CHAPTER

19

Circles

Chapter Highlights

Circle, Standard Equation of a Circle, General Equation of a Circle, Conditions for an equation to represent a circle, Equation of a Circle in some Special cases, equation of a circle in diameter form, Intercepts made by a circle on the axes, Parametric Equations of a Circle, Position of a Point with respect to a circle, Intersection of a Line and a Circle, Length of intercept made by a circle on a line, The least and Greatest Distance of a Point from a Circle, Contact of Two Circles, Tangent to a Circle at a Given Point, Equation of the Tangent in Slope Form, Condition of Tangency, Tangents From a Point Outside the Circle, Length of the Tangent From a Point to a Circle, Normal to the circle at a given Point, Pair of Tangents, Common Tangent S to two circles, Power of a Point with respect to a circle, Director Circle, Equation of Chord of Contact, Equation of chord if its Mid Point is known, Common chord of Two Circles, Diameter of a circle, Angle of intersection of two circles, Orthogonal Intersection of two Circles, Family of Circles, Image of the Circle by the Line Mirror.

CIRCLE

A circle is the locus of a point which moves in a plane such that its distance from a fixed point always remains constant. The fixed point is called the *centre* and the constant distance is called the *radius* of the circle.

STANDARD EQUATION OF A CIRCLE

The equation of a circle with the centre at (h, k) and radius a, is

$$(x - h)^2 + (y - k)^2 = a^2$$

If the centre of the circle is at the origin and radius is a, then the equation of circle is $x^2 + y^2 = a^2$.

GENERAL EQUATION OF A CIRCLE

The general equation of a circle is of the form

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad (1)$$

where g, f and c are constants.

The coordinates of its centre are $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

CONDITIONS FOR AN EQUATION TO REPRESENT A CIRCLE

A general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

in x, y represents a circle if

1. coefficient of x^2 = coefficient of y^2, i.e., $a = b$,
2. coefficient of xy is zero, i.e., $h = 0$.

To Find the Centre and Radius of a Circle whose Equation is GIven

SHORT-CUT METHOD

- Make the coefficients of x^2 and y^2 equal to 1 and right hand side equal to zero.
- The coordinates of centre will be (h, k), where

$$h = -\frac{1}{2}\text{(coefficient of } x)$$

and

$$k = -\frac{1}{2}\text{(coefficient of } y)$$

- Radius $= \sqrt{h^2 + k^2 - \text{constant term}}$

Nature of the Circle

1. If $g^2 + f^2 - c > 0$, then the general eqn. (1) represents real circle with centre $(-g, -f)$.
2. If $g^2 + f^2 - c = 0$, then the general eqn. (1) represents a circle whose centre is $(-g, -f)$ and radius is zero i.e., the circle coincides with the centre represented by a point $(-g, -f)$. It is, therefore called a *point circle.*
3. If $g^2 + f^2 - c < 0$, the radius of the circle is imaginary but the centre is real. Such a circle is called a *virtual circle or imaginary circle* as it is not possible to draw such a circle.

EQUATION OF A CIRCLE IN SOME SPECIAL CASES

1. If the centre of the circle is (h, k) and it passes through origin then its equation is $(x - h)^2 + (y - k)^2 = h^2 + k^2$ $\Rightarrow \quad x^2 + y^2 - 2hx - 2ky = 0$

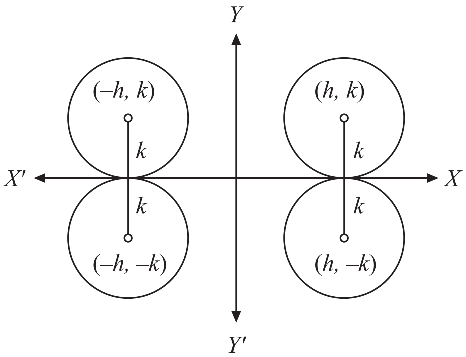


Fig. 19.1

2. If the circle touches x-axis then its equation is $(x \pm h)^2 + (y \pm k)^2 = k^2$. (Four cases)
3. If the circle touches y-axis then its equation is $(x \pm h)^2 + (y \pm k)^2 = h^2$. (Four cases)

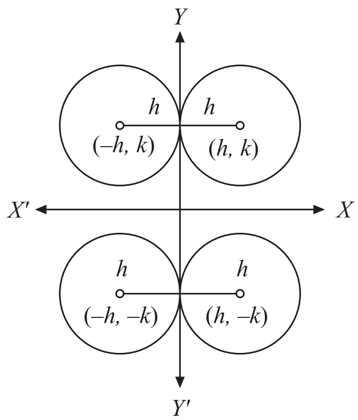


Fig. 19.2

4. If the circle touches both the axes then its equation is $(x \pm r)^2 + (y \pm r)^2 = r^2$. (Four cases)

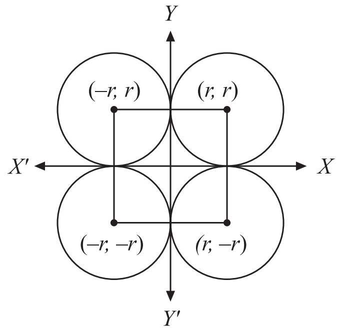


Fig. 19.3

5. If the circle touches x-axis at origin then its equation is $x^2 + (y \pm k)^2 = k^2 \Rightarrow x^2 + y^2 \pm 2ky = 0$. (Two cases)

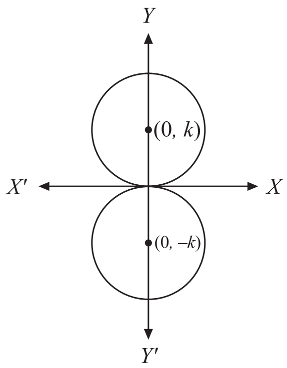


Fig. 19.4

6. If the circle touches y-axis at origin, the equation of circle is $(x \pm h)^2 + y^2 = h^2 \Rightarrow x^2 + y^2 \Rightarrow 2xh = 0$. (Two cases)

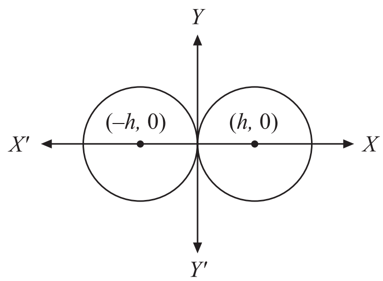


Fig. 19.5

7. If the circle passes through origin and cuts intercepts a and b on the axes, the equation of circle is $x^2 + y^2 - ax - by = 0$ and centre is $C(a/2, b/2)$. (Four cases)

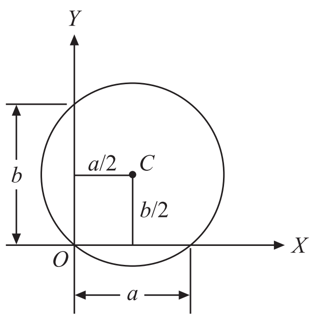


Fig. 19.6

SOLVED EXAMPLES

1. The tangent to the circle $x^2 + y^2 = 9$, which is parallel to y-axis and does not lie in third quadrant, touches the circle at the point

(A) $(-3, 0)$ (B) $(3, 0)$
(C) $(0, 3)$ (D) $(0, -3)$

Solution: (B)

Any line parallel to y-axis is $x = k$.

If it touches the circle $x^2 + y^2 = 9$, then ⊥ distance from the centre (0, 0) of the circle to the line $x = k$, must be equal to radius 3.

i.e., $$\frac{|0-k|}{1} = 3 \Rightarrow k = \pm 3$$

$$\therefore \quad k = 3$$

($\because$ line does not lie in the IIIrd quadrant)

$\therefore$ The equation of the tangent line is $x = 3$.

This meets the circle when $9 + y^2 = 9 \Rightarrow y = 0$.

$\therefore$ Point of contact is (3, 0).

2. The equation of circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length $3a$ is

(A) $x^2 + y^2 = 9a^2$ (B) $x^2 + y^2 = 16a^2$
(C) $x^2 + y^2 = 4a^2$ (D) $x^2 + y^2 = a^2$

Solution: (C)

The centroid of an equilateral triangle is the centre of its circum centre and the radius of the circle is the distance of any vertex from the centroid i.e., radius of the circle

= distance of centroid from any vertex

$$= \frac{2}{3}(\text{Median}) = \frac{2}{3}(3a) = 2a$$

Hence, equation of circle whose centre is (0, 0) and radius $2a$ is

$$(x-0)^2 + (y-0)^2 = (2a)^2 \quad \text{or} \quad x^2 + y^2 = 4a^2$$

3 The equation of the circle inscribed in the triangle, formed by the coordinate axes and the line $12x + 5y = 60$, is given by

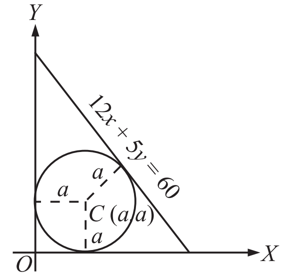
(A) $x^2 + y^2 + 4x + 4y + 4 = 0$
(B) $x^2 + y^2 - 4x - 4y + 4 = 0$
(C) $x^2 + y^2 - 4x - 4y - 4 = 0$
(D) none of these

Solution: (B)

Let the radius of the circle be a.

Then the centre is $C \equiv (a, a)$.

Also, the distance of $C(a, a)$ from the line $12x + 5y = 60$ is a.



$$\therefore \quad \frac{|12a + 5a - 60|}{\sqrt{12^2 + 5^2}} = a \quad \text{or} \quad \frac{|17a - 60|}{13} = a.$$

or $\quad 17a - 60 = \pm 13a$ or $a = 15, 2$

It is clear from the figure that $a \neq 15$.

$$\therefore \quad a = 2$$

$\therefore$ The equation of the incircle is

$$(x-2)^2 + (y-2)^2 = 2^2$$

or $$x^2 + y^2 - 4x - 4y + 4 = 0$$

4. The area of an equilateral triangle inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

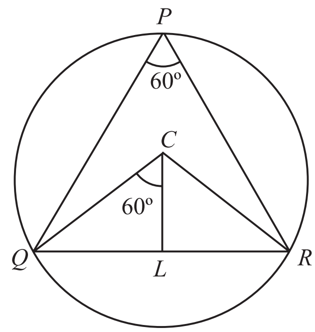
(A) $\frac{3\sqrt{3}}{2}(g^2 + f^2 - c)$ (B) $\frac{3\sqrt{3}}{4}(g^2 + f^2 - c)$
(C) $\frac{3\sqrt{3}}{4}(g^2 + f^2 + c)$ (D) none of these

Solution: (B)

Given Circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

Let C be its centre and PQR be an equilateral triangle inscribed in the circle, then $C \equiv (-g, -f)$ and radius of the circle $CQ = \sqrt{g^2 + f^2 - c}$.



From ΔQLC, $QL = CQ \sin 60° = \frac{\sqrt{3}}{2}\sqrt{g^2 + f^2 - c}$

$$\therefore \quad QR = 2QL = \sqrt{3} \cdot \sqrt{g^2 + f^2 - c}$$

Now, area of $\Delta PQR = \frac{\sqrt{3}}{4} \cdot QR^2 = \frac{\sqrt{3}}{4} \cdot 3(g^2 + f^2 - c)$

$$= \frac{3\sqrt{3}}{4}(g^2 + f^2 - c)$$

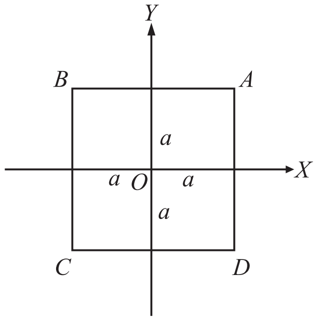
5. A point moves so that the sum of the squares of its distances from the four sides of a square is constant. The locus of the point is

(A) a circle (B) an ellipse
(C) a hyperbola (D) none of these

Solution: (A)

Take the centre of the square as origin and axes parallel to its sides.

Let side of square be $2a$.



The equations of sides are

$$AD: x = a,\ BC: x = -a$$
$$AB: y = a,\ CD: y = -a$$

Let $P(x, y)$ be any point on locus.

Distances of P from the sides of square are

$$x - a,\ x + a,\ y - a \text{ and } y + a$$

By the given condition,

$$(x - a)^2 + (x + a)^2 + (y - a)^2 + (y + a)^2 = \text{constant}$$
$$= 4c^2 \text{ (say)}$$

$\therefore$ $2x^2 + 2y^2 = 4c^2 - 4a^2$

or $x^2 + y^2 = 2(c^2 - a^2)$, which is a circle.

6. If the equation of the incircle of an equilateral triangle is $x^2 + y^2 + 4x - 6y + 4 = 0$, then the equation of the circumcircle of the triangle is

(A) $x^2 + y^2 + 4x + 6y - 23 = 0$
(B) $x^2 + y^2 + 4x - 6y - 23 = 0$
(C) $x^2 + y^2 - 4x - 6y - 23 = 0$
(D) none of these

Solution: (B)

Given equation of the incircle is

$$x^2 + y^2 + 4x - 6y + 4 = 0$$

Its incentre is $(-2, 3)$ and inradius $= \sqrt{4 + 9 - 4} = 3$.

Since in an equilateral triangle, the incentre and the circumcentre coincide,

$\therefore$ Circumcentre $\equiv (-2, 3)$

Also, in an equilateral triangle, circumradius = 2 (inradius)

$\therefore$ Circumradius $= 2 \cdot 3 = 6$.

$\therefore$ The equation of the circumcircle is

$$(x + 2)^2 + (y - 3)^2 = (6)^2$$

or $x^2 + y^2 + 4x - 6y - 23 = 0$

7. The ΔPQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates $(3, 4)$ and $(-4, 3)$ respectively, then $\angle QPR$ is equal to

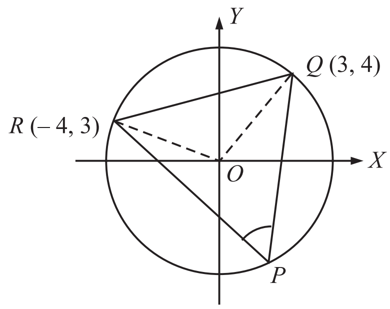
(A) $\pi/2$ (B) $\pi/3$
(C) $\pi/4$ (D) $\pi/6$

Solution: (C)

Let $m_1 = $ slope of $OQ = \dfrac{4}{3}$ and $m_2 = $ slope of $OR = \dfrac{-3}{4}$.

As $m_1 m_2 = -1, \angle QOR = \dfrac{\pi}{2}$.

Thus, $\angle QPR = \dfrac{\pi}{4}$



($\because$ angle subtended at the centre of a circle is double the angle subtended in the alternate segment).

EQUATION OF A CIRCLE IN DIAMETER FORM

The equation of the circle drawn on the line segment joining two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter is

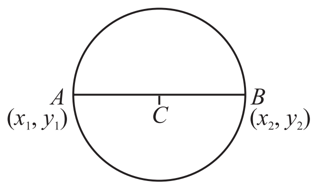


Fig. 19.7

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

Its Centre $= \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$ and

Radius $= \sqrt{\left(\dfrac{x_1 - x_2}{2}\right)^2 + \left(\dfrac{y_1 - y_2}{2}\right)^2}$

SOLVED EXAMPLE

8. Extremities of a diagonal of a rectangle are (0, 0) and (4, 3). The equations of the tangents to the circumcircle of the rectangle which are parallel to this diagonal are

(A) $16x + 8y \pm 25 = 0$ (B) $6x - 8y \pm 25 = 0$
(C) $8x + 6y \pm 25 = 0$ (D) none of these

Solution: (B)

Extremities of the diagonal OA of the rectangle are O (0, 0) and A(4, 3). Then OA is the diameter of the circumcircle, so equation of the circumcircle is

$$x(x-4) + y(y-3) = 0 \text{ i.e., } x^2 + y^2 - 4x - 3y = 0$$

i.e., $$(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{5}{2}\right)^2 \quad (1)$$

m = slope of $OA = 3/4$ (2)

$\therefore$ Tangents parallel to the diagonal OA are

$$y - \frac{3}{2} = \frac{3}{4}(x-2) \pm \frac{5}{2}\sqrt{1 + \frac{9}{16}}$$

That is $6x - 8y \pm 25 = 0$

INTERCEPTS MADE BY A CIRCLE ON THE AXES

1. The length of the intercept made by the circle $x^2 + y^2 + 2gx + 2fc + c = 0$ on
 x-axis $= AB = 2\sqrt{g^2 - c}$
 y-axis $= CD = 2\sqrt{f^2 - c}$
2. Intercepts are always positive.
3. If the circle touches x-axis then $|AB| = 0$ Thus, $c = g^2$
4. If the circle touches y-axis, then $|CD| = 0$ Thus, $c = f^2$
5. If the circle touches both the axes, then $|AB| = 0 = |CD|$ Thus, $c = g^2 = f^2$

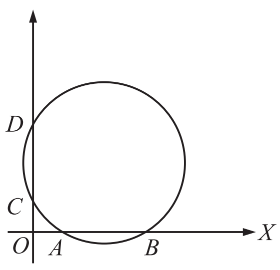


Fig. 19.8

TRICK(S) FOR PROBLEM SOLVING

- The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the x-axis in real and distinct points, touches or does not meet in real points according as $g^2 > 0$ = or < c.
- Similarly, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the y-axis in real and distinct points, touches or does not meet in real points according as f^2 >, = or < c.

SOLVED EXAMPLES

9. A variable circle passes through the point P(1, 2) and touches the x-axis. The locus of the other end of the diameter through P is

(A) $(x-1)^2 = 8y$ (B) $(x+1)^2 = 8y$
(C) $(y-1)^2 = 8x$ (D) none of these

Solution: (A)

The equation of any circle touching x-axis is of the form

$$(x-h)^2 + (y-k)^2 = k^2$$

Let the coordinates of the other end of the diameter through P be (α, β)

Then, $$\frac{\alpha+1}{2} = h \text{ and } \frac{\beta+2}{2} = k$$

i.e., $$\alpha = 2h - 1 \text{ and } \beta = 2k - 2 \quad (1)$$

Also, $(h-1)^2 + (k-2)^2 = (\text{radius})^2 = k^2$

$$\Rightarrow \left(\frac{\alpha+1}{2} - 1\right)^2 + \left(\frac{\beta+2}{2} - 2\right)^2 = \left(\frac{\beta+2}{2}\right)^2$$

$$\Rightarrow (\alpha-1)^2 + (\beta-2)^2 = (\beta+2)^2$$

$$\Rightarrow (\alpha-1)^2 = 8\beta$$

$\therefore$ Locus of (α, β) is $(x-1)^2 = 8y$

10. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y + 3 = 0$. Its sides are parallel to the coordinate axes. Then one vertex of the square is

(A) $(1+\sqrt{2}, -2)$ (B) $(1-\sqrt{2}, -2)$
(C) $(1, -2+\sqrt{2})$ (D) none of these

Solution: (D)

The centre of the given circle is (1, –2). Since the sides of the square inscribed in the circle are parallel to the coordinate axes, so the x-coordinate of any vertex cannot be equal to 1 and its y-coordinate cannot be equal to –2. Hence none of the points given in (A), (B) and (C) can be the vertex of the square.

PARAMETRIC EQUATIONS OF A CIRCLE

(a) The parametric equations of a circle $x^2 + y^2 = a^2$ are $x = a\cos\theta$, $y = a\sin\theta$, $0 \le \theta < 2\pi$. θ is called parameter and the point $P(a\cos\theta, a\sin\theta)$ is called the point 'θ' on the circle $x^2 + y^2 = a^2$. Thus, the coordinates of any point on the circle $x^2 + y^2 = a^2$ may be taken as $(a\cos\theta, a\sin\theta)$.

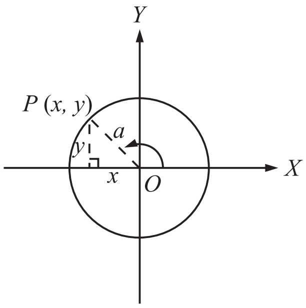


Fig. 19.9

(b) The parametric equations of a circle $(x-h)^2+(y-k)^2=a^2$ are $x=h+a\cos\theta$, $y=k+a\sin\theta$, $0\le\theta<2\pi$. is called the point 'θ' on this circle. Thus the coordinates of any point on this circle may be taken as $(h+a\cos\theta, k+a\sin\theta)$.

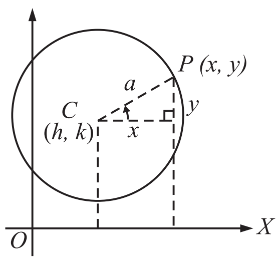


Fig. 19.10

(c) The parametric coordinates of any point on the circle $x^2+y^2+2gx+2fy+c=0$ are

$$x=-g+\sqrt{(g^2+f^2-c)}\cos\theta \text{ and}$$

$$y=-f+\sqrt{(g^2+f^2-c)}\sin\theta \qquad (0\le\theta<2\pi)$$

POSITION OF A POINT WITH RESPECT TO A CIRCLE

Let $S\equiv x^2+y^2+2gx+2fy+c=0$, be a circle and $P(x_1, y_1)$ be a point in the plane of S, then $S_1\equiv x_1^2+y_1^2+2gx_1+2fy_1+c$. The point $P(x_1, y_1)$ lies outside, on or inside the circle S according as $S_1>, =$ or <0, respectively.

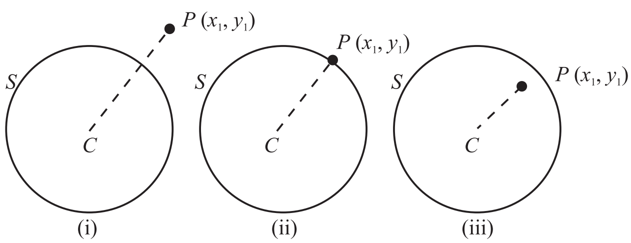


Fig. 19.11

TRICK(S) FOR PROBLEM SOLVING

Let S be a circle and $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane of S, then they lie

- on the same side of S iff S_1 and S_2 have same sign and
- on the opposite sides of S iff S_1 and S_2 have opposite signs

SOLVED EXAMPLES

11. If (α, β) is a point on the chord PQ of the circle $x^2+y^2=19$, where the coordinates of P and Q are $(3, -4)$ and $(4, 3)$ respectively, then

(A) $\alpha\in[3, 4], \beta\in[-4, 3]$
(B) $\alpha\in[-4, 3], \beta\in[3, 4]$
(C) $\alpha\in[3, 3], \beta\in[-4, 4]$
(D) none of these

Solution: (A)

Clearly, the point (α, β) is either an internal point or one of the end points of the line segment joining $P(3, -4)$ and $Q(4, 3)$.

$\therefore \quad 3\le\alpha\le 4$ and $-4\le\beta\le 3$

12. If the point $(2, k)$ lies outside the circles

$x^2+y^2+x-2y-14=0$ and $x^2+y^2=13$, then

(A) $k\in(-3, -2)\cup(3, 4)$
(B) $k\in(-3, 4)$
(C) $k\in(-\infty, -3)\cup(4, \infty)$
(D) $k\in(-\infty, -2)\cup(3, \infty)$

Solution: (C)

Since the point $(2, k)$ lies outside the circle

$$x^2+y^2+x-2y-14=0$$

$\therefore \quad 4+k^2+2-2k-14>0 \quad$ or $\quad k^2-2k-8>0$

or $\quad (k+2)(k-4)>0$

or $\quad k\in(-\infty, -2)\cup(4, \infty)$ (1)

Also, the point $(2, k)$ lies outside the circle

$$x^2+y^2=13.$$

$\therefore \quad 4+k^2-13>0$ or $k^2-9>0$

or $\quad (k-3)(k+3)>0$

or $\quad k\in(-\infty, -3)\cup(3, \infty)$ (2)

The common solution of (1) and (2) is given by,

$$k\in(-\infty, -3)\cup(4, \infty)$$

13. If the point $(k+1, k)$ lies inside the region bounded by the curve and y-axis, then k belongs to the interval $x=\sqrt{25-y^2}$.

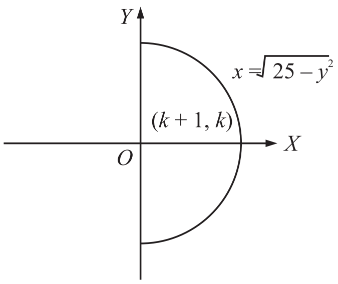
(A) $(-1, 3)$ (B) $(-4, 3)$
(C) $(-\infty, -4)\cup(3, \infty)$ (D) none of these

Solution: (A)

Since the point $(k+1, k)$ lies inside the region bounded by $x = \sqrt{25 - y^2}$ and y-axis,

$\therefore \quad (k+1)^2 + k^2 - 25 < 0$

and $\quad k + 1 > 0$



$\Rightarrow \quad 2k^2 + 2k - 24 < 0$ and $k > -1$

$\Rightarrow \quad k^2 + k - 12 < 0$ and $k > -1$

$\Rightarrow \quad (k+4)(k-3) < 0$ and $k > -1$

$\Rightarrow \quad -4 < k < 3$ and $k > -1$

$\Rightarrow \quad -1 < k < 3$

CIRCLE THROUGH THREE POINTS

We can find a unique circle through three non-collinear points. To find the unique equation of circle, we can follow the following method.

Step I: Assume the general equation of the circle as

$$S : x^2 + y^2 + 2gx + 2fy + c = 0$$

Step II: The coordinates of three points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ (if they lie on the circle) will satisfy the equation of the circle and thus we shall get three simultaneous equations in g, f and c such that

$$S_2 : x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

$$S_2 : x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$

$$S_3 : x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$$

Step III: Solve the above three simultaneous equations in three variables to obtain the values of g, f and c.

Step IV: Substitute the values of g, f and c obtained from step III in the equation assumed in step I to get the desired equation of the circle.

TRICK(S) FOR PROBLEM SOLVING

- The equation of the circle through three non-collinear points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

- From given three points taking any two as extremities of diameter of a circle $S = 0$ and equation of straight line passing through these two points is $L = 0$. then required equation of circle is $S + \lambda L = 0$, where λ is a parameter, which can be found out by putting third point in the equation.
- If the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ meet the coordinate axes in four distinct points, then those points are concyclic if $a_1a_2 = b_1b_2$.

  Also, the equation of the circle passing through those concyclic points is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) - (a_1b_2 + a_2b_1)xy = 0$.
- The equation of the circumcircle of the triangle formed by the line $ax + by + c = 0$ with the coordinate axes is $ab(x^2 + y^2) + c(bx + ay) = 0$.

SOLVED EXAMPLES

14. If a circle passes through the points of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is

(A) 2 (B) 1
(C) −1 (D) −2

Solution: (A)

Let the lines cuts the x-axis at A and B, then

$$OA = -\frac{1}{\lambda} \text{ and } OB = -3 \cdot$$

Also, if the lines cut the y-axis at C and D, then

$$OC = 1 \text{ and } OB = \frac{3}{2}$$

Now if the circle passes through A, B, C and D then

$$OA \times OB = OC \times OD \quad \Rightarrow \quad \left(-\frac{1}{\lambda}\right)(-3) = 1 \times \frac{3}{2}$$

$$\Rightarrow \quad \lambda = 2$$

15. If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, then

(A) $a_1a_2 = b_1b_2$
(B) $a_1b_1 = a_2b_2$
(C) $a_1b_2 = a_2b_1$
(D) none of these

Solution: (A)

The line $a_1x + b_1y + c_1 = 0$ cuts the coordinate axes at $A(-c_1/a_1, 0)$ and $B(0, -c_1/b_1)$ and the line $a_2x + b_2y + c_2 = 0$ cuts the axes at $C(-c_2/a_2, 0)$ and $D(0, -c_2/b_2)$.

So, AC and BD are chords along x-axis and y-axis respectively, intersecting at origin O.

Since *A, B, C, D* are concyclic, therefore

$$OA \cdot OC = OB \cdot OD$$

$$\Rightarrow \quad \left(\frac{-c_1}{a_1}\right)\cdot\left(\frac{-c_2}{a_2}\right) = \left(\frac{-c_1}{b_1}\right)\cdot\left(\frac{-c_2}{b_2}\right)$$

or $$a_1a_2 = b_1b_2$$

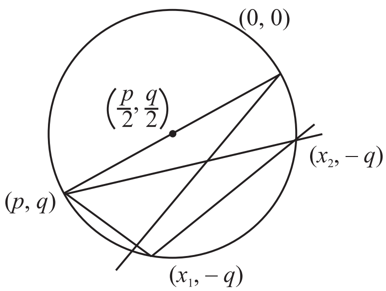
16. Two distinct chords drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$, where $pq \neq 0$, are bisected by the *x*-axis. Then

(A) $|p| = |q|$ (B) $p^2 = 8q^2$
(C) $p^2 < 8q^2$ (D) $p^2 > 8q^2$

Solution: (D)

Given circle is $x^2 + y^2 = px + qy$.

Since the centre of the circle is, $\left(\frac{p}{2}, \frac{q}{2}\right)$, so (p, q) and $(0, 0)$ are the end points of a diameter. As the two chords are bisected by *x*-axis, the chords will cut the circle at the points $(x_1, -q)$ and $(x_2, -q)$, where x_1, x_2 are real.



The equation of the line joining these points is $y = -q$. Solving $y = -q$ and $x^2 + y^2 = px + qy$, we get

$$x^2 - px + 2q^2 = 0$$

The roots of this equation are x_1 and x_2. Since the roots are real and distinct, ∴ discriminent > 0

i.e., $$p^2 - 8q^2 > 0 \quad \text{or} \quad p^2 > 8q^2$$

INTERSECTION OF A LINE AND A CIRCLE

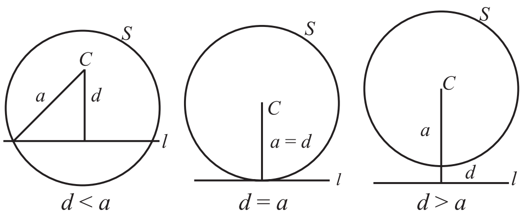


Fig. 19.12

Let *S* be a circle with centre *C* and radius *a*. Let *l* be any line in the plane of the circle and *d* be the perpendicular distance from *C* to the line *l*, then

1. *l* intersects *S* in two distinct points iff $d < a$.
2. *l* intersects *S* in one and only point iff $d = a$, i.e., *the line l touches the circle if perpendicular distance from the centre to the line l must be equal to radius of the circle.*
3. *l* does not intersect *S* iff $d > a$.

LENGTH OF INTERCEPT MADE BY A CIRCLE ON A LINE

If a line *l* meets a circle *S*, with centre *C* and radius *a*, in two distinct points and if *d* is the perpendicular distance of centre *C* from the line *l*, then the length of the intercept made by the circle on the line $= |AB| = 2\sqrt{a^2 - d^2}$.

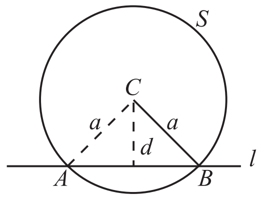


Fig. 19.13

TRICK(S) FOR PROBLEM SOLVING

- ***Note:*** If the points of intersection of a line *l* and a circle *S* are known, then the distance between these points is the required length of intercept and there is no need of using the above formula.
- The length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is $2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$.
- If $a^2(1 + m^2) - c^2 > 0$, line will meet the circle at two real and different points.
- If $c^2 = a^2(1 + m^2)$, line will touch the circle.
- If $a^2(1 + m^2) - c^2 < 0$, line will meet the circle at two imaginary points.

SOLVED EXAMPLES

17. If a chord of the circle $x^2 + y^2 = 32$ makes equal intercepts of length *l* on the coordinate axes, then

(A) $|l| < 8$ (B) $|l| < 16$
(C) $|l| > 8$ (D) none of these

Solution: (A)

Since the chord makes equal intercepts of length *l* on the coordinate axes, so its equation can be written in the form $x \pm y = \pm l$.

