

97. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - \sqrt{3}i}{2}$ are the vertices of a triangle which is

(A) of area zero
(B) right angled isosceles
(C) equilateral
(D) obtuse angled isosceles

98. z_1, z_2 are two non-real complex numbers such that $\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1$. Then, z_1, z_2 and the origin

(A) are collinear
(B) form right angled triangle
(C) form right angle isosceles triangle
(D) form an equilateral triangle

99. If z_1 and z_2 (± 0) are two complex numbers such that $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$, then

(A) $z_2 = ikz_1, k \in R$ (B) $z_2 = kz_1, k \in R$
(C) $z_2 = z_1$ (D) None of these

100. If a, b, c are real, $a^2 + b^2 + c^2 = 1$ and $b + ic = (1 + a)z$, then $\frac{1 + iz}{1 - iz} =$

(A) $\frac{a - ib}{1 + c}$ (B) $\frac{a + ib}{1 + c}$
(C) $\frac{a + ib}{1 - c}$ (D) $\frac{a - ib}{1 - c}$

101. If z_1, z_2 are two complex numbers and w^k, $k = 0, 1, \ldots, n - 1$ are the nth roots of unity, then $\sum_{k=0}^{n-1} |\,z_1 + z_2 w^k\,|^2$

(A) $< n\,(|z_1|^2 + |z_2|^2)$ (B) $= n\,(|z_1|^2 + |z_2|^2)$
(C) $> n\,(|z_1|^2 + |z_2|^2)$ (D) can't say

102. The equation $|z - z_1|^2 + |z - z_2|^2 = k$, $k \in R$ represents a circle if

(A) $k \geq \frac{1}{2}|\,z_1 - z_2\,|^2$ (B) $k \leq \frac{1}{2}|\,z_1 - z_2\,|^2$
(C) $k \geq \frac{1}{2}|\,z_1 + z_2\,|^2$ (D) $k \leq \frac{1}{2}|\,z_1 + z_2\,|^2$

103. $f(z)$ when divided by $z - i$ gives remainder i; when divided by $z + i$ gives remainder $i + 1$. When $f(z)$ is divided by $z^2 + 1$, the remainder is

(A) $\frac{i}{2}z + \left(i - \frac{1}{2}\right)$ (B) $\frac{i}{2}z - \left(i + \frac{1}{2}\right)$
(C) $\frac{i}{2}z + \left(i + \frac{1}{2}\right)$ (D) $\frac{-i}{2}z + \left(i + \frac{1}{2}\right)$

104. The value of the expression $(\omega - 1)(\omega - \omega^2)(\omega - \omega^3) \ldots (\omega - \omega^{n-1})$, where w is the nth root of unity, is

(A) $n\omega^{n-1}$ (B) $n\omega^n$
(C) $(n - 1)\,\omega^n$ (D) $(n - 1)\,\omega^{n-1}$

105. If $|z - i| = 1$ and $\arg(z) = \theta, \theta \in \left(0, \frac{\pi}{2}\right)$, then the value of $\cot\theta - \frac{2}{z}$ is equal to

(A) 0 (B) i (C) $-i$ (D) 1

106. The reflection of the complex number $\frac{4 + 3i}{1 + 2i}$ in the straight line $iz =$ is

(A) $2 + i$ (B) $2 - i$
(C) $1 + 2i$ (D) $1 - 2i$

107. If $i = \sqrt{-1}$, then

$$4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365} \text{ is equal to}$$

(A) $1 - i\sqrt{3}$ (B) $-1 + i\sqrt{3}$
(C) $i\sqrt{3}$ (D) $-\sqrt{3}\,i$

108. Let $\bar{b}\,z + b\,\bar{z} = c$, $b \neq 0$, be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflection of a point z_2 through the line, then $\bar{z}_1 b + z_2 \bar{b} =$

(A) $4c$
(B) $2c$
(C) c
(D) None of these

109. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, then $p^2 = k \cos^2 \dfrac{\alpha}{2}$, where $k =$

(A) q
(B) $2q$
(C) $4q$
(D) None of these

110. If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is

(A) equal to 1
(B) less than 1
(C) greater than 3
(D) equal to 3

111. If $|z| \leq 1$, $|\omega| \leq 1$, then $|z - \omega|^2$

(A) $\leq (|z| - |\omega|)^2 - (\text{Arg } z - \text{Arg } \omega)^2$
(B) $\leq (|z| - |\omega|)^2 + (\text{Arg } z - \text{Arg } \omega)^2$
(C) $\leq (|z| - |\omega|)^2 + 2\,(\text{Arg } z - \text{Arg } \omega)^2$
(D) None of these

112. Suppose, z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$ then z_2 and z_3 are equal to

(A) $-2, 1 - i\sqrt{3}$
(B) $2, 1 - i\sqrt{3}$
(C) $-2, 1 + i\sqrt{3}$
(D) None of these

113. If $k = \dfrac{3n}{2}$, where n is an even positive integer, then

$$\sum_{r=1}^{k} (-3)^{r-1} \cdot {}^{3n}C_{2r-1} =$$

(A) 0
(B) 1
(C) -1
(D) None of these

114. If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then a and b are

(A) $2 + \sqrt{3}, 2 - \sqrt{3}$
(B) $2 - \sqrt{3}, 2 - \sqrt{3}$
(C) $2 - \sqrt{3}, 2 + \sqrt{3}$
(D) None of these

115. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\dfrac{z_1 + z_2}{z_1 - z_2}$ may be

(A) 0
(B) real and positive
(C) real and negative
(D) purely imaginary

116. If the complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C, then

$$(z_1 - z_2)^2 = k\,(z_1 - z_3)\,(z_3 - z_2), \text{ where } k =$$

(A) 1
(B) 2
(C) 4
(D) None of these

117. If the origin and the two points represented by complex numbers A and B form vertices of an equilateral triangle, then $\dfrac{A}{B} + \dfrac{B}{A} =$

(A) 1
(B) -1
(C) 2
(D) None of these

118. If $2\sqrt{2x^4} = (\sqrt{3} - 1) + i\,(\sqrt{3} + 1)$, then

$$x = \cos \frac{1}{4}\,(2n\pi + k) + i \sin \frac{1}{4}\,(2n\pi + k);$$

$n = 0, 1, 2, 3$, where $k =$

(A) $\dfrac{\pi}{12}$
(B) $\dfrac{5\pi}{12}$
(C) $\dfrac{7\pi}{12}$
(D) None of these

119. $$\sum_{p=1}^{32} (3p + 2)\left[\sum_{q=1}^{10}\left(\sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11}\right)\right]^p =$$

(A) $8\,(1 - i)$
(B) $16\,(1 - i)$
(C) $48\,(1 - i)$
(D) None of these

120. The three vertices of a triangle are represented by the complex numbers 0, z_1 and z_2. If the triangle is equilateral, then

(A) $z_1^2 + z_2^2 + z_1 z_2 = 0$
(B) $z_1^2 + z_2^2 = z_1 z_2$
(C) $z_2^2 - z_1^2 = z_1 z_2$
(D) $z_1^2 - z_2^2 = z_1 z_2$

121. If $|z - 25i| \leq 15$, then $|$maximum amp (z) – minimum amp $(z)|$ is equal to

(A) $\sin^{-1}\left(\dfrac{3}{5}\right) - \cos^{-1}\left(\dfrac{3}{5}\right)$

(B) $\dfrac{\pi}{2} + \cos^{-1}\left(\dfrac{3}{5}\right)$

(C) $\pi - 2\cos^{-1}\left(\dfrac{3}{5}\right)$

(D) $\cos^{-1}\left(\dfrac{3}{5}\right)$

122. If $z^2 + (p + iq)\,z + r + is = 0$ where p, q, r, s are non-zero, has real roots, then

(A) $pqs = s^2 + q^2r$ (B) $pqr = r^2 + p^2s$
(C) $prs = q^2 + r^2p$ (D) $qrs = p^2 + s^2q$

123. If z_1 and z_2 are any two complex numbers, then $\left|z_1 + \sqrt{z_1^2 - z_2^2}\right| + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right|$ is equal to

(A) $|z_1 + z_2|$ (B) $|z_1|$
(C) $|z_2|$ (D) None of these

124. If $z = x + iy$ lies in IIIrd quadrant, then $\frac{\bar{z}}{z}$ also lies in the IIIrd quadrant if

(A) $y > x > 0$ (B) $y < x < 0$
(C) $x < y < 0$ (D) $x > y > 0$

125. If in an argand plane points z_1, z_2, z_3 are the vertices of an isosceles triangle right angled at z_2, then

(A) $z_1^2 + 2z_2^2 + z_3^2 = 2\,z_2\,(z_1 + z_3)$
(B) $z_1^2 + z_2^2 + z_3^2 = 2\,z_2\,(z_1 + z_3)$
(C) $z_1^2 + z_2^2 + 2z_3^2 = 2\,z_2\,(z_1 + z_3)$
(D) $2z_1^2 + z_2^2 + z_3^2 = 2\,z_2\,(z_1 + z_3)$

126. In the Argand diagram, if O, P and Q represent respectively the origin and the complex numbers z and $z + iz$, then the $\angle OPQ$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{2\pi}{3}$

127. If z satisfies $|z + 1| < |z - 2|$, and $\omega = 3z + 2 + i$, then

(A) $|\omega + 1| < |\omega - 8|$ (B) $|\omega + 1| < |\omega - 7|$
(C) $\omega + \bar{\omega} > 7$ (D) $|\omega + 5| < |\omega - 4|$

128. If $P(x)$ and $Q(x)$ are two polynomials such that $f(x) = P(x^3) + x\,Q(x^3)$ is divisible by $x^2 + x + 1$, then

(A) $P(x)$ is divisible by $(x - 1)$ but $Q(x)$ is not divisible by $x - 1$
(B) $Q(x)$ is divisible by $(x - 1)$ but $P(x)$ is not divisible by $x - 1$
(C) Both $P(x)$ and $Q(x)$ are divisible by $x - 1$
(D) $f(x)$ is divisible by $x - 1$

129. If $\alpha = \cos\left(\frac{8\pi}{11}\right) + i\sin\left(\frac{8\pi}{11}\right)$, then

Re $(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$ is equal to

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$
(C) 0 (D) None of these

130. Let p be a complex number such that $|a| < 1$ and $z_1, z_2, \ldots, z_n$ be the vertices of a polygon such that $z_k = 1 + a + a^2 + \ldots a^k$, then the vertices of the polygon lie within the circle

(A) $|z - a| = \frac{1}{|1-a|}$ (B) $|z - 1| = \frac{1}{|1-a|}$
(C) $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$ (D) None of these

131. If A, B, C are the angles of a triangle and e^{iA}, e^{iB}, e^{iC} are in A.P., then the triangle must be

(A) right angle (B) isosceles triangle
(C) equilateral (D) None of these

132. $e^{2mi\cot^{-1}p}\cdot\left(\frac{pi+1}{pi-1}\right)^m =$

(A) 0 (B) 1
(C) -1 (D) None of these

133. If z_1 and $\bar{z}_1$ represent adjacent vertices of a regular polygon of n sides and if $\frac{\text{Im}\,(z_1)}{\text{Re}\,(z_1)} = \sqrt{2} - 1$, then n is equal to

(A) 4 (B) 8
(C) 16 (D) None of these

134. If z_1, z_2, z_3 are non-zero, non-collinear complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, then the points z_1, z_2, z_3 lie

(A) in the interior of a circle
(B) on a circle passing through origin
(C) in the exterior of a circle
(D) None of these

135. If $|z - 25\,i| \le 15$, then the least positive value of arg z is

(A) $\pi - \tan^{-1}\frac{4}{3}$ (B) $\tan^{-1}\frac{4}{3}$
(C) $-\pi + \tan^{-1}\frac{4}{3}$ (D) None of these

136. If $|z - 4 + 3i| \le 2$, then the least and the greatest values of $|z|$ are

(A) 3, 7 (B) 4, 7
(C) 3, 9 (D) None of these

137. If z_1, z_2 are two complex numbers and $c > 0$ such that $|z_1 + z_2|^2 \le (1 + c)\,|z_1|^2 + k\,|z_2|^2$, then $k =$

(A) $1 - c$ (B) $c - 1$ (C) $1 + c^{-1}$ (D) $1 - c^{-1}$

138. If $|z - 4 + 3i| \le 1$ and m and n are the least and greatest values of $|z|$ and k is the least value of $\frac{x^4 + x^2 + 4}{x}$ on the interval $(0, \infty)$, then k is equal to

(A) m (B) n
(C) $m + n$ (D) None of these

139. If $n > 1$, then the roots of $z^n = (z + 1)^n$ lie on a

(A) circle
(B) straight line
(C) parabola
(D) None of these

140. Let z be a complex number satisfying $z^2 + z + 1 = 0$. If n is not a multiple of 3, then the value of $z^n + z^{2n} =$

(A) 2 (B) -2
(C) 0 (D) -1

141. If 1, α_1, α_2, α_3 and α_4 be the roots of $x^5 - 1 = 0$, then

$$\frac{\omega-\alpha_1}{\omega^2-\alpha_1}\cdot\frac{\omega-\alpha_2}{\omega^2-\alpha_2}\cdot\frac{\omega-\alpha_3}{\omega^2-\alpha_3}\cdot\frac{\omega-\alpha_4}{\omega^2-\alpha_4} =$$

(A) 1 (B) ω
(C) ω^2 (D) None of these

142. If z_1 and z_2 both satisfy the relation $z + \bar{z} = 2|z - 1|$ and arg $(z_1 - z_2) = \frac{\pi}{4}$, then the imaginary part of $(z_1 + z_2)$ is

(A) 0 (B) 1
(C) 2 (D) None of these

143. All the roots of the equation $a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 3$, where $|a_i| \le 1$, $i = 1, 2, 3, 4$, lie outside the circle with centre origin and radius

(A) 1 (B) $\frac{1}{3}$
(C) $\frac{2}{3}$ (D) None of these

144. If $\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega} =$, where a, b, c are real and ω is a non-real cube root of unity, then

(A) $a + b + c + d = -2abcd$
(B) $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d} = 2$
(C) $\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}+\frac{1}{d+\omega^2} = -\frac{2}{\omega^2}$
(D) $abc + bcd + abd + acd = 4$

145. If $z_1 + z_2 + z_3 = A$, $z_1 + z_2\,\omega + z_3\,\omega^2 = B$ and $z_1 + z_2\,\omega^2 + z_3\,\omega = C$, where 1, ω, ω^2 are the three cube roots of unity, then $|A|^2 + |B|^2 + |C|^2 =$

(A) $3\,(|z_1|^2 + |z_2|^2 + |z_3|^2)$
(B) $2\,(|z_1|^2 + |z_2|^2 + |z_3|^2)$
(C) $(|z_1|^2 + |z_2|^2 + |z_3|^2)$
(D) None of these

146. If α, β are the roots of $z + \frac{1}{z} = 2(\cos\theta + \sin\theta)$ Then,

(A) $|\alpha - i| > |\beta - i|$ (B) $|\alpha - i| < |\beta - i|$
(C) $|\alpha - i| = |i - \beta|$ (D) $|\alpha - i| = |\beta - i|$

147. If at least one value of the complex number $z = x + iy$ satisfies the condition $\left|z + \sqrt{2}\right| = a^2 - 3a + 2$ and the inequality $\left|z + i\sqrt{2}\right| < a^2$, then

(A) $a > 2$ (B) $a = 2$
(C) $a < 2$ (D) None of these

148. If α is the nth root of unity, then $1 + 2\alpha + 3\alpha^2 + \ldots$ to n terms is equal to

(A) $-\frac{n}{(1-\alpha)^2}$ (B) $-\frac{n}{1-\alpha}$
(C) $-\frac{2n}{1-\alpha}$ (D) $-\frac{2n}{(1-\alpha)^2}$

149. Let O, A, B be three collinear points such that $OA \cdot OB = 1$. If O and B represent the complex numbers o and z, then A represents

(A) $\frac{1}{\bar{z}}$ (B) $\frac{1}{z}$
(C) $\bar{z}$ (D) z^2

150. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1 + i$ and $2 - i$, respectively, then A represents the complex number

(A) $3 - \frac{i}{2}$ or $3 + \frac{i}{2}$ (B) $3 + \frac{i}{2}$ or $1 + \frac{3}{2}i$
(C) $3 - i$ or $1 - 3i$ (D) None of these

151. The locus represented by the complex equation $|z - 2 - i| = |z|\sin\left(\frac{\pi}{4} - \arg z\right)$ is the part of

(A) a pair of straight lines
(B) a circle
(C) a parabola
(D) a rectangular hyperbola

152. If z_1, z_2, z_3 are three points lying on the circle $|z| = 2$, then the minimum value of $|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2$ is equal to

(A) 6 (B) 12
(C) 15 (D) 24

More than One Option Correct Type

153. The centre of a regular polygon of n sides is located at the point $z = 0$, and one of its vertex z_1 is known. If z_2 be the vertex adjacent to z_1, then z_2 is equal to

(A) $z_1\left(\cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}\right)$

(B) $z_1\left(\cos\dfrac{\pi}{n} + i\sin\dfrac{\pi}{n}\right)$

(C) $z_1\left(\cos\dfrac{2\pi}{n} - i\sin\dfrac{2\pi}{n}\right)$

(D) $z_1\left(\cos\dfrac{\pi}{n} - i\sin\dfrac{\pi}{n}\right)$

154. $\sqrt{i} - \sqrt{-i}$ is equal to

(A) $i\sqrt{2}$ (B) $\dfrac{1}{i\sqrt{2}}$ (C) 0 (D) $-\,i\sqrt{2}$

155. If z_1, z_2, z_3, z_4 are the four complex numbers represented by the vertices of a quadrilateral taken in order such that $z_1 - z_4 = z_2 - z_3$ and amp $\dfrac{z_4 - z_1}{z_2 - z_1} = \dfrac{\pi}{2}$ then the quadrilateral is a

(A) square

(B) rhombus

(C) rectangle

(D) a cyclic quadrilateral

156. The sum

$$\sum_{m=1}^{4n+1}\left[\sum_{k=1^m}^{m+1}\left\{\sin\left(\frac{2\pi k}{m}\right) - i\cos\left(\frac{2\pi k}{m}\right)\right\}\right]^m \text{ is}$$

(A) independent of n

(B) purely imaginary

(C) purely real

(D) a root of $x^{4n+1} + 1 = 0$

157. $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\operatorname{Re}(z_1\bar{z}_2) = 0$. If $w_1 = a + ic$ and $w_2 = b + id$ $(a, b, c, d \in R)$, then

(A) $|w_1| = 1$ (B) $|w_2| = 1$

(C) $\operatorname{Re}(w_1\bar{w}_2) = 0$ (D) $\operatorname{Re}(w_1\bar{w}_2) = 1$

158. If arg $(z^{3/8}) = \dfrac{1}{2}$ arg $(z^2 + \bar{z}\, z^{1/2})$, then

(A) $|z| = 1$ (B) $z =$

(C) $Re\,(z) = 0$ (D) $Im\,(z) = 0$

159. If $z_1^{\,2} + 2z_2^{\,2} + z_3^{\,2} = 2z_2(z_1 + z_3)$, where z_1, z_2, z_3 are the vertices of a triangle, then the triangle is

(A) isosceles (B) right angled

(C) equilateral (D) obtuse angled

160. If $|z_1 - z_2| = |z_1| + |z_2|$, then

(A) $\arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{\pi}{2}$

(B) $\arg\left(\dfrac{z_1}{z_2}\right) = (2n + 1)\pi,\ n \in I$

(C) $z_1\bar{z}_2 + \bar{z}_1 z_2 \le 0$

(D) $z_1 = l\, z_2,\ l \in R$

161. If $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers such that $|z_1| = |z_2| = 1$ and Re $(z_1 \,.\, \bar{z}_2) = 0$ then for the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$

(A) Re $(\omega_1\bar{\omega}_2) = 0$ (B) Re $(\omega_1\bar{\omega}_2) = 1$

(C) $|\omega_1| = 1$ (D) None of these

162. If z_1, z_2, z_3 are the vertices of an equilateral triangle in the complex plane and z_0 is the centroid, then

(A) $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$

(B) $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$

(C) $z_1^2 + z_2^2 + z_3^2 = 3\,z_0^2$

(D) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

163. If $a, b, c, \ldots, k$ are the roots of the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0$ $(p_1, p_2, \ldots, p_n$ are real) and $(1 + a^2)(1 + b^2) \ldots (1 + k^2) = x^2 + y^2$, then

(A) $x = 1 - p_2 + p_4 \ldots$

(B) $y = p_1 - p_3 + p_5 - \ldots$

(C) $x = 1 + p_2 + p_4 + \ldots$

(D) $y = p_1 + p_3 + p_5 + \ldots.$

164. If z_1, z_2, z_3 and z_4 are the vertices of a square $PQRS$ in order, then

(A) $z_4 + z_2 = z_3 + z_1$

(B) $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_4| = |z_4 - z_1|$

(C) $|z_3 - z_1| = |z_4 - z_2|$

(D) The real part of $\dfrac{z_1 - z_3}{z_2 - z_4}$ is zero

165. If z_1, z_2, z_3 are the vertices of an isosceles triangle and right angled at z_2, then

(A) $z_1^{\,2} + z_3^{\,2} + 2z_2^{\,2} = 2\,(z_1 + z_3)\,z_2$

(B) $z_1^{\,2} + z_3^{\,2} = 2\,z_2(z_1 + z_3 - z_2)$

(C) $(z_1 - z_2)^2 + (z_2 - z_3)^2 = 0$

(D) $\dfrac{z_1 - z_2}{z_2 - z_3}$ is imaginary

166. A, B, C are the points representing the complex numbers z_1, z_2, z_3, respectively on the complex plane and the circumcentre of the triangle ABC lies at the origin. If the altitude AD of the triangle ABC meets circumcircle again at P, then P represents the complex number

(A) $-\bar{z}_1 z_2 z_3$ (B) $-\dfrac{\bar{z}_1 z_2}{\bar{z}_3}$

(C) $-\dfrac{\bar{z}_1 z_3}{\bar{z}_2}$ (D) $-\dfrac{z_2 z_3}{z_1}$

167. If the points A and B are represented by the non-zero complex numbers z_1 and z_2 on the argand plane such that $|z_1 + z_2| = |z_1 - z_2|$ and O is the origin, then

(A) orthocentre of ΔOAB lies at O

(B) circumcentre of ΔOAB is $\dfrac{z_1 + z_2}{2}$

(C) $\arg\left(\dfrac{z_1}{z_2}\right) = \pm\dfrac{\pi}{2}$

(D) ΔOAB is isosceles

168. If $f(x)$ and $g(x)$ are two polynomials such that the polynomial $h(x) = x f(x^3) + x^2 g(x^6)$ is divisible by $x^2 + x + 1$, then

(A) $f(1) = g(1)$ (B) $f(1) = -g(1)$

(C) $h(1) = 0$ (D) $h(-1) = 0$

169. If α is the fifth root of unity, then

(A) $|1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4| = 0$

(B) $|1 + \alpha + \alpha^2 + \alpha^3| = 1$

(C) $|1 + \alpha + \alpha^2| = 2\cos\dfrac{\pi}{5}$

(D) $|1 + \alpha| = 2\cos\dfrac{\pi}{10}$

Passage Based Questions

Passage 1

If n be a rational number, then

$$\cos n\theta + i \sin n\theta$$

is the value or one of the values of $(\cos\theta + i\sin\theta)^n$.

If p and q be the integers prime to each other ($q \neq 0$), then all the values of $(\cos\theta + i\sin\theta)^{p/q}$ are given by

$$\cos\left[(2k\pi + \theta)\frac{p}{q}\right] + i\sin\left[(2k\pi + \theta)\frac{p}{q}\right],$$

where $k = 0, 1, 2, \ldots q - 1$.

There are apparently two different ways of calculating the values of $(\cos\theta + i\sin\theta)^{p/q}$, (where p and q are integers prime to each other and $q \neq 0$), namely,

1. by writing $(\cos\theta + i\sin\theta)^{p/q}$ as $\{(\cos\theta + i\sin\theta)^p\}^{1/q}$,
2. by writing $(\cos\theta + i\sin\theta)^{p/q}$ as $\{(\cos\theta + i\sin\theta)^{1/q}\}^p$.

It can be easily seen that if p and q are prime to each other, the q values in each case are the same, so that each of the two ways will yield the same result.

170. The value of $\displaystyle\sum_{r=1}^{16}\left(\sin\frac{2r\pi}{17} + i\cos\frac{2r\pi}{17}\right)$ is

(A) 1 (B) -1 (C) i (D) $-i$

171. One of the values of $(a + ib)^{m/n} + (a - ib)^{m/n}$ is

(A) $2(a^2 + b^2)^{m/n}\cos\left(\dfrac{m}{n}\tan^{-1}\dfrac{b}{a}\right)$

(B) $2(a^2 + b^2)^{m/2n}\cos\left(\dfrac{m}{n}\tan^{-1}\dfrac{b}{a}\right)$

(C) $(a^2 + b^2)^{m/2n}\cos\left(\dfrac{m}{n}\tan^{-1}\dfrac{b}{a}\right)$

(D) None of these

172. The values of $(16)^{1/4}$ are

(A) $\pm 2, \pm 2i$ (B) $\pm 4, \pm 4i$

(C) $\pm 1, \pm i$ (D) None of these

173. The roots of the equation $z^4 + 1 = 0$ are

(A) $(\pm 1 \pm i)$ (B) $(\pm 2 \pm 2i)$

(C) $\dfrac{1}{\sqrt{2}}(\pm 1 \pm i)$ (D) None of these

Passage 2

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right],$$

where $k = 0, 1, 2, 3, \ldots (n - 1)$, gives n, nth roots of the complex number z.

In particular, since $1 = \cos 0 + i \sin 0$, therefore

$$(1)^{1/n} = (\cos 0 + i\sin 0)^{1/n}$$

$$= \cos\frac{2\pi r + 0}{n} + i\sin\frac{2\pi r + 0}{n};\ r = 0, 1, 2, \ldots, (n - 1)$$

$$= \cos\frac{2\pi r}{n} + i\sin\frac{2\pi r}{n};\ r = 0, 1, 2, \ldots, (n - 1)$$

$$= e^{i\frac{2r\pi}{n}};\ r = 0, 1, 2, \ldots, (n - 1)$$

$$= 1, e^{(i 2\pi/n)}, e^{(i 4\pi/n)}, \ldots, e^{(i 2(n-1)\pi/n)}$$

$$= 1, \alpha, \alpha^2, \alpha^3, \ldots \alpha^{n-1}, \text{ where } \alpha = e^{(i 2\pi/n)}$$

These are the n, nth roots of unity.

Clearly,

1. $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$
2. $1 \cdot \alpha \cdot \alpha \ldots \alpha^{n-1} = (-1)^{n-1}$
3. nth roots of unity form a G.P. with common ratio $e^{(i2\pi/n)}$.

174. If $1, \omega, \omega^2, \ldots \omega^{n-1}$ are the n, nth roots of unity and z_1 and z_2 are any two complex numbers, then

$$\sum_{k=0}^{n-1} |z_1 + \omega^k z_2|^2 =$$

(A) $n\,[|z_1|^2 + |z_2|^2]$
(B) $(n-1)\,[|z_1|^2 + |z_2|^2]$
(C) $(n+1)\,[|z_1|^2 + |z_2|^2]$
(D) None of these

175. If $1, a_1, a_2, \ldots, a_{n-1}$ are the n, nth roots of unity, then $(1 - a_1)(1 - a_2)(1 - a_3) \ldots (1 - a_{n-1}) =$
(A) $n + 1$
(B) n
(C) $n - 1$
(D) None of these.

176. If $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are the n nth roots of unity then $\sum_{i=1}^{n-1} \dfrac{1}{2 - \alpha^i}$ is equal to

(A) $\dfrac{(n-2)2^{n-1} + 1}{2^n - 1}$ (B) $(n-2) \times 2^n$

(C) $\dfrac{(n-2) \cdot 2^{n-1}}{2^n - 1}$ (D) None of these

177. If $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the n, nth roots of unity, then $(1 - \omega)(1 - \omega)^2 \ldots (1 - \omega^{n-1})$ is equal to
(A) 0 (B) 1
(C) n (D) n^2

Passage 3

Solution of Equations

Certain types of algebraic equations can be solved with the help of De'Moivre's theorem

Equations of the type $pz^n + q = 0$:

If $pz^n + q = 0$, where p and q are complex numbers, and $p \neq 0$, then

$$z^n = -q/p$$

The roots of the given equation are, therefore, the n values of $(-q/p)^{1/n}$. For example, consider the equation $z^7 + 1 = 0$. $z^7 + 1 = 0 \Rightarrow z^7 = -1 = \text{cis}\,(2p + 1)\pi$, where p is an integer.

Therefore, $z = \text{cis}\,[(2p + 1)\pi/7]$, $p = 0, 1, \ldots, 6$

On putting $p = 0, 1, 2, 3, 4, 5, 6$, the roots are seen to be $\cos(\pi/7) \pm i \sin(\pi/7)$, $\cos(3\pi/7) \pm i \sin(3\pi/7)$, $\cos(5\pi/7) \pm i \sin(5\pi/7)$, -1.

