CHAPTER 21

Vector Algebra

Chapter Highlights

Scalars and vectors, Representation of Vectors, Types of vectors, Equal Vectors, Fixed Vectors, Free Vectors, Angle between Two Vectors, Addition (sum or resultant) of Two Vectors, Position Vector, Component of a Vector, Linear Combination, Linearly Dependent and Independent System of Vectors, Collinearity of Three Points, Coplanarity of Four Points, Some Results on linearly dependent and independent Vectors, Product Of Two Vectors, Scalar Product of Two Vectors, Some Useful identities, Work done by a Force, Vector Product of Two Vectors, Moment of a Force about a point, Scalar Triple Product.

SCALARS AND VECTORS

Scalar Quantity A quantity which has only magnitude and no direction is called a *scalar quantity* or simply a *scalar*.

Examples of scalars are mass, temperature, volume, work and so on. To specify a scalar, two things are needed.

1. a unit in terms of which it is measured
2. a real number (+ve, –ve or zero)

Vector Quantity A quantity which has magnitude as well as direction is called a *vector quantity* or simply a *vector*.

Examples of vectors are displacement, velocity, acceleration, force and so on. To specify a vector, three things are needed.

1. a unit in terms of which it is measured
2. a real number (+ve, –ve or zero)
3. a particular direction

REPRESENTATION OF VECTORS

The best way to represent a vector is with the help of a directed line segment. Suppose A and B are two points, then by the vector $\vec{AB}$. , we mean a quantity whose magnitude is the length AB and whose direction is from A to B.

A and B are called the end points of the vector $\vec{AB}$. In particular A is called the initial point and B is called the terminal point.

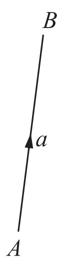


Fig 21.1

Sometimes a vector $\vec{AB}$. is expressed by a single letter **a** (which is always written in bold type, to distinguish it from a scalar). Sometimes, however, we write the vector a as $\vec{a}$ **or** $\bar{a}$ **.**

IMPORTANT POINTS

Every vector ($\vec{AB}$) has the following three characteristics:

(a) **Length:** The length of $\vec{AB}$ is denoted by $|\vec{AB}|$ or simply AB.

(b) **Support:** The line of unlimited length of which AB is segment is called the support of vector $\vec{AB}$.

(c) **Sense:** The sense of $\vec{AB}$ is from A to B and that of $\vec{BA}$ will be from B to A. The sense of directed line segment is from its initial point to the terminal point.

Modulus (or Magnitude) of a Vector

The positive real number which is the measure of the length of the vector, is called the *modulus, length, magnitude, absolute value* or *norm* of the vector.

The modulus of a vector a or OA is usually denoted by $| a |$ or $|\vec{OA}|$ or by the corresponding letter 'a' (not in bold-faced type), i.e.,

$$|\vec{OA}| = OA \text{ and } |a| = a$$

Multiplication of a Vector by a Scalar

The product of a scalar m and a vector a, is defined as a vector ma or am whose magnitude is the product of the magnitudes of m and a and whose direction is that of a or opposite to a accordingly as m is positive or negative.

TYPES OF VECTORS

Equal Vectors Two vectors a and b are equal when they have (1) the same magnitude and (2) the same direction. Symbolically such vectors are written as: $a = b$.

Unit Vectors A vector whose magnitude is unity is called a *unit vector*. The unit vector having the same direction as that of given vector a is usually denoted by the symbol $\hat{a}$ (read as 'a cap'), i.e.,

A vector = Modulus of vector × Unit vector in its direction

or $$a = |a|\,\hat{a}$$

Also, Unit vector in a direction

$$= \frac{\text{vector in that direction}}{\text{modulus of vector}}$$

$$\hat{a} = \frac{a}{|a|}$$

> **CAUTION**
>
> No units are to be attached with a unit vector, i.e., unit vector is dimensionless physical quantity

Zero or Null Vector A vector whose magnitude is zero, is called a *zero vector*. For such a vector, initial and terminal points are coincident so that its direction is indeterminate. A zero vector is denoted by the bold-faced symbol O or O.

Collinear (or Parallel Vectors) The vectors which are parallel to the same straight line are called *collinear vectors*.

Vectors which are not parallel to the same line are called *non-collinear vectors.*

Like and Unlike Vectors Collinear vectors having the same direction are called *like* vectors and those having the opposite directions are called *unlike* vectors.

Remark: If two vectors a and b are collinear, then there exists a scalar m such that $b = ma$, m being positive or negative according as a and b are like or unlike vectors.

Conversely, if $b = ma$ be given, then a and b must be collinear (or parallel) vectors such that $|b| = |m|\,|a|$.

Reciprocal Vector Let $|a|$ be the modulus of the given vector a. Then a vector whose direction is that of a but modulus is $1/|a|$ (reciprocal of the modulus of a) is called the reciprocal of a and is written as a^{-1}. Thus,

$$a^{-1} = \frac{1}{|a|}\hat{a} = \frac{|a|}{|a|^2}\hat{a} = \frac{a}{|a|^2}$$

Coplanar and Non-coplanar Vectors Three or more vectors are said to be *coplanar* when they are parallel to the same plane. Otherwise they are said to be *non-coplanar vectors*.

Co-initial Vectors The vectors which have the same initial point are called *co-initial vectors*.

Negative of a Vector A vector having the same modulus as that of a given vector a and the direction opposite to that of a, is called the negative of a and is denoted by $-a$. Clearly, if $OA = a$, then

$$AO = -a, \text{ and therefore, } OA = -AO$$

EQUAL VECTORS

Two vectors a and b are said to be equal when they have equal magnitudes and same direction. Geometrically, if head of one vector coincides with, the head of other and so do the tails coincide then the vectors are said to be equal.

> **CAUTION**
>
> If $a = b$, then $a = b$, always.
> But if, $a = b$ doesn't always imply $a = b$

FIXED VECTORS

Fixed vector is that vector whose initial point or tail is fixed. It is also called localised vector. For example, position vector and displacement vector are fixed vectors.

FREE VECTORS

Free vector is that whose initial point or tail is not fixed. It is also known as non localised vector, For example, velocity vector of a particular moving particle along a straight line is a free vector.

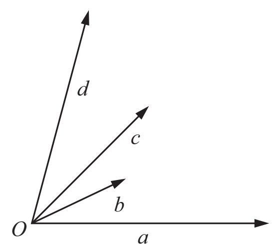


Fig. 21.2

ANGLE BETWEEN TWO VECTORS

The angle between two vectors a and b represented by OA and OB, is defined as the angle AOB which does not exceed π. This is also known as the *inclination of given vectors a* and b. If the angle AOB be θ, then $0 \le \theta \le \pi$.

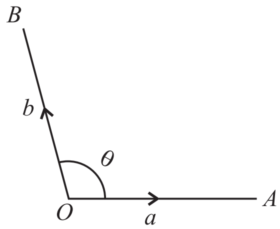


Fig 21.3

If $\theta = \frac{\pi}{2}$, then vectors are said to be *perpendicular* or *orthogonal* and if $\theta = 0$ or π, then vectors are said to be *parallel* or *coincident.*

IMPORTANT POINTS

Whenever finding angle between two vectors, make sure that either their heads coincide or their tails coincide.

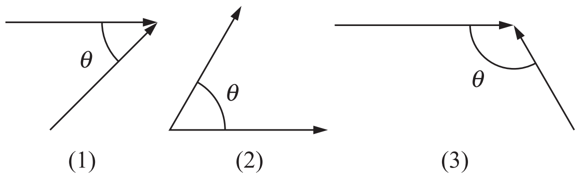


Fig 21.4

i.e., if heads coincide or tails coincide then internal angle is the angle between two vectors (whether acute or obtuse) as in (1), (2), (3) and (4).

If heads coincide with tall then external angle is the angle between the two vectors as in (5) and (6).

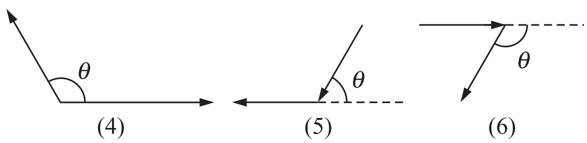


Fig 21.5

ADDITION (SUM OR RESULTANT) OF TWO VECTORS

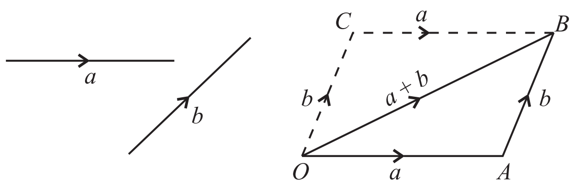


Fig 21.6

Let a, b be two vectors. Take any point O and draw the vectors $OA = a$ and $AB = b$ such that the terminal point of the vector a is the initial point of vector b. Join OB. Then the vector OB is defined as the sum of a and b and is written as

$$OB = OA + AB = a + b \qquad (1)$$

This method of addition of vectors is known as the **triangle law of addition**.

Completing the parallelogram $OABC$. Since

$$AB = OC = b$$

$$OB = OA + AB = OA + OC \qquad (2)$$

That is, the sum of two co-initial vectors is the vector represented by the diagonal of the parallelogram formed with the component vectors as adjacent sides.

This method of addition of vectors is known as the **parallelogram law of addition**.

Remark From Equation (1), $-BO = OA + AB$ or $OA + AB + BO = O$, showing that *the sum of vectors determined by the sides of a triangle, taken in order, is zero.*

Properties of Vector Addition

1. **Vector addition is commutative** For any two vectors a and b, we have
$$a + b = b + a$$
2. **Vector addition is associative** For any three vectors a, b and c, we have
$$(a + b) + c = a + (b + c)$$
3. **Existence of additive identity** For every vector a, we have
$$a + O = a = O + a$$
where O is the null vector.
4. **Existence of additive inverse** For a given vector a, there exists a vector $-a$ such that
$$a + (-a) = (-a) + a = 0$$
The vector $-a$ is called the additive inverse of a.

Properties of Multiplication of Vector by a Scalar

1. If $m = o$, then $ma = o$
2. If m and n be two scalars, then
$$m(na) = mna = n(ma)$$
3. If m and n be two scalars, then
$$(m + n)a = ma + na$$
4. If a, b are any two vectors and m be any scalar, then
$$m(a + b) = ma + mb$$

Subtraction (or Difference) of Two Vectors

Subtraction of vectors If a and b are two vectors, then their subtration $a - b$ is defined as $a - b = a + (-b)$ where $-b$ is the negative of b having magnitude equal to that of b and direction opposite to b.

Then $a - b = (a_1 - b_1)\, i + (a_2 - b_2)\, j + (a_3 - b_3)k$

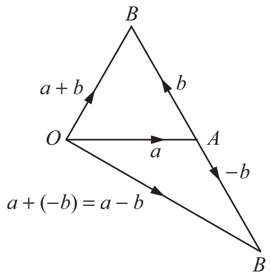


Fig 21.7

Properties of Vector Subraction

(i) $a - b \neq b - a$ (ii) $(a - b) - c \neq a - (b - c)$

(iii) Since any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors a and b, we have

(A) $|a + b| \leq |a| + |b|$
(B) $|a + b| \geq |a| - |b|$
(C) $|a - b| \leq |a| + |b|$
(D) $|a - b| \geq |a| - |b|$

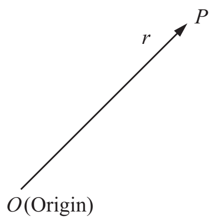


Fig 21.8

POSITION VECTOR

If a point O is fixed as the origin in space (or plane) and P is any point, the OP is called the position vector of P with respect to O.

If we say that P is the point r, then we mean that the position vector of P is r with respect to some origin O.

1. **AB in terms of the position vectors of points A and B:** If a and b are position vectors of points A and B respectively. Then $OA = a$, $OB = b$

$\therefore\ AB$ = (Position vector of B) – (Position vector of A)
$= OB - OA = b - a$

2. **Position vector of a dividing point** The position vectors of the points dividing the line AB in the ratio $m : n$ internally or externally are $\dfrac{mb + na}{m + n}$ or $\dfrac{mb - na}{m - n}$

TRICK(S) FOR PROBLEM SOLVING

- If P is the mid-point of AB, then it divides AB in the ratio 1 : 1. Thus position vector of P is given by

$$OP = \frac{a + b}{2}$$

- Since $OP = r = \dfrac{mb + na}{m + n} = \dfrac{m}{m + n} b + \dfrac{n}{m + n} a$
$= \lambda a + \mu b$

where $\lambda = \dfrac{n}{m + n}$ and $\mu = \dfrac{m}{m + n}$

Thus, p.v. of any point P on AB can always be taken as $r = \lambda a + \mu b$, where $\lambda + \mu = 1$.

- Since $OP = r = \dfrac{mb + na}{m + n}$

$\Rightarrow\ (m + n)\, r = mb + na$

$\Rightarrow\ n \cdot OA + m \cdot OB = (n + m)\, OP$,

where P is a point on AB dividing it in the ratio $m : n$.

In particular, if P is the mid-point of AB, then

$$OA + OB = 2OP$$

That is, twice the position vector of the middle point is equal to the sum of the vectors of the ends.

SOLVED EXAMPLES

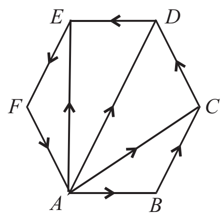
1. In a regular hexagon $ABCDEF$,

$AB + AC + AD + AE + AF = kAD$, where k is equal to

(A) 1 (B) 2
(C) 3 (D) none of these

Solution (C)

$AB + AC + AD + AE + AF$
$= ED + AC + AD + AE + CD$ ($\because AB = ED, AF = CD$)



$= (AC + CD) + (AE + ED) + AD$
$= AD + AD + AD = 3AD$

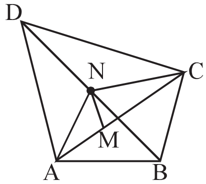
2. If M and N are the mid points of the diagonals AC and BD respectively of a quadrilateral $ABCD$, then
$AB + AD + CB + CD =$

(A) $4NM$ (B) $4MN$
(C) $2MN$ (D) none of these

Solution (B)

In ΔABD, N is the mid-point of BD,

$\therefore \quad AB + AD = 2AN \quad (1)$



In ΔCBD, N is the mid point of BD,

$\therefore \quad CB + CD = 2CN \quad (2)$

Adding (1) and (2), we have

$$AB + AD + CB + CD = 2(AN + CN) \quad (3)$$

In ΔANC, M is the mid-point of AC

$\therefore \quad AN + CN = 2MN$

From (3), we get

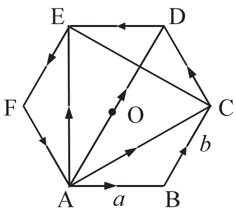
$$AB + AD + CB + CD = 2(2MN) = 4MN$$

3. Five forces AB, AC, AD, AE, AF act at the vertex A of a regular hexagon $ABCDEF$. If O is the centroid of the hexagon, then their resultant is a force given by

(A) $4AO$ (B) $5AO$
(C) $6AO$ (D) none of these

Solution (C)

If R is the resultant of given forces, then



$$\begin{aligned} R &= AB + AC + AD + AE + AF \\ &= ED + AC + AD + AE + CD \\ &\qquad (\because AB = ED \text{ and } AF = CD) \\ &= (AC + CD) + (AE + ED) + AD \\ &= AD + AD + AD = 3AD = 6AO. \end{aligned}$$

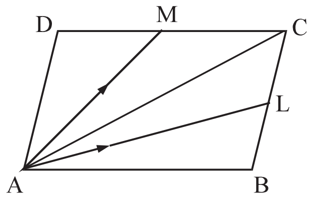
4. $ABCD$ is parallelogram. If L and M are the middle points of BC and CD, then $AL + AM =$

(A) $\frac{1}{2}AC$ (B) $\frac{3}{2}AC$
(C) AC (D) none of these

Solution (B)

$$AL = AB + BL = AB + \frac{1}{2}BC = AB + \frac{1}{2}AD$$

$$AM = AD + DM = AD + \frac{1}{2}DC = AD + \frac{1}{2}AB$$



$$\text{Adding, } AL + AM = \frac{3}{2}(AB + AD)$$

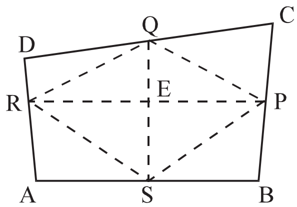
$$= \frac{3}{2}(AB + BS) = \frac{3}{2}AC$$

5. $ABCD$ is a quadrilateral and E the point of intersection of the lines joining the middle points of opposite sides. If O is any point, then the resultant of OA, OB, OC and OD is equal to

(A) $2OE$ (B) OE
(C) $4OE$ (D) none of these

Solution (C)

Let P, Q, R, S be the mid-points of sides BC, CD, DA, AB respectively of a quadrilateral $ABCD$. By geometry the figure formed by joining the mid-points P, Q, R, S will be a parallelogram. Hence, its diagonals will bisect each other, say at E.



Now, P is the mid-point of BC

$\therefore \quad OB + OC = 2OP \quad (1)$

And R is the mid-point of AD

$\therefore \quad OA + OD = 2OR \quad (2)$

Adding (1) and (2), we have

$$OA + OB + OC + OD = 2(OP + OR) = 2 \cdot 2OE = 4OE$$

$(\because E \text{ is mid-point of } PR, \therefore OP + OR = 2OE)$

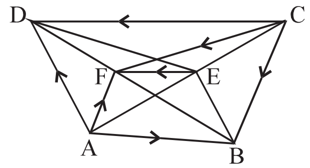
6. Two forces act at the vertex A of a quadrilateral $ABCD$ represented by AB, AD and two at C represented by CB and CD. If E and F are the middle points of AC and BD respectively, then their resultant is represented by

(A) EF (B) $2EF$
(C) $\frac{3}{2}EF$ (D) $4EF$

Solution (D)

We have,

$AB + AD = 2AF$, where F is mid-point of BD



Also, $CB + CD = 2CF$;

$\therefore \quad AB + AD + CB + CD$

$= 2(AF + CF) = -2(FA + FC) = -2\,[2FE]$,

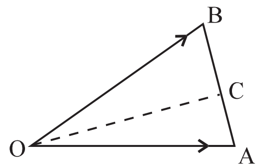
where E is the mid-point of $AC = -4FE = 4EF$.

7. Let $OA = i + 3j - 2k$ and $OB = 3i + j - 2k$. The vector OC bisecting the angle AOB where C is a point on the line AB is

(A) $2(i + j - k)$ (B) $4(i + j - k)$
(C) $i + j - k$ (D) none of these

Solution (A)

Taking O as origin, the position vectors of A and B are $a = i + 3j - 2k$ and $b = 3i + j - 2k$ respectively. We have $|a| = |b| = \sqrt{14}$



So, the bisector OC of $\angle AOB$ meets AB at its mid-point C.

$$\therefore \quad OC = \frac{1}{2}(OA + OB) = 2(i + j - k)$$

8. The vector c, directed along the internal bisector of the angle between the vectors $a = 7i - 4j - 4k$ and $b = -2i - j + 2k$ with $|c| = 5\sqrt{6}$ is

(A) $\frac{5}{3}(5i + 5j + 2k)$ (B) $\frac{5}{3}(i + 7j + 2k)$

(C) $\frac{5}{3}(-5i + 5j + 2k)$ (D) $\frac{5}{3}(i - 7j + 2k)$

Solution (D)

The required vector **c** is given by

$$c = \lambda(a + b) = \lambda\left(\frac{a}{|a|} + \frac{b}{|b|}\right)$$

$$= \lambda\left\{\frac{1}{9}(7i - 4j - 4k) + \frac{1}{3}(-2i - j + 2k)\right\}$$

or, $$c = \frac{\lambda}{9}(\hat{i} - 7j + 2k)$$

$$\Rightarrow \quad |c| = \pm\frac{\lambda}{9}\sqrt{1 + 49 + 4} = \pm\frac{\lambda}{9}\sqrt{54}.$$

But $|c| = 5\sqrt{6}$ (given)

$$\therefore \quad \pm\frac{\lambda}{9}\sqrt{54} = 5\sqrt{6} \quad \Rightarrow \quad \lambda = \pm 15.$$

Hence, $c = \pm\frac{15}{9}(i - 7j + 2k) = \pm\frac{5}{3}(i - 7j + 2k)$

9. If the points P, Q, R, S have position vectors p, q, r, s such that $p - q = 2(s - r)$, then

(A) PQ and RS bisect each other
(B) PQ and PR bisect each other
(C) PQ and RS trisect each other
(D) QS and PR trisect each other

Solution (D)

We have, $p - q = 2(s - r)$

$$\Rightarrow \quad p + 2r = q + 2s \quad \Rightarrow \quad \frac{p + 2r}{1 + 2} = \frac{q + 2s}{1 + 2}$$

$\therefore$ Point dividing PR in the ratio 2 : 1 is same as the point dividing QS in the ratio 2 : 1

Hence QS and PR trisect each other.