Since the chord intersects the given circle at two distinct points, therefore, the length of the ⊥ from the centre (0, 0) of the given circle to the chord must be less than the radius

i.e., $$\left|\frac{\pm l}{\sqrt{2}}\right| < \sqrt{32} \Rightarrow l^2 < 64 \Rightarrow |l| < 8$$

18. The equation of the circle whose centre is (3, –1) and which cuts off an intercept of length 6 from the line $2x - 5y + 18 = 0$, is

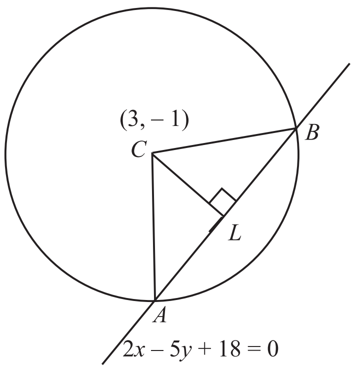
(A) $x^2 + y^2 - 6x + 2y + 28 = 0$
(B) $x^2 + y^2 + 6x + 2y - 28 = 0$
(C) $x^2 + y^2 - 6x - 2y + 28 = 0$
(D) $x^2 + y^2 - 6x + 2y - 28 = 0$

Solution: (D)

Let C be the centre of the circle, then $C \equiv (3, -1)$.

Equation of line AB is $2x - 5y + 18 = 0$ and $AB = 6$

$\therefore \quad AL = 3$



CL = length of the ⊥ from C on AB

$$= \frac{|2\times 3 - 5(-1) + 18|}{\sqrt{(2)^2 + (-5)^2}} = \sqrt{29}$$

$\therefore$ radius of the circle $AC = \sqrt{AL^2 + CL^2}$

$$= \sqrt{3^2 + 19} = \sqrt{38}$$

Thus, equation of the required circle is

$$(x - 3)^2 + (y + 1)^2 = 38$$

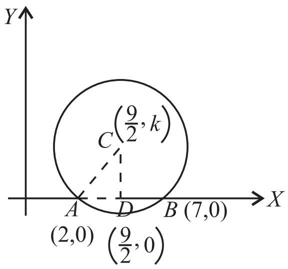
or $\quad x^2 + y^2 - 6x + 2y - 28 = 0$

19. Circles are drawn through the point (2, 0) to cut intercept of length 5 units on the x-axis. If their centres lie in the first quadrant, then their equation is

(A) $x^2 + y^2 - 9x + 2ky + 14 = 0$
(B) $3x^2 + 3y^2 + 27x - 2ky + 42 = 0$
(C) $x^2 + y^2 - 9x - 2ky + 14 = 0$
(D) $x^2 + y^2 - 2kx - 9y + 14 = 0$

Solution: (C)

It is clear from the figure that the coordinates of centre of such circles are $\left(\frac{9}{2}, k\right)$.



$\therefore$ Equation of such circles is

$$\left(x - \frac{9}{2}\right)^2 + (y - k)^2$$

$$= \left(\frac{9}{2} - 2\right)^2 + (k - 0)^2 = \frac{25}{4} + k^2$$

or $\quad x^2 + y^2 - 9x - 2ky + 14 = 0$

20. The line $y = mx + c$ intersects the circle $x^2 + y^2 = r^2$ at the two real distinct points if

(A) $-r\sqrt{1+m^2} < c < r\sqrt{1+m^2}$
(B) $-c\sqrt{1-m^2} < r < c\sqrt{1+m^2}$
(C) $-r\sqrt{1-m^2} < c < r\sqrt{1+m^2}$
(D) none of these

Solution: (A)

Given line is $\quad y = mx + c \quad (1)$

and the given circle is $\quad x^2 + y^2 = r^2 \quad (2)$

Solving (1) and (2), we get

$$(1 + m^2)x^2 + 2mcx + c^2 - r^2 = 0 \quad (3)$$

For two real distinct points of intersection, both the roots of (3) must be real and distinct.

$\therefore \quad 4m^2c^2 - 4(1 + m^2)(c^2 - r^2) > 0$

$\Rightarrow \quad c^2 < r^2(1 + m^2)$

$\Rightarrow \quad -r\sqrt{1+m^2} < c < r\sqrt{1+m^2}$

THE LEAST AND GREATEST DISTANCE OF A POINT FROM A CIRCLE

Let $S = 0$ be a circle and $A(x_1, y_1)$ be a point. If the diameter of the circle through A is passing through the circle at P and Q, then $AP = |AC - r| =$ least distance; $AQ = AC + r =$ greatest distance where 'r' is the radius and C is the centre of the circle.

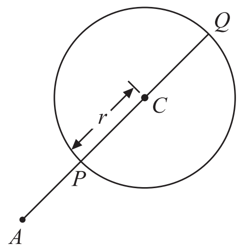


Fig. 19.14

CONTACT OF TWO CIRCLES

The two circles having centres at $A(x_1, y_1)$ and $B(x_2, y_2)$ and radii r_1 and r_2 respectively will

1. Intersect in two real distinct points if and only if $|r_1 - r_2| < AB < r_1 + r_2$

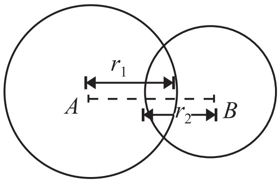


Fig. 19.15

2. Touch each other externally $AB = r_1 + r_2$ and their point of contact C is given by,

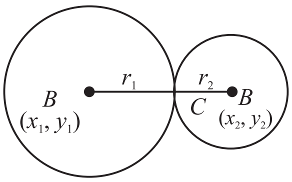


Fig. 19.16

$$C \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$

3. Touch each other internally if $AB = |r_1 - r_2|$, and their point of contact C is given by,

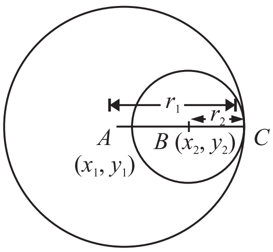


Fig. 19.17

$$C \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$$

4. One circle lies outside the other if $AB > r_1 + r_2$.

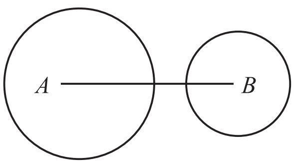


Fig. 19.18

5. One circle is contained in the other if $AB < |r_1 - r_2|$.

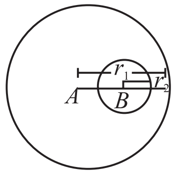


Fig. 19.19

SOLVED EXAMPLES

21. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 8x + 12 = 0$ is

(A) 1 (B) 2
(C) 3 (D) 4

Solution: (C)

The equations of the circles are

$$x^2 + y^2 = 4 \quad (1)$$

and $$x^2 + y^2 - 8x + 12 = 0 \quad (2)$$

Centre of (1) is $C_1 \equiv (0, 0)$ and radius $r_1 = 2$
Centre of (2) is $C_2 \equiv (4, 0)$ and radius $r_2 = 2$
d = distance between centres = $C_1C_2 = 4$.
Since $C_1C_2 = r_1 + r_2$, $\therefore$ the two circles touch each other externally. Hence 3 common tangents can be drawn to the two circles.

22. The equation of a circle of radius 2 touching the circles $x^2 + y^2 - 4|x| = 0$ is

(A) $x^2 + y^2 + 2\sqrt{3}y + 2 = 0$
(B) $x^2 + y^2 + 4\sqrt{3}y + 8 = 0$
(C) $x^2 + y^2 - 4\sqrt{3}y + 8 = 0$
(D) none of these

Solution: (B, C)

The given circles are

$$x^2 + y^2 - 4x = 0, x > 0$$

i.e., $$(x - 2)^2 + y^2 = 2^2, x > 0$$

and $$x^2 + y^2 + 4x = 0, x < 0$$

i.e., $$(x + 2)^2 + y^2 = 2^2, x < 0$$

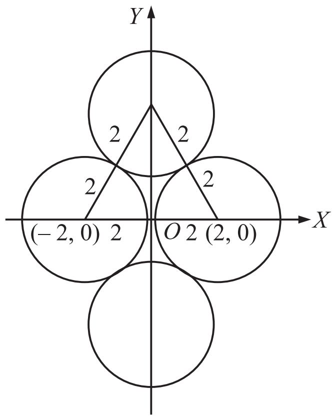
Clearly, from the figure, the centres of the required circles are at $(0, \sqrt{12})$ and $(0, \sqrt{12})$.

$\therefore$ Equations of the required circles are

$$(x - 0)^2 + (y \mp \sqrt{12})^2 = 2^2$$

i.e., $$x^2 + y^2 + 2\sqrt{12}y + 8 = 0$$

and $$x^2 + y^2 - 2\sqrt{12}y + 8 = 0$$



23. The coordinates of the point at which the circles $x^2 + y^2 - 4x - 2y - 4 = 0$ and $x^2 + y^2 - 12x - 8y - 36 = 0$ touch each other, are

(A) (3, –2) (B) (–2, 3)
(C) (3, 2) (D) none of these

Solution: (D)

Given circles are

$$x^2 + y^2 - 4x - 2y - 4 = 0 \quad (1)$$

and $$x^2 + y^2 - 12x - 8y - 36 = 0 \quad (2)$$

Centre of circle (1) is $C_1 \equiv (2, 1)$

and radius $= r_1 = \sqrt{4+1+4} = 3$

Centre of circle (2) is $C_2 \equiv (6, 4)$

and radius $= r_2 = \sqrt{36+16+36} = \sqrt{88}$

Also, d = distance between C_1 and C_2

$= C_1C_2 = \sqrt{16+9} = 5$

Since $d \neq r_1 \pm r_2$

$\therefore$ the two circles do not touch each other.

TANGENT TO A CIRCLE AT A GIVEN POINT

1. Equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) on it is $xx_1 + yy_1 = a^2$.
2. Equation of the tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

at the point (x_1, y_1) on it is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

3. Equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $(a\cos\theta, a\sin\theta)$ on it is

$$x\cos\theta + y\sin\theta = a$$

(Parametric form of equation of tangent)

Notation: The equation of the tangent at the point (x_1, y_1) on the circle $S = 0$ is $T = 0$.

EQUATION OF THE TANGENT IN SLOPE FORM

The equation of a tangent of slope m to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{1+m^2}$.

The coordinates of the point of contact are

$$\left(\pm\frac{am}{\sqrt{1+m^2}}, \mp\frac{a}{\sqrt{1+m^2}}\right)$$

CONDITION OF TANGENCY

The straight line $y = mx + c$ will be a tangent to the circle

$$x^2 + y^2 = a^2 \text{ if } c = \pm a\sqrt{1+m^2}$$

TRICK(S) FOR PROBLEM SOLVING

- A line will touch a circle if and only if the length of the perpendicular from the centre of the circle to the line is equal to the radius of the circle.
- The condition that the line $lx + my + n = 0$ touches the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$(lg + mf - n)^2 = (l^2 + m^2)(g^2 + f^2 - c)$$

- Equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of slope is $y = mx + mg - f \pm\sqrt{(g^2 + f^2 - c)}\sqrt{(1+m^2)}$.
- If the line $lx + my + n = 0$ is a tangent to the circle $(x - h)^2 + (y - k)^2 = a^2$, then $(hl + km + n)^2 = a^2(l^2 + m^2)$.

TANGENTS FROM A POINT OUTSIDE THE CIRCLE

WORKING RULE

- Let the point be (x_1, y_1).
  Write the equation of a straight line passing through the point (x_1, y_1) and having slope m i.e.,

$$(y - y_1) = m(x - x_1) \quad (1)$$

- Find the length of the perpendicular from the centre of the circle to the line (1) and equate it to the radius of the circle. Call this equation as (2).
- Obtain the value of m from the Eq. (2).
- Substitute this value of m in Eq. (1) to obtain the required equation of tangent.

SOLVED EXAMPLES

24. The equation of the circle which has a tangent $2x - y - 1 = 0$ at (3, 5) on it and with the centre on $x + y = 5$, is

(A) $x^2 + y^2 + 6x - 16y + 28 = 0$
(B) $x^2 + y^2 - 6x + 16y - 28 = 0$
(C) $x^2 + y^2 + 6x + 6y - 28 = 0$
(D) $x^2 + y^2 - 6x - 6y - 28 = 0$

Solution: (A)

Clearly, the centre of the circle lies on the line through the point (3, 5) and ⊥ to the tangent $2x - y - 1 = 0$. The equation of such line is

$$(y-5) = \frac{-1}{2}(x-3) \quad \text{i.e.,} \quad x+2y=13 \qquad (1)$$

Also, it is given that centre lies on the line

$$x + y = 5 \qquad (2)$$

Solving (1) and (2), we obtain the coordinates of the centre of the circle as $C \equiv (-3, 8)$.

Also, radius of the circle $= \sqrt{36+9} = \sqrt{45}$.

∴ Equation of the circle is

$$(x+3)^2 + (y-8)^2 = (\sqrt{45})^2$$

That is, $x^2 + y^2 + 6x - 16y + 28 = 0$

25. The tangent to the circle $x^2 + y^2 = 5$ at the point (1, –2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$. Then its point of contact is

(A) (3, –1) (b) (–3, 0)
(C) (–1, –1) (d) (–2, 1)

Solution: (A)

Equation of tangent to the circle

$$x^2 + y^2 = 5 \text{ at } (1, -2) \text{ is } x - 2y - 5 = 0 \qquad (1)$$

Let this line touches the circle

$$x^2 + y^2 - 8x + 6y + 20 = 0 \text{ at } (x_1, y_1)$$

∴ Equation of tangent at (x_1, y_1) is

$$xx_1 + yy_1 - 4(x + x_1) + 3(y + y_1) + 20 = 0$$

or $x(x_1 - 4) + y(y_1 + 3) - 4x_1 + 3y_1 + 20 = 0 \qquad (2)$

Now (1) and (2) represent the same line

$$\therefore \frac{x_1-4}{1} = \frac{y_1+3}{-2} = \frac{-4x_1+3y_1+20}{-5}$$

$\Rightarrow -2x_1 + 8 = y_1 + 3$ or $2x_1 + y_1 - 5 = 0$

Only the point (3, –1) satisfies it. Hence, the point of contact is (3, –1).

LENGTH OF THE TANGENT FROM A POINT TO A CIRCLE

The length of the tangent that can be drawn from the point $P(x_1, y_1)$ to the circle $S = 0$ is $\sqrt{S_1}$, where the coefficients of x^2 and y^2 in the equation of the circle are unity. Thus, the length of the tangent from the point $P(x_1, y_1)$ to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$$|T| = PT = \sqrt{x_1^2 + y_1^2 + 2gx, + 2fy_1 + c} = \sqrt{S_1}$$

TRICK(S) FOR PROBLEM SOLVING

- While calculating the length of tangent using the above formula, it must be noted that the coefficients of x^2 and y^2 must be unity.
- If the point lies inside the circle, then $S_1 < 0$ and $|T|$ is imaginary therefore we cannot have any tangent from a point inside the circle.
- If the point lies on the circle, then $S_1 = 0$ and hence $|T| = 0$. So, if the point lies on the circle, then we have only one tangent having zero length.
- If the point lies outside the circle, then $S_1 > 0$ and $|T|$ is finite.

CAUTION

No tangent can be drawn to a circle from a point lying within the circle.

SOLVED EXAMPLE

26. If the distances from the origin of the centres of the three circles $x^2 + y^2 + 2\alpha_i x = a^2 (i = 1, 2, 3)$ are in G.P., then the lengths of the tangents drawn to them from any point on the circles $x^2 + y^2 = a^2$ are in

(A) A.P. (B) G.P.
(C) H.P. (D) none of these

Solution: (B)

The centres of the three given circles are $(-\alpha_1, 0)$, $(-\alpha_2, 0)$ and $(-\alpha_3, 0)$.

The distances of the three points from the origin are α_1, α_2 and α_3.

Given: α_1, α_2 and a_3 are in G.P.

i.e., $\alpha_2^2 = \alpha_1\alpha_3 \qquad (1)$

Now, coordinates of any point on the circle $x^2 + y^2 = a^2$ are $(a \cos\theta, a \sin\theta)$.

∴ The lengths of the tangents drawn from the point $(a \cos\theta, a \sin\theta)$ to the three given circles are

$$\sqrt{2\alpha_1 a\cos\theta}, \sqrt{2\alpha_2 a\cos\theta} \text{ and } \sqrt{2\alpha_3 a\cos\theta}$$

which, in view of (1), are in G.P.

NORMAL TO THE CIRCLE AT A GIVEN POINT

The normal of a circle at any point is a striaght line perpendicular to the tangent at the point and always passes through the centre of the circle.

1. **Equation of normal:**
   - The equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) is $y - y_1$

$$= \frac{y_1 + f}{x_1 + g}(x - x_1) \text{ or } \frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}.$$

   - The equation of normal to the circle $x^2 + y^2 = a^2$ at any point (x_1, y_1) is $xy_1 - x_1y = 0$ or $\frac{x}{x_1} = \frac{y}{y_1}$.

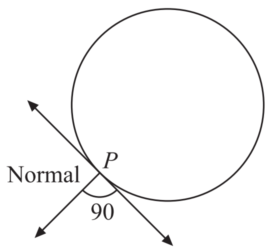


Fig. 19.20

2. **Parametric form:** Since parametric coordinates of any point on the circle $x^2 + y^2 = a^2$ is $(a \cos\theta, a \sin\theta)$.
   $\therefore$ equation of normal at $(a \cos\theta, a \sin\theta)$ is

$$\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = \text{or } y = x \tan\theta \text{ or } y = mx,$$

   where $m = \tan\theta$, which is slope form of normal.

PAIR OF TANGENTS

The equation of the pair of tangents drawn from the point $P(x_1, y_1)$ to the circle

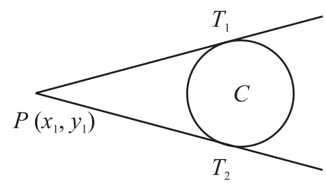


Fig. 19.21

$S = 0$ is $SS_1 = T^2$, where

$$S : x^2 + y^2 + 2gx + 2fy + c,$$

$$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

and $$T : xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c.$$

TRICK(S) FOR PROBLEM SOLVING

The pair of tangents from (0, 0) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are at right angles if $g^2 + f^2 = 2c$.

COMMON TANGENTS TO TWO CIRCLES

Direct Common Tangents

The direct common tangents to the two circles meet at a point (say P) which lies on the line joining the centres C_1 and C_2 of the two circles and divide C_1C_2 externally in the ratio of their radii say (r_1 and r_2)

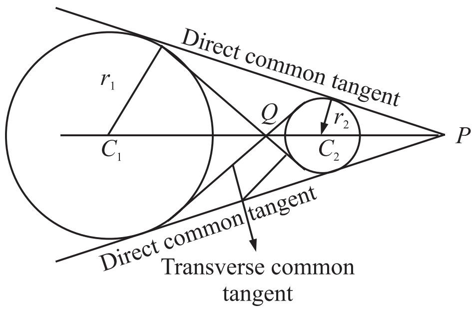


Fig. 19.22

SHORT-CUT METHOD

- Find the coordinates of centres C_1, C_2 and radii r_1, r_2 of two given circles.
- Find the coordinates of the point P dividing C_1C_2 in the ratio $r_1 : r_2$ externally. Let $P \equiv (h, k)$.
- Write the equation of any line through the point $P(h, k)$, i.e.,

$$(y - k) = m\ (x - h) \qquad (1)$$

- Find the two values of m, using the fact that the length of perpendicular on line (1) from the centre C_1 of one circle is equal to its radius r_1.
- Substituting these values of m in equation (1), the equations of two direct common tangents are obtained.

Transverse Common Tangents

The transverse common tangents to the two circles intersect at a point (say P) which lies on the line joining the centres C_1 and C_2 of the two circles and divide C_1C_2 internally in the ratio of their radii r_1 and r_2.

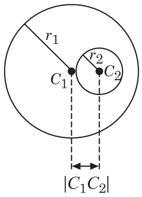
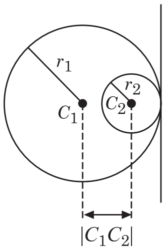
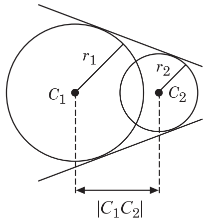
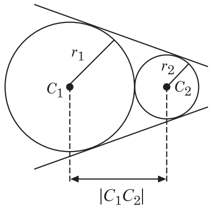
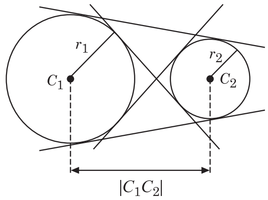
SHORT-CUT METHOD

- Find the coordinates of centres C_1, C_2 and radii r_1, r_2 of two given circles.
- Find the coordinates of the point P dividing C_1C_2, in the ratio $r_1 : r_2$ internally. Let $P \equiv (h, k)$.
- Write the equation of any line through the point $P\ (h, k)$

$$(y - k) = m\ (x - h) \qquad (1)$$

- Find the two values of m, using the fact that the length of perpendicular on (1) from the centre C_1 of one circle is equal to its radius r_1.
- Substituting these values of m in eqn. (1), the equations of two transverse common tangents are obtained.

TRICK(S) FOR PROBLEM SOLVING

Two or More Circles in a plane					
Direct common Tangents	0	1	2	3	2
Transverse common Tangents	0	0	0	0	2

SOLVED EXAMPLES

26. If the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 24x - 10y + a^2 = 0$, $a \in I$, have exactly two common tangents, then the number of possible values of a is

(A) 11 (B) 13
(C) 0 (D) 2

Solution: (B)

The equations of the circles are

$$x^2 + y^2 = 4 \quad (1)$$

and $$x^2 + y^2 - 24x - 10y + a^2 = 0 \quad (2)$$

Centre of (1) is $C_1 \equiv (0, 0)$ and radius $r_1 = 2$

Centre of (2) is $C_2 \equiv (12, 5)$ and radius $r_2 = \sqrt{169 - a^2}$

d = distance between centres = C_1C_2

$$= \sqrt{144 + 25} = 13$$

If the two circles have exactly two common tangents, then

$$169 - a^2 > 0 \text{ and } r_1 + r_2 > d$$

$\Rightarrow \quad (a - 13)(a + 13) < 0$ and $2 + \sqrt{169 - a^2} > 13$

$\Rightarrow \quad -13 < a < 13$ and $169 - a^2 > 121$

$\Rightarrow \quad -13 < a < 13$ and $a^2 - 48 < 0$

$\Rightarrow \quad -13 < a < 13$ and $-\sqrt{48} < a < \sqrt{48}$

$\Rightarrow \quad -\sqrt{48} < a < \sqrt{48}$

Since a is an integer,

$\therefore \quad a = -6, -5, -4, \ldots, 4, 5, 6$

$\therefore$ The number of possible values of a is 13.

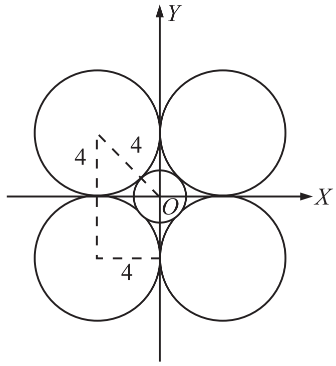
27. If the equations of four circles are $(x \pm 4)^2 + (y \pm 4)^2 = 4^2$, then the radius of the smallest circle touching all the four circles is

(A) $4(\sqrt{2} + 1)$ (B) $4(\sqrt{2} - 1)$

(C) $2(\sqrt{2} - 1)$ (D) none of these

Solution: (B)

Clearly, from the figure, the radius of the smallest circle touching the given circles is



$$= \sqrt{4^2 + 4^2} - 4 \quad \text{i.e.,} \quad 4\sqrt{2} - 4$$

28. Locus of the centre of a circle of radius 4 which touches the circle $x^2 + y^2 - 4x + 2y - 4 = 0$ externally is

(A) $x^2 + y^2 - 4x + 2y - 44 = 0$
(B) $x^2 + y^2 + 4x + 2y - 44 = 0$
(C) $x^2 + y^2 - 4x - 2y - 44 = 0$
(D) none of these

Solution: (A)

Let the centre of the circle S_1 be $C_1(x_1, y_1)$

Its radius $= r_1 = 4$.

Given circle is $S_2 \equiv x^2 + y^2 - 4x + 2y - 4 = 0$

Its centre is $C_2(2, -1)$ and radius

$$= r_2 = \sqrt{4 + 1 + 4} = 3$$

Also, d = distance between the centres.

$$= \sqrt{(x_1 - 2)^2 + (y_1 + 1)^2}$$

Since the two circles touch each other externally,

$\therefore \quad d = r_1 + r_2$

$\Rightarrow \quad \sqrt{(x_1 - 2)^2 + (y_1 + 1)^2} = 4 + 3$

$\Rightarrow \qquad x_1^2 + y_1^2 - 4x_1 + 2y_1 + 5 = 49$

$\therefore$ locus of (x_1, y_1) is

$$x^2 + y^2 - 4x + 2y - 44 = 0$$

29. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y - 24 = 0$ is

(A) 0 (B) 1
(C) 3 (D) 4

Solution: (B)

Given circles are $x^2 + y^2 - 4 = 0$ (1)

and $x^2 + y^2 - 6x - 8y - 24 = 0$ (2)

Centre of circle (1) is $C_1 \equiv (0, 0)$ and radius $r_1 = 2$
Centre of circle (2) is $C_2 \equiv (3, 4)$ and radius $r_2 = 7$
Also d = distance between the centres = $C_1C_2 = 5$
Since $d = r_2 - r_1$, therefore the given circles touch internally, as such they can have just one common tangent at the point of contact.

30. The number of tangents to the circle

$$x^2 + y^2 - 8x - 6y + 9 = 0$$

which pass through the point (3, –2) is

(A) 2 (B) 1
(C) 0 (D) none of these

Solution: (A)

Let $S \equiv x^2 + y^2 - 8x - 6y + 9 = 0$.
Now S for $(3, -2) = 9 + 4 - 24 + 12 + 9 > 0$,
$\therefore$ the point (3, –2) lies outside the circle.
$\therefore$ Two tangents can be drawn to the circle from the point (3, –2).

POWER OF A POINT WITH RESPECT TO A CIRCLE

If from a point $P(x_1, y_1)$, inside or outside the circle a secant be drawn intersecting the circle in two points A and B then $PA \cdot PB$ = constant. The product $PA \cdot PB$ is called power of the point $P(x_1, y_1)$ w.r.t. the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

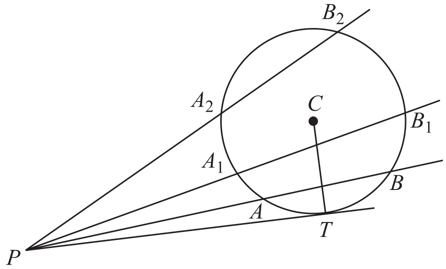


Fig. 19.23

That is for a number of secants

$$PA \cdot PB = PA_1 \cdot PB_1 = PA_2 \cdot PB_2 = \ldots = PT^2 = S_1$$

where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

TRICK(S) FOR PROBLEM SOLVING

- If the two lines $a_1x + b_1x + c_1 = 0$ and $a_2x + b_2x + c_2 = 0$ meet the coordinate axes in four distinct points, then these points are concyclic if $a_1a_2 = b_1b_2$.
  Also, the equation of the circle passing through these concyclic points is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) - (a_1b_2 + a_2b_1)xy = 0$.
- The equation of the circumcircle of the triangle formed by the line $ax + by + c = 0$ with the coordinate axes is $ab(x^2 + y^2) + c(bx + ay) = 0$

DIRECTOR CIRCLE

The locus of the point of intersection of two perpendicular tangents to a circle is called the Director circle.

Let the circle be $x^2 + y^2 = a^2$, then equation of director circle is $x^2 + y^2 = 2a^2 = (\sqrt{2a})^2$.

Clearly, director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the given circle.

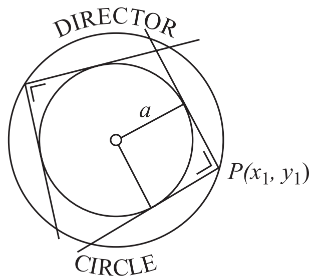


Fig. 19.24

REMEMBER

Director circle of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

SOLVED EXAMPLE

31. The coordinates of a point on the line $y = 2$ from which the tangents drawn to the circle $x^2 + y^2 = 25$ are perpendicular, are

(A) $(\sqrt{46}, 2)$ (B) $(-\sqrt{46}, 2)$
(C) $(\sqrt{37}, 2)$ (D) $(-\sqrt{37}, 2)$

Solution: (A, B)

Let the point on the given line be $(x_1, 2)$.
Since the tangents drawn from $(x_1, 2)$ to the given circle are at right angles, so the point $(x_1, 2)$ must also lie on the director circle whose equation is

$$x^2 + y^2 = 2.25 \quad \text{i.e.,} \quad x^2 + y^2 = 50$$

$\therefore \quad x_1^2 + 4 = 50 \quad \Rightarrow \quad x_1 = \pm\sqrt{46}$

So, the points are $(\sqrt{46}, 2)$ and $(-\sqrt{46}, 2)$.

EQUATION OF CHORD OF CONTACT

The chord joining the points of contact of the two tangents to a conic drawn from a given point outside it, is called the chord of contact of tangents

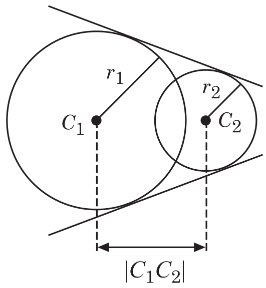


Fig. 19.25

1. **Equation of chord of contact**
   The equation of the chord of contact of tangents drawn from the point $(x_1\ y_1)$ to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.
2. The equation of chord of contact of tangent drawn from the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy - c = 0$ is $T = 0$.
   $\Rightarrow \quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

TRICK(S) FOR PROBLEM SOLVING

- It is clear from above that the equation to the chord of contact coincides with the equation of the tangent, if the point (x_1, y_1) lies on the circle.
- The length of chord of contact $= 2\sqrt{r^2 - p^2}$, p being length of perpendicular from centre to the chord.
- Area of ΔAPQ is given by $\dfrac{a(x_1^2 + y_1^2 - a^2)^{3/2}}{x_1^2 + y_1^2}$.

SOLVED EXAMPLE

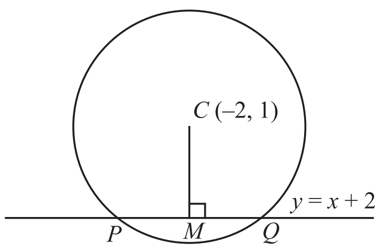
32. The coordinates of the middle point of the chord which the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ cuts off on the line $y = x + 2$, are

(A) $\left(\frac{-3}{2}, \frac{1}{2}\right)$ (B) $\left(\frac{3}{2}, \frac{1}{2}\right)$

(C) $\left(\frac{-3}{2}, \frac{-1}{2}\right)$ (D) $\left(\frac{3}{2}, \frac{-1}{2}\right)$

Solution: (A)

Equation of chord PQ is $y = x + 2$



or $\quad x - y + 2 = 0 \quad (1)$

Centre of circle is $C(-2, 1)$.
Draw $CM \perp PQ$, then M is the mid point of PQ.
Equation of any line $\perp$ to PQ is $x + y + k = 0$
If it passes through $C(-2, 1)$ then

$$-2 + 1 + k = 0 \quad \text{or} \quad k = 1$$

$\therefore$ Equation of CM is $x + y + 1 = 0$. (2)

Solving (1) and (2), we obtain $x = -\frac{3}{2}$ and $y = \frac{1}{2}$.

$\therefore$ Coordinates of M are $\left(\frac{-3}{2}, \frac{1}{2}\right)$.

EQUATION OF CHORD IF ITS MID POINT IS KNOWN

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1, y_1) is given by $T = S_1$.