Equations of the type $pz^{2n} + qz^n + r = 0$, where p, q and r are complex numbers and $p \neq 0$.

$$z^n = \frac{-q \pm \sqrt{q^2 - 4pr}}{2p}$$

Denoting these values of z^n by α and β, we have two equations $z^n = \alpha$ and $z^n = \beta$, each of which can be solved by the method given in the above example.

Equations of the type $a\,(pz + q)^n + b\,(rz + s)^n = 0$:

The substitution $\dfrac{pz + q}{rz + s} = w$ reduces the given equation to the form

$$aw^n + b = 0 \qquad \text{(i)}$$

which can be solved by the method given above. If w_k be a root of the equation (i), the corresponding root z_k of the given equation is obtained by solving the equation

$$\frac{pz_k + q}{rz_k + s} = w_k.$$

178. The roots of the equation $x^6 + x^3 + 1 = 0$ are $\cos\left(\dfrac{p\pi}{9}\right) \pm i \sin\left(\dfrac{p\pi}{9}\right)$, where $p =$

(A) 2 (B) 8 (C) 14 (D) 20

179. The roots of the equation $z^4 + 4z^2 + 16 = 0$ are
(A) $\pm 1 + i\sqrt{3}$ (B) $\pm 1 - i\sqrt{3}$
(C) $\pm 2 + 2\sqrt{3}\,i$ (D) $\pm 2 - 2\sqrt{3}\,i$

180. The roots of the equation $(2 + z)^6 + (2 - z)^6 = 0$ are
(A) $\pm 2i \tan \pi/12$
(B) $\pm 2i \tan 5\pi/12$
(C) $\pm 2i$
(D) ± 2

181. The roots of the equation $z^4 - z^3 + z^2 - z + 1 = 0$ are $\cos\left(\dfrac{p\pi}{5}\right) + i \sin\left(\dfrac{p\pi}{5}\right)$ where $p =$
(A) 1, 3, 5, 7, 9
(B) 1, 3, 7, 9
(C) 3, 5, 7, 9
(D) None of these

Match the Column Type

182.

Column-I	Column-II
(I) If $z_r = \cos\left(\frac{\pi}{3^r}\right) + i\sin\left(\frac{\pi}{3^r}\right)$ $r = 1, 2, 3, \ldots$, then $z_1z_2z_3 \ldots \infty =$	(A) $i - 1$
(II) If $iz^3 + z^2 - z + i = 0$, then $\lvert z\rvert =$	(B) 1
(III) If $\frac{z-2}{z+2}$ $(z \neq -2)$ is purely imaginary, then $\lvert z\rvert =$	(C) 2
(IV) The value of the sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals	(D) $i - 1$

183. If 1, ω, ω^2 be the three cube roots of unity, then

Column-I	Column-II
(I) $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots$ to $2n$ factors	(A) $\sqrt{3}$
(II) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \ldots$ to $2n$ factors =	(B) 1
(III) If $\left(\sqrt{3} + i\right)^{100} = 2^{99}(a + ib)$, then $b =$	(C) 2^{2n}
(IV) $1(2 - \omega)(2 - \omega^2) + 2(3 - \omega)(3 - \omega^2) + \ldots + (n - 1)(n - \omega)(n - \omega^2) =$	(D) $\frac{n^2(n+1)^2}{4} - n$

184.

Column-I	Column-II
(I) $\left(\frac{\sqrt{3}+i}{2}\right)^6 + \left(\frac{i-\sqrt{3}}{2}\right)^6$	(A) $-\frac{7}{2}$
(II) If $\frac{z-1}{z+1}$ is purely imaginary, then $\lvert z\rvert =$	(B) 0
(III) $\left(i + \sqrt{3}\right)^{100} + \left(i - \sqrt{3}\right)^{100} + 2^{100} =$	(C) –2
(IV) Let $z_k = (k = 0, 1, 2, \ldots 6)$ be the roots of the equation $(z + 1)^7 + z^7 = 0$, then $\sum_{k=0}^{6} \text{Re}(z_k)$ is equal to	(D) 1

Assertion-Reason Type

Instructions: *In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:*

(A) Assertion(A) is True and Reason(R) is True; Reason(R) **is** a correct explanation for Assertion(A)
(B) Assertion(A) is True, Reason(R) is True; Reason(R) **is NOT** a correct explanation for Assertion(A)
(C) Assertion(A) is True, Reason(R) is False
(D) Assertion(A) is False, Reason(R) is True

185. **Assertion:** If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$ and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = -1$, then $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -1$

Reason: $(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2) = \cos(\alpha_1 + \alpha_2) + i\sin(\alpha_1 + \alpha_2)$

186. **Assertion:** The locus of the point z satisfying the condition $\arg\frac{z-1}{z+1} = \frac{\pi}{3}$ is a parabola

Reason: $\text{Arg}\,\frac{z_1}{z_2} = \text{Arg}\,z_1 - \text{Arg}\,z_2$

187. **Assertion:** If the area of the triangle on the argand plane formed by the complex numbers $-z$, iz, $z - iz$ is 600 square units, then $\lvert z\rvert = 20$

Reason: Area of the triangle on the argand plane formed by the complex numbers $-z$, iz, $z - iz$ is $\frac{3}{2}\lvert z^2\rvert$.

188. **Assertion:** If $\lvert z - 1\rvert + \lvert z + 3\rvert \leq 8$, then the range of values of $\lvert z - 4\rvert$ is [1, 9]

Reason: $\lvert z - 1\rvert + \lvert z + 3\rvert \leq 8 \Rightarrow z$ lies inside or on the ellipse whose foci are (1, 0) and (–3, 0) and vertices are (–5, 0) and (3, 0).

189. **Assertion:** The greatest value of the moduli of complex numbers z satisfying the equation $\left|z - \frac{4}{z}\right| = 2$ is $\sqrt{5}+1$

Reason: For any two complex numbers z_1 and z_2, $|z_1 - z_2| \geq |z_1| - |z_2|$

190. **Assertion:** The locus of the points representing the complex numbers satisfying $|z| - 2 = 0$, $|z - i| - |z + 5i| = 0$ is the single point $(0, -2)$

Reason: If z is a variable point and z_1, z_2 are two fixed points in the argand plane, then $|z - z_1| = |z - z_2| \Rightarrow$ locus of z is the perpendicular bisector of the line segment joining z_1 and z_2.

191. **Assertion:** If $z_0 = \frac{1}{2}(1+i)$, then $P_n(z) = (1 + z_0)(1 + z_0^{2})(1 + z_0^{22}) \ldots (1 + z_0^{2n}) = (1+i)\left(1 - \frac{1}{2^{2^n}}\right)$, where $n > 1$ is a positive integer.

Reason: $P_n(z) = \dfrac{1 - z_0^{2^{n+1}}}{1 - z_0}$

192. **Assertion:** If amp . $[z_1(z_3 - z_2)] =$ amp . $[z_3(z_2 - z_1)]$, then $0, z_1, z_2, z_3$ are concyclic.

Reason: For four concyclic points z_1, z_2, z_3, z_4, $\dfrac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$ is purely real.

Previous Year's Questions

193. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ equals: **[2002]**

(A) $128\,\omega$
(B) $-128\,\omega$
(C) $128\,\omega^2$
(D) $-128\,\omega^2$

194. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further, assume that the origin, z_1 and z_2 form an equilateral triangle, then **[2003]**

(A) $a^2 = b$
(B) $a^2 = 2b$
(C) $a^2 = 3b$
(D) $a^2 = 4b$

195. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$, and Arg (z) – Arg $(\omega) = \frac{\pi}{2}$, then $\bar{Z}\omega$ is equal to **[2003]**

(A) 1
(B) –1
(C) i
(D) $-i$

196. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then **[2003]**

(A) $x = 4n$, where n is any positive integer
(B) $x = 2n$, where n is any positive integer
(C) $x = 4n + 1$, where n is any positive integer
(D) $x = 2n + 1$, where n is any positive integer

197. Let z, w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and arg $zw = \pi$. Then arg z equals **[2004]**

(A) $\frac{\pi}{4}$
(B) $\frac{5\pi}{4}$
(C) $\frac{3\pi}{4}$
(D) $\frac{\pi}{2}$

198. If $z = x - iy$ and $z^{\frac{1}{3}} = p + iq$, then $\dfrac{\left(\frac{x}{p} + \frac{y}{q}\right)}{(p^2 + q^2)}$ is equal to **[2004]**

(A) 1
(B) –2
(C) 2
(D) –1

199. If $|z^2 - 1| = |z|^2 + 1$, then z lies on **[2004]**

(A) the real axis
(B) an ellipse
(C) a circle
(D) the imaginary axis.

200. If the cubes roots of unity are $1, \omega, \omega^2$ then the roots of the equation **[2005]**
$(x - 1)^3 + 8 = 0$, are

(A) $-1, -1 + 2\omega, -1 - \omega^2$
(B) $-1, -1, -1$
(C) $-1, 1 - 2\omega, 1 - 2\omega^2$
(D) $-1, 1 + 2\omega, 1 + 2\omega^2$

201. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$ then $\arg z_1 - \arg z_2$ is equal to **[2005]**

(A) $\frac{\pi}{2}$
(B) $-\pi$
(C) 0
(D) $-\frac{\pi}{2}$

202. If $w = \dfrac{z}{z - \frac{1}{3}i}$ and $|w| = 1$, then z lies on **[2005]**

(A) an ellipse
(B) a circle
(C) a straight line
(D) a parabola

203. The value of $\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11}+i\cos\frac{2k\pi}{11}\right)$ is **[2006]**

(A) i (B) 1
(C) –1 (D) $-i$

204. If $z^2 + z + 1 = 0$, where z is a complex number, then the value of

$$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+\ldots+\left(z^6+\frac{1}{z^6}\right)^2$$

is **[2006]**

(A) 18 (B) 54
(C) 6 (D) 12

205. If $|z + 4| \le 3$, then the maximum value of $|z + 1|$ is **[2007]**

(A) 4 (B) 10
(C) 6 (D) 0

206. The conjugate of a complex number is $\frac{1}{i-1}$ Then the complex number is **[2008]**

(A) $\frac{-1}{i-1}$ (B) $\frac{1}{i+1}$
(C) $\frac{-1}{i+1}$ (D) $\frac{1}{i+1}$

***207.** If $\left|Z-\frac{4}{Z}\right|=2$, then the maximum value of $|Z|$ is equal to **[2009]**

(A) $\sqrt{3}+1$ (B) $\sqrt{5}+1$
(C) 2 (D) $2+\sqrt{2}$

208. The number of complex numbers z such that $|z-1| = |z+1| = |z-i|$ equals [2010]

(A) 1 (B) 2
(C) ∞ (D) 0

209. Let α, β be real numbers and z a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line $\text{Re}(z) = 1$, then it is necessary that **[2011]**

(A) $\beta \in (-1, 0)$ (B) $|\beta| = 1$
(C) $\beta \in (1, \infty)$ (D) $\beta \in (0, 1)$

210. If $\omega(\ne 1)$ is a cube root of unity, and $(1 + \omega)^7 = A + B\omega$. Then (A, B) equals **[2011]**

(A) (1, 1) (B) (1, 0)
(C) (–1, 1) (D) (0, 1)

211. If $z \ne 1$ and $\frac{z^2}{z-1}$ is real, then the point which is represented by the complex number z lies **[2012]**

(A) either on the real axis or on a circle passing through the origin
(B) on a circle with centre at the origin
(C) either on the real axis or on a circle not passing through the origin
(D) on the imaginary axis

212. If z is a complex number of unit modulus and argument θ, then $\left(\frac{1+z}{1+\bar{z}}\right)$ equals **[2013]**

(A) $\frac{\pi}{2}-\theta$ (B) θ
(C) $\pi - \theta$ (D) $-\theta$

213. If z is a complex number such that $|z| \ge 2$, then the minimum value of $\left|z+\frac{1}{2}\right|$ **[2014]**

(A) is equal to $\frac{5}{2}$
(B) lies in the interval (1, 2)
(C) is strictly greater than $\frac{5}{2}$
(D) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

214. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a **[2015]**

(A) straight line parallel to y-axis.
(B) circle of radius 2.
(C) circle of radius $\sqrt{2}$.
(D) straight line parallel to x-axis.

215. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is **[2016]**

(a) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (B) $\frac{\pi}{3}$
(c) $\frac{\pi}{6}$ (D) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

ANSWER KEYS

Single Option Correct Type

1. (A) **2.** (D) **3.** (A, C) **4.** (D) **5.** (B) **6.** (B) **7.** (D) **8.** (C) **9.** (A) **10.** (C)
11. (D) **12.** (D) **13.** (C) **14.** (B) **15.** (B) **16.** (C) **17.** (B) **18.** (C) **19.** (D) **20.** (B)
21. (B) **22.** (C) **23.** (B) **24.** (B) **25.** (B) **26.** (C) **27.** (C) **28.** (A) **29.** (C) **30.** (A)
31. (C) **32.** (B) **33.** (B) **34.** (C) **35.** (C) **36.** (A) **37.** (C) **38.** (C) **39.** (B) **40.** (C)
41. (C, D) **42.** (A) **43.** (A) **44.** (C) **45.** (B) **46.** (C) **47.** (C) **48.** (A) **49.** (C)
50. (C) **51.** (B) **52.** (B) **53.** (B) **54.** (A, C) **55.** (B) **56.** (C) **57.** (A) **58.** (A) **59.** (B)
60. (A) **61.** (C) **62.** (B) **63.** (A) **64.** (C) **65.** (A) **66.** (C) **67.** (A) **68.** (C) **69.** (C)
70. (B) **71.** (D) **72.** (A) **73.** (C) **74.** (B) **75.** (A) **76.** (B) **77.** (C) **78.** (B) **79.** (C)
80. (C) **81.** (B) **82.** (B) **83.** (B) **84.** (B) **85.** (A) **86.** (A) **87.** (B) **88.** (C) **89.** (C)
90. (A) **91.** (C) **92.** (C) **93.** (B) **94.** (B) **95.** (B) **96.** (A) **97.** (C) **98.** (D) **99.** (A)
100. (B) **101.** (B) **102.** (A) **103.** (C) **104.** (A) **105.** (B) **106.** (D) **107.** (C) **108.** (C) **109.** (C)
110. (A) **111.** (B) **112.** (A) **113.** (A) **114.** (B) **115.** (D) **116.** (B) **117.** (A) **118.** (B) **119.** (C)
120. (B) **121.** (C) **122.** (A) **123.** (D) **124.** (C) **125.** (B) **126.** (C) **127.** (A) **128.** (D) **129.** (B)
130. (C) **131.** (C) **132.** (B) **133.** (B) **134.** (B) **135.** (B) **136.** (A) **137.** (C) **138.** (B) **139.** (B)
140. (D) **141.** (B) **142.** (C) **143.** (C) **144.** (B) **145.** (A) **146.** (D) **147.** (A) **148.** (B) **149.** (A)
150. (A) **151.** (C) **152.** (B)

More than One Option Correct Type

153. (A) and (C) **154.** (A) and (D) **155.** (C) and (D) **156.** (A) and (B) **157.** (A), (B) and (C)
158. (A), (B) and (D) **159.** (A) and (B) **160.** (B) and (C) **161.** (A) and (C)
162. (A), (B), (C) and (D) **163.** (A) and (B) **164.** (A), (B), (C) and (D)
165. (A), (B), (C) and (D) **166.** (B), (C) and (D) **167.** (A), (B) and (C) **168.** (A), (B) and (C)
169. (A), (B) and (C)

Passage Based Questions

170. (D) **171.** (B) **172.** (A) **173.** (C) **174.** (A) **175.** (B) **176.** (A) **177.** (C) **178.** (A), (B) and (C)
179. (A) and (B) **180.** (A), (B) and (C) **181.** (B)

Match the Column Type

182. (I) → (D); (II) → (B); (III) → (C); (IV) → (A)
183. (I) → (B); (II) → (C); (III) → (A); (IV) → (D)
184. (I) → (C); (II) → (D); (III) → (B); (IV) → (A)

Assertion-Reason Type

185. (A) **186.** (D) **187.** (A) **188.** (A) **189.** (A) **190.** (A) **191.** (A) **192.** (A)

Previous Year's Questions

193. (D) **194.** (C) **195.** (D) or (C) **196.** (A) **197.** (C) **198.** (B) **199.** (D) **200.** (C) **201.** (C)
202. (C) **203.** (D) **204.** (D) **205.** (C) **206.** (C) **207.** (B) **208.** (A) **209.** (C) **210.** (A)
211. (A) **212.** (B) **213.** (B) **214.** (B) **215.** (A)

HINTS AND SOLUTIONS

Single Option Correct Type

1. We have, $\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1+i$

$\Rightarrow \quad \left(\frac{p}{a}+\frac{q}{b}+\frac{r}{c}\right)^2 = 1-1+2i=2i$

$\Rightarrow \quad \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}+2\left(\frac{qr}{bc}+\frac{rp}{ca}+\frac{pq}{ab}\right)=2i$

$\Rightarrow \quad \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}+\frac{2pqr}{abc}\left(\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)=2i$

$\Rightarrow \quad \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=2i$

The correct option is (A)

2. $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate of each other

if $\quad \sin x+i\cos 2x=\overline{\cos x-i\sin 2x}$

$\Rightarrow \quad \sin x+i\cos 2x=\cos x+i\sin 2x$

$\Rightarrow \quad \sin x=\cos x$ and $\cos 2x=\sin 2x$

$\Rightarrow \quad \tan x=1$ and $\tan 2x=1$,

which is not possible for any value of x.

The correct option is (D)

3. Since $|z_1+z_2|=|z_1|+|z_2|$,

$\therefore \quad P, O, Q$ where P is the affix of z_1, O is the affix of origin and Q the affix of z_2 must lie in the same straight line. Thus, $\arg z_1-\arg z_2=\pm\pi$.

The correct option is (A) and (C)

4. Let $z=x+iy$, then

$z^2+|z|^2=0 \Rightarrow (x+iy)^2+|x+iy|^2=0$

$\Rightarrow \quad x^2-y^2+2ixy+x^2+y^2=0$

$\Rightarrow \quad 2x^2+2ixy=0$

$\Rightarrow \quad 2x^2=0$ and $2xy=0$

$\Rightarrow \quad x=0$ and $xy=0$

Clearly y can be any real number. Hence, we will get infinitely many solutions.

The correct option is (D)

5. As ω is the nth root of unity so, $\omega^n-1=0$

$\Rightarrow \quad (\omega-1)(1+\omega+\omega^2+,\ldots,+\omega^{n-1})=0$

Hence, $1+\omega+\omega^2+,\ldots,+\omega^{n-1}=0$

or $\quad \omega-1=0$ i.e., $\omega=1$

The correct option is (B)

6. Since $z+\sqrt{2}\;|z+1|+i=0$

$\therefore \quad x+i(y+1)+\sqrt{2}\;|x+iy+1|=0$

$\therefore \quad y+1=0 \qquad (\because |x+iy+1|$ is real)

$\therefore \quad y=-1$

$\therefore \quad x+\sqrt{2}\;|x-i+1|=0$

$\Rightarrow \quad x^2=2[(x+1)^2+1]=2(x^2+2x+2)$

$\therefore \quad x^2+4x+4=0$

$\Rightarrow \quad (x+2)^2=0$

or $\quad x=-2$

$\therefore \quad z=-2-i$.

The correct option is (B)

7. If $\frac{z_1}{z_2}=z$, the given equation becomes

$z^2-z+1=0$ i.e., $z=-\omega$ and $-\omega^2$

or, $\quad \frac{z_1}{z_2}=-\omega \Rightarrow z_1=-z_2\omega$

$OB=|z_2-0|=|z_2|$

$OA=|z_1-0|=|-z_2\omega|=|z_2|\,|-\omega|=|z_2|$

and $AB=|z_2-z_1|=|z_2+z_2\omega|$

$\qquad =|z_2|\,|1+\omega|=|z^2|\,|-\omega^2|=|z_2|$

Thus z_1, z_2 and origin form an equilateral triangle.

The correct option is (D)

8. Let $a=r\cos\theta$ and $b=r\sin\theta$

$\therefore \quad \tan\theta=\frac{b}{a}$

Now $\frac{a-ib}{a+ib}=\frac{r(\cos\theta-i\sin\theta)}{r(\cos\theta+i\sin\theta)}=(\cos\theta-i\sin\theta)^2$

$\qquad =\cos 2\theta-i\sin 2\theta=e^{-2i}\theta$

$\therefore \quad i\log\frac{a-ib}{a+ib}=i\log e^{-2i}\theta=i(-2i\theta)=2\theta$

$\therefore \quad \tan\left[i\log\frac{a-ib}{a+ib}\right]=\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2\frac{b}{a}}{1-\frac{b^2}{a^2}}$

$\qquad =\frac{2ab}{a^2-b^2}$

The correct option is (C)

9. Since $(\sqrt{3}+i)^{100}=2^{99}(a+ib)$

$\therefore \quad (\sqrt{3}-i)^{100}=2^{99}(a-ib)$

$\therefore \quad (\sqrt{3}+i)^{100}-(\sqrt{3}-i)^{100}=2^{99}(2ib)=2^{100}(ib)$

$\Rightarrow \quad i^{100}[1-\sqrt{3}\,i]^{100}-i^{100}[-1+\sqrt{3}\,i]^{100}=2^{100}(ib)$

or $\quad (-2\omega)^{100}-(2\omega^2)^{100}=2^{100}(ib)$

or $\quad \omega-\omega^2=(ib)$ or $\sqrt{3}\;i=ib$

$\therefore \quad b=\sqrt{3}$

The correct option is (A)

10. We have,

$\frac{1-i\sin\alpha}{1+2i\sin\alpha}=\frac{(1-i\sin\alpha)(1-2i\sin\alpha)}{(1+2i\sin\alpha)(1-2i\sin\alpha)}$

$\qquad =\frac{1-3i\sin\alpha-2\sin^2\alpha}{1-4i^2\sin^2\alpha}$

$$= \frac{(1-2\sin^2\alpha)-3i\sin\alpha}{1+4^2\sin\alpha},$$

which is purely real if $\sin\alpha = 0$, i.e., if $\alpha = n\pi$, where n is an integer.

The correct option is (C)

11. By the given condition, $|z-1| < |z-i|$

$\Rightarrow \quad |x-1+iy| < |x+i(y-1)|$

$\Rightarrow \quad (x-1)^2 + y^2 < x^2 + (y-1)^2$

$\Rightarrow \quad -2x < -2y \Rightarrow x > y \Rightarrow x - y > 0$

The correct option is (B)

12. Perimeter = 6 × distance of vertex $1 + 2i$ from the centre (0, 0).

$= 6\,[(1-0) + i\,(2-0)] = 6\sqrt{5}$

The correct option is (D)

13. Let $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$z_3 = x_3 + iy_3$

$\bar{z}_2 z_3 = (x_2 - iy_2)(x_3 + iy_3)$

$\quad = (x_2x_3 + y_2y_3) + i\,(x_2y_3 - x_3y_2)$

$\Rightarrow \quad \text{Im}\,(\bar{z}_2 z_3) = x_2y_3 - x_3y_2$

$\therefore \quad z_1\,\text{Im}\,(\bar{z}_2 z_3) = (x_1 + iy_1)(x_2y_3 - x_3y_2)$

Similarly

$z_2\,\text{Im}\,(\bar{z}_3 z_1) = (x_2 + iy_2)(x_3y_1 - x_1y_3)$

$z_3\,\text{Im}\,(\bar{z}_1 z_2) = (x_3 + iy_3)(x_1y_2 - x_2y_1)$

Adding we get the result = 0

The correct option is (C)

14. $\because \dfrac{2z_1}{3z_2}$ is purely an imaginary number,

$$\therefore \quad \text{let } \frac{2z_1}{3z_2} = ib \quad \text{or} \quad \frac{z_1}{z_2} = \frac{3}{2}\,ib \qquad (1)$$

$$\therefore \quad \left|\frac{z_1 - z_2}{z_1 + z_2}\right|^4 = \left|\frac{\frac{z_1}{z_2} - 1}{\frac{z_1}{z_2} + 1}\right|^4 = \left|\frac{\frac{3}{2}ib - 1}{\frac{3}{2}ib + 1}\right|^4$$

$$= \frac{\left|\frac{3}{2}ib - 1\right|^4}{\left|\frac{3}{2}ib + 1\right|^4} = \frac{\left(\sqrt{\frac{9b^2}{4} + 1}\right)^4}{\left(\sqrt{\frac{9b^2}{4} + 1}\right)^4} = 1$$

The correct option is (B)

15. $x^6 = (4-3i)^5$

$$x^6 = 5^6\left(\frac{4}{5} - \frac{3i}{5}\right) = 5^5\,(\cos\theta + i\sin\theta)^5$$

$$\text{where} \quad \theta = -\tan^{-1}\left(\frac{3}{4}\right) = 5^5\,(\cos 5\theta + i\sin 5\theta)$$

$x = 5^{5/6}\,(\cos 5\theta + i\sin 5\theta)^{1/6}$

$$= 5^{5/6}\left[\cos\left(\frac{2k\pi + 5\theta}{6}\right) + i\sin\left(\frac{2k\pi + 5\theta}{6}\right)\right]$$

$x_1x_2 \ldots x_6 = 5^5\,[\cos(5\pi + 5\theta) + i\sin(5\pi + 5\theta)]$

$\quad = 5^5\,(-\cos 5\theta - i\sin 5\theta)$

$\quad = -5^5(\cos 5\theta + i\sin 5\theta)$

The correct option is (B)

16. Given $|z_1 + z_2| = |z_1| + |z_2|$

Squaring, $|z_1 + z_2|^2 = (|z_1| + |z_2|)^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2\text{Re}\,|z_1|\,|z_2| = |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2|$

$\Rightarrow \quad 2\text{Re}\,|z_1|\,|\bar{z}_2| = 2\,|z_1|\,|z_2|$

$\Rightarrow \quad |z_1|\,|\bar{z}_2|\cos(\theta_1 - \theta_2) = |z_1|\,|z_2|$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \quad \theta_1 - \theta_2 = 0 \quad \text{or} \quad \theta_1 = \theta_2$

or $\quad \arg z_1 = \arg z_2$

The correct option is (C)

17. For every $a \in R$, $|a| = \sqrt{a^2}$

$\therefore \quad |a|^2 = a^2$

Now $(|x| - |y|)^2 \geq 0$

$\Rightarrow \quad |x|^2 + |y|^2 - 2\,|x|\,|y| \geq 0$

$\Rightarrow \quad 2\,|x|\,|y| \leq |x|^2 + |y|^2$

$\Rightarrow \quad |x|^2 + |y|^2 + 2\,|x|\,|y| \leq 2\,|x|^2 + 2\,|y|^2$

$\Rightarrow \quad (|x| + |y|)^2 \leq 2\,(x^2 + y^2)$

$\Rightarrow \quad (|x| + |y|)^2 \leq 2\,|z|^2$

$\therefore \quad |x| + |y| \leq \sqrt{\mathbf{2}}\,|z|.$

The correct option is (B)

18. We have, $|1+i|^2 \times |1+2i|^2 \times |1+3i|^2 \ldots |1+ni|^2$

$= |\alpha + i\beta|^2$

$\Rightarrow \quad (1+1) \times (1+4) \times (1+9) \ldots (1+n^2) = \alpha^2 + \beta^2$

$\Rightarrow \quad 2 \times 5 \times 10 \ldots (1+n^2) = \alpha^2 + \beta^2.$

The correct option is (C)

19. $|z_1| = 1$, $|z_2| = 1$

$$\Rightarrow \quad a^2 + b^2 = 1 \qquad (1)$$

$$p^2 + q^2 = 1 \qquad (2)$$

$\text{Im}\,(z_1\,\overline{z_2}) = \text{Im}\,(a + ib)(p - iq) = bp - aq$

$$\therefore \quad bp - aq = 1 \qquad (3)$$

$\omega_1\overline{\omega_2} = (a + ip)(b - iq) = (ab + pq) + i\,(bp - aq)$

$\quad = (ab + pq) + i\,(1)$

$\therefore \quad \text{Im}\left(\omega_1\overline{\omega_2}\right) = 1$

The correct option is (D)

20. We have, $\sqrt[3]{a + ib} = x + iy$

$a + ib = (x + iy)^3 = (x^3 - 3xy^2) + i\,(3x^2y - y^3)$

$\therefore \quad a = x^3 - 3xy^2$ and $b = 3x^2y - y^3$

$\therefore \quad \dfrac{a}{x} = x^2 - 3y^2$ and $\dfrac{b}{y} = 3x^2 - y^2$

$\therefore \quad \dfrac{a}{x} + \dfrac{b}{y} = x^2 - 3y^2 + 3x^2 - y^2 = 4\,(x^2 - y^2)$

The correct option is (B)

21. As $(a-b)^2 \ge 0$, $a^2 + b^2 \ge 2ab$ (1)

But $|z| = \sqrt{a^2+b^2}$; so from Eq. (1), $|z|^2 \ge 2ab$

$\therefore \quad |z|^2 + a^2 + b^2 \ge a^2 + b^2 + 2ab$

or $\quad |z|^2 + |z|^2 \ge (a+b)^2$;

$\therefore \quad 2\,|z|^2 \ge (a+b)^2$

$\therefore \quad \sqrt{2}\,|z| \ge a + b$ as $|z|$ is positive.