10. If a and b are position vectors of A and B respectively, then the position vector of a point C in AB produced such that $AC = 3AB$ is

(A) $3a - b$ (B) $3b - a$
(C) $3a - 2b$ (D) $3b - 2a$

Solution (D)

$AC = 3AB \quad \Rightarrow \quad c - a = 3(b - a) \quad \Rightarrow \quad c = 3b - 2a.$

11. A vector a has components $2p$ and 1 w.r.t a rectangular cartesian system. This system is rotated through a certain angle about the origin in the counter-clockwise sense. If w.r.t the new system, a has components $p + 1$ and 1, then

(A) $p = 0$ (B) $p = 1$ or $p = -\frac{1}{3}$

(C) $p = -1$ or $p = \frac{1}{3}$ (D) $p = 1$ or $p = -1$

Solution (B)

Let i, j be unit vectors along the co-ordinate axes

$\therefore \quad a = 2pi + 1 \cdot j \quad (1)$

On rotation, let b be the vector having components $p + 1$ and 1.

$\therefore \quad b = (p + 1)\,i + 1 \cdot j \quad (2)$

where i, j are unit vectors along the new co-ordinate axes.

But on rotation $|b| = |a| \quad \Rightarrow \quad |b|^2 = |a|^2$

$\Rightarrow \quad (p + 1)^2 + 1 = (2p)^2 + 1 \quad \Rightarrow \quad 3p^2 - 2p - 1 = 0$

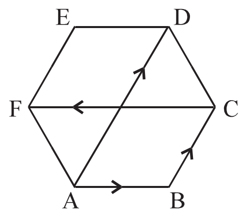
$$\Rightarrow \quad (3p + 1)(p - 1) = 0 \quad \Rightarrow \quad p = 1 \text{ or } -\frac{1}{3}$$

12. Let $ABCDEF$ be a regular hexagon. If $AD = xBC$ and $CF = yAB$, then $xy =$

(A) 4 (B) -4
(C) 2 (D) -2

Solution (B)

Since $ABCDEF$ is a regular hexagon, from plane geometry, we have



$AD = 2\ BC$ and $FC = 2AB$

$\therefore \quad AD = 2BC$ and $FC = 2AB \quad (1)$

Given that $AD = xBC$.

$\therefore \quad 2BC = xBC$, by (1)

$\Rightarrow \quad x = 2 \quad (2)$

Again, given that $CF = yAB$ or $-FC = yAB$.

$\therefore \quad -2AB = yAB$, using (2)

$\Rightarrow \quad y = -2 \quad (3)$

From (2) and (3), $xy = 2\,(-2) = -4$.

13. A vector a is collinear with vector $6i - 8j - \frac{15}{2}k$ of magnitude 50 making an obtuse angle with z-axis is

(A) $24i - 32j - 30k$ (B) $-24i + 32j + 30k$
(C) $24i + 32j - 30k$ (D) none of these

Solution (A)

Let $b = 6i - 8j - \frac{15}{2}k$.

A unit vector along b is $\pm\frac{2}{25}(6i - 8j - \frac{15}{2}k)$.

$\therefore \quad a =$ a vector of length 50 along b

$$= \pm 4\,(6i - 8j - \frac{15}{2}k).$$

Since a makes obtuse angle with z-axis, so we must have

$$a \cdot k < 0$$

Thus, $a = 24i - 32j - 30k$

14. Given a cube $ABCDA_1B_1C_1D_1$ with lower base $ABCD$, upper base $A_1B_1C_1D_1$ and the lateral edges AA_1, BB_1, CC_1, and DD_1; M and M_1 are the centres of the faces $ABCD$ and $A_1B_1C_1D_1$ respectively. O is a point on line MM_1, such that

$OA + OB + OC + OD = OM_1$, then $OM = \lambda.\ OM_1$ if $\lambda =$

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$
(C) $\frac{1}{6}$ (D) $\frac{1}{8}$

Solution (A)

$OM_1 = OA + OB + OC + OD$ (given)

$= OM + MA + OM + MB + OM + MC + OM + MD$

$= 4OM + (MA + MC) + (MB + MD)$

$= 4OM\ (\because\ MA = -MC,\ MB = -MD)$

$\therefore \quad OM = \frac{1}{4}OM_1 \quad \therefore \quad \lambda = \frac{1}{4}$

15. In a trapezoid the vector $BC = \lambda AD$ and $p = \mu AD$ and $p = AC + BD$ is collinear with AD. Then

(A) $\mu = \lambda + 1$ (B) $\lambda = \mu + 1$
(C) $\lambda + \mu = 1$ (D) $\mu = 2 + \lambda$

Solution (A)

We have,

$p = AC + BD = AC + BC + CD = AC + \lambda AD + CD$

$= (AC + CD) + \lambda AD = AD + \lambda AD = (1 + \lambda)AD$

Since $p = \mu AD \quad \therefore \quad \mu = 1 + \lambda$

16. $AB = 3i + j - k$ and $AC = i - j + 3k$. If the point P on the line segment BC is equidistant from AB and AC, then AP is

(A) $2i - k$ (B) $i - 2k$
(C) $2i + k$ (D) none of these

Solution (C)

A point equidistant from AB and AC is on the bisector of the angle BAC.

A vector along the internal bisector of the angle BAC

$$= \frac{AB}{|AB|} + \frac{AC}{|AC|}$$

$$= \frac{3i + j - k}{\sqrt{9+1+1}} + \frac{i - j + 3k}{\sqrt{1+1+9}} = \frac{1}{\sqrt{11}}(4i + 2k)$$

$\therefore \quad AP = t\,(2i + k)$

$\therefore \quad BP = AP - AB = t\,(2i + k) - (3i + j - k)$

$= (2t - 3)\,i - j + (t + 1)\,k$

Also $\quad BC = AC - AB = (i - j + 3k) - (3i + j - k)$

$= -2i - 2j + 4k.$

But $\quad BP = s\ BC.$

$\therefore \quad (2t - 3)i - j + (t + 1)\,k = s(-2i - 2j + 4k)$

$\therefore \quad 2t - 3 = -2s,\ -1 = -2s,\ t + 1 = 4s$

$\therefore \quad s = \frac{1}{2}$ and $t = 1 \quad \therefore \quad AP = 2i + k.$

17. The position vectors a, b, c and d of four distinct points A, B, C and D lie on a plane are such that $|a - d| = |b - d| = |c - d|$ then the point D is the

(A) centroid of ΔABC
(B) orthocentre of ΔABC
(C) circumcentre of ΔABC
(D) none of these

Solution (C)

We have, $a - d = -AD$, $b - d = -BD$ and $c - d = -CD$. According to the given condition $|AD| = |BD| = |CD|$. Thus, D is the circumcentre of ΔABC.

COMPONENT OF A VECTOR

The component of a vector PQ on a line l is RS, where R and S are the feet of perpendiculars from P and Q on the line l.

The vector component of PQ on l will be denoted by RS.

1. If θ is the angle between PQ and RS, then the component of PQ on $l = PQ \cos\theta = |PQ| \cos\theta$ and the vector component of PQ on $l = PQ \cos\theta$.
2. If r is the position vector of a point P, having coordinates (x, y, z) then $r = xi + yj + zk$, where i, j, k are unit vectors along x, y and z axes respectively.
3. xi, yj, zk are the vector components of r on x, y and z axes respectively.
4. If a point P in space has coordinates (x, y, z), then its $p.v.$ r is $xi + yj + zk$ and $|r| = |r| = \sqrt{x^2 + y^2 + z^2}$.

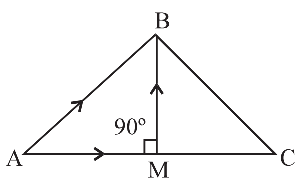
SOLVED EXAMPLE

18. The triangle ABC is defined by the vertices $A(1, -2, 2)$, $B(1, 4, 0)$ and $C(-4, 1, 1)$. Let M be the foot of the altitude drawn from the vertex B to side AC. Then BM =

(A) (–20/7, –30/7, 10/7) (B) (–20, –30, 10)
(C) (2, 3, –1) (D) none of these

Solution (A)

Since MB is the component of $AB \perp$ to AC



$MB = AB - AM$

$$= AB - \frac{(AB \cdot AC)AC}{(AC)^2}$$

$$= (6j - 2k) - \frac{\{(6j - 2k)\cdot(-5i + 3j - k)\}(-5i + 3j - k)}{(25 + 9 + 1)}$$

$$= (6j - 2k) - \frac{20}{35}(-5i + 3j - k) = \frac{10}{7}(2i + 3j - k).$$

$$\therefore \quad BM = -\frac{10}{7}(2i + 3j - k).$$

LINEAR COMBINATION

A vector r is said to be a linear combination of the given vectors a, b, c ..., and so on if there exist a system of scalars x, y, z, ..., and so on such that

$$r = xa + yb + zc + \ldots$$

LINEARLY DEPENDENT AND INDEPENDENT SYSTEM OF VECTORS

The system of n vectors $a_1, a_2, \ldots, a_n$ is said to be linearly dependent, if there exist scalars $x_1, x_2, \ldots, x_n$ not all zero such that

$$x_1a_1 + x_2a_2 + \ldots + x_na_n = 0$$

The same system of vectors is said to be linearly independent, if all scalers are zero, i.e. $x_1 = x_2 = \ldots = x_n = 0$.

REMARK

When the system of n vectors $a_1, a_2, \ldots, a_n$ is linearly dependent (or independent), then n vectors $a_1, a_2, \ldots, a_n$ are said to be linearly dependent (or independent).

COLLINEARITY OF THREE POINTS

The necessary and sufficient condition for three points with position vectors a, b and c to be collinear is that there exist three scalars x, y, z, not all zero, such that

$$xa + yb + zc = 0, \text{ where } x + y + z = 0$$

Test of Collinearity of Two Vectors To prove that two vectors a and b are collinear, find a scalar m such that one of the vectors is m times the other. In case no such scalar m exists, then the two vectors will be non-collinear vectors.

Test of Collinearity of Three Points

Method 1: To prove that three points A, B, C are collinear, find the vectors AB and AC and show that there exists a scalar m such that $AB = mAC$.

If no such scalar m exists, then the points are not collinear.

Method 2: To prove that three points A, B, C with position vectors a, b, c respectively are collinear, find three scalars x, y, z (not all zero) such that

$$xa + yb + zc = 0, \text{ where } x + y + z = 0$$

If no such scalars x, y, z exist, then the points are not collinear.

SOLVED EXAMPLES

19. If the vectors a and b are non-collinear, then the value of x, for which, the vectors $c = (x - 2)\, a + b$ and $d = (2x + 1)\, a - b$ are collinear is

(A) $\frac{4}{3}$ (B) $\frac{2}{3}$

(C) $\frac{1}{3}$ (D) none of these

Solution (C)

The vector c is non-zero since the coefficient in b is different from zero, and so the vectors c and d are collinear if for some number y we have

$d = yc$ that is

$(2x + 1)a - b = y\,(x - 2)a + yb$

or $(yx - 2y - 2x - 1)a + (y + 1)b = 0$

Since a, b are non-collinear, we must have

$$yx - 2y - 2x - 1 = 0 \text{ and } y + 1 = 0.$$

Solving these equations, we get $y = -1$ and $x = 1/3$

20. With reference to a right handed system of mutually perpendicular unit vectors i, j, k $\alpha = 3i - j$ and $\beta = 2i + j - 3k$

If $\beta = \beta_1 + \beta_2$, where β_1 is parallel to α and β_2 is perpendicular to α, then

(A) $\beta_1 = \frac{3}{2}i + \frac{1}{2}j$

(B) $\beta_1 = \frac{3}{2}i - \frac{1}{2}j$

(C) $\beta_2 = \frac{1}{2}i + \frac{3}{2}j - 3k$

(D) $\beta_2 = \frac{1}{2}i - \frac{3}{2}j - 3k$

Solution (B, C)

Since β_1 is parallel to α,

let $\beta_1 = \lambda\alpha$, where λ is a scalar.

$\beta = \beta_1 + \beta_2$ (Given)

$\therefore\ \beta_2 = \beta - \beta_1$

$= 2i + j - 3k - \lambda\alpha = 2i + j - 3k - \lambda(3i - j)$

$= (2 - 3\lambda)i + (1 + \lambda)i - 3k$

Since β_2 is perpendicular to α

$\therefore\ \beta_2 \cdot a = 0$

$\Rightarrow\ [(2 - 3\lambda)i + (1 + \lambda)i - 3i] \cdot (3i - i) = 0$

$\Rightarrow\ 3(2 - 3\lambda) - (1 + \lambda) = 0$

$$\Rightarrow\ 6 - 9\lambda - 1 - \lambda = 0 \ \Rightarrow\ 5 - 10\lambda = 0, \ \therefore\ \lambda = \frac{1}{2}$$

$$\therefore\ \beta_1 = \lambda\alpha = \frac{1}{2}(3i - i) = \frac{3}{2}i - \frac{1}{2}j$$

$$\beta_2 = \left(2 - \frac{3}{2}\right)i + \left(1 + \frac{1}{2}\right)j - 3k = \frac{1}{2}i + \frac{3}{2}j - 3k$$

$\therefore$ $2i + j - 3k = \lambda\alpha + \beta_2$

Hence, $\beta = \beta_1 + \beta_2$

COPLANARITY OF FOUR POINTS

The necessary and sufficient condition for four points with position vectors a, b, c and d to be coplanar is that there exist scalars x, y, z and w, not all zero, such that

$$xa + yb + zc + wd = 0$$

where $x + y + z + w = 0$.

Test of Coplanarity of Three Vectors To prove that three vectors a, b and c are coplanar, express one of these vectors as the linear combination of the other two such as $c = xa + yb$.

Now, compare the coefficients from the two sides and find the values of x and y. If real values of scalars x and y exist, then the vectors are coplanar otherwise non-coplanar.

Test of Coplanarity of Four Points

Method 1: To prove that four points A, B, C and D are coplanar, find the vectors AB, AC and AD and show that these three vectors are coplanar.

Method 2: To prove that four points A, B, C and D with position vectors a, b, c and d respectively are coplanar, find four scalars x, y, z, w (not all zero) such that

$$xa + yb + zc + wd = 0 \text{ where } x + y + z + w = 0$$

If no such scalars x, y, z, w exist, then the points are non-coplanar.

SOME RESULTS ON LINEARLY DEPENDENT AND INDEPENDENT VECTORS

1. If a, b, c are non-coplanar vectors, then these are linearly independent and conversely if a, b, c are linearly independent, then they are non-coplanar.
2. If $a = a_1 i + a_2 j + a_3 k$, $b = b_1 i + b_2 j + b_3 k$ and $c = c_1 i + c_2 j + c_3 k$ are three linearly dependent vectors, then
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$
3. Let a, b, c be three non-coplanar vectors. Then, vectors $x_1 a + y_1 b + z_1 c$, $x_2 a + y_2 b + z_2 c$ and $x_3 a + y_3 b + z_3 c$ will be coplanar if

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

4. Two non-zero, non-collinear vectors are linearly independent.
5. Any two collinear vectors are linearly dependent.
6. Any three non-coplanar vectors are linearly independent.
7. Any three coplanar vectors are linearly dependent.
8. Any four vectors in 3-dimensional space are linearly dependent.

PRODUCT OF TWO VECTORS

Between two vectors, two distinct kinds of products are defined. One being a pure number is called the *scalar product* while the other being a vector quantity is called the *vector product*.

SCALAR PRODUCT OF TWO VECTORS

The scalar product or dot product of two vectors a and b is defined as: $|a||b|\cos\theta$, where θ is the angle between them such that $0 \le \theta \le \pi$. It is denoted by placing a dot between the vectors a and b. Thus,

$$a \cdot b = |a||b|\cos\theta$$

If either a or b is O, we define $a \cdot b = O$.

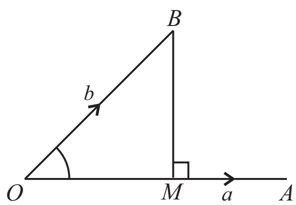


Fig 21.9

Key Results on Scalar Product

1. Scalar product is commutative. For any two vectors a and b we have
$$a \cdot b = b \cdot a.$$
2. If m is any scalar and a, b be any two vectors, then $(ma) \cdot b = m(a \cdot b) = a \cdot (mb)$
3. Scalar product is distributive with respect to vector addition, i.e., for any three vectors a, b and c, we have
$$a \cdot (b + c) = a \cdot b + a \cdot c.$$
4. Magnitude of a vector as a scalar product: For any vector a
$$a \cdot a = |a|^2 = a^2$$
5. Scalar product of two perpendicular vectors is zero, i.e., if a and b are two perpendicular vectors, then $a \cdot b = 0$.
   However, if $a \cdot b = 0 \Rightarrow$ Either $a = 0$ or $b = 0$ or $a \perp b$.
6. Scalar product of mutually orthogonal unit vectors i, j, k:
$$i \cdot i = 1 = j \cdot j = k \cdot k$$
   and $$i \cdot j = j \cdot k = k \cdot i = 0$$
7. Scalar porduct of two vectors in terms of components: If
   $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$, then $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$
   Thus, the scalar product of two vectors is equal to the sum of the products of their corresponding components.
8. Angle between two vectors in terms of the components of the given vectors.
   If θ is the angle between two vectors
   $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$, then
$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

IMPORTANT POINTS

If θ is acute, $a.b$ is positive and if θ is obtuse, $a.b$ is negative

9. Components of a vector b along and perpendicular to vector a
   Component of b along $a = \left(\dfrac{a \cdot b}{|a|^2}\right) a$
   Component of b perpendicular to $a = b - \left(\dfrac{a \cdot b}{|a|^2}\right) a$
10. Any vector r can be expressed as:
$$r = (r \cdot i)i + (r \cdot j)j + (r \cdot k)k$$

SOME USEFUL IDENTITIES

Since scalar product satisfies commutative and distributive laws, we have

1. $(a + b)^2 = a^2 + b^2 + 2a \cdot b$
2. $(a - b)^2 = a^2 + b^2 - 2a \cdot b$
3. $(a + b) \cdot (a - b) = a^2 - b^2$

WORK DONE BY A FORCE

Work done by a force F in displacing a particle from A to B is defined by

$$W = F \cdot AB$$

IMPORTANT POINTS

- $(a + b + c)^2 = |a|^2 + |b|^2 + |c|^2 + 2|(a \cdot b + b \cdot c + c \cdot a)$
- $|\hat{i} + \hat{j} + \hat{k}| = \sqrt{3}$
- Cauchy - Schwarz Inequality
  $(a \cdot b) \leq |a|^2|b|^2$
- If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force

SOLVED EXAMPLES

21. If $\vec{a}$ and $\vec{b}$ are unit vectors, then the greatest value of $|\vec{a}+\vec{b}|+|\vec{a}-\vec{b}|$ is

(A) $2\sqrt{2}$ (B) $\sqrt{2}$
(C) 2 (D) $4\sqrt{2}$

Solution (A)

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then, $\vec{a}.\vec{b} = \cos\theta$

Now, $|\vec{a}+\vec{b}| = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = 2 + 2\cos\theta$

$$= 4\cos^2\frac{\theta}{2}$$

$\Rightarrow \quad |\vec{a}+\vec{b}| = 2\cos\frac{\theta}{2}.$

Similarly, $|\vec{a}-\vec{b}| = 2\sin\frac{\theta}{2}$

$\therefore \quad |\vec{a}+\vec{b}|+|\vec{a}-\vec{b}| = 2\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right) \leq 2\sqrt{2}$

22. If $a + b + c = 0$ and $|a| = 3$, $|b| = 5$, $|c| = 7$, then the angle between a and b is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$
(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

Solution (B)

Let θ be the angle between a and b

$\therefore \quad a \cdot b = |a|\,|b| \cos\theta$

$\therefore \quad \cos\theta = \frac{a \cdot b}{|a||b|} = \frac{a \cdot b}{(3)(5)} = \frac{a \cdot b}{15} \qquad (1)$

Now $\quad a + b + c = 0$

$\therefore \quad a + b = -c$

$\Rightarrow \quad |a + b| = |-c| = |c| \Rightarrow |a + b|^2 = |c|^2$

$\Rightarrow \quad (a + b) \cdot (a + b) = (7)^2 \Rightarrow |a|^2 + 2a \cdot b + |b|^2 = 49$

$\Rightarrow \quad (3)^2 + 2a \cdot b + (5)^2 = 49$

$\Rightarrow \quad 2a \times b = 49 - 9 - 25 = 15 \quad \Rightarrow \quad a \times b = \frac{15}{2}$

$\therefore$ From (1), we get

$$\cos\theta = \frac{15}{2\times 15} = \frac{1}{2} = \cos 60° \quad \Rightarrow \quad \theta = 60°$$

23. $\vec{a}$ and $\vec{c}$ are unit vectors and $|\vec{b}| = 4$. The angle between $\vec{a}$ and $\vec{c}$ is $\cos^{-1}\left(\frac{1}{4}\right)$. If $\vec{b} - 2\vec{c} = \lambda\vec{a}$, then λ is equal to

(A) 3, 4 (B) –3, 4
(C) 3, –4 (D) $\frac{1}{4}, \frac{3}{4}$

Solution (C)

Given: $|\vec{a}| = 1, |\vec{c}| = 1$ and $|\vec{b}| = 4$.