That is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

SOLVED EXAMPLES

33. The locus of the mid point of the chord of the circle $x^2 + y^2 - 2x - 2y - 2 = 0$ which makes an angle of 120º at the centre is

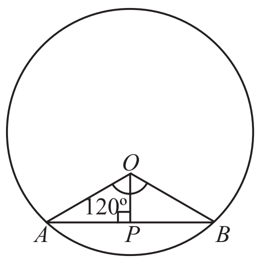
(A) $x^2 + y^2 - 2x - 2y + 1 = 0$
(B) $x^2 + y^2 + x + y - 1 = 0$
(C) $x^2 + y^2 - 2x - 2y - 1 = 0$
(D) none of these

Solution: (A)

Given equation of circle is

$$x^2 + y^2 - 2x - 2y - 2 = 0$$

Let mid point of chord AB be (h, k)
Its centre is (1, 1) and radius $= \sqrt{1+1+2} = 2. = OB$.



In ΔOPB, $\angle OBP = 30º$.

$\therefore \quad \sin 30º = OP/2$ or $OP = 1$

Since, $\quad OP = 1 \quad \Rightarrow \quad (h - 1)^2 + (k - 1)^2 = 1$

or $\quad h^2 + k^2 - 2h - 2k + 1 = 0$

$\therefore$ Locus of mid point of chord is

$$x^2 + y^2 - 2x - 2y + 1 = 0$$

34. The locus of the centres of circles passing through the origin and cutting the circle $x^2 + y^2 + 6x - 4y + 2 = 0$ orthogonally is

(A) $2x - 3y + 1 = 0$ (B) $2x + 3y + 1 = 0$
(C) $3x - 2y + 1 = 0$ (D) none of these

Solution: (C)

Let the equation of one of the circles be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Since it passes through origin,

$\therefore \quad c = 0.$

So, the equation becomes

$$x^2 + y^2 + 2gx + 2fy = 0$$

Since it cuts the circle $x^2 + y^2 + 6x - 4y + 2 = 0$ orthogonally,

$\therefore \quad 2g(3) + 2f(-2) = 0 + 2$

$\Rightarrow \quad -6(-g) + 4(-f) = 2$

Thus, the locus of the centre $(-g, -f)$ is

$$-6x + 4y = 2 \text{ or } 3x - 2y + 1 = 0$$

COMMON CHORD OF TWO CIRCLES

The chord joining the points of intersection of two given circles is called their common chord.

1. **Equation of common chord:** The equation of the common chord of two circles

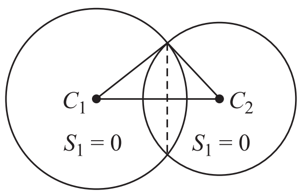


Fig. 19.26

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \qquad (1)$$

and $$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \qquad (2)$$

is $$2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$$

i.e., $$S_1 - S_2 = 0$$

2. **Length of the common chord**

$$PQ = 2(PM) = 2\sqrt{C_1P^2 - C_1M^2}$$

where C_1P = radius of the circle $S_1 = 0$ and C_1M = length of the perpendicular from the centre C_1 to the common chord PQ.

TRICK(S) FOR PROBLEM SOLVING

- The coefficients of x^2 and y^2 in both the equations $S_1 = 0$ and $S_2 = 0$ must be unity.
- If the circles $S_1 = 0$ and $S_2 = 0$ touch, then the common chord $S_1 - S_2 = 0$ becomes the tangent to both of the circles and hence perpendicular from the centre of the either circle to it should be equal to the corresponding radius.

SOLVED EXAMPLE

35. If the circle $x^2 + y^2 + 6x + 8y + a = 0$ bisects the circumference of the circle $x^2 + y^2 + 2x - 6y - b = 0$, then $a + b$ is equal to

(A) 38 (B) −38
(C) 42 (D) none of these

Solution: (B)

Given circles are

$$S_1 : x^2 + y^2 + 6x + 8y + a = 0 \qquad (1)$$

and $$S_2 : x^2 + y^2 + 2x - 6y - b = 0 \qquad (2)$$

The equation of common chord of the two circles is

$$S_1 - S_2 = 0 \text{ i.e., } 4x + 14y + (a + b) = 0 \qquad (3)$$

Since the circle S_1 bisects the circumference of circle S_2, therefore, (1) passes through the centre of circle S_2, i.e., $(-1, 3)$

$\therefore \quad 4(-1) + 14(3) + a + b = 0 \quad \Rightarrow \quad a + b = -38$

DIAMETER OF A CIRCLE

The locus of the middle points of a system of parallel chords of a circle is called a diameter of the circle.

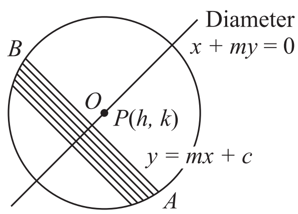


Fig. 19.27

The equation of the diameter bisecting parallel chords $y = mx + c$ (c is a parameter) of the circle $x^2 + y^2 = a^2$ is $x + my = 0$.

REMEMBER

The diameter corresponding to a system of parallel chords of a circle always passes through the centre of the circle and is perpendicular to the parallel chords.

TRICK(S) FOR PROBLEM SOLVING

- The length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{c - c_1}$
- If two tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are perpendicular to each other, then $g^2 + f^2 = 2c$.
- The angle between the tangents from (α, β) to the circle $x^2 + y^2 = a^2$ is $2\tan^{-1}\left(\dfrac{a}{\sqrt{\alpha^2 + \beta^2 - a^2}}\right)$.

- If OA and OB are the tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and C is the centre of the circle, then the area of the quadrilateral $OACB$ is $\sqrt{c(g^2 + f^2 - c)}$.
- The length of the common chord of the circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + bx + ay + c = 0$ is $\sqrt{\frac{1}{2}(a+b)^2 - 4c}$.
- If O is the origin and OP, OQ are tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the circumcentre of the triangle OPQ is $\left(\frac{-g}{2}, \frac{-f}{2}\right)$.
- The length of the chord intercepted by the circle $x^2 + y^2 = r^2$ on the line $\frac{x}{a} + \frac{y}{b} = 1$ is $2\sqrt{\left(\frac{r^2(a^2+b^2) - a^2b^2}{a^2+b^2}\right)}$

ANGLE OF INTERSECTION OF TWO CIRCLES

The angle between the two circles is the angle between their tangents at their point of intersection.

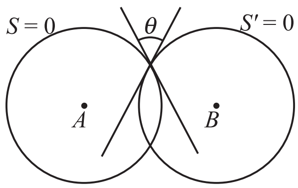


Fig. 19.28

The angle of intersection θ of two circles

$$S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

is given by $$\cos\theta = \pm\frac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}.\sqrt{g_2^2 + f_2^2 - c_2}}$$

or $\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$, where r_1 and r_2 are radii of the two circles and d is the distance between their centres.

TRICK(S) FOR PROBLEM SOLVING

- $\cos\theta \in (-\infty, -1) \cup (1, \infty) \Rightarrow$ Circles do not intersect
- $\cos\theta \in (-1, 1) \Rightarrow$ Circles intersect each other
- $\cos\theta = 0 \Rightarrow$ Circles intersect each other orthogonally
- $\cos\theta \in \{-1, 1\} \Rightarrow$ Circles touch each other internally or externally.

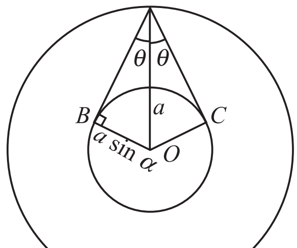
SOLVED EXAMPLE

36. From any point on the circle $x^2 + y^2 = a^2$ tangents are drawn to the circle $x^2 + y^2 = a^2 \sin^2 \alpha$. The angle between them is

(a) $\alpha/2$ (b) α
(c) 2α (d) none of these

Solution: (C)

Let the angle between the tangents be 2θ.
From the figure,



$$\sin\theta = \frac{a\sin\alpha}{a} = \sin\alpha$$

$\Rightarrow$ $\theta = \alpha$

Thus, the required angle
$= 2\theta = 2\alpha$

ORTHOGONAL INTERSECTION OF TWO CIRCLES

Two circles are said to intersect orthogonally when they intersect at right angles.

The condition for the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ to intersect orthogonally is given by,

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

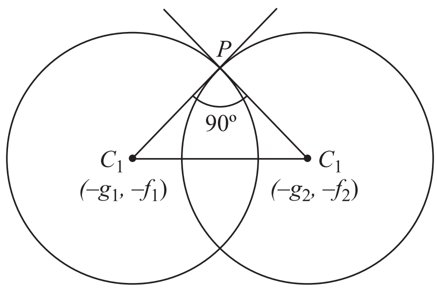


Fig. 19.29

SOLVED EXAMPLE

37. The locus of the centres of the circles which cut the circles $x^2 + y^2 + 4x - 6y + 9 = 0$ and $x^2 + y^2 - 4x + 6y + 4 = 0$ orthogonally is

(A) $8x - 12y + 5 = 0$ (B) $8x + 12y - 5 = 0$
(C) $12x - 8y + 5 = 0$ (D) none of these

Solution: (A)

Let the equation of one of the circles be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Since it cuts the given circles orthogonally,

$\therefore \quad 2g(2) + 2f(-3) = c + 9$

and $\quad 2g(-2) + 2f(3) = c + 4$

i.e., $\quad 4g - 6f = c + 9 \quad$ and $\quad -4g + 6f = c + 4$

On subtracting, we get, $\quad 8g - 12f = 5$

i.e., $\quad -8(-g) + 12(-f) = 5$

So the locus of $(-g, -f)$ is $-8x + 12y = 5$.

FAMILY OF CIRCLES

1. The equation of the family of circles passing through the point of intersection of two given circles $S = 0$ and $S' = 0$ is given as $S + \lambda S' = 0$, (where λ is a parameter, $\lambda \neq -1$)

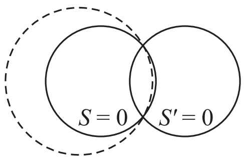


Fig. 19.30

2. The equation of the family of circles passing through the point of intersection of circle $S = 0$ and a line $L = 0$ is given as

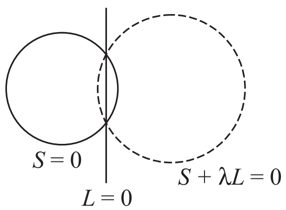


Fig. 19.31

$$S + \lambda L = 0, \text{ (where } \lambda \text{ is a parameter)}$$

3. The equation of the family of circles touching the circle $S = 0$ and the line $L = 0$ at their point of contact P is

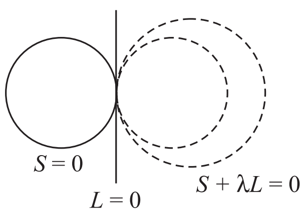


Fig. 19.32

$$S + \lambda L = 0, \text{ (where } \lambda \text{ is a parameter)}$$

4. The equation of the family of circles passing through two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be written in the form

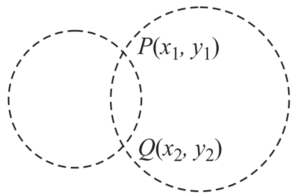


Fig. 19.33

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

(where λ is a parameter)

SOLVED EXAMPLES

38. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as a diameter is

(A) $x^2 + y^2 + x + y = 0$ (B) $x^2 + y^2 - x - y = 0$
(C) $x^2 + y^2 + x - y = 0$ (D) none of these

Solution: (B)

Equation of any circle passing through the point of intersection of $x^2 + y^2 - 2x = 0$ and $y = x$ is

$$x^2 + y^2 - 2x + \lambda(y - x) = 0$$

or
$$x^2 + y^2 - (2 + \lambda)x + \lambda y = 0$$

Its centre is $\left(\dfrac{2+\lambda}{2}, \dfrac{-\lambda}{2}\right)$.

For AB to be the diameter of the required circle, the centre must lie on AB, i.e.,

$$\frac{2+\lambda}{2} = \frac{-\lambda}{2} \quad \Rightarrow \quad \lambda = -1$$

Thus, equation of required circle is

$$x^2 + y^2 - x - y = 0$$

39. The distance from the centre of the circle $x^2 + y^2 = 2x$ to straight line passing though the points of intersection of the two circles $x^2 + y^2 + 5x - 8y + 1 = 0$ and $x^2 + y^2 - 3x - 7y - 25 = 0$ is

(A) $\dfrac{1}{3}$ (B) 2
(C) 3 (D) 1

Solution: (B)

The equation of the straight line passing through the points of intersection of given circles is

$$(x^2 + y^2 + 5x - 8y + 1) - (x^2 + y^2 - 3x + 7y - 25) = 0$$

i.e., $\quad 8x - 15y + 26 = 0 \quad (1)$

Also, centre of the circle $x^2 + y^2 - 2x = 0$ is (1, 0).

$\therefore$ Distance of the point (1, 0) from the straight line (1) is

$$= \frac{|8(1) - 15(0) + 26|}{\sqrt{64 + 225}} = \frac{34}{17} = 2$$

IMAGE OF THE CIRCLE BY THE LINE MIRROR

Let the circle be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

and the line be $L \equiv lx + my + n = 0$

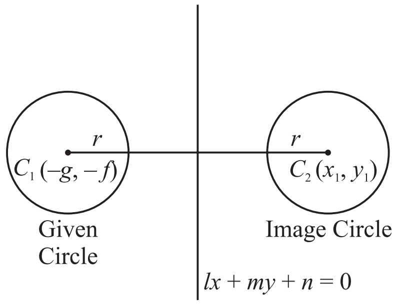


Fig. 19.34

The radius of the image circle remains unchanged but centre changes. Let the centre of image circle be (x_1, y_1). Then,

Slope of C_1C_2 × Slope of the line $L = -1$ (1)

and mid point of C_1 $(-g, -f)$ and C_2 (x_1, y_1) lies on the line

$lx + my + n = 0$

i.e., $$l\left(\frac{x_1 - g}{2}\right) + m\left(\frac{y_1 - f}{2}\right) + n = 0 \quad (2)$$

Solving eqns. (1) and (2), to get value of (x_1, y_1). Then the required image circle is

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

where $$r = \sqrt{g^2 + f^2 - c}$$

EXERCISES

Single Option Correct Type

1. An isosceles ΔABC is inscribed in a circle $x^2 + y^2 = a^2$ with the vertex A at $(a, 0)$ and the base angles B and C each equal to 75° then length of the base BC is

(A) $\frac{a}{2}$ (B) a

(C) $\frac{2a}{\sqrt{3}}$ (D) $\frac{\sqrt{3}a}{2}$

2. Let a_n, $n = 1, 2, 3, 4$ represent four distinct positive real numbers other than unity such that each pair of the logarithm of a_n and the reciprocal of logarithm denotes a point on a circle, whose centre lies on y-axis. The product of these four numbers is

(A) 0 (B) 1

(C) 2 (D) 13

3. If the tangents PA and PB are drawn from the point $P(-1, 2)$ to the circle $x^2 + y^2 + x - 2y - 3 = 0$ and C is the centre of the circle, then the area of the quadrilateral $PACB$ is

(A) 4 (B) 16

(C) does not exist (D) none of these

4. If the line $(y - 2) = m(x + 1)$ intersects the circle $x^2 + y^2 + 2x - 4y - 3 = 0$ at two real distinct points, then the number of possible values of m is

(A) 2 (B) 1

(C) any real value of m (D) none of these

5. The number of points on the circle $x^2 + y^2 - 4x - 10y + 13 = 0$ which are at a distance 1 from the point $(-3, 2)$ is

(A) 1 (B) 2

(C) 3 (D) none of these

6. If the equations of four circles are $(x \pm 4)^2 + (y \pm 4)^2 = 4^2$, then the radius of the smallest circle touching all the four circles is

(A) $4(\sqrt{2} + 1)$ (B) $4(\sqrt{2} - 1)$

(C) $2(\sqrt{2} - 1)$ (D) none of these

7. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as a diameter is

(A) $x^2 + y^2 + x + y = 0$ (B) $x^2 + y^2 - x - y = 0$
(C) $x^2 + y^2 + x - y = 0$ (D) none of these

8. The locus of the mid-point of the chord of the circle $x^2 + y^2 - 2x - 2y - 2 = 0$ which makes an angle of 120º at the centre is

(A) $x^2 + y^2 - 2x - 2y + 1 = 0$
(B) $x^2 + y^2 + x + y - 1 = 0$
(C) $x^2 + y^2 - 2x - 2y - 1 = 0$
(D) none of these

9. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y + 3 = 0$. Its sides are parallel to the coordinate axes. Then, one vertex of the square is

(A) $(1+\sqrt{2}, -2)$ (B) $(1-\sqrt{2}, -2)$
(C) $(1, -2+\sqrt{2})$ (D) none of these

10. If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, then

(A) $a_1a_2 = b_1b_2$ (B) $a_1b_1 = a_2b_2$
(C) $a_1b_2 = a_2b_1$ (D) none of these

11. The circle $x^2 + y^2 = 4$ cuts the line joining the points $A(1, 0)$ and $B(3, 4)$ in two points P and Q. Let $\frac{BP}{PA} = \alpha$ and $\frac{BQ}{QA} = \beta$. Then, α and β are roots of the quadratic equation

(A) $3x^2 + 2x - 21 = 0$ (B) $3x^2 + 2x + 21 = 0$
(C) $2x^2 + 3x - 21 = 0$ (D) none of these

12. If the equation of the incircle of an equilateral triangle is $x^2 + y^2 + 4x - 6y + 4 = 0$, then the equation of the circumcircle of the triangle is

(A) $x^2 + y^2 + 4x + 6y - 23 = 0$
(B) $x^2 + y^2 + 4x - 6y - 23 = 0$
(C) $x^2 + y^2 - 4x - 6y - 23 = 0$
(D) none of these

13. Two distinct chords drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$, where $pq \neq 0$, are bisected by the x-axis. Then,

(A) $|p| = |q|$ (B) $p^2 = 8q^2$
(C) $p^2 < 8q^2$ (D) $p^2 > 8q^2$

14. For the two circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2y = 0$, there is/are

(A) one pair of common tangents
(B) two pairs of common tangents
(C) three common tangents
(D) no common tangent

15. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then, the locus of the centroid of the ΔPAB as P moves on the circle is

(A) a parabola
(B) a circle
(C) an ellipse
(D) a pair of straight lines

16. The equation of the smallest circle passing through the intersection of the line $x + y = 1$ and the circle $x^2 + y^2 = 9$ is

(A) $x^2 + y^2 + x + y - 8 = 0$
(B) $x^2 + y^2 - x - y - 8 = 0$
(C) $x^2 + y^2 - x - y + 8 = 0$
(D) none of these

17. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circles $x^2 + y^2 + 2g'x + 2f'y + c' = 0$, then

(A) $2g'(g - g') + 2f'(f - f') = c - c'$
(B) $g'(g - g') + f'(f - f') + c - c' = 0$
(C) $2g(g - g') + 2f(f - f') = c - c'$
(D) none of these

18. If (a, b) is a point on the circle whose centre is on the x-axis and which touches the line $x + y = 0$ at $(2, -2)$, then the greatest value of a is

(A) $4+2\sqrt{2}$ (B) $2+2\sqrt{2}$
(C) $4+\sqrt{2}$ (D) none of these

19. The equation $(x + y - 6)(xy - 3x - y + 3) = 0$ represents the sides of a triangle then the equation of the circumcircle of the triangle is

(A) $x^2 + y^2 - 5x - 9y + 20 = 0$
(B) $x^2 + y^2 - 4x - 8y + 18 = 0$
(C) $x^2 + y^2 - 3x - 5y + 8 = 0$
(D) $x^2 + y^2 + 2x - 3y - 1 = 0$

20. If $a > 2b > 0$ then the positive value of m for which $y = mx - b\sqrt{1+m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is

(A) $\frac{2b}{\sqrt{a^2 - 4b^2}}$ (B) $\frac{\sqrt{a^2 - 4b^2}}{2b}$
(C) $\frac{2b}{a - 2b}$ (D) $\frac{b}{a - 2b}$

21. If the locus of a point which moves so that the line joining the points of contact of the tangents drawn from it to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = a^2$, is the circle $x^2 + y^2 = c^2$, then a, b, c are in

(A) A. P. (B) G. P.
(C) H. P. (D) none of these

22. A variable circle passes through the fixed point $A(p, q)$ and touches x-axis. The locus of the other end of the diameter through A is

(A) $(y - q)^2 = 4px$ (B) $(x - q)^2 = 4py$
(C) $(y - p)^2 = 4qx$ (D) $(x - p)^2 = 4qy$

23. The point (1, 4) lies inside the circle $x^2 + y^2 - 6x - 10y + p = 0$ which does not touch or intersect the coordinate axes, then

(A) $0 < p < 29$ (B) $25 < p < 29$
(C) $9 < p < 25$ (D) $9 < p < 29$

24. A circle C_1 of radius 2 touches both x-axis and y-axis. Another circle C_2 whose radius is greater than 2 touches circle C_1 and both the axes. Then, the radius of circle C_2 is

(A) $6-4\sqrt{2}$ (B) $6+4\sqrt{2}$
(C) $6-4\sqrt{3}$ (D) $6+4\sqrt{3}$

25. The equation to the sides AB, BC, CA of a ΔABC are $x + y = 1$, $4x - y + 4 = 0$ and $2x + 3y = 6$. Circles are drawn on AB, BC, CA as diameters. The point of concurrence of the common chords is

(A) centroid of the triangle
(B) orthocentre
(C) circumcentre
(D) incentre

26. The coordinates of the point on the circle $x^2 + y^2 - 2x - 4y - 11 = 0$ farthest from the origin are

(A) $\left(2+\frac{8}{\sqrt{5}},\ 1+\frac{4}{\sqrt{5}}\right)$

(B) $\left(1+\frac{4}{\sqrt{5}},\ 2+\frac{8}{\sqrt{5}}\right)$

(C) $\left(1+\frac{8}{\sqrt{5}},\ 2+\frac{4}{\sqrt{5}}\right)$

(D) none of these

27. If the line $3x + ay - 20 = 0$ cuts the circle $x^2 + y^2 = 25$ at real, distinct or coincident points, then a belongs to the interval

(A) $[-\sqrt{7}, \sqrt{7}]$
(B) $(-\sqrt{7}, \sqrt{7})$
(C) $(-\infty - \sqrt{7}] \cup [\sqrt{7}, \infty)$
(D) none of these

28. The locus of centre of the circle which touches the circle $x^2 + (y - 1)^2 = 1$ externally and also touches x-axis is

(A) $\{(x, y): x^2 + (y - 1)^2 = 4\} \cup \{(x, y): y < 0\}$
(B) $\{(x, y): x^2 = 4y\} \cup \{(0, y): y < 0\}$
(C) $\{(x, y): x^2 = y\} \cup \{(0, y): y < 0\}$
(D) $\{(x, y): x^2 = 4y\} \cup \{(x, y): y < 0\}$

29. Let PQ and RS be tangents at the extremeties of the diameter PR of a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals

(A) $\sqrt{PQ\cdot RS}$ (B) $\frac{PQ+RS}{2}$

(C) $\frac{2PQ\cdot RS}{PQ+RS}$ (D) $\sqrt{\frac{PQ^2+RS^2}{2}}$

30. Circles are drawn through the point (–5, 0) to cut the x-axis on the positive side and making an intercept of 10 units on the x-axis. The equation of the locus of the centre of these circles is

(A) $x + y = 0$ (B) $x - y = 0$
(C) $x = 0$ (D) $y = 0$

31. The circle $x^2 + y^2 - 4x - 8y + 16 = 0$ rolls up the tangent to it at $(2+\sqrt{3}, 3)$ by 2 units, assuming the x–axis as horizontal, the equation of the circle in the new position is

(A) $x^2+y^2-6x-2(4+\sqrt{3})y+24+8\sqrt{3}=0$
(B) $x^2+y^2+6x-2(4+\sqrt{3})y+24+8\sqrt{3}=0$
(C) $x^2+y^2-6x+2(4+\sqrt{3})y+24+8\sqrt{3}=0$
(D) none of these

32. The equation of the circle, passing through the point (2, 8), touching the lines $4x - 3y - 24 = 0$ and $4x + 3y - 42 = 0$ and having x coordinate of the centre of the circle numerically less then or equal to 8, is

(A) $x^2 + y^2 + 4x - 6y - 12 = 0$
(B) $x^2 + y^2 - 4x + 6y - 12 = 0$
(C) $x^2 + y^2 - 4x - 6y - 12 = 0$
(D) none of these

33. If the line $\frac{x}{a}+\frac{y}{b}=1$ moves in such a way that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$, where c is a constant, then the locus of the foot of the perpendicular from the origin on the straight line describes the circle

(A) $x^2 + y^2 = 4c^2$ (B) $x^2 + y^2 = 2c^2$
(C) $x^2 + y^2 = c^2$ (D) none of these

34. A circle touches both the x-axis and the line $4x - 3y + 4 = 0$. If its centre is in the third quadrant and lies on the line $x - y - 1 = 0$, then the equation of the circle is

(A) $9(x^2 + y^2) + 6x + 24y - 1 = 0$
(B) $9(x^2 + y^2) + 6x - 24y + 1 = 0$
(C) $9(x^2 + y^2) + 6x + 24y + 1 = 0$
(D) none of these

35. The line $Ax + By + C = 0$ cuts the circle $x^2 + y^2 + ax + by + c = 0$ in P and Q. The line $A'x + B'y + C' = 0$ cuts the circle $x^2 + y^2 + a'x + b'y + c' = 0$ in R and S. If P, Q, R, S are concyclic points, then

(A) $\begin{vmatrix} a+a' & b+b' & c+c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$

(B) $\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$

(C) $\begin{vmatrix} A(a+a') & B(b+b') & C(c+c') \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$

(D) none of these

36. If θ_1, θ_2 be the inclinations of tangents drawn from the point P to the circle $x^2 + y^2 = a^2$ and $\cot\theta_1 + \cot\theta_2 = k$, then the locus of P is

(A) $k(y^2 + a^2) = 2xy$ (B) $k(y^2 - a^2) = 2xy$
(C) $k(y^2 - a^2) = xy$ (D) none of these

37. A line meets the coordinate axes in A and B. A circle is circumscribed about the ΔAOB. If m, n are the distances of the tangent to the circle at the origin from the points A and B, respectively, the diameter of the circle is

(A) $m(m + n)$ (B) $m + n$
(C) $n(m + n)$ (D) none of these

38. If the chord of contact of tangents from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$, then a, b, c are in

(A) A. P. (B) G. P.
(C) H. P. (D) none of these

39. To which of the following circles, the line $y - x + 3 = 0$ is normal at the point $\left(3+\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$?

(A) $\left(x-3-\frac{3}{\sqrt{2}}\right)^2 + \left(y-\frac{3}{\sqrt{2}}\right)^2 = 9$

(B) $\left(x-\frac{3}{\sqrt{2}}\right)^2 + \left(y-\frac{3}{\sqrt{2}}\right)^2 = 9$

(C) $x^2 + (y - 3)^2 = 9$
(D) $(x - 3)^2 + y^2 = 9$

40. If a circle passes through the points where the lines $3kx - 2y - 1 = 0$ and $4x - 3y + 2 = 0$ meet the coordinate axes then $k =$

(A) 1 (B) -1
(C) $\frac{1}{2}$ (D) $\frac{-1}{2}$

41. Let C be any circle with centre $(0, \sqrt{2})$. Then, on the circle C, there can be

(A) at the most one rational point
(B) at the most two rational points
(C) at the most three rational points
(D) none of these

42. If the tangents PQ and PR are drawn to the circle $x^2 + y^2 = a^2$ from the point $P(x_1, y_1)$, then the equation of the circumcircle of ΔPQR is

(A) $x^2 + y^2 - xx_1 - yy_1 = 0$
(B) $x^2 + y^2 + xx_1 + yy_1 = 0$
(C) $x^2 + y^2 - 2xx_1 - 2yy_1 = 0$
(D) none of these

43. A ray of light, incident at the point $(-2, -1)$, gets reflected from the tangent at $(0, -1)$ to the circle $x^2 + y^2 = 1$. The reflected ray touches the circle. The equation of the line along which the incident ray moved is

(A) $4x - 3y + 11 = 0$ (B) $4x + 3y + 11 = 0$
(C) $3x + 4y + 11 = 0$ (D) none of these

44. The coordinates of a point P on the circle $x^2 + y^2 - 4x - 6y + 9 = 0$ such that $\angle POX$ is minimum, where O is the origin and OX is the x-axis, are

(A) $\left(\frac{36}{13}, \frac{15}{13}\right)$ (B) $\left(\frac{-36}{13}, \frac{15}{13}\right)$

(C) $\left(\frac{14}{27}, \frac{12}{27}\right)$ (D) none of these

45. The locus of the centre of a circle which passes through the point $(0, 0)$ and cuts off a length $2b$ from the line $x = c$, is

(A) $y^2 + 2cx = b^2 + c^2$ (B) $x^2 + cx = b^2 + c^2$
(C) $y^2 + 2cy = b^2 + c^2$ (D) none of these

46. If P, Q is a pair of conjugate points with respect to a circle S, then the circle on PQ as diameter

(A) touches the circle S
(B) cuts the circle S at an angle $\frac{\pi}{4}$
(C) cuts the circle S orthogonally
(D) none of these

47. The common chord of the circle $x^2 + y^2 + 8x + 4y - 5 = 0$ and a circle passing through the origin and touching the line $y = x$, passes through the fixed point

(A) $\left(\frac{5}{12}, \frac{5}{12}\right)$ (B) $\left(\frac{5}{12}, \frac{-5}{12}\right)$

(C) $\left(\frac{-5}{12}, \frac{5}{12}\right)$ (D) none of these

48. If the circles $x^2 + y^2 = 1$ and $x^2 + y^2 - 4x - 6y + 12 = 0$ cut off equal intercepts on a line which passes through the point (1, 1), then the slope of the line is

(A) 1 (B) −1

(C) $\frac{3}{2}$ (D) $-\frac{3}{2}$

49. Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line drawn from the point P intersects the curve at points Q and R. If the product $PQ.PR$ is independent of the slope of the line, then the curve is

(A) an ellipse (B) a hyperbola

(C) a circle (D) none of these

50. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the interecpts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then which of the following equations can represent L_1?