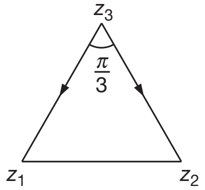
$\therefore \quad |z| \ge \dfrac{1}{\sqrt{2}}(a+b)$

The correct option is (B)

22. $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2}$

$= \cos\left(\dfrac{-\pi}{3}\right) + \sin\left(\dfrac{-\pi}{3}\right) = e^{-i\pi/3}$

$\therefore \quad \left|\dfrac{z_1 - z_3}{z_2 - z_3}\right| = \left|e^{-i\pi/3}\right| = 1$



and angle between $z_1 - z_3$ and $z_2 - z_3$ is $\dfrac{\pi}{3}$.

$\therefore$ triangle is equilateral.

The correct option is (C)

23. Putting $x = 1, \omega, \omega^2$ respectively,

$3^n = a_0 + a_1 + a_2 + \ldots + a_{2n}$

$(1 + \omega + \omega^2)^n = a_0 + a_1\,\omega + a_2\,\omega^2 + \ldots + a_{2n}\,\omega^{2n}$

$(1 + \omega^2 + \omega^4)^n = a_0 + a_1\,\omega^2 + a_2\,\omega^4 + \ldots + a_{2n}\,\omega^{4n}$

Adding these, $3^n + (1 + \omega + \omega^2)^n + (1 + \omega^2 + \omega^4)^n$

$= 3a_0 + a_1(1 + \omega + \omega^2) + a_2(1 + \omega^2 + \omega^4)$

$+ a_3(1 + \omega^3 + \omega^6) + \ldots$

$\therefore \quad 3^n + 0^n + 0^n = 3a_0 + 3a_3 + 3a_6 + \ldots$

$\therefore \quad 3^{n-1} = a_0 + a_3 + a_6 + \ldots$

The correct option is (B)

24. Let $\sqrt[n]{1} = x$; $\therefore x^n = 1$; $\therefore x^n - 1 = 0$

$\therefore \quad x^n - 1 = (x-1)(x-a_1)(x-a_2)\ldots(x-a_{n-1})$

$\therefore \quad (x-a_1)(x-a_2)(x-a_3)\ldots(x-a_{n-1})$

$= \dfrac{x^n - 1}{x-1} = \dfrac{1-x^n}{1-x} = 1 + x + x^2 +, \ldots, + x^{n-1}.$

Putting $x = 1$, we get

$(1-a_1)(1-a_2)(1-a_3)\ldots(1-a_{n-1}) = n.$

The correct option is (B)

25. The closest distance = length of the perpendicular from the origin on the line $a\bar{z} + \bar{a}z + a\bar{a} = 0$

$= \dfrac{a(0) + \bar{a}|0| + a\bar{a}}{2|a|} = \dfrac{|a|^2}{2|a|} = \dfrac{|a|}{2}$

The correct option is (B)

26. Given $|z-1| + |z+3| \le 8$

$\therefore$ z lies inside or on the ellipse whose foci are (1, 0) and (–3, 0). Vertices are (–5, 0) and (3, 0). Now, $|z-4|$ is distance of z from (4, 0).

Minimum distance is 1 and maximum is 9.

The correct option is (C)

27. We have, $z^4 + 1 = 0 \Rightarrow z^4 = -1$

$\Rightarrow \quad z = (\cos\pi + i\sin\pi)^{1/4}$

$\Rightarrow \quad z = \cos\dfrac{1}{4}(2k\pi + \pi) + i\sin\dfrac{1}{4}(2k\pi + \pi), k = 0, 1, 2, 3.$

$\Rightarrow \quad z = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}, \cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4},$

$\cos\dfrac{5\pi}{4} + i\sin\dfrac{5\pi}{4}, \cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}$

$= \dfrac{1}{\sqrt{2}}(1+i), \dfrac{1}{\sqrt{2}}(-1+i), \dfrac{1}{\sqrt{2}}(-1-i), \dfrac{1}{\sqrt{2}}(1-i).$

Hence, the four roots of $z^4 + 1 = 0$ are $\dfrac{1}{\sqrt{2}}(\pm 1 \pm i)$.

The correct option is (C)

28. Put $1 = r\cos\theta$ and $-1 = r\sin\theta$

$\Rightarrow \quad r = \sqrt{2}$ and $\theta = -\dfrac{\pi}{4}$

Then given equation takes the form

$(\sqrt{2})^n\left[\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right]^n = 2^n$

or $\quad 2^{n/2}\left[\cos\dfrac{n\pi}{4} - i\sin\dfrac{n\pi}{4}\right] = 2^n$

Equating real and imaginary parts, we get

$\cos\dfrac{n\pi}{4} = 2^{n/2}$ and $-\sin\dfrac{n\pi}{4} = 0$

These are satisfied only for $n = 0$.

Hence, $\boldsymbol{n = 0}$ is the only solution.

The correct option is (A)

29. We have, $2 = \left|z - \dfrac{4}{z}\right| \ge |z| - \left|\dfrac{4}{z}\right|$

$\Rightarrow \quad |z| - \left|\dfrac{4}{z}\right| \le 2$

$\Rightarrow \quad |z|^2 - 2\,|z| - 4 \le 0 \quad$ or $\quad (|z| - 1)^2 - 5 \le 0$

$\Rightarrow \quad (|z| - 1)^2 \le 5 \quad$ or $\quad |z| - 1 \le \Rightarrow |z| \le + 1$

Hence the greatest value of $|z|$ is $\sqrt{5} + 1$.

The correct option is (C)

30. We have, $\operatorname{Arg}(z+i) - \operatorname{Arg}(z-i) = \dfrac{\pi}{2}$

$\Rightarrow \quad \operatorname{Arg}\left(\dfrac{z+i}{z-i}\right) = \dfrac{\pi}{2}$

$\therefore \quad \operatorname{Re}\left(\dfrac{z+i}{z-i}\right) = 0$

$\Rightarrow \quad \dfrac{\left(\dfrac{z+i}{z-i}\right) + \overline{\left(\dfrac{z+i}{z-i}\right)}}{2} = 0 \Rightarrow \left(\dfrac{z+i}{z-i}\right) + \left(\dfrac{\bar{z} + \bar{i}}{\bar{z} - \bar{i}}\right) = 0$

$\Rightarrow \quad (z+i)(\bar{z}+i) + (z-i)(\bar{z}-i) = 0$

$\Rightarrow \quad z\bar{z} + i(z + \bar{z}) - 1 + z\,\bar{z} - i(z + \bar{z}) - 1 = 0$

$\Rightarrow \quad 2(z\bar{z}) = 2$

$\Rightarrow \quad z\bar{z} = 1 \quad$ or $\quad |z|^2 = 1$

$\Rightarrow \quad |z| = 1$

The equation represents a **circle** centred at origin and radius 1 unit.

The correct option is (A)

31. Let $z = x + iy$

Since $\dfrac{z^2}{z-1}$ is real = k (say), where $k \in R$

$\therefore \quad \dfrac{(x+iy)^2}{x+iy-1} = k$

$\Rightarrow \quad x^2 + 2i\,xy - y^2 = k\,(x-1) + i\,ky$

$\Rightarrow \quad x^2 - y^2 = k\,(x-1) \qquad (1)$

$2xy = ky \qquad (2)$

Eq. (2) gives either $y = 0$ or $k = 2x$

If $k = 2x$, then $x^2 - y^2 = 2x^2 - 2x$

$\Rightarrow \quad x^2 + y^2 - 2x = 0$, which is a circle.

Thus, lies either on the real axis $y = 0$ or on a circle.

The correct option is (C)

32. Clearly $\left|\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$

$$\Rightarrow \left(\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right)\left(\frac{\bar{z}_1 - 2\bar{z}_2}{2 - \bar{z}_1 z_2}\right) = 1$$

$\Rightarrow \quad z_1\bar{z}_1 - 2z_1\bar{z}_2 - 2\bar{z}_1 z_2 + 4z_2\bar{z}_2$

$= 4 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 + z_1 z_2 \bar{z}_1\bar{z}_2$

$\Rightarrow \quad |z_1|^2 + 4\,|z_2|^2 = 4 + |z_1|^2\,|z_2|^2$

$\Rightarrow \quad |z_1|^2\,[1 - |z_2|^2] = 4\,[1 - |z_2|^2]$

$\Rightarrow \quad |z_1|^2 = 4 \qquad (\because |z_2| \neq 1)$

$\Rightarrow \quad |z_1| = 2$

The correct option is (B)

33. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Now $z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$

and $z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

As $|z_1 + z_2| = |z_1 - z_2|$, we get

$(x_1 + x_2)^2 + (y_1 + y_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

or, $\quad x_1 x_2 + y_1 y_2 = 0 \qquad (1)$

$$\text{Now amp } z_1 - \text{amp } z_2 = \tan^{-1}\frac{y_1}{x_1} - \tan^{-1}\frac{y_2}{x_2}$$

$$= \tan^{-1}\frac{\dfrac{y_1}{x_1} - \dfrac{y_2}{x_2}}{1 + \dfrac{y_1}{x_1}\cdot\dfrac{y_2}{x_2}}$$

$$= \tan^{-1}\frac{x_2 y_1 - y_2 x_1}{x_1 x_2 + y_1 y_2}$$

$= \tan^{-1}\infty$, by (1)

$\therefore \quad |\text{amp } z_1 - \text{amp } z_2| = \dfrac{\pi}{2}.$

The correct option is (B)

34. We have,

$$\log_{1/2}\left(\frac{|z-1|+4}{3\,|z-1|-2}\right) > 1 = \log_{1/2}\left(\frac{1}{2}\right)$$

$\Rightarrow \quad \dfrac{|z-1|+4}{3\,|z-1|-2} < \dfrac{1}{2} < 1$

($\because \log_a x$ is a decreasing function if $a < 1$)

$\Rightarrow \quad |z-1| + 4 < 3\,|z-1| - 2$

$\Rightarrow \quad 2\,|z-1| > 6 \Rightarrow |z-1| > 3$

which is an exterior of a circle.

The correct option is (C)

35. Suppose x is a real root

Then $x^3 + ix - 1 = 0 \Rightarrow \ x^3 - 1 = 0$ and $x = 0$

There is no real number satisfying these two equations.

The correct option is (C)

36. Let α, β, γ be the roots

$\therefore \quad \alpha + \beta + \gamma = -a$

$\therefore \quad |-a| = |\alpha + \beta + \gamma| \leq |\alpha| + |\beta| + |\gamma| = 1 + 1 + 1$

$\therefore \quad |a| \leq 3$

The correct option is (A)

37. We have, $\dfrac{z_1}{z_2} + \dfrac{z_2}{z_1} = 1$

$\Rightarrow \quad z_1^2 + z_2^2 = z_1 z_2$

$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_{1,}$

where $z_3 = 0$.

$\Rightarrow \quad z_1, z_2$ and the origin form an equilateral triangle.

The correct option is (C)

38. We have, $S\,(n) = i^n + i^{-n} = i^n + \dfrac{1}{i^n} = \dfrac{i^{2n}+1}{i^n}$

$= \dfrac{(-1)^n + 1}{i^n}$, $n = 1, 2, 3, 4, \ldots$

$\therefore \quad$ values of $S\,(n)$ are $0, -2, 2, 0, -2, 2, \ldots$

$\therefore \quad$ Total number of distinct values of $S\,(n)$ is 3.

The correct option is (C)

39. We have, $|z_1 + z_2| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2}\right| = \left|\dfrac{z_1 + z_2}{z_1 z_2}\right|$

$\Rightarrow \quad |z_1 + z_2|\left(1 - \dfrac{1}{|z_1 z_2|}\right) = 0$

$\Rightarrow \quad |z_1 z_2| = 1. \qquad (\because z_1 \neq -z_2)$

The correct option is (B)

40. We have, amp $(z - 1)$ = amp $(z + 3i)$

$\Rightarrow \quad \tan^{-1}\dfrac{y}{x-1} = \tan^{-1}\dfrac{y+3}{x}$

$\Rightarrow \quad xy = (x-1)\,(y+3)$

$\Rightarrow \quad 3\,(x-1) = y$

$\therefore \quad (x-1) : y = 1 : 3.$

The correct option is (C)

41. We have, $z_1 - z_4 = z_2 - z_3$ or $\dfrac{z_1 + z_3}{2} = \dfrac{z_2 + z_4}{2}$

i.e., the diagonals bisect each other.

$\therefore \quad$ It is a parallelogram.

Also, $\operatorname{amp} \frac{z_4 - z_1}{z_2 - z_1} = \frac{\pi}{2}$

⇒ angle at z_1 is a right angle.

∴ It is a rectangle and hence a cyclic quadrilateral.

The correct option is (C) and (D)

42. Given: $|z| = 2$ and $\arg(z) = \frac{2\pi}{3}$.

∴ If $z = r(\cos\theta + i\sin\theta)$, then $r = 2$ and $\theta = \frac{2\pi}{3}$

$$\therefore \quad z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= (-1 + i\sqrt{3})$$

The correct option is (A)

43. $\log_{\sqrt{3}}\left(\frac{|z|^2 - |z| + 1}{2 + |z|}\right) < 2$

$$\Rightarrow \quad \frac{|z|^2 - |z| + 1}{2 + |z|} < (\sqrt{3})^2$$

$\Rightarrow \quad |z|^2 - |z| + 1 < 6 + 3|z|$

$\Rightarrow \quad |z|^2 - 4|z| - 5 < 0 \Rightarrow (|z| - 5)(|z| + 1) < 0$

$\Rightarrow \quad |z| - 5 < 0$, since $|z| + 1 > 0 \Rightarrow \mathbf{|z| < 5}$

The correct option is (A)

44. We have, $|z| = 1$, or $\sqrt{x^2 + y^2} = 1$

or $x^2 + y^2 - 1 = 0$.

$$\text{Now} \left(\frac{z-1}{z+1}\right) = \left[\frac{(x-1)+iy}{(x+1)+iy}\right] \times \left[\frac{(x+1)-iy}{(x+1)-iy}\right]$$

$$= \frac{(x^2 + y^2 - 1) + 2iy}{(x+1)^2 + y^2} = \frac{2iy}{(x+1)^2 + y^2}$$

$(x^2 + y^2 - 1 = 0)$

The correct option is (C)

45. Let $z_1 = x_1 + iy_1$

∴ $\bar{z}_1 = x_1 - iy_1$

Now $z_2 = \bar{z}_1 \Rightarrow z_1 + z_2 = z_1 + \bar{z}_1$

$= x_1 + iy_1 + x_1 - iy_1$

$= 2x_1$, which is real.

Hence, result holds if $\mathbf{z_2 = \bar{z}_1}$.

The correct option is (B)

46. $|z - i| = |z + 5i|$ represents equation of perpendicular bisector of points (0, 1) and (0, –5) i.e., $y = -2$, now $|z| = 2$ is $x^2 + y^2 = 4$

$\Rightarrow \quad x^2 + 4 = 4 \Rightarrow x = 0$

∴ z represents a single point (0, –2).

The correct option is (C)

47. If a complex number z is rotated through an angle 180º, then it's new position is $-z$. So, $2 - 4i$ is $-2 + 4i$ and stretching it 5/2 times means modulus 5/2 times of previous complex number

i.e., $\frac{5}{2}(-2 + 4i) = -5 + 10i$

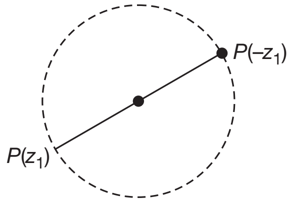
The correct option is (C)

48. Midpoint of P and P' is centre of circle C such that

$$\frac{z_1 + (-z_1)}{2} = 0$$

∴ Centre of circle lies at origin.

Now, the equation of circle with centre at origin and radius $|z_1|$ or $|-z_1|$ is



$|z - 0| = |z_1|$

$\Rightarrow \quad |z|^2 = |z_1|^2$

$\Rightarrow \quad z \cdot \bar{z} = z_1 \cdot \bar{z}_1$

$$\therefore \quad \frac{z}{z_1} = \frac{\bar{z}_1}{\bar{z}} = \overline{\left(\frac{z_1}{z}\right)}$$

The correct option is (A)

49. We have

$$(1 + i)^2 = \left(\frac{p}{a} + \frac{q}{b} + \frac{r}{c}\right)^2$$

$$\Rightarrow \quad 1 - 1 + 2i = \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} + 2\left(\frac{qr}{bc} + \frac{rp}{ca} + \frac{pq}{ab}\right)$$

$$\Rightarrow \quad 2i = \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} + \frac{2abc}{pqr}\left(\frac{a}{p} + \frac{b}{q} + \frac{c}{r}\right)$$

$$= \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} + \frac{2abc}{pqr}(0)$$

$$\Rightarrow \quad \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = 2i$$

The correct option is (C)

50. Interpretating according to Coni's Theorem. Let the angle between the lines joining

z_1, z_2 and $z_1, -z_2$ be α

$$\therefore \quad \frac{z_1 - z_2}{z_1 + z_2} = \cos\alpha + t\sin\alpha$$

Using Componendo and Dividendo, we have

$$\frac{2z_1}{-2z_2} = \frac{1 + \cos\alpha + i\sin\alpha}{\cos\alpha - 1 + i\sin\alpha}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{2\cos^2\left(\frac{\alpha}{2}\right) + i\,2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{-2\sin^2\left(\frac{\alpha}{2}\right) + i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = i\cot\frac{\alpha}{2}$$

$$\Rightarrow \quad iz_1 = -\cot\frac{\alpha}{2}z_2$$

But $iz_1 = kz_2$ $\qquad \left[\text{where, } k = -\cot\frac{\alpha}{2} \text{ (say)}\right]$

Now, $k = -\cot\frac{\alpha}{2}$

$$\Rightarrow \quad \cot\frac{\alpha}{2} = -k \Rightarrow \tan\alpha = \frac{2k}{k^2 - 1}$$

$$\Rightarrow \quad \tan\alpha = -\frac{2k}{1 - k^2}$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(-\frac{2k}{1 - k^2}\right) = -2\tan^{-1}k$$

Now, $\dfrac{z_1 - z_2}{z_1 + z_2} = \cos\alpha + i\sin\alpha$

where α is the angle between $(z_1 - z_2)$ and $(z_1 + z_2)$.

$\Rightarrow\quad \alpha = -2\tan^{-1}k$ is the angle between $(z_1 - z_2)$ and $(z_1 + z_2)$.

The correct option is (C)

51. $x^2 - \sqrt{3}x + 1 = 0$

$$\Rightarrow\quad x = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{\sqrt{3}}{2} \pm \frac{i}{2} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$$

$$\left(x^n - \frac{1}{x^n}\right)^2 = \left(x^{2n} + \frac{1}{x^{2n}} - 2\right) = -2 + 2\cos\frac{n\pi}{3}$$

$$\Rightarrow\quad \sum_{n=1}^{24}\left(x^n - \frac{1}{x^n}\right)^2 = \sum_{n=1}^{24}\left(-2 + 2\cos\frac{n\pi}{3}\right)$$

$$= -48 + 2\left[\cos\frac{\pi}{3} + \cos\frac{2\pi}{3} + \ldots + \cos\frac{24\pi}{3}\right]$$

$$= \frac{+2\left[\cos\left(\frac{2\pi}{3} + \frac{23\pi}{6}\right).\sin\left(\frac{24\pi}{6}\right)\right]}{\sin\frac{\pi}{6}} - 48$$

$= 0 - 48 = -48$

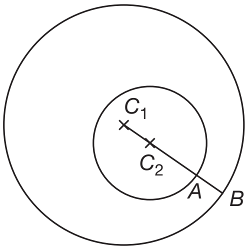
The correct option is (B)

52. $C_1(0, 0)$ is centre of bigger circle and C_2 (3, 4) is centre of smaller circle

$C_1B = r_1 = 12$ (radius of bigger circle)

$C_2A = r_2 = 5$ (radius of smaller circle)

$C_1C_2 = 5$, minimum value of $|z_1 - z_2| = AB$



Now, $C_1C_2 + C_2A + AB = C_1B$

$\Rightarrow\quad AB = 12 - 5 - 5 = 2$

The correct option is (B)

53. $\left|\sqrt{2}z_1 + i\sqrt{3}\bar{z}_2\right|^2 + \left|\sqrt{3}\bar{z}_1 + i\sqrt{2}\,z_2\right|^2$

$= \left(\sqrt{2}z_1 + i\sqrt{3}\,\bar{z}_2\right)\left(\sqrt{2}\,\bar{z}_1 - i\sqrt{3}\,z_2\right)$

$+ \left(\sqrt{3}\,\bar{z}_1 + i\sqrt{2}\,z_2\right)\left(\sqrt{3}\,z_1 - i\sqrt{2}\,\bar{z}_2\right)$

$= 5\,(|z_1|^2 + |z_2|^2) > 5 \cdot 2\sqrt{|z_1|^2|z_2|^2} = 10\,|z_1\,z_2|$

Since $A \cdot M > G \cdot M$ for $|z_1| \neq |z_2|$

The correct option is (B)

54. Let z_1, z_2, z_3 be affixes of points A, B, C respectively. Since, z_1, z_2, z_3 are in A.P., therefore

$2z_2 = z_1 + z_3$

$\Rightarrow\quad z_2 = \dfrac{z_1 + z_3}{2}$

$\Rightarrow\quad B$ is the mid point of the line AC

$\Rightarrow\quad A, B, C$ are collinear

$\therefore\quad z_1, z_2, z_3$ lie on a line

The correct option is (A) and (C)

55. If z is a root of $(z - 1)^{25} = 2\omega^2(z + 1)^{25}$, then

$$\left(\frac{z-1}{z+1}\right)^{25} = 2\omega^2 \Rightarrow \left|\frac{z-1}{z+1}\right|^{25} = 2\,|\omega^2| = 2$$

$$\Rightarrow\quad \left|\frac{z-1}{z+1}\right| = 2^{1/25}$$

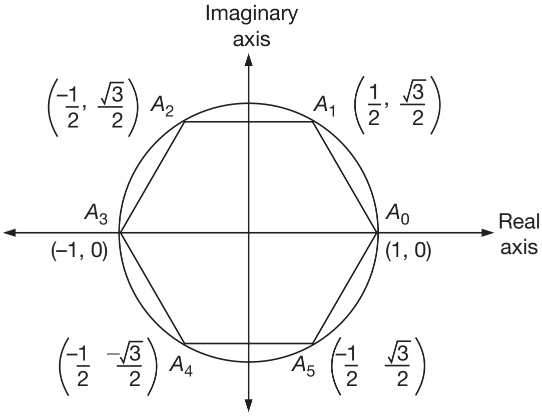
As $2^{1/25} \neq 1$, we get z lies on a circle.

The correct option is (B)

56. We have $(A_0A_1)^2 = \dfrac{1}{4} + \dfrac{3}{4} = 1$

$\Rightarrow\quad A_0A_1 = 1$

$$A_0A_1 = \left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{9+3}{4} = 3$$



$\Rightarrow\quad A_0A_2 = \sqrt{3}$

Similarly, $A_0A_4 = \sqrt{3}$

Thus, $(A_0A_1)\,(A_0A_2)\,(A_0A_4) = 3$

The correct option is (C)

57. $z_1 + z_2 = -\dfrac{b}{a}$ (1)

$z_1z_2 = \dfrac{c}{a}$ (2)

$z_2 = iz_1$ (3)

From Eq. (1) and (2)

$z_1(1 + i) = \dfrac{-b}{a}$

$$\Rightarrow\quad z_1 = \frac{-b}{2a}(1 - i) \Rightarrow z_1^2 = \frac{b^2}{4a^2}(-2i) = \frac{-b^2}{2a^2}i$$

From Eq. (2) and (3)

$$z_1^2 = \frac{c}{ai} = \frac{-c}{a}i \Rightarrow \frac{-b^2}{2a^2}\,i = \frac{c}{a}\,i \Rightarrow a = \frac{b^2}{2c}$$

The correct option is (A)

58. $z^3 + 2z^2 + 2z + 1 = 0$

$\Rightarrow \quad (z^3 + 1) + 2z\,(z + 1) = 0$

$\Rightarrow \quad (z + 1)\,(z^2 - z + 1 + 2z) = 0$

$\Rightarrow \quad (z + 1)\,(z^2 + z + 1) = 0$

$\Rightarrow \quad z = -1, \omega, \omega^2$

Now equation $z^{1985} + z^{100} + 1 = 0$ gets satisfied by w and w^2 but not by -1. So, common roots are ω and ω^2.

The correct option is (A)

59. $\dfrac{z-1}{i}$ should be real

i.e., $\dfrac{x + iy - 1}{i} = y - i\,(x - 1)$ is real

$\Rightarrow \quad x - 1 = 0$ i.e., $x = 1$

$\therefore \quad \sin^{-1} \dfrac{z-1}{i} = \sin^{-1} y$

$\therefore \quad -1 \le y \le 1$

The correct option is (B)

60. Since, $x^2 - x + 1 = 0$ (given)

$\therefore$ Solving for x, we have $x = -\omega$ and $x = -\omega^2$

Case I: $x = \omega$

$$\therefore \quad S = \sum_{n=1}^{5} \left(\omega^n + \frac{1}{\omega^n}\right)^2$$

$$\Rightarrow \quad S = \sum_{n=1}^{5} \left(\omega^n + \omega^{2n}\right)^2$$

$$\Rightarrow \quad S = (-1)^2 + (-1)^2 + 2^2 + (-1)^2 + (-1)^2 \qquad (\therefore S = 8)$$

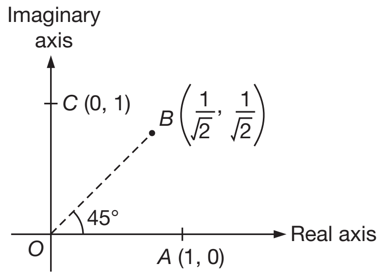
Case II: $x = \omega^2$

$$\therefore \quad S = \sum_{n=1}^{5} \left[\omega^{2n} + \left(\frac{1}{\omega^2}\right)^n\right]^2$$

$$\Rightarrow \quad S = \sum_{n=1}^{5} \left(\omega^{2n} + \omega^n\right)^2 = 8$$

The correct option is (A)

61. Points A, B and C are at equal distance from origin O and as $\angle BOA = 45°$ clearly $BC = AB$. So, triangle is isoceles.



The correct option is (C)

62. Let α be a real root of the given equation.

Then, $\alpha^2 + (a + ib)\,\alpha + c + id = 0$

$\Rightarrow \quad \alpha^2 + a\alpha + c = 0$ and $b\alpha + d = 0$

$$\Rightarrow \quad \left(-\frac{d}{b}\right)^2 + a\left(-\frac{b}{d}\right) + c = 0$$

$$\Rightarrow \quad \frac{d^2}{b^2} - \frac{ad}{b} + c = 0$$

or $\quad d^2 - abd + b^2c = 0$

The correct option is (B)

63. $|iz + z_0| = |iz - i^2 + z_0 - 1|$

$= |i\,(z - i) + 5 + 3i - 1|$

$= |i\,(z - i) + (4 + 3i)|$

$\therefore \quad |iz + z_0| \le |i\,(z - i)| + |4 + 3i|$

$\le 1.2 + 5 = 7$

$\therefore$ Maximum value of $|iz + z_0|$ is 7

The correct option is (A)

64. $z\left(\overline{z - 2i}\right) = z\bar{z} + 2iz = 2\,(2 + i)$ gives

$x^2 + y^2 - 2y = 4$ and $2x = 2$,

on equating the real and imaginary parts.

$\therefore \quad y^2 - 2y - 3 = 0$ giving $y = 3, -1$

The solutions are $1 + 3i$ and $1 - i$.

The correct option is (C)

65. Let the roots of the given equation be $1 + ip$ and $1 - ip$, where $p \in \mathbb{R}$

$\Rightarrow \quad \beta =$ product of roots

$= (1 + ip)(1 - ip) = 1 + p^2 > 1, \forall p \in \mathbb{R}$

$\Rightarrow \quad \beta \in (1, \infty)$

The correct option is (A)

66. $(1 + \omega)^7 = A + B\omega$

$\Rightarrow \quad (-\omega^2)^7 = A + B\omega$

$\Rightarrow \quad -\omega^2 = A + B\omega$

$\Rightarrow \quad 1 + \omega = A + B\omega$

$\Rightarrow \quad A = 1, B = 1$

$\Rightarrow \quad (A, B) = (1, 1)$

The correct option is (C)

67. $\dfrac{z^2}{z-1} = \dfrac{\bar{z}^2}{\bar{z}-1}$

$\Rightarrow \quad z\bar{z}z - z^2 = z\bar{z}\bar{z} - \bar{z}^2$

$\Rightarrow \quad |z|^2\,(z - \bar{z}) - (z - \bar{z})(z + \bar{z}) = 0$

$\Rightarrow \quad (z - \bar{z})[|z|^2 - (z + \bar{z})] = 0$

Either $z = \bar{z} \Rightarrow$ real axis

or $\quad |z|^2 = z + \bar{z} \Rightarrow z\bar{z} - z - \bar{z} = 0$

represents a circle passing through origin.