$\Rightarrow \quad |\vec{a}\cdot\vec{c}| = 1\cdot 1\cdot\frac{1}{4} = \frac{1}{4}$

Now, $\vec{b} - 2\vec{c} = \lambda\vec{a} \quad \Rightarrow \quad \vec{a}\cdot\vec{b} - 2\vec{a}\cdot\vec{c} = \lambda a^2$

$\Rightarrow \quad \vec{a}\cdot\vec{b} - 2\cdot\frac{1}{4} = \lambda$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = \lambda + \frac{1}{2}$

Again, $\vec{b} - 2\vec{c} = \lambda\vec{a} \quad \Rightarrow \quad \vec{b}\cdot\vec{b} - 2\vec{b}\cdot\vec{c} = \lambda\vec{b}\cdot\vec{a}$

$\Rightarrow \quad 16 - 2\vec{b}\cdot\vec{c} = \lambda\left(\lambda + \frac{1}{2}\right)$

$\Rightarrow \quad \vec{b}\cdot\vec{c} = 8 - \frac{\lambda^2}{2} - \frac{\lambda}{4}$

Also, $\vec{b} - 2\vec{c} = \lambda\vec{a} \quad \Rightarrow \quad \vec{b}\cdot\vec{c} - 2\vec{c}\cdot\vec{c} = \lambda\vec{b}\cdot\vec{a}$

$\Rightarrow \quad 8 - \frac{\lambda^2}{2} - \frac{\lambda}{4} - 2(1) = \lambda\left(\frac{1}{4}\right)$

$\Rightarrow \quad \lambda^2 + \lambda - 12 = 0$

$\therefore \quad \lambda = -4, 3$

24. The vectors $a = xi - 3j - k$ and $b = 2xi + xj - k$ include an acute angle and b and positive y-axis include an obtuse angle. Then values of x may be

(A) –2 (B) –3
(C) all $x < 0$ (D) all $x > 0$

Solution (A, B, C)

According to the question

$a \cdot b > 0$ and $b \cdot j < 0 \Rightarrow 2x^2 - 3x + 1 > 0$ and $x < 0$

$\Rightarrow \quad (2x - 1)(x - 1) > 0$ and $x < 0$

i.e., $\left(x < \frac{1}{2} \text{ or } x > 1\right)$ and $x < 0 \quad \Rightarrow \quad x < 0$

25. If the unit vectors a and b are inclined at angle $2\theta\,(0 \leq \theta \leq \pi)$ and $|a - b| < 1$, then θ lies in the interval

(A) $\left[0, \frac{\pi}{6}\right)$ (B) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

(C) $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$ (D) none of these

Solution (A)

$a \cdot b = |a||b| \cos 2\theta$

$\Rightarrow a \cdot b = (1)(1) \cos 2\theta = \cos 2\theta.$

$|a - b| < 1$

$\Rightarrow a^2 + b^2 - 2a \cdot b < 1 \Rightarrow 1 + 1 - 2\cos\theta < 1$

$\Rightarrow 2(1 - \cos 2\theta) < 1 \Rightarrow 2(2\sin^2\theta) < 1$

$\Rightarrow \sin^2\theta < \frac{1}{4} \Rightarrow \theta$ lies in $\left[0, \frac{\pi}{6}\right)$

26. If a, b, c are three vectors such that $|a| = 3$, $|b| = 4$, $|c| = 5$ and a, b, c are perpendicular to $b + c$, $c + a$, $a + b$ respectively, then $|a + b + c| =$

(A) $6\sqrt{2}$ (B) $4\sqrt{2}$

(C) $3\sqrt{2}$ (D) $5\sqrt{2}$

Solution (D)

$\because a \perp (b + c)$

$\therefore \quad a \cdot (b + c) = 0 \Rightarrow a \cdot b + a \cdot c = 0$ (1)

Similarly $b \perp (c + a) \Rightarrow b \cdot c + b \cdot a = 0$ (2)

and $c \perp (a + b) = 0 \Rightarrow c \cdot a + c \cdot b = 0$ (3)

Adding (1), (2), (3), we get

$$2(a \cdot b + b \cdot c + c \cdot a) = 0$$

Now, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(a \cdot b + b \cdot c + c \cdot a)$

$= |a|^2 + |b|^2 + |c|^2 + 0$

$= (3)^2 + (4)^2 + (c)^2 = 50$

Hence, $|a + b + c| = 5\sqrt{2}$

27. A parallelogram is constructed on the vector $a = 3p - q$ and $b = p + 3q$, given that $|p| = |q| = 2$ and the angle between p and q is $\frac{\pi}{3}$. The length of a diagonal is

(A) $4\sqrt{5}$ (B) $4\sqrt{3}$

(C) $4\sqrt{7}$ (D) none of these

Solution (B)

The diagonals of the parallelogram are represented by the vectors.

$a + b = (3p - q) + (p + 3q) = 4p + 2q$

and $a - b = (3p - q) - (p + 3q) = 2p - 4q$

Now, $|a + b|^2 = |4q + 2q|^2 = 16|p|^2 + 4|q|^2 + 16p \cdot q$

$$= 16(2)^2 + 4(2)^2 + 16(2)(2)\cos\frac{\pi}{3}$$

$$= 64 + 16 + 12 = 112 \quad \left(\because \cos\frac{\pi}{3} = \frac{1}{2}\right)$$

$\Rightarrow \quad |a + b| = \sqrt{112} = 4\sqrt{7}$

Similarly, $|a - b| = 4\sqrt{3}$

Hence the lengths of the diagonals are $4\sqrt{3}$ and $4\sqrt{3}$.

28. Let $a = 2i - j + k$, $b = i + 2j - k$ and $c = i + j - 2k$ be three vectors. A vector in the plane of b and c whose projection on a is of magnitude $\sqrt{\frac{2}{3}}$ is

(A) $2i + 3j - 3k$ (B) $2i + 3j + 3k$

(C) $-2i - j + 5k$ (D) $2i + j + 5k$

Solution (A, C)

Any vector in the plane of b and c is

$r = b + \lambda c = (1 + 2j + k) + \lambda(i + j - 2k)$

$= (1 + \lambda)i + (2 + \lambda)j + (-1 - 2\lambda)k$

Projection of r on a is $= \frac{r \cdot a}{|a|}$

$$= \frac{2(1+\lambda) - (2+\lambda) - (1+2\lambda)}{\sqrt{4+1+1}} = \frac{-\lambda - 1}{\sqrt{6}}$$

$$\therefore \quad \frac{-\lambda - 1}{\sqrt{6}} = \pm\sqrt{\frac{2}{3}} \Rightarrow -\lambda - 1 = \pm 2; \lambda = -3 \text{ or } 1$$

Hence, $r = -2i - j + 5k$ or $r = 2i + 3j - 3k$

29. If the moduli of vectors a, b, c are 3, 4, 5 are respectively and a and $b + c$, b and $c + a$, c and $a + b$ are mutually perpendicular then the modulus of $a + b + c$ is

(A) $\sqrt{12}$ (B) 12

(C) $5\sqrt{2}$ (D) 50

Solution (C)

According to the given condition, we have

$a \cdot (b + c) = 0$ (1)

$b \cdot (c + a) = 0$ (2)

$c \cdot (a + b) = 0$ (3)

Now adding (1), (2) and (3), we get

$2(a \cdot b + b \cdot c + c \cdot a) = 0 \quad (\because \; a \cdot b = b \cdot a \text{ etc.})$

Hence, $|a + b + c|^2$

$= a^2 + b^2 + c^2 + 2(a \cdot b + b \cdot c + c \cdot a)$

$= 3^2 + 4^2 + 5^2 = 9 + 16 + 25 = 50$

$\Rightarrow \quad |a + b + c| = \sqrt{50} = 5\sqrt{2}$

30. If $p = i - 2j + 3k$ and $q = 3i + j + 2k$, than a vector r which is linear combination of p and q and also perpendicular to q is

(A) $i + 5j - 4k$ (B) $i - 5j + 4k$

(C) $\frac{-1}{2}(i + 5j - 4k)$ (D) none of these

Solution (C)

We have, $r = p + \lambda q$

$\Rightarrow \ r \cdot q = p \cdot q + \lambda q \cdot q$

$\therefore \ 0 = 7 + 14\lambda$

$(\because \ p \cdot q = 3 - 2 + 6 = 7$ and $q \cdot q = 9 + 1 + 4 = 14)$

$$\Rightarrow \ \lambda = -\frac{7}{14} = -\frac{1}{2}. \quad \therefore \quad r = -\frac{1}{2}(i + 5j - 4k)$$

31. A unit vector in XY plane that makes an angle of 45º with the vector $i + j$ and an angle of 60º with the vector $3i - 4j$ is

(A) i (B) $\frac{i+j}{\sqrt{2}}$

(C) $\frac{i-j}{\sqrt{2}}$ (D) none of these

Solution (D)

Let the required unit vector in the x-y plane be

$$r = xi + yi, \quad \therefore \quad |r| = \sqrt{(x^2 + y^2)} = 1$$

or $x^2 + y^2 = 1$ (1)

Now the angle between r and vector $3i - 4j$ is 60º.

$$\cos 60° = \frac{(xi + yj)\cdot(3i - 4j)}{|xi + yj||3i - 4j|} \quad \Rightarrow \quad x + y = 1 \quad (2)$$

and $3x - 4y = \frac{5}{2}$ (3)

There exists no real values of x and y, satisfying equations (1), (2) and (3).

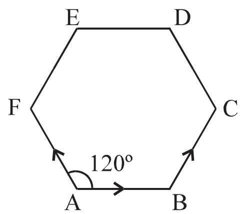
32. If $ABCDEF$ is a regular hexagon, then $AB \cdot AF$ is equal to

(A) $\frac{1}{2}BC^2$ (B) $-\frac{1}{2}BC^2$

(C) $\frac{1}{2}AC^2$ (D) $\frac{-1}{2}AC^2$

Solution (B)

Let a be the length of each side of the hexagon $ABCDEF$ so that $AB = AF = BC = a$. Also from plane geometry, $\angle BAF = 120°$.



Hence, we have

$$AB \cdot AF = |AB||AF|\cos 120°$$
$$= (AB)(AF)(-1/2)$$
$$= -(a^2/2) = -\frac{1}{2}BC^2.$$

33. Let a, b, c be the three vectors such that $a \cdot (b + c) = b \cdot (c + a) = 0$ and $|a| = 1, |b| = 4, |c| = 8$, then $|a + b + c| =$

(A) 13 (B) 81

(C) 9 (D) 5

Solution (C)

Adding $a \cdot (b + c) = b \cdot (c + a) = c \cdot (a + b) = 0$,

we get $2(a \cdot b + b \cdot c + c \cdot a) = 0$

$\Rightarrow \ |a + b + c|^2 = |a|^2 + |b|^2 + |c|^2 + 2(a\cdot b + b\cdot c + c\cdot a)$

$= 1 + 16 + 64 = 81$

Hence, $|a + b + c| = 9$

34. If e_1, e_2 are two unit vectors and θ is the angle between them, then $\cos\frac{\theta}{2}$ is

(A) $\frac{1}{2}|e_1 + e_2|$ (B) $\frac{1}{2}|e_1 - e_2|$

(C) $\frac{|e_1 e_2|}{2}$ (D) $\frac{|e_1 + e_2|}{2|e_1||e_2|}$

Solution (A)

$(e_1 + e_2)^2 = e_1 \cdot e_1 + 2e_1 \cdot e_2 + e_2 \cdot e_2$

$\Rightarrow \ |e_1 + e_2| = |e|^2 + 2|e_1||e_2|\cos\theta + |e_2|^2$

$\Rightarrow \ |e_1 + e_2|^2 = 1 + 2 \cdot 1 \cdot 1\cos\theta + 1$

$(\because \ |e_1| = |e_2| = 1)$

$$\Rightarrow \ |e_1 + e_2|^2 = 2(1 + \cos\theta) = 2\left(2\cos^2\frac{\theta}{2}\right)$$

$$\Rightarrow \ |e_1 + e_2|^2 = 4\cos^2\frac{\theta}{2} \ \Rightarrow \ \cos\frac{\theta}{2} = \frac{1}{2}|e_1 + e_2|$$

35. The points O, A, B, C, D are such that $OA = a$, $OB = b$, $OC = 2a + 3b$ and $OD = a - 2b$. If $|a| = 3|b|$, then the angle between BD and AC is

(A) π (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ (D) none of these

Solution (B)

$BD \cdot AC = |BD|\,|AC|\cos\theta$ where θ is the angle between BD and AC.

$\Rightarrow \ (OD - OB)\cdot(OC - OA)$

$= |OD - OB|\,|OC - OA|\cos\theta$

$\Rightarrow \ (a - 2b - b)\cdot(2a + 3b - a) = |a - 3b|\,|a + 3b|\cos\theta$

$\Rightarrow \ a^2 - 9b^2 = |a - 3b|\,|a + 3b|\cos\theta$

$\Rightarrow \ 0 = |a - 3b|\,|a + 3b|\cos\theta \ (\because \ |a| = 3|b|, \therefore \ a^2 = 9b^2)$

$$\Rightarrow \quad \cos\theta = 0, \quad \therefore \quad \theta = \frac{\pi}{2}$$

36. A, B, C, D are four points on a plane with position vectors a, b, c, d respectively such that $(a - d) \cdot (b - c) = (b - d) \cdot (c - a) = 0$. For ΔABC, D is the

(A) Incentre (B) orthocentre
(C) centroid (D) none of these

Solution (B)

Since $(a - d) \cdot (b - c), = 0 \quad \therefore \quad DA \cdot CB = 0$

$\therefore \quad AD \perp BC$

Since $(b - d) \cdot (c - a) = 0, \quad \therefore \quad DB \cdot AC = 0$

$\therefore \quad BD \perp CA$

Then D is the intersection of the altitudes through A and B. Therefore, D is the orthocentre of the triangle ABC.

37. Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}, \vec{b} = \hat{i} - \hat{j} + \hat{k}, \vec{i} = \hat{i} + \hat{j} - \hat{k}$. A vector coplanar to $\vec{a}$ and $\vec{b}$ has a projection along $\vec{c}$ of magnitude $\frac{1}{\sqrt{3}}$, then the vector is:

(A) $2\hat{i} + \hat{j} + 2\hat{k}$ (B) $4\hat{i} - \hat{j} + 4\hat{k}$
(C) $2\hat{i} - \hat{j} + 4\hat{k}$ (D) none of these

Solution (A, B)

Let vector $\vec{r}$ be coplanar to $\vec{a}$ and $\vec{b}$

$\therefore \quad \vec{r} = \vec{a} + t\vec{b}$

$\Rightarrow \quad \vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + t(\hat{i} - \hat{j} + \hat{k})$

$= \hat{i}(1+t) + \hat{j}(2-t) + \hat{k}(1+t)$

Then projection of $\vec{r}$ on $\vec{c} = \frac{1}{\sqrt{3}}$

$$\Rightarrow \quad \frac{\vec{r} \cdot \vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{|1 \cdot (1+t) + 1.(2-t) - 1.(1+t)|}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$\Rightarrow \quad |2 - t| = \pm 1 \quad \Rightarrow \quad t = 1$ or 3

When $t = 1$, we have $\vec{r} = 2\hat{i} + \hat{j} + 2\hat{k}$

When $t = 3$, we have $\vec{r} = 4\hat{i} - \hat{j} + 4\hat{k}$

38. The values of x for which the angle between the vectors $a = xi - 3j - k$ and $b = 2xi + xj - k$ is acute and the angle between the vector b and the y-axis lies between $\frac{\pi}{2}$ and π are

(A) $1, \frac{1}{2}$ (B) $0 < x < \frac{1}{2}$
(C) all $x < 0$ (D) $x < 0$ or $x > 1$

Solution (C)

$a \cdot b > 0$ and $b \cdot j < 0$.

$\Rightarrow \quad 2x^2 - 3x + 1 > 0$ and $x < 0$

$\Rightarrow \quad (2x - 1)(x - 1) > 0$ and $x < 0$

$\Rightarrow \quad (x < 1/2$, or $x > 1)$ and $x < 0$

Hence, $x < 0$ is the required solution.

39. A unit vector in xy plane that makes an angle of 45º with the vector $i + j$ and an angle of 60º with the vector $3i - 4j$ is

(A) i (B) $(i + j)/\sqrt{2}$
(C) $(i - j)/\sqrt{2}$ (D) none of these

Solution (D)

Let the required unit vector in the x-y plane be

$$r = xi + yj, \quad \therefore \quad |r| = \sqrt{(x^2 + y^2)} = 1$$

or $\qquad x^2 + y^2 = 1 \qquad (1)$

Since angle between r and vector $i + j$ is 45º and the angle between r and vector $3i - 4j$ is 60º.

$$\therefore \quad \cos 45° = \frac{(xi + yj) \cdot (i + j)}{|xi + yj||i + j|}$$

$$\text{and} \quad \cos 60° = \frac{(xi + yj) \cdot (3i - 4j)}{|xi + yj||3i - 4j|}$$

$\Rightarrow \qquad x + y = 1 \qquad (2)$

$\qquad 3x - 4y = 5/2 \qquad (3)$

No real values of x and y exist satisfying equations (1), (2), and (3).

VECTOR PRODUCT OF TWO VECTORS

The vector product or cross product of two vectors a and b is defined as

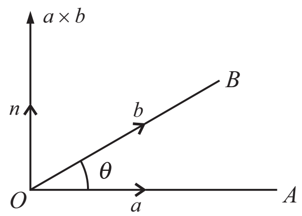


Fig. 21.10

$$a \times b = |a||b| \sin \theta \, n$$

1. $|a||b| \sin \theta$ is the modulus of $a \times b$, θ being the angle between the directions of a and b and $0 \le \theta \le \pi$;
2. direction of $a \times b$ is that of the unit vector n which is perpendicular to both a and b such that a, b and n form a right handed system.