(A) $x + y = 0$ (B) $x - y = 0$

(C) $7y + 2x = 0$ (D) $x - 7y = 0$

51. A triangle has two of its sides along the axes. If the third side touches the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$, then the equation of the locus of the circumcentre of the triangle is

(A) $2a(x + y) = 2xy + a^2$

(B) $2a(x - y) = 2xy + a^2$

(C) $2a(x + y) = 2xy - a^2$

(D) none of these

52. A point moves such that the sum of the squares of its distances from the sides of a square of side unity is equal to 9. The locus of the point is a circle such that

(A) centre of the circle coincides with that of square

(B) centre of the circle is $\left(\frac{1}{2}, \frac{1}{2}\right)$

(C) radius of the circle is 2

(D) all the above are true

53. The range of values of a for which the line $y + x = 0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2 + 2y^2 - (1+\sqrt{2}a)x - (1-\sqrt{2}a)y = 0$ is

(A) $(-\infty, -2) \cup (2, \infty)$ (B) $(-2, 2)$

(C) $(2, \infty)$ (D) none of these

54. The locus of the centres of the circles which touch the two circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 4ax$ externally is

(A) $12x^2 - 4y^2 - 24ax + 9a^2 = 0$

(B) $12x^2 + 4y^2 - 24ax + 9a^2 = 0$

(C) $12x^2 - 4y^2 + 24ax + 9a^2 = 0$

(D) none of these

55. If a circle passes through the points of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is

(A) 2 (B) 1

(C) −1 (D) −2

56. A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$, where O is the origin. The circle contains the point (−10, 2) in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. The equation of the circle is

(A) $x^2 + y^2 + 18x - 2y + 32 = 0$

(B) $x^2 + y^2 - 18x - 2y + 32 = 0$

(C) $x^2 + y^2 + 18x + 2y + 32 = 0$

(D) none of these

57. If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is a given circle, then the locus of the foot of the perpendicular drawn from origin upon any chord of S which subtends a right angle at the origin, is

(A) $2(x^2 + y^2) + 2gx + 2fy + c = 0$

(B) $2(x^2 + y^2) + 2gx + 2fy - c = 0$

(C) $x^2 + y^2 + gx + fy + c = 0$

(D) none of these

58. The equation of the circle, having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normals and having size just sufficient to contain the circle $x(x - 4) + y(y - 3) = 0$, is

(A) $x^2 + y^2 + 6x + 3y - 45 = 0$

(B) $x^2 + y^2 + 6x - 3y - 45 = 0$

(C) $x^2 + y^2 + 6x - 3y + 45 = 0$

(D) none of these

59. The equation of the system of coaxal circles that are tangent at $(\sqrt{2}, 4)$ to the locus of the point of intersection of mutually ⊥ tangents to the circle $x^2 + y^2 = 9$, is

(A) $(x^2 + y^2 - 18) + \lambda(\sqrt{2}x + 4y - 18) = 0$

(B) $(x^2 + y^2 - 18) + \lambda(4x + \sqrt{2y} - 18) = 0$

(C) $(x^2 + y^2 - 16) + \lambda(\sqrt{2x} + 4y - 16) = 0$

(D) none of these

60. The point on the straight line $y = 2x + 11$ which is nearest to the circle $16(x^2 + y^2) + 32x - 8y - 50 = 0$ is

(A) $\left(\frac{9}{2}, 2\right)$ (B) $\left(-\frac{9}{2}, 2\right)$

(C) $\left(\frac{9}{2}, -2\right)$ (D) none of these

61. Extremities of a diagonal of a rectangle are (0, 0) and (4, 3). The equations of the tangents to the circumcircle of the rectangle which are parallel to this diagonal are

(A) $16x + 8y \pm 25 = 0$ (B) $6x - 8y \pm 25 = 0$

(C) $8x + 6y \pm 25 = 0$ (D) none of these

62. The base AB of a triangle is fixed and its vertex C moves such that $\sin A = k \sin B (k \neq 1)$. If a is the length of the base AB, then the locus of C is a circle whose radius is equal to

(A) $\frac{ak}{(2-k^2)}$ (B) $\frac{ak}{(1-k^2)}$

(C) $\frac{2ak}{1-k^2}$ (D) none of these

63. The equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$ is

(A) $x^2 + y^2 + 32x + 4y + 235 = 0$
(B) $x^2 + y^2 - 32x + 4y + 235 = 0$
(C) $x^2 + y^2 + 32x + 4y - 235 = 0$
(D) none of these

64. The locus of the centre of a circle touching the circle $x^2 + y^2 - 4y - 2x = 2\sqrt{3} - 1$ internally and tangents on which from (1, 2) is making a 60º angle with each other, is

(A) $(x - 1)^2 + (y - 2)^2 = 3$
(B) $(x-2)^2 + (y-1)^2 = 1 + 2\sqrt{3}$
(C) $x^2 + y^2 = 1$
(D) none of these

65. The equation of locus of the point of intersection of tangents to the circle $x^2 + y^2 = 1$ at the points whose parametric angles differ by 60º is

(A) $3x^2 + 3y^2 = 1$ (B) $x^2 + y^2 = 3$
(C) $3x^2 + 3y^2 = 4$ (D) none of these

66. If a square is inscribed in the circle $x^2 + y^2 - 2x + 4y + 3 = 0$ and its sides are parallel to the coordinate axes, then one vertex of the square is

(A) $(1+\sqrt{2}, -2)$ (B) $(1-\sqrt{2}, -2)$
(C) $(1, -2+\sqrt{2})$ (D) none of these

67. The equation of the chord of the circle $x^2 + y^2 = a^2$ passing through the point (2, 3) and farthest from the centre is

(A) $2x + 3y = 13$ (B) $3x + 2y = 13$
(C) $2x - 3y = 13$ (D) none of these

68. The range of values of p such that the angle θ between the pair of tangents drawn from the point $(p, 0)$ to the circle $x^2 + y^2 = 1$ lies in $\left(\frac{\pi}{3}, \pi\right)$ is

(A) $(-2, -1) \cup (1, 2)$ (B) $(-3, -2) \cup (2, 3)$
(C) $(0, 2)$ (D) none of these

69. A circle whose centre coincides with the origin having radius 'a' cuts x-axis at A and B. If P and Q are two points on the circle whose parametric angles differ by 2θ, then the locus of the intersection point of AP and BQ is

(A) $x^2 + y^2 + 2ay \tan\theta = a^2$
(B) $x^2 + y^2 - 2ay \tan\theta = a^2$
(C) $x^2 + y^2 + 2ay \cot\theta = a^2$
(D) none of these

70. If a chord AB subtends a right angle at the centre of a given circle, then the locus of the centroid of the triangle PAB as P moves on the circle is a/an

(A) parabola (B) ellipse
(C) hyperbola (D) circle

71. If $-3l^2 - 6l - 1 + 6m^2 = 0$ then the equation of the circle for which $lx + my + 1 = 0$ is a tangent is

(A) $(x + 3)^2 + y^2 = 6$ (B) $(x - 3)^2 + y^2 = 6$
(C) $x^2 + (y - 3)^2 = 6$ (D) $x^2 + (y + 3)^2 = 6$

72. Let S_1 and S_2 be two circles with S_2 lying inside S_1. A circle S lying inside S_1 touches S_1 internally and S_2 externally. The locus of the centre of S is a/an

(A) parabola (B) ellipse
(C) hyperbola (D) circle

73. $S(x, y) = 0$ represents a circle. The equation $S(x, 2) = 0$ gives two identical solutions $x = 1$ and the equation $S(1, y) = 0$ gives two distinct solutions $y = 0, 2$. The equation of the circle is

(A) $x^2 + y^2 + 2x + 2y + 1 = 0$
(B) $x^2 + y^2 + 2x + 2y - 1 = 0$
(C) $x^2 + y^2 - 2x - 2y + 1 = 0$
(D) none of these

More than One Option Correct Type

74. The equation of a circle of radius 2 touching the circles $x^2 + y^2 - 4|x| = 0$ is

(A) $x^2 + y^2 + 2\sqrt{3}y + 2 = 0$
(B) $x^2 + y^2 + 4\sqrt{3}y + 8 = 0$
(C) $x^2 + y^2 - 4\sqrt{3}y + 8 = 0$
(D) none of these

75. The coordinates of a point on the line $y = 2$ from which the tangents drawn to the circle $x^2 + y^2 = 25$ are perpendicular, are

(A) $(\sqrt{46}, 2)$ (B) $(-\sqrt{46}, 2)$
(C) $(\sqrt{37}, 2)$ (D) $(-\sqrt{37}, 2)$

76. Extremities of a diagonal of a rectangle are (0, 0) and (4, 3). The equations of the tangents to the circumcircle of the rectangle which are parallel to this diagonal are

(A) $16x + 8y + 25 = 0$ (B) $6x - 8y + 25 = 0$
(C) $8x + 6y - 25 = 0$ (D) $6x - 8y - 25 = 0$

77. The equation of the circle, touching the axis of x at the origin and the line $3y = 4x + 24$, is

(A) $x^2 + y^2 + 24y = 0$ (B) $x^2 + y^2 - 6y = 0$
(C) $x^2 + y^2 - 24y = 0$ (D) $x^2 + y^2 + 6y = 0$

78. If the equation of the common tangent at the point (1, −1) to the two circles, each of radius 13, is $12x + 5y - 7 = 0$, then the centres of the two circles are

(A) (13, 4) (B) (13, −4)
(C) (−11, 6) (D) (−11, −6)

79. The coordinates of two points on the circle $x^2 + y^2 - 12x - 16y + 75 = 0$, one nearest to the origin and the other farthest from it, are

(A) (3, 4) (B) (3, 2)
(C) (9, 12) (D) (9, −12)

80. The equation of a circle of equal radius, touching both the circles $x^2 + y^2 = a^2$ and $(x - 2a)^2 + y^2 = a^2$ is given by

(A) $x^2 + y^2 - 2ax - 2\sqrt{3}ay + 3a^2 = 0$
(B) $x^2 + y^2 - 2ax + 2\sqrt{3}ay + 3a^2 = 0$
(C) $x^2 + y^2 + 2ax - 2\sqrt{3}ay + 3a^2 = 0$
(D) none of these

81. With respect to the circle $x^2 + y^2 + 6x - 8y - 10 = 0$,

(A) The chord of contact of tangents from (2, 1) is $5x - 3y - 8 = 0$
(B) the pole of the line $5x - 3y - 8 = 0$ is (2, 1)
(C) the polar of the point (2, 1) is $5x - 3y - 8 = 0$
(D) all of these

82. For the circles $S_1 \equiv x^2 + y^2 - 4x - 6y - 12 = 0$ and $S_2 \equiv x^2 + y^2 + 6x + 4y - 12 = 0$ and the line $L \equiv x + y = 0$

(A) L is the common tangent of S_1 and S_2
(B) L is the common chord of S_1 and S_2
(C) L is radical axis of S_1 and S_2
(D) L is perpendicular to the line joining the centres of S_1 and S_2

83. Two circles, each of radius 5 units, touch each other at (1, 2). If the equation of their common tangent is $4x + 3y = 10$, then the equations of the circles are

(A) $x^2 + y^2 + 10x + 10y + 25 = 0$
(B) $x^2 + y^2 - 10x - 10y + 25 = 0$
(C) $x^2 + y^2 + 6x + 2y - 15 = 0$
(D) $x^2 + y^2 - 6x - 2y + 15 = 0$

84. If the circle $C_1 : x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope equal to $\frac{3}{4}$, then the coordinates of the centre of C_2 are

(A) $\left(\frac{9}{5}, -\frac{12}{5}\right)$ (B) $\left(-\frac{9}{5}, \frac{12}{5}\right)$
(C) $\left(\frac{9}{5}, \frac{12}{5}\right)$ (D) $\left(-\frac{9}{5}, -\frac{12}{5}\right)$

85. The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which have two of its sides along the coordinate axes. If the locus of the circumcentre of the triangle is $x + y - xy + k\sqrt{x^2 + y^2} = 0$, then k is equal to

(A) 1 (B) −1
(C) 2 (D) none of these

86. From a point on the line $4x - 3y = 6$, tangents are drawn to the circle $x^2 + y^2 - 6x - 4y + 4 = 0$ which make an angle of $\tan^{-1}\frac{24}{7}$ between them, the coordinates of such points are

(A) (0, 2) (B) (0, −2)
(C) (6, 6) (D) (−6, 6)

87. A tangent drawn from the point (4, 0) to the circle $x^2 + y^2 = 8$ touches it at a point A in the first quadrant. The coordinates of another point B on the circle such that $AB = 4$, are

(A) (2, −2) (B) (−2, 2)
(C) (2, 2) (D) (−2, −2)

88. Two vertices of an equilateral triangle are (−1, 0) and (1, 0). An equation of its circumcentre is

(A) $x^2 + y^2 + \frac{2}{\sqrt{3}}y - 1 = 0$
(B) $x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 = 0$
(C) $x^2 + y^2 + \frac{2}{\sqrt{3}}y + 1 = 0$
(D) none of these

89. If one of the circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2bx + c = 0$ lies within the other, then

(A) $ab < 0$ (B) $ab > 0$
(C) $c < 0$ (D) $c > 0$

90. A tangent to the circle $x^2 + y^2 = 1$ through the point (0, 5) cuts the circle $x^2 + y^2 = 4$ at A and B. The tangents for the circle $x^2 + y^2 = 4$ at A and B meet at C. The coordinates of C are

(A) $\left(\frac{8\sqrt{6}}{5}, \frac{4}{5}\right)$ (B) $\left(-\frac{8\sqrt{6}}{5}, \frac{4}{5}\right)$
(C) $\left(\frac{8\sqrt{6}}{5}, -\frac{4}{5}\right)$ (D) $\left(-\frac{8\sqrt{6}}{5}, -\frac{4}{5}\right)$

91. The equation $r = |\cos\theta|$ represents

(A) two circles of radii $\frac{1}{2}$ each

(B) two circles centered at $\left(\frac{1}{2}, 0\right)$ and $\left(-\frac{1}{2}, 0\right)$

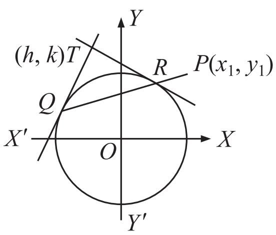
(C) two circles touching each other at the origin

(D) pair of straight lines

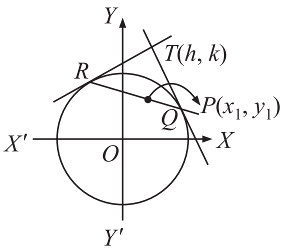
Passage Based Questions

Passage 1

Let a straight line be drawn from a point P to meet the circle in Q and R. Let the tangents at Q and R meet at T. The locus of T is called the polar of P with respect to the circle. The given point P is called the pole of the polar line.



(i)



(ii)

Let P (x_1, y_1) be the given point lying outside the circle in fig. (*i*) and inside the circle in fig. (*ii*).

Through P, draw a line to meet the circle in Q and R. Let the tangents to the circle at Q and R meet in $T(h, k)$.

It is required to find the polar of P, i.e., the locus of T.

Equation of QR, the chord of contact of the tangents drawn from T to the circle $x^2 + y^2 = a^2$

is $$xh + yk = a^2 \quad (1)$$

$\because$ (1) passes through $P(x_1, y_1)$, $\therefore$ $x_1h + y_1k = a^2$.

$\therefore$ the locus of (h, k) is $xx_1 + yy_1 = a^2$, which is the equation of polar of P.

92. If the polar of P with respect to the circle $x^2 + y^2 = a^2$ touches the circle $(x - f)^2 + (y - g)^2 = b^2$, then its locus is given by the equation

(A) $(fx + gy - a^2)^2 = a^2(x^2 + y^2)$

(B) $(fx + gy - a^2)^2 = b^2(x^2 + y^2)$

(C) $(fx - gy - a^2)^2 = a^2(x^2 + y^2)$

(D) none of these

93. The pole of the line $3x + 4y = 45$ with respect to the circle $x^2 + y^2 - 6x - 8y + 5 = 0$ is

(A) (6, 8) (B) (6, –8)

(C) (–6, 8) (D) (–6, –8)

94. The pole of the chord of the circle $x^2 + y^2 = 16$ which is bisected at the point (–2, 3), with respect to the circle is

(A) $\left(\frac{-32}{13}, \frac{48}{13}\right)$ (B) $\left(\frac{32}{13}, \frac{48}{13}\right)$

(C) $\left(\frac{-32}{13}, \frac{-48}{13}\right)$ (D) none of these

95. The coordinates of the poles of the common chord of the circles $x^2 + y^2 = 12$ and $x^2 + y^2 - 5x + 2y - 2 = 0$ with respect to the circle $x^2 + y^2 = 12$ are

(A) $\left(6, \frac{-12}{5}\right)$ (B) $\left(-6, \frac{12}{5}\right)$

(C) $\left(6, \frac{12}{5}\right)$ (D) none of these

Passage 2

A system of circles, every two of which have the same radical axis, is said to be *coaxal*.

The simplest equation of a coaxal system of circles is
$$x^2 + y^2 + 2gx + c = 0$$
where g is different for different circles of the system and c is the same for all the circles, the common radical axis being the axis of y and the line of centres, the axis of x. The equation of a system of coaxal circles, having given the equation of radical axis and one circle of the system $S = 0$ is $S + \lambda u = 0$. The equation of a system of coaxal circles, having given equations of two circles of the system.
$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$
$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$
is $$S_1 + \lambda S_2 = 0$$
The members of the coaxal system of circles which are of zero radius are called the limiting points of the system.

96. If A, B, C be the centres and r_1, r_2, r_3 the radii of three coaxal circles, then $r_1^2 \cdot BC + r_2^2 \cdot CA + r_2^3 \cdot AB =$

(A) $BC{\cdot}CA{\cdot}AB$ (B) $-BC{\cdot}CA{\cdot}AB$

(C) $2BC{\cdot}CA{\cdot}AB$ (D) none of these

97. If, A, B, C be the centres of three coaxal circles and t_1, t_2, t_3 be the tangents to them from any point, then $BC \cdot t_1^2 + CA \cdot t_2^2 + AB \cdot t_3^2$

(A) 0 (B) 1

(C) –1 (D) none of these

98. The limiting points of the coaxal system determined by the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$ are

(A) $(-1, 2), \left(\frac{3}{5}, \frac{-14}{5}\right)$
(B) $(-1, 2), \left(\frac{3}{5}, \frac{14}{5}\right)$
(C) $(-1, 2), \left(\frac{-3}{5}, \frac{14}{5}\right)$
(D) none of these

99. The equation of the circle which passes through the origin and belongs to the coaxal system whose limiting points are (1, 2) and (4, 3), is

(A) $2x^2 + 2y^2 - x - 7y = 0$
(B) $2x^2 + 2y^2 + x - 7y = 0$
(C) $2x^2 + 2y^2 + x + 7y = 0$
(D) none of these

Match the Column Type

100.

Column-I	Column-I
I. If the circle $x^2 + y^2 - 4x - 6y + k = 0$ does not touch or intersect the axes and the point (2, 2) lies inside the circle, then k belongs to	(A) $(-1, 3)$
II. If the line $3x + ay - 20 = 0$ cuts the circle $x^2 + y^2 = 25$ at real, distinct or coincident points, then a belongs to the interval	(B) $(-\infty, -3) \cup (4, \infty)$
III. If the point $(2, k)$ lies outside the circles $x^2 + y^2 + x - 2y - 14 = 0$ and $x^2 + y^2 = 13$, then k belongs to the interval	(C) $(-\infty, -\sqrt{7}] \cup [\sqrt{7}, \infty)$
IV. If the point $(k + 1, k)$ lies inside the region bounded by the curve $x = \sqrt{25 - y^2}$, then k belongs to the interval	(D) $(9, 12)$

101.

Column-I	Column-I
I. The number of common tangents to the circles $x^2 + y^2 - 6x - 2y + 9 = 0$ and $x^2 + y^2 - 14x - 8y + 61 = 0$ is	(A) 3
II. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 8x + 12 = 0$ is	(B) 4
III. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y - 24 = 0$ is	(C) 2
IV. The number of tangents to the circle $x^2 + y^2 - 8x - 6y + 9 = 0$ which pass through the point $(3, -2)$ is	(D) 1

102.

Column-I	Column-I
I. The tangent to the circle $x^2 + y^2 = 9$, which is parallel to y-axis and does not lie in third quadrant, touches the circle at the point	(A) $\left(-\frac{3}{2}, \frac{1}{2}\right)$
II. The coordinates of the middle point of the chord which the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ cuts off on the line $y = x + 2$, are	(B) (1, 1)
III. The circle passing through three distinct points $(1, t)$, $(t, 1)$ and (t, t) passes through the point	(C) (3, 0)
IV. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the circle $x^2 + y^2 = 1$ pass through the fixed point	(D) $\left(\frac{1}{2}, \frac{1}{4}\right)$

Assertion-Reason Type

Instructions *In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options*:

(A) Assertion (A) is True and Reason (R) is True; Reason (R) is a correct explanation for Assertion (A)
(B) Assertion (A) is True, Reason (R) is True; Reason (R) is not a correct explanation for Assertion (A)
(C) Assertion (A) is True, Reason (R) is False
(D) Assertion (A) is False, Reason (R) is True

103. Assertion: The locus of the centres of circles passing through the origin and cutting the circle $x^2 + y^2 + 6x - 4y + 2 = 0$ orthogonally is $3x - 2y + 1 = 0$.

Reason: The two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ cut each other orthogonally if $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

104. Assertion: The tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$ also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$. Then its point of contact is $(3, -1)$.

Reason: The equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

105. Assertion: If the centroid of an equilateral triangle is $(2, 2)$ and its one vertex is $(-3, 4)$, then the equation of its circumcircle is $x^2 + y^2 - 4x - 4y - 21 = 0$.

Reason: Circumcentre coincides with the centroid of an equilateral triangle.

106. Assertion: If the point $(2, 4)$ is interior to the circle $x^2 + y^2 - 6x - 10y + k = 0$ and the circle does not cut the axes at any point, then $25 < k < 32$.

Reason: If the point (x_1, y_1) lies inside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ then $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$.

107. Assertion: The equation of the circle passing through the point $(2a, 0)$ and whose radical axis is $x = \frac{a}{2}$ with respect to the circle $x^2 + y^2 = a^2$, will be $x^2 + y^2 - 2ax = 0$.

Reason: The equation of radical axis of two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.

108. Assertion: The chord of contact of tangents from a point P to a circle passes through Q. If l_1 and l_2 are the lengths of the tangents from P and Q to the circle, then PQ is equal to $\sqrt{l_1^2 + l_2^2}$.

Reason: The equation of chord of contact of tangents drawn from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

109. Assertion: If the point on a circle nearest to the point $P(2, 1)$ is at 4 unit distance and the farthest is $(6, 5)$, then the equation of the circle is

$$(x-6)(x-2-2\sqrt{2}) + (y-5)(y-1-2\sqrt{2}) = 0.$$

Reason: The equation of a circle having end points of the diameter as (x_1, y_1) and (x_2, y_2) is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Previous Year's Questions

110. The greatest distance of the point $P(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is **[2002]**

(A) 10 unit (B) 15 unit
(C) 5 unit (D) none of these

111. The equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which make equal intercepts on the positive co-ordinate axes, is **[2002]**

(A) $x + y = 2$ (B) $x + y = 2\sqrt{2}$
(C) $x + y = 4$ (D) $x + y = 8$

112. If the two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then **[2003]**

(A) $2 < r < 8$ (B) $r < 2$
(C) $r = 2$ (D) $r > 2$

113. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle having area as 154 sq units. Then the equation of the circle is **[2003]**

(A) $x^2 + y^2 + 2x - 2y = 62$
(B) $x^2 + y^2 + 2x - 2y = 47$
(C) $x^2 + y^2 - 2x + 2y = 47$
(D) $x^2 + y^2 - 2x + 2y = 62$

114. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is **[2004]**

(A) $2ax + 2by + (a^2 + b^2 + 4) = 0$
(B) $2ax + 2by - (a^2 + b^2 + 4) = 0$
(C) $2ax - 2by + (a^2 + b^2 + 4) = 0$
(D) $2ax - 2by - (a^2 + b^2 + 4) = 0$

115. A variable circle passes through the fixed point A (p, q) and touches x-axis. The locus of the other end of the diameter through A is **[2004]**

(A) $(x - p)^2 = 4qy$ (B) $(x - q)^2 = 4py$
(C) $(y - p)^2 = 4qx$ (D) $(y - q)^2 = 4px$

116. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is **[2004]**

(A) $x^2 + y^2 - 2x + 2y - 23 = 0$
(B) $x^2 + y^2 - 2x - 2y - 23 = 0$
(C) $x^2 + y^2 + 2x + 2y - 23 = 0$
(D) $x^2 + y^2 + 2x - 2y - 23 = 0$

117. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle on AB as a diameter is **[2004]**

(A) $x^2 + y^2 - x - y = 0$ (B) $x^2 + y^2 - x + y = 0$
(C) $x^2 + y^2 + x + y = 0$ (D) $x^2 + y^2 + x - y = 0$

118. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q then the line $5x + by - a = 0$ passes through P and Q for **[2005]**

(A) exactly one value of a
(B) no value of a
(C) infinitely many values of a
(D) exactly two values of a

119. A circle touches the x-axis and also touches the circle with centre at (0, 3) and radius 2. The locus of the centre of the circle is **[2005]**

(A) an ellipse (B) a circle
(C) a hyperbola (D) a parabola

120. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = p^2$ orthogonally, then the equation of the locus of its centre is **[2005]**

(A) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - p^2) = 0$
(B) $2ax + 2by - (a^2 - \beta^2 + p^2) = 0$
(C) $x^2 + y^2 - 2ax - 3by - (a^2 - \beta^2 - p^2) = 0$
(D) $2ax + 2by - (a^2 + b^2 + p^2) = 0$

121. If the lines $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ are two diameters of a circle of area 49π square units, the equation of the circle is **[2006]**

(A) $x^2 + y^2 + 2x - 2y - 47 = 0$
(B) $x^2 + y^2 + 2x - 2y - 62 = 0$
(C) $x^2 + y^2 - 2x + 2y - 62 = 0$
(D) $x^2 + y^2 - 2x + 2y - 47 = 0$

122. Let C be the circle with centre (0, 0) and radius 3 units. The equation of the locus of the mid points of the chords of the circle C that subtend an angle of $\frac{2\pi}{3}$ at its centre is **[2006]**

(A) $x^2 + y^2 = \frac{3}{2}$ (B) $x^2 + y^2 = 1$
(C) $x^2 + y^2 = \frac{27}{4}$ (D) $x^2 + y^2 = \frac{9}{4}$

123. Consider a family of circles which are passing through the point $(-1, 1)$ and are tangent to x-axis. If (h, k) are the co-ordinates of the centre of the circles, then the set of values of k is given by the interval **[2007]**

(A) $0 < k < \frac{1}{2}$ (B) $k \geq \frac{1}{2}$
(C) $-\frac{1}{2} \leq k \leq \frac{1}{2}$ (D) $k \leq \frac{1}{2}$

124. The point diametrically opposite to the point $P(1, 0)$ on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is **[2008]**

(A) $(3, -4)$ (B) $(-3, 4)$
(C) $(-3, -4)$ (D) $(3, 4)$

125. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and (1, 1) for **[2009]**

(A) all values of p
(B) all except one value of p
(C) all except two values of p
(D) exactly one value of p

126. Three distinct points A, B and C are given in the 2-dimensional coordinate plane such that the ratio of the distance of anyone of them from the point (1, 0) to the distance from the point $(-1, 0)$ is equal to $\frac{1}{3}$ Then the circumcentre of the triangle ABC is at the point **[2009]**

(A) (0, 0) (B) $\left(\frac{5}{4}, 0\right)$
(C) $\left(\frac{5}{2}, 0\right)$ (D) $\left(\frac{5}{3}, 0\right)$

127. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points then m satisfies **[2010]**

(A) $-35 < m < 15$ (B) $15 < m < 65$
(C) $35 < m < 85$ (D) $-85 < m < -35$

128. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ $(c > 0)$ touch each other if **[2011]**

(A) $|a| = c$ (B) $a = 2c$
(C) $|a| = 2c$ (D) $2|a| = c$

129. The length of the diameter of the circle which touches the x-axis at the point (l, 0) and passes through the point (2, 3) is **[2012]**

(A) $\frac{10}{3}$ (B) $\frac{3}{5}$

(C) $\frac{6}{5}$ (D) $\frac{5}{3}$

130. The circle passing through (1, –2) and touching the x-axis at (3, 0) also passes through the point **[2013]**

(A) (2, –5) (B) (5, –2)

(C) (–2, 5) (D) (–5, 2)

131. Let C be the circle with centre at (l, 1) and with radius 1. If T is the circle centered at (0, y), passing through origin and touching the circle C externally, then the radius of T is equal to **[2014]**

(A) $\frac{\sqrt{3}}{\sqrt{2}}$ (B) $\frac{\sqrt{3}}{2}$

(C) $\frac{1}{2}$ (D) $\frac{1}{4}$

132. The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$, is **[2015]**

(A) 2 (B) 3

(C) 4 (D) 1

133. If one of the diameters of the circle, given by the equation. $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is **[2016]**

(A) 10 (B) $5\sqrt{2}$

(C) $5\sqrt{3}$ (D) 5

ANSWER KEYS

Single Option Correct Type

1. (B)	**2.** (B)	**3.** (C)	**4.** (C)	**5.** (D)	**6.** (B)	**7.** (B)	**8.** (A)	**9.** (D)	**10.** (A)
11. (A)	**12.** (B)	**13.** (D)	**14.** (D)	**15.** (B)	**16.** (B)	**17.** (A)	**18.** (A)	**19.** (B)	**20.** (A)
21. (B)	**22.** (D)	**23.** (B)	**24.** (B)	**25.** (B)	**26.** (B)	**27.** (C)	**28.** (B)	**29.** (A)	**30.** (C)
31. (A)	**32.** (C)	**33.** (C)	**34.** (C)	**35.** (B)	**36.** (B)	**37.** (B)	**38.** (B)	**39.** (D)	**40.** (C)
41. (B)	**42.** (A)	**43.** (B)	**44.** (A)	**45.** (A)	**46.** (C)	**47.** (A)	**48.** (C)	**49.** (C)	**50.** (B)
51. (A)	**52.** (D)	**53.** (A)	**54.** (A)	**55.** (A)	**56.** (A)	**57.** (A)	**58.** (B)	**59.** (A)	**60.** (B)
61. (B)	**62.** (B)	**63.** (A)	**64.** (A)	**65.** (C)	**66.** (D)	**67.** (A)	**68.** (A)	**69.** (B)	**70.** (D)
71. (B)	**72.** (B)	**73.** (C)							

More than One Option Correct Type

74. (B, C) **75.** (A, B) **76.** (B, D) **77.** (A, B) **78.** (A, D) **79.** (A, C) **80.** (A, B)
81. (B, C) **82.** (B, C, D) **83.** (B, C) **84.** (A, B) **85.** (A, B) **86.** (B, C) **87.** (A, B)
88. (A, B) **89.** (B, D) **90.** (A, B) **91.** (A, B, C)

Passage Based Questions

92. (B) **93.** (A) **94.** (A) **95.** (A) **96.** (B) **97.** (A) **98.** (B) **99.** (A)

Match the Column Type

100. I $\leftrightarrow$ (D), II $\leftrightarrow$ (C), III $\leftrightarrow$ (B), IV $\leftrightarrow$ (A) **101.** I $\leftrightarrow$ (B), II $\leftrightarrow$ (A), III $\leftrightarrow$ (D), IV $\leftrightarrow$ (C)
102. I $\leftrightarrow$ (C), II $\leftrightarrow$ (A), III $\leftrightarrow$ (B), IV $\leftrightarrow$ (D)

Assertion-Reason Type

103. (A) **104.** (A) **105.** (A) **106.** (A) **107** (A) **108.** (A) **109.** (A)

Previous Year's Questions

110. (B)	**111.** (B)	**112.** (A)	**113.** (C)	**114.** (B)	**115.** (A)	**116.** (A)	**117.** (A)	**118.** (B)	**119.** (D)
120. (D)	**121.** (D)	**122.** (D)	**123.** (B)	**124.** (C)	**125.** (A)	**126.** (B)	**127.** (A)	**128.** (A)	**129.** (A)
130. (B)	**131.** (D)	**132.** (B)	**133.** (C)						

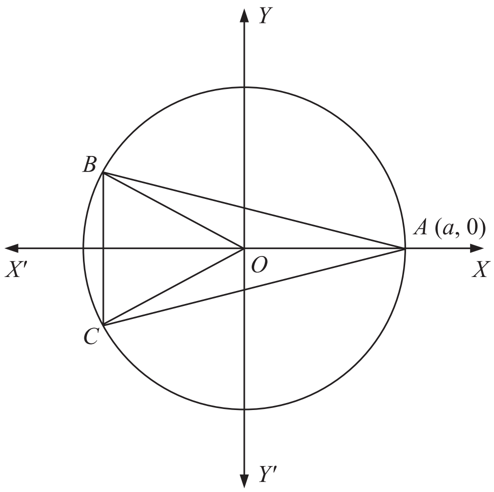
HINTS AND SOLUTIONS

Single Option Correct Type

1. $\angle B = \angle C = 75^\circ$

$\Rightarrow \quad \angle BAC = 30^\circ$

$\Rightarrow \quad \angle BOC = 60^\circ$



$\Rightarrow \quad BOC$ is an equilateral triangle

$\Rightarrow \quad BC = OB =$ the radius of the cirlce

$\Rightarrow \quad BC = a.$

2. Let $(0, a)$ be the centre and r be the radius of the given circle, then its equation is

$$(x - 0)^2 + (y - a)^2 = r^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ay + a^2 - r^2 = 0 \qquad (1)$$

Since the point $\left(\log a_n, \dfrac{1}{\log a_n}\right)$; $n = 1, 2, 3, 4$ lie on the above circle, therefore $(\log a_n)^2 + \dfrac{1}{(\log a_n)^2} - \dfrac{2a}{\log a_n} + a^2 - r^2 = 0$, $n = 1, 2, 3, 4$

$\Rightarrow \quad \log a_1, \log a_2, \log a_3, \log a_4$ are the roots of the equation

$$\lambda^4 + (a^2 - r^2)l^2 - 2a\lambda + 1 = 0.$$

$\therefore \quad$ Sum of the roots $= 0$

$\Rightarrow \quad \log a_1 + \log a_2 + \log a_3 + \log a_4 = 0$

$\Rightarrow \quad \log(a_1a_2a_3a_4) = 0 \quad$ or $\quad a_1a_2a_3a_4 = 1.$

3. The given circle is $S : x^2 + y^2 + x - 2y - 3 = 0.$

Since $S]_{P\,(-1,\,2)} = 1 + 4 - 1 - 4 - 3 = -3 < 0$, the point $P(-1, 2)$ lies inside the circle. Consequently, the tangents from the point $P(-1, 2)$ to the circle do not exist. Thus, the quadrilateral $PACB$ cannot be formed.