The correct option is (A)

68. C_1: $|z - i| = 2 \Rightarrow x^2 + (y - 1)^2 = 2^2$

C_2: $|z - 1 - 2i| = 4 \Rightarrow (x - 1)^2 + (y - 2)^2 = 4^2$

$C_1C_2 = \sqrt{2}$

$r_1 = 2, r_2 = 4$

$\Rightarrow \quad C_1C_2 < |r_1 - r_2|$

$\Rightarrow$ one circle lies inside the other. Now point (2, 2) lies outside circle C_1 and inside circle C_2.

$\Rightarrow \quad C_1$ lies inside C_2

The correct option is (C)

69. Put $-i$ in place of i

Hence $\frac{-1}{i+1}$.

The correct option is (C)

70. $|z| = \left|\left(z-\frac{4}{z}\right)+\frac{4}{z}\right| \Rightarrow |z| = \left|z-\frac{4}{z}+\frac{4}{z}\right|$

$\Rightarrow \quad |z| \le \left|z-\frac{4}{z}\right| + \frac{4}{|z|} \Rightarrow |z| \le 2+\frac{4}{|z|}$

$\Rightarrow \quad |z|^2 - 2|z| - 4 \le 0$

$[|z| - (\sqrt{5}+1)][|z| - (1-\sqrt{5})] \le 0$

$\Rightarrow \quad 1-\sqrt{5} \le |z| \le \sqrt{5}+1$

The correct option is (B)

71. We have, $z_1(z^2{}_1 - 3z^2{}_2) = 2$ (1)

and, $z_2(3z^2{}_1 - z^2{}_2) = 11$ (2)

Multiplying (2) by i and adding it to (1), we get

$z^3{}_1 - 3z^2{}_2 z_1 + i(3z^2{}_1 z_2 - z^3{}_2) = 2 + 11i$

$\Rightarrow \quad (z_1 + iz_2)^3 = 2 + 11i$ (3)

Again, multiplying (2) by i and subtracting it from (1), we get

$z^3{}_1 - 3z^2{}_2 z_1 - i(3z^2{}_1 z_2 - z^3{}_2) = 2 - 11i$

$\Rightarrow \quad (z_1 - iz_2)^3 = 2 - 11i$ (4)

Multiplying (3) and (4), we get

$(z^2{}_1 + z^2{}_2)^3 = 4 + 121 \Rightarrow z^2{}_1 + z^2{}_2 = 5.$

The correct option is (D)

72. We have, $\sqrt{1-c^2} = nc - 1$

$\Rightarrow \quad 1 - c^2 = n^2c^2 + 1 - 2nc$

$\Rightarrow \quad \frac{c}{2n} = \frac{1}{1+n^2}$ (1)

Now, $\frac{c}{2n}(1+nz)\left(1+\frac{n}{z}\right)$

$= \frac{1}{1+n^2}\left\{1+n^2+n\left(z+\frac{1}{z}\right)\right\}$

$= \frac{1}{1+n^2}\left(1+n^2+n(2\cos\theta)\right)$

$= \quad 1+\left(\frac{2n}{1+n^2}\right)\cos\theta = 1 + c\cos\theta$ (Using (1))

The correct option is (A)

73. We have, $z_k = 1 + a + a^2 + \ldots a^k = \frac{1-a^{k+1}}{1-a}$

$\Rightarrow \quad z_k - \frac{1}{1-a} = \frac{-a^{k+1}}{1-a}$

$\Rightarrow \quad \left|z_k - \frac{1}{1-a}\right| = \frac{|a|^{k+1}}{|1-a|} < \frac{1}{|1-a|}$ $(\because |a| < 1)$

Therefore, vertices $z_1, z_2, \ldots, z_n$ of the polygon lie within the circle.

$\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$

The correct option is (C)

74. We have,

$|3| = |a_1z^3 + a_2z^2 + a_3z + a_4|$

$\Rightarrow \quad 3 \le |a_1z^3| + |a_2z^2| + |a_3z| + |a_4|$

$\Rightarrow \quad 3 \le |a_1|\,|z^3| + |a_2|\,|z^2| + |a_3|\,|z| + |a_4|$

$\Rightarrow \quad 3 \le |z|^3 + |z|^2 + |z| + 1$ $(\because |a_i| \le 1)$

$\Rightarrow \quad 3 < 1 + |z| + |z|^2 + |z|^3 < 1 + |z| + |z|^2 + \ldots \infty$

$\Rightarrow \quad 3 < \frac{1}{1-|z|} \Rightarrow 1 - |z| < \frac{1}{3}$

$\therefore \quad |z| > \frac{2}{3}$

The correct option is (B)

75. We have, $z^4 = (z-1)^4$

$\Rightarrow \quad \left(\frac{z-1}{z}\right) = 1^{1/4} = e^{\frac{2n\pi i}{4}}$, $n = 0, 1, 2, 3.$

Since for all these values of z,

$\left|\frac{z-1}{z}\right| = 1$ so they lie on the line bisecting perpendicularly the join of $z = 1$ and $z = 0$.

The correct option is (A)

76. We have, $|z| = \left|z+\frac{2}{z}-\frac{2}{z}\right| \le \left|z+\frac{2}{z}\right| + \frac{2}{|z|}$.

$\Rightarrow \quad |z| \le 2 + \frac{2}{|z|} \Rightarrow |z|^2 \le 2|z| + 2$

$\Rightarrow \quad |z|^2 - 2|z| + 1 \le 1 + 2 \Rightarrow (|z| - 1)^2 \le 3$

$\Rightarrow \quad -\sqrt{3} \le |z| - 1 \le \sqrt{3} \Rightarrow 1 - \sqrt{3} \le |z| \le 1 + \sqrt{3}$

That is, the maximum value of $|z|$ is $1 + \sqrt{3}$

The correct option is (B)

77. We have, $|z + \bar{z}| + |z - \bar{z}| = 8$

$\Rightarrow \quad 2|x| + 2|y| = 8 \quad$ or $\quad |x| + |y| = 4$

The correct option is (C)

78. Since $z + \sqrt{2}\,|z+1| + i = 0$

$\therefore \quad x + i(y+1) + \sqrt{2}\,|x + iy + 1| = 0$

$\therefore \quad y + 1 = 0 \qquad [\because |x + iy + 1| \text{ is real}]$

$\therefore \quad y = -1$

$\therefore \quad x + \sqrt{2}\,|x - i + 1| = 0$

$\Rightarrow \quad x^2 = 2[(x+1)^2 + 1] = 2(x^2 + 2x + 2)$

$\therefore \quad x^2 + 4x + 4 = 0$

$\therefore \quad (x+2)^2 = 0$

$\therefore \quad x = -2$

$\therefore \quad z = -2 - i$

The correct option is (B)

79. Let $a = r\cos\theta$ and $b = r\sin\theta$

$\therefore \quad \tan\theta = \frac{b}{a}$

Now, $\dfrac{a-ib}{a+ib} = \dfrac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)} = (\cos\theta - i\sin\theta)^2$

$= \cos 2\theta - i\sin 2\theta = e^{-2i}\theta$

$\therefore \quad i\log\dfrac{a-ib}{a+ib} = i\log e^{-2i}\theta = i(-2i\theta) = 2\theta$

$\therefore \quad \tan\left[i\log\dfrac{a-ib}{a+ib}\right] = \tan 2\theta$

$$= \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\frac{b}{a}}{1-\frac{b^2}{a^2}}$$

$$= \frac{2ab}{a^2-b^2}$$

The correct option is (C)

80. Given $|z_1+z_2| = |z_1| + |z_2|$

Squaring, $|z_1+z_2|^2 = (|z_1| + |z_2|)^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2Re\,|z_1|\,|z_2|$

$\quad = |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2|$

$\Rightarrow \quad 2\text{Re}\,|z_1|\,|\bar{z}_2| = 2\,|z_1|\,|z_2|$

$\Rightarrow \quad |z_1|\,|\bar{z}_2|\cos(\theta_1-\theta_2) = |z_1|\,|z_2|$

$\Rightarrow \quad \cos(\theta_1-\theta_2) = 1 \Rightarrow \theta_1-\theta_2 = 0 \text{ or } \theta_1 = \theta_2$

or, $\quad \arg z_1 = \arg z_2$

The correct option is (C)

81. For every $a \in R$, $|a| = \sqrt{a^2}$

$\therefore \quad |a|^2 = a^2$

Now, $(|x| - |y|)^2 \geq 0$

$\Rightarrow \quad |x|^2 + |y|^2 - 2\,|x|\,|y| \geq 0$

$\Rightarrow \quad 2\,|x|\,|y| \leq |x|^2 + |y|^2$

$\Rightarrow \quad |x|^2 + |y|^2 + 2\,|x|\,|y| \leq 2\,|x|^2 + 2\,|y|^2$

$\Rightarrow \quad (|x| + |y|)^2 \leq 2(x^2+y^2) \Rightarrow (|x|+|y|)^2 \leq 2\,|z|^2$

$\therefore \quad |x| + |y| \leq \sqrt{2}\,|z|$

The correct option is (B)

82. As $(a-b)^2 \geq 0$, $a^2 + b^2 \geq 2ab$ (i)

But $|z| = \sqrt{a^2+b^2}$; so from (i), $|z|^2 \geq 2ab$

$\therefore \quad |z|^2 + a^2 + b^2 \geq a^2 + b^2 + 2ab$

or, $\quad |z|^2 + |z|^2 \geq (a+b)^2$;

$\therefore \quad 2\,|z|^2 \geq (a+b)^2$

$\therefore \quad \sqrt{2}\,|z| \geq a + b$ as $|z|$ is positive.

$\therefore \quad |z| \geq \dfrac{1}{\sqrt{2}}(a+b)$

The correct option is (B)

83. Putting $x = 1, \omega, \omega^2$, respectively,

$3^n = a_0 + a_1 + a_2 + \ldots + a_{2n}$

$(1+\omega+\omega^2)^n = a_0 + a_1\,\omega + a_2\,\omega^2 + \ldots + a_{2n}\,\omega^{2n}$

$(1+\omega^2+\omega^4)^n = a_0 + a_1\,\omega^2 + a_2\,\omega^4 + \ldots + a_{2n}\,\omega^{4n}$

Adding these, $3^n + (1+\omega+\omega^2)^n + (1+\omega^2+\omega^4)^n$

$= 3a_0 + a_1(1+\omega+\omega^2) + a_2(1+\omega^2+\omega^4)$
$+ a_3(1+\omega^3+\omega^6) + \ldots$

$\therefore \quad 3^n + 0^n + 0^n = 3a_0 + 3a_3 + 3a_6 + \ldots$

$\therefore \quad 3^{n-1} = a_0 + a_3 + a_6 + \ldots$

The correct option is (B)

84. The closest distance = length of the perpendicular from the origin on the line $a\bar{z} + \bar{a}z + a\bar{a} = 0$

$$= \frac{a(0) + \bar{a}\,|0| + a\bar{a}}{2|a|} = \frac{|a|^2}{2|a|} = \frac{|a|}{2}$$

The correct option is (B)

85. Put $1 = r\cos\theta$ and $-1 = r\sin\theta$

$\Rightarrow \quad r = \sqrt{2}$ and $\theta = -\dfrac{\pi}{4}$

Then, given equation takes the form

$$(\sqrt{2})^n\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]^n = 2^n$$

or, $\quad 2^{n/2}\left[\cos\dfrac{n\pi}{4} - i\sin\dfrac{n\pi}{4}\right] = 2^n$

Equating real and imaginary parts, we get

$\cos\dfrac{n\pi}{4} = 2^{n/2}$ and $-\sin\dfrac{n\pi}{4} = 0$

These are satisfied only for $n = 0$

Hence, $n = 0$ is the only solution.

The correct option is (A)

86. We have, $\text{Arg}(z+i) - \text{Arg}(z-i) = \dfrac{\pi}{2}$

$\Rightarrow \quad \text{Arg}\left(\dfrac{z+i}{z-i}\right) = \dfrac{\pi}{2}$

$\therefore \quad \text{Re}\left(\dfrac{z+i}{z-i}\right) = 0$

$$\Rightarrow \quad \frac{\left(\frac{z+i}{z-i}\right) + \overline{\left(\frac{z+i}{z-i}\right)}}{2} = 0$$

$$\Rightarrow \quad \left(\frac{z+i}{z-i}\right) + \left(\frac{\bar{z}+\bar{i}}{\bar{z}-\bar{i}}\right) = 0$$

$\Rightarrow \quad (z+i)(\bar{z}+i) + (z-i)(\bar{z}-i) = 0$

$\Rightarrow \quad z\bar{z} + i(z+\bar{z}) - 1 + z\bar{z} - i(z+\bar{z}) - 1 = 0$

$\Rightarrow \quad 2(z\bar{z}) = 2 \Rightarrow z = 1 \text{ or } |z|^2 = 1$

$\Rightarrow \quad |z| = 1$

The equation represents a circle centered at origin and radius 1 unit

The correct option is (A)

87. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Now $z_1 + z_2 = (x_1+x_2) + i(y_1+y_2)$

and, $z_1 - z_2 = (x_1-x_2) + i(y_1-y_2)$

As $|z_1+z_2| = |z_1-z_2|$, we get

$(x_1+x_2)^2 + (y_1+y_2)^2 = (x_1-x_2)^2 + (y_1-y_2)^2$

or, $\quad x_1x_2 + y_1y_2 = 0$ (1)

Now, amp z_1 – amp z_2 = $\tan^{-1}\dfrac{y_1}{x_1} - \tan^{-1}\dfrac{y_2}{x_2}$

$$= \tan^{-1}\frac{\dfrac{y_1}{x_1}-\dfrac{y_2}{x_2}}{1+\dfrac{y_1}{x_1}\cdot\dfrac{y_2}{x_2}} = \tan^{-1}\frac{x_2 y_1 - y_2 x_1}{x_1 x_2 + y_1 y_2}$$

$= \tan^{-1}\infty$, by (1)

$\therefore$ $|\text{amp } z_1 - \text{amp } z_2| = \dfrac{\pi}{2}$

The correct option is (B)

88. We have,

$$-\sqrt{3} > 1 = \log_{1/2}\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{|z-1|+4}{3|z-1|-2} < \frac{1}{2} < 1$$

[$\because$ $\log_a x$ is a decreasing function if $a < 1$]

$\Rightarrow$ $|z-1| + 4 < 3|z-1| - 2$

$\Rightarrow$ $2|z-1| > 6 \Rightarrow |z-1| > 3$

which is an exterior of a circle.

The correct option is (C)

89. We have, $\dfrac{z_1}{z_2} + \dfrac{z_2}{z_1} = 1$

$\Rightarrow$ $z_1^2 + z_2^2 = z_1 z_2$

$\Rightarrow$ $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1,$

where $z_3 = 0$.

$\Rightarrow$ z_1, z_2 and the origin form an equilateral triangle.

The correct option is (C)

90. Mid-point of P and P' is centre of circle C such that

$$\frac{z_1 + (-z_1)}{2} = 0$$

$\therefore$ Centre of circle lies at origin.

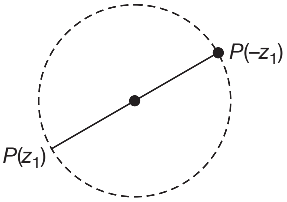
Now, the equation of circle with centre at origin and radius $|z_1|$ or $|-z_1|$ is

$|z - 0| = |z_1|$

$\Rightarrow$ $|z|^2 = |z_1|^2$

$\Rightarrow$ $z \cdot \bar{z} = z_1 \cdot \bar{z}_1$

$\therefore$ $\dfrac{z}{z_1} = \dfrac{\bar{z}_1}{\bar{z}} = \overline{\left(\dfrac{\mathbf{z_1}}{z}\right)}$



The correct option is (A)

91. We have,

$$(1+i)^2 = \left(\frac{p}{a}+\frac{q}{b}+\frac{r}{c}\right)^2$$

$$\Rightarrow 1 - 1 + 2i = \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}+2\left(\frac{qr}{bc}+\frac{rp}{ca}+\frac{pq}{ab}\right)$$

$$\Rightarrow 2i = \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}+\frac{2abc}{pqr}\left(\frac{a}{p}+\frac{q}{b}+\frac{r}{c}\right)$$

$$= \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}+\frac{2abc}{pqr}\ (0)$$

$$\Rightarrow \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2} = \mathbf{2i}$$

The correct option is (C)

92. Interpretating according to Coni's theorem. Let the angle between the lines joining

z_1, z_2 and z_1, $-z_2$ be a

$$\therefore \frac{z_1 - z_2}{z_1 + z_2} = \cos\alpha + t\sin\alpha$$

Using Componendo and Dividendo, we have

$$\frac{2z_1}{-2z_2} = \frac{1+\cos\alpha + t\sin\alpha}{\cos\alpha - 1 + t\sin\alpha}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{2\cos^2\left(\dfrac{a}{2}\right)+t\,2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}{-2\sin^2\left(\dfrac{\alpha}{2}\right)+t\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}$$

$$\Rightarrow \frac{z_1}{z_2} = t\cot\frac{\alpha}{2} \Rightarrow tz_1 = -\cot\frac{\alpha}{2}z_2$$

But $tz_1 = kz_2\left\{\text{where, } k = -\cot\dfrac{\alpha}{2}\text{ (say)}\right\}$

Now, $k = -\cot\dfrac{\alpha}{2} \Rightarrow \cot\dfrac{\alpha}{2} = -k$

$$\Rightarrow \tan\alpha = \frac{2k}{k^2-1} \Rightarrow \tan\alpha = -\frac{2k}{1-k^2}$$

$$\Rightarrow \alpha = \tan^{-1}\left(-\frac{2k}{1-k^2}\right) = -2\tan^{-1}k$$

Now, $\dfrac{z_1 - z_2}{z_1 + z_2} = \cos\alpha + i\sin\alpha$

where α is the angle between $(z_1 - z_2)$ and $(z_1 + z_2)$

$\Rightarrow$ $\alpha = -2\tan^{-1}k$ is the angle between $(z_1 - z_2)$ and $(z_1 + z_2)$.

The correct option is (C)

93. $x^2 - \sqrt{3}x + 1 = 0$

$$\Rightarrow x = \frac{\sqrt{3}\pm\sqrt{3-4}}{2} = \frac{\sqrt{3}}{2}\pm\frac{i}{2} = \cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$$

$$\left(x^n - \frac{1}{x^n}\right)^2 = \left(x^{2n}+\frac{1}{x^{2n}}-2\right)$$

$$= -2 + 2\cos\frac{n\pi}{3}$$

$$\Rightarrow \sum_{n=1}^{24}\left(x^n - \frac{1}{x^n}\right)^2 = \sum_{n=1}^{24}\left(-2+2\cos\frac{n\pi}{3}\right)$$

$$= -48 + 2\left[\cos\frac{\pi}{3}+\cos\frac{2\pi}{3}+\ldots+\cos\frac{24\pi}{3}\right]$$

$$= \frac{+2\left[\cos\left(\dfrac{2\pi}{3}+\dfrac{23\pi}{6}\right).\sin\left(\dfrac{24\pi}{6}\right)\right]}{\sin\dfrac{\pi}{6}} - 48$$

$= 0 - 48 = -48$

The correct option is (B)

94. $\left|\sqrt{2}z_1 + i\sqrt{3}\bar{z}_2\right|^2 + \left|\sqrt{3}\bar{z}_1 + i\sqrt{2}\,z_2\right|^2$

$= \left(\sqrt{2}z_1 + i\sqrt{3}\,\bar{z}_2\right)\left(\sqrt{2}\,\bar{z}_1 - i\sqrt{3}\,z_2\right)$

$+ \left(\sqrt{3}\,\bar{z}_1 + i\sqrt{2}\,z_2\right)\left(\sqrt{3}\,z_1 - i\sqrt{2}\,\bar{z}_2\right)$

$= 5\,(|z_1|^2 + |z_2|^2) > 5 \cdot 2\,\sqrt{|z_1|^2|z_2|^2} = 10\,|z_1\,z_2|$

Since $A \cdot M > G \cdot M$ for $|z_1| \neq |z_2|$

The correct option is (B)

95. If z is a root of $(z-1)^{25} = 2\omega^2(z+1)^{25}$, then

$\left(\frac{z-1}{z+1}\right)^{25} = 2\omega^2 \Rightarrow \left|\frac{z-1}{z+1}\right|^{25} = 2\,|\omega^2| = 2$

$\Rightarrow \left|\frac{z-1}{z+1}\right| = 2^{1/25}$

As $2^{1/25} \neq 1$, we get z lies on a circle.

The correct option is (B)

96. $z_1 + z_2 = -\frac{b}{a}$ (1)

$z_1 z_2 = \frac{c}{a}$ (2)

$z_2 = iz_1$ (3)

From equations (1) and (3), we have

$z_1(1+i) = \frac{-b}{a} \Rightarrow z_1 = \frac{-b}{2a}\,(1-i)$

$\Rightarrow\ z_1^2 = \frac{b^2}{4a^2}\,(-2i) = \frac{-b^2}{2a^2}\,i$

From equations (2) and (3), we have

$z_1^2 = \frac{c}{ai} = \frac{-c}{a}\,i$

$\Rightarrow\ \frac{-b^2}{2a^2}\,i = \frac{-c^2}{a^2}\,i$

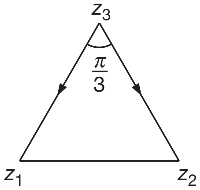
$\therefore\ a = \frac{b^2}{2c}$

The correct option is (A)

97. $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$

$= \cos\left(\frac{-\pi}{3}\right) + \sin\left(\frac{-\pi}{3}\right) = e^{-i\pi/3}$

$\therefore\ \left|\frac{z_1 - z_3}{z_2 - z_3}\right| = \left|e^{-i\pi/3}\right| = 1$



and angle between $z_1 - z_3$ and $z_2 - z_3$ is $\frac{\pi}{3}$.

$\therefore$ triangle is equilateral.

The correct option is (C)

98. If $\frac{z_1}{z_2} = z$, the given equation becomes

$z^2 - z + 1 = 0$ i.e., $z = -\,\omega$ and $-\,\omega^2$

or, $\frac{z_1}{z_2} = -\,\omega \Rightarrow z_1 = -\,z_2\omega$

$OB = |z_2 - 0| = |z_2|$

$OA = |z_1 - 0| = |-z_2\omega| = |z_2|\,|-\omega|$

$= |z_2|$

and, $AB = |z_2 - z_1| = |z_2 + z_2\omega|$

$= |z_2|\,|1 + \omega| = |z^2|\,|-\omega^2| = |z_2|$

Thus, z_1, z_2 and origin form an equilateral triangle.

The correct option is (D)

99. We have, $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1 \Rightarrow \left|\frac{z_1/z_2 - 1}{z_1/z_2 + 1}\right| = 1$

$\Rightarrow\ \left|\frac{z_1}{z_2} - 1\right| = \left|\frac{z_1}{z_2} + 1\right|$

$\Rightarrow\ \frac{z_1}{z_2}$ lies on the perpendicular bisector of the segment joining $A\,(-1 + 0i)$ and $B\,(1 + 0i)$.

$\therefore\ \frac{z_1}{z_2} = ai$ for some $a \in R$

$\Rightarrow\ \frac{z_2}{z_1} = \frac{1}{ai} = \frac{-i}{a}$

$\therefore\ z_2 = i\,kz_1$ for some $k \in R$

The correct option is (A)

100. We have,

$$\frac{1+iz}{1-iz} = \frac{1 + i\,\frac{b+ic}{1+a}}{1 - i\,\frac{b+ic}{1+a}} = \frac{1+a-c+ib}{1+a+c-ib}$$

$$= \frac{[1+a-c+ib]\,[1+a+c+ib]}{[1+a+c-ib]\,[1+a+c+ib]}$$

$$= \frac{z_0}{(1+a+c)^2 + b^2} \qquad (1)$$

Now, $R\,(z_0) = (1+a-c)\,(1+a+c) - b^2$

$= (1+a)^2 - c^2 - b^2 = 1 + a^2 + 2a - (1 - a^2)$

$= 2a^2 + 2a = 2a\,(1+a)$

and, $I_m(z_0) = 2b\,(1+a)$.

Thus, $z_0 = 2\,(1+a)\,(a+ib)$

Also, denominator of (1) $= 1 + a^2 + c^2 + 2a + 2c + 2ac + b^2$

$= 2 + 2a + 2c + 2ac$

$= 2\,(1+a)\,(1+c)$

Therefore, $\frac{1+iz}{1-iz} = \frac{a+ib}{1+c}$

The correct option is (B)

101. We have,

$|z_1 + z_2\,\mathrm{w}^k|^2 = (z_1 + z_2\,w^k)\left(\overline{z_1 + z_2\,w^k}\right)$

$= (z_1 + z_2 w^k)\,(\bar{z}_1 + \bar{z}_2 w^{-k})$

$\left[w^k = e^{i(2\pi k/n)} \Rightarrow \overline{w^k} = e^{-i(2\pi k/n)} = w^{-k}\right]$

$= |\,z_1\,|^2 + |\,z_2\,|^2 + \bar{z}_1\,z_2 w^k + z_1\bar{z}_2\,w^{-k}$

Therefore, we have,

$$\sum_{k=0}^{n-1} | z_1 + z_2 w^k |^2 = n(| z_1 |^2 + | z_2 |^2) + \bar{z}_1 z_2 \sum_{k=0}^{n-1} w^k$$

$$+ z_1 \bar{z}_2 \sum_{k=0}^{n-1} w^{-k}$$

$$= n(| z_1 |^2 + | z_2 |^2) \left[\sum_{k=0}^{n-1} w^k = 0 = \sum_{k=0}^{n-1} w^{-k} \right].$$

The correct option is (B)

102. We have, $|z - z_1|^2 + |z - z_2|^2 = k$

$$\Rightarrow \quad 2| z |^2 + | z_1 |^2 + | z_2 |^2 - 2\,\mathrm{Re}(z \bar{z}_1) - 2\mathrm{Re}(z \bar{z}_2) = k$$

$$\Rightarrow \quad 2| z |^2 - 2\mathrm{Re}\{z(\bar{z}_1 + \bar{z}_2)\} = k - (| z_1 |^2 + | z_2 |^2)$$

$$\Rightarrow \quad | z |^2 - \mathrm{Re}\{z(\bar{z}_1 + \bar{z}_2)\} = \frac{1}{2}(k - | z_1 |^2 - | z_2 |^2)$$

$$\Rightarrow \quad \left| z - \frac{z_1 + z_2}{2} \right|^2 - \frac{1}{4} | z_1 + z_2 |^2 = \frac{1}{2}(k - | z_1 |^2 - | z_2 |^2)$$

$$\Rightarrow \quad \left| z - \frac{z_1 + z_2}{2} \right|^2 = \frac{1}{2} k - \frac{1}{4}\{| z_1 |^2 + | z_2 |^2 - 2\mathrm{Re}(z_1 \bar{z}_2)\}$$

$$= \frac{1}{2} k - \frac{1}{4} | z_1 - z_2 |^2$$

$$\Rightarrow \quad \left| z - \frac{z_1 + z_2}{2} \right|^2 = \frac{1}{4}(2k - | z_1 - z_2 |^2)$$

which will represent a real circle having centre at $\frac{z_1 + z_2}{2}$

and radius $= \frac{1}{2}\sqrt{2k - | z_1 - z_2 |^2}$, provided $k \geq \frac{1}{2} | z_1 - z_2 |^2$

The correct option is (A)

103. Let $\alpha z + \beta$ be the remainder when $f(z)$ is divided by $z^2 + 1$. Then, we have

$$f(z) = (z^2 + 1)\, g(z) + Az + B$$

Given: $f(z)$ when divided by $z - i$ gives remainder i

$\Rightarrow \quad f(i) = i$

$\Rightarrow \quad (i^2 + 1)\, g(i) + Ai + B = i$

$\Rightarrow \quad Ai + B = i \qquad (1)$

Also, $f(z)$ when divided by $z + i$ gives remainder $i + 1$

$\Rightarrow \quad f(-i) = i + 1$

$\Rightarrow \quad (i^2 + 1)\, g(-i) - B = i + 1$

$\Rightarrow \quad -Ai + B = i + 1 \qquad (2)$

Solving equations (1) and (2), we get

$A = i/2$ and $B = i + 1/2$

$$\therefore \quad \text{remainder is } \left(\frac{i}{2}\right) z + \left(i + \frac{1}{2}\right)$$

The correct option is (C)

104. We have,

$$x^n - 1 = (x - 1)(x - w)(x - w^2) \ldots (x - w^{n-1})$$

$$\Rightarrow \quad \frac{x^n - 1}{x - w} = (x - 1)(x - w^2) \ldots (x - w^{n-1})$$

Putting $x = w$ on both sides, we have

$$(w - 1)(w - w^2) \ldots (w - w^{n-1}) = \lim_{x \to w} \frac{x^n - 1}{x - w} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to w} \frac{n x^{n-1}}{1} = n w^{n-1}$$

The correct option is (A)

105. We have,

$$z - i = e^{i}\alpha \Rightarrow z = i + e^{i}\alpha$$

$$= \cos\alpha + i(1 + \sin\alpha)$$

$$\therefore \quad \theta = \arg(z) = \tan^{-1}\left(\frac{1 + \sin\alpha}{\cos\alpha}\right)$$

$$\Rightarrow \quad \tan\theta = \frac{1 + \sin\alpha}{\cos\alpha}$$

$$\therefore \quad \cot\theta - \frac{2}{z} = \frac{\cos\alpha}{1 + \sin\alpha} - \frac{2}{\cos\alpha + i(1 + \sin\alpha)}$$

$$= \frac{\cos\alpha}{1 + \sin\alpha} - \frac{2[\cos\alpha - i(1 + \sin\alpha)]}{\cos^2\alpha + (1 + \sin\alpha)^2}$$

$$= \frac{\cos\alpha}{1 + \sin\alpha} - \frac{2[\cos\alpha - i(1 + \sin\alpha)]}{2(1 + \sin\alpha)}$$

$$= i.$$

The correct option is (B)

106. We have,

$$z_1 = \frac{4 + 3i}{1 + 2i} = \frac{(4 + 3i)(1 - 2i)}{(1 + 2i)(1 - 2i)}$$

$$= \frac{10 - 5i}{5} = 2 - i$$

which represents the point whose coordinates are (2, –1)

Also, we have,

$i z = \bar{z}$

$\Rightarrow \quad i(x + iy) - (x - iy) = 0 \qquad$ [Putting $z = x + iy$]

$\Rightarrow \quad i(x + y) - (x + y) = 0$

$\Rightarrow \quad (i - 1)(x + y) = 0$

which represents the line $y = -x$

Hence, reflection of the point (2, –1) in the line $y = -x$ gives the point (1, –2) which is equivalent to $1 - 2i$ in the argand plane.