REMEMBER

- By right handed system we mean that as the first vector a is turned towards the second vector b through an angle θ, n will point in the direction in which a right handed screw would advance if turned in a similar manner.

- If either a or b is O, we have $a \cdot b = O$.
- $|a \times b| = |a|\,|b| \sin\theta$
- A unit vector perpendicular to the plane of two given vectors a and b is given as $n = \dfrac{a \times b}{|a \times b|}$.
- $a \times b$ is perpendicular to the plane of a and b.

Key Results on Vector Product

1. Vector product is not commutative. For any two vectors a and b
$$a \times b \neq b \times a$$
In fact, $a \times b = -b \times a$
2. Vector product is associative with respect to a scalar. If m and n be any scalars and a, b any vectors, then
$$m(a \times b) = (ma) \times b = a \times (mb) = (a \times b)m;$$
$$(ma) \times (nb) = (na) \times (mb) = (mna) \times b = a \times (mnb) = mn(a \times b)$$
3. Vector product is distributive with respect to addition. For any three vectors a, b and c
$$a \times (b + c) = a \times b + a \times c$$
4. If two vectors a and b are parallel, then $a \times b = 0$. In particular, $a \times a = 0$.
5. Vector product of mutually orthogonal unit vectors i, j, k:
$$i \times i = j \times j = k \times k = O$$
and $i \times j = k = -j \times i$,
$$j \times k = i = -k \times j,\ k \times i = j = -i \times k.$$
6. Vector product in terms of components. Let
$$a = a_1 i + a_2 j + a_3 k$$
and $b = b_1 i + b_2 j + b_3 k$, then
$$a \times b = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$$
$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
7. Angle between two vectors: If θ is the angle between two vectors a and b, then $\sin\theta = \dfrac{|a \times b|}{|a||b|}$.

Geometrical Interpretation of Cross Product

1. The area of a parallelogram with adjacent sides a and b is given by $|a \times b|$.
2. The area of a triangle with adjacent sides a and b is given by $\frac{1}{2}|a \times b|$.
3. The area of a triangle ABC is
$$\frac{1}{2}|AB \times AC| \text{ or } \frac{1}{2}|BC \times BA| \text{ or } \frac{1}{2}|CB \times CA|$$
4. The area of a parallelogram with diagonals a and b is given by $\frac{1}{2}|a \times b|$.
5. The area of a quadrilateral $ABCD$ is given by $\frac{1}{2}|AC \times BD|$, where AC and BD are its diagonals.
6. Vector area of a ΔABC, when a, b, c are the position vectors of A, B, C respectively is given by
$$\Delta ABC = \frac{1}{2}(a \times b + b \times c + c \times a)$$

MOMENT OF A FORCE ABOUT A POINT

The vector moment or torque M of a force F acting at a point A about the point O is given by
$$M = r \times F = OA \times F$$
where $r = OA$ is the position vector of the point A with respect to the point O.

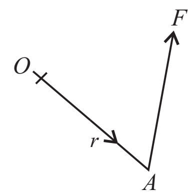


Fig 21.11

NOTE

The algebraic sum of the moments of a system of forces about any point is equal to the moment of their resultant about the same point.

TRICK(S) FOR PROBLEM SOLVING

Lagrange's Identity

If a and b are any two vectors, then
$$|a \times b|^2 + (a \cdot b)^2 = |a|^2\,|b|^2$$

SOLVED EXAMPLES

40. If $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors and $\vec{r}$ be any arbitrary vector, then

$$(\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})+(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})+(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})$$

is equal to

(A) 0 (B) $[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$

(C) $2[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$ (D) $3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$

Solution (C)

We have, $(\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})$

$$=\left((\vec{a}\times\vec{b})\cdot\vec{c}\right)\vec{r}-\left((\vec{a}\times\vec{b})\cdot\vec{r}\right)\vec{c}$$

$$=[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-[\vec{a}\,\vec{b}\,\vec{r}]\vec{c}$$

Similarly, $(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})=[\vec{b}\,\vec{c}\,\vec{a}]\vec{r}-[\vec{b}\,\vec{c}\,\vec{r}]\vec{c}$

and, $(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})=[\vec{c}\,\vec{a}\,\vec{b}]\vec{r}-[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}$

$\therefore\ (\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})+(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})+(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})$

$$=3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-([\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\vec{c})$$

$$=3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$$

$$=2[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$$

41. If $A = 2i + k$, $B = i + j + k$ and $C = 4i - 3j + 7k$, then a vector R which satisfies $R \times B = C \times B$ and $R \cdot A = O$, is

(A) $-i - 8j + 2k$ (B) $i - 8j + 2k$

(C) $i + 8j + 2k$ (D) none of these

Solution (A)

Let $R = xi + yj + zk$

$\therefore\ R\cdot A = 0 \Rightarrow 2x + z = 0$ (1)

$$R\times B = C\times B \Rightarrow \begin{vmatrix} i & j & k \\ x & y & z \\ 1 & 1 & 1\end{vmatrix} = \begin{vmatrix} i & j & k \\ 4 & -3 & 7 \\ 1 & 1 & 1\end{vmatrix}$$

$\Rightarrow\ (y - z)i + (z - x)j + (x - y)k = -10i + 3j + 7k$

$\Rightarrow\ y - z = -10$ (2)

$z - x = 3$ (3)

and $x - y = 7$

Solving (1) and (2), we get $x = -1$, $z = 2$

$\therefore$ From (2), $y = -8$.

Hence $R = -i - 8j + 2k$

42. Let $A(0, 0, 0)$, $B(1, 1, 1)$, $C(3, 2, 1)$ and $D(3, 1, 2)$ be four points. The angle between the planes through the points A, B, C and through the points A, B, D is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{3}$

Solution (D)

Let n_1 and n_2 be the vectors normal to the planes ABC and ABD respectively.

$$n_1 = AB \times AC = -i + 2j - k$$
$$n_2 = AB \times AD = i + j - 2k$$

Let θ be the acute angle between the planes, then θ is the acute angle between their normals n_1 and n_2

$$\therefore\ \cos\theta = \frac{|-1+2+2|}{\sqrt{6}\cdot\sqrt{6}} = \frac{3}{2} = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\Rightarrow\ = \frac{\pi}{3}$$

43. Given $A = ai + bj + ck$, $B = di + 3j + 4k$ and $C = 3i + j - 2k$. If the vectors A , B and C form a triangle such that $A = B + C$, then

(A) $a = -8, b = -4, c = 2, d = -11$

(B) $a = -8, b = 4, c = -2, d = -11$

(C) $a = -8, b = 4, c = 2, d = -11$

(D) none of these

Solution (C)

Here A, B, C are the vectors which represent the sides of the triangle ABC where

$$A = ai + bj + ck$$
$$B = di + 3j + 4k$$
$$C = 3i + j - 2k$$

Given that, $A = B + C$

$\therefore\ ai + bj + ck = (d + 3)i + 4j + 2k$

$\Rightarrow\ a = d + 3, b = 4, c = 2.$

$$B\times C = \begin{vmatrix} i & j & k \\ d & 3 & 4 \\ 3 & 1 & -2\end{vmatrix}$$

$$= -10i + (2d + 12)j + (d - 9)k$$

$$\therefore\ \text{Area of the } \Delta ABC = \frac{1}{2}|B\times C|$$

$$= \frac{1}{2}\sqrt{[100 + (2d+12)^2 + (d-9)^2]}$$

$$= 5\sqrt{6}\ \text{(Given)}$$

$$\Rightarrow\ \sqrt{(5d^2 + 30d + 325)} = 10\sqrt{6}$$

$\Rightarrow\ 5d^2 + 30d + 325 = 600 \Rightarrow 5d^2 + 30d - 275 = 0$

$\Rightarrow\ d^2 + 6d - 55 = 0 \Rightarrow (d + 11)(d - 5) = 0$

$\Rightarrow\ d = 5$ or -11

When $d = 5$, $a = 8$, $b = 4$, $c = 2$

and when $d = -11$, $a = -8$, $b = 4$, $c = 2$.

44. If $x + y = a$, $x \times y = b$ and $x \cdot a = 1$, then

(A) $x = \frac{a + a\times b}{a^2}$ (B) $y = \frac{(a^2-1)a - a\times b}{a^2}$

(C) $x = \frac{b + a\times b}{a^2}$ (D) $y = \frac{(b^2-1)b - a\times b}{a^2}$

Solution (A, B)

Given $x + y = a$

$\Rightarrow \quad y = a - x \quad (1)$

$x \times y = b \quad (2)$

$x \cdot a = 1 \quad (3)$

From (1) and (2), we get

$x \times (a - x) = b$

$\Rightarrow x \times a - x \times x = b \Rightarrow x \times a = b$

$\Rightarrow a \times (x \times a) = a \times b \Rightarrow (a \cdot a)\, x - (a \cdot x)\, a = a \times b$

$\Rightarrow |a|^2 x - 1 \cdot a = a \times b \quad$ [From (3)]

$$\Rightarrow \quad x = \frac{(a + a \times b)}{a^2}$$

and $y = a - x = \dfrac{(a^2 - 1)a - a \times b}{a^2}$

45. If $a \times (b \times c) + (a \cdot b)\, b = (4 - 2\beta - \sin\alpha)\, b + (\beta^2 - 1)\, c$ and $(c \cdot c)\, a = c$, while b and c are non-collinear, then

(A) $\alpha = \dfrac{\pi}{2}, \beta = -1$ (B) $\alpha = \dfrac{\pi}{2}, \beta = 1$

(C) $\alpha = \dfrac{\pi}{3}, \beta = -1$ (D) $\alpha = \dfrac{\pi}{3}, \beta = -1$

Solution (B)

We have, $a \times (b \times c) + (a \cdot b)\, b$

$= (4 - 2\beta - \sin\alpha)\, b + (b^2 - 1)c \quad (1)$

and $(c \cdot c)\, a = c \quad (2)$

where b and c are non-collinear vectors and α, β are scalars

From (2), $(c \cdot c)\, a \cdot c = c \cdot c$

$\therefore \quad a \cdot c = 1 \quad (3)$

From (1), we get

$(a \cdot c)b - (a \cdot b)c + (a \cdot b)b$

$= (4 - 2\beta - \sin\alpha)\, b + (\beta^2 - 1)c$

or $[1 + (a \cdot b)]\, b - (a \cdot b)\, c$

$= (4 - 2\beta - \sin\alpha)\, b + (\beta^2 - 1)\, c$

$\Rightarrow \quad 1 + (a \cdot b) = 4 - 2\beta - \sin\alpha \quad (4)$

and $a \cdot b = -(b^2 - 1) \quad (5)$

$\therefore \quad \sin\alpha = 1 + (1 - \beta)^2 \Rightarrow \beta = 1, \sin\alpha = 1$

i.e., $\alpha = \dfrac{\pi}{2} + 2n\pi, n \in I.$

46. If $u = a - b$, $v = a + b$ and $|a| = |b| = 2$, then $|u \times v|$ =

(A) $2\sqrt{[16 - (a \cdot b)^2]}$ (B) $\sqrt{[16 - (a \cdot b)^2]}$

(C) $2\sqrt{[4 - (a \cdot b)^2]}$ (D) $\sqrt{[4 - (a \cdot b)^2]}$

Solution (A)

$u \times v = (a - b) \times (a + b) = 2a \times b$

$\therefore \quad |u \times v| = 2|a \times b|$

$= 2\sqrt{[a^2b^2 \sin^2\theta]} = 2\sqrt{(a^2b^2 - a^2b^2\cos^2\theta)}$

$= 2\sqrt{[16 - (a \cdot b)^2]} \quad (\because \; |a| = |b| = 2)$

47. If $a \cdot i = 4$, then $(a \cdot j) \times (2j - 3k) =$

(A) 12 (B) 2

(C) 0 (D) −12

Solution (D)

We have, $(a \cdot j) \times (2j - 3k)$

$= a \cdot (j \times (2j - 3k)) = a \cdot (-3(j \times k)$

$= -3(a \cdot i) \quad (\because \; j \times k = i)$

$= -3(4) = -12.$

48. If $a \cdot b = \beta$ and $a \times b = c$, then b is equal to

(A) $(\beta a - a \times c)/a^2$ (B) $(\beta a + a \times c)/a^2$

(C) $(\beta c - a \times c)/a^2$ (D) $(\beta c + a \times c)/a^2$

Solution (A)

Here a and $c = a \times b$ are non-collinear vectors.

$\therefore \quad$ Let $b = xa + y(a \times c) \quad (1)$

$\therefore \quad \beta = a \cdot b = a \cdot [xa + y(a \times c)]$

$= x|a|^2 + ya \cdot (a \times c) = xa^2 \Rightarrow x = \beta/a^2$

And $\quad c = a \times b = a \times [xa + y(a \times c)]$

$= xa \times a + ya \times (a \times c)$

$= 0 + y(a \cdot c)\, a - y(a \cdot a)c$

$= y[a \cdot (a \times b)]a - ya^2c = -ya^2c$

$\Rightarrow \quad y = -1/a^2$

$\therefore$ from (1), $b = (\beta a - a \times c)/a^2$

49. Let $OA = a$, $OB = 10a + 2b$ and $OC = b$ where O, A, C are non-collinear. Let p denote the area of the quadrilateral $OABC$ and q denote the area of the parallelogram with OA and OC as adjacent sides. Then $\dfrac{p}{q}$ is equal to

(A) 4 (B) 6

(C) $\dfrac{1}{2}\dfrac{|a - b|}{|a|}$ (D) none of these

Solution (B)

Given, $|a \times b| = q$

and $\dfrac{1}{2}|b \times (10a + 2b)| + \dfrac{1}{2}|a \times (10a + 2b)| = p$

$\therefore \quad 5|b \times a| + |a \times b| = p$

$$\Rightarrow \quad 6|a \times b| = p \quad \Rightarrow \quad 6q = p \quad \Rightarrow \quad \frac{p}{q} = 6.$$

50. Two planes are perpendicular to one another. One of them contains a and b and the other contains vector c and d, then $(a \times b) \cdot (c \times d)$ is equal to

(A) 1 (B) 0

(C) $[a\, b\, c]\, d$ (D) $[b\, c\, d]\, a$

Solution (B)

$a \times b$ is a vector perpendicular to the plane of a and b and $c \times d$ is a vector perpendicular to the plane of c and d.

Since, these planes are $\perp$ to one another

$\therefore \quad (a \times b) \cdot (c \times d) = 0$

51. If the vectors c, $a = xi + yj + zk$ and $b = j$ are such that a, c and b form a right handed system then c is

(A) $zi - xk$ (B) 0

(C) yj (D) $-zi - xk$

Solution (A)

Since a, c, b form a right-handed system, so

$c = \lambda(b \times a), \lambda > 0$

i.e., $c = \lambda i \times (xi + yj + zk) = \lambda(zi - xk)$

Taking $\lambda = 1$, $c = zi - xk$

52. $ABCD$ is a quadrilateral with $AB = a$, $AD = b$ and $AC = 2\boldsymbol{a} + 3b$. If its area is α times the area of the parallelogram with AB, AD as adjacent sides, then α is equal to

(A) 5 (B) $\frac{5}{2}$

(C) 1 (D) $\frac{1}{2}$

Solution (B)

Area of quadrilateral $ABCD$

$= \text{Area of } \Delta ABC + \text{Area of } \Delta ACD$

$$= \frac{1}{2}[a \times (2a + 3b)] + \frac{1}{2}[(2a + 3b) \times b]$$

$$= \frac{1}{2}[3a \times b + 2a \times b] = \frac{5}{2} a \times b$$

$$= \frac{5}{2} \text{ Area of } \|_{gm} ABCD. \quad \therefore \quad \alpha = \frac{5}{2}$$

53. Consider a tetrahedron with faces F_1, F_2, F_3, F_4. Let V_1, V_2, V_3, V_4 be the vectors whose magnitudes are respectively equal to areas of F_1, F_2, F_3, F_4 and whose directions are perpendicular to their faces in outward direction. Then $| V_1 + V_2 + V_3 + V_4 |$ equals

(A) 1 (B) 4

(C) 0 (D) none of these

Solution (C)

We have,

$$v_1 = \frac{1}{2}(a \times b), v_2 = \frac{1}{2}(b \times c)$$

$$v_3 = \frac{1}{3}(c \times a) \text{ and } v_4 = \frac{1}{2}[(c - a) \times (b - a)]$$

$\therefore \quad v_1 + v_2 + v_3 + v_4 = 0$

$\therefore \quad | v_1 + v_2 + v_3 + v_4 | = 0$

SCALAR TRIPLE PRODUCT

Scalar triple product of three vectros If a, b, c are three vectors, then their scalar triple product is defined as the dot product of two vectors a and $b \times c$. It is generally denoted by $a \cdot (b \times c)$ or $[a\ b\ c]$.

Properties of Scalar Triple Product

(i) If a, b, c are cyclicallly permuted, the value of scalar triple product remains the same, i.e., $(a \times b).\ c = (b \times c) \cdot a = (c \times a).\ b$ or $[a\ b\ c] = [b\ c\ a] = [c\ a\ b]$

(ii) The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude i.e., $[a\ b\ c] = -[b\ a\ c] = -[c\ b\ a] = -[a\ c\ b]$

(iii) In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same i.e., $(a \times b) .\ c = a .\ (b \times c)$

(iv) The scalar triple product of three vectors is zero if any two of them are equal.

(v) For any three vectors a, b, c and scalar λ, $[\lambda a\ b\ c] = \lambda [a\ b\ c]$

(vi) The scalar triple product of three vectors is zero if any two of them are parallel or collinear.

(vii) If a, b, c, d are four vectors, then $[(a + b)\ c\ d] = [a\ c\ d] + [b\ c\ d]$

(viii) The necessary and sufficient condition for three non-zero non-collinear vectors a, b, c to be coplanar is that [a b c] = 0.

(ix) Four points with position vectors a, b, c and d will be coplanar, if $[a\ b\ c] + [d\ c\ a] + [d\ a\ b] = [\text{a b c}]$.

(x) Volume of parallelopiped whose coterminous edges are a, b, c is $[a\ b\ c]$ or $a \cdot (b \times c)$.

Scalar Triple Product in Terms of Components

(i) If $a = a_1 i + a_2 j + a_3 k$, $b = b_1 i + b_2 j + b_3 k$ and $c = c_1 i + c_2 j + c_2 j + c_3 k$ be three vectros

$$\text{then, } [a\ b\ c] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(ii) If $a = a_1 l + a_2 m + a_3 n$, $b = b_1 l + b_2 m + b_3 n$ and $c = c_1 l + c_2 m + c_3 n$, then

$$[a\ b\ c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [l\ m\ n]$$

(iii) For any three vectors a, b and c

(A) $[a + b\ b + c\ c + a] = 2[a\ b\ c]$

(B) $[a - b\ b - c\ c - a] = 0$

(C) $[a \times b\ b \times c\ c \times a] = [a\ b\ c]^2$

Tetrahedron A tetrahendron is a three-dimensional figure formed by four triangles. $OABC$ is a tetrahedron with ΔABC as the base. OA, OB, OC, AB, BC and CA are known as edges of the tetrahedron. OA, BC; OB, CA and OC, AB are known as the pairs of oppposite edges. A tetrahedron in which all edges are equal, is called a *regular tetrahedron*. Any two edges of regular tetrahedron are perpendicular to each other.

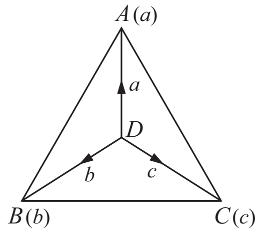


Fig. 21.12

Volume of Tetrahedron

(i) The volume of a tetrahendron

$$= \frac{1}{3}(\text{area of the base})(\text{corresponding altitude})$$

$$= \frac{1}{6}[AB\ BC\ AD]$$

(ii) If a, b, c are position vectors of vertices A, B and C with respect to O, then volume of tetrahedron $OABC = \frac{1}{6}[a\ b\ c]$

(iii) If a, b, c, d are position vectors of vertices A, B, C, D of a tetrahedron $ABCD$, then its volume

$$= \frac{1}{6}[b - a\ c - a\ d - a]$$

Reciprocal system of vectors Let a, b, c be three non-coplanar vectors, and let $a' = \frac{b \times c}{[a\,b\,c]}, b' = \frac{c \times a}{[a\,b\,c]}, c' = \frac{a \times b}{[a\,b\,c]}$. a', b', c' are said to form a reciprocal system of vectors for the vectors a, b, c.