4. The given line passes through the point $(-1, 2)$.

Given circle is $S \equiv x^2 + y^2 + 2x - 4y - 3 = 0.$

Since $S]_{(-1,2)} = 1 + 4 - 2 - 8 - 3 < 0,$

$\therefore$ $(-1, 2)$ is an interior point of the circle. Thus, m can have any real value.

5. Given circle is $x^2 + y^2 - 4x - 10y + 13 = 0$

Its centre is $C \equiv (2, 5)$ and radius $= 4$

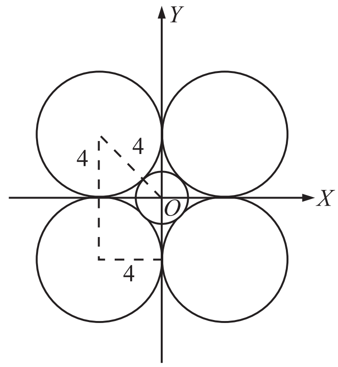
Also, $AC = \sqrt{(2-5)^2 + (-3-2)^2}$

$= \sqrt{9 + 25} = \sqrt{34} = 5.83.$

$\therefore \quad AB = AC - BC = 5.83 - 4 = 1.83 > 1.$

$\therefore \quad$ There is no point on the circle at a distance 1 from the point $(-3, 2)$.

6. Clearly, from the figure, the radius of the smallest circle touching the given circles is



$= \sqrt{4^2 + 4^2} - 4 \quad$ i.e., $\quad 4\sqrt{2} - 4.$

7. Equation of any circle passing through the point of intersection of $x^2 + y^2 - 2x = 0$ and $y = x$ is

$$x^2 + y^2 - 2x + \lambda(y - x) = 0$$

or, $\quad x^2 + y^2 - (2 + \lambda)\, x + \lambda y = 0.$

Its centre is $\left(\dfrac{2+\lambda}{2}, \dfrac{-\lambda}{2}\right).$

For AB to be the diameter of the required circle, the centre must lie on AB, i.e.,

$$\frac{2+\lambda}{2} = \frac{-\lambda}{2} \quad \Rightarrow \quad \lambda = -1.$$

Thus, equation of required circle is

$$x^2 + y^2 - x - y = 0.$$

8. Given equation of circle is

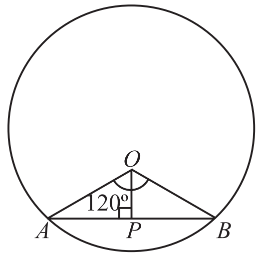
$$x^2 + y^2 - 2x - 2y - 2 = 0.$$

Let mid-point of chord AB be (h, k)

Its centre is $(1, 1)$ and radius $= \sqrt{1 + 1 + 2} = 2 = OB$

In ΔOPB, $\angle OBP = 30^\circ$.

$\therefore \quad \sin 30^\circ = OP/2$ or $OP = 1.$



Since, $OP = 1 \Rightarrow (h-1)^2 + (k-1)^2 = 1$

or, $h^2 + k^2 - 2h - 2k + 1 = 0$

$\therefore$ Locus of mid point of chord is

$$x^2 + y^2 - 2x - 2y + 1 = 0.$$

9. The centre of the given circle is $(1, -2)$. Since the sides of the square inscribed in the circle are parallel to the coordinate axes, so the x-coordinate of any vertex cannot be equal to 1 and its y-coordinate cannot be equal to -2. Hence, none of the points given in (a), (b) and (c) can be the vertex of the square.

10. The line $a_1x + b_1y + c_1 = 0$ cuts the coordinate axes at $A(-c_1/a_1, 0)$ and $B(0, -c_1/b_1)$ and the line $a_2x + b_2y + c_2 = 0$ cuts the axes at $C(-c_2/a_2, 0)$ and $D(0, -c_2/b_2)$.

So, AC and BD are chords along x-axis and y-axis, respectively, intersecting at origin O.

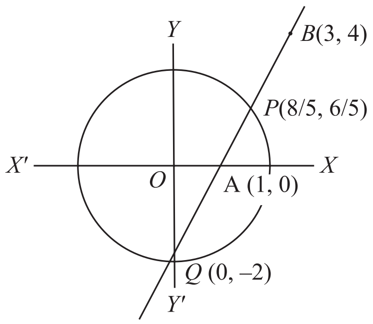
Since A, B, C, D are concyclic, therefore

$$OA \cdot OC = OB \cdot OD$$

$$\Rightarrow \quad \left(\frac{-c_1}{a_1}\right) \cdot \left(\frac{-c_2}{a_2}\right) = \left(\frac{-c_1}{b_1}\right) \cdot \left(\frac{-c_2}{b_2}\right)$$

or, $a_1a_2 = b_1b_2$.

11. The equation of the line joining $A(1, 0)$ and $B(3, 4)$ is $y = 2x - 2$. This cuts the circle $x^2 + y^2 = 4$ at $Q(0, -2)$ and $P\left(\frac{8}{5}, \frac{6}{5}\right)$.



We have, $BQ = 3\sqrt{5}, \; QA = \sqrt{5}, \; BP = \frac{7}{\sqrt{5}}$ and $PA = \frac{3}{\sqrt{5}}$

$$\therefore \quad \alpha = \frac{BP}{PA} = \frac{7/\sqrt{5}}{3/\sqrt{5}} = \frac{7}{3} \text{ and } \beta = \frac{BQ}{QA} = \frac{3\sqrt{5}}{-\sqrt{5}} = -3$$

$\therefore$ α, β are roots of the equation $x^2 - x(\alpha + \beta) + a\beta = 0$

i.e., $$x^2 - x\left(\frac{7}{3} - 3\right) + \frac{7}{3}(-3) = 0$$

or, $$3x^2 + 2x - 21 = 0.$$

12. Given equation of the incircle is

$$x^2 + y^2 + 4x - 6y + 4 = 0.$$

Its incentre is $(-2, 3)$ and inradius $= \sqrt{4 + 9 - 4} = 3$.

Since in an equilateral triangle, the incentre and the circumcentre coincide,

$\therefore$ Circumcentre $\equiv (-2, 3)$.

Also, in an equilateral triangle, circumradius = 2 (inradius)

$\therefore$ Circumradius $= 2 \cdot 3 = 6$.

$\therefore$ The equation of the circumcircle is

$$(x + 2)^2 + (y - 3)^2 = (6)^2$$

or, $$x^2 + y^2 + 4x - 6y - 23 = 0.$$

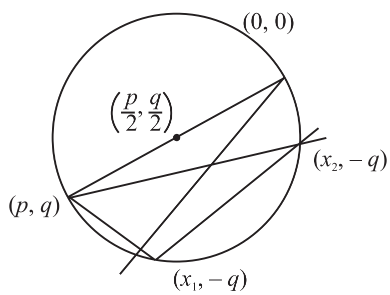
TRICK(S) FOR PROBLEM SOLVING

In an equilateral triangle

- the incentre and the circumcentre coincide
- circum radius = 2 (in radius)

13. Given circle is $x^2 + y^2 = px + qy$.

Since the centre of the circle is $\left(\frac{p}{2}, \frac{q}{2}\right)$, so (p, q) and $(0, 0)$ are the end points of a diameter. As the two chords are bisected by x-axis, the chords will cut the circle at the points $(x_1, -q)$ and $(x_2, -q)$, where x_1, x_2 are real.



The equation of the line joining these points is $y = -q$.

Solving $y = -q$ and $x^2 + y^2 = px + qy$, we get

$$x^2 - px + 2q^2 = 0.$$

The roots of this equation are x_1 and x_2. Since the roots are real and distinct, $\therefore$ discriminent > 0

i.e., $p^2 - 8q^2 > 0$ or $p^2 > 8q^2$.

14. Given circles are

$$S_1: x^2 + y^2 - 16 = 0 \quad (1)$$

and, $$S_2: x^2 + y^2 - 2y = 0 \quad (2)$$

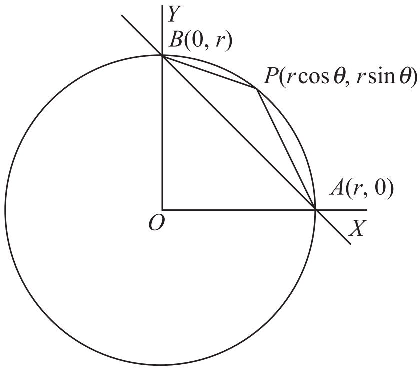
Centre of S_1 is C_1: (0, 0) and radius $r_1 = 4$

Centre of S_2 is C_2: (0, 1) and radius $r_2 = 1$

$$\therefore \quad C_1C_2 = \sqrt{0 + 1} = 1$$

Since $|C_1C_2| < |r_1 - r_2|$, $\therefore$ S_2 is completely within S_1 and hence there are no common tangents to the two circles.

15. Let the centroid ≡ (α, β). Then,



$$\alpha = \frac{r + r\cos\theta}{3}, \beta = \frac{r + r\sin\theta}{3}$$

or, $$\left(\alpha - \frac{r}{3}\right)^2 + \left(\beta - \frac{r}{3}\right)^2 = \frac{r^2}{9}$$

∴ The locus is $\left(x - \frac{r}{3}\right)^2 + \left(y - \frac{r}{3}\right)^2 = \left(\frac{r}{3}\right)^2$, which is a circle.

16. The equation of a circle passing through the intersection of the given line and the circle is

$$(x^2 + y^2 - 9) + k(x + y - 1) = 0$$

Its centre is $\left(-\frac{k}{2}, -\frac{k}{2}\right)$.

The circle is the smallest if the centre $\left(-\frac{k}{2}, -\frac{k}{2}\right)$ lies on the chord $x + y = 1$

∴ $-\frac{k}{2} - \frac{k}{2} = 1 \Rightarrow k = -1$

Thus, the equation of the smallest circle is

$$(x^2 + y^2 - 9) - 1(x + y - 1) = 0$$

i.e., $$x^2 + y^2 - x - y - 8 = 0$$

17. The given circles are

$$S_1: x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

and, $$S_2: x^2 + y^2 + 2g'x + 2f'y + c' = 0 \quad (2)$$

The equation of common chord of (1) and (2) is

$$S_1 - S_2 = 0$$

i.e., $$2(g - g')x + 2(f - f')y + (c - c') = 0 \quad (3)$$

Since (1) bisects the circumference of (2), therefore common chord will be the diameter of circle (2)

∴ Centre $(-g', -f')$ of circle (2) lies on (3).

∴ $-2(g - g')g' - 2(f - f')f' + c - c' = 0$

or, $2g'(g - g') + 2f'(f - f') = c - c'$.

TRICK(S) FOR PROBLEM SOLVING

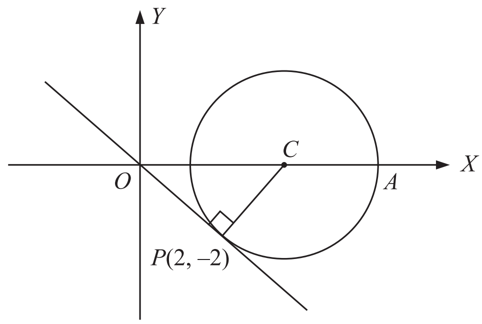
If a circle bisects the circumference of another circle, then their common chord is the diameter of second circle

18. Since the slope of the given line is –1,

∴ $\angle COP = 45°$

∴ $OP = 2\sqrt{2} = CP$

∴ $OC = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$



The point on the circle with the greatest x-coordinate is A.

∴ $a = OA = OC + CA = 4 + 2\sqrt{2}$.

19. We have, $x + y = 6$

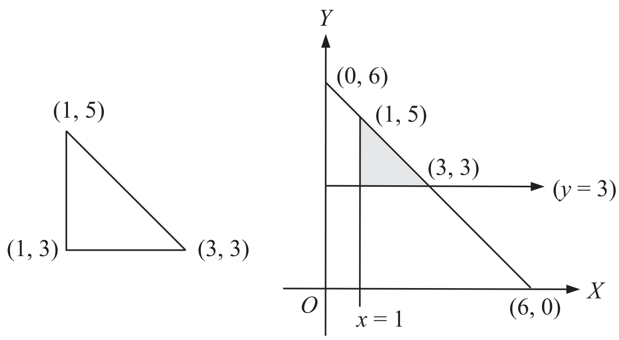
and, $xy - y - 3x + 3 = 0$

⇒ $y(x - 1) - 3(x - 1) = 0$

⇒ $(x - 1)(y - 3) = 0$

Equations of the sides of the triangle are

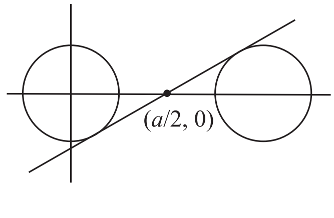
$$x + y = 6 \quad (1)$$
$$y = 3 \quad (2)$$
$$x = 1 \quad (3)$$



Shaded triangle is right angled at (1, 3). ∴ the circumcircle is the circle on (3, 3) and (1, 5) as ends of a diameter and its equation is $(x - 3)(x - 1) + (y - 3)(y - 5) = 0$, i.e., $x^2 + y^2 - 4x - 8y + 18 = 0$.

20. Clearly, the line $y = mx - b\sqrt{1 + m^2}$ will pass from the point $(a/2, 0)$ (mid-point of the centres of the circles)

$$\Rightarrow m = \frac{2b}{\sqrt{a^2 - 4b^2}}$$



21. Let $P(h, k)$ be any point on the locus. Equation of the chord of contact of P with respect to the circle $x^2 + y^2 = b^2$ is $hx + ky = b^2$. If it touches the circle $x^2 + y^2 = a^2$, then

$$\left|\frac{-b^2}{\sqrt{h^2+k^2}}\right| = a \Rightarrow a^2(h^2+k^2) = b^4$$

So that the locus of $P(h, k)$ is $x^2 + y^2 = (b^2/a)^2$

$$\therefore \quad c^2 = \left(\frac{b^2}{a}\right)^2 \quad \Rightarrow \quad ac = b^2$$

$\Rightarrow$ a, b, c are in G. P.

22. Let the variable circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

$$\therefore \quad p^2 + q^2 + 2gp + 2fq + c = 0 \quad (2)$$

Circle (1) touches x-axis,

$\therefore$ $g^2 - c = 0 \Rightarrow c = g^2$. From (2)

$$p^2 + q^2 + 2gp + 2fq + g^2 = 0 \quad (3)$$

Let the other end of diameter through (p, q) be (h, k), then

$$\frac{h+p}{2} = -g \text{ and } \frac{k+q}{2} = -f$$

Put in (3)

$$p^2 + q^2 + 2p\left(-\frac{h+p}{2}\right) + 2q\left(-\frac{k+q}{2}\right) + \left(\frac{h+p}{2}\right)^2 = 0$$

$\Rightarrow$ $h^2 + p^2 - 2hp - 4kq = 0$

$\therefore$ Locus of (h, k) is

$x^2 + p^2 - 2xp - 4yq = 0 \quad \Rightarrow \quad (x - p)^2 = 4qy$

23. Since the circle does not touch or intersect the coordinates axes, the absolute values of x and y coordinates of the centre are greater than the radius of the circle. Coordinates of the centre of the circle are (3, 5) and the radius is $\sqrt{9+25-p}$

so that $3 > \sqrt{9+25-p} \quad \Rightarrow \quad p > 25 \quad (1)$

$$5 > \sqrt{9+25-p} \quad \Rightarrow \quad p > 9 \quad (2)$$

and the point (1, 4) lies inside the circle

$\Rightarrow$ $1 + 16 - 6 - 10 \times 4 + p < 0 \quad \Rightarrow \quad p < 29 \quad (3)$

From (1), (2), (3) we get

$$25 < p < 29.$$

24. First circle touches both axes and radius is 2 unit.

Hence, centre of circle is (2, 2).

Let radius of other circle be a and this circle also touches both the axes.

Hence, centre of circle is (a, a).

This circle touches first circle

Hence, $\sqrt{(a-2)^2 + (a-2)^2} = a + 2$

On squaring both the sides, we get

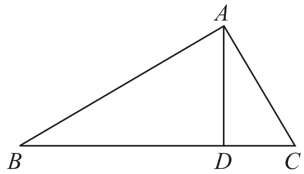
$a^2 - 12a + 4 = 0$

$$\Rightarrow \quad a = \frac{12 \pm \sqrt{(12)^2 - 4\times 4\times 1}}{2} = \frac{12 \pm \sqrt{128}}{2} = 6 \pm 4\sqrt{2}$$

But $a > 2$, therefore $a = 6 - 4\sqrt{2}$ is neglected.

Hence, $a = 6 + 4\sqrt{2}$

25. Since $\lfloor ADB = \lfloor ADC = 90°$, circles on AB and AC as diameters pass through D and therefore the altitude AD is the common chord. Similarly, the other two common chords are the other two altitudes and hence they concur at the ortho-centre.



26. The equation of the given circle can be written as

$$(x - 1)^2 + (y - 2)^2 = 16 = 4^2$$

So, the coordinates of any point P on the circle are $(1 + 4\cos\theta, 2 + 4\sin\theta)$ whose distance from the origin is

$$d = \sqrt{(1+4\cos\theta)^2 + (2+4\sin\theta)^2}$$
$$= \sqrt{21 + 8\cos\theta + 16\sin\theta}$$

$$\Rightarrow \quad d = \sqrt{21 + 8(\cos\theta + 2\sin\theta)}$$
$$= \sqrt{21 + 8r\cos(\theta - \alpha)}$$

where $r\cos\alpha = 1$, $r\sin\alpha = 2$

$$= \sqrt{21 + 8\sqrt{5}\cos(\theta - \alpha)}$$

which is maximum when $\cos(\theta - \alpha) = 1$,

i.e., $\theta = \alpha = \tan^{-1} 2$.

$$\Rightarrow \quad \tan\theta = 2 \text{ so } \sin\theta = \frac{2}{\sqrt{5}} = \text{ and } \cos\theta = \frac{1}{\sqrt{5}}$$

and, the coordinates of the required point are

$$\left(1 + \frac{4}{\sqrt{5}},\ 2 + \frac{8}{\sqrt{5}}\right).$$

27. The length of the $\perp$ from the centre (0, 0) of the given circle to the line $3x + ay - 20 = 0$ is

$$\frac{|3(0) + a(0) - 20|}{\sqrt{9+a^2}} = \frac{20}{\sqrt{9+a^2}}.$$

Radius of the given circle = 5

Since the line cuts the circle at real, distinct or coincident points

$$\therefore \quad \frac{20}{\sqrt{9+a^2}} \le 5 \quad \Rightarrow \quad a^2 + 9 \ge 16 \quad \Rightarrow \quad a^2 - 7 \ge 0$$

$$\Rightarrow \quad (a + \sqrt{7})(a - \sqrt{7}) \ge 0$$

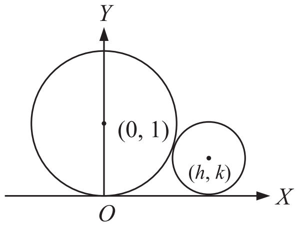
$$\Rightarrow \quad a \in (-\infty, -\sqrt{7}] \cup [\sqrt{7}, \infty)$$

28. According to given condition

$$\sqrt{(h-0)^2 + (k-1)^2} = 1 + |k|$$

$\Rightarrow$ $h^2 + (k - 1)^2 = (1 + |k|)^2$

$\Rightarrow$ $h^2 = 2k + 2|k|$



Hence locus is $x^2 = 2y + 2|y|$

Clearly, for $y > 0$, $x^2 = 4y$

and for $y < 0$, $x^2 = 0 \Rightarrow x = 0$.

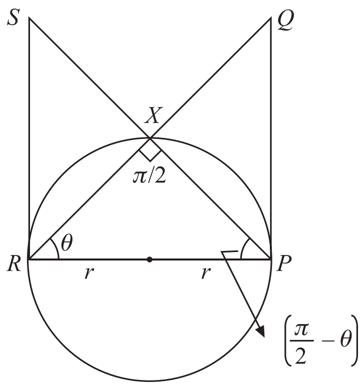
29. $\tan\theta = \dfrac{PQ}{PR} = \dfrac{PQ}{2r}$

Also, $\tan\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{RS}{2r}$

i.e., $\cot\theta = \dfrac{RS}{2r}$

$\therefore \quad \tan\theta \cot\theta = \dfrac{PQ \cdot RS}{4r^2}$

$\Rightarrow \quad 4r^2 = PQ \cdot RS$



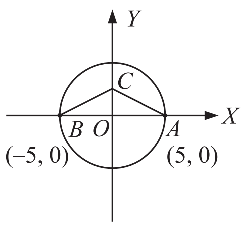
$\Rightarrow \quad 2r = \sqrt{(PQ)(RS)}$.

30. Since the circles pass through (–5, 0) and makes an intercept of 10 units on the positive side of the x-axis, it also passes through (5, 0).

$\therefore$ if $C(x, y)$ is the centre of such a circle, $(x + 5)^2 + y^2 = (x - 5)^2 + y^2$

($\because$ $CA = CB =$ radius)

$\Rightarrow \quad x = 0$.



31. Given circle is

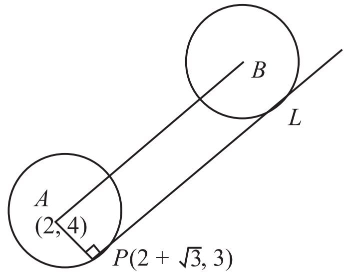
$$x^2 + y^2 - 4x - 8y + 16 = 0 \qquad (1)$$

Let $P = (2 + \sqrt{3}, 3)$.

Equation of tangent to the circle (1) at $P(2 + \sqrt{3}, 3)$ is $(2 + \sqrt{3})x + 3y - 2(x + 2 + \sqrt{3}) - 4(y + 3) + 16 = 0$

or, $\quad \sqrt{3}x - y - 2\sqrt{3} = 0 \qquad (2)$

If line (2) makes an angle θ with the positive direction of x-axis, then $\tan\theta = \sqrt{3}$, $\quad \therefore \quad \theta = 60°$.



Let A and B be the centres of the circles in old and new positions, respectively, then

$A \equiv (2, 4)$ and $B \equiv (2 + 2\cos 60°, 4 + 2\sin 60°)$ ($\because$ $AB = 2$)

Thus, $B \equiv (3, 4 + \sqrt{3})$.

Radius of the circle $= \sqrt{2^2 + 4^2 - 16} = 2$.

$\therefore$ Equation of the circle in the new position is

$(x - 3)^2 + (y - 4 - \sqrt{3})^2 = 2^2$

or, $\quad x^2 + y^2 - 6x - 2(4 + \sqrt{3})y + 24 + 8\sqrt{3} = 0$

32. Let $C(\alpha, \beta)$ be the centre of the circle.

Since the circle passes through the point (2, 8),

$\therefore$ radius of the circle $= \sqrt{(\alpha - 2)^2 + (\beta - 8)^2}$.

Since the circle touches the lines $4x - 3y - 24 = 0$

and, $4x - 3y - 42 = 0$,

$$\therefore \quad \underset{(i)}{\frac{|4\alpha - 3\beta - 24|}{5}} = \underset{(ii)}{\frac{|4\alpha + 3\beta - 42|}{5}} = \underset{(iii)}{\sqrt{(\alpha - 2)^2 + (\beta - 8)^2}}$$

From (i) and (ii), we get

$4\alpha - 3\beta - 24 = \pm(4\alpha + 3\beta - 42)$

$$\therefore \begin{cases} 6\beta = 18 \Rightarrow \beta = 3 & \text{(taking positive sign)} \\ 8\alpha = 66 \Rightarrow \alpha = \dfrac{33}{4} & \text{(taking negative sign).} \end{cases}$$

Given, $|\alpha| \le 8$ or $-8 \le \alpha \le 8$, $\quad \therefore \quad a \ne \dfrac{33}{4}$.

Putting $\beta = 3$ in equations (i) and (iii) and equating, we get

$(4\alpha - 33)^2 = 25\,[(\alpha - 2)^2 + 25]$

or, $\quad 16\alpha^2 - 264\alpha + 1089 = 25\alpha^2 + 725 - 100\alpha$

or, $\quad 9\alpha^2 + 164\alpha - 364 = 0$

$$\therefore \quad \alpha = \frac{-164 \pm \sqrt{(164)^2 + 36 \times 364}}{18}$$

$$= \frac{-164 \pm 200}{18} = 2, \frac{-182}{9}.$$

But $-8 \le \alpha \le 8$, $\quad \therefore \quad \alpha = 2$.

Now, $(\text{radius})^2 = (\alpha - 2)^2 + (3 - 8)^2$

$$= (2 - 2)^2 + (3 - 8)^2 = 25.$$

Hence, the equation of the required circle is

$$(x - 2)^2 + (y - 3)^2 = 25$$

or, $x^2 + y^2 - 4x - 6y - 12 = 0$.

33. The equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$ (1)

where, $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ (2)

Here, a, b are parameters while c is a constant.

Any line $\perp r$ to (1) is $\frac{x}{b} - \frac{y}{a} + k = 0$.

If it passes through the origin then $k = 0$.

$\therefore$ Equation of the line through the origin and $\perp_r$ to (1) is

$$\frac{x}{b} - \frac{y}{a} = 0 \quad (3)$$

The locus of the foot of the $\perp_r$ from origin on (1), i.e., locus of the point of intersection of (1) and (3) is obtained by eliminating the parameters a and b between them.

Squaring (1) and (3) and adding, we get

$$\left(\frac{1}{a^2} + \frac{1}{b^2}\right)x^2 + \left(\frac{1}{b^2} + \frac{1}{a^2}\right)y^2 = 1$$

or, $$\frac{1}{c^2}x^2 + \frac{1}{c^2}y^2 = 1 \quad \text{(Using (2)}$$

or $x^2 + y^2 = c^2$, which is clearly a circle with centre at origin and radius c.

34. Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

Its centre is $C(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

Since circle (1) touches the x-axis

$\therefore$ $g^2 - c = 0$ or $c = g^2$ (2)

Again, since circle (1) touches the line

$4x - 3y + 4 = 0$ (3)

$\therefore$ $\frac{|-4g + 3f + 4|}{5} = \sqrt{g^2 + f^2 - c} = \sqrt{f^2} = |f|$

[from (2)]

or, $-4g + 3f + 4 = \pm 5f$

$\therefore$ $4g + 2f = 4$ or $2g + f = 2$ (4)

$-4g + 8f = -4$ or $g - 2f = 1$ (5)

Again, since centre $C(-g, -f)$ lies on the line

$x - y - 1 = 0$

$\therefore$ $-g + f = 1$ (6)

Solving (4) and (6), we get $g = \frac{1}{3}, f = \frac{4}{3}$.

Thus, $C = \left(\frac{-1}{3}, \frac{-4}{3}\right)$ which lies in the third quadrant.

Also, from (2), $c = g^2 = \frac{1}{9}$.

Solving (5) and (6), we get $f = -2$, $g = -3$

$\therefore$ $C \equiv (3, 2)$ which lies in the first quadrant.

Thus, for the required circle $g = \frac{1}{3}, f = \frac{4}{3}, c = \frac{1}{9}$.

$\therefore$ Equation of the required circle is

$$x^2 + y^2 + \frac{2}{3}x + \frac{8}{3}y + \frac{1}{9} = 0$$

or, $9(x^2 + y^2) + 6x + 24y + 1 = 0$.

35. Equation of any circle through the points of intersection P and Q of line

$$Ax + By + C = 0 \quad (1)$$

and circle $x^2 + y^2 + ax + by + c = 0$ (2)

is $x^2 + y^2 + ax + by + c + \lambda(Ax + By + C) = 0$

or, $x^2 + y^2 + (a + \lambda A)x + (b + \lambda B)y + c + \lambda C = 0$ (3)

Again, equation of any circle through the points of intersection R and S of line

$$A'x + B'y + C' = 0 \quad (4)$$

and circle $x^2 + y^2 + a'x + b'y + c' = 0$ (5)

is $x^2 + y^2 + a'x + b'y + c' + \mu(A'x + B'y + C') = 0$

or, $x^2 + y^2 + (a' + \mu A')x + (b' + \mu B')y + c' + \mu C' = 0$ (6)

If circles (3) and (6) are same, then points P, Q, R, S will lie on the same circle, i.e., points P, Q, R, S will be concyclic.

Comparing the coefficients in (3) and (6), we get

$$\frac{1}{1} = \frac{1}{1} = \frac{a + \lambda A}{a' + \mu A'} = \frac{b + \lambda B}{b' + \mu B'} = \frac{c + \lambda C}{b' + \mu C'}$$

(*i*) (*ii*) (*iii*) (*iv*) (*v*)

From (*i*) and (*iii*), we get $a - a' + \lambda A - \mu A' = 0$ (7)

From (*i*) and (*iv*), we get $b - b' + \lambda B - \mu B' = 0$ (8)

From (*i*) and (*v*), we get $c - c' + \lambda C - \mu C' = 0$ (9)

Eliminating λ and $-\mu$ from equations (7), (8) and (9) and writing the result in determinant form, we get

$$\begin{vmatrix} a - a' & A & A' \\ b - b' & B & B \\ c - c' & C & C' \end{vmatrix} = 0$$

or, $$\begin{vmatrix} a - a' & b - b' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$$

36. Equation of the circle is

$$x^2 + y^2 = a^2 \quad (1)$$

Let P be the point (x_1, y_1).

Equation of any tangent to (1) is $y = mx + a\sqrt{1 + m^2}$

If it passes through $P(x_1, y_1)$, then

$$y_1 = mx_1 + a\sqrt{1 + m^2}$$

or, $y_1 - mx_1 = a\sqrt{1 + m^2}$.

Squaring $+ 2mx_1y_1 + m^2 = a^2(1 + m^2)$

or, $(x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 - a^2) = 0$ (2)

This is a quadratic in m. If m_1 and m_2 are its roots, then these are the slopes of the tangents from P.

Since inclinations of tangents are given to be θ_1 and θ_2,

$\therefore$ let $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.

Here, $\cot\theta_1 + \cot\theta_2 = k$ i.e., $\dfrac{1}{\tan\theta_1} + \dfrac{1}{\tan\theta_2} = k$

or, $\dfrac{1}{m_1} + \dfrac{1}{m_2} = k$ or $m_1 + m_2 = km_1m_2$.

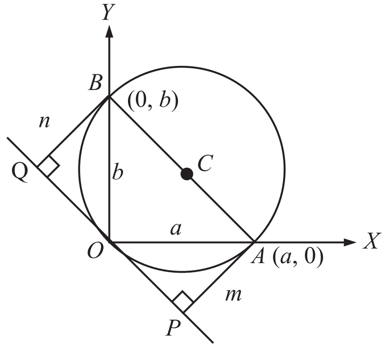
$\therefore$ $\dfrac{2x_1y_1}{x_1^2 - a^2} = k \cdot \dfrac{y_1^2 - a^2}{x_1^2 - a^2}$ or $2x_1y_1 = k(y_1^2 - a^2)$

$\therefore$ Locus of P is $k(y^2 - a^2) = 2xy$.

37. Let $A \equiv (a, 0)$ and $B \equiv (0, b)$.

Since $\angle AOB = 90°$, $\therefore$ AB is the diameter.