The correct option is (D)

107. We know that $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$$\text{Thus, } 4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$$

$$= 4 + 5\omega^{334} + 3\omega^{365}$$

$$= 4 + 5(\omega^3)^{111}\,\omega + 3(\omega^3)^{123}\,\omega^2$$

$$= 4 + 5(1)^{111}\,\omega + 3(1)^{123}\,\omega^2$$

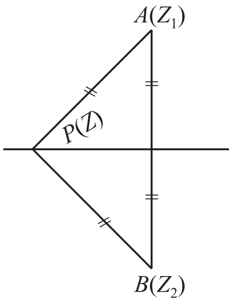
$$= 4 + 5\omega + 3\omega^2 = 1 + 2\omega + 3(1 + \omega + \omega^2)$$

$$= 1 + 2\omega + 3(0) = 1 + 2\omega = 1 - 1 + \sqrt{3}\, i = \sqrt{3}\, i.$$

The correct option is (C)

108. The given line is

$$\bar{b}\,z + b\,\bar{z} = c \qquad (1)$$



Let $A\ (z_1)$ be a reflection of $B\ (z_2)$ in the line (1).

Let $P\ (z)$ be any point on the line (1). We have,

$AP = BP$

$\Rightarrow\ |AP|^2 = |BP|^2$

$\Rightarrow\ |z - z_1|^2 = |z - z_2|^2$

$\Rightarrow\ (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$$\Rightarrow\ (\bar{z}_2 - \bar{z}_1)\,z + (z_2 - z_1)\,\bar{z} + z_1\bar{z} - z_2\bar{z}_2 = 0 \qquad (2)$$

Since (1) and (2) represent the same line, we get

$$\frac{\bar{b}}{\bar{z}_2 - \bar{z}_1} = \frac{b}{z_2 - z_1} = \frac{c}{z_1\bar{z}_1 - z_2\bar{z}_2} = k \text{ (say)}$$

$\Rightarrow\ k(\bar{z}_2 - \bar{z}_1) = \bar{b},\ k(z_2 - z_1) = b,\ k(z_1\bar{z}_1 - z_2\bar{z}_2) = c$

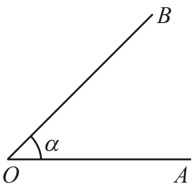
Now, $\bar{z}_1 b + z_2\,\bar{b}$

$$= \bar{z}_1\{k(z_2 - z_1)\} + z_2\{k(\bar{z}_2 - \bar{z}_1)\}$$
$$= k\{\bar{z}_1 z_2 - z_1\bar{z}_1 + \bar{z}_2 z_2 - z_2\bar{z}_1\}$$
$$= k(z_2\bar{z}_2 - z_1\bar{z}_1) = c.$$

The correct option is (C)

109. We have, $z_1 + z_2 = -p$ and $z_1 z_2 = q$

We know that



$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}(\cos\alpha + i\sin\alpha)$$

Since $|z_1| = |z_2|$ $\quad(\because \text{OA} = \text{OB})$

we get $\dfrac{z_1}{z_2} = \dfrac{\cos\alpha + i\sin\alpha}{1}$

Applying Componendo and Dividendo, we get

$$\frac{z_1 + z_2}{z_1 - z_2} = \frac{\cos\alpha + i\sin\alpha + 1}{\cos\alpha + i\sin\alpha - 1}$$

$$= \frac{2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{-2\sin^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$= \frac{2\cos\frac{\alpha}{2}\left[\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right]}{2i\sin\frac{\alpha}{2}\left[\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right]} = -i\cot\frac{\alpha}{2}$$

$$\Rightarrow\ \frac{p}{z_1 - z_2} = i\cot\frac{\alpha}{2}$$

Squaring we obtain

$$\frac{p^2}{(z_1 + z_2)^2 - 4z_1z_2} = -\cot^2\frac{\alpha}{2}$$

$$\Rightarrow\ \frac{p^2}{p^2 - 4q} = -\cot^2\frac{\alpha}{2}$$

$$\Rightarrow\ p^2 = -p^2\cot^2\frac{\alpha}{2} + 4q\cot^2\frac{\alpha}{2}$$

$$\Rightarrow\ p^2\left(1 + \cot^2\frac{\alpha}{2}\right) = 4q\cot^2\frac{\alpha}{2}$$

$$\Rightarrow\ p^2\,\text{cosec}^2 = 4q\cot^2\frac{\alpha}{2}$$

$$\Rightarrow\ p^2 = 4q\cos^2\frac{\alpha}{2}$$

$\therefore\ k = 4q$

The correct option is (C)

110. Since $|z_1| = |z_2| = |z_3| = 1$,

we get, $z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = 1$.

$$\text{Now, } 1 = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = |\bar{z}_1 + \bar{z}_2 + \bar{z}_3|$$

$$= \left|\overline{z_1 + z_2 + z_3}\right| = |z_1 + z_2 + z_3|.$$

The correct option is (A)

111. Let $z = r_1(\cos\alpha + i\sin\alpha)$

and, $\omega = r_2(\cos\beta + i\sin\beta) \Rightarrow \text{Arg}\,z = \alpha,\ \text{Arg}\,\omega = \beta$

$|z| = r_1,\ |\omega| = r_2$.

Given: $|z| \le 1,\ |\omega| \le 1 \Rightarrow r_1 \le 1,\ r_2 \le 1$

Now, consider $|z - \omega|^2 = |(r_1\cos\alpha - r_2\cos\beta) + i(r_1\sin\alpha - r_2\sin\beta)^2|$

$= (r_1\cos\alpha - r_2\cos\beta)^2 + (r_1\sin\alpha - r_2\sin\beta)^2$

$= r_1^2(\cos^2\alpha + \sin^2\alpha) + r_2^2(\cos^2\beta + \sin^2\beta)$
$-2r_1r_2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\alpha - \beta)$

$= (r_1 - r_2)^2 + 2r_1r_2[1 - \cos(\alpha - \beta)]$

$$= (r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\frac{\alpha - \beta}{2}\right)$$

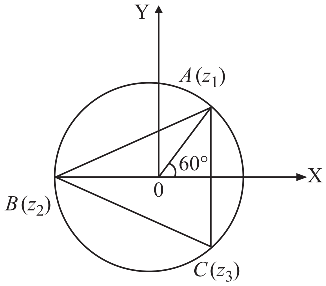
$$\le (r_1 - r_2)^2 + 4 \times 1 \times 1\left(\frac{\alpha - \beta}{2}\right)^2$$

or, $|z - \omega|^2 \le (|z| - |\omega|)^2 + (\text{Arg}\,z - \text{Arg}\,\omega)^2$.

The correct option is (B)

112. $A(z_1), B(z_2), C(z_3)$ lie on $|z| = 2$ whose centre is at $O(0, 0)$ and radius 2.

$z_1 = 1 + \sqrt{3}\,i$ hence $|z| = 2$ and $\text{Arg}(z_1) = \dfrac{\pi}{3}$



In turn $|z_2| = |z_3| = 2$ and $\text{Arg}(z_2) = \text{Arg}(z_1) + 120° = 180°$

$\therefore\ z_2 = -2$

Further, $\text{Arg}(z_3) = \text{Arg}(z_2) + 120° = 300°$

Hence, $z_3 = 2\left[\cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right)\right]$

$$= 2\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right] = 2\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

$$= 1 - \sqrt{3}\,i$$

Thus, $z_2 = -2$ and $z_3 = 1 - i\sqrt{3}$

The correct option is (A)

113. We know that,

$$(1+x)^{3n} = 1 + {}^{3n}C_1\,x + {}^{3n}C_2\,x^2 + \ldots + {}^{3n}C_{3n}\,x^{3n} \quad (1)$$

$$(1-x)^{3n} = 1 - {}^{3n}C_1\,x + {}^{3n}C_2\,x^2 + \ldots + (-1)^{3n}\,{}^{3n}C_{3n}\,x^{3n} \quad (2)$$

Subtracting (2) from (1) gives

$$(1+x)^{3n} - (1-x)^{3n} = 2\,[{}^{3n}C_1\,x + {}^{3n}C_3\,x^3 + {}^{3n}C_5\,x^5 + \ldots]$$

$$= 2x\,[{}^{3n}C_1 + {}^{3n}C_3\,x^2 + {}^{3n}C_5\,x^4 + \ldots]$$

Putting $x = i\sqrt{3}$, we get

$$(1 + i\sqrt{3}\,)^{3n} - (1 - i\sqrt{3}\,)^{3n}$$

$$= 2i\,\sqrt{3}\,[{}^{3n}C_1 - 3 \times {}^{3n}C_3 + 3^2 \times {}^{3n}C_5 \ldots]$$

Therefore, ${}^{3n}C_1 - 3\,{}^{3n}C_3 + 3^2 \times {}^{3n}C_5 \ldots$

$$= \frac{1}{2i\sqrt{3}}\,[(1 + i\sqrt{3}\,)^{3n} - (1 - i\sqrt{3}\,)^{3n}]$$

$$= \frac{1}{2i\sqrt{3}} \times 2^{3n}\left[\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{3n}\right]$$

$$= \frac{2^{3n-1}}{i\sqrt{3}}\,[(\cos n\pi + i\sin n\pi) - (\cos n\pi - i\sin n\pi)]$$

$$= \frac{2^{3n-1}}{i\sqrt{3}}\,2i\sin n\pi = 0 \text{ as } n \text{ is an integer.}$$

The correct option is (A)

114. We know that the triangle with vertices z_1, z_2, z_3 is an equilateral if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$\therefore$ The triangle with vertices $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ will be equilateral if

$$(a+i)^2 + (1+bi)^2 + 0 = (a+i)(1+bi) + 0 + 0$$

$$\Rightarrow a^2 - 1 + 2ai + 1 - b^2 + 2bi = (a-b) + i(1+ab)$$

$$\Rightarrow a^2 - b^2 = a - b \quad (1)$$

and $2(a+b) = 1 + ab$ (2)

[Equating real and imaginary parts]

(1) $\Rightarrow (a-b)\,[a+b-1] = 0$

$\Rightarrow a = b$ or $a = 1 - b$.

Subsitituting the value of $a - b$ in (2), we get

$$2(a+a) = 1 + a^2 \Rightarrow a^2 - 4a + 1 = 0$$

$$\Rightarrow a = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Since $0 < a < 1$ and $0 < b < 1$,

$\therefore\ a = b = 2 - \sqrt{3}$.

Substituting $a + b = 1$ in (2), we get

$$2 = 1 + a(1-a) \Rightarrow a^2 - a + 1 = 0$$

which gives imaginary values of a and b.

Hence, $a = b = 2 - \sqrt{3}$

The correct option is (B)

115. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

where, $x_1 \neq x_2$, $y_1 \neq y_2$ and $x_1^2 + y_1^2 = x_2^2 + y_2^2$

Now, $\dfrac{z_1 + z_2}{z_1 - z_2} = \dfrac{(x_1 + x_2) + i(y_1 + y_2)}{(x_1 - x_2) + i(y_1 - y_2)}$

$$= \frac{[(x_1 + x_2) + i(y_1 + y_2)]\,[(x_1 - x_2) - i(y_1 - y_2)]}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \frac{[(x_1^2 - x_2^2) + (y_1^2 - y_2^2)] + i[x_1y_1 - y_1x_2 + y_2x_1 - y_2x_2 - x_1y_1 + x_1y_2 - x_2y_1 + x_2y_2]}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \frac{2i(x_1y_2 - y_1x_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

= a purely imaginary or 0 if $\dfrac{x_1}{x_2} = \dfrac{y_1}{y_2}$.

If $\dfrac{x_1}{x_2} = \dfrac{y_1}{y_2}$ then $x_1 + iy_1 = k(x_2 + iy_2)$

If $k = 1$, $z_1 = z_2$, which is not true and if $k \neq 1$, $|z_1| \neq |z_2|$.

$\therefore\ \dfrac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary.

The correct option is (D)

116. Since $|CA| = |CB|$ and $\angle ACB = 90^\circ$

$\therefore\ (z_2 - z_3) = \pm i(z_1 - z_3)$

$\Rightarrow (z_2 - z_3)^2 = -(z_1 - z_3)^2$

$\Rightarrow z_2^2 + z_3^2 - 2z_2 z_3 = -z_1^2 - z_3^2 - 2z_1 z_3$

$z_1^2 + z_2^2 - 2z_1 z_2 = 2(z_1 z_3 - z_1 z_2 - z_2 z_3 + z_2^2)$

$\Rightarrow (z_1 - z_2)^2 = 2\,[z_1 - z_3]\,[z_3 - z_2]$

The correct option is (B)

117. Let $A = z_1$, $B = z_2$ and $C = z_3$,

where A, B, C are vertices of equilateral triangle.

Given that third point C is origin, so $z_3 = 0$.

Let $z_2 - z_3 = \alpha$, $z_3 - z_1 = \beta$, $z_1 - z_2 = \gamma$

or, $z_2 = \alpha$, $-z_1 = \beta$, $z_1 - z_2 = \gamma$

$\therefore\ \alpha + \beta + \gamma = z_2 - z_1 + z_1 - z_2 = 0$

or, $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$ (1)

Since the triangle is equilateral triangle,

$\therefore\ BC = CA = AB$

or, $|(z_2 - 0)| = |0 - z_1| = |z_1 - z_2|$

or, $|\alpha| = |\beta| = |\gamma|$

or, $|\alpha|^2 = |\beta|^2 = |\gamma|^2$

or, $\alpha\bar{\alpha} = \beta\bar{\beta} = \gamma\bar{\gamma} = k$ (say)

$\therefore\ \bar{\alpha} = \dfrac{k}{\alpha}$, $\bar{\beta} = \dfrac{k}{\beta}$, $\bar{\gamma} = \dfrac{k}{\gamma}$

Substituting values of $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ in (1), we get

$$\frac{k}{\alpha} + \frac{k}{\beta} + \frac{k}{\gamma} = 0$$

or, $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0$

or, $\frac{1}{z_2} + \frac{1}{-z_1} + \frac{1}{z_1 - z_2} = 0$

or, $\frac{z_1(z_1 - z_2) - z_2(z_1 - z_2) + z_1 z_2}{z_1 z_2 (z_1 - z_2)} = 0$

or, $z_1^2 - z_1 z_2 - z_1 z_2 + z_2^2 + z_1 z_2 = 0$

or, $z_1^2 + z_2^2 = z_1 z_2$

or, $\frac{z_1^2}{z_1 z_2} + \frac{z_2^2}{z_1 z_2} = 1$

or, $\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1$

or, $\frac{A}{B} + \frac{B}{A} = 1.$

The correct option is (A)

118. Given $2\sqrt{2}\, x^4 = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$

$$\Rightarrow \quad x^4 = \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

$$\Rightarrow \quad |x^4|^2 = \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2 = 1$$

$\therefore \quad |x^4| = 1$

and, $\arg(x^4) = \tan^{-1}\left[\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) \Big/ \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\right]$

$$= \tan^{-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) = 75° = \frac{5\pi}{12}$$

$$\therefore \quad x^4 = 1\left[\cos\left(2n\pi + \frac{5\pi}{12}\right) + i\sin\left(2n\pi + \frac{5\pi}{12}\right)\right]$$

Using $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, we have

$$x = \cos\frac{1}{4}\left(2n\pi + \frac{5\pi}{12}\right) + i\sin\frac{1}{4}\left(2n\pi + \frac{5\pi}{12}\right);$$

$n = 0, 1, 2, 3$

The correct option is (B)

119. $\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)$

$$= \left\{\sin\frac{2\pi}{11} + \sin\frac{4\pi}{11} + \ldots + 10 \text{ terms}\right\}$$

$$i\left\{\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \ldots + 10 \text{ terms}\right\}$$

$$= \frac{\sin\left(\frac{2\pi}{11} + \frac{9\pi}{11}\right)\sin\frac{10\pi}{11}}{\sin\frac{\pi}{11}} - i\,\frac{\cos\left(\frac{2\pi}{11} + \frac{9\pi}{11}\right)\sin\frac{10\pi}{11}}{\sin\frac{\pi}{11}}$$

$$= 0 - i(-1) = i \qquad (1)$$

$$\therefore \quad S = \sum_{p=1}^{32}(3p+2)\left[\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right]^p$$

$$= \sum_{p=1}^{32}(3p+2)\, i^p = 3\sum_{p=1}^{32} p i^p + 2\sum_{p=1}^{32} i^p$$

$$= 3A + 2B \qquad (2)$$

Now, $A = i + 2i^2 + 3i^3 + \ldots + 32i^{32}$

$\Rightarrow \quad Ai = i^2 + 2i^3 + \ldots + 31i^{32} + 32i^{33}$

$\Rightarrow \quad A(1 - i) = i + i^2 + \ldots + i^{32} - 32i^{33}$

$$= \frac{i^{32} - 1}{i - 1} - 32i = -32i \qquad [\because i^{32} = 1]$$

$$\therefore \quad A = \frac{-32i}{1-i} = \frac{32i(1+i)}{2} = 16(1 - i) \qquad (3)$$

and, $B = i + i^2 + \ldots + i^{32} = 0 \qquad (4)$

Hence, $S = 3 \times 16(1 - i) = 48(1 - i).$

The correct option is (C)

120. Since the triangle is equilateral

$\therefore \quad |z_1 - 0| = |z_2 - z_1| = |0 - z_2|$

$\Rightarrow \quad |z_1|^2 = |z_2 - z_1|^2 = |z_2|^2$

$\Rightarrow \quad z_1\overline{z}_1 = (z_2 - z_1) \times (\overline{z}_2 - \overline{z}_1) = z_2\overline{z}_2$

Now, $z_1\overline{z}_1 = z_2\overline{z}_2 \Rightarrow \frac{z_1}{z_2} = \frac{\overline{z}_2}{\overline{z}_1} = \frac{z_1 - z_2}{\overline{z}_2 - \overline{z}_1}.$

Also, $z_2\overline{z}_2 = (z_2 - z_1)(\overline{z}_2 - \overline{z}_1) = (z_2 - z_1)\frac{(z_1 - z_2)\overline{z}_2}{z_1}$

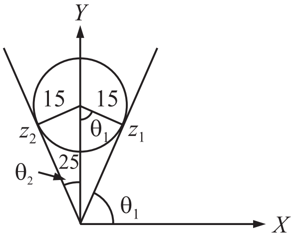
$\therefore \quad z_1 z_2 = (z_2 - z_1)(z_1 - z_2) = z_2 z_1 - z_1^2 - z_2^2 + z_1 z_2$

$\Rightarrow \quad z_1^2 + z_2^2 = z_1 z_2.$

The correct option is (B)

121. We have, max. amp (z) = amp (z_2),

min. amp (z) = amp (z_1).



Now, amp $(z_1) = \theta_1 = \cos^{-1}\left(\frac{15}{25}\right) = \cos^{-1}\left(\frac{3}{5}\right)$

and, amp $(z_2) = \frac{\pi}{2} + \theta_2$

$$= \frac{\pi}{2} + \sin^{-1}\left(\frac{15}{25}\right)$$

$$= \frac{\pi}{2} + \sin^{-1}\left(\frac{3}{5}\right)$$

$\therefore\quad |\text{max. amp}\,(z) - \text{min. amp}\,(z)| = \left|\frac{\pi}{2} + \sin^{-1}\frac{3}{5} - \cos^{-1}\frac{3}{5}\right|$

$= \left|\frac{\pi}{2} + \frac{\pi}{2} - -\cos^{-1}\frac{3}{5} - \cos^{-1}\frac{3}{5}\right|$

$= \pi - 2\cos^{-1}\left(\frac{3}{5}\right)$

The correct option is (C)

122. Putting $z = x + iy$, we get

$(x + iy)^2 + (p + iq)(x + iy) + r + is = 0$

$\Rightarrow\quad (x^2 - y^2 + px - q\,y + r) + i\,(2xy + py + qx + s) = 0$

$\Rightarrow\quad x^2 - y^2 + px - q\,y + r = 0 \qquad (1)$

and, $2xy + py + qx + s = 0 \qquad (2)$

If the roots are real, then $y = 0$

$\therefore$ (1) gives $x^2 + px + r = 0 \qquad (3)$

and (2) gives $pqx + s = 0 \qquad (4)$

From (4), $x = -\dfrac{s}{q}$

Putting in (3), we get $\dfrac{s^2}{q^2} - \dfrac{ps}{q} + r = 0$

or, $r^2 - pqs + rq^2 = 0 \Rightarrow pqs = s^2 + rq^2$.

The correct option is (A)

123. We know that $|z_1 + z_2|^2 + |z_1 - z_2|^2$

$= 2\,[|z_1|^2 + |z_2|^2] \qquad (1)$

Now, $\left[\left|z_1 + \sqrt{z_1^2 - z_2^2}\right| + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right|\right]^2$

$= \left|z_1 + \sqrt{z_1^2 - z_2^2}\right|^2 + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right|^2 + 2\,|z_1^2 - (z_1^2 - z_2^2)|$

$= 2\,|z_1|^2 + 2\,|z_1^2 - z_2^2| + 2\,|z_2^2| \qquad$ [By (1)]

$= 2\,|z_1|^2 + 2\,|z_2|^2 + 2\,|z_1^2 - z_2^2|$

$= |z_1 + z_2|^2 + |z_1 - z_2|^2 + 2\,|z_1 + z_2|\,|z_1 - z_2|$

$= (|z_1 + z_2| + |z_1 - z_2|)^2$

Taking square root of both sides, we get

$$\left|z_1 + \sqrt{z_1^2 - z_2^2}\right| + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right| = |z_1 + z_2| + |z_1 - z_2|.$$

The correct option is (D)

124. Since $z = x + iy$ lies in the third quadrant

$\therefore\quad x < 0, y < 0$. Again $\bar{z} = x - iy$

$\therefore\quad \dfrac{\bar{z}}{z} = \dfrac{x - iy}{x + iy} = \dfrac{(x - iy)(x - iy)}{x^2 + y^2} = \dfrac{x^2 - y^2 - 2ixy}{x^2 + y^2}$

$= \dfrac{x^2 - y^2}{x^2 + y^2} - \dfrac{2xy}{x^2 + y^2}\,i = A + iB$

where, $A = \dfrac{x^2 - y^2}{x^2 + y^2}$ and $B = -\dfrac{2xy}{x^2 + y^2}$

Since $x < 0, y < 0$

$\therefore\quad \dfrac{-2xy}{x^2 + y^2} < 0$

$\therefore\quad B < 0$

Now, $\dfrac{\bar{z}}{z}$ lies in the IIIrd quadrant if $A < 0$

i.e., if $x^2 - y^2 < 0$ or $x^2 < y^2$ i.e., if $x < y < 0$.

The correct option is (C)

125. Let $BA = BC$

$\Rightarrow\quad |z_1 - z_2|^2 = |z_3 - z_2|^2$

$\Rightarrow\quad (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = (z_3 - z_2)(\bar{z}_3 - \bar{z}_2) \qquad (1)$

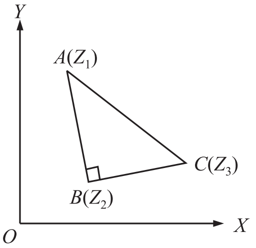
Again,

$\therefore\quad \angle ABC = 90^\circ$

$\therefore\quad \arg\dfrac{BA}{BC} = 90^\circ$

$\Rightarrow\quad \arg\dfrac{z_1 - z_2}{z_3 - z_2} = 90^\circ \Rightarrow$ real part of $\dfrac{z_1 - z_2}{z_3 - z_2} = 0$

$\Rightarrow\quad \dfrac{1}{2}\left[\dfrac{z_1 - z_2}{z_3 - z_2} + \dfrac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_2}\right] = 0$



$\Rightarrow\quad \dfrac{z_1 - z_2}{z_3 - z_2} = -\dfrac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_2} \Rightarrow \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \dfrac{z_2 - z_3}{\bar{z}_3 - \bar{z}_2} \qquad (2)$

$(1) \times (2) \Rightarrow\quad (z_1 - z_2)^2 = -(z_2 - z_3)^2$

$\Rightarrow\quad z_1^2 + z_2^2 + z_3^2 = 2\,z_2\,(z_1 + z_3)$

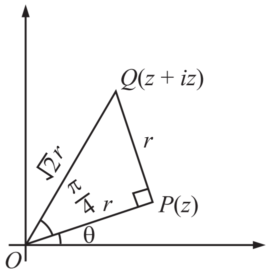
The correct option is (B)

126. Let $z = r\,(\cos\theta + i\sin\theta)$, then

$z + iz = r\,(\cos\theta + i\sin\theta) + ir\,(\cos\theta + i\sin\theta)$

$= r\,[(\cos\theta - \sin\theta) + i\,(\sin\theta + \cos\theta)]$

$= \sqrt{2}\,r\left[\cos\left(\theta + \dfrac{\pi}{4}\right) + i\sin\left(\theta + \dfrac{\pi}{4}\right)\right]$



In ΔOPQ,

$PQ^2 = r^2 + (\sqrt{2}r)^2 - 2r(\sqrt{2}r)\cos\dfrac{\pi}{4}$

$= r^2 + 2r^2 - 2r^2 = r^2$

$\therefore\quad PQ = r,$

$\therefore\quad \angle OPQ = \dfrac{\pi}{2}$

The correct option is (C)

127. Given $|z+1| < |z-2|$

and, $\omega = 3z + 2 + i$

$\therefore \quad \bar{\omega} = 3\bar{z} + 2 - i$

$\therefore \quad \omega + \bar{\omega} = 3(z + \bar{z}) + 4 \qquad \text{(i)}$

Now, $|z+1|^2 < |z-2|^2$

$\Rightarrow \quad (z+1)\,\overline{(z+1)} < (z-2)\,\overline{(z-2)}$

$\Rightarrow \quad (z+1)(\bar{z}+1) < (z-2)(\bar{z}-2)$

$\Rightarrow \quad z\bar{z} + z + \bar{z} + 1 < z\bar{z} - 2z - 2\bar{z} + 4$

$\Rightarrow \quad 3z + 3\bar{z} < 3 \Rightarrow \quad z + \bar{z} < 1 \qquad \text{(ii)}$

From (i) and (ii), we get

$\omega + \bar{\omega} < 3 \times 1 + 4 = 7 \Rightarrow \omega + \bar{\omega} < 7$

Clearly, $|\omega + 1| < |\omega - 8|$ gives

$|\omega + 1|^2 < |\omega - 8|^2$

$\Rightarrow \quad (\omega + 1)(\bar{\omega} + 1) < (\omega - 8)(\bar{\omega} - 8)$

$\Rightarrow \quad \omega\bar{\omega} + \omega + \bar{\omega} + 1 < \omega\bar{\omega} - 8\bar{\omega} - 8\omega + 64$

$\Rightarrow \quad 9(\omega + \bar{\omega}) < 63 \Rightarrow \omega + \bar{\omega} < 7$

The correct option is (A)

128. We have, $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

Since $f(x)$ is divisible by $x^2 + x + 1$, $f(\omega) = 0$, $f(\omega^2) = 0$

$\therefore \quad P(\omega^3) + \omega Q(\omega^3) = 0$ or $P(1) + \omega Q(1) = 0 \qquad (1)$

and, $P(\omega^6) + \omega^2 Q(\omega^6) = 0$ or $P(1) + \omega^2 Q(1) = 0 \qquad (2)$

From (1) and (2) we obtain

$P(1) = 0$ and $Q(1) = 0$.