If a, b, c and a', b', c' form a reciprocal system of vectors then

(i) $a, a' = b\ b' = c.c' = 1$

(ii) $a.\ b' = a.\ c' = 0$; $b.c' = b.a' = 0$; $c.\ a' = c.b' = 0$

(iii) $[a'b'c'] = \frac{1}{[a\,b\,c]}$

(iv) a, b, c are non-coplanar iff so are a', b', c'.

SOLVED EXAMPLES

54. If i, j, k are the unit vectors and mutually perpendicular, then $[i - j\ j - k\ k - i] =$

(A) 0 (B) 1
(C) −1 (D) none of these

Solution (A)

$(i - j) \cdot [(j - k) \times (k - i)]$
$= (i - j) \cdot (j \times k - j \times i - k \times k + k \times i)$
$= (i - j) \cdot (i + k - 0 + j)$
$= i \cdot i + i \cdot k + i \cdot j - j \cdot i - j \cdot k - j \cdot j$
$= i^2 + 0 + 0 - 0 - 0 - j^2 = i^2 - j^2 = 1 - 1 = 0.$

55. If e_1', e_2', e_3' are vectors reciprocal to the non-coplanar vectors e_1, e_2, e_3 then $[e_1', e_2', e_3'][e_1\ e_2\ e_3] =$

(A) $\frac{-1}{2}$ (B) 1
(C) 0 (D) 4

Solution (B)

Since $[e_1' e_2' e_3'] = \frac{1}{[e_1 e_2 e_3]}$

$\therefore\ [e'_1\ e'_2\ e'_3]\ [e_1\ e_2\ e_3] = 1$

56. If $r = \lambda(a \times b) + \mu(b \times c) + v(c \times a)$ and $[a\,b\,c] = \frac{1}{8}$, then $\lambda + \mu + v$ is equal to

(A) $r \cdot (a + b + c)$ (B) $8r \cdot (a + b + c)$
(C) $4r \cdot (a + b + c)$ (D) none of these

Solution (B)

Clearly $r \cdot c = \lambda[a\ b\ c] = \frac{1}{8}\lambda$

$r \cdot a = \mu\ [a\ b\ c] = \frac{1}{8}\mu$

$r \times b = v\ [a\ b\ c] = \frac{1}{8}v$

$\therefore\ r \cdot (a + b + c) = \frac{1}{8}(\lambda + \mu + v)$

$\therefore\ \lambda + \mu + v = 8r \cdot (a + b + c)$

Vector triple product

Let a, b, c be any three vectors, then the vectors $a \times (b \times c)$ and $(a \times b) \times c$ are called vector triple product of a, b, c.

Thus, $a \times (b \times c) = (a.c)\ b - (a.b)\ c$

Properties of Vector Triple Product

(i) The vector triple product $a \times (b \times c)$ is a linear combination of those two vectors which are within brackets.

(ii) The vector $r = a \times (b \times c)$ is perpendicular to a and lies in the plane of b and c.

(iii) The formula $a \times (b \times c) = (a\,.c)\,b - (a.b)\,c$ is true only when the vector outside the bracket is on the left most side. If it is not, we first shift on left by using the properties of cross product and then apply the same formula. Thus, $(b \times c) \times a = -\{a \times (b \times c)\} = -\{(a \cdot c)\,b - (a \cdot b)\,c\} = (a \cdot b)\,c - (a \cdot c)\,b$

(iv) Vector triple product is a vector quantity.

(v) $a \times (b \times c) \neq (a \times b) \times c$

Rotation of a Vector About an Axis

Let $a = (a_1, a_2, a_3)$. If system is rotated about

(i) x-axis through an angle α, then the new components of a are $(a_1, a_2 \cos\alpha + a_3 \sin\alpha, -a_2 \sin\alpha + a_3 \cos\alpha)$.

(ii) y-axis through an angle α, then the new components of a are $(-a_3 \sin\alpha + a_1 \cos\alpha, a_2, a_3 \cos\alpha + a_1 \sin\alpha)$.

(iii) z-axis through an angle α, then the new components of a are $(a_1 \cos\alpha + a_2 \sin\alpha, -a_1 \sin\alpha + a_2 \cos\alpha, a_3)$.

SOLVED EXAMPLES

57. Given three unit vectors a, b, c no two of which are collinear satisfying $a \times (b \times c) = \frac{1}{2}b$. The angle between a and b is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{2}$ (D) none of these

Solution (C)

We have, $a \times (b \times c) = \frac{1}{2}b$

$\Rightarrow (a \cdot c)\,b - (a \cdot b)\,c = 1/2b$

$\Rightarrow (a \cdot c - 1/2)\,b = (a \cdot b)\,c$

But since b and c are non-parallel, so the only possibility is $a \cdot c = 1/2$ and $a \cdot b = 0$ Hence the angle between a and b is $\pi/2$.

58. If a and b are two unit vectors, then the vector $(a + b) \times (a \times b)$ is parallel to the vector

(A) $a - b$ (B) $a + b$

(C) $2a - b$ (D) $2a + b$

Solution (A)

We have, $(a + b) \times (a \times b)$

$= a \times (a \times b) + b \times (a \times b)$

$= (a \cdot b)\,a - (a \cdot a)\,b + (b \cdot b)\,a - (b \cdot a)\,b$

$= (a \cdot b)(a - b) + a - b$ $(b \cdot b = b^2 = 1, a \cdot a = a^2 = 1$ as a, b are unit vectors)

$= (a \cdot b + 1)(a - b)$

$= x(a - b)$ where $x = a \cdot b + 1$ is a scalar.

$\therefore$ The given vector is parallel to $a - b$.

EXERCISES

Single Option Correct Type

1. a and b are mutually perpendicular unit vectors. If r is a vector satisfying $r \cdot a = 0$, $r \cdot b = 1$ and $[r\ a\ b] = 1$, then r is

(A) $a \times b + b$ (B) $a + (a \times b)$

(C) $b + (a \times b)$ (D) $a \times b + a$

2. a, b, c are three vectors of magnitude, $\sqrt{3}, 1, 2$ such that $a \times (a \times c) + 3b = O$. If θ is the angle between a and c, then $\cos^2\theta$ is equal to

(A) $\frac{1}{4}$ (B) $\frac{3}{4}$

(C) $\frac{1}{2}$ (D) none of these

3. A vector a has components $2p$ and 1 w.r.t. a rectangular cartesian system. This system is rotated through a certain angle about the origin in the counter-clockwise sense. If w.r.t. the new system, a has components $p + 1$ and 1, then

(A) $p = 0$ (B) $p = 1$ or $p = -\frac{1}{3}$

(C) $p = -1$ or $p = \frac{1}{3}$ (D) $p = 1$ or $p = -1$

4. If $a \cdot b = \beta$ and $a \times b = c$, then β is equal to

(A) $(\beta a - a \times c)/a^2$ (B) $(\beta a + a \times c)/a^2$

(C) $(\beta c - a \times c)/a^2$ (D) $(\beta c + a \times c)/a^2$

5. Let $ABCDEF$ be a regular hexagon. If $AD = x\,BC$ and $CF = y\,AB$, then $xy =$

(A) 4 (B) -4

(C) 2 (D) -2

6. Given a cube $ABCDA_1B_1C_1D_1$ with lower base $ABCD$, upper base $A_1B_1C_1D_1$ and the lateral edges AA_1, BB_1, CC_1 and DD_1; M and M_1 are the centres of the faces $ABCD$ and $A_1B_1C_1D_1$ respectively. O is a point on line MM_1, such that
$OA + OB + OC + OD = OM_1$, then $OM = \lambda\ OM_1$ if $\lambda =$

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$
(C) $\frac{1}{6}$ (D) $\frac{1}{8}$

7. The triangle ABC is defined by the vertices $A(1, -2, 2)$, $B(1, 4, 0)$ and $C(-4, 1, 1)$. Let M be the foot of the altitude drawn from the vertex B to side AC. Then, $BM =$

(A) $(-20/7, -30/7, 10/7)$ (B) $(-20, -30, 10)$
(C) $(2, 3, -1)$ (D) none of these

8. If $AB = 3i + j - k$ and $AC = i - j + 3k$. If the point P on the line segment BC is equidistant from AB and AC, then AP is

(A) $2i - k$ (B) $i - 2k$
(C) $2i + k$ (D) none of these

9. A, B, C, D are four points on a plane with position vectors a, b, c, d, respectively, such that $(a - d)\cdot(b - c) = (b - d)\cdot(c - a) = 0$. For ΔABC, D is the

(A) incentre (B) orthocentre
(C) centroid (D) none of these

10. If a and b are two unit vectors, then the vector $(a + b) \times (a \times b)$ is parallel to the vector

(A) $a - b$ (B) $a + b$
(C) $2a - b$ (D) $2a + b$

11. If $r = \lambda(a \times b) + \mu(b \times c) + \nu(c \times a)$ and $[a\ b\ c] = \frac{1}{8}$, then $\lambda + \mu + \nu$ is equal to

(A) $r \cdot (a + b + c)$ (B) $8r \cdot (a + b + c)$
(C) $4r \cdot (a + b + c)$ (D) none of these

12. In a parallelogram $ABCD$, $|AB| = a$, $|AD| = b$ and $|AC| = c$. Then, $DB \cdot AB$ has the value

(A) $\frac{3a^2 + b^2 - c^2}{2}$ (B) $\frac{a^2 + 3b^2 - c^2}{2}$
(C) $\frac{a^2 - b^2 + 3c^2}{2}$ (D) $\frac{a^2 + 3b^2 + c^2}{2}$

13. Let $a = a_1i + a_2j + a_3k$, $b = b_1i + b_2j + b_3k$, and $c = c_1i + c_2j + c_3k$ be three non-zero vectors such that c is a unit vector perpendicular to both vectors a and b. If the angle between vectors a and b is $\pi/6$, then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$$ is equal to

(A) 0
(B) 1
(C) $\frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$
(D) $\frac{3}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2)$
(E) none of these

14. If a, b, c are non-coplanar unit vectors such that $a \times (b \times c) = \frac{b + c}{\sqrt{2}}$, then the angle between a and b is

(A) $\frac{3\pi}{4}$ (B) $\frac{\pi}{4}$
(C) $\frac{\pi}{2}$ (D) π

15. If the vectors a and b are perpendicular to each other, then a vector v in terms of a and b satisfying the equations $v \cdot a = 0$, $v \cdot b = 1$ and $[v\ a\ b] = 1$ is

(A) $\frac{1}{|b|^2}b + \frac{1}{|a \times b|^2}a \times b$
(B) $\frac{b}{|b|} + \frac{a \times b}{|a \times b|^2}$
(C) $\frac{b}{|b|^2} + \frac{a \times b}{|a \times b|}$
(D) none of these

16. Let the unit vectors a and b be perpendicular to each other and the unit vector c be inclined at an angle θ to both a and b. If $c = xa + yb + z(a \times b)$, then

(A) $x = \cos\theta$, $y = \sin\theta$, $z = \cos 2\theta$
(B) $x = \sin\theta$, $y = \cos\theta$, $z = -\cos 2\theta$
(C) $x = y = \cos\theta$, $z^2 = \cos 2\theta$
(D) $x = y = \cos\theta$, $z^2 = -\cos 2\theta$

17. If S is the circumcentre, O is the orthocentre of ΔABC, then $SA + SB + SC =$

(A) SO (B) $2SO$
(C) OS (D) $2OS$

18. If a, c, d are non-coplanar vectors and $d \cdot \{a \times [b \times (c \times d)]\}$ is equal to

(A) $(b \cdot d)[a\ c\ d]$ (B) $(a \cdot d)[a\ c\ d]$
(C) $(c \cdot d)[a\ c\ d]$ (D) none of these

19. If $4a + 5b + 9c = 0$, then $(a \times b) \times [(b \times c) \times (c \times a)]$ is equal to

(A) A vector perpendicular to the plane of a, b and c
(B) A scalar quantity
(C) 0
(D) none of these

20. If $\sum_{i=1}^{n} a_i = 0$ where $|a_i| = 1 \forall i$, then the value of is $\sum_{1 \le i < j \le n} \sum a_i \cdot a_j$

(A) $-\frac{n}{2}$ (B) $-n$

(C) $\frac{n}{2}$ (D) n

21. Forces P, Q act at O and have a resultant R. If any transversal cuts their lines of action at A, B, C, respectively, then

(A) $\frac{P}{OA} + \frac{Q}{OB} + \frac{R}{OC} = 0$

(B) $\frac{P}{OA} + \frac{Q}{OB} + \frac{R}{OC} = 1$

(C) $\frac{P}{OA} + \frac{Q}{OB} - \frac{R}{OC} = 0$

(D) $\frac{P}{OA} + \frac{Q}{OB} - \frac{R}{OC} = 1.$

22. A vector A has components A_1, A_2, A_3 in a right-handed rectangular cartesian coordinate system Ox, Oy, Oz. The coordinate system is rotated about the z-axis through an angle $\frac{\pi}{2}$. The components of A in the new coordinate system are

(A) $A_1, -A_2, A_3$ (B) A_2, A_1, A_3

(C) $A_1, A_2, -A_3$ (D) $A_2, -A_1, A_3$.

23. In a $\Delta\ OAB$, E is the mid-point of OB and D is a point on AB such that $AD : DB = 2 : 1$. If OD and AE intersect at P, then the ratio $OP : PD$ is

(A) 1 : 2 (B) 2 : 1

(C) 3 : 2 (D) 2 : 3.

24. If a, b, c are three non-parallel unit vectors such that $a \times (b \times c) = \frac{1}{2} b$, then the angles which **a** makes with b and c are

(A) 90º, 60º (B) 45º, 60º

(C) 30º, 60º (D) none of these

25. If $a = i + j - k$, $b = i - j + k$ and c is a unit vector perpendicular to the vector a and coplanar with a and b, then a unit vector d perpendicular to both a and c is

(A) $\frac{1}{\sqrt{6}}(2i - j + k)$ (B) $\frac{1}{\sqrt{2}}(i + j)$

(C) $\frac{1}{\sqrt{2}}(j + k)$ (D) $\frac{1}{\sqrt{2}}(i + k)$

26. If the pth, qth and rth terms of a G. P. are positive numbers a, b and c, respectively, then the angle between the vectors $il_n a + jl_n b + kl_n c$ and $i(q - r) + j(r - p) + k(p - q)$ is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$

(C) $\frac{\pi}{2}$ (D) none of these

27. A vector a is collinear with vector $b = \left(6, -8, -7\frac{1}{2}\right)$ and make an acute angle with the positive direction of z-axis. If $|a| = 50$, then $a =$

(A) (24, 32, 30) (B) (24, –32, 30)

(C) (–24, 32, 30) (D) none of these

28. The perpendicular distance of a corner of a unit cube form a diagonal not passing through it is

(A) $\sqrt{6}$ (B) $\frac{\sqrt{6}}{3}$

(C) $\frac{3}{\sqrt{6}}$ (D) none of these

29. The vectors a, b and c are equal in length and taken pairwise, they make equal angles. If $a = i + j$, $b = j + k$, and c makes an obtuse angle with the base vector i, then c is equal to

(A) $i + k$ (B) $-i + 4j - k$

(C) $\frac{-1}{3}i + \frac{4}{3}j - \frac{1}{3}k$ (D) $\frac{1}{3}i + \frac{-4}{3}j + \frac{1}{3}k.$

30. If the four points a, b, c, d are coplanar, then $[b\ c\ d] + [c\ a\ d] + [a\ b\ d] =$

(A) 0 (B) 1

(C) –1 (D) $[a\ b\ c]$

31. A tetrahedron has vertices at $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. Then, the angle between the faces OAB and ABC will be

(A) $\cos^{-1}\left(\frac{19}{35}\right)$ (B) $\cos^{-1}\left(\frac{71}{31}\right)$

(C) 30° (D) 90°

32. If a quadrilateral $ABCD$ is such that $AB = b$, $AD = d$ and $AC = mb + pd$ $(m + p \geq 1)$, then the area of the quadrilateral is $k(p + m)\ |b \times d|$, where k is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{8}$

(C) $\frac{1}{2}$ (D) none of these

33. Let a be a unit vector and b be a non-zero vector not parallel to a. If two sides of the triangle are represented by the vectors $\sqrt{3}\,(a \times b)$ and $b - (a \cdot b)\, a$, then the angles of the triangle are

(A) 30°, 90°, 60° (B) 45°, 45°, 90°
(C) 60°, 60°, 60° (D) none of these

34. Let u and v be unit vectors. If w is a vector such that $w + (w \times u) = v$, then $|(u \times v) \cdot w|$

(A) $\le \frac{1}{3}$ (B) $\le \frac{1}{2}$
(C) $> \frac{1}{3}$ (D) $\ge \frac{1}{2}$

35. If b and c are any two non-collinear unit vectors and a is any vector, then $(a \cdot b)b + (a \cdot c)c + \frac{a \cdot (b+c)}{|b+c|^2}(b \times c) =$

(A) a (B) b
(C) c (D) none of these

36. If the vector $-i + j - k$ bisects the angle between $3i + 4j$ and vector c, then the unit vector along c is

(A) $\frac{-11i - 10j - 2k}{15}$ (B) $\frac{-11i + 10j + 2k}{15}$
(C) $\frac{-11i + 10j - 2k}{15}$ (D) none of these

37. If a, b and c are three unit vectors such that $a + b + c$ is also a unit vector and θ_1, θ_2 and θ_3 are angles between the vectors a, b; b, c and c, a, respectively, then among θ_1, θ_2 and θ_3.