$\therefore$ Centre of the circle is $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ and radius $= \dfrac{1}{2}\sqrt{a^2 + b^2}$.



$\therefore$ Equation of the circle is

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{1}{4}(a^2 + b^2)$$

or, $x^2 + y^2 - ax - by = 0$.

Equation of tangent to the circle at $O(0, 0)$ is

$$ax + by = 0 \qquad (1)$$

$\therefore$ m = length of $\perp$ from $A(a, 0)$ on (1) $= \dfrac{a^2}{\sqrt{a^2 + b^2}}$

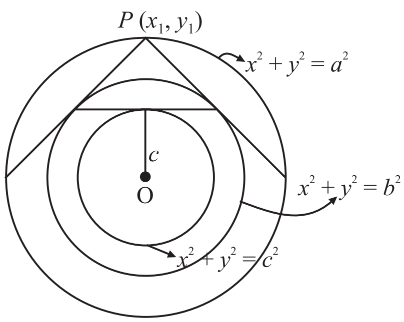
and, n = length of $\perp$ from $(0, b)$ on (1) $= \dfrac{b^2}{\sqrt{a^2 + b^2}}$.

$\therefore$ Diameter $= \sqrt{a^2 + b^2} = m + n$.

38. Let (x_1, y_1) be any point on the circle $x^2 + y^2 = a^2$.

then, $x_1^2 + y_1^2 = a^2$ (1)

Equation of chord of contact of tangents from (x_1, y_1) to the circle $x^2 + y^2 = b^2$ is $xx_1 + yy_1 = b^2$ (2)



$\because$ (2) touches the circle $x^2 + y^2 = c^2$,

$\therefore$ the length of $\perp$ from centre (0, 0) on (2) is = radius c.

$$\Rightarrow \quad \frac{b^2}{\sqrt{x_1^2 + y_1^2}} = c$$

or, $\dfrac{b^2}{\sqrt{a^2}} = c$ [Using (1)]

or, $b^2 = ac$. Hence, a, b, c are in G.P.

39. The point $\left(3 + \dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$ does not satisfy circles given in (a) and (c).

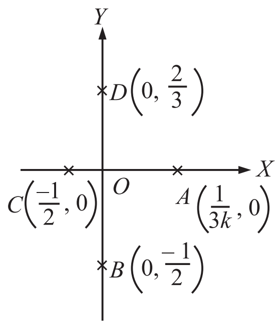
$\therefore$ (a) and (c) cannot be the correct choices. The centre of circle given in (b) is $\left(\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}\right)$ which does not lie on the line $y - x + 3 = 0$.

$\therefore$ The circle given in (b) cannot be the correct choice. The centre (3, 0) of circle given in (d) lie on the line

$$y - x + 3 = 0.$$

Thus, the line is normal at the given point on the circle given in (d).

40. The line $3kx - 2y - 1 = 0$ meets x-axis and y-axis at $A\left(\dfrac{1}{3k}, 0\right)$ and $B\left(0, -\dfrac{1}{2}\right)$ respectively and the line $4x - 3y + 2 = 0$ cuts x-axis and y-axis at $C\left(-\dfrac{1}{2}, 0\right)$ and $D\left(0, \dfrac{2}{3}\right)$ respectively.



Since the four points are concyclic, therefore

$$OA \cdot OC = OB \cdot OD$$

$$\Rightarrow \quad \frac{1}{3|k|} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{2}{3} \quad \Rightarrow \quad |k| = \frac{1}{2}.$$

According to the given geometrical position (see figure), k must be positive,

$\therefore$ $k = \dfrac{1}{2}$.

41. Let the equation of the circle be

$x^2 + (y - \sqrt{2})^2 = a^2$

$\Rightarrow$ $x^2 + y^2 - 2\sqrt{2}y = c$,

where, $c = a^2 - 2 \to$ Rational number.

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be three distinct rational points on the circle, then

$$x_1^2 + y_1^2 - 2\sqrt{2}y_1 = c \qquad (1)$$

$$x_2^2 + y_2^2 - 2\sqrt{2}y_2 = c \qquad (2)$$

$$x_3^2 + y_3^2 - 2\sqrt{2}y_3 = c \qquad (3)$$

Comparing the irrational parts of the equations, we get

$$y_1 = y_2 = y_3 \qquad (4)$$

Comparing the rational parts of the equations, we get

$$x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2$$

$\because \quad y_1 = y_2 = y_3,$

$\therefore \quad x_1^2 = x_2^2 = x_3^2.$

$\therefore$ The only possible values of x are $\pm x_1, \pm x_2, \pm x_3$.

$\therefore$ There can be at the most two rational points on the circle C.

42. Given circle is $x^2 + y^2 - a^2 = 0$ (1)

Since PQ and PR are tangents to the circle (1), therefore QR is chord of contact of point $P(x_1, y_1)$ and hence equation of QR is

$$xx_1 + yy_1 - a^2 = 0 \qquad (2)$$

Now, equation of any circle through the point of intersection Q and R of circle (1) and line (2) is

$$x^2 + y^2 - a^2 + k(xx_1 + yy_1 - a^2) = 0 \qquad (3)$$

Circle (3) will be circumcircle of ΔPQR if it passes through the point $P(x_1, y_1)$.

i.e., if $x_1^2 + y_1^2 - a^2 + k(x_1^2 + y_1^2 - a^2) = 0 \quad \Rightarrow \quad k = -1.$

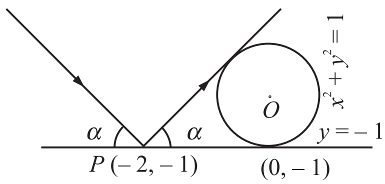
Hence, from (3), equation of required circle is

$$x^2 + y^2 - a^2 - (xx_1 + yy_1 - a^2) = 0$$

or, $\quad x^2 + y^2 - xx_1 - yy_1 = 0.$

43. The equation of the reflected ray is $(y + 1) = m(x + 2)$

or, $mx - y + 2m - 1 = 0$ (1)



Since it touches the circles $x^2 + y^2 = 1$.

$\therefore$ length of $\perp$ from (0, 0) on (1) = radius 1

$$\Rightarrow \quad \frac{|m(0) - 0 + 2m - 1|}{\sqrt{1 + m^2}} = 1 \quad \Rightarrow \quad \frac{2m - 1}{\sqrt{1 + m^2}} = \pm 1$$

$$\Rightarrow \quad (2m - 1)^2 = (1 + m^2) \Rightarrow 3m^2 - 4m = 0 \Rightarrow m = 0, \frac{4}{3}.$$

$\therefore$ Equation of the reflected ray is

$$(y + 1) = \frac{4}{3}(x + 2) \quad \text{or} \quad 4x - 3y + 5 = 0.$$

Let α be the angle between the reflected ray and the line $y = -1$.

$$\text{Then, } \tan\alpha = \frac{\left|\frac{4}{3} - 0\right|}{1 + \frac{4}{3}\cdot 0} = \pm\frac{4}{3}.$$

$\therefore$ Slope of the incident ray $= -\frac{4}{3}$.

Hence, equation of the incident ray is

$$(y + 1) = \frac{-4}{3}(x + 2) \text{ i.e., } 3(y + 1) = -4(x + 2)$$

or, $\quad 4x + 3y + 11 = 0.$

44. Given circle is $x^2 + y^2 - 4x - 6y + 9 = 0$ (1)

Its centre is $C(2, 3)$ and radius is 2.

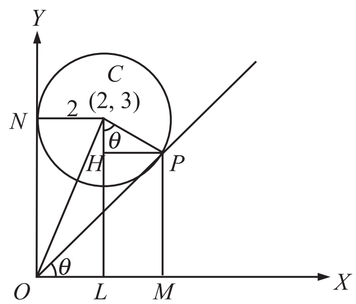
Let OP and ON be the two tangents from 0 to circle (1), then $\angle POX$ will be minimum when OP is tangent to the circle at P. Let $\angle POX = \theta$, then $\angle LCP = \theta$.

Now, $CP = 2$, $OC = \sqrt{2^2 + 3^2} = \sqrt{13}$.

$\therefore \quad OP = \sqrt{OC^2 - CP^2} = \sqrt{13 - 4} = 3.$

$\because \quad C \equiv (2, 3), \quad \therefore \quad OL = 2.$

From the figure, $OM = OL + LM = OL + HP$



$\therefore \quad OP\cos\theta = 2 + 2\sin\theta$ or $3\cos\theta = 2 + 2\sin\theta$

or, $\quad 3 = 2\sec\theta + 2\tan\theta$ or $3 - 2\tan\theta = 2\sec\theta$

or, $\quad 9 + 4\tan^2\theta - 12\tan\theta = 4(1 + \tan^2\theta)$

or, $\quad 5 = 12\tan\theta \quad \therefore \quad \tan\theta = \frac{5}{12}$

$\therefore \quad \cos\theta = \frac{12}{13}$ and $\sin\theta = \frac{5}{13}$.

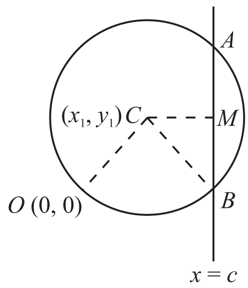
$\therefore \quad P \equiv (OP\cos\theta, OP\sin\theta)$ i.e., $P \equiv \left(\frac{36}{13}, \frac{15}{13}\right)$.

45. Let the centre of the circle be $C(x_1, y_1)$.

As it passes through (0, 0), its radius $= OC = \sqrt{x_1^2 + y_1^2}$.

Let AB be the line $x = c$ meeting the circle in A and B. Draw $CM \perp AB$.

Join CB.



$$CB = \text{radius} = \sqrt{x_1^2 + y_1^2}.$$

CM = length of ⊥ from C on $AB = x_1 - c$.

Now, $AB = 2b$ (given).

∴ $2BM = 2b$ or $\sqrt{CB^2 - CM^2} = b$

or, $CB^2 - CM^2 = b^2$ or $x_1^2 + y_1^2 - (x_1 - c)^2 = b^2$

or, $y_1^2 + 2cx_1 - c^2 = b^2$.

∴ Locus of (x_1, y_1) is $y^2 + 2cx = b^2 + c^2$.

46. Let the coordinates of P and Q be (x_1, y_1) and (x_2, y_2), respectively.

Let the given circle be

$$S \equiv x^2 + y^2 - a^2 = 0 \quad (1)$$

Polar of P w.r.t. (1) is $xx_1 + yy_1 = a^2$.

∵ P, Q are conjugate points,

∴ polar of P passes through Q

$\Rightarrow x_1x_2 + y_1y_2 = a^2$ (2)

Equation of circle on PQ as diameter is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

or, $x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + (x_1x_2 + y_1y_2) = 0$

$x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + a^2 = 0$ (3)

[Using (2)]

Clearly, circles (1) and (3) cut each other orthogonally.

47. Given circle is $x^2 + y^2 + 8x + 4y - 5 = 0$ (1)

Let the equation of the second circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Since it passes through origin, ∴ $c = 0$.

So, the equation becomes

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (2)$$

The equation of common chord of (1) and (2) is

$$2(g - 4)x + 2(f - 2)y + 5 = 0 \quad (3)$$

Since the line $y = x$ touches the circle (2)

∴ $x^2 + x^2 + 2gx + 2fx = 0$ has equal roots

i.e., $f + g = 0$.

∴ From (3), the equation of common chord is

$2(g - 4)x + 2(-g - 2)y + 5 = 0$

or, $(-8x - 4y + 5) + g(2x - 2y) = 0$,

which passes through the point of intersection of

$8x + 4y - 5 = 0$ and $x = y$, i.e., the point $\left(\frac{5}{12}, \frac{5}{12}\right)$.

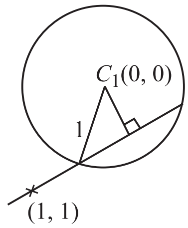
48. Let the equation of line be $y = mx + c$. Since it passes through (1, 1),

∴ $1 = m + c$, i.e., $c = 1 - m$.

So, the line becomes $y = mx + 1 - m$

i.e., $mx - y - m + 1 = 0$ (1)

Length of ⊥ from the centre (0, 0) of first circle to the line (1)

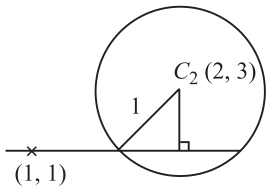


$$= \frac{|0 - 0 - m + 1|}{\sqrt{m^2 + 1}} = \frac{|1 - m|}{\sqrt{m^2 + 1}}.$$

So, the length of the intercept

$$= 2\sqrt{1^2 - \frac{(1 - m)^2}{m^2 + 1}} = 2\sqrt{\frac{2m}{m^2 + 1}}.$$

Also, the length of ⊥ from the centre (2, 3) of second circle to the line (1) $= \frac{|2m - 3 - m + 1|}{\sqrt{m^2 + 1}} = \frac{|m - 2|}{\sqrt{m^2 + 1}}$



∴ the length of intercept in this case

$$= 2\sqrt{1 - \frac{(m - 2)^2}{m^2 + 1}} = 2\sqrt{\frac{4m - 3}{m^2 + 1}}$$

Given, $2\sqrt{\frac{2m}{m^2 + 1}} = 2\sqrt{\frac{4m - 3}{m^2 + 1}}$

$$\Rightarrow \frac{2m}{m^2 + 1} = \frac{4m - 3}{m^2 + 1} \Rightarrow 2m = 3 \text{ or } m = \frac{3}{2}.$$

49. Let the coordinates of point P be (x_1, y_1). Equation of any line through P can be written as

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \quad (1)$$

$\Rightarrow x = x_1 + r\cos\theta,\ y = y_1 + r\sin\theta$.

Coordinates of any point an (1) is of the form $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. This point will lie on $ax^2 + 2hxy + by^2 = 1$ if

$a(x_1 + r\cos\theta)^2 + 2h(x_1 + r\cos\theta)(y_1 + r\sin\theta)$
$+ b(y_1 + r\sin\theta)^2 - 1 = 0$

$\Rightarrow r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)$
$+ 2r[x_1(a\cos\theta + h\sin\theta) + y_1(h\cos\theta + b\sin\theta)]$
$+ax_1^2 + 2hx_1y_1 + by_1^2 - 1 = 0$ (2)

Let $PQ = r_1$ and $PR = r_2$. Then r_1, r_2 are the roots of (2).

$$\therefore\ PQ \cdot PR = r_1r_2 = \frac{ax_1^2 + 2hx_1y_1 + by_1^2 - 1}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}.$$

We know rewrite the denominator. We have
$D = a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta.$

$$= \frac{1}{2}[(a+b)+(a-b)\cos 2\theta] + h\sin 2\theta$$

$$= \frac{a+b}{2} + \frac{1}{2}(a-b)\cos 2\theta + h\sin 2\theta$$

Put $\frac{1}{2}(a-b) = k\ \sin\alpha, h = k\cos\alpha.$

$$\Rightarrow \quad k = \sqrt{\left(\frac{a-b}{2}\right)^2 + h^2} \text{ and } \tan\ a = \frac{a-b}{2h}.$$

$$\therefore \quad D = \frac{1}{2}(a+b) + \sqrt{\left(\frac{a-b}{2}\right)^2 + h^2}\ \sin(2\theta+\alpha)$$

$$\text{Thus, } PQ\cdot PR = \frac{ax_1^2 + 2hx_1y_1 + by_1^2 - 1}{\frac{1}{2}(a+b) + \sqrt{\left(\frac{a-b}{2}\right)^2 + h^2}\ \sin(2\theta+\alpha)}$$

For this to be independent of θ, we must have

$$\left(\frac{a-b}{2}\right)^2 + h^2 = 0 \quad \Rightarrow \quad a = b \text{ and } h = 0.$$

But this is the condition for the given curve to represent a circle.

50. Let the equation of line L_1 be $y = mx$. Intercpts made by L_1 and L_2 on the circle will be equal if L_1 and L_2 are at the same distance from the centre of the circle. Centre of the given circle is $\left(\frac{1}{2}, \frac{-3}{2}\right)$. Therefore,

$$\frac{\left|\frac{1}{2} - \frac{3}{2} - 1\right|}{\sqrt{1+1}} = \frac{\left|\frac{m}{2} + \frac{3}{2}\right|}{\sqrt{1+m^2}} \quad \Rightarrow \quad \frac{4}{\sqrt{2}} = \frac{m+3}{\sqrt{1+m^2}}$$

$$\Rightarrow 7m^2 - 6m - 1 = 0$$

$$\Rightarrow (7m+1)(m-1) = 0$$

$$\Rightarrow m = 1, \ \frac{-1}{7}.$$

Thus, two chords are $y = x$ and $7y + x = 0$.

51. Let the third side be $\frac{x}{\alpha} + \frac{y}{\beta} = 1.$

For the circle, centre $\equiv (a, a)$ and radius $= a$.
Since the third side touches the circle,

$$\therefore \quad a = \frac{\frac{a}{\alpha} + \frac{a}{\beta} - 1}{\sqrt{\frac{1}{\alpha^2} + \frac{1}{\beta^2}}} \qquad (1)$$

Vertices of the triangle are $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$,
$\therefore$ if the circumcentre is (γ, δ) then

$$\gamma = \frac{\alpha}{2} \text{ and } \delta = \frac{\beta}{2}.$$

$$\therefore \text{ From (1), } \quad a^2\left(\frac{1}{4r^2} + \frac{1}{4\delta^2}\right) = \left(\frac{a}{2\gamma} + \frac{a}{2\delta} - 1\right)^2$$

$$\Rightarrow \quad 2a(\gamma + \delta) - a^2 = 2\gamma\delta$$

So, the locus of (γ, δ) is $2a(x + y) = 2xy + a^2$.

52. Let the sides of the square be $y = 0$,

$$y = 1, x = 0 \text{ and } x = 1.$$

Let the moving point be (x, y).
Then, $y^2 + (y - 1)^2 + x^2 + (x - 1)^2 = 9$ is the equation of the locus.

$$\Rightarrow \quad 2x^2 + 2y^2 - 2x - 2y - 7 = 0,$$

which represents a circle having centre $\left(\frac{1}{2}, \frac{1}{2}\right)$ (which is also the centre of the square) and radius 2.

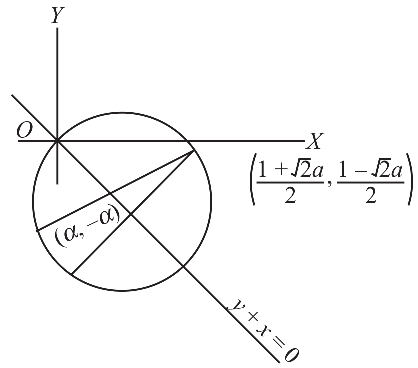
53. Equation of the given circle can be written as

$$x^2 + y^2 - \left(\frac{1+\sqrt{2}a}{2}\right)x - \left(\frac{1-\sqrt{2}a}{2}\right)y = 0.$$

Since $y + x = 0$ bisects two chords of this circle, mid-points of the chords must be of the form $(\alpha, -\alpha)$.
Equation of the chord having $(\alpha, -\alpha)$ as mid-point is

$$\text{T} = S_1 \text{ i.e., } ax + (-\alpha)y - \left(\frac{1+\sqrt{2}a}{4}\right)(x+\alpha) - \left(\frac{1-\sqrt{2}a}{4}\right)(y+\alpha)$$

$$= \alpha^2 + a^2 - \left(\frac{1+\sqrt{2}a}{2}\right)\alpha - \left(\frac{1-\sqrt{2}a}{2}\right)(-\alpha)$$



$$\Rightarrow 4ax - 4ay - (1+\sqrt{2}a)x - (1-\sqrt{2}a)y$$
$$= 8a^2 - (1+\sqrt{2}a)a + (1-\sqrt{2}a)a.$$

This chord will pass through the point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ if

$$4\alpha\left(\frac{1+\sqrt{2}a}{2}\right) - 4\alpha\left(\frac{1-\sqrt{2}a}{2}\right) - \frac{(1+\sqrt{2}a)(1+\sqrt{2}a)}{2} - \frac{(1-\sqrt{2}a)(1-\sqrt{2}a)}{2}$$

$$= 8\alpha^2 - 2\sqrt{2}a\alpha$$

$$\Rightarrow \quad 2\alpha\left[1+\sqrt{2}a-1+\sqrt{2}a\right]-\frac{1}{2}\left[(1+\sqrt{2}a)^2+(1-\sqrt{2}a)^2\right]$$

$$= 8\alpha^2 - 2\sqrt{2}a\alpha$$

$$\Rightarrow 4\sqrt{2}a\alpha - \frac{1}{2}\left[2(1)^2+2(\sqrt{2}a)^2\right] = 8\alpha^2 - 2\sqrt{2}a\alpha$$

[$\because$ $(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$]

$$\Rightarrow \quad 8a^2 - 6\sqrt{2}a\,a + 1 + 2a^2 = 0.$$

This quadratic equation will have two distinct roots if

$$(6\sqrt{2}a)^2 - 4(8)(1+2a^2) > 0.$$

$\Rightarrow$ $72a^2 - 32\,(1 + 2a^2) > 0 \Rightarrow 9a^2 - 4 - 8a^2 > 0$

$\Rightarrow$ $a^2 - 4 > 0 \Rightarrow (a-2)\,(a+2) > 0$

$\Rightarrow$ $a \in (-\infty, -2) \cup (2, \infty)$.

54. Let $x^2 + y^2 + 2g\,x + 2f\,y + c = 0$ be the variable circle. Since it touches the given circles externally

$$\therefore \quad \sqrt{(-g-0)^2+(-f-0)^2} = \sqrt{g^2+f^2-c} + a \qquad (1)$$

$$\text{and,} \quad \sqrt{(-g-2a)^2+(-f-0)^2} = \sqrt{g^2+f^2-c} + 2a \qquad (2)$$

Subtracting (1) from (2), we get

$$\sqrt{(g+2a)^2+f^2} = \sqrt{g^2+f^2} + a.$$

Squaring both sides, we get

$$(g+2a)^2 + f^2 = a^2 + g^2 + f^2 + 2a\sqrt{g^2+f^2}$$

$$\Rightarrow \quad 4ag + 4a^2 = a^2 + 2a\sqrt{g^2+f^2}$$

$\Rightarrow$ $(4g + 3a)^2 = 4\,(g^2 + f^2)$

or, $(-4\,(-g) + 3a)^2 = 4\,[(-g)^2 + (-f)^2]$.

$\therefore$ Locus of centre $(-g, -f)$ is $(-4x + 3a)^2 = 4\,(x^2 + y^2)$

or, $12x^2 - 4y^2 - 24ax + 9a^2 = 0$.

55. Let the lines cuts the x-axis at A and B, then

$$OA = -\frac{1}{\lambda} \text{ and } OB = -3.$$

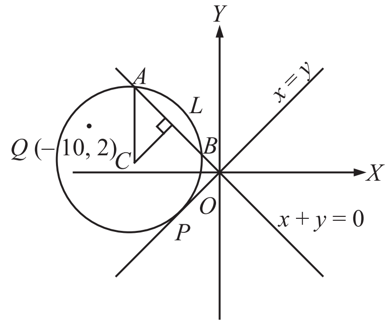
Also, if the lines cut the y-axis at C and D, then

$$OC = 1 \text{ and } OB = \frac{3}{2}.$$

Now, if the circle passes through A, B, C and D then

$$OA \times OB = OC \times OD \Rightarrow \left(-\frac{1}{\lambda}\right)(-3) = 1 \times \frac{3}{2} \Rightarrow \lambda = 2.$$

56. Let $C(\alpha, \beta)$ be the centre of the circle touching OP at P and making intercept $AB = 6\sqrt{2}$ on the line $x + y = 0$ as shown in the figure. If r is the radius of the circle then



$$r^2 = AC^2 = CL^2 + LA^2 = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)^2 + (3\sqrt{2})^2 \qquad (1)$$

Since $\quad OP = 4\sqrt{2}, \quad \therefore \quad OC^2 = CP^2 + PO^2$

$$\Rightarrow \quad \alpha^2 + \beta^2 = r^2 + (4\sqrt{2})^2 \qquad (2)$$

From (1), and (2), we get $\alpha^2 + \beta^2 = \left(\dfrac{\alpha+\beta}{\sqrt{2}}\right)^2 + 18 + 32$

$$\Rightarrow \quad (\alpha^2+\beta^2) - \frac{(\alpha+\beta)^2}{2} = 50 \Rightarrow (\alpha-\beta)^2 = 100$$

$$\Rightarrow \quad \alpha - \beta = \pm 10 \qquad (3)$$

Also, $CP = r \Rightarrow r = \left|\dfrac{\alpha-\beta}{\sqrt{2}}\right| \Rightarrow r^2 = \dfrac{(\alpha-\beta)^2}{2}$ (4)

From (3) and (4), we get $r^2 = (5\sqrt{2})^2 \quad \Rightarrow \quad r = (5\sqrt{2})^2$.

Substituting $r = 5\sqrt{2}$ in (1), we get

$$(5\sqrt{2})^2 = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)^2 + 18$$

$$\Rightarrow \quad 32 = \frac{(\alpha+\beta)^2}{2} \Rightarrow (\alpha+\beta)^2 = 64$$

$$\Rightarrow \quad \alpha + \beta = \pm 8 \qquad (5)$$

Form (3) and (5), we get

$\alpha = 9, \beta = -1$ or $\alpha = -9, \beta = 1$

or $\alpha = 1, \beta = -9$ or $\alpha = -1, \beta = 9$.

Since $Q(-10, 2)$ lies in the interior of the circle,

$\therefore$ CQ must be less than $5\sqrt{2}$.

Thus, centre of the circle must be $(-9, 1)$.

$\therefore$ The equation of the required circle is

$$(x+9)^2 + (y-1)^2 = (5\sqrt{2})^2$$

$\Rightarrow$ $x^2 + y^2 + 18x - 2y + 32 = 0$.

57. Let P be the foot of $\perp$ from origin O on any chord of the circle S whose coordinates are (α, β). Then, the slope of OP is $\dfrac{\beta}{\alpha}$ and thus the slope of chord is $-\dfrac{\alpha}{\beta}$ and its equation passing through (α, β) is

$$y-\beta=-\frac{\alpha}{\beta}(x-a) \quad \Rightarrow \quad \beta y-\beta^2=-\alpha x+\alpha^2$$

$\Rightarrow \quad \alpha x+\beta y=\alpha^2+\beta^2$ (1)

Now, homogenizing the equation of the given circle

$$x^2+y^2+2g\,x+2f\,y+c=0$$

with the help of (1), we get

$$x^2+y^2(2gx+2fy)\left(\frac{\alpha x+\beta y}{\alpha^2+\beta^2}\right)+c\left(\frac{\alpha x+\beta y}{\alpha^2+\beta^2}\right)^2=0. \qquad (2)$$

Now, equation (2) represents a pair of straight lines passing through origin. These lines will be at right angle if sum of the coefficients of x^2 and y^2 is zero.

i.e., $(\alpha^2+\beta^2)^2+(\alpha^2+\beta^2)^2+2g\alpha(\alpha^2+\beta^2)$
$+2\beta f(\alpha^2+\beta^2)+c(\alpha^2+\beta^2)$

$\Rightarrow \quad 2(\alpha^2+\beta^2)+2g\alpha+2f\beta+c=0$ (3)

From equation (3), the locus of $P(\alpha, \beta)$ is

$$2(x^2+y^2)+2gx+2fy+c=0$$

which is the required locus.

58. Given equation of line is

$x^2+2xy+3x+6y=0$ (1)

$\Rightarrow \quad x(x+2y)+3(x+2y)=0 \Rightarrow (x+2y)(x+3)=0$

So, equations of normals are $x+3=0$ (2)

and, $x+2y=0$ (3)

Solving (2) and (3), we get $x=-3,\ y=\frac{3}{2}$.

$\therefore$ Coordinates of centre of the circle are $\left(-3, \frac{3}{2}\right)$.

Given equation of circle is $x(x-4)+y(y-3)=0$

$\Rightarrow \quad x^2+y^2-4x-3y=0.$ (4)

Radius of the circle $=\sqrt{(-2)^2+\left(\frac{-3}{2}\right)^2}=\frac{5}{2}$ and coordinates of centre are $\left(2, \frac{3}{2}\right)$.

Since the required circle is just sufficient to contain the circle (4), therefore the distance between the centres

$\left(-3, \frac{3}{2}\right)$ and $\left(2, \frac{3}{2}\right)$ = the difference of their radius.

Let the radius of the required circle be a.

$$\therefore \quad \sqrt{(-3-2)^2+\left(\frac{3}{2}-\frac{3}{2}\right)^2}=a-\frac{5}{2} \quad \Rightarrow \quad 5=a-\frac{5}{2}$$

$$\Rightarrow \quad a=5+\frac{5}{2}=\frac{15}{2}.$$

Therefore, equation of required circle is

$$(x+3)^2\left(y-\frac{3}{2}\right)^2=\left(\frac{15}{2}\right)^2$$

$$\Rightarrow \quad x^2+y^2+6x-3y+9+\frac{9}{4}-\frac{225}{4}=0$$

$\Rightarrow \quad x^2+y^2+6x-3y-45=0.$

59. Centre of the circle $x^2+y^2=9$ is (0, 0) and any tangent to the circle is

$x\cos\alpha+y\sin\alpha=3$ (1)

Its distance from centre (0, 0) is equal to radius 3.

Any tangent to $x^2+y^2=9$ but $\perp$ to (1) is obtained by replacing α by $(\alpha-90°)$ and its equation is

$x\cos(\alpha-90°)+y\sin(\alpha-90°)=3$

or, $x\cos(90°-\alpha)-y\sin(90°-\alpha)=3$

or, $x\sin\alpha-y\cos\alpha=3$ (2)

Squaring and adding (1) and (2) we get $x^2+y^2=18$ which is a circle concentric with the given circle.

$\therefore$ Locus is $S \equiv x^2+y^2-18=0$ (3)

Equation of tangent to (3) at $(\sqrt{2}, 4)$ is

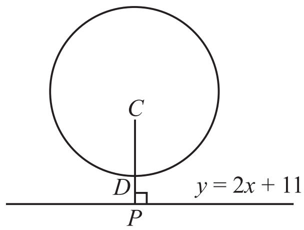
$$p \equiv \sqrt{2}x+4y-18=0.$$

$\therefore$ System of coaxal circles is $S+\lambda P=0$.

60. Let P be the point on the line

$2x-y+11=0$ (1)

which is nearest to the circle



$x^2+y^2+2x-\frac{1}{2}y-\frac{25}{8}=0$ (2)

with centre $C\left(-1, \frac{1}{4}\right)$.

Then, CP is $\perp$ to the line (1) and $CP >$ radius.

[Note that if $CP \le r$, the line intersects or touches the circle and then the point of intersection or point of contact are required points]

Here, $CP=\frac{\left|-2-\frac{1}{4}+11\right|}{\sqrt{5}}=\frac{35}{4\sqrt{5}}>\frac{\sqrt{67}}{4}$ (radius).

Now, equation of CP [$\perp$ to line (1)] is

$$x+2y=\lambda, \text{ where } -1+\frac{1}{2}=\lambda \;=\lambda \text{ or } \lambda=-\frac{1}{2}.$$

$\therefore$ Equation of CP is $2x+4y+1=0$ (3)

Solving (1) and (3), we get $y=2,\ x=-9/2$.

Hence, the required point is $\left(\frac{-9}{2}, 2\right)$.

61. Extremities of the diagonal OA of the rectangle are $O(0, 0)$ and $A(4, 3)$. Then, OA is the diameter of the circumcircle, so equation of the circumcircle is

$x(x-4)+y(y-3)=0$ i.e., $x^2+y^2-4x-3y=0$

i.e., $(x-2)^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{5}{2}\right)^2$ (1)

m = slope of $OA=\frac{3}{4}$ (2)

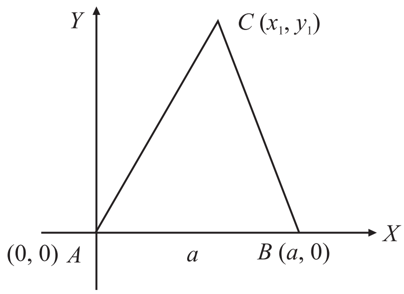
$\therefore$ Tangents parallel to the diagonal OA are

$$y-\frac{3}{2}=\frac{3}{4}(x-2)\pm\frac{5}{2}\sqrt{1+\frac{9}{16}}$$

i.e., $6x-8y\pm 25=0$.