$\therefore$ Both $P(x)$ and $Q(x)$ are divisible by $x - 1$

Since $f(x) = P(x) + x\,Q(x)$, we get $f(x)$ is divisible by $x - 1$.

The correct option is (D)

129. We have, $\alpha = \cos\left(\dfrac{8\pi}{11}\right) + i\sin\left(\dfrac{8\pi}{11}\right) = e^{\frac{8\pi}{11}i}$

$\text{Re}\,(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$

$$= \frac{\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^3 + \bar{\alpha}^4 + \bar{\alpha}^5}{2}$$

$$= \frac{-1 + (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^3 + \bar{\alpha}^4 + \bar{\alpha}^5)}{2}$$

$= \dfrac{-1+0}{2}$ (sum of 11, 11th roots of unity)

$= -\dfrac{1}{2}$

The correct option is (B)

130. We have, $z_k = 1 + a + a^2 + \ldots + a^k = \dfrac{1 - a^{k+1}}{1 - a}$

$\Rightarrow \quad z_k - \dfrac{1}{1-a} = \dfrac{-a^{k+1}}{1-a}$

$\Rightarrow \quad \left|z_k - \dfrac{1}{1-a}\right| = \dfrac{|a|^{k+1}}{|1-a|} < \dfrac{1}{|1-a|} \qquad (\because |a| < 1)$

$\therefore$ Vertices of the polygon $z_1, z_2, \ldots, z_n$ lie within the circle

$\left|z - \dfrac{1}{1-a}\right| = \dfrac{1}{|1-a|}$

The correct option is (C)

131. We have, e^{iA}, e^{iB}, e^{iC} are in A. P.

$\Rightarrow \quad 2e^{iB} = e^{iA} + e^{iC}$

$\Rightarrow \quad 2(\cos B + i\sin B)$

$= (\cos A + \cos C) + i(\sin A + \sin C)$

$\Rightarrow \quad 2\cos B = \cos A + \cos C$

and, $2\sin B = \sin A + \sin C$

$\Rightarrow \quad 2\cos B = 2\cos\dfrac{A+C}{2}\cos\dfrac{A-C}{2} \qquad (1)$

and, $2\sin B = 2\sin\dfrac{A+C}{2}\cos\dfrac{A-C}{2} \qquad (2)$

Dividing (1) by (2), we get

$\cot B = \cot\left(\dfrac{A+C}{2}\right) = \tan\dfrac{B}{2} \Rightarrow \cos\dfrac{3B}{2} = 0$

$\Rightarrow \quad \dfrac{3B}{2} = \dfrac{\pi}{2}$ or $B = \dfrac{\pi}{3} \Rightarrow A + C = \dfrac{2\pi}{3}$.

Putting this value in (1), we get

$2\cos\dfrac{\pi}{3} = 2\cos\dfrac{\pi}{3}\cos\left(\dfrac{A-C}{2}\right) \Rightarrow \cos\left(\dfrac{A-C}{2}\right) = 1$

$\Rightarrow \quad \left(\dfrac{A-C}{2}\right) = 0$ or $A = C \Rightarrow \quad A = B = C = \dfrac{\pi}{3}$.

The correct option is (C)

132. Let $\cot^{-1} p = \theta$, then $p = \cot\theta$

$$\therefore \quad e^{2mi\cot^{-1}p}\cdot\left(\frac{pi+1}{pi-1}\right)^m = e^{2mi\theta}\cdot\left(\frac{i\cot\theta+1}{i\cot\theta-1}\right)^m$$

$$= e^{2mi\theta}\cdot\left(\frac{i(\cot\theta-i)}{i(\cot\theta+i)}\right)^m = e^{2mi\theta}\cdot\left(\frac{\cot\theta-i}{\cot\theta+i}\right)^m$$

$$= e^{2mi\theta}\cdot\left(\frac{\cos\theta-i\sin\theta}{\cos\theta+i\sin\theta}\right)^m = e^{2mi\theta}\cdot\left(\frac{e^{-i\theta}}{e^{i\theta}}\right)^m$$

$$= e^{2mi}\theta\,(e^{-2i}\theta)^m = e^{2mi}\theta \times e^{-2mi}\theta = e^0 = 1$$

The correct option is (B)

133. Since z_1 and $\bar{z}_1$ are the adjacent vertices of a regular polygon of n sides, we have, $\angle z_1 0\, \bar{z}_1 = \dfrac{2\pi}{n}$

and, $|z_1| = |\bar{z}_1|$

Thus, $\quad z_1 = \bar{z}_1\, e^2\pi^{i/n}$

Let $z_1 = r(\cos\theta + i\sin\theta) = re^i\theta$

$\Rightarrow \quad \bar{z}_1 = re^{-i}\theta$

Since $\quad z_1 = \bar{z}_1\, e^2\pi^{i/n}$

$\Rightarrow \quad re^i\theta = re^{-i}\theta\, e^2\pi^{i/n} = re^2\pi^{i/n - i}\theta$

$\Rightarrow \quad \theta = \dfrac{2\pi}{n} - \theta$ or $\theta = \dfrac{\pi}{n}$

Therefore, $z_1 = r(\cos\theta + i\sin\theta) = r\left[\cos\dfrac{\pi}{n} + i\sin\dfrac{\pi}{n}\right]$

Now, $\dfrac{\text{Im}\,(z_1)}{\text{Re}\,(z_1)} = \sqrt{2} - 1 \Rightarrow \dfrac{r\sin\left(\dfrac{\pi}{n}\right)}{r\cos\left(\dfrac{\pi}{n}\right)} = \sqrt{2} - 1$

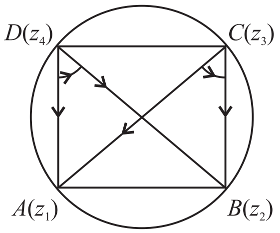
$\Rightarrow \quad \tan\dfrac{\pi}{n} = \sqrt{2} - 1 = \tan\dfrac{\pi}{8} \Rightarrow n = 8$

The correct option is (B)

134. We have,

$$\frac{2}{z_1}=\frac{1}{z_2}+\frac{1}{z_3}=\frac{z_3+z_2}{z_2 z_3}$$

$$\Rightarrow\quad z_1=\frac{2z_2z_3}{z_2+z_3}.$$



Now, $\left(\dfrac{z_2-z_4}{z_1-z_4}\right)\left(\dfrac{z_1-z_3}{z_2-z_3}\right)$

$$=\left(\frac{z_2-z_4}{\dfrac{2z_2z_3}{z_2+z_3}-z_4}\right)\left(\frac{\dfrac{2z_2z_3}{z_2+z_3}-z_3}{z_2-z_3}\right)$$

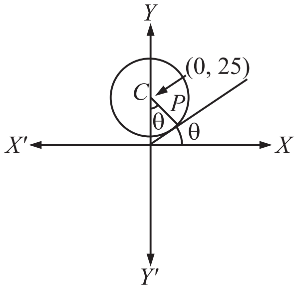
$$=\frac{z_2}{\dfrac{2z_2z_3}{z_2+z_3}}\,\frac{z_3(z_2-z_3)}{(z_2+z_3)(z_2-z_3)}\qquad[\text{taking } z_4=0]$$

$$=\frac{1}{2}\ \text{(a real number).}$$

Hence, points z_1, z_2, z_3 and origin are concyclic and therefore, z_1, z_2, z_3 lie on a circle passing through the origin.

The correct option is (B)

135. Since $|z-25i|\le 15$, therefore, distance between z and $25i$ is less than or equal to 15.



Thus, point z will lie in the interior and boundary of the circle whose centre is (0, 25) and radius is 15.

Let OP be tangent to the circle at point P.

Let $\angle POX=\theta$. Then, $\angle OCP=\theta$

Now, $OC=25$, $CP=15$

∴ $OP=20$.

Now, $\tan\theta=\dfrac{OP}{CP}=\dfrac{20}{15}=\dfrac{4}{3}$

∴ Least positive value of $\arg z=\theta=\tan^{-1}\left(\dfrac{4}{3}\right)$

The correct option is (B)

136. Let $z=x+iy$

Given, $|z-4+3i|\le 2$.

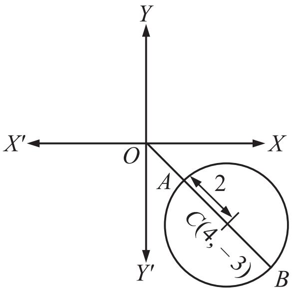
∴ $|x+iy-4+3i|\le 2$

or, $\sqrt{(x-4)^2+(y+3)^2}\le 2$

or, $(x-4)^2+(y+3)^2\le 2^2$

Thus, z lies in the interior or on the boundary of the circle whose centre is (4, –3) and radius is 2.

Least value of $|z|=OA=OC-AC=5-2=3$.



Greatest value of $|z|=OB=OC+CB=5+2=7$.

Thus, $3\le|z|\le 7$.

The correct option is (A)

137. Since $\text{Re}\,(z_1\bar{z}_2)\le|z_1\bar{z}_2|$

∴ $|z_1|^2+|z_2|^2+2\,\text{Re}\,(z_1\bar{z}_2)\le|z_1|^2+|z_2|^2+2\,|z_1\bar{z}_2|$

$\Rightarrow\ |z_1+z_2|^2\le|z_1|^2+|z_2|^2+2\,|z_1|\,|z_2|$ (1)

Also, Since A.M. ≥ G.M.

$$\therefore\quad \frac{(\sqrt{c}\,|z_1|)^2+\left(\dfrac{1}{\sqrt{c}}|z_2|\right)^2}{2}\ge\left\{c\cdot|z_1|^2\cdot\frac{1}{c}|z_2|^2\right\}^{\frac{1}{2}}$$

(∵ $c>0$)

$\Rightarrow\ c\,|z_1|^2+\dfrac{1}{c}\,|z_2|^2\ge 2\,|z_1|\,|z_2|$

∴ $|z_1|^2+|z_1|^2+2\,|z_1|\,|z_2|\le|z_1|^2+|z_2|^2+c\,|z_1|^2+\dfrac{1}{c}\,|z_2|^2$

$\Rightarrow\ |z_1|^2+|z_2|^2+2\,|z_1|\,|z_2|\le(1+c)\,|z_1|^2$
$+(1+c^{-1})\,(|z_2|^2)$ (2)

From (1) and (2), we get

$|z_1+z_2|^2\le(1+c)\,|z_1|^2+(1+c^{-1})\,|z_2|^2$

∴ $k=1+c^{-1}$

The correct option is (C)

138. We have, $1\ge|z-(4-3i)|$

$$\ge\left\{\begin{array}{l}|z|-|4-3i|\\|4-3i|-|z|\end{array}\right.\Rightarrow 1\ge\left\{\begin{array}{l}|z|-5\\5-|z|\end{array}\right.$$

$\Rightarrow\ |z|\le 6$ and $|z|\ge 4$

$\Rightarrow\ 4\le|z|\le 6\Rightarrow m=4,\ n=6$

Let $y=\dfrac{x^4+x^2+4}{x}=x^3+x+\dfrac{4}{x}$

$$=x^3+x+\frac{1}{x}+\frac{1}{x}+\frac{1}{x}+\frac{1}{x}$$

Since $x\in(0,\infty)$, ∴ $x^3, x, \dfrac{1}{x}, \dfrac{1}{x}, \dfrac{1}{x}, \dfrac{1}{x}$ are all positive numbers whose product is 1.

∴ their sum y will be least when

$x^3=x=\dfrac{1}{x}\Rightarrow x=1$

∴ least value of $y=6$

∴ $k=6$

Hence, $k=n$

The correct option is (B)

139. We have, $z^n = (z+1)^n$

$\Rightarrow \left(\frac{z+1}{z}\right)^n = 1 = \cos 0 + i \sin 0$

$\Rightarrow \frac{z+1}{z} = (\cos 2\pi r + i \sin 2\pi r)^{1/n}$

$= \cos \frac{2\pi r}{n} + i \sin \frac{2\pi r}{n}$

where, $r = 0, 1, 2, \ldots, n-1$.

$\Rightarrow 1 + \frac{1}{z} = \cos \frac{2\pi r}{n} + i \sin \frac{2\pi r}{n}$

$\Rightarrow 1 + \frac{1}{z} = 1 - 2 \sin^2 \frac{r\pi}{n} + i \times 2 \sin \frac{\pi r}{n} \times \cos \frac{\pi r}{n}$

$\Rightarrow \frac{1}{z} = -2 \sin^2 \frac{r\pi}{n} + 2i \times \sin \frac{r\pi}{n} \cos \frac{r\pi}{n}$

$\Rightarrow z = \frac{1}{i\left(2\sin\frac{r\pi}{n}\right)\left[\cos\frac{r\pi}{n} + i \sin\frac{r\pi}{n}\right]}$

$= \frac{1}{i\left(2\sin\frac{r\pi}{n}\right)}\left[\cos\frac{r\pi}{n} - i \sin\frac{r\pi}{n}\right]$

$\Rightarrow x + iy = -\frac{i}{2}\cot\frac{r\pi}{n} - \frac{1}{2} \Rightarrow x = -\frac{1}{2}$

Hence, all the points lie on the line $x = -\frac{1}{2}$

The correct option is (B)

140. We have, $z^2 + z + 1 = 0$ (1)

$\Rightarrow (z-1)(z^2+z+1) = 0,$

$\therefore z^3 = 1.$

If n is not a multiple of 3, then we can write

$n = 3m + r$, where $m \in$ I and $r = 1$ or 2,

then $2n = 6m + 2r$

If $r = 1$, then $2r = 2$

$\therefore z^n + z^{2n} = (z^3)^m \times z^r + (z^3)^{2n} \times z^{2r} = z^r + z^{2r}$

$= z + z^2 = -1$ [Using (1)]

If $r = 2$, then $2r = 4$

$\therefore 2n = 3(m+1) + 1$

$\therefore z^n + z^{2n} = (z^3)^m \times z^r + (z^3)^{m+1} \times z^1 = z^2 + z = -1$

Hence, $z^n + z^{2n} = -1$

The correct option is (D)

141. Since $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of the equation $x^5 - 1 = 0$.

$\therefore (x^5 - 1) = (x-1)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$

$\Rightarrow \frac{x^5-1}{x-1} = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$ (1)

Putting $x = \omega$ in (1), we get

$\frac{\omega^5-1}{\omega-1} = (\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)(\omega-\alpha_4)$

$\Rightarrow \frac{\omega^2-1}{\omega-1} = (\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)(\omega-\alpha_4)$ (2)

and putting $x = \omega^2$ in (1), we get

$\frac{\omega^{10}-1}{\omega^2-1} = (\omega^2-\alpha_1)(\omega^2-\alpha_2)(\omega^2-\alpha_3)(\omega^2-\alpha_4)$

$\Rightarrow \frac{\omega-1}{\omega^2-1} = (\omega^2-\alpha_1)(\omega^2-\alpha_2)(\omega^2-\alpha_3)(\omega^2-\alpha_4)$ (3)

Dividing (2) by (3), we get

$\frac{\omega-\alpha_1}{\omega^2-\alpha_1} \cdot \frac{\omega-\alpha_2}{\omega^2-\alpha_2} \cdot \frac{\omega-\alpha_3}{\omega^2-\alpha_3} \cdot \frac{\omega-\alpha_4}{\omega^2-\alpha_4} = \frac{(\omega^2-1)^2}{(\omega-1)^2}$

$= \frac{\omega^4+1-2\omega^2}{\omega^2+1-2\omega} = \frac{\omega+1-2\omega^2}{\omega^2+1-2\omega}$

$= \frac{-\omega^2-2\omega^2}{-\omega-2\omega} = \frac{-3\omega^2}{-3\omega} = \omega$

The correct option is (B)

142. Let $z = x + iy$

We have, $z + \bar{z} = 2|z-1|$

$\Rightarrow \frac{z+\bar{z}}{2} = |z-1|$

$\Rightarrow x = |x + iy - 1| \Rightarrow x = |(x-1) + iy|$

$\Rightarrow x^2 = (x-1)^2 + y^2 \Rightarrow 2x = 1 + y^2.$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

then, $2x_1 = 1 + y_1^2$ (1)

and, $2x_2 = 1 + y_2^2$ (2)

Subtracting (2) from (1), we get

$2(x_1 - x_2) = y_1^2 - y_2^2$

$\Rightarrow 2(x_1 - x_2) = (y_1 + y_2)(y_1 - y_2)$ (3)

But given $\arg(z_1 - z_2) = \frac{\pi}{4}$

i.e., $\tan^{-1}\left(\frac{y_1-y_2}{x_1-x_2}\right) = \frac{\pi}{4} \Rightarrow \frac{y_1-y_2}{x_1-x_2} = 1$

$\therefore y_1 - y_2 = x_1 - x_2,$ (4)

$\therefore$ From (3) and (4) we get

$y_1 + y_2 = 2$

$\therefore \text{Im}(z_1 + z_2) = 2.$

The correct option is (C)

143. We have,

$a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 3$

$\Rightarrow |3| = |a_1 z^3 + a_2 z^2 + a_3 z + a_4|$

$\Rightarrow 3 \le |a_1 z^3| + |a_2 z^2| + |a_3 z| + |a_4|$

$\Rightarrow 3 \le |a_1||z^3| + |a_2||z^2| + |a_3||z| + |a_4|$

$\Rightarrow 3 \le |z|^3 + |z|^2 + |z| + 1 \quad (\because |a_i| \le 1)$

$\Rightarrow 3 \le 1 + |z| + |z|^2 + |z|^3 < 1 + |z| + |z|^2 + |z|^3 + \ldots \infty$

$\Rightarrow 3 < 1 + |z| + |z|^2 + |z|^3 + \ldots \infty$

$\Rightarrow 3 < \frac{1}{1-|z|} \Rightarrow 1 - |z| < \frac{1}{3}$

$\Rightarrow \frac{2}{3} - |z| < 0 \Rightarrow |z| > \frac{2}{3}$

The correct option is (C)

144. Since, $\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} + \frac{1}{d+\omega} = \frac{2}{\omega}$

$\therefore$ ω is the root of the equation

$$\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{c+x}+\frac{1}{d+x} = \frac{2}{x}$$

$\Rightarrow\ 2x^4 + (\Sigma a)\,x^3 + 0 \times x^2 - (\Sigma abc)\,x - 2\,abcd = 0$

Let α, β, γ be the other roots, then

$$\omega + \alpha + \beta + \gamma = -\frac{\Sigma a}{2} \tag{1}$$

$\Sigma\alpha\beta = 0$

$$\alpha\beta + \alpha\omega + \beta\omega + \gamma\omega + \beta\gamma + \gamma\alpha = 0 \tag{2}$$

$$\Sigma\alpha\beta\gamma = \frac{\Sigma abc}{2} \tag{3}$$

$$\alpha\beta\gamma\omega = -abcd \tag{4}$$

Since complex roots occurs in conjugate pairs

$\therefore\ \gamma = \bar{\omega} = \omega^2.$

$\therefore$ From (2),

$\alpha\beta + \omega(\alpha+\beta) + \omega \times \omega^2 + \omega^2(\alpha+\beta) = 0$

$\Rightarrow\ \alpha\beta + (\omega + \omega^2)(\alpha+\beta) + \omega^3 = 0$

$\Rightarrow\ \alpha\beta + (-1)(\alpha+\beta) + 1 = 0$

$\Rightarrow\ \alpha\beta - \alpha - \beta + 1 = 0$

$\Rightarrow\ \alpha(\beta - 1) - (\beta - 1) = 0 \Rightarrow (\alpha - 1)(\beta - 1) = 0$

$\therefore$ either $\alpha = 1$ or $\beta = 1$

Hence one root is unity

$$\therefore\ \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1} = 2$$

The correct option is (B)

145. We have, $|A|^2 + |B|^2 + |C|^2 = A\bar{A} + B\bar{B} + C\bar{C}$ (1)

But
$$\begin{aligned} A\bar{A} &= (z_1 + z_2 + z_3)(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) \\ &= z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 + \bar{z}_1(z_2 + z_3) \\ &\quad + \bar{z}_2(z_3 + z_1) + \bar{z}_3(z_1 + z_2) \\ &= |z_1|^2 + |z_2|^2 + |z_3|^2 + \bar{z}_1(z_2 + z_3) \\ &\quad + \bar{z}_2(z_3 + z_1) + \bar{z}_3(z_1 + z_2) \end{aligned}$$

$$\begin{aligned} B\bar{B} &= (z_1 + z_2\,\omega + z_3\,\omega^2)\,(\overline{z_1} + \overline{z_2\omega} + \overline{z_3\omega^2}) \\ &= (z_1 + z_2\,\omega + z_3\,\omega^2)\,(\bar{z}_1 + \bar{z}_2\omega^2 + \bar{z}_3\omega) \\ &\qquad [\because\ \bar{\omega} = \omega^2 \text{ and } \overline{(\omega^2)} = \omega] \\ &= z_1\bar{z}_1 + z_2\bar{z}_2\omega^3 + z_3\bar{z}_3\omega^3 + \bar{z}_1\,(z_2\,\omega + z_3\,\omega^2) \\ &\quad + \bar{z}_2\,(z_3\,\omega^4 + z_1\,\omega^2) + \bar{z}_3\,(z_1\,\omega + z_2\,\omega^2) \\ &= |z_1|^2 + |z_2|^2 + |z_3|^2 + \bar{z}_1(z_2\omega + z_3\omega^2) \\ &\quad + \bar{z}_2(z_3\omega + z_1\omega^2) + \bar{z}_3(z_1\omega + z_2\omega^2) \end{aligned} \tag{2}$$

Similarly,
$$\begin{aligned} C\bar{C} &= |z_1|^2 + |z_2|^2 + |z_3|^2 + \bar{z}_1(z_2\omega^2 + z_3\omega) \\ &\quad + \bar{z}_2(z_3\omega^2 + z_1\omega) + \bar{z}_3(z_1\omega^2 + z_2\omega) \end{aligned} \tag{3}$$

Adding (1), (2) and (3), we get

$$\begin{aligned} A\bar{A} + B\bar{B} + C\bar{C} &= 3\,[|z_1|^2 + |z_2|^2 + |z_3|^2] \\ &\quad + \bar{z}_1\,[z_2(1 + \omega + \omega^2) + z_3(1 + \omega^2 + \omega)] \\ &\quad + \bar{z}_2\,[z_3(1 + \omega + \omega^2) + z_1(1 + \omega + \omega^2)] \\ &\quad + \bar{z}_3\,[z_1(1 + \omega + \omega^2) + z_2(1 + \omega^2 + \omega)] \\ &= 3\,[|z_1|^2 + |z_2|^2 + |z_3|^2] \qquad [\because\ 1 + \omega + \omega^2 = 0] \end{aligned}$$

$\therefore$ From (1) and (2), we conclude

$|A|^2 + |B|^2 + |C|^2 = 3\,[|z_1|^2 + |z_2|^2 + |z_3|^2].$

The correct option is (A)

146. $z + \dfrac{1}{z} = 2(\cos\theta + \text{i}\sin\theta) = 2e^{i}\theta$

$\therefore\ z^2 - 2e^{i}\theta z + 1 = 0$

$\Rightarrow\ z = e^{i\theta} \pm \sqrt{e^{2i\theta} - 1}$

$$\begin{aligned} e^{2i}\theta - 1 &= \cos 2\theta + \text{i}\sin 2\theta - 1 \\ &= 2\sin\theta(-\sin\theta + i\cos\theta) \\ &= 2\sin\theta\left[\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right] \end{aligned}$$

$$\begin{aligned} \therefore\ \text{Let } \alpha &= e^{i\theta} + \sqrt{e^{2i\theta} - 1} \\ &= \cos\theta + \sqrt{\sin\theta}\,\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \\ &\quad + i\left[\sin\theta + \sqrt{2\sin\theta}\,\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right] \end{aligned}$$

$$\therefore\ |\alpha - i|^2 = \left\{\cos\theta + \sqrt{2\sin\theta}\,\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}^2$$

$$+\left\{\sin\theta + \sqrt{2\sin\theta}\,\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) - 1\right\}^2$$

$$= 2 + 2\sqrt{2\sin\theta}\,\cos\left(\frac{\pi}{4} + \frac{\theta}{2} - \theta\right)$$

$$- 2\sqrt{2\sin\theta}\,\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = 2$$

Similarly, $|\beta - i|^2 = 2$

$\therefore\ |\alpha - i| = |\beta - i|$

The correct option is (D)

147. If $z = x + iy$ is a complex number satisfying the given conditions, then

$$\begin{aligned} a^2 - 3a + 2 &= \left|z + \sqrt{2}\right| \\ &= \left|z + i\sqrt{2} + \sqrt{2} - i\sqrt{2}\right| \\ &\le \left|z + i\sqrt{2}\right| + \sqrt{2}\,|1 - i| \\ &< a^2 + 2 \end{aligned}$$

$\Rightarrow\ -3a < 0 \Rightarrow a > 0$

Since $\left|z + \sqrt{2}\right| = a^2 - 3a + 2$ represents a circle with centre at $A\left(-\sqrt{2}, 0\right)$ and radius $\sqrt{a^2 - 3a + 2}$ and $\left|z + \sqrt{2}i\right| < a^2$ represents the interior of the circle with centre at $B\left(0, \sqrt{2}\right)$ and radius a. Therefore, there will be a complex number satisfying the given condition and the given inequality if the distance AB is less than the sum or difference of the radii of the two circles, i.e., if

$$\sqrt{\left(-\sqrt{2}-0\right)^2+\left(0+\sqrt{2}\right)^2} < \sqrt{a^2-3a+2} \pm a$$

⇒ $2 \pm a < \sqrt{a^2-3a+2}$

⇒ $4 + a^2 \pm 4a < a^2 - 3a + 2$

⇒ $-a < -2$ or $7a < -2$

⇒ $a > 2$ or $a < \dfrac{-2}{7}$

But, $a > 0$,

∴ $a > 2$

The correct option is (A)

148. Let $S_n = 1 + 2\alpha + 3\alpha^2 + \ldots + n\alpha^{n-1}$

∴ $\alpha S_n = \alpha + 2\alpha^2 + \ldots + (n-1)\alpha^{n-1} + n\alpha^n$

Subtracting, we get,

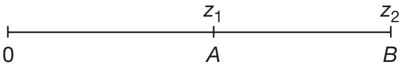
$$S_n(1-\alpha) = 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} - n\alpha^n$$

$$= \frac{1(1-\alpha^n)}{1-\alpha} - n\alpha^n$$

∴ $S_n = \dfrac{1-\alpha^n}{(1-\alpha)^2} - \dfrac{n\alpha^n}{1-\alpha} = \dfrac{-n}{1-\alpha}$.

The correct option is (B)

149. Let A represent z_1



Since $OA \cdot OB = 1$, ∴ $|z_1 - 0| \times |z - 0| = 1$

⇒ $|z_1| = \dfrac{1}{|z|}$

Also, $\arg\left(\dfrac{z_1 - 0}{z - 0}\right) = 0 \Rightarrow \arg\left(\dfrac{z_1}{z}\right) = 0$

⇒ $\arg z_1 = \arg z$

If θ is the argument of z, then

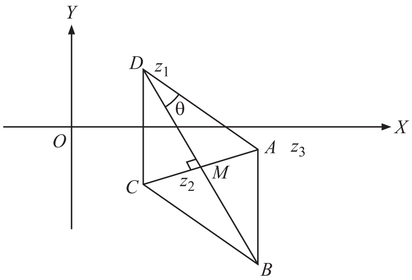
$z = |z|\, e^{i}\theta$

∴ $z_1 = \dfrac{1}{|z|}e^{i\theta} = \dfrac{1}{|z|^2}|z|\,e^{i\theta} = \dfrac{z}{z\bar{z}} = \dfrac{1}{\bar{z}}$

∴ A is $\dfrac{1}{\bar{z}}$

The correct option is (A)

150. Let $ABCD$ be the rhombus and M be the point of intersection of the diagonals AC and BD



Let point D be $z_1 = 1 + i$ and point M be $z_2 = 2 - i$

Also, let point A be z_3

Then, $z_2 - z_1 = 1 - 2i$ and $|z_2 - z_1| = \sqrt{5} = MD$

As given,

$AC = \dfrac{1}{2}BD \Rightarrow AM = \dfrac{1}{2}DM \Rightarrow AM = \dfrac{\sqrt{5}}{2}$

⇒ $AD = |z_3 - z_1| = \sqrt{DM^2 + AM^2}$

$$= \sqrt{\left(\sqrt{5}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2} = \frac{5}{2}$$

Therefore, in Δ AMD,

$\cos\theta = \dfrac{\sqrt{5}}{5/2} = \dfrac{2}{\sqrt{5}}$ and $\sin\theta = \dfrac{\sqrt{5}/2}{5/2} = \dfrac{1}{\sqrt{5}}$

Now, by rotation of complex numbers we know that $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\theta}$ (anticlockwise rotation)

⇒ $\dfrac{z_3 - (1+i)}{1-2i} = \dfrac{5/2}{\sqrt{5}}(\cos\theta + i\sin\theta)$

⇒ $\dfrac{z_3 - (1+i)}{1-2i} = 1 + \dfrac{i}{2}$ (using values of $\cos\theta$ and $\sin\theta$)

⇒ $z_3 = \dfrac{2+i}{2}(1-2i) + (1+i) \Rightarrow z_3 = 3 - \dfrac{i}{2}$

Similarly, taking clockwise rotation we get another possible position of A as

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{-i\theta}$$

⇒ $z_3 = \left(\dfrac{1-i}{2}\right)(1-2i) + (1+i) \Rightarrow z_3 = 1 - \dfrac{3}{2}i$

So, A represents the complex numbers $3 - \dfrac{i}{2}$ or $1 - \dfrac{3}{2}i$

The correct option is (A)

151. Let $z = x + iy = r(\cos\theta + i\sin\theta)$, then the equation is

$$|(x-2) + i(y-1)| = r\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)$$

$$= \frac{1}{\sqrt{2}}(r\cos\theta - r\sin\theta)$$

or, $\sqrt{(x-2)^2 + (y-1)^2} = \dfrac{1}{\sqrt{2}}(x - y)$

which is the part of a parabola with focus (2, 1) and directrix $x - y = 0$.