(A) all are acute angles
(B) all are right angles
(C) at least one is obtuse angle
(D) none of these

38. If x and y are two non-collinear vectors and ABC is a triangle with side lengths a, b, c satisfying $(20a - 15b)x + (15b - 12c)y + (12c - 20a)(x \times y) = \vec{0}$, then ΔABC is

(A) an acute-angled triangle
(B) an obtuse-angled triangle
(C) a right-angled triangle
(D) an isosceles triangle

39. If $\alpha(a \times b) + \beta(b \times c) + \gamma(c \times a) = 0$, then

(A) a, b, c are coplanar if all of $\alpha, \beta, \gamma \neq 0$
(B) a, b, c are coplanar if any one of $\alpha, \beta, \gamma \neq 0$
(C) a, b, c are non-coplanar for any α, β, γ
(D) none of these

40. If $p \times q = r$ and $p \,.\, q = c$, then $q =$

(A) $\frac{cp - p \times r}{|p|^2}$ (B) $\frac{cp + p \times r}{|p|^2}$
(C) $\frac{cr - p \times r}{|p|^2}$ (D) $\frac{cr + p \times r}{|p|^2}$

41. Let $a = i + j$ and $b = 2i - k$. The point of intersection of the lines $r \times a = b \times a$ and $r \times b = a \times b$ is

(A) $-i + j + k$ (B) $3i - j + k$
(C) $3i + j - k$ (D) $i - j - k$

42. The sides of a parallelogram are $2i + 4j - 5k$ and $i + 2j + 3k$. The unit vector parallel to one of the diagonals size is

(A) $\frac{1}{7}(3i + 6j - 2k)$ (B) $\frac{1}{7}(3i - 6j - 2k)$
(C) $\frac{1}{7}(-3i + 6j - 2k)$ (D) $\frac{1}{7}(3i + 6j + 2k)$

More than One Option Correct Type

43. If $x + y = a$, $x \times y = b$ and $x \cdot a = 1$, then

(A) $x = \frac{a + a \times b}{a^2}$

(B) $y = \frac{(a^2 - 1)a - a \times b}{a^2}$

(C) $x = \frac{b + a \times b}{a^2}$

(D) $y = \frac{(b^2 - 1)b - a \times b}{a^2}$

44. If $a \times (b \times c) + (a \cdot b)b = (4 - 2\beta - \sin\alpha)b + (\beta^2 - 1)c$ and $(c \cdot c)a = c$, while b and c are non-collinear, then

(A) $\alpha = \frac{\pi}{2}$ (B) $\alpha = \frac{\pi}{3}$
(C) $\beta = 1$ (D) $\beta = -1$

45. If $x \times b = c \times b$ and $x \perp a$ then x is equal to

(A) $\frac{b \times (a \times c)}{b \cdot c}$ (B) $\frac{(b \times c) \times a}{b \cdot a}$
(C) $\frac{a \times (c \times b)}{a \cdot b}$ (D) none of these

46. Let $b = 4i + 3j$ and c be two vectors perpendicular to each other in xy-plane, then the vector in the same plane having projections 1 and 2 along b and c respectively is

(A) $2i - j$ (B) $-2i + j$
(C) $2i + j$ (D) none of these

47. A vector of magnitude 2 along a bisector of the angle between the two vectors $2i - 2j + k$ and $i + 2j - 2k$ is

(A) $\frac{2}{\sqrt{10}}(3i - k)$ (B) $\frac{1}{\sqrt{26}}(i - 4j + 3k)$

(C) $\frac{2}{\sqrt{26}}(i - 4j + 3k)$ (D) none of these

48. The value of λ such that $(x, y, z) \neq (0, 0, 0)$ and $(i + j + 3k)x + (3i - 3j + k)y + (-4i + 5j)z = \lambda(xi + yj + zk)$ is

(A) 0 (B) 1
(C) -1 (D) none of these

49. A unit vector a makes an angle $\frac{\pi}{4}$ with i and $\frac{\pi}{3}$ with j. If the angle between a and k is θ, where θ is acute, then

(A) $a = \frac{1}{2}i + \frac{1}{\sqrt{2}}j + \frac{1}{2}k$

(B) $a = \frac{1}{\sqrt{2}}i + \frac{1}{2}j + \frac{1}{2}k$

(C) $\theta = \frac{\pi}{3}$

(D) $\theta = \frac{\pi}{6}$

50. If the three vectors $a = (12, 4, 3)$, $b = (8, -12, -9)$ and $c = (33, -4, -24)$ define a parallelopiped, then

(A) the lengths of the edges are 13, 17, 41
(B) areas of the faces are 220, 435, 455
(C) volume of parallelopiped is 3696
(D) all of these

51. A vector of magnitude $\sqrt{51}$ which makes equal angles with the vectors $a = \frac{1}{3}(i - 2j + 2k)$,

$b = \frac{1}{5}(-4i - 3k)$ and $c = j$ is given by

(A) $5i - j - 5k$ (B) $-5i + j + 5k$
(C) $5i + j + 5k$ (D) none of these

52. A, B, C and D are four points such that $AB = m(2i - 6j + 2k)$, $BC = i - 2j$ and $CD = n(-6i + 15j - 3k)$. If AB and CD intersect at some point E, then

(A) $m \geq \frac{1}{2}$

(B) $n \geq \frac{1}{3}$

(C) area of $\Delta BCE = \frac{1}{2}\sqrt{6}$

(D) all of these

53. The position vectors of two points A and C are $9i - j + 7k$ and $7i - 2j + 7k$, respectively. The point of intersection of vectors $AB = 4i - j + 3k$ and $CD = 2i - j + 2k$ is P. If vector PQ is perpendicular to AB and CD and $PQ = 15$ units, the position vector of Q is

(A) $6i - 9j - 9k$ (B) $-4i + 11j + 11k$
(C) $6i + 9j - 9k$ (D) none of these

54. If $DA = a$, $AB = b$ and $CB = ka$, where $k > 0$ and X, Y are the mid-points of DB and AC respectively such that $|a| = 17$ and $|XY| = 4$, then k is equal to

(A) $\frac{8}{17}$ (B) $\frac{9}{17}$

(C) $\frac{25}{17}$ (D) 1

55. Let a and b be two non-collinear unit vectors. If $u = a - (a \cdot b)b$ and $v = a \times b$, then $|v|$ is

(A) $|u|$ (B) $|u| + |u \cdot a|$
(C) $|u| + |u \cdot b|$ (D) $|u| + u \cdot (a + b)$

56. A non-zero vector a is parallel to the line of intersection of the plane determined by the vectors i, $i + j$ and the plane determined by the vectors $i - j$, $i + k$. The angle between a and the vector $i - 2j + 2k$ is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$

(C) $\frac{3\pi}{4}$ (D) none of these

57. A unit vector coplanar with $i + j + 2k$ and $i + 2j + k$ and perpendicular to $i + j + k$ is

(A) $\frac{j - k}{\sqrt{2}}$ (B) $\frac{-j + k}{\sqrt{2}}$

(C) $\frac{j + k}{\sqrt{2}}$ (D) $\frac{-(j + k)}{\sqrt{2}}$

58. The vectors a, b, c are of same length and taken pairwise, they form equal angles. If $a = i + j$ and $b = j + k$, then $c =$

(A) $i + k$ (B) $j + k$

(C) $i + k$ (D) $-\frac{i}{3} + \frac{4}{3}j - \frac{k}{3}$

59. The vector $a = -i + 2j + k$ is rotate D through a right angle Passing through the y-axis in its way. The new vector is

(A) $\sqrt{3}\,(i + j - k)$ (B) $-\sqrt{3}\,(i + j - k)$

(C) $\sqrt{2}\,(i + j - k)$ (D) $-\sqrt{2}\,(i + j - k)$

60. a and c are unit vectors and $|b| = 4$ with $a \times b = 2a \times c$. The angle between a and c is $\cos^{-1}\left(\frac{1}{4}\right)$. Then, $b - 2c$ = la, if λ is

(A) 3 (B) −3

(C) 4 (D) −4

61. The vector c directed along the bisectors of the angle between the vectors $a = 7i - 4j - 4k$ and $\hat{b} = -2i - j + 2k$ if $|c| = 3\sqrt{6}$, is given by

(A) $i - 7j + 2k$ (B) $2i + 7j - 3k$

(C) $-i + 7j - 2k$ (D) $4i + 7j - 4k$

Passage Based Questions

Passage 1

The vector differential operator DEL, written ∇, is defined by $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$, where $\frac{\partial}{\partial x}$ represents the derivative w.r.t. x regarding y and z as constant. Similarly, $\frac{\partial}{\partial y}$ represents the derivative w.r.t. y regarding x and z as constant and $\frac{\partial}{\partial z}$ represents the derivative w.r.t. z regarding x and y as constant. The operator ∇ is also known as nabla.

Let $\phi(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then, the gradient of ϕ, written $\nabla\phi$ or grad ϕ, is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k.$$

Let r be any vector such that $r = xi + yj + zk$

62. If $\phi = ln\,|\,r\,|$ then $\nabla\phi =$

(A) $\frac{r}{r^2}$ (B) $\frac{r}{r^3}$

(C) $\frac{r}{r^4}$ (D) $\frac{r}{r}$

63. If $\phi = \frac{1}{r}$, then $\nabla\phi =$

(A) $\frac{r}{r}$ (B) $\frac{r}{r^2}$

(C) $\frac{r}{r^3}$ (D) $-\frac{r}{r^3}$

64. $\nabla r^n =$

(A) $n\,rn{-}^1\,r$ (B) $n\,rn{-}^2\,r$

(C) $n\,rn\,r$ (D) none of these

Passage 2

Let $\nabla(x, y, z) = V_1 i + V_2 j + V_3 k$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then, the divergence of V, written $\nabla \cdot V$ or div V is defined by

$$\nabla \cdot V = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot(V_1 i + V_2 j + V_3 k)$$
$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Here, $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$ is the del operator.

Note the analogy with $A \cdot B = A_1B_1 + A_2B_2 + A_3B_3$. Also, note that $\vec{\nabla} \cdot V \neq V \cdot \nabla$.

65. If $\phi = 2x^3y^2z^4$, then $\nabla \,.\, \nabla\phi = k\,(3xy^2z^4 + x^3z^4 + 6x^3y^2z^2)$, where $k =$

(A) 2 (B) 3

(C) 4 (D) 6

66. If $r = xi + yj + zk$ then $\nabla^2\left(\frac{1}{r}\right) =$

(A) 1 (B) −1

(C) 0 (D) none of these

67. If $r = xi + yj + zk$, then $\nabla \cdot \left(\frac{r}{r^3}\right) =$

(A) 0 (B) 1

(C) − 1 (D) none of these

Passage 3

Let $V(x, y, z) = V_1 i + V_2 j + V_3 k$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then, the curl or roation of $\vec{V}$, written $\nabla \times \vec{V}$, curl $\vec{V}$ or rot $\vec{V}$, is defined by

$$\nabla \times V = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times (V_1 i + V_2 j + V_3 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z_2}\right) i + \left(\frac{\partial V_2}{\partial z} - \frac{\partial V_3}{\partial x}\right) j + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) k$$

Note that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must precede V_1, V_2, V_3.

68. For a scalar function ϕ, possessing continuous second order partial derivatives $\nabla \times (\nabla \phi) =$

(A) ϕ (B) 0
(C) $\nabla \phi$ (D) none of these

69. For a vector function A possessing continuous second order partial derivatives, $\nabla \cdot (\nabla \times A) =$

(A) A (B) $\nabla \times A$
(C) 0 (D) none of these

Match the Column Type

70.

Column-I	Column-I
I. The points O, A, B, C, D are such that $OA = a$, $OB = b$, $OC = 2a + 3b$ and $OD = a - 2b$. If $\lvert a \rvert = 3 \lvert b \rvert$, then the angle between BD and AC is	(A) 0
II. a, b, c are three unit vectors such that $a \times (b \times c) = \frac{1}{2}(b + c)$. If the vectors b and c are non-parallel, then the angle between a and b is	(B) $\frac{2\pi}{3}$
III. Let the vectors a, b, c and d be such that $(a \times b) \times (c \times d) = 0$. Let P_1 and P_2 be planes determined by the pairs of vectors a, b and c, d respectively, then the angle between P_1 and P_2 is	(C) $\frac{3\pi}{4}$
IV. If a and b are two vectors such that $a \cdot b < 0$ and $\lvert a \cdot b \rvert = \lvert a \times b \rvert$, then the angle between vectors a and b is	(D) $\frac{\pi}{2}$

Assertion-Reason Type

Instructions: In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:

(A) Assertion(A) is True and Reason(R) is True; Reason(R) is a correct explanation for Assertion(A)
(B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)
(C) Assertion(A) is True, Reason(R) is False
(D) Assertion(A) is False, Reason(R) is True

71. Assertion: i, j, k are orthonormal unit vectors and a is any vector. If $a \times r = j = \hat{j}$, then $a \cdot r$ is arbitrary scalar.

Reason: For any two arbitrary vectors a and r, $|a \times r|^2 + |a \cdot r|^2 = |a|^2 \, |r|^2$

72. Assertion: If a, b, c are three non-coplanar, non-zero vectors, then

$$(a \cdot a) b \times c + (a \cdot b) c \times a + (a \cdot c) a \times b = [bca] a$$

Reason: If the vectors a, b, c are non-coplanar, then so are $b \times c, c \times a, a \times b$

73. Assertion: If

$$i \times [(a - j) \times i] + j \times [(a - k) \times j] + k \times [(a - i) \times k] = 0, \text{ then}$$

$$a = \frac{1}{2}(i + j + k)$$

Reason: $(a \cdot i) i + (a \cdot j) j + (a \cdot k) k = a$ for any vector a.

Previous Year's Questions

74. Given two vectors are $\hat{i}-\hat{j}$ and $\hat{i}+2\hat{j}$ the unit vector coplanar with the two vectors and perpendicular to first is: **[2002]**

(A) $\frac{}{\sqrt{}}(\hat{i}\quad\hat{j})$ (B) $\frac{1}{\sqrt{5}}(2\hat{i}+\hat{j})$

(C) $\pm\frac{1}{\sqrt{2}}(\hat{i}+\hat{k})$ (D) none of these

75. The vector $\hat{i}+x\hat{j}+3\hat{k}$ is rotated through an angle θ and doubled in magnitude, then it becomes $4\hat{i}+(4x-2)\hat{j}+2\hat{k}$. The value of x are: **[2002]**

(A) $\left\{-\frac{2}{3},2\right\}$ (B) $\left\{\frac{1}{3},2\right\}$

(C) $\left\{\frac{2}{3},0\right\}$ (D) $\{2, 7\}$

76. If the vectors $\vec{a},\vec{b}$ and $\vec{c}$ from the sides BC, CA and AB respectively of a triangle ABC, then: **[2002]**

(A) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{b}=0$

(B) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

(C) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$

(D) $\vec{a}\times\vec{a}+\vec{a}\times\vec{c}+\vec{c}\times\vec{a}=0$

77. If the vectors $\vec{a}-x\hat{i}+y\hat{j}+z\hat{k}$ and such that $\vec{a},\vec{c}$ and $\vec{b}$ form a right handed system, then $\vec{c}$ is: **[2002]**

(A) $z\hat{i}-x\hat{k}$ (B) $\vec{0}$

(C) $y\hat{j}$ (D) $-z\hat{i}+x\hat{k}$

78. The centre of the circle given by $\vec{r}\cdot(\hat{i}+2\hat{j}+2\hat{k})=15$ and $|\vec{r}\cdot(\hat{j}+2\hat{k})|=4$ is: **[2002]**

(A) (0, 1, 2) (B) (1, 3, 4)

(C) (–1, 3, 4) (D) none of these

79. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}=0$ and vectors $(1, a, a^2)$ $(1, b, b^2)$ and $(1, c, c^2)$ are non-coplanar, then the product abc equals **[2003]**

(A) 2 (B) –1

(C) 1 (D) 0

80. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of these are collinear. If the vector $\vec{a}+2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b}+3\vec{c}$ is collinear with $\vec{a}$ (λ being some non-zero scalar) then $\vec{a}+2\vec{b}+6\vec{c}$ equals **[2004]**

(A) $\lambda\vec{a}$ (B) $\lambda\vec{b}$

(C) $\lambda\vec{c}$ (D) 0

81 A particle is acted upon by constant forces $4\hat{i}+\hat{j}-3\hat{k}$ and $3\hat{i}+\hat{j}-\hat{k}$ which displace it from a point $\hat{i}+2\hat{j}+3\hat{k}$ to the point $5\hat{i}+4\hat{j}+\hat{k}$. The work done in standard units by the forces is given by **[2004]**

(A) 40 (B) 30

(C) 25 (D) 15

82. If $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\bar{a}+2\bar{b}+3\bar{c},\lambda\bar{b}+4\bar{c}$ and $(2\lambda-1)\bar{c}$ are non-coplanar for **[2004]**

(A) all values of λ

(B) all except one value of λ

(C) all except two values of λ

(D) no value or λ

83. Let $\bar{u},\bar{v},\bar{w}$ be such that $|\bar{u}|=1,|\bar{v}|=2,|\bar{w}|=3$. If the projection $\bar{v}$ along $\bar{u}$ is equal to that of $\bar{w}$ along $\bar{u}$ and $\bar{v},\bar{w}$ are perpendicular to each other then $|\bar{u}-\bar{v}+\bar{w}|$ equals **[2004]**

(A) 2 (B) $\sqrt{7}$

(C) $\sqrt{14}$ (D) 14

84. Let $\bar{a},\bar{b}$ and $\bar{c}$ be non-zero vectors such that $(\bar{a}\times\bar{b})\times\bar{c}=-|\bar{b}||\bar{c}|\bar{a}$. If θ is the acute angle between the vectors $\bar{b}$ and $\bar{c}$ then $\sin\theta$ equals **[2004]**

(A) $\frac{1}{3}$ (B) $\frac{\sqrt{2}}{3}$

(C) $\frac{2}{3}$ (D) $\frac{2\sqrt{2}}{3}$

85. If C is the mid point of AB and P is any point outside AB, then **[2005]**

(A) $\overrightarrow{PA}+\overrightarrow{PB}=2\overrightarrow{PC}$

(B) $\overrightarrow{PA}+\overrightarrow{PB}=\overrightarrow{PC}$

(C) $\overrightarrow{PA}+\overrightarrow{PB}+2\overrightarrow{PC}=0$

(D) $\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}=0$

86. The distance between the line $\vec{r}=2\hat{i}-2\hat{j}+3\hat{k}+\lambda(\hat{i}+\hat{j}+4\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}+5\hat{j}+\hat{k})=5$ is **[2005]**

(A) $\frac{10}{9}$ (B) $\frac{10}{3\sqrt{3}}$

(C) $\frac{3}{10}$ (D) $\frac{10}{3}$

87. For any vector $\vec{a}$, the value of $(\vec{a}\times\hat{i})^2+(\vec{a}\times\hat{j})^2+(\vec{a}\times\hat{k})^2$ is equal to **[2005]**

(A) $3\vec{a}^2$ (B) $\vec{a}^2$
(C) $2\vec{a}^2$ (D) $4\vec{a}^2$

88. If non-zero numbers a, b, c are in H.P., then the straight line $\frac{x}{a}+\frac{y}{b}+\frac{1}{c}=0$ always passes through a fixed point. That point is **[2005]**

(A) (–1, 2) (B) (–1, –2)
(C) (1, –2) (D) $\left(1,-\frac{1}{2}\right)$

89. Let a, b and c be distinct non-negative numbers. If the vectors $a\hat{i}+a\hat{j}+c\hat{k}, \hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is **[2005]**

(A) the Geometric Mean of a and b
(B) the Arithmetic Mean of a and b
(C) equal to zero
(D) the Harmonic Mean of a and b

90. If $\vec{a},\vec{b},\vec{c}$ are non-coplanar vectors and λ, is a real number then $[\lambda(\vec{a}+\vec{b})\,\lambda^2\vec{b}\,\lambda\vec{c}]=[\vec{a}\vec{b}+\vec{c}\vec{b}]$ for **[2005]**

(A) exactly one value of λ
(B) no value of λ
(C) exactly three values of λ
(D) exactly two values of λ

91. Let $\vec{a}=\hat{i}-\hat{k}, \vec{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$ and $\vec{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$. Then $[\vec{a},\vec{b},\vec{c}]$ depends on **[2005]**

(A) only y
(B) only x
(C) both x and y
(D) neither x nor y

92. If $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$, where $\vec{a},\vec{b}$ and $\vec{c}$ are any three vectors such that $\vec{a}\cdot\vec{b}\neq 0, \vec{b}\cdot\vec{c}\neq 0$, then $\vec{a}$ and $\vec{c}$ are **[2006]**

(A) inclined at an angle of $\frac{\pi}{3}$ between them
(B) inclined at an angle of $\frac{\pi}{6}$ between them
(C) perpendicular
(D) parallel

93. The values of a, for which the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}, \hat{i}-3\hat{j}-5\hat{k}$ and $a\hat{i}-3\hat{j}+\hat{k}$ respectively are the vertices of a right-angled triangle with $C=\frac{\pi}{2}$ are **[2006]**

(A) 2 and 1 (B) –2 and –1
(C) –2 and 1 (D) 2 and –1

94. If $\hat{u}$ and $\hat{v}$ are unit vectors and θ is the acute angle between them, then $2\hat{u}\times 3\hat{v}$ is a unit vector for **[2007]**

(A) exactly two values of θ
(B) more than two values of θ
(C) no value of θ
(D) exactly one value of θ

95. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{c}=x\hat{i}+(x-2)\hat{j}-\hat{k}$. If the vector $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$, then x equals **[2007]**

(A) 0 (B) 1
(C) –4 (D) –2

96. The vector $\vec{a}=\alpha\hat{i}+2\hat{j}+\beta\hat{k}$ lies in the plane of the vectors $\vec{b}=\hat{i}+\hat{j}$ and $\vec{c}=\hat{j}+\hat{k}$ and bisects the angle between $\vec{b}$ and $\vec{c}$. Then which one of the following gives possible values of α and β? **[2008]**