62. Let the coordinates of C be (x_1, y_1) and the coordinates of A and B be $(0, 0)$ and $(a, 0)$, respectively.

Given, $k=\frac{\sin A}{\sin B}=\frac{BC}{AC}$



$\Rightarrow BC^2=k^2AC^2$

$\Rightarrow (x_1-a)^2+y_1^2=k^2(x_1^2+y_1^2)$

$\Rightarrow (1-k^2)x_1^2+(1-k^2)y_1^2-2ax_1+a^2=0$

$\Rightarrow x_1^2+y_1^2-\frac{2ax_1}{1-k^2}x+\frac{a^2}{1-k^2}=0,$ $[\because k\neq 1]$

Hence, locus of C is

$$x^2+y^2-\frac{2a}{1-k^2}+\frac{a^2}{1-k^2}=0,$$

which is a circle whose centre is $\left(\frac{a}{1-k^2}, 0\right)$

and radius $=\sqrt{\frac{a^2}{(1-k^2)^2}-\frac{a^2}{(1-k^2)}}=\frac{ak}{1-k^2}$.

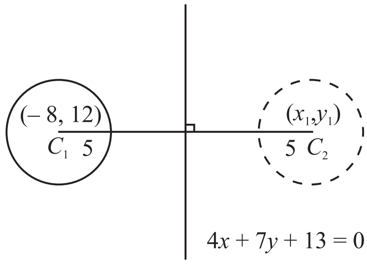
63. The given circle and line are

$x^2+y^2+16x-24y+183=0$ (1)

and, $4x+7y+13=0$ (2)

Centre and radius of circle (1) are $C_1(-8, 12)$ and 5 respectively. Let the centre of the imaged circle be $C_2(x_1, y_1)$.

Then, slope of $C_1C_2\times$ slope of given line



$\Rightarrow \frac{y_1-12}{x_1+8}\times-\frac{4}{7}=-1 \Rightarrow 7x_1-4y_1+104=0$ (3)

and mid point of C_1C_2 i.e., $\left(\frac{x_1-8}{2}, \frac{y_1+12}{2}\right)$ lie on (2)

i.e., $4\left(\frac{x_1-8}{2}\right)+7\left(\frac{y_1+12}{2}\right)+13=0$

or, $4x_1+7y_1+78=0$ (4)

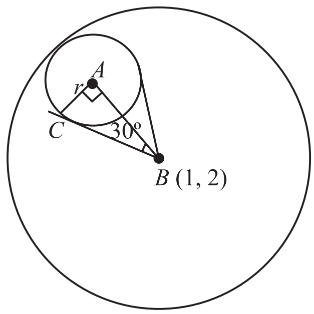
Solving (3) and (4), we get $(x_1, y_1)\equiv(-16, -2)$.

$\therefore$ Equation of the imaged circle is

$(x+16)^2+(y+2)^2=5^2$

or, $x^2+y^2+32x+4y+235=0$.

64. Let r and R be radius of required and given circle, respectively and let centre is (h, k), By given condition



$\sqrt{(h-1)^2+(k-2)^2}=R-r$

Now, $\frac{r}{AB}=\tan 30°$

$r=AB\tan 30°=(R-r)\frac{1}{\sqrt{3}}$ $(AB=R-r)$

$\Rightarrow \sqrt{(h-1)^2+(k-2)^2}=R-\frac{R}{1+\sqrt{3}}=R\left(\frac{\sqrt{3}}{1+\sqrt{3}}\right)$

Now, $R=1+\sqrt{3}$

$\Rightarrow \sqrt{(h-1)^2+(k-2)^2}=\sqrt{3}$

$\therefore$ Locus is $(x-1)^2+(y-2)^2=3$

65. The point of intersection of the tangents at the points $P(\theta_1)$ and $Q(\theta_2)$ on the circle $x^2+y^2=1$ is given by

$$x = \frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)} = \frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{60^\circ}{2}\right)}$$

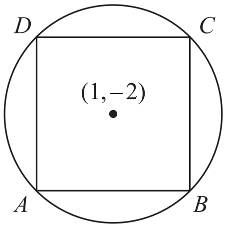
$$y = \frac{a\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)} = \frac{\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{60^\circ}{2}\right)}$$

$\Rightarrow$ $(x \cos 30^\circ)^2 + (y \cos 30^\circ)^2 = 1$

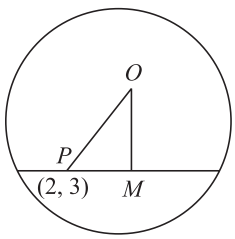
$\Rightarrow$ $(x^2 + y^2)\frac{3}{4} = 1 \Rightarrow x^2 + y^2 = \frac{4}{3}$

$\Rightarrow$ $3x^2 + 3y^2 = 4$

66. Centre is (1, –2). Radius = $\sqrt{1+4-3} = \sqrt{2}$. Since the sides are parallel to coordinate axes, vertices do not lie on horizontal and vertical lines through. (1, 2). $\therefore$ the given points are not vertices.



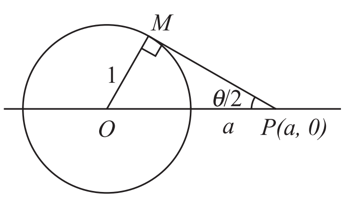
67. Let (2, 3) be given point, M be the middle point of a chord of the circle $x^2 + y^2 = a^2$ through P. Then, the distance of the centre O of the circle from the chord is OM. and $(OM)^2 = (OP)^2 - (PM)^2$ which is maximum when PM is minimum, i.e., P coincides with M, the middle point of the chord.



Hence, the equation of the chord is $T = S_1$.

i.e., $2x + 3y - a^2 = (2)^2 + (3)^2 - a^2 \Rightarrow 2x + 3y = 13$.

68. We have $\frac{\pi}{3} < \theta < \pi$



$$\frac{\pi}{6} < \frac{\theta}{2} < \frac{\pi}{2} \Rightarrow \frac{1}{2} < \sin\left(\frac{\theta}{2}\right) < 1$$

$$\Rightarrow \frac{1}{2} < \frac{1}{a} < 1 \qquad \left[\because \sin\left(\frac{\theta}{2}\right) = \frac{1}{a}\right]$$

$\therefore$ $1 < a < 2$

There can be symmetrical points on the negative x-axis too. Hence, we have $a \in (-2, -1) \cup (1, 2)$.

69. Let $P \equiv (a \cos \alpha, a \sin \alpha)$ and $Q \equiv (a \cos \beta, a \sin \beta)$, where $\beta - \alpha = 2\theta$

Also, $A \equiv (a, 0)$ and $B \equiv (-a, 0)$

If $R(h, k)$ be the intersection point of AP and BQ, the slope of AR = slope of AP [$\because$ R is lies on AP]

$$\Rightarrow \frac{k}{h-a} = \frac{\sin\alpha}{\cos\alpha - 1} \Rightarrow \tan\left(\frac{\alpha}{2}\right) = \frac{a-h}{h} \quad (1)$$

$$\Rightarrow \frac{k}{h+a} = \frac{\sin\beta}{\cos\beta + 1} \Rightarrow \tan\left(\frac{\beta}{2}\right) = \frac{k}{h+a} \quad (2)$$

Since, $\beta - \alpha = 2\theta$, we have $\frac{\beta}{2} - \frac{\alpha}{2} = \theta$

$$\Rightarrow \frac{\tan\left(\frac{\beta}{2}\right) - \tan\left(\frac{\alpha}{2}\right)}{1 + \tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\alpha}{2}\right)} = \tan\theta$$

$$\Rightarrow \frac{\frac{k}{h+a} - \frac{a-h}{k}}{1 + \left(\frac{k}{h+a}\right)\left(\frac{a-h}{k}\right)} = \tan\theta$$

$\Rightarrow$ $h^2 + k^2 - 2ak \tan\theta = a^2$

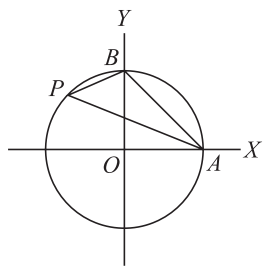
Hence, the locus of R is $x^2 + y^2 - 2ay \tan \theta = a^2$.

70. We choose the centre O of the circle as the origin and the lines OA, OB as the x-axis and the y-axis, respectively. If a be a radius of the given circle, then

$$A \equiv (a, 0) \text{ and } B \equiv (0, a)$$

and a variable point on the circle

$$P \equiv (a \cos\theta, a \sin\theta)$$



If (h, k) be the coordinates of the centroid of triangle PAB, then we have

$3h = a\,(1 + \cos\theta)$ (1)

and, $3k = a\,(1 + \sin\theta)$ (2)

Eliminating θ from equations (1) and (2), we have

$$(3h - a)^2 + (3k - a)^2 = a^2.$$

Putting (x, y) in place of (h, k) gives the equation of the required locus as

$(3x - a)^2 + (3y - a)^2 = a^2$, which is a circle.

71. The given expression can be written as

$6\,(l^2 + m^2) = 9l^2 + 6l + 1$

$$\Rightarrow \quad \frac{3l+1}{\sqrt{l^2+m^2}} = \sqrt{6}$$

$\Rightarrow$ the perpendicular distance of the point (3, 0) from the line $lx + my + 1 = 0$ is $\sqrt{6}$.

Therefore, the given line is a tangent to the circle $(x - 3)^2 + y^2 = 6$.

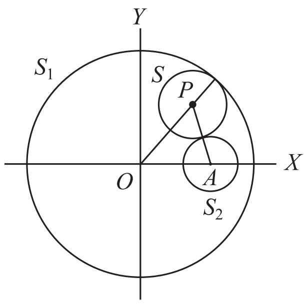
72. We choose centre of S_1 as the origin and the line joining the centres of S_1, S_2 as the x-axis. Let the centre of S_2 be $A(a, 0)$ and b, $c(b > c)$ be the radii of S_1, S_2, respectively. If $P(h, k)$ be the centre of the variable circle S and r be its radius then we have

$$OP + r = b$$

i.e., $\sqrt{h^2 + k^2} = b - r$ (1)

and, $AP = r + c$

i.e., $\sqrt{(h-a)^2 + k^2} = r + c$ (2)



Eliminating r from equations (1) and (2), we have

$$OP + AP = b + c$$

i.e., sum of distances of P from two fixed points O and A = constant. Hence, P lies on an ellipse having foci at O and A.

73. We have,

$S(x, 2) = 0$ gives two identical solutions $x = 1$

$\Rightarrow$ line $y = 2$ is a tangent to the circle $S(x, y) = 0$ at the point (1, 2)

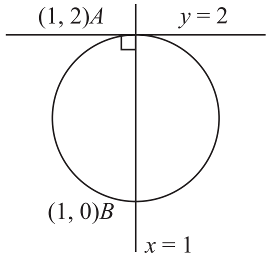
and, $S(1, y) = 0$ gives two distinct solutions $y = 0, 2$

$\Rightarrow$ line $x = 1$ cuts the circle $S(x,y) = 0$ at the points, (1, 0) and (1.2)

Clearly, from the fig., the points $A(1, 2)$ and $B(1, 0)$ are diametrically opposite points.

Thus, equation of the circle, is

$$(x - 1)^2 + y(y - 2) = 0$$



i.e., $x^2 + y^2 - 2x - 2y + 1 = 0$.

More than One Option Correct Type

74. The given circles are

$$x^2 + y^2 - 4x = 0, x > 0$$

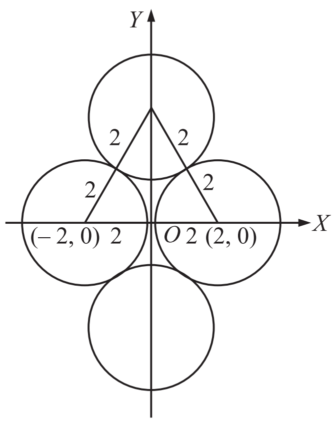
i.e., $(x - 2)^2 + y^2 = 2^2, x > 0$.

and, $x^2 + y^2 + 4x = 0, x < 0$

i.e., $(x + 2)^2 + y^2 = 2^2, x < 0$.

Clearly, from the figure, the centres of the required circles are at $(0, \sqrt{12})$ and $(0, -\sqrt{12})$.

$\therefore$ Equations of the required circles are



$$(x - 0)^2 + (y \pm \sqrt{12})^2 = 2^2$$

i.e., $x^2 + y^2 + 2\sqrt{12}y + 8 = 0$

and, $x^2 + y^2 - 2\sqrt{12}y + 8 = 0$.

75. Let the point on the given line be $(x_1, 2)$.

Since the tangents drawn from $(x_1, 2)$ to the given circle are at right angles, so the point $(x_1, 2)$ must also lie on the director circle whose equation is

$$x^2 + y^2 = 2.25 \text{ i.e., } x^2 + y^2 = 50.$$

$\therefore \quad x_1^2 + 4 = 50 \quad \Rightarrow \quad x_1 = \pm\sqrt{46}$.

So, the points are $(\sqrt{46}, 2)$ and $(-\sqrt{46}, 2)$.

76. Extremities of the diagonal OA of the rectangle are $O(0, 0)$ and $A(4, 3)$. Then, OA is the diameter of the circumcircle, so equation of the circumcircle is

$x(x - 4) + y(y - 3) = 0$ i.e., $x^2 + y^2 - 4x - 3y = 0$

i.e., $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{5}{2}\right)^2$ (1)

m = slope of $OA = 3/4$ (2)

$\therefore$ Tangents parallel to the diagonal OA are

$$y - \frac{3}{2} = \frac{3}{4}(x-2) \pm \frac{5}{2}\sqrt{1 + \frac{9}{16}}$$

i.e., $6x - 8y \pm 25 = 0.$

77. Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

Equation of tangent to (1) at origin (0, 0) is

$$x \cdot 0 + y \cdot 0 + g(x+0) + f(y+0) + c = 0$$

or, $gx + fy + c = 0.$

But it is given to be axis of x, i.e., $y = 0$

$\therefore$ $g = 0, c = 0.$

$\therefore$ Equation (1) becomes $x^2 + y^2 + 2fy = 0$ (2)

If it touches the line $3y = 4x + 24$ i.e., $4x - 3y + 24 = 0$, then the length of $\perp$ from centre $(0, -f)$ on the line is numerically = radius f.

$$\therefore \frac{4(0) - 3(-f) + 24}{\sqrt{16+9}} = \pm f$$

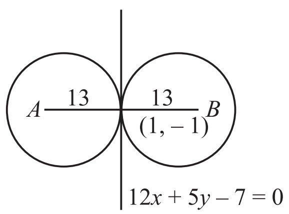
$\Rightarrow$ $3f + 24 = \pm 5f \Rightarrow f = 12, -3.$

Putting these values of f in (2), the equations of circles are $x^2 + y^2 + 24y = 0$ and $x^2 + y^2 - 6y = 0.$

78. Let A, B, be the centres of the two circles. Slope of the common tangent $= -\frac{12}{5}$

$\therefore$ Slope of AB is

$$\tan\theta = \frac{1}{-12/5} = \frac{5}{12}.$$



The point (1, –1) lies on the line AB and the points A and B are at a distance 13 from the point (1, –1).

$\therefore$ Coordinates of A and B are

$(1 \pm 13\cos\theta, -1 \pm 13\sin\theta)$, where $\tan\theta = \frac{5}{12}$.

i.e., $\left(1 \pm 13 \cdot \frac{12}{13}, -1 \pm 13 \cdot \frac{5}{13}\right)$ or $(1 \pm 12, -1 \pm 5)$

i.e., (13, 4) and (–11, –6).

79. The equation of the circle is

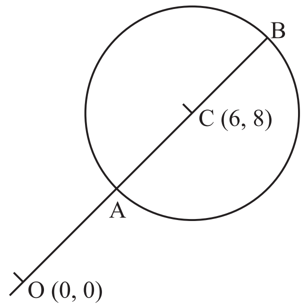
$x^2 + y^2 - 12x - 16y + 75 = 0.$

Join its centre C with O.

Let OC meets the circle in A and B.

Centre of the circle is $C \equiv (6, 8)$ and radius

$$AC = \sqrt{36 + 64 - 75} = 5.$$



Also, $OC = \sqrt{36+64} = 10.$

$\therefore$ $OA = OC - AC = 5.$

Now, the point on the circle nearest to the origin is A and farthest from it is B.

$\because$ $OA : AC = 5 : 5 = 1 : 1$

$\Rightarrow$ A is the mid-point of OC.

$\therefore$ the coordinates of A are $\left(\frac{0+6}{2}, \frac{0+8}{2}\right)$ i.e., (3, 4).

Let the coordinate of B be (h, k).

Since C is the mid point of AB

$\therefore$ $\frac{3+h}{2} = 6, \frac{4+k}{2} = 8 \Rightarrow h = 9, k = 12.$

$\therefore$ The coordinates of B are (9, 12).

80. Given circles are $x^2 + y^2 = a^2$ (1)

and, $(x - 2a)^2 + y^2 = a^2$ (2)

Let A and B be the centres and r_1 and r_2 the radii of the two circles (1) and (2), respectively, then

$$A \equiv (0, 0), B \equiv (2a, 0), r_1 = a, r_2 = a$$

Let the equation of equal circle touching circles (1) and (2) be

$(x - \alpha)^2 + (y - \beta)^2 = a^2$ (3)

Its centre C is (α, β) and radius $r_3 = a$.

Since (3) touches (1),

$\therefore$ $AC = r_1 + r_3 = 2a$

[Here $AC \neq |r_1 - r_3|$ as $r_1 - r_3 = a - a = 0$]

or, $AC^2 = 4a^2$ or $a^2 + b^2 = 4a$ (4)

Again, since circle (3) touches circle (2)

$\therefore$ $BC = r_2 + r_3$ or $BC^2 = (r_2 + r_3)^2$

$\therefore$ $(2a - \alpha)^2 + b^2 = (a + a)^2$ or $a^2 + b^2 - 4a\alpha = 0$

or, $4a^2 - 4a\alpha = 0$ [from (4)]

$\therefore$ $\alpha = a$ and from (4), $\beta = \pm\sqrt{3}a$.

Thus, required circles are

$$(x - a)^2 + (y \mp \sqrt{3}a)^2 = a^2.$$

or, $x^2 + y^2 - 2ax \mp 2\sqrt{3}ay + 3a^2 = 0.$

81. Given circle is $S \equiv x^2 + y^2 + 6x - 8y - 9 = 0$

Since $S]_{(2,1)} = 4 + 1 + 12 - 8 - 10 = -1 < 0$,

so the point (2, 1) lies inside the given circle.

$\therefore$ Chord of contact of tangents from (2, 1) does not exist.

Also, polar of (2, 1) w.r.t. the circle is

$$2x + y + 3(x + 2) - 4(y + 1) - 10 = 0$$

i.e., $\quad 5x - 3y - 8 = 0.$

82. Centre of S_1 is $C_1 \equiv (2, 3)$ and radius of S_1 is $r_1 = 5$. Centre of S_2 is $C_2 \equiv (-3, -2)$ and radius of S_2 is $r_2 = 5$.

Also, d = distance between centres

$$= C_1C_2 = \sqrt{25+25}\ \sqrt{50}$$

$\therefore \quad |r_1 - r_2| = 0 \quad$ and $\quad r_1 + r_2 = 10.$

Since $|r_1 - r_2| < C_1C_2 < r_1 + r_2$, therefore circles S_1 and S_2 intersect.

Equation of the common chord of S_1 and S_2 is

$$S_1 - S_2 = 0 \quad \text{i.e., } x + y = 0,$$

which is also the equation of the radical axis and hence it is $\perp$ to the line joining the centres C_1 and C_2 of the two circles.

83. Let (α, β) be the centre of on of the circles. Then, centre must lie on the line $\perp$ to the common tangent $4x + 3y = 10$ and passing through the point (1, 2). Thus, the equation of the radius is

$$3x - 4y + k = 0 \qquad (1)$$

Since it passes through (1, 2), therefore $3 - 8 + k = 0$

i.e., $k = 5$

Substituting $k = 5$ in (1), the equation of the line joining centres is

$$3x - 4y + 5 = 0 \qquad (2)$$

As centre lies on (2), we have $3\alpha - 4\beta + 5 = 0$

$$\beta = \frac{3\alpha + 5}{4} \qquad (3)$$

Since the radius of circle is 5, therefore

$$(\alpha - 1)^2 + (\beta - 2)^2 = 25 \qquad (4)$$

Substituting the value of β from (3) in (4), we get

$$(\alpha - 1)^2 + \left(\frac{3\alpha + 5}{4} - 2\right)^2 = 25$$

$$\Rightarrow \quad (\alpha - 1)^2 + \frac{(3\alpha - 3)^2}{16} = 25$$

$$\Rightarrow \quad (\alpha - 1)^2 + \frac{9}{16}(\alpha - 1)^2 = 25$$

$$\Rightarrow \quad (\alpha - 1)^2\left(1 + \frac{9}{16}\right) = 25 \quad \Rightarrow \quad (\alpha - 1)^2 = 16$$

$$\Rightarrow \quad (\alpha - 1)^2 = \pm 4 \Rightarrow \alpha = 5 \quad \text{or} \quad \alpha = -3 \qquad (5)$$

From (3) and (5), we have $\alpha = 5, \beta = 5$

or, $\quad \alpha = -3, \beta = -1.$

Thus, the cerntres of the circles are (5, 5) and (–3, –1).

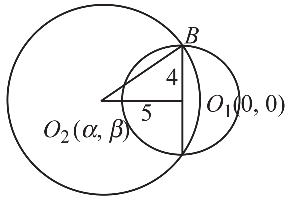
Therefore, the equations of the required circles are

$$(x - 5)^2 + (y - 5)^2 = 5^2 \text{ and } (x + 3)^2 + (y + 1)^2 = 5^2.$$

$\Rightarrow \quad x^2 + y^2 - 10x - 10y + 25 = 0$

and, $x^2 + y^2 + 6x + 2y - 15 = 0.$

84. When two circles intersect, the common chord of maximum length will be the diameter of the smaller circle.



Let $O_2(\alpha, \beta)$ be the centre of the circle C_2 of radius 5 and O_1 (0, 0) be the centre of the circle C_1 of radius 4.

Then, $O_1B = 4$ and $O_2B = 5$.

$$\therefore \quad O_1O_2 = \sqrt{25 - 16} = 3 = \sqrt{\alpha^2 + \beta^2} \qquad (1)$$

Since slope of O_1B is $\frac{3}{4}$ and O_1B is $\perp$ to O_2O_1,

$$\therefore \quad -\frac{\alpha}{\beta} = \frac{3}{4} \quad \Rightarrow \quad \alpha = \frac{-3\beta}{4} \qquad (2)$$

From (1) and (2), we get $9 = \beta^2 + \frac{9\beta^2}{16}$

$$\Rightarrow \quad \frac{25}{16}\beta^2 = 9 \quad \Rightarrow \quad \beta^2 = \frac{144}{25} \quad \Rightarrow \quad \beta = \pm\frac{12}{5}$$

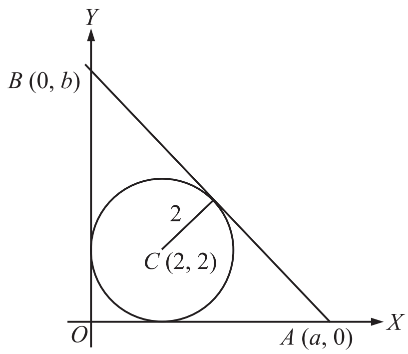
When $\beta = \frac{12}{5};\ \alpha = \frac{-3}{4} \times \frac{12}{5} = \frac{-9}{5}$

When $\beta = \frac{12}{5};\ \alpha = \frac{-3}{4} \times \frac{12}{5} = \frac{-9}{5}$

Thus, the coordinates of the centre of circle C_2 are

$$\left(\frac{9}{5}, \frac{-12}{5}\right) \text{ or } \left(\frac{-9}{5}, \frac{12}{5}\right).$$

85. Let OAB be the triangle in which the given circle of radius 2 with centre $C(2, 2)$ is inscribed and let the equation of AB be



$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (1)$$

$\therefore \quad OA = a, OB = b$

$\Rightarrow$ the centre of the circumscribed circle of ΔOAB is $\left(\frac{a}{2}, \frac{b}{2}\right)$.

Let $\alpha = \frac{a}{2}$ and $\beta = \frac{b}{2} \qquad (2)$

Now, the length of the $\perp$ from $C(2, 2)$ on (1) must be equal to the radius 2.

$$\therefore \quad \pm 2 = \frac{\left|\frac{2}{a}+\frac{2}{b}-1\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} \qquad (3)$$

Eliminating a and b from (2) and (3), we get

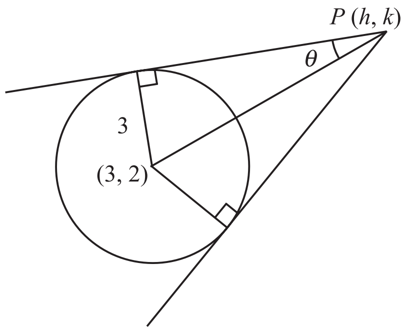
$$\therefore \quad \pm 2 = \frac{\left|\frac{1}{\alpha}+\frac{1}{\beta}-1\right|}{\sqrt{\frac{1}{4\alpha^2}+\frac{1}{4\beta^2}}} \Rightarrow \pm 1 = \frac{\alpha+\beta-\alpha\beta}{\sqrt{\alpha^2+\beta^2}}$$

$$\Rightarrow \quad \alpha+\beta-\alpha\beta \pm \sqrt{\alpha^2+\beta^2} = 0 \qquad (4)$$

$\therefore$ Locus of (α, β) is $x + y - xy \pm \sqrt{x^2+y^2} = 0$.

$\therefore \quad k = \pm 1$.

86. The given circle is $x^2 + y^2 - 6x - 4y + 4 = 0$.



Its centre is (3, 2) and radius = 3.

The given line is $4x - 3y = 6$ (1),

which passes through the centre of the circle, so tangents from any point on this line are equally inclined to this line. Let (h, k) be a point on it.

Given, $\tan 2\theta = \frac{24}{7}, 0 < \theta < \frac{\pi}{2} \Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{24}{7}$

$\Rightarrow \quad 12\tan^2\theta + 7\tan\theta - 12 = 0$

$\Rightarrow \quad \tan\theta = \frac{3}{4}$ or $\tan\theta = -\frac{4}{3}$

$\therefore \quad \tan\theta = \frac{3}{4}$ $(\tan\theta > 0)$ (2)

$$\therefore \quad \frac{3}{\sqrt{(h-3)^2+(k-2)^2}} = \sin\theta = \frac{3}{5}.$$

or, $(h-3)^2 + (k-2)^2 = 25$

$\Rightarrow \quad (h-3)^2 + \frac{16}{9}(h-3)^2 = 25$

[$\because$ the point (h, k) lies on the line (1), $\therefore$ $4h - 3k = 6$]

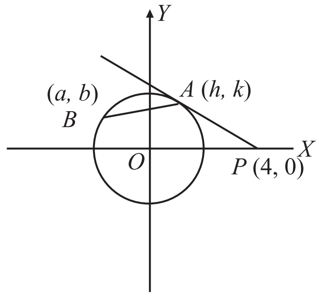
$\Rightarrow \quad \frac{25}{9}(h-3)^2 = 25$ or $(h-3)^2 = 9$.

or, $h - 3 = \pm 3$. $\therefore$ $h = 0, 6$.

$\therefore \quad (h, k) \equiv (0, -2), (6, 6)$.

87. Equation of given circle is $x^2 + y^2 = 8$ (1)

Let $A(h, k)$ be the point of contact, in the first quadrant, of tangent from $P(4, 0)$ to the circle (1).



Equation of tangent at $A(h, k)$ is $hx + ky = 8$.

It passes through $P(4, 0)$, $\therefore$ $4h = 8$ or $h = 2$.

Since, $A(h, k)$ lies on the circle, we get

$h^2 + k^2 = 8$ or $4 + k^2 = 8$ or $k = 2$ ($\therefore$ $k > 0$)

$\therefore \quad A \equiv (2, 2)$.

Let the coordinates of point B on circle (1), be (a, b) such that $AB = 4$.

$\therefore \quad a^2 + b^2 = 8$ (2)

and, $AB^2 = (a-2)^2 + (b-2)^2 = 16$ (3)

Solving (2) and (3), we get $a = 2, b = -2$

or, $a = -2, b = 2$

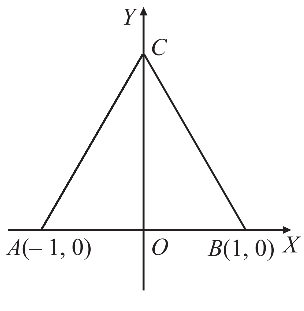
Hence, the coordinates of B are $(2, -2)$ or $(-2, 2)$.

88. Since two of the vertices $A(-1, 0)$ and $B(1, 0)$ lie on x-axis, the third vertex C will lie on y-axis. Let the coordinates of C be $(0, \alpha)$.

Then, $CA^2 = AB^2 = BC^2 = 4$

$\Rightarrow 1 + \alpha^2 = 4 \Rightarrow \alpha^2 = 3 \Rightarrow \alpha = \pm\sqrt{3} \quad \Rightarrow \quad OC = \pm\sqrt{3}$.

Since triangle is equilateral, circumcentre of the triangle coincides with its centroid, i.e., with $\left(0, \pm\frac{1}{\sqrt{3}}\right)$ and circumradius is $\frac{2}{\sqrt{3}}$.



Hence possible equations of the circumcircle are

$$x^2 + \left(y \pm \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{2}{\sqrt{3}}\right)^2$$

or, $x^2 + y^2 \pm \frac{2}{\sqrt{3}}y - 1 = 0$.

TRICK(S) FOR PROBLEM SOLVING

- For an equilateral triangle, the circumcentre of the triangle coincides with its centroid.

89. Centres of given circles are $(-a, 0)$ and $(-b, 0)$. Radical axis is $2x(a-b) = 0$ i.e., $x = 0$. Since one of the circles lies within the other. $\therefore$ their points of intersection are imaginary, $\therefore$ the radical axis does not intersect them. Thus, both the circles lie on the same side of radical axis i.e., $x = 0$. Hence, a and b are of the same sign. $\therefore\ ab > 0$.

Also, since the origin lies on the radical axis $x = 0$, therefore $(0, 0)$ lies outside the two circles, $\therefore\ c > 0$. Hence $ab > 0$, $c > 0$.

90. If $C \equiv (\alpha, \beta)$ then chord of contact of the tangents from it to the circle $x^2 + y^2 = 4$ is $ax + by = 4$ which passes through the point $(0, 5)$

$\therefore\quad 5\beta = 4$ i.e., $\beta = 4/5$

Also, $ax + by = 4$ is tangent to the circle $x^2 + y^2 = 1$

$$\therefore\quad \frac{4}{\sqrt{\alpha^2+\beta^2}} = 1 \Rightarrow\ 16 = \alpha^2 + \left(\frac{4}{5}\right)^2$$

$$\Rightarrow\quad \alpha^2 = 16 - \frac{16}{25} = \frac{384}{25}\ \therefore\ \alpha = \pm\frac{8\sqrt{6}}{5}$$

$$\therefore\quad \text{Point } C \text{ is } \left(\frac{8\sqrt{6}}{5}, \frac{4}{5}\right), \left(\frac{-8\sqrt{6}}{5}, \frac{4}{5}\right)$$

91. Multiplying by r, the given equation becomes $r^2 = |r\cos\theta|$ (1)

In Cartesian form equation (1) can be written as

$x^2 + y^2 = |x|$

$\Rightarrow\quad x^2 + y^2 = +x$ if $x > 0$

and $x^2 + y^2 = -x$ if $x < 0$

Equation (2) represents a circle of radius $\frac{1}{2}$ and centered at $\left(\frac{1}{2}, 0\right)$. Equation (3) represents a circle of radius $\frac{1}{2}$ and centered at $\left(-\frac{1}{2}, 0\right)$. Hence, the given equation represents two circles touching each other at the origin.