The correct option is (C)

152. $|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2$

$= 2(|z_1|^2 + |z_2|^2 + |z_3|^2) + \left(z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_3 + \bar{z}_2 z_3 + z_3\bar{z}_1 + z_1\bar{z}_3\right)$

$= 24 + \left(z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_3 + \bar{z}_2 z_3 + z_3\bar{z}_1 + z_3\bar{z}_1\right)$ (1)

Also,

$|z_1 + z_2 + z_3|^2 \geq 0$

⇒ $z_1\bar{z}_2 + \bar{z}_1 z_2 + z_2\bar{z}_3 + \bar{z}_2 z_3 + z_3\bar{z}_1 + \bar{z}_3 z_1 \geq -12$

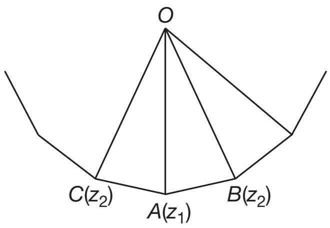
∴ $|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2 \geq 12$

The correct option is (B)

More than One Option Correct Type

153. Let A be the vertex with affix z_1. There are two possibilities and can be obtained by rotating z_1 through either $\frac{2\pi}{n}$ in clockwise or in anticlockwise direction.

$$z_2 = z_1 e^{\pm\frac{2\pi}{n}} \qquad \{\because |z_2| = |z_1|\}$$



The correct option is (A) and (C)

154. We have, $i = 0 + i \times 1 = \frac{1}{2}(0 + 2i)$

$$= \frac{1}{2}(1 + i^2 + 2 \times 1 \times i) = \frac{1}{2}(1 + i)^2$$

$$\therefore \quad \sqrt{i} = \pm\frac{1}{\sqrt{2}}(1 + i)$$

$$\therefore \quad \sqrt{-i} = \pm\frac{1}{\sqrt{2}}(1 - i)$$

Hence, $\sqrt{i} - \sqrt{-i} = \pm\frac{1}{\sqrt{2}}[(1 + i) - (1 - i)] = \pm\sqrt{2}i$

The correct option is (A) and (D)

155. We have, $z_1 - z_4 = z_2 - z_3$ or $\frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$

i.e., the diagonals bisect each other.

$\therefore$ It is a parallelogram.

Also, amp $\frac{z_4 - z_1}{z_2 - z_1} = \frac{\pi}{2}$

$\Rightarrow$ angle at z_1 is a right angle.

$\therefore$ It is a rectangle and hence a cyclic quadrilateral.

The correct option is (C) and (D)

156. Let

$$S = \sum_{m=1}^{4n+1}\left[\sum_{k=1^m}^{m+1}\left\{\sin\left(\frac{2\pi k}{m}\right) - i\cos\left(\frac{2\pi k}{m}\right)\right\}\right]^m$$

Now, let $t_k = \sin\left(\frac{2\pi k}{m}\right) - i\cos\left(\frac{2\pi k}{m}\right)$

$$\Rightarrow \quad t_k = -i\left\{\cos\left(\frac{2\pi k}{m}\right) + i\sin\left(\frac{2\pi k}{m}\right)\right\}$$

Assuming, $\alpha = \cos\left(\frac{2\pi}{m}\right) + i\sin\left(\frac{2\pi}{m}\right)$ (1)

$\Rightarrow \quad t_k = -i\alpha^k$

Now, $S_1 = \sum_{k=1}^{m+1}(t_k)$

$$\Rightarrow \quad S_1 = -i\sum_{k=1}^{m+1}\alpha^k \Rightarrow S_1 = -i(\alpha + \alpha^2 + \ldots + \alpha^{m+1})$$

$$\Rightarrow \quad S_1 = \frac{-i\alpha(\alpha^{m+1} - 1)}{\alpha - 1} = -i\left\{\frac{\alpha^m - \alpha}{\alpha - 1}\right\}$$

Since, $\alpha^m = (\cos 2\pi + i\sin 2\pi) = 1$ {using (1)}

$$\Rightarrow \quad S_1 = i\,\frac{\alpha - 1}{\alpha - 1} = i$$

Now, $S = \sum_{m=1}^{4n+1} S_1$

Thus, $S = \sum_{m=1}^{4n+1} i^m = \frac{i(i^{n+1} - 1)}{i - 1} = \frac{i(i - 1)}{i - 1} = i$

which is purely imaginary and independent of n. Also, i is not a root of $x^{4n+1} + 1 = 0$.

The correct option is (A) and (B)

157. We have,

$|a + ib| = 1 \Rightarrow a + ib = \cos\theta + i\sin\theta$

$|c + id| = 1 \Rightarrow c + id = \cos\phi + i\sin\phi$

$\Rightarrow \quad a = \cos\theta, b = \sin\theta, c = \cos\phi, d = \sin\phi$

Now, $z_1\bar{z}_2 = (\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi)$

$\Rightarrow \quad \text{Re}(z_1\bar{z}_2) = \cos\theta\cos\phi + \sin\theta\sin\phi$

Thus, $= 0 \Rightarrow \cos(\theta - \phi) = 0$ (1)

$\Rightarrow \quad \theta - \phi = \frac{\pi}{2}$ or $-\frac{\pi}{2}$

Now, $|w_1|^2 = a^2 + c^2 = \cos^2\theta + \cos^2\phi$

$$= \cos^2\left(\phi \pm \frac{\pi}{2}\right) + \cos^2\phi$$

$$= \sin^2\phi + \cos^2\phi = 1$$

Thus, $|w_1| = 1$. Similarly, $|w_2| = 1$

Now, $w_1\bar{w}_2 = (\cos\theta + i\sin\theta)\cdot(\cos\phi - i\sin\phi)$

$\Rightarrow \quad \text{Re}(w_1\bar{w}_2) = (\cos\theta\sin\theta + \cos\phi\sin\phi)$

$$= \frac{1}{2}(\sin 2\theta + \sin 2\phi)$$

$$= \sin(\theta + \phi)\cos(\theta - \phi)$$

$= 0$ [From (1)]

The correct option is (A), (B) and (C)

158. We have,

$$\arg(z^{3/8}) = \frac{1}{2}\arg(z^2 + \bar{z}z^{1/2})$$

$$\Rightarrow \quad 2\arg(z^{3/8}) = \arg(z^2 + \bar{z}\,z^{1/2})$$

$$\Rightarrow \quad \arg(z^{3/4}) - \arg(z^2 + \bar{z}\,z^{1/2}) = 0$$

$[\because 2\arg(z) = \arg(z^2)]$

$$\Rightarrow \quad \arg\left(\frac{z^2 + \bar{z}\,z^{1/2}}{z^{3/4}}\right) = 0$$

$$\Rightarrow \quad Im\left(z^{5/4} + \frac{\bar{z}}{z^{1/4}}\right) = 0$$

$$\Rightarrow \quad z^{5/4} + \frac{\bar{z}}{z^{1/4}} = \left(\overline{z^{5/4} + \frac{\bar{z}}{z^{1/4}}}\right)$$

$$\Rightarrow \quad z^{5/4} + \frac{\bar{z}}{z^{1/4}} = (\bar{z})^{5/4} + \frac{z}{(\bar{z})^{1/4}}$$

$$\Rightarrow \quad z^{5/4} + \frac{\bar{z}\,(\bar{z})^{1/4}}{|z|^{1/2}} = (\bar{z})^{5/4} + \frac{zz^{1/4}}{|z|^{1/2}}$$

$$\Rightarrow \quad z^{5/4} - (\bar{z})^{5/4} = \frac{z^{5/4} - (\bar{z})^{5/4}}{|z|^{1/2}}$$

$$\Rightarrow \quad \left\{z^{5/4} - (\bar{z})^{5/4}\right\}\left(1 - \frac{1}{|z|^{1/2}}\right) = 0$$

$\therefore \quad z = \bar{z}$ or $|z| = 1$.

The correct option is (A), (B) and (D)

159. We have, $z_1{}^2 + 2z_2{}^2 + z_3{}^2 = 2z_2\,(z_1 + z_3)$

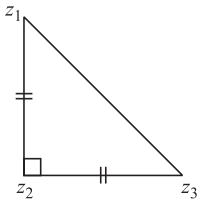
$\Rightarrow \quad (z_1{}^2 - 2z_1z_2 + z_2{}^2) + (z_2{}^2 + z_3{}^2 - 2z_2z_3) = 0$

$\Rightarrow \quad (z_1 - z_2)^2 + (z_3 - z_2)^2 = 0$

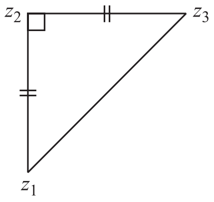
$\Rightarrow \quad (z_1 - z_2) = \pm\, i\,(z_3 - z_2)$

$\Rightarrow \quad (z_1 - z_2) = (z_3 - z_2)\, e^{i(\pi/2)},\ (z_3 - z_2)e^{-i(\pi/2)}$

i.e.,



or



Thus, in either case the triangle is an isosceles right angled triangle, right angled at z_2.

The correct option is (A) and (B)

160. We have,

$|z_1 - z_2|^2 = (|z_1| + |z_2|)^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 - 2\,|z_1|\,|z_2| \cos\theta$

$= |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2|$

$\Rightarrow \quad \cos\theta = -1$ where $\theta = \arg\,(z_1/z_2)$

$\Rightarrow \quad \arg\,(z_1/z_2) = (2n + 1)\,\pi,\ n \in I$

$\Rightarrow \quad z_1/z_2$ lies on the negative real axis

Hence, we can write $z_1 = l\,z_2,\ l \in R^-$

Also, we have,

$Re\,(z_1/z_2) \le 0$

$$\Rightarrow \quad \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} \le 0 \text{ or } z_1\bar{z}_2 + z_2\bar{z}_1 \le 0$$

The correct option is (B) and (C)

161. Let $z_1 = a + ib,\ z_2 = c + id$.

Then, $|z_1| = |z_2| = 1 \Rightarrow a^2 + b^2 = 1$ and $c^2 + d^2 = 1$ (1)

Also, $\text{Re}\,(z_1\ \bar{z}_2) = 0 \Rightarrow \text{Re}\,\{(a + ib)\,(c - id)\} = 0$

or, $\quad ac - bd = 0$ (2)

or, $\quad ac = -\,bd$

$$\text{or} \quad \frac{a}{b} = \frac{a}{-c} = \lambda \quad \text{(say)}$$

$\therefore \quad a^2 + b^2 = 1 \Rightarrow b^2 + b^2\,\lambda^2 = 1$

or, $\quad b^2\,(1 + \lambda^2) = 1$ (3)

and, $\quad c^2 + d^2 = 1 \Rightarrow d^2\left(1 + \dfrac{1}{\lambda^2}\right) = 1$

or, $\quad d^2\,(1 + \lambda^2) = \lambda^2$ (4)

$$\therefore \quad \frac{b^2}{d^2} = \frac{1}{\lambda^2} \text{ or } d^2 = b^2\,\lambda^2$$

$$\text{Now,} \quad |\omega_1| = a^2 + c^2 = b^2\,\lambda^2 + \frac{d^2}{\lambda^2}$$

$$= b^2\,\lambda^2 + \frac{b^2\,\lambda^2}{\lambda^2} = b^2\,(1 + \lambda^2) = 1$$

Also, $\quad \text{Re}\,(\omega_1\ \bar{\omega}_2) = \text{Re}\,\{(a + ic)\,(b - id)\}$

$$= (ab + cd) = b^2\,\lambda + d\left(-\frac{d}{\lambda}\right)$$

$$= b^2\,\lambda - \frac{b^2\,\lambda^2}{\lambda} = 0.$$

Hence, $|\omega_1| = 1$ and $\text{Re}\,(\omega_1\ \bar{\omega}_2) = 0$

The correct option is (A) and (C)

162. Let A, B, C be the vertices of the equilateral triangle represented by the complex numbers z_1, z_2, z_3, respectively.

Then, $AB = BC = AC$ and $\angle A = \angle B = \angle C = \dfrac{\pi}{3}$.

$$\therefore \quad \frac{z_3 - z_1}{z_2 - z_1} = e^{i\frac{\pi}{3}} = \frac{z_1 - z_2}{z_3 - z_2} = \frac{z_2 - z_3}{z_1 - z_3} \tag{1}$$

$\Rightarrow \quad (z_3 - z_1)\,(z_3 - z_2) = -\,(z_1 - z_2)^2$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1\,z_2 + z_2\,z_3 + z_3\,z_1 \tag{2}$$

$$\Rightarrow \quad [(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2] = 0 \tag{3}$$

Now, $3\,z_0 = z_1 + z_2 + z_3$, [z_0 is centroid]

$\therefore \quad z_1^2 + z_2^2 + z_3^2 + 2\,(z_1\,z_2 + z_2\,z_3 + z_3\,z_1) = 9\,z_0^2$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = 3\,z_0^2 \tag{4}$$

From (1), we also have

$(z_3 - z_1)\,(z_3 - z_2) + (z_1 - z_2)\,(z_1 - z_2)\,(z_1 - z_3 + z_3 - z_2) = 0$

$\Rightarrow \quad (z_2 - z_1)\,(z_3 - z_1) + (z_1 - z_2)\,(z_3 - z_2)$

$\quad + (z_1 - z_2)\,(z_1 - z_3) = 0$

Dividing by $(z_1 - z_2)\,(z_2 - z_3)\,(z_3 - z_1)$, we get

$$\frac{1}{z_2 - z_1} + \frac{1}{z_1 - z_3} + \frac{1}{z_3 - z_2} = 0$$

The correct option is (A), (B), (C) and (D)

163. Since a, b, c, … k are the roots of the given equation, we have the identity

$x^n + p_1\,x^{n-1} + p_2\,x^{n-2} + \ldots + p_{n-1}\,x + p_n$

$\equiv (x - a)\,(x - b)\,(x - c)\,\ldots\,(x - k)$ (1)

In the identity (1), put $x = i$

Then, $i^n + p_1\,i^{n-1} + p_2\,i^{n-2} + \ldots + p_{n-1}\,i + p_n$

$\equiv (i - a)\,(i - b)\,(i - c)\,\ldots\,(i - k)$

or, $\quad i^n\,[1 + p_1\,i^{-1} + p_2\,i^{-2} + p_3\,i^{-3} + p_4\,i^{-4} + \ldots$

$\quad + p_{n-1}\,i^{-(n-1)} + p_n\,i^{-n}]$

$\equiv (i - a)\,(i - b)\,(i - c)\,\ldots\,(i - k)$.

But, $i^{-1} = \dfrac{1}{i} = -i,\ i^{-2} = \dfrac{1}{i^2} = -1$

$i^{-3} = \dfrac{1}{i^3} = i,\ i^{-4} = \dfrac{1}{i^4} = 1$ etc.

∴ The above identity may be written as

$i^n [(1 - p_2 + p_4 - \ldots) - i\,(p_1 - p_3 + p_5 - \ldots)]$

$\equiv (-1)^n (a - i)(b - i)(c - i) \ldots (k - i)$ (2)

Similarly, putting $x = -i$ in (1), we shall obtain

$(-i)^n [(1 - p_2 + p_4 + \ldots) + i\,(p_1 - p_3 + p_5 \ldots)]$

$\equiv (-1)^n (a + i)(b + i)(c + i) \ldots (k + i)$ (3)

Multiplying (2) and (3), we get

$(-1)^n \times i^{2n} [(1 - p_2 + p_4 \ldots)^2 - i^2 (p_1 - p_3 + p_5 \ldots)^2]$

$= (-1)^{2n} (a^2 - i^2)(b^2 - i^2)(c^2 - i^2) \ldots (k^2 - i^2)$

∵ $(-1)^n i^{2n} = (-1)^n (-1)^n = (-1)^{2n} = 1$, this gives

$(1 - p_2 + p_4 \ldots)^2 + (p_1 - p_3 + p_5 \ldots)^2$

$= (a^2 + 1)(b^2 + 1)(c^2 + 1) \ldots (k^2 + 1)$

The correct option is (A) and (B)

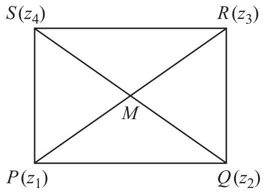
164. Since the diagonals of the square bisect each other,

∴ mid-point of PR $= \dfrac{z_1 + z_3}{2}$ (1)

mid-point of QS $= \dfrac{z_2 + z_4}{2}$ (2)

From (1) and (2), we get $z_1 + z_3 = z_2 + z_4$

⇒ (a) is correct



Since all the sides of a square are equal

⇒ $PQ = QR = RS = SP$

⇒ $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_4| = |z_4 - z_1|$

⇒ (b) is correct

We know that the diagonals of a square are equal

∴ $PR = QS$

⇒ $|z_1 - z_3| = |z_2 - z_4|$ ⇒ (c) is correct

Since the diagonals are perpendicular to each other

∴ $\dfrac{z_1 - z_3}{z_2 - z_4} = \left|\dfrac{z_1 - z_3}{z_2 - z_4}\right| (\cos\alpha + i\sin\alpha)$ where α is 90º

This is purely imaginary. Therefore, real part of $\dfrac{z_1 - z_3}{z_2 - z_4}$ is zero.

⇒ (d) is correct

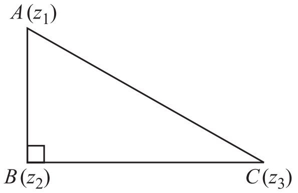
The correct option is (A), (B), (C) and (D)

165. Given $AB = BC$ (isosceles triangle), $\angle B = 90º$

∴ $\angle C = \angle A = \dfrac{\pi}{4}$

∴ $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{AC}{AB} e^{i\pi/4} = \sqrt{2}\, e^{i\pi/4}$ (1)

$\dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{BC}{AC} e^{i\pi/4} = \dfrac{1}{\sqrt{2}} e^{i\pi/4}$ (2)



From equations (1) and (2), we get

$$\frac{\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)}{\left(\dfrac{z_2 - z_3}{z_1 - z_3}\right)} = \frac{\sqrt{2}\, e^{i\pi/4}}{\dfrac{1}{\sqrt{2}} e^{i\pi/4}} = 2$$

⇒ $\dfrac{z_3 - z_1}{z_2 - z_1} \cdot \dfrac{z_1 - z_3}{z_2 - z_3} = 2$

⇒ $-(z_3 - z_1)^2 = 2\,(z_2^{\,2} - z_2 z_3 - z_1 z_2 + z_1 z_3)$

⇒ $z_1^{\,2} + z_3^{\,2} - 2\,z_1 z_3 = -2\,z_2^{\,2} + 2\,z_2 z_3 + 2\,z_1 z_2 - 2\,z_1 z_3$

⇒ $z_1^{\,2} + z_3^{\,2} + 2\,z_2^{\,2} = 2\,z_2 (z_1 + z_3)$

or, $z_1^{\,2} + z_3^{\,2} = 2 z_2\,(z_1 + z_3 - z_2)$

or, $z_1^{\,2} + z_2^{\,2} + z_2^{\,2} + z_3^{\,2} - 2z_1 z_2 - 2z_2 z_3 = 0$

or, $(z_1 - z_2)^2 + (z_2 - z_3)^2 = 0$

or, $\left(\dfrac{z_1 - z_2}{z_2 - z_3}\right)^2 = -1 = \cos\pi + i\sin\pi$

or, $\dfrac{z_1 - z_2}{z_2 - z_3} = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = i$ (imaginary)

The correct option is (A), (B), (C) and (D)

166. We have, $\angle DAC = \dfrac{\pi}{2} - C$ and $OC = OD$

∴ $\dfrac{z}{z_3} = \cos(\pi - 2C) + i\sin(\pi - 2C)$

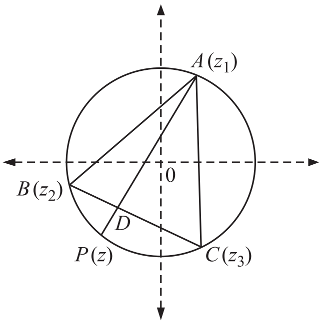
or, $\dfrac{z}{z_3} = -\cos 2C + i\sin 2C$ (1)

Again, $\angle AOB = 2C$ and $OA = OB$

∴ $\dfrac{z_1}{z_2} = \cos 2C + i\sin 2C$ (2)

Multiply (1) and (2), we get

$\dfrac{z z_1}{z_2 z_3} = -1$



$$\Rightarrow \quad z = \frac{-z_2 z_3}{z_1} = \frac{-\bar{z}_1 \bar{z}_2 z_2 z_3}{z_1 \bar{z}_1 \bar{z}_2} = \frac{-\bar{z}_1 z_3}{\bar{z}_2}$$

$$(\because z_1 \bar{z}_1 = z_2 \bar{z}_2)$$

$$= \frac{-\bar{z}_1 z_2}{\bar{z}_3}.$$

The correct option is (B), (C) and (D)

167. We have,

$|z_1 + z_2| = |z_1 - z_2|$

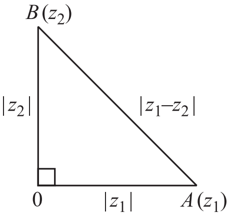
$\Rightarrow \quad (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow \quad z_1\bar{z}_2 + z_2\bar{z}_1 = 0 \qquad (1)$

$\Rightarrow \quad \dfrac{z_1}{z_2} = -\left(\overline{\dfrac{z_1}{z_2}}\right)$

$\Rightarrow \quad \dfrac{z_1}{z_2}$ is purely imaginary



Also, from (1),

$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow$ ΔOAB is a right angled triangle, right angled at O.

So, orthocentre lies at O and circumcentre $= \dfrac{z_1 + z_2}{2}$.

The correct option is (A), (B) and (C)

168. The roots of $x^2 + x + 1 = 0$ are w and w^2

So, $h(w) = 0$ and $h(w)^2 = 0$

$\Rightarrow \quad w f(1) + w^2 g(1) = 0$ and $w^2 f(1) + w g(1) = 0$

$\Rightarrow \quad f(1) = g(1) = 0$

$\therefore \quad h(1) = f(1) + g(1) = 0.$

The correct option is (A), (B) and (C)

169. We have, $\alpha = \cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}$

and, $\quad 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

$\therefore \quad |1 + \alpha + \alpha^2 + \alpha^3| = |-\alpha^4| = |\alpha|^4 = 1$

Also, $\quad |1 + \alpha + \alpha^2| = |-\alpha^3(1 + \alpha)| = |1 + \alpha| \qquad (1)$

$$= \left|1 + \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right|$$

$$= \left|2\cos\frac{\pi}{5}\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right|$$

$$= 2\cos\frac{\pi}{5}$$

Again, from (i), $|1 + \alpha| = |1 + \alpha + \alpha^2| = 2\cos\dfrac{\pi}{5}$.

The correct option is (A), (B) and (C)

Passage Based Questions

170. It is obvious that we must consider the 17th root of unity. Consider the equation $z^{17} - 1 = 0$. The roots of this equation are

$$z = \cos\frac{2r\pi}{17} + i\sin\frac{2r\pi}{17}, \; r = 0, 1, \ldots, 16 \qquad \text{(i)}$$

Since the sum of the roots of (i) is zero, therefore,

$$\sum_{r=0}^{16}\left(\cos\frac{2r\pi}{17} + i\sin\frac{2r\pi}{17}\right) = 0$$

Equating real and imaginary parts on both sides, we have

$$\sum_{r=0}^{16}\cos\frac{2r\pi}{17} = 0, \quad \sum_{r=0}^{16}\sin\frac{2r\pi}{17} = 0$$

$$\therefore \quad \sum_{r=1}^{16}\cos\frac{2r\pi}{17} = -\cos 0 = -1,$$

$$\sum_{r=1}^{16}\sin\frac{2r\pi}{17} = -\sin 0 = 0$$

Hence, the desired sum $= 0 + i(-1) = -i$

The correct option is (D)

171. Let $a + ib = r(\cos\theta + i\sin\theta) \qquad$ (i)

Then, $r\cos\theta = a$ and $r\sin\theta = b$

$$\Rightarrow \quad r^2 = a^2 + b^2, \; \tan\theta = \frac{b}{a} \text{ or } \theta = \tan^{-1}\frac{b}{a}. \qquad \text{(ii)}$$

Taking conjugates of both sides of (i), we get

$a - ib = r(\cos\theta - i\sin\theta)$.

Now, by De Moivre's theorem, one of the values of

$(a + ib)^{m/n}$ is $r^{m/n}\left(\cos\dfrac{m\theta}{n} + i\sin\dfrac{m\theta}{n}\right)$

and one of the values of

$(a - ib)^{m/n}$ is $r^{m/n}\left(\cos\dfrac{m\theta}{n} - i\sin\dfrac{m\theta}{n}\right)$

Hence, one of the values of $(a + ib)^{m/n} + (a - ib)^{m/n}$ is

$$= 2r^{m/n}\cos\frac{m\theta}{n}$$

$$= 2(a^2 + b^2)^{m/2n}\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right), \text{ using (ii).}$$

The correct option is (B)

172. We have, $(16)^{1/4} = (2^4)^{1/4} = 2(1)^{1/4}$

$$= 2(\cos 0 + i\sin 0)^{1/4}$$

$$= 2\left\{\cos\frac{1}{4}(2k\pi + 0) + i\sin\frac{1}{4}(2k\pi + 0)\right\}, \; k = 0, 1, 2, 3$$

$= 2 \times 1, 2 \times i, 2 \times -1, 2 \times -i = \pm 2, \pm 2i$.

The correct option is (A)

173. We have, $z^4 + 1 = 0 \Rightarrow z^4 = -1$

$\Rightarrow \quad z = (\cos\pi + i\sin\pi)^{1/4}$

$$\Rightarrow \quad z = \cos\frac{1}{4}(2k\pi + \pi) + i\sin\frac{1}{4}(2k\pi + \pi),$$

$$k = 0, 1, 2, 3.$$

$$\Rightarrow \quad z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4},\ \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4},$$

$$\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4},\ \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$$

$$= \frac{1}{\sqrt{2}}(1+i),\ \frac{1}{\sqrt{2}}(-1+i),\ \frac{1}{\sqrt{2}}(-1-i),\ \frac{1}{\sqrt{2}}(1-i).$$

Hence, the four roots of $z^4 + 1 = 0$ are $(\pm 1 \pm i)$.