(A) $\alpha=2, \beta=2$ (B) $\alpha=1, \beta=2$
(C) $\alpha=2, \beta=1$ (D) $\alpha=1, \beta=1$

97. The non-zero vectors $\vec{a},\vec{b}$ and $\vec{c}$ are related by $\vec{a}=8\vec{b}$ and $\vec{c}=7\vec{b}$. Then the angle between $\vec{a}$ and $\vec{c}$ is **[2008]**

(A) 0 (B) $\pi/4$
(C) $\pi/2$ (D) π

98. If $\vec{u},\vec{v},\vec{w}$ are non-coplanar vectors and p, q are real numbers, then the equality $[3\vec{u}\ \ p\vec{v}\ \ p\vec{w}]-[p\vec{v}\ \ \vec{w}\ \ q\vec{u}]-[2\vec{w}\ \ q\vec{v}\ \ q\vec{u}]=0$ holds for **[2009]**

(A) exactly one value of (p, q)
(B) exactly two values of (p, q)
(C) more than two but not all values of (p, q)
(D) all values of (p, q)

99. The projections of a vector on the three coordinate axis are 6, −3, 2 respectively. The direction cosines of the vector are **[2009]**

(A) 6, −3, 2 (B) $\frac{6}{5},-\frac{3}{5},\frac{2}{5}$
(C) $\frac{6}{7},-\frac{3}{7},\frac{2}{7}$ (D) $-\frac{6}{7},-\frac{3}{7},\frac{2}{7}$

100. Let $\vec{a}=\vec{j}-\vec{k}$ and $\vec{c}=\vec{i}-\vec{j}-\vec{k}$. Then, the vector $\vec{b}$ satisfying $\vec{a}\times\vec{b}+\vec{c}=\vec{0}$ and $a\cdot\vec{b}=3$ is **[2010]**

(A) $2\hat{i}-\hat{j}+2\hat{k}$ (B) $\hat{i}-\hat{j}-2\hat{k}$
(C) $\hat{i}+\hat{j}-2\hat{k}$ (D) $-\hat{i}+\hat{j}-2\hat{k}$

101. If the vectors $\vec{a}=\hat{i}-\hat{j}+2\hat{k}, \vec{b}=2\hat{i}+4\hat{j}+\hat{k}$ and $\vec{c}=\lambda\hat{i}+\hat{j}+\mu\hat{k}$ are mutually orthogonal, then the tuple (λ, μ) = **[2010]**

(A) (2, −3) (B) (−2, 3)
(C) (3, −2) (D) (−3, 2)

102. If $a = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k})$ and $b = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$, then the value of the expression $(2z - b).[(a \times b) \times (a + 2b)]$ is **[2011]**

(A) −3 (B) 5
(C) 3 (D) −5

103. The vectors $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying: $\vec{b} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a} \cdot \vec{d} = 0$. Then, the vector $\vec{d}$ is equal to **[2011]**

(A) $c + \left(\frac{a.c}{a.b}\right)b$ (B) $b + \left(\frac{b.c}{a.b}\right)c$

(C) $c - \left(\frac{a.c}{a.b}\right)b$ (D) $b - \left(\frac{b.c}{a.b}\right)c$

104. Let $\hat{a}$ and $\hat{b}$ be two unit vectors. If the vectors $\vec{c} = \hat{a} + 2\hat{b}$ and $d = 5\hat{a} - 4\hat{b}$ are perpendicular to each other, then the angle between $\hat{a}$ and $\hat{b}$ is **[2012]**

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

105. Let $ABCD$ be a parallelogram such that $\overrightarrow{AB} = \vec{q}, \overrightarrow{AD} = \vec{p}$ and $\square BAD$ be an acute angle. If $\vec{r}$ is the vector which coincides with the altitude directed from the vertex B to the side AD, then $\vec{r}$ is given by **[2012]**

(A) $\vec{r} = 3\vec{q} - \frac{3(\vec{p} \cdot \vec{q})}{(\vec{p} \cdot \vec{p})}\vec{p}$

(B) $\vec{r} = -\vec{q} + \left(\frac{\vec{p} \cdot \vec{q}}{\vec{p} \cdot \vec{p}}\right)\vec{p}$

(C) $\vec{r} = \vec{q} - \left(\frac{\vec{p} \cdot \vec{q}}{\vec{p} \cdot \vec{p}}\right)\vec{p}$

(D) $\vec{r} = -3\vec{q} + \frac{3(\vec{p} \cdot \vec{q})}{(\vec{p} \cdot \vec{p})}\vec{p}$

106. If the vectors $\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} + 2\hat{j} + 4\hat{k}$ represent the sides of a triangle ABC, then the length of the median through A is **[2013]**

(A) $\sqrt{72}$ (B) $\sqrt{33}$
(C) $\sqrt{45}$ (D) $\sqrt{18}$

107. If $[\vec{a} \times \vec{b}\vec{b} \times \vec{c}\vec{c} \times \vec{a}] = \lambda[\vec{a}\vec{b}\vec{c}]^2$, then the value of λ is equal to **[2014]**

(A) 2 (B) 3
(C) 0 (D) 1

108. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$. If θ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of $\sin\theta$ **[2015]**

(A) $\frac{-\sqrt{2}}{3}$ (B) $\frac{2}{3}$

(C) $\frac{-2\sqrt{3}}{3}$ (D) $\frac{2\sqrt{2}}{3}$

109. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$. if $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is **[2016]**

(A) $\frac{5\pi}{6}$ (B) $\frac{3\pi}{4}$

(C) $\frac{\pi}{2}$ (D) $\frac{2\pi}{3}$

ANSWER KEYS

Single Option Correct Type

1. (A)	**2.** (B)	**3.** (B)	**4.** (A)	**5.** (B)	**6.** (A)	**7.** (A)	**8.** (C)	**9.** (B)	**10.** (A)
11. (B)	**12.** (A)	**13.** (C)	**14.** (A)	**15.** (A)	**16.** (D)	**17.** (A)	**18.** (A)	**19.** (C)	**20.** (A)
21. (C)	**22.** (D)	**23.** (C)	**24.** (A)	**25.** (C)	**26.** (C)	**27.** (C)	**28.** (B)	**29.** (C)	**30.** (D)
31. (A)	**32.** (C)	**33.** (A)	**34.** (B)	**35.** (A)	**36.** (A)	**37.** (C)	**38.** (C)	**39.** (B)	**40.** (A)
41. (C)	**42.** (A)								

More than One Option Correct Type

43. (A, B) **44.** (A, C) **45.** (B, C) **46.** (A, B) **47.** (A, C) **48.** (A, C) **49.** (B, D) **50.** (A, B, C, D)
51. (A, B) **52.** (A, B, C, D) **53.** (A, B) **54.** (B, C)
55. (A, C) **56.** (B, C) **57.** (A, B) **58.** (A, D) **59.** (C, D) **60.** (A, D)
61. (A, C)

Passage Based Questions

62. (A) **63**. (D) **64**. (B) **65**. (C) **66**. (C) **67**. (A) **68**. (B) **69**. (C)

Match the Column Type

70. I ↔ (D), II ↔ (B), III ↔ (A), IV ↔ (C)

Assertion-Reason Type

71. (A) **72**. (A) **73**. (A)

Previous Year's Questions

74. (A)	**75.** (A)	**76.** (B)	**77.** (A)	**78.** (B)	**79.** (B)	**80.** (D)	**81.** (A)	**82.** (C)	**83.** (C)
84. (A)	**85.** (A)	**86.** (B)	**87.** (C)	**88.** (C)	**89.** (C)	**90.** (B)	**91.** (D)	**92.** (D)	**93.** (A)
94. (D)	**95.** (D)	**96.** (D)	**97.** (D)	**98.** (A)	**99.** (C)	**100.** (B)	**101.** (D)	**102.** (D)	**103.** (C)
104. (C)	**105.** (B)	**106.** (B)	**107.** (D)	**108.** (D)	**109.** (A)				

HINTS AND SOLUTIONS

Single Option Correct Type

1. Let $r = x_1 a + x_2 b + x_3 (a \times b)$

Then, $r \cdot a = x_1 |a|^2$, $r \cdot b = x_2 |b|^2$

and, $r \cdot (a \times b) = x_3 |a \times b|^2$

$\Rightarrow \; x_1 = 0, x_2 = 1, x_3 = 1 \;\therefore\; r = a \times b + b$

2. We have, $a \times (a \times c) + 3b = 0$

$\Rightarrow \; (a \cdot c)a - (a \cdot a)c + 3b = 0$

$\Rightarrow \; (2\sqrt{3}\cos\theta)a - 3c + 3b = 0$

$\Rightarrow \; (2\cos\theta)a - \sqrt{3}c + \sqrt{3}b = 0$

$\Rightarrow \; |(2\cos\theta)a - \sqrt{3}c|^2 = |-\sqrt{3}b|^2$

$\Rightarrow \; 4\cos^2\theta\,|a|^2 + 3|c|^2 - 4\sqrt{3}\cos\theta\,(a \cdot c) = 3|b|^2$

$\Rightarrow \; 12\cos^2\theta + 12 - 4\sqrt{3}\cos\theta \times \sqrt{3} \times 2\cos\theta = 3$

$\Rightarrow \; 12\cos^2\theta + 9 - 24\cos^2\theta = 0$

$\Rightarrow \; 12\cos^2\theta = 9 \;\Rightarrow\; \cos^2\theta = \frac{9}{12} = \frac{3}{4}.$

3. Let i, j be unit vectors along the coordinate axes

$\therefore \; a = 2pi + 1 \cdot j$ (1)

On rotation, let b be the vector having components $p + 1$ and 1.

$\therefore \; b = (p + 1)\,i + 1 \cdot j$ (2)

where i, j are unit vectors along the new coordinate axes.

But on rotation $|b| = |a| \Rightarrow |b|^2 = |a|^2$

$\Rightarrow \; (p + 1)^2 + 1 = (2p)^2 + 1 \Rightarrow 3p^2 - 2p - 1 = 0$

$\Rightarrow \; (3p + 1)(p - 1) = 0 \;\Rightarrow\; p = 1 \text{ or } -\frac{1}{3}.$

4. Here, a and $c = a \times b$ are non-collinear vectors.

$\therefore$ Let $b = xa + y\,(a \times c)$ (1)

$\therefore \; \beta = a \cdot b = a \cdot [xa + y(a \times c)]$

$= x\,|a|^2 + ya \cdot (a \times c) = xa^2 \Rightarrow x = \beta/a^2.$

And, $c = a \times b = a \times [xa + y(a \times c)]$

$= xa \times a + ya \times (a \times c)$

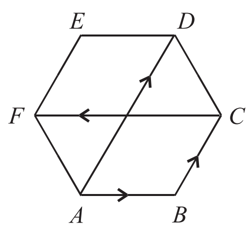
$= 0 + y(a \cdot c)\,a - y\,(a \cdot a)\,c$

$= y\,\{a \cdot (a \times b)\}\,a - ya^2c = -ya^2c$

$\Rightarrow \; y = -1/a^2$

$\therefore$ from (1), $b = (\beta a - a \times c)/a^2$.

5. Since $ABCDEF$ is a regular hexagon, from plane geometry, we have



$AD = 2BC$ and $FC = 2AB$

$\therefore \quad AD = 2BC$ and $FC = 2AB$. (1)

Given that $AD = xBC$.

$\therefore \quad 2BC = xBC$, by (i)

$\Rightarrow \quad x = 2$. (2)

Again, given that $CF = yAB$ or $-FC = yAB$.

$\therefore \quad -2AB = yAB$, using (ii)

$\Rightarrow \quad y = -2$. (3)

From (ii) and (iii), $xy = 2\,(-2) = -4$.

6. $OM_1 = OA + OB + OC + OD$ (given)

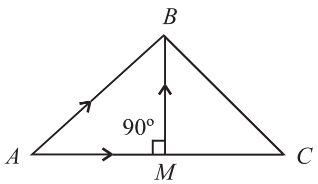
$= OM + MA + OM + MB + OM + MC + OM + MD$

$= 4OM + (MA + MC) + (MB + MD)$

$= 4OM$ $\quad (\because MA = -MC, MB = -MD)$

$\therefore \quad OM = \dfrac{1}{4} OM_1, \quad \therefore \quad \lambda = \dfrac{1}{4}$.

7. Since MB is the component of $AB \perp$ to AC



$MB = AB - AM$

$= AB - \dfrac{(AB \cdot AC)\,AC}{(AC)^2}$

$= (6j - 2k) - \dfrac{\{(6j-2k)\cdot(-5i+3j-k)\}\,(-5i+3j-k)}{(25+9+1)}$

$= (6j - 2k) - \dfrac{20}{35}(-5i + 3j - k) = \dfrac{10}{7}(2i + 3j - k)$.

$\therefore \quad BM = -\dfrac{10}{7}(2i + 3j - k)$.

8. A point equidistant from AB and AC is on the bisector of the angle BAC.

A vector along the internal bisector of the angle BAC

$= \dfrac{AB}{|AB|} + \dfrac{AC}{|AC|}$

$= \dfrac{3i + j - k}{\sqrt{9+1+1}} + \dfrac{i - j + 3k}{\sqrt{1+1+9}} = \dfrac{1}{\sqrt{11}}(4i + 2k)$

$\therefore \quad AP = t(2i + k)$

$\therefore \quad BP = AP - AB = t(2i + k) - (3i + j - k)$

$= (2t - 3)\,i - j + (t + 1)\,k$

Also, $BC = AC - AB = (i - j + 3k) - (3i + j - k)$

$= -2i - 2j + 4k$.

But $BP = s\,BC$.

$\therefore \quad (2t - 3)\,i - j + (t + 1)\,k = s\,(-2i - 2j + 4k)$

$\therefore \quad 2t - 3 = -2s, -1 = -2s, t + 1 = 4s$

$\therefore \quad s = \dfrac{1}{2}$ and $t = 1 \quad \therefore \quad AP = 2i + k$.

9. Since $(a - d)\cdot(b - c) = 0 \therefore DA \cdot CB = 0$

$\therefore \quad AD \perp BC$.

Since $(b - d)\cdot(c - a) = 0$, $\therefore DB \cdot AC = 0$

$\therefore \quad BD \perp CA$.

Then, D is the intersection of the altitudes through A and B. Therefore, D is the orthocentre of the triangle ABC.

10. We have, $(a + b) \times (a \times b)$

$= a \times (a \times b) + b \times (a \times b)$

$= (a \cdot b)\,a - (a \cdot a)\,b + (b \cdot b)\,a - (b \cdot a)\,b$

$= (a \cdot b)\,(a - b) + a - b$ ($b \cdot b = b^2 = 1$, $a \cdot a = a^2 = 1$ as a, b are unit vectors)

$= (a \cdot b + 1)\,(a - b)$

$= x(a - b)$, where $x = a \cdot b + 1$ is a scalar.

$\therefore \quad$ The given vector is parallel to $a - b$.

11. Clearly, $r \cdot c = \lambda[a\,b\,c] = \dfrac{1}{8}\lambda$

$r \cdot a = \mu[a\,b\,c] = \dfrac{1}{8}\mu$

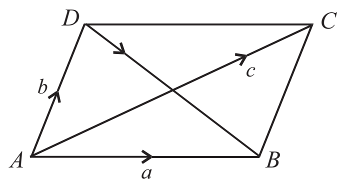
$r \cdot b = v\,[abc] = \dfrac{1}{8}v$

$\therefore \quad r \cdot (a + b + c) = \dfrac{1}{8}(\lambda + \mu + v)$

$\therefore \quad \lambda + \mu + v = 8r \cdot (a + b + c)$.

12. $\because \quad DB = DA + AB$

or, $\quad DA = DB - AB$



$\therefore \quad (DA)^2 = (DB)^2 + (AB)^2 - 2DB \cdot AB$ (1)

In parallelogram $2(a^2 + b^2) = c^2 + DB^2$

$\therefore \quad (DB)^2 = 2a^2 + 2b^2 - c^2$

$\therefore \quad$ From (1), $b^2 = 2a^2 + 2b^2 - c^2 + a^2 - 2AB \cdot DB$

$\therefore \quad AB \cdot DB = \dfrac{3a^2 + b^2 - c^2}{2}$.

13. We have,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = [a\,b\,c]^2 = [(a \times b)\cdot c]^2$$

$= [|\,a\,|\,|\,b\,|\,\sin \pi/6\; c \cdot c]^2$

$= |\,a\,|^2\,|\,b\,|^2 \sin^2 \pi/6 \cdot 1$ $\quad [\,|c| = 1]$

$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \times 1/4$

$= 1/4\,(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$.

14. $\frac{1}{\sqrt{2}}(b+c) = a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$

$\Rightarrow \left(a\cdot c - \frac{1}{\sqrt{2}}\right)b - \left(a\cdot b + \frac{1}{\sqrt{2}}\right)c = 0$

Since a, b, c are non-coplanar so a, b, c are linearly independent. Hence,

$a\cdot b = -\frac{1}{\sqrt{2}}.$

$\therefore \cos\theta = \frac{a\cdot b}{|a||b|} = a\cdot b = -\frac{1}{\sqrt{2}}$

(θ is the angle between a and b)

$\Rightarrow \theta = 3\pi/4.$

15. We have $a \perp b \Rightarrow a, b, a\times b$ are linearly independent

$\Rightarrow$ v can be expressed uniquely in terms of a, b and $a\times b$.

Let $v = xa + yb + za\times b$ (1)

Given that, : $a\cdot b = 0$, $v\cdot a = 0$, $v\cdot b = 1$, $[v, a, b] = 1$

$\therefore v\cdot a = xa^2$ or $xa^2 = 0$ or $x = 0$ (2)

$v\cdot b = xv\cdot b + yb^2 + zb\cdot(a\times b)$

or, $yb^2 = 1$ or $y = \frac{1}{b^2}$ (3)

$v\cdot a\times b = x\cdot 0 + y\cdot 0 + z|a\times b|^2$

or, $z|a\times b|^2 = 1$

or, $z = \frac{1}{|a\times b|^2}$ (4)

From (1), (2), (3) and (4), we get

$v = \frac{1}{|b|^2}b + \frac{1}{|a\times b|^2}a\times b.$

16. We have, $c = xa + yb + z(a\times b)$

$\Rightarrow c\cdot a = x$ and $c\cdot b = y \Rightarrow x = y = \cos\theta$

Now, $c\cdot c = |c|^2$

$\Rightarrow [xa + yb + z(a\times b)]\cdot[xa + yb + z(a\times b)] = |c|^2$

$\Rightarrow 2x^2 + z^2|a\times b|^2 = 1$

$\Rightarrow 2x^2 + z^2[|a|^2|b|^2 - (a\cdot b)^2] = 1$

$\Rightarrow 2x^2 + z^2[1-0] = 1 \quad [\because a\perp b \therefore a\cdot b = 0]$

$\Rightarrow 2x^2 + z^2 = 1 \Rightarrow z^2 = 1 - 2x^2 = 1 - 2\cos^2\theta = -\cos 2\theta.$

17. From geometry

$2\,SD = AO$, where D is the mid-point of BC.

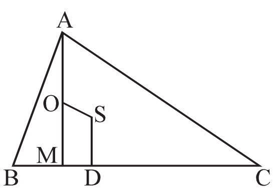
$\therefore SA + SB + SC$

$= SA + SD + DB + SD + DC$

$= SA + 2\,SD$ $[\because DB + DC = 0]$

$= SA + AO = SO$

$\therefore SA + SB + SC = SO.$



18. We have, $d\cdot\{a\times[b\times(c\times d)]\}$

$= d\cdot(a\times((b\cdot d)c - (b\cdot c)d))$

$= d\cdot((b\cdot d)(a\times c) - (b\cdot c)(a\times d))$

$= (b\cdot d)\,d\cdot(a\times c) - (b\cdot c)\,d\cdot(a\times d)$

$= (b\cdot d)[a\;c\;d]$ $[\because d\cdot(a\times d) = 0].$

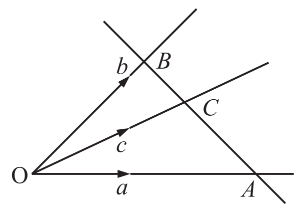
19. $4a + 5b + 9c = 0 \Rightarrow$ Vectors a, b and c are collinear $\Rightarrow (b\times c)\times(c\times a) = \vec{0}.$

20. $\left(\sum_{i=1}^{n} a_i\right)\cdot\left(\sum_{i=1}^{n} a_i\right) = \sum_{i=1}^{n}|a_i|^2 + 2\sum_{1\le i<j\le n}\sum a_i\cdot a_j$

$\Rightarrow 0 = n + 2\sum_{1\le i<j\le n}\sum a_i\cdot a_j \Rightarrow \sum_{1\le i<j\le n}\sum a_i\cdot a_j = -\frac{n}{2}$

21. Let O be taken as the origin of reference. Let a, b, c be the position vectors of A, B, C respectively so that

$OA = a$, $OB = b$, $OC = c$.