Passage Based Questions

92. Let P be the point (x_1, y_1).

Polar of P w.r.t. the circle $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2 \qquad (1)$$

$\because$ (1) touches the circle $(x-f)^2 + (y-g)^2 = b^2$

$\therefore$ length of $\perp$ from centre (f, g) on (1) = radius b

$$\Rightarrow\quad \pm\frac{fx_1 + gy_1 - a^2}{\sqrt{x_1^2 + y_1^2}} = b$$

or, $(fx_1 + gy_1 - a^2)^2 = b^2(x_1^2 + y_1^2)$.

$\therefore$ Locus of $P(x_1, y_1)$ is $(fx + gy - a^2)^2 = b^2(x^2 + y^2)$.

93. The equation of the line is $3x + 4y - 45 = 0$ (1)

and that of the circle is $x^2 + y^2 - 6x - 8y + 5 = 0$ (2)

Let (x_1, y_1) be the pole of (1) w.r.t. (2).

The equation of polar of (x_1, y_1) w.r.t. (2) is

$$xx_1 + yy_1 - 3(x + x_1) - 4(y + y_1) + 5 = 0$$

or, $(x_1 - 3)x + (y_1 - 4)y - 3x_1 - 4y_1 + 5 = 0$ (3)

As (1) and (3) both represent the same line i.e., polar of (x_1, y_1) w.r.t. (2)

$\therefore$ Comparing coefficients

$$\underset{(a)}{\frac{x_1 - 3}{3}} = \underset{(b)}{\frac{y_1 - 4}{4}} = \underset{(c)}{\frac{-3x_1 - 4y_1 + 5}{-45}} \qquad (4)$$

From (a) and (b), we have

$$4x_1 - 12 = 3y_1 - 12$$

or, $4x_1 = 3y_1$ or $y_1 = \frac{4}{3}x_1$ (5)

From (b) and (c), we have $-45y_1 + 180 = -12x_1 - 16y_1 + 20$

or, $12x_1 - 29y_1 + 160 = 0$

or, $12x_1 - 29\left(\frac{4}{3}x_1\right) + 160 = 0$ [Using (5)]

or, $36x_1 - 116x_1 + 480 = 0$ or $80x_1 = 480$

$\therefore\quad x_1 = 6$.

From (5), $y_1 = \frac{4}{3} \times 6 = 8$.

Hence, the required pole is $(6, 8)$.

94. Equation of the circle is $x^2 + y^2 = 16$ (1)

Equation of the chord having $(-2, 3)$ as its mid point is

$T = S_1$

i.e., $x(-2) + y(3) - 16 = (-2)^2 + (3)^2 - 16$

or, $-2x + 3y = 4 + 9$ or $2x - 3y + 13 = 0$ (2)

Let (x_1, y_1) be the pole of (2) w.r.t. (1)

Polar of (x_1, y_1) w.r.t. the circle (1) is $xx_1 + yy_1 = 16$ (3)

Since (2) and (3) both represent the polar of (x_1, y_1)

$$\therefore\quad \frac{x_1}{2} = \frac{y_1}{-3} = -\frac{16}{13} \text{ or } x_1 = -\frac{32}{13}, y_1 = \frac{48}{13}.$$

Hence, pole of (2) is $\left(-\frac{32}{13}, \frac{48}{13}\right)$.

95. Equations of the circles are

$S_1: x^2 + y^2 - 12 = 0$ (1)

and, $S_2: x^2 + y^2 - 5x + 2y - 2 = 0$ (2)

Common chord of (1) and (2) is

$$5x - 2y - 10 = 0 \qquad (3)$$

$$(S_1 - S_2 = 0)$$

Let (x_1, y_1) be its pole w.r.t. (1)

Polar of (x_1, y_1) w.r.t. (1) is $xx_1 + yy_1 = 12$ (4)

As (3) and (4) both represent the polar of (x_1, y_1) w.r.t. (1), $\therefore$ comparing coefficients

$$\frac{x_1}{5} = \frac{y_1}{-2} = \frac{-12}{-10} \Rightarrow\ x_1 = 6, y_1 = -\frac{12}{5}.$$

Hence, pole of (3) w.r.t. (1) is $\left(6, \frac{-12}{5}\right)$.

96. Let the three coaxial circles be

$$x^2 + y^2 + 2g_r x + \lambda = 0 \ (r = 1, 2, 3).$$

Coordinates of A, B, C, the centres, are

$$(-g_1, 0), (-g_2, 0), (-g_3, 0).$$

Also, $r_1 = \sqrt{g_1^2 - \lambda}, r_2 = \sqrt{g_2^2 - \lambda}, r_3 = \sqrt{g_3^2 - \lambda}.$

Now, $BC \cdot CA \cdot AB = (g_2 - g_3) \cdot (g_3 - g_1) \cdot (g_1 - g_2).$

Also, $r_1^2 \cdot BC + r_2^2 \cdot CA + r_3^2 \cdot AB$

$= (g_1^2 - \lambda)(g_2 - g_3) + (g_2^2 - \lambda)(g_3 - g_1) + (g_3^2 - \lambda) \cdot (g_1 - g_2)$

$= g_1^2 (g_2 - g_3) + g_2^2 (g_3 - g_1) + (g_1 - g_2)$

$\quad - \lambda(g_2 - g_3 + g_3 - g_1 + g_1 - g_2)$

$= -(g_1 - g_2)(g_2 - g_3)(g_3 - g_1)$

$\quad [\because \Sigma a^2 (b - c) = -(a - b)(b - c)(c - a)]$

$\quad = -BC \cdot CA \cdot AB.$

97. Let the equations of the three coaxal circles be

$$x^2 + y^2 + 2g_r x + \lambda = 0 \ [r = 1, 2, 3].$$

Coordinates of A, B, C, the centres, are $(-g_1, 0), (-g_2, 0)$ and $(-g_3, 0)$, respectively.

$\therefore \quad BC = g_2 - g_3; CA = g_3 - g_1; AB = g_1 - g_2.$

Let (x', y') be any point from which tangents to the three circles have been drawn. Then,

$t_1^2 = x'^2 + y'^2 + 2g_1 x' + \lambda$

$t_2^2 = x'^2 + y'^2 + 2g_2 x' + \lambda$

$t_3^2 = x'^2 + y'^2 + 2g_3 x' + \lambda$

$\therefore \quad BC \cdot t_1^2 + CA \cdot t_2^2 AB \cdot t_2^3$

$\quad = (g_2 - g_3)(x'^2 + y'^2 + 2g_1 x' + \lambda)$

$\quad + (g_3 - g_1)(x'^2 + y'^2 + 2g_2 x' + \lambda)$

$+ (g_1 - g_2)(x'^2 + y'^2 + 2g_3 x' + \lambda)$

$= (x'^2 + y'^2 + \lambda)(g_2 - g_3 + g_3 - g_1 + g_1 - g_2)$

$+ 2x' [g_1 (g_2 - g_3) + g_2 (g_3 - g_1) + g_3 (g_1 - g_2)]$

$= 0.$

98. The equations of two circles are

$$x^2 + y^2 - 2x - 6y + 9 = 0 \qquad (1)$$

and, $$x^2 + y^2 + 6x - 2y + 1 = 0 \qquad (2)$$

Their radical axis is $8x + 4y - 8 = 0$ or $2x + y - 2 = 0$ (3)

The equation of any circle coaxal with the given circles is

$$x^2 + y^2 - 2x - 6y + 9 + \lambda(2x + y - 2) = 0$$

or, $x^2 + y^2 + (2\lambda - 2)x + (\lambda - 6)y + (9 - 2\lambda) = 0$ (4)

The centre of this circle is $\left[(1 - \lambda), \left(3 - \frac{\lambda}{2}\right)\right]$ (5)

Its radius $= \sqrt{(1-\lambda)^2 + (3 - \frac{\lambda}{2})^2 - (9 - 2\lambda)}$

$= \sqrt{\frac{5\lambda^2}{4} - 3\lambda + 1}.$

For limiting points, its radius = 0

i.e., $\frac{5\lambda^2}{4} - 3\lambda + 1 = 0$ or $5\lambda^2 - 12\lambda + 4 = 0$

or, $5\lambda^2 - 10\lambda - 2\lambda + 4 = 0$ or $(\lambda - 2)(5\lambda - 2) = 0$

$\therefore \quad \lambda = 2, \frac{2}{5}.$

Substituting these values in (5), the limiting points are

$(-1, 2)$ and $\left(\frac{3}{5}, \frac{14}{5}\right).$

99. Since the limiting points are point circles belonging to the coaxal system,

$\therefore$ their equations are $(x - 1)^2 + (y - 2)^2 = 0$

and, $(x - 4)^2 + (y - 3)^2 = 0$

i.e., $x^2 + y^2 - 2x - 4y + 5 = 0$ (1)

and, $x^2 + y^2 - 8x - 6y + 25 = 0$ (2)

Equation of any circle coaxal with the circles (1) and (2) is

$$x^2 + y^2 - 2x - 4y + 5 + \lambda(x^2 + y^2 - 8x - 6y + 25) = 0 \quad (3)$$

(3) will pass through the origin, if

$$5 + 25\lambda = 0 \text{ i.e., } \lambda = -\frac{1}{5}$$

$\therefore$ (3) becomes, $x^2 + y^2 - 2x - 4y$

$$+5 - \frac{1}{5}(x^2 + y^2 - 8x - 6y + 25) = 0$$

or, $2x^2 + 2y^2 - x - 7y = 0$, which is the required equation.

Match the Column Type

100. The centre of the given circle is (2, 3) and the radius $= \sqrt{4 + 9 - k}$, i.e., $\sqrt{13 - k}$.

Since the given circle does not touch or intersect the coordinate axes and the point (2, 2) lies inside the circle

$\therefore$ x-coordinate of centre > radius i.e., $2 > \sqrt{13 - k}$, ,

y-coordinate of centre > radius i.e., $3 > \sqrt{13 - k}$

and, $4 + 4 - 8 - 12 + k < 0$

$\Rightarrow \quad 4 > 13 - k, 9 > 13 - k$ and $-12 + k < 0$

$\Rightarrow \quad k > 9, k > 4$ and $k < 12$

$\Rightarrow \quad 9 < k < 12$

II. The length of the $\perp$ from the centre (0, 0) of the given circle to the line $3x + ay - 20 = 0$ is

$$= \frac{|3(0) + a(0) - 20|}{\sqrt{9 + a^2}} = \frac{20}{\sqrt{9 + a^2}}$$

Radius of the given circle = 5.

Since the line cuts the circle at real, distinct or coincident points,

$\therefore \quad \frac{20}{\sqrt{9 + a^2}} \le 5 \quad \Rightarrow a^2 + 9 \ge 16 \Rightarrow a^2 - 7 \ge 0$

$\Rightarrow (a - \sqrt{7})(a - \sqrt{7}) \ge 0$

$\Rightarrow \alpha \in (-\infty, -\sqrt{7}] \cup [\sqrt{7}, \infty)$.

III. Since the point (2, k) lies outside the circle

$$x^2 + y^2 + x - 2y - 14 = 0$$

$\therefore$ $4 + k^2 + 2 - 2k - 14 > 0$ or $k^2 - 2k - 8 > 0$

or, $(k + 2)(k - 4) > 0$

or, $k \in (-\infty, -2) \cup (4, \infty)$ (1)

Also, the point (2, k) lies outside the circle

$x^2 + y^2 = 13$.

$\therefore$ $4 + k^2 - 13 > 0$ or $k^2 - 9 > 0$

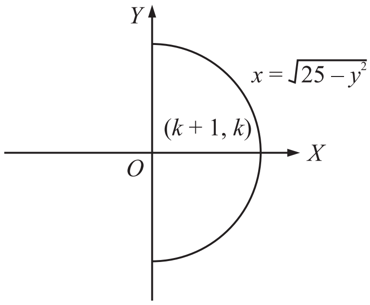
or, $(k - 3)(k + 3) > 0$

or, $k \in (-\infty, -3) \cup (3, \infty)$ (2)

The common solution of (1) and (2) is given by

$k \in (-\infty, -3) \cup (4, \infty)$.

IV. Since the point (k + 1, k) lies inside the region bounded by $x = \sqrt{25 - y^2}$ and y-axis,



$\therefore$ $(k + 1)^2 + k^2 - 25 < 0$

and, $k + 1 > 0$

$\Rightarrow$ $2k^2 + 2k - 24 < 0$ and $k > -1$

$\Rightarrow$ $k^2 + k - 12 < 0$ and $k > -1$

$\Rightarrow$ $(k + 4)(k - 3) < 0$ and $k > -1$

$\Rightarrow$ $-4 < k < 3$ and $k > -1$

$\Rightarrow$ $-1 < k < 3$.

101 **I.** The equations of the circles are

$$x^2 + y^2 - 6x - 2y + 9 = 0 \quad (1)$$

and, $$x^2 + y^2 - 14x - 8y + 61 = 0 \quad (2)$$

Centre of (1) is $C_1 \equiv (3, 1)$ and radius $r_1 = 1$

Centre of (2) is $C_2 \equiv (7, 4)$ and radius $r_2 = 2$

d = distance between centres $= C_1C_2 = \sqrt{16 + 9} = 5$.

So, $r_1 + r_2 < d$, $\therefore$ the two circles do not cut each other and hence the number of common tangents is 4.

II. The equations of the circles are

$$x^2 + y^2 = 4 \quad (1)$$

and, $$x^2 + y^2 - 8x + 12 = 0 \quad (2)$$

Centre of (1) is $C_1 \equiv (0, 0)$ and radius $r_1 = 2$

Centre of (2) is $C_2 \equiv (4, 0)$ and radius $r_2 = 2$

d = distance between centres $= C_1C_2 = 4$.

Since $C_1C_2 = r_1 + r_2$, $\therefore$ the two circles touch each other externally. Hence, 3 common tangents can be drawn to the two circles.

III. Given circles are $x^2 + y^2 - 4 = 0$ (1)

and, $$x^2 + y^2 - 6x - 8y - 24 = 0 \quad (2)$$

Centre of circle (1) is $C_1 \equiv (0, 0)$ and radius $r_1 = 2$

Centre of circle (2) is $C_2 \equiv (3, 4)$ and radius $r_2 = 7$

Also, d = distance between the centres $= C_1C_2 = 5$

Since $d = r_2 - r_1$, therefore the given circles touch internally, as such they can have just one common tangent at the point of contact.

IV. Let $S \equiv x^2 + y^2 - 8x - 6y + 9 = 0$.

Now, S for $(3, -2) = 9 + 4 - 24 + 12 + 9 > 0$,

$\therefore$ the point (3, –2) lies outside the circle.

$\therefore$ Two tangents can be drawn to the circle from the point (3, –2).

102. **I.** Any line parallel to y-axis is $x = k$.

If it touches the circle $x^2 + y^2 = 9$, then $\perp$ distance from the centre (0, 0) of the circle to the line $x = k$, must be equal to radius 3.

i.e., $$\frac{|0 - k|}{1} = 3 \Rightarrow k = \pm 3$$

$\therefore$ $k = 3$. ($\because$ line does not lie in the IIIrd quadrant)

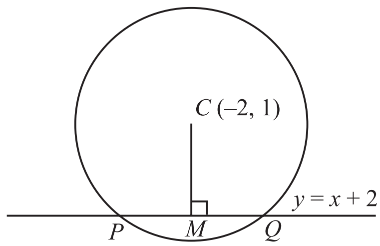
$\therefore$ The equation of the tangent line is $x = 3$.

This meets the circle when $9 + y^2 = 9 \Rightarrow y = 0$.

$\therefore$ Point of contact is (3, 0).

II. Equation of chord PQ is

$y = x + 2$



or, $$x - y + 2 = 0 \quad (1)$$

Centre of circle is $C(-2, 1)$.

Draw $CM \perp PQ$, then M is the mid point of PQ.

Equation of any line $\perp$ to PQ is $x + y + k = 0$

If it passes through $C(-2, 1)$ then

$$-2 + 1 + k = 0 \text{ or } k = 1.$$

$\therefore$ Equation of CM is $x + y + 1 = 0$. (2)

Solving (1) and (2), we obtain $x = -\frac{3}{2}$ and $y = \frac{1}{2}$.

$\therefore$ Coordinates of M are $\left(\frac{-3}{2}, \frac{1}{2}\right)$.

III. Let the equation of the circle passing through the given ponits be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

then, $$1 + t^2 + 2g + 2ft + c = 0 \quad (2)$$

$$1 + t^2 + 2gt + 2f + c = 0 \quad (3)$$

and, $$t^2 + t^2 + 2gt + 2ft + c = 0 \quad (4)$$

Solving (2), (3) and (4) we get

$$g = f = -(t + 1)/2 \text{ and } c = 2t.$$

and the equation (i) becomes

$$x^2 + y^2 - (t + 1)x - (t + 1)y + 2t = 0$$

which is satisfied by (1, 1) for all values of t.

IV. The chord of contact of tangents from (α, β) is

$$ax + by = 1 \quad (1)$$

Clearly, (1) passes through $\left(\frac{1}{2}, \frac{1}{4}\right)$.

Assertion-Reason Type

103. Let the equation of one of the circles be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Since it passes through origin, $\therefore c = 0$.

So, the equation becomes $x^2 + y^2 + 2gx + 2fy = 0$.

Since it cuts the circle $x^2 + y^2 + 6x - 4y + 2 = 0$ orthogonally,

$\therefore \quad 2g(3) + 2f(-2) = 0 + 2 \quad \Rightarrow \quad -6(-g) + 4(-f) = 2.$

Thus, the locus of the centre $(-g, -f)$ is

$-6x + 4y = 2$ or $3x - 2y + 1 = 0$.

104. Equation of tangent to the circle

$$x^2 + y^2 = 5 \text{ at } (1, -2) \text{ is } x - 2y - 5 = 0 \quad (1)$$

Let this line touches the circle

$$x^2 + y^2 - 8x + 6y + 20 = 0 \text{ at } (x_1, y_1)$$

$\therefore$ Equation of tangent at (x_1, y_1) is

$$xx_1 + yy_1 - 4(x + x_1) + 3(y + y_1) + 20 = 0$$

or, $\quad x(x_1 - 4) + y(y_1 + 3) - 4x_1 + 3y_1 + 20 = 0 \quad (2)$

Now, (1) and (2) represent the same line

$$\therefore \quad \frac{x_1 - 4}{1} = \frac{y_1 + 3}{-2} = \frac{-4x_1 + 3y_1 + 20}{-5}$$

$\Rightarrow \quad -2x_1 + 8 = y_1 + 3$ or $2x_1 + y_1 - 5 = 0$.

Only the point (3, –1) satisfies it. Hence, the point of contact is (3, –1).

105. Since the circumcentre coincides with the centroid of an equilateral triangle,

$\therefore$ Circumcentre is (2, 2).

Radius $= \sqrt{(2+3)^2 + (2-4)^2} = \sqrt{29}$

$\therefore$ The equation of the circumcircle is

$(x - 2)^2 + (y - 2)^2 = 29 \Rightarrow x^2 + y^2 - 4x - 4y - 21 = 0.$

106. Since the point (2, 4) is interior to the given circle,

$\therefore \quad 2^2 + 4^2 - 6 \times 2 - 10 \times 4 + k < 0$

$\Rightarrow \quad k - 32 < 0 \Rightarrow k < 32 \quad (1)$

Solving $y = 0$, $x^2 + y^2 - 6x - 10y + k = 0$

We get $x^2 - 6x + k = 0$,

which must have imaginary roots

$\therefore \quad$ Discriminant $= 36 - 4k < 0 \Rightarrow k > 9 \quad (2)$

Again, solving $x = 0$, $x^2 + y^2 - 6x - 10y + k = 0$,

we get $y^2 - 10y + k = 0$

which must have imaginary roots

$\therefore$ Discriminant $= 100 - 4k < 0 \Rightarrow k > 25 \quad (3)$

From (1), (2) and (3), we get $25 < k < 32$.

107. Let equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Since it passes through $(2a, 0)$, so

$$4a^2 + 4ag + c = 0 \quad (1)$$

Also, its radical axis with $x^2 + y^2 = a^2$ is

$$2gx + 2fy + c + a^2 = 0$$

But the radical axis is $x = \frac{a}{2}$, so we get

$$\frac{2g}{1} = \frac{c + a}{-a/2} \text{ and } f = 0$$

or, $\quad ag + c + a = 0 \quad (2)$

From (1) and (2), we get g and c.

Hence, equation is $x^2 + y^2 - 2ax = 0$

108. Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$

Let the equation of given circle be $x^2 + y^2 = a^2$

The equation of chord of contact of tangents drawn from the point $P(x_1, y_1)$ to the given circle is

$$xx_1 + yy_1 = a^2$$

Since it passes through $Q(x_2, y_2)$

$\therefore \quad x_1x_2 + y_1y_2 = a^2 \quad (1)$

Now, $l_1 = \sqrt{x_1^2 + y_1^2 - a^2}$, $l_2 = \sqrt{x_2^2 + y_2^2 - a^2}$

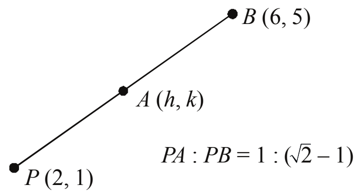
and, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2(x_1x_2 + y_1y_2)}$

$= \sqrt{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2a^2}$ [Using (1)]

$= \sqrt{(x_1^2 + y_1^2 - a^2) + (x_2^2 + y_2^2 - a^2)} = \sqrt{l_1^2 + l_2^2}$.

109. Let $A\ (h, k)$ be the nearest point lying on the circle.



We have $PB = \sqrt{(6-2)^2 + (5-1)^2} = 4\sqrt{2}$ and $PA = 4$ (given)

Therefore, $AB = PB - PA = 4(\sqrt{2} - 1)$

Thus, $\frac{AB}{AP} = \frac{\sqrt{2} - 1}{1}$

Hence, $h = \frac{6 + 2(\sqrt{2} - 1)}{1 + (\sqrt{2} - 1)} = \frac{2\sqrt{2} + 4}{\sqrt{2}} = 2 + 2\sqrt{2}$ and

$$k = \frac{5 + (\sqrt{2} - 1)}{1 + (\sqrt{2} - 1)} = \frac{\sqrt{2} + 4}{\sqrt{2}} = 1 + 2\sqrt{2}$$

The required circle has AB as its diameter.

Hence, its equation is

$$(x - 6)(x - 2 - 2\sqrt{2}) + (y - 5)(y - 1 - 2\sqrt{2}) = 0$$

Previous Year's Questions

110. Equation of circle is

$x^2 + y^2 - 4x - 2y - 20 = 0.$

whose centre is $C(2, 1)$ and radius is 5 units.

Since $S_1 = 10^2 + 7^2 - 4 \times 10 - 2 \times 7 - 20 > 0$

So P lies outside the circle.

Now, $PC = \sqrt{(2-10)^2 + (1-7)^2}$

$= \sqrt{8^2 + 6^2} = \sqrt{10^2} = 10$

$\therefore$ Greatest distance between circle and the point $P = 10 + 5 = 15$ unit.

111. Equation of circle is $x^2 + y^2 + 4x - 4y + 4 = 0$ with center at $(-2, 2)$. and radias 2. The equation of tangent is $x + y = 2\sqrt{2}$.

112. $(x - 1)^2 + (y - 3)^2 = r^2$

$(x - 4)^2 + (y + 1)^2 - 16 - 1 + 8 = 0$

$(x - 4)^2 + (y + 2)^2 = 9.$

Distance between their centers = 5

If the circle with center at (1, 3) touches the second circle then the radius of the first circle = 2.

But it is given that they touch at two distinct points so $r > 2$ as long as $r < 8$.

Again when $r = 8$, they intersect at only one point and for $r > 8$, no intersection point can be found.

113. Intersection of diameters is the point $(1, -1)$

Also, $\pi s^2 = 154$

$\Rightarrow s^2 = 49$

$\therefore$ Equation becomes $(x - 1)^2 + (y + 1)^2 = 49$

114. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ then $c = 4$ and it passes through (a, b) $\Rightarrow$ $a^2 + b^2 + 2ga + 2fb + 4 = 0$.

Hence locus of the centre is $2ax + 2by - (a^2 + b^2 + 4) = 0$.

115. Let the other end of diameter is (h, k) then equation of circle will be $(x - h)(x - p) + (y - k)(y - q) = 0$ Put $y = 0$ (since the circle touches the x-axis)

$\Rightarrow$ $x^2 - (h + p)x + (hp + kq) = 0$ $\Rightarrow$ $(h + p)^2 = 4(hp + kq)$ $(D = 0)$

$\Rightarrow$ $(x - p)^2 = 4qy$.

116. The intersection of the given lines is the centre of the circle i.e., $(1, -1)$

And circumference = 10π $\Rightarrow$ radius $r = 5$

$\Rightarrow$ Equation of circle is $x^2 + y^2 - 2x + 2y - 23 = 0$.

117. Points of intersection of line $y = x$ with $x^2 + y^2 - 2x = 0$ are (0, 0) and (1, 1).

Hence equation of circle having end points of diameter (0, 0) and (1, 1) is $x^2 + y^2 - x - y = 0$.

118. Let $S_1 : x^2 + y^2 + 2ax + cy + a = 0$

$S_2 : x^2 + y^2 - 3ax + dy - 1 = 0$

Equation of radical axis of S_1 and S_2 is $S_1 - S_2 = 0$

$\Rightarrow$ $5ax + (c - d)y + a + 1 = 0$

Given that $5x + by - a = 0$ passes through P and Q

$\Rightarrow$ $\frac{a}{1} = \frac{c-d}{b} = \frac{a+1}{-a}$

$\Rightarrow$ $a + 1 = -a^2$

$\Rightarrow$ $a^2 + a + 1 = 0$

No real value of a.

119. Equation of circle with centre (0, 3) and radius 2 is $x^2 + (y - 3)^2 = 4$.

Let locus of the variable circle be (α, β)

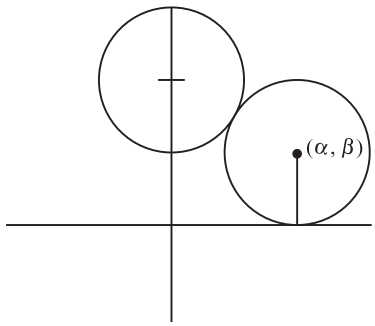
$\because$ It touches x-axis.

$\therefore$ Its equation is $(x - \alpha)^2 + (y - \beta)^2 = b^2$

Circles touch externally

$\therefore$ $\sqrt{\alpha^2 + (\beta - 3)^2} = 2 + \beta$

$\therefore$ Locus is $x^2 = 10\left(y - \frac{1}{2}\right)$ which is a parabola.



120. Let the centre be (α, β)

$\because$ It cuts the circle $x^2 + y^2 = p^2$ orthogonally

We write $2(-\alpha) \times 0 + 2(-\beta) \times 0 = c_1 - p^2$

$\Rightarrow$ $c_1 = p^2$

Let equation of circle be $x^2 + y^2 - 2\alpha x - 2\beta y + p^2 = 0$

It passes through (a, b) therefore $a^2 + b^2 - 2\alpha a - 2\beta b + p^2 = 0$

$\therefore$ The locus is $2ax + 2by - (a^2 + b^2 + p^2) = 0$.

121. The point of intersection of $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ is $(1, -1)$, which is the centre of the circle and radius of the circle = 7.

$\therefore$ Equation is $(x - 1)^2 + (y + 1)^2 = 49 \Rightarrow x^2 + y^2 - 2x + 2y - 47 = 0$.

122. The locus (h, k) is given by

$$\cos\frac{\pi}{3} = \frac{\sqrt{h^2 + k^2}}{3} \Rightarrow h^2 + k^2 = \frac{9}{4}$$

123. According to question

The equation of circle is $(x - h)^2 + (y - k)^2 = k^2$

Since, it passes through point $(-1, 1)$ we have

$(-1-h)^2 + (1-k)^2 = k^2$

$\Rightarrow \quad h^2 + 2h - 2k + 2 = 0$

Now, $D \geq 0$

$\Rightarrow \quad 2k - 1 \geq 0$

$\Rightarrow \quad k \geq \frac{1}{2}$

124. Centre $(-1, -2)$

Let (α, β) be the required point, then

$\frac{\alpha+1}{2} = -1$ and $\frac{\beta+0}{2} = -2$

125. Let the given circles be

$S: x^2 + y^2 + 3x + 7y + 2p - 5 = 0$

$S': x^2 + y^2 + 2x + 2y - p^2 = 0$

Now the equation of the required circle is $S + \lambda S' = 0$

As it passes through (1, 1), the value of $\lambda = \frac{-(7+2p)}{(6-p^2)}$

If $7 + 2p = 0$, it becomes the first circle

$\therefore$ it is true for all values of p

126. Given points $P(1, 0)$ and $Q(-1, 0)$.

Let $A = (x, y)$, then

$$\frac{AP}{AQ} = \frac{BP}{BQ} = \frac{CP}{CQ} = \frac{1}{3} \qquad (1)$$

$\Rightarrow \quad 3AP = AQ \Rightarrow 9AP^2 = AQ^2 \Rightarrow 9(x-1)^2 + 9y^2 = (x+1)^2 + y^2$

$\Rightarrow \quad 9x^2 - 18x + 9 + 9y^2 = x^2 + 2x + 1 + y^2 \Rightarrow 8x^2 - 20x + 8y^2 + 8 = 0$

$$\Rightarrow \quad x^2 + y^2 - \frac{5}{2}x + 1 = 0 \qquad (2)$$

$\therefore$ A lies on the circle

Similarly B, C are also lies on the same circle

$\therefore$ Circum centre of ABC = Centre of Circle (1) $= \left(\frac{5}{4}, 0\right)$

127. Circle $x^2 + y^2 - 4x - 8y - 5 = 0$

Centre = (2, 4), Radius $= \sqrt{4+16+5} = 5$

If circle is intersecting line $3x - 4y = m$ at two distinct points.

$\Rightarrow$ length of perpendicular from centre < radius

$\Rightarrow \quad \frac{|6-16-m|}{5} < 5$

$\Rightarrow \quad |10 + m| < 25$

$\Rightarrow \quad -25 < m + 10 < 25$

$\Rightarrow -35 < m < 15.$

128. $c_1 = \left(\frac{a}{2}, 0\right); c_2 = (0, 0)$

$r_1 = \frac{a}{2}; r_2 = c$

$c_1c_2 = r_1 - r_2 \quad \Rightarrow \quad \frac{a}{2} = c - \frac{a}{2} \quad \Rightarrow \quad c = a$

129. Let (h, k) be centre.

$(h-1)^2 + (k-0)^2 = k^2 \quad \Rightarrow \quad h = 1$

$(h-1)^2 + (k-3)^2 = k^2 \quad \Rightarrow \quad k = \frac{5}{3}$

$\therefore$ diameter is $2k = \frac{10}{3}$

130. Assume that the equation of circle be

$(x-3)^2 + y^2 + \lambda y = 0$

The circle passes through $(1, -2)$

$\Rightarrow \quad 4 + 4 - 2\lambda = 0 \quad \Rightarrow \quad \lambda = 4$

$(x-3)^2 + y^2 + 4y = 0 \quad \Rightarrow$ Clearly $(5, -2)$ satisfies.

131. According to the figure

$(1+y)^2 = (1-y)^2 + 1 \ (y > 0)$

$\Rightarrow \quad y = \frac{1}{4}$

132. $c_1(2, 3)$;

$r_1 = \sqrt{4+9+12} = 5$

And $c_2(-3, -9)$;

$r_2 = \sqrt{9+81-26} = 8$

$\therefore$ Distance $c_1c_2 = \sqrt{25+144} = 13$

$\therefore$ $c_1c_2 = r_1 + r_2$ touching externally.

$\Rightarrow$ 3 common tangents.

133.