The correct option is (C)

174. Since $1, \omega, \omega^2, \ldots \omega^{n-1}$ are the n, nth roots of unity

$$\therefore \quad \sum_{k=0}^{n-1} \omega^k = 0 \text{ and } \sum_{k=0}^{n-1} (\bar{\omega})^k = 0$$

$$\text{Now, } \sum_{k=0}^{n-1} |z_1 + \omega^k z_2|^2 = \sum_{k=0}^{n-1} (z_1 + \omega^k z_2)(\bar{z}_1 + (\bar{\omega})^k \bar{z}_2)$$

$$= \sum_{k=0}^{n-1} z_1\bar{z}_1 + z_1\bar{z}_2(\bar{\omega})^k + \bar{z}_1 z_2\,\omega^k + z_2\bar{z}_2(\omega^k)(\bar{\omega})^k$$

$$= \sum_{k=0}^{n-1} |z_1|^2 + \sum_{k=0}^{n-1} z_1\bar{z}_2(\bar{\omega})^k + \sum_{k=0}^{n-1} \bar{z}_1 z_2 \omega^k + \sum_{k=0}^{n-1} |z_2|^2$$

$$= n\,|z_1|^2 + 0 + 0 + n\,|z_2|^2 = n\,(|z_1|^2 + |z_2|^2)$$

The correct option is (A)

175. Let $= \sqrt[n]{1} = x$;

$\therefore \quad x^n = 1;$

$\therefore \quad x^n - 1 = 0$

$\therefore \quad x^n - 1 = (x-1)(x-a_1)(x-a_2)\ldots(x-a_{n-1})$

$\therefore \quad (x-a_1)(x-a_2)(x-a_3)\ldots(x-a_{n-1})$

$$= \frac{x^n - 1}{x-1} = \frac{1-x^n}{1-x} = 1 + x + x^2 + \ldots + x^{n-1}$$

Putting $x = 1$, we get

$(1-a_1)(1-a_2)(1-a_3)\ldots(1-a_{n-1}) = n$

The correct option is (B)

176. Since $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are the n, nth roots of unity,

$\therefore \quad x^n - 1 = (x-1)(x-\alpha)(x-\alpha^2)\ldots(x-\alpha^{n-1})$

$\Rightarrow \quad \log(x^n - 1) = \log(x-1) + \log(x-\alpha)$

$\quad + \log(x-\alpha^2) + \ldots + \log(x-\alpha^{n-1})$

Differentiating both sides with respect to 'x', we get

$$\frac{x^{n-1}}{x^n - 1} = \frac{1}{x-1} + \frac{1}{x-\alpha} + \frac{1}{x-\alpha^2} + \ldots + \frac{1}{x-\alpha^{n-1}}$$

Putting $x = 2$, we get

$$\frac{n2^{n-1}}{2^n - 1} = \frac{1}{1} + \frac{1}{2-\alpha} + \frac{1}{2-\alpha^2} + \ldots + \frac{1}{2-\alpha^{n-1}}$$

$$\therefore \quad \frac{n\cdot 2^{n-1}}{2^n - 1} - 1 = \sum_{i=1}^{n-1} \frac{1}{2-\alpha^i}$$

$$\text{Hence,} \quad \sum_{i=1}^{n-1} \frac{1}{2-\alpha^i} = \frac{n\cdot 2^{n-1} - 2^n + 1}{2^n - 1} = \frac{(n-2)2^{n-1} + 1}{2^n - 1}$$

The correct option is (A)

177. Let $a^n = 1$

$(a^n - 1) = (a-1)(a-\omega)(a-\omega^2)\ldots(a-\omega^{n-1})$

$$\Rightarrow \quad (a-\omega)(a-\omega^2)\ldots(a-\omega^{n-1}) = \frac{a^n - 1}{a-1}$$

$$\Rightarrow \quad \lim_{a\to 1}(a-\omega)(a-\omega^2)\ldots(a-\omega^{n-1}) = \lim_{a\to 1}\frac{a^n - 1}{a-1}$$

$\Rightarrow \quad (1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1}) = n$

The correct option is (C)

178. Solving $x^6 + x^3 + 1 = 0$ as a quadratic in x^3, we get

$$x^3 = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2} = r(\cos\theta \pm i\sin\theta)$$

$$\therefore \quad r\cos\theta = \frac{-1}{2},\ r\sin\theta = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad r^2 = 1 \text{ and } \tan\theta = -\sqrt{3}$$

$$\Rightarrow \quad r = 1 \text{ and } \theta = \frac{2\pi}{3} \Rightarrow x^3 = \cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}$$

$$\therefore \quad x = \left[\cos\left(2k\pi + \frac{2\pi}{3}\right) \pm i\sin\left(2k\pi + \frac{2\pi}{3}\right)\right]^{1/3}$$

$$= \cos(6k+2)\frac{\pi}{9} \pm i\sin(6k+2)\frac{\pi}{9},\ k = 0, 1, 2$$

Hence, $\cos\left(\frac{p\pi}{9}\right) \pm i\sin\left(\frac{p\pi}{9}\right)$, $p = 2, 8, 14$ are the required roots.

The correct option is (A), (B) and (C)

179. Solving $z^4 + 4z^2 + 16 = 0$ as a quadratic in z^2, $z^2 = \dfrac{-4 \pm \sqrt{16-64}}{2} = -2 \pm 2\sqrt{3}\,i$

Let $z^2 = -2 \pm 2\sqrt{3}\,i = r(\cos\theta \pm i\sin\theta)$ (1)

$\therefore \quad r\cos\theta = -2$ and $r\sin\theta = 2\sqrt{3}$

$\Rightarrow \quad r^2 = 16$ and $\tan\theta = -\sqrt{3} \Rightarrow r = 4,\ \theta = \dfrac{2\pi}{3}$ (2)

$$z = 2\left(\cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}\right)^{1/2}, \text{ by (1) and (2)}$$

$$= 2\left[\cos\left(2k\pi + \frac{2\pi}{3}\right)\cdot\frac{1}{2} \pm i\sin\left(2k\pi + \frac{2\pi}{3}\right)\cdot\frac{1}{2}\right]$$

$k = 0, 1$

$$= 2\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right),\ 2\left(\cos\frac{4\pi}{3} \pm i\sin\frac{4\pi}{3}\right)$$

$$= 2\left(\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right),\ -2\left(\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)$$

Hence, $\pm 1 \pm i\sqrt{3}$ are the required roots.

The correct option is (A) and (B)

180. Let $\dfrac{2+z}{2-z} = w$ (i)

Then, $\quad w^6 = -1 = \text{cis}\,(2p+1)\pi$, where π is an integer.

$\therefore \quad w = \text{cis}\,[(2p+1)\,\pi/6]$, where $p = 0, 1, \ldots 5$

From (i), we have,

$$2 + z = w\,(2 - z) \text{ or } z = \frac{2(w-1)}{w+1}$$

$$= \frac{2\,[\cos\{(2p+1)\pi/6\} + i\sin\{(2p+1)\pi/6\} - 1]}{\cos\{(2p+1)\pi/6\} + i\sin\{(2p+1)\pi/6\} + 1}$$

$$= \frac{2\,[2i\sin\{(2p+1)\pi/12\}\cos\{(2p+1)\pi/12\} - 2\sin^2\{(2p+1)\pi/12\}]}{2i\sin\{(2p+1)\pi/12\}\cos\{(2p+1)\pi/12\} + 2\cos^2\{(2p+1)\pi/12\}}$$

$= 2i \tan[(2p + 1)\,\pi/12]$, $p = 0, 1, \ldots, 5$

On giving the values 0, 1, …, 5 the roots are seen to be $\pm 2i \tan \pi/12$, $\pm 2i \tan 5\pi/12$, $\pm 2i$.

The correct option is (A), (B) and (C)

181. Multiplying the given equation by $(z + 1)$ and simplifying, we obtain, $z^5 + 1 = 0$, whose roots are $z = \cos(p\pi/5) + i\sin(p\pi/5)$, $p = 1, 3, 5, 7, 9$.

For $p = 5$, the root $z = -1$ corresponds to $z + 1 = 0$

Hence, the required roots are

$\cos(p\pi/5) + i\sin(p\pi/5)$; $p = 1, 3, 7, 9$.

The correct option is (B)

Match the Column Type

182. **I.** Since $z_r = \cos\left(\frac{\pi}{3^r}\right) + i\sin\left(\frac{\pi}{3^r}\right)$, $r = 1, 2, 3, \ldots$

we have, $z_1 \cdot z_2 \cdot z_3 \ldots\infty$

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}\right)\ldots\infty$$

$$= \cos\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots\right)$$

$$= \cos\left(\frac{\frac{\pi}{3}}{1 - \frac{1}{3}}\right) + i\sin\left(\frac{\frac{\pi}{3}}{1 - \frac{1}{3}}\right)$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i \times 1 = \boldsymbol{i}$$

The correct option is (D)

II. Given, $i z^3 + z^2 - z + i = 0$

$\Rightarrow\ i z^2 (z - i) - (z - i) = 0$

$\Rightarrow\ (z - i)(i z^2 - 1) = 0 \Rightarrow z = i$ or $z^2 = \frac{1}{i} = -i$.

Now, $z = -i \Rightarrow |z| = |i| = 1$

and, $z^2 = -i \Rightarrow |z^2| = |-i| \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$

Thus, in both cases $|\boldsymbol{z}| = 1$

The correct option is (B)

III. Let $z = x + iy$

Then, $\frac{z-2}{z+2} = \frac{x + iy - 2}{x + iy + 2} = \frac{(x-2) + iy}{(x+2) + iy}$

$$= \frac{[(x-2) + iy][(x+2) - iy]}{(x+2)^2 + y^2}$$

$$= \frac{(x^2 + y^2 - 4) + i(4y)}{(x+2)^2 + y^2}$$

Since $\frac{z-2}{z+2}$ is purely imaginary,

$\therefore\ x^2 + y^2 - 4 = 0$

$\Rightarrow\ x^2 + y^2 = 4 \Rightarrow |z|^2 = 4 \Rightarrow |z| = 2$

The correct option is (C)

IV. $$\sum_{n=1}^{13} (i^n + i^{n+1}) = \sum_{n=1}^{13} i^n (1+i) = (1+i)\left\{\frac{i(1 - i^{13})}{1 - i}\right\}$$

$$= (1+i)\left\{\frac{i(1-i)}{1-i}\right\}$$

$$= (1+i)\,i = \boldsymbol{-1 + i}$$

The correct option is (A)

183. **I.** We have, $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots$ to $2n$ factors

$= (1 + \omega)(1 + \omega^2)(1 + \omega^3 \times \omega)(1 + \omega^6 \times \omega^2)$ … to $2n$ factors

$= (1 + \omega)(1 + \omega^2)(1 + \omega)(1 + \omega^2)$ … to $2n$ factors

[$\because\ \omega^3 = \omega^6 = \ldots = 1$]

$= [(1 + \omega)(1 + \omega)$ … to n factors]

[$(1 + \omega^2)(1 + \omega^2)$ … to n factors]

$= (1 + \omega)^n (1 + \omega^2)^n = [(1 + \omega)(1 + \omega^2)]^n$

$= (1 + \omega + \omega^2 + \omega^3)^n = (0 + 1)^n = 1$

[$\because\ 1 + \omega + \omega^2 = 0,\ \omega^3 = 1$]

The correct option is (B)

II. We have,

$(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$

$(1 - \omega^8 + \omega^{16})$… to $2n$ factors

$= (1 - \omega + \omega^2)(1 - \omega^2 + \omega)(1 - \omega + \omega^2)$

$(1 - \omega^2 + \omega)$ … to $2n$ factors.

[$\because\ \omega^4 = \omega,\ \omega^8 = \omega^2,\ \omega^{16} = \omega$ and so on]

$= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2)$ … to $2n$ factors

$= (2^2\,\omega^3)(2^2\,\omega^3)$ … to n factors

[$\because\ (-2\omega)(-2\omega^2) = 2^2\,\omega^3 = 2^2$]

$= (2^2)^n = \boldsymbol{2^{2n}}$

The correct option is (C)

III. Since $(\sqrt{3}\ + i)^{100} = 2^{99}\ (a + ib)$

$\therefore\quad (\sqrt{3}\ - i)^{100} = 2^{99}\ (a - ib)$

$\therefore\quad (\sqrt{3}\ + i)^{100} - (\sqrt{3}\ - i)^{100} = 2^{99}\ (2ib) = 2^{100}\ (ib)$

$\Rightarrow\quad i^{100}\ [1 - \sqrt{3}\ i]^{100} - i^{100}\ [-1 + \sqrt{3}\ i]^{100} = 2^{100}\ (ib)$

or, $\quad (-2\omega)^{100} - (2\omega^2)^{100} = 2^{100}\ (ib)$

or, $\quad \omega - \omega^2 = (ib)$ or $\sqrt{3}\ i = ib$

$\therefore\quad b = \sqrt{3}$

The correct option is (A)

IV. The given expression

$$= \sum_{2}^{n} (n-1)(n-w)(n-w^2)$$

$$= \sum_{1}^{n} (n-1)(n-w)(n-w^2)$$

$$[\because (1-1)(1-w)(1-w^2) = 0]$$

$$= \sum_{1}^{n} (n^3 - 1) = \frac{n^2(n+1)^2}{4} - n$$

The correct option is (D)

184. **I.** We have,

$$\frac{\sqrt{3}+i}{2} = \frac{i\sqrt{3}+i^2}{2i} = -i\left(\frac{-1+\sqrt{3}\,i}{2}\right) = -i\omega$$

and, $$\frac{i-\sqrt{3}}{2} = \frac{i^2 - i\sqrt{3}}{2i} = -i\left(\frac{-1-\sqrt{3}\,i}{2}\right) = -i\omega^2$$

$$\left[\because \frac{1}{i} = -i\right]$$

Hence, $$\left(\frac{\sqrt{3}+i}{2}\right)^6 + \left(\frac{i-\sqrt{3}}{2}\right)^6 = (-i\omega)^6 + (-i\omega^2)^6$$

$$= i^6\,(\omega^6 + \omega^{12})$$

$$= -1\,(1+1) = \mathbf{-2.}$$

The correct option is (C)

II. Let $\dfrac{z-1}{z+1} = iy$, where y is real

$$\Rightarrow\quad \frac{z+1}{z-1} = \frac{1}{iy}$$

$$\Rightarrow\quad \frac{2z}{2} = \frac{1+iy}{1-iy}\qquad \text{[by componendo and dividendo]}$$

$$\Rightarrow\quad z = \frac{1+iy}{1-iy} \Rightarrow |z| = \frac{\sqrt{1+y^2}}{\sqrt{1+y^2}} = \mathbf{1}$$

The correct option is (D)

III. We have, $i + \sqrt{3} = \dfrac{-1+\sqrt{3}\,i}{2}\cdot\dfrac{2}{i} = \dfrac{2\omega}{i}$

and, $\quad i - \sqrt{3} = \dfrac{-1-\sqrt{3}\,i}{2}\cdot\dfrac{2}{i} = \dfrac{2\omega^2}{i}$

$\therefore\quad (i + \sqrt{3}\ )^{100} + (i - \sqrt{3}\ )^{100} + 2^{100}$

$$= \left(\frac{2\omega}{i}\right)^{100} + \left(\frac{2\omega^2}{i}\right)^{100} + 2^{100}$$

$$= \frac{2^{100}}{i^{100}}\ (\omega^{100} + \omega^{200}) + 2^{100}$$

$$= 2^{100}\,(\omega + \omega^2) + 2^{100}$$

$$= -2^{100} + 2^{100} = \mathbf{0}$$

The correct option is (B)

IV. Let $z_k = x_k + iy_k$, we have $(z_k + 1)^7 + z_k^7 = 0$

$\Rightarrow\quad (z_k + 1)^7 = -z_k^7 \Rightarrow |z_k + 1|^7 = |z_k|^7$

$\Rightarrow\quad |z_k + 1| = |z_k| \Rightarrow |x_k + iy_k + 1|^2 = |x_k + iy_k|^2$

$\Rightarrow\quad (x_k + 1)^2 + y_k^2 = x_k^2 + y_k^2$

$\Rightarrow\quad 2x_k + 1 = 0$ or $x_k = -\dfrac{1}{2}$

Thus, $\displaystyle\sum_{k=0}^{6} \operatorname{Re}(z_k) = \sum_{k=0}^{6} x_k = -\frac{7}{2}$

The correct option is (A)

Assertion-Reasoning Type

185. We have,

$\dfrac{1}{a} = \cos\alpha - i\sin\alpha,\ \dfrac{1}{b} = \cos\beta - i\sin\beta$

Now, $\quad \dfrac{a}{b} = (\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)$

or, $\quad \dfrac{a}{b} = \cos(\alpha - \beta) + i\sin(\alpha - \beta).$

Similarly, $\dfrac{b}{c} = \cos(\beta - \gamma) + i\sin(\beta - \gamma)$

and, $\quad \dfrac{c}{a} = \cos(\gamma - \alpha) + i\sin(\gamma - \alpha).$

Putting these values in $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} = -1$, we get

$[\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha)]$

$+\, i\,[\sin(\alpha - \beta) + \sin(\beta - \gamma) + \sin(\gamma - \alpha)]$

$= -1 = -1 + 0\,i.$

Comparing real and imaginary parts, we get

$\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -1$

The correct option is (A)

186. We have, $\arg\dfrac{z-1}{z+1} = \dfrac{\pi}{3}$

$$\Rightarrow\quad \arg\frac{x+iy-1}{x+iy+1} = \frac{\pi}{3}\qquad [\text{Putting } z = x+iy]$$

$$\Rightarrow\quad \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = \frac{\pi}{3}$$

$$\left(\because \operatorname{Arg}\frac{z_1}{z_2} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2\right)$$

$$\Rightarrow \quad \tan^{-1}\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\frac{y^2}{x^2-1}} = \frac{\pi}{3}$$

$$\Rightarrow \quad \frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{3} = \sqrt{3}$$

$\Rightarrow \quad x^2+y^2-\frac{2}{\sqrt{3}}\,y-1=0$, which is a **circle.**

The correct option is (D)

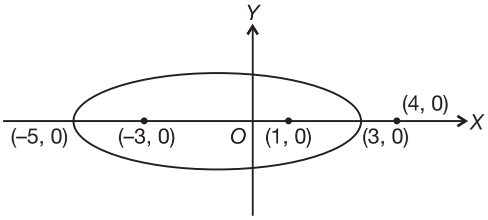
187. Area of the triangle on the argand plane formed by the complex numbers $-z$, iz, $z-iz$ is $\frac{3}{2}\,|z|^2$.

$\therefore \quad \frac{3}{2}\,|z|^2 = 600 \Rightarrow |z| = \mathbf{20.}$

The correct option is (A)

188. Given, $|z-1|+|z+3| \le 8$

$\therefore$ z lies inside or on the ellipse whose foci are (1, 0) and (–3, 0) and vertices are (–5, 0) and (3, 0).



Now, $|z-4|$ is distance of z from (4, 0). Minimum distance is 1 and maximum is 9.

The correct option is (A)

189. We have, $2 = \left|z-\frac{4}{z}\right| \ge |z| - \left|\frac{4}{z}\right|$

$\Rightarrow \quad |z| - \left|\frac{4}{z}\right| \le 2$

$\Rightarrow \quad |z|^2 - 2\,|z| - 4 \le 0$ or $(|z|-1)^2 - 5 \le 0$

$\Rightarrow \quad (|z|-1)^2 \le 5 \quad$ or $\quad |z|-1 \le \sqrt{5} \Rightarrow |z| \le \sqrt{5}+1$

Hence, the greatest value of $|z|$ is $\sqrt{\mathbf{5}}+1$.

The correct option is (A)

190. $|z-i| = |z+5i|$ represents equation of perpendicular bisector of points (0, 1) and (0, –5), i.e., $y=-2$, now $|z|=2$ is $x^2+y^2=4$

$\Rightarrow \quad x^2+4=4 \Rightarrow x=0$

z represents a single point (0, –2).

The correct option is (A)

191. We have,

$$\begin{aligned}(1-z_0)\,P_n(z) &= (1-z_0{}^2)\,(1+z_0{}^2)\,(1+z_0{}^{22})\,\dots\,(1+z_0{}^{2n})\\ &= (1-z_0{}^{22})\,(1+z_0{}^{22})\,(1+z_0{}^{23})\,\dots\,(1+z_0{}^{2n})\\ &= (1-z_0{}^{23})\,(1+z_0{}^{23})\,\dots\,(1+z_0{}^{2n})\\ &= (1-z_0{}^{2n+1}) \qquad (2^{2m}\cdot 2^{2m} = 2^{2m+1})\end{aligned}$$

Now, $z_0{}^2 = \frac{i}{2} \Rightarrow z_0{}^{2n+1} = (z_0{}^2)^{2n} = \left(\frac{i}{2}\right)^{2^n} = \frac{i^{2^n}}{2^{2^n}}$

Now, since 2^n is divisible by 4, if $n>1 \Rightarrow i^{2n}=1$

Thus, $1-z_0{}^{2n+1} = 1-\frac{1}{2^{2^n}} \Rightarrow P_n(z) = \frac{1}{1-z_0}\left(1-\frac{1}{2^{2^n}}\right)$

$$= (1+i)\left(1-\frac{1}{2^{2^n}}\right).$$

The correct option is (A)

192. Since amp $\cdot$ $[z_1\,(z_3-z_2)]$ = amp $\cdot$ $[z_3\,(z_2-z_1)]$

$\therefore \quad \text{amp.}\left(\frac{z_1\,(z_3-z_2)}{z_3\,(z_2-z_1)}\right) = 0$

$\Rightarrow \quad \frac{z_1(z_3-z_2)}{z_3(z_2-z_1)}$ is purely real.

Hence, 0, z_1, z_2, z_3 are concyclic

$\left(\because\right.$ for four concyclic points $\frac{(z_1-z_3)\,(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is purely real$\left.\right)$

The correct option is (A)

Previous Year's Questions

193. Key Idea : If ω is a cube root of unity, then $1+\omega+\omega^2=0$ and $\omega^3=1$

$(1+\omega+\omega^2)=0$ and $\omega^3=1$

$$\begin{aligned}(1+\omega-\omega^2)^7 &= (-\omega^2-\omega^2)^7 && (\because 1+\omega+\omega^2=0)\\ &= (-2\omega^2)^7\\ &= -2^7\cdot\omega^{14}\\ &= -128(\omega^3)^4\omega^2\\ &= -128\,\omega^2 && (\because \omega^3=1)\end{aligned}$$

The correct option is (D)

194. Given $z_1^2+z_2^2-z_1z_2=0$

$\Rightarrow (z_1+z_2)^2 - 3z_1z_2 = 0$

$\Rightarrow a^2 = 3b.$

The correct option is (C)

195. $\arg(z)-\arg(\omega) = \arg\left(\frac{z}{\omega}\right) = \frac{\pi}{2}$

$\Rightarrow |z\omega| = 1$

$\Rightarrow \bar{z}\omega = -i$ or $+i$.

The correct option is (D) or (C)

196. $\frac{1+i}{1-i}=\frac{(1+i)^2}{2}=i$

$\Rightarrow \left(\frac{1+i}{1-i}\right)^x = i^x$

$\Rightarrow x = 4n$.

The correct option is (A)

197. Here $\omega = \frac{Z}{i} \Rightarrow \arg\left(z\frac{z}{i}\right)=\pi$

$\Rightarrow 2\arg(z)-\arg(i)=\pi$

$\Rightarrow 2\arg(z)-\frac{\pi}{2}=\pi$

$\Rightarrow \arg(z)=\frac{3\pi}{4}$.

The correct option is (C)

198. $z=(p+iq)^3=p(p^2-3q^2)-iq(q^2-3p^2)$

$\Rightarrow \frac{x}{p}=p^2-3q^2$ and $\frac{y}{q}=q^2-3p^2 \Rightarrow \frac{\frac{x}{p}+\frac{y}{q}}{(p^2+q^2)}=-2$.

The correct option is (B)

199. Since $|z^2-1|^2=\left(|z|^2+1\right)^2$, we have

$\left(z^2-1\right)\left(\bar{z}^2-1\right)=|z|^4+2|z|^2+1$

$\Rightarrow z^2+\bar{z}^2+2z\bar{z}=0 \Rightarrow z+\bar{z}=0$

$\Rightarrow R(z)=0 \Rightarrow z$ lies on the imaginary axis.

The correct option is (D)

200. Given equation $(x-1)^3+8=0$ implies that $(x-1)=(-2)(1)^{1/3}$

$\Rightarrow x-1=-2$ or -2ω or $-2\omega^2$

Or $n=-1$ or $1-2\omega$ or $1-2\omega^2$.

The correct option is (C)

201. $|z_1+z_2|=|z_1|+|z_2| \Rightarrow z_1$ and z_2 are collinear and are to the same side of origin; hence $\arg z_1-\arg z_2=0$.

The correct option is (C)

202. As given $w=\frac{z}{z-\frac{1}{3}i} \Rightarrow |w|=\frac{|z|}{\left|z-\frac{1}{3}i\right|}=1 \Rightarrow$ distance of z from origin and point $\left(0,\frac{1}{3}\right)$ is same.

Hence z lies on the bisector of the line joining points (0, 0) and (0, 1/3).

Hence z lies on a straight line.

The correct option is (C)

203. Given sum

$$\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11}+i\cos\frac{2k\pi}{11}\right)$$

$$=\sum_{k=1}^{10}\sin\frac{2k\pi}{11}+i\sum_{k=1}^{10}\cos\frac{2k\pi}{11}$$

$=0+i(-1)=-i$

The correct option is (D)

204. The given equation $z^2+z+1=0$

$\Rightarrow \quad z=\omega$ or ω^2.

So, $\quad z+\frac{1}{z}=\omega+\omega^2=-1,\ z^2+\frac{1}{z^2}=\omega^2+\omega=-1,\ z^3+\frac{1}{z^3}=\omega^3+\omega^3=2,$

$z^4+\frac{1}{z^4}=-1,\ z^5+\frac{1}{z^5}=-1$ and $z^6+\frac{1}{z^6}=2.$

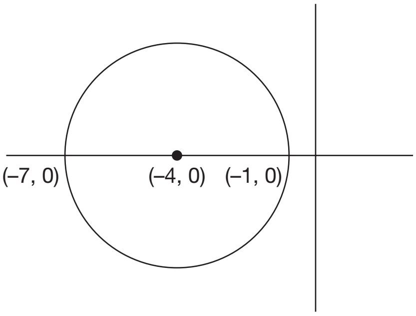
∴ The given sum = 1 + 1 + 4 + 1 + 1 + 4 = 12

The correct option is (D)

205. From the Argand diagram, maximum value of $|z+1|$ is 6.

Alternative:

$|z+1|=|z+4-3|\le|z+4|+|-3|=6.$



The correct option is (C)

206. Put $-i$ in place of i

Hence, the complex conjugate is $\frac{-1}{i+1}$

The correct option is (C)

207. One can write $|Z|=\left|\left(Z-\frac{4}{Z}\right)+\frac{4}{Z}\right| \Rightarrow |Z|=\left|Z-\frac{4}{Z}+\frac{4}{Z}\right|$

$\Rightarrow |Z|\le\left|Z-\frac{4}{Z}\right|+\frac{4}{|Z|} \Rightarrow |Z|\le 2+\frac{4}{|Z|}$

$\Rightarrow |Z|^2-2|Z|-4\le 0$

$\therefore \left(|Z|-\left(\sqrt{5}+1\right)\right)\left(|Z|-\left(1-\sqrt{5}\right)\right)\le 0$

$\Rightarrow 1-\sqrt{5}\le|Z|\le\sqrt{5}+1$

The correct option is (B)

208. Let $\quad z=x+iy$

$|z-1|=|z+1| \Rightarrow \text{Re } z=0 \Rightarrow x=0$

$|z-1|=|z-i| \Rightarrow x=y$

$|z+1|=|z-i| \Rightarrow y=-x$

Only (0, 0) will satisfy all conditions.

⇒ Number of complex number $z=1$

The correct option is (A)

209. Suppose roots are $1 + pi$, $1 + qi$

Sum of roots $1 + pi + 1 + qi = -\alpha$ which is real

$\Rightarrow$ roots of $1 + pi, 1 - pi$

Product of roots $= \beta = 1 + p^2 \in (1, \infty)$

$p \neq 0$ since roots are distinct.

The correct option is (C)

210. $1 + \omega = -\omega^2$

$$(1+\omega)^7 = (-\omega^2) = -\omega^{14} = -\omega^2$$

$$= 1 + \omega = A + B\omega \Rightarrow (A, B) = (1,1)$$

The correct option is (A)

211. Let $z = x + iy$ $\quad (\therefore x \neq 1 \text{ as } z \neq 1)$

$$z^2 = (x^2 - y^2) + i(2xy)$$

$\dfrac{z^2}{z-1}$ is real $\Rightarrow$ its imaginary part $= 0$

$$\Rightarrow 2xy(x-1) - y(x^2 - y^2) = 0$$

$$\Rightarrow y(x^2 + y^2 - 2x) = 0$$

$$\Rightarrow y = 0; x^2 + y^2 - 2x = 0$$

$\therefore z$ lies either on real axis or on a circle through origin.

The correct option is (A)

212. Given $|z| = 1 \Rightarrow z\bar{z} = 1$

$$\therefore \quad \frac{1+z}{1+\bar{z}} = \frac{1+z}{1+\frac{1}{z}} = z\,.$$

The correct option is (B)

213. $|z| \geq 2$

$$\left|z + \frac{1}{2}\right| \geq \left||z| - \frac{1}{2}\right| \geq 2 - \frac{1}{2} \geq \frac{3}{2}\,.$$

Hence, minimum distance between z and $\left(-\frac{1}{2}, 0\right)$ is $\frac{3}{2}$

The correct option is (B)

214. Given that $\left|\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$

$$\Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2)$$

$$= (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$$

$$\Rightarrow |z_1|^2 - 2z_2 - \bar{z}_1 - 2\bar{z}_2 z_1 + 4|z_2|^2$$

$$= 4 - 2z_1\bar{z}_2 - 2\bar{z}_1 z_2 + |z_1|^2 |z_2|^2$$

$$\Rightarrow |z_1|^2 + 4|z_2|^2 - |z_1|^2|z_2|^2 - 4 = 0$$

$$\Rightarrow |z_1|^2\left(1 - |z_2|^2\right) - 4\left(1 - |z_2|^2\right) = 0$$

$$\Rightarrow |z_1| = 2(as|z_2| \neq 1)$$

The correct option is (B)

215. We have,

$$z = \frac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$$

$$\Rightarrow \quad z = \frac{(2 + 3i\sin\theta)(1 + 2i\sin\theta)}{1 + 4\sin^2\theta}$$

$$= \frac{(2 - 6\sin^2\theta) + 7i\sin\theta}{1 + 4\sin^2\theta}$$

For z to be purely imaginary, we have Re $(z) = 0$

$$\Rightarrow 2 - 6\sin^2\theta = 0 \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \pm\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

The correct option is (A)