Then, the vector representing the force P along OA

$= P$ (unit vector along OA) $= P\frac{OA}{OA} = \frac{P}{OA}a.$

Similarly, the vector representing the force Q along OB $= \frac{Q}{OB}b$ and the vector representing the force R along OC

$= \frac{R}{OC}c.$

Since R is the resultant of P and Q, we have

$(P/OA)\,a + (Q/OB)\,b = (R/OC)\,c$

or, $(P/OA)\,a + (Q/OB)\,b - (R/OC)\,a = 0.$ (1)

Since, a, b, c are the position vectors of three collinear points, therefore, we have

the algebraic sum of the coefficients of a, b, c in (i) = 0

i.e., $(P/OA) + (Q/OB) - (R/OC) = 0$

Hence, $\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}.$

22. When rotated through $\frac{\pi}{2}$, the new x-axis is along old y-axis and new y-axis is along the old negative x-axis; z-axis remains same as before. Hence, the components of A in the new system are $A_2, -A_1, A_3$.

23. Let a and b be the position vectors of A and B respectively w.r.t. O as the origin.

Then p.v. of $E = OE$

$= \frac{1}{2}OB = \frac{1}{2}b$

and, p.v. of $D = OD = \frac{1}{3}(2b + a), \because AD:DB = 2:1$

Now let OD and AE intersect at P such that $OP/OD = \lambda$ and $AP/AE = \mu$.

$\therefore \quad OP = \lambda OD$ and $AP = \mu AE$.

or, $\quad OP = \lambda OD = \frac{1}{3}(2b + a)\lambda$.

and, $AP = \mu AE = \mu(OE - OA) = \mu\left(\frac{1}{2}b - a\right)$.

In $\Delta\, OAP$, $OA + AP = OP$

$\Rightarrow \quad a + \mu\left(\frac{1}{2}b - a\right) = \frac{1}{3}(2b + a)\lambda$

$\Rightarrow \quad \left(1 - \mu - \frac{1}{3}\lambda\right)a + \left(\frac{1}{2}\mu - \frac{2}{3}\lambda\right)b = 0$

$\Rightarrow \quad 1 - \mu - \frac{1}{3}\lambda = 0$ and $\frac{1}{2}\mu - \frac{2}{3}\lambda = 0$,

$\because \quad a$ and b are non-collinear.

Solving these equations, we get $\mu = \frac{4}{5}, \lambda = \frac{3}{5}$.

Now, $OP = \lambda OD$, $PD = OD - OP = (1 - \lambda)\, OD$

$\therefore \quad \frac{OP}{PD} = \frac{\lambda}{1-\lambda} = \frac{3/5}{1 - 3/5} = \frac{3}{2}$

$\Rightarrow \quad OP = \frac{3}{2}PD \quad \Rightarrow \quad OP = \frac{3}{2}PD \quad \Rightarrow \quad OP : PD = 3:2.$

24. We have, $a \times (b \times c) = \frac{1}{2}b$

$\therefore \quad (a\cdot c)b - (a\cdot b)c = \frac{1}{2}b$

$\Rightarrow \quad \left(a\cdot c - \frac{1}{2}\right)b - (a\cdot b)c = 0 \qquad (1)$

Since b and c are non-parallel, (Given)

$\therefore \quad$ (i) exists if coefficients of b and c vanish separately

i.e., $a\cdot c - \frac{1}{2} = 0$ and $a\cdot b = 0$.

$\Rightarrow \quad a\cdot c = \frac{1}{2}$ and $a\cdot b = 0$.

Let θ_1 and θ_2 be the angles which a makes with b and c, respectively. Also a, b, c are unit vectors,

$\therefore \quad \cos\theta_2 = \frac{1}{2}$ and $\cos\theta_1 = 0$

i.e., $\theta_2 = 60^\circ$ and $\theta_1 = 90^\circ$

Hence, $\theta_1 = 90^\circ$ and $\theta_2 = 60^\circ$.

25. Let $c = xa + yb$

$= x(i + j - k) + y(i - j + k)$

$= (x + y)i + (x - y)j - (x - y)k$

Since c is $\perp$ to a

$\therefore \quad x + y + x - y + x - y = 0 \Rightarrow 3x = y$

Since $|c| = 1$,

$\therefore \quad (x + y)^2 + (x - y)^2 + (x - y)^2 = 1$

$\therefore \quad (4x)^2 + (-2x)^2 + (-2x)^2 = 1.$

$\Rightarrow \quad 24x^2 = 1. \quad \therefore \quad x = \frac{1}{2\sqrt{6}}$

$\therefore \quad c = 4xi + 2xj + 2xk = \frac{2}{\sqrt{6}}i - \frac{1}{\sqrt{6}}j + \frac{1}{\sqrt{6}}k$

i.e., $c = \frac{1}{\sqrt{6}}(2i - j - k)$

Let $d = d_1 i + d_2 j + d_3 k$

Since d is $\perp$ to a and c

$\therefore \quad d\cdot a = 0 \Rightarrow d_1 + d_2 - d_3 = 0$

and, $d\cdot c = 0 \Rightarrow 2d_1 - d_2 + d_3 = 0$

$\Rightarrow 3d_1 = 0, \therefore d_1 = 0, \therefore d_2 = d_3$

Since $|d| = 1$,

$\therefore \quad d_1^2 + d_2^2 + d_3^2 = 1$

$\Rightarrow \quad 0 + d_2^2 + d_2^2 = 1 \quad \Rightarrow \quad 2d_2^2 = 1$

$\Rightarrow \quad d_2^2 = \frac{1}{2}$, or $d_2 = \frac{1}{\sqrt{2}}, \therefore \quad d_2 = d_3 = \frac{1}{\sqrt{2}}$.

Hence, $d = \frac{1}{\sqrt{2}}(j + k)$.

26. Let x be the first term and y the common ratio of G.P.

Then, $a = p$th term $= xy^{p-1}$

$b = q$th term $= xy^{q-1}$

$c = r$th term $= xy^{r-1}$

Now, $(il_n a + jl_n b + kl_n c)\cdot[i(q - r) + j(r - p) + k(p - q)]$

$= (q - r)\, l_n a + (r - p)\, l_n b + (p - q)\, l_n c$

$= (q - r)\, l_n(xy^{p-1}) + (r - p)\, l_n(xy^{q-1}) + (p - q) l_n(xy^{r-1})$

$= (q - r)\, \{l_n x - (p - 1)\, l_n y\} + \ldots$

$= \{(q - r) + (r - p) + (p - q)\}\, l_n x + \{(q - r)(p - 1)$

$+ (r - p)(q - 1) + (p - q)(r - 1)\}\, l_n y = 0.$

Hence, the vectors are perpendicular.

27. Since a is collinear with $b = 6i - 8j - (15/2)\,k$.

$\therefore \quad$ there exists a scalar t, s.t.

$a = tb = 6ti - 8tj - (15/2)\, tk$.

Now, if a makes an angle θ with positve z-axis, then

$\cos\theta = \frac{a\cdot k}{|a||k|} = \frac{-(15/2)t}{|a|}$.

$\because \quad \theta < 90^\circ \quad \therefore \quad \cos\theta = \frac{-15t}{2|a|} > 0 \quad \Rightarrow \quad t < 0.$

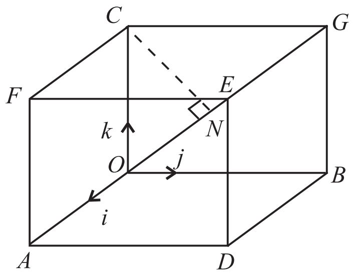
Now, $|a| = \sqrt{[\{36 + 64 + (225/4)\}t^2]} = 50$

$\Rightarrow \quad t^2 = 16 \Rightarrow t = -4$,

[Rejecting $t = 4$, $\because \; t < 0$]

Hence, $a = -24i + 32j + 30k = (-24, 32, 30)$.

28. Let the unit vectors i, j and k be denoted by the coterminous edges OA, OB and OC, respectively of the unit cube. Let CN be the perpendicular drawn from C on the diagonal OE of the cube which does not pass through C. Here, $OE = i + j + k$. Let e be the unit vector along OE. Then,



$$e = \frac{i+j+k}{|i+j+k|} = \frac{i+j+k}{\sqrt{(1+1+1)}}$$

$$= \frac{1}{\sqrt{3}}(i+j+k).$$

$\therefore$ ON = projection of OC on OE

$$= k \cdot e = k \cdot \frac{1}{\sqrt{3}}(i+j+k) = \frac{1}{\sqrt{3}}.$$

$\therefore$ $CN^2 = OC^2 - ON^2$ [In right triangle $\Delta\, OCN$]

$$= 1^2 - \left(\frac{1}{\sqrt{3}}\right)^2 = 1 - \frac{1}{3} = \frac{2}{3}.$$

$$\therefore \quad CN = \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

29. The length of a is $|a| = \sqrt{1^2+1^2} = \sqrt{2}$. Similarly, $|b| = \sqrt{2}$. Since the three vectors have equal length, $\therefore$ $|c| = \sqrt{2}$. Let $c = c_1 i + c_2 j + c_3 k$. Then, since c makes an obtuse angle with i, we must have $c \cdot i = c_1 < 0$. We are also given that the angles between the vectors are equal, i.e.,

$$\cos^{-1}\frac{a \cdot b}{|a||b|} = \cos^{-1}\frac{a \cdot c}{|a||c|} = \cos^{-1}\frac{b \cdot c}{|b||c|}$$

Now, $a \cdot b = 1$, $a \cdot c = c_1 + c_2$ and $b \cdot c = c_2 + c_3$, so

$$\cos^{-1}\frac{a \cdot b}{|a||b|} = \cos^{-1}\frac{1}{2} = \cos^{-1}\frac{c_1+c_2}{2}$$

$$= \cos^{-1}\frac{c_2+c_3}{2}.$$

This gives the equations $c_1 + c_2 = 1$ and $c_2 + c_3 = 1$, from which we get $c_3 = c_1$ and $c_2 = 1 - c_1$. Putting these values in $c_1^2 + c_2^2 + c_3^2 = 2$, we get

$$c_1^2 + (1-c_1)^2 + c_1^2 = 2$$

$$\Rightarrow \quad 3c_1^2 - 2c_1 - 1 = 0$$

$$\Rightarrow \quad (c_1 - 1)(3c_1 + 1) = 0 \quad \Rightarrow \quad c_1 = 1, -\frac{1}{3}.$$

Since c_1 must be less than zero, we rule out the solution $c_1 = 1$, giving $c_1 = -1/3 = c_3$ and $c_2 = 1 - c_1 = \frac{4}{3}$. Hence, the required vector is

$$c = -\frac{1}{3}i + \frac{4}{3}j - \frac{1}{3}k.$$

30. Let the given points be A, B, C, D, respectively. Now, if the points A, B, C, D are coplanar, then AB, AC, AD are coplanar.

$\therefore$ $AB \cdot (AC \times AD) = 0$

$\Rightarrow$ $(b-a) \cdot \{(c-a) \times (d-a)\} = 0.$

$\Rightarrow$ $(b-a) \cdot \{c \times d - c \times a - a \times d\} = 0$ $\quad \because a \times a = 0$

$\Rightarrow$ $b \cdot (c \times d) - b \cdot (c \times a) - b \cdot (a \times d) - a \cdot (c \times d)$
$+ a \cdot (c \times a) + a \cdot (a \times d) = 0$

$\Rightarrow$ $[b\ c\ d] - [b\ c\ a] - [b\ a\ d] - [a\ c\ d] + 0 + 0 = 0$

$\Rightarrow$ $[b\ c\ d] - [a\ b\ c] + [a\ b\ d] + [c\ a\ d] = 0$

$\Rightarrow$ $[b\ c\ d] + [c\ a\ d] + [a\ b\ d] = [a\ b\ c].$

31. Vector $\perp$ to face OAB

$$= OA \times OB = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5i - j - 3k \qquad (1)$$

Vector $\perp$ to the face ABC

$$= AB \times AC = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = i - 5j - 3k \qquad (2)$$

Since the angle between the faces = angle between their normals

$$\therefore \quad \cos\theta = \left|\frac{5+5+9}{\sqrt{35}\sqrt{35}}\right| = \frac{19}{35}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{19}{35}\right).$$

32. We have,

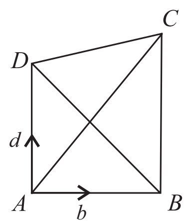
$BC = BA + AC = -b + mb + pd$

$= (m-1)\, b + pd$

and, $CD = CA + AD = -mb - pd + d$

$= -mb + (1-p)\, d$

Area of the triangle $ABD = \frac{1}{2}|b \times d|$.



Area of the triangle BCD

$$= \frac{1}{2}|(m-1)b + pd) \times (-mb + (1-p)d)|$$

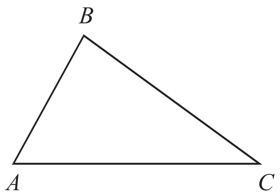
$$= \frac{1}{2}|-pmd \times b + (1-p)(m-1)b \times d|$$

$$= \frac{1}{2}|m-1+p||b \times d| = \frac{1}{2}(p+m-1)\,|b \times d|$$

Hence, the area of the quadrilateral $ABCD$

$= \frac{1}{2}(p+m)|b \times d|$. $\therefore$ $k = \frac{1}{2}$.

33. Let ABC be a triangle in which the given vectors are represented by the sides AB and AC.



i.e., $AB = \sqrt{3}(a \times b)$

and, $AC = b - (a \cdot b)a$

$\therefore$ $AB \cdot AC = \sqrt{3(a \times b) \cdot [b - (a \cdot b)a]}$

$= \sqrt{3[(a \times b) \cdot b - (a \cdot b)(a \times b) \cdot a]}$

$= \sqrt{3[0 - 0]} = 0$.

Therefore, $\angle BAC = 90°$

$AB^2 = [\sqrt{3}(a \times b)]^2 = 3(a \times b)^2$ (1)

$AC^2 = [b - (a \cdot b)a]^2$

$= (b)^2 + (a \cdot b)^2 a^2 - 2(b \cdot a)(a \cdot b)$

$= (b)^2 + (a \cdot b)^2 - 2(a \cdot b)^2$

$= (b)^2 - (a \cdot b)^2$

$= (b)^2 - |a|^2 |b|^2 \cos^2\theta$

$= (b)^2[1 - |a|^2 \cos^2\theta] = (b)^2(1 - \cos^2\theta)$

$= (b)^2 \sin^2\theta = |a|^2 |b|^2 \sin^2\theta = (a \times b)^2$. (2)

Dividing (i) by (ii), we get

$$\frac{AB^2}{AC^2} = \frac{3(a \times b)^2}{(a \times b)^2}$$

$\Rightarrow$ $AB^2 = 3 \cdot AC^2$ $\Rightarrow$ $AB = \sqrt{3}\, AC$.

$\tan C = \frac{AB}{AC} = \frac{\sqrt{3}\, AC}{AC} = \sqrt{3}$, $\therefore$ $\angle C = 60°$

So $\therefore$ $\angle A = 180 - 90° - 60° = 30°$.

Hence, angles of the triangle are 30°, 90° and 60°.

34. Given equation is $w + (w \times u) = v$ (1)

Taking cross-product with u, we get

$u \times [w + (w \times u)] = u \times v$

$\Rightarrow$ $u \times w + u \times (w \times u) = u \times v$

$\Rightarrow$ $u \times w + (u \cdot u)w - (u \cdot w)u = u \times v$

$\Rightarrow$ $u \times w + w - (u \cdot w)u = u \times v$ (2)

Taking scalar product of (1) with u we get

$u \cdot w + u \cdot (w \times u) = u \cdot v$

$\Rightarrow$ $u \cdot w = u \cdot v$ [$\because$ $u \cdot (w \times u) = 0$] (3)

Taking scalar product of (1) with v we get

$v \cdot w + v \cdot (w \times u) = v \cdot v$

$\Rightarrow$ $v \cdot w + [vwu] = 1$

$\Rightarrow$ $(u \times v) \cdot w = 1 - v \cdot w$ (4)

Taking scalar product of (2) with w we get

$(u \times w) \cdot w + w \cdot w - (u \cdot w)(u \cdot w) = (u \times v) \cdot w$

$\Rightarrow$ $0 + |w|^2 - (u \cdot w)^2 = (u \times v) \cdot w$ (5)

$\Rightarrow$ $(u \times v) \cdot w = |w|^2 - (u \cdot w)^2$.

Taking scalar product of (1) with w we get

$w \cdot w + (w \times u) \cdot w = v \cdot w$

$\Rightarrow$ $|w|^2 = v \cdot w$

$\Rightarrow$ $|w|^2 = 1 - (u \times v) \cdot w$ [using (4)] (6)

From (5) we get

$(u \times v) \cdot w = |w|^2 - (u \cdot w)^2$

$= 1 - (u \times v) \cdot w - (u \cdot w)^2$

$\Rightarrow$ $2(u \times v) \cdot w = 1 - (u \cdot v)^2$ [Using (3)]

Thus, $|(u \times v) \cdot w| = \frac{1}{2}|1 - (u \cdot v)^2|$

$\leq \frac{1}{2}$ [$\therefore$ $(u \cdot v)^2 \geq 0$].

35. Let I be a unit vector in the direction of b, J in the direction of c. Note that $b = I$ and $c = J$.

We have $b \times c = |b||c| \sin\alpha\, k = \sin\alpha\, k$,

where k is a unit vector perpendicular to b and c.

$\Rightarrow$ $|b \times c| = \sin\alpha$ $\Rightarrow$ $k = \frac{b \times c}{|b \times c|}$

Any vector a can be written as a linear combination of I, J and k. Let

$a = a_1 i + a_2 J + a_3 k$

Now, $a \cdot b = a \cdot I = a_1$, $a \cdot c = a \cdot J = a_2$

and, $a \cdot \frac{b \times c}{|b \times c|} = a \cdot k = a_3$

Thus, $(a \cdot b)b + (a \cdot c)c + \frac{a \cdot (b \times c)}{|b \times c|^2}(b \times c)$

$= a_1 b + a_2 c + a_3 \frac{b \times c}{|b \times c|}$

$= a_1 I + a_2 J + a_3 k = a$.

36. Let $c = xi + yj + zk$, where $x^2 + y^2 + z^2 = 1$

Unit vector along $3i + 4j = \frac{1}{5}(3i + 4j)$

$\therefore$ equation of the bisector of these two is

$$r = t\left[(xi + yj + zk) + \left(\frac{3i + 4j}{5}\right)\right]$$

But the bisector is $-i + j - k$.

$\therefore$ $t\left(x + \frac{3}{5}\right)i + t\left(y + \frac{4}{5}\right)j + tz = -i + j - k$

$\Rightarrow$ $x = -\frac{1}{t} - \frac{3}{5} = -\frac{3t + 5}{5t}$,

$y = \frac{1}{t} - \frac{4}{5} = \frac{5 - 4t}{5t}$ and $z = -\frac{1}{t}$

But $x^2 + y^2 + z^2 = 1$

$\therefore$ $\left(-\frac{3t + 5}{5t}\right)^2 + \left(\frac{5 - 4t}{5t}\right) + \frac{1}{t^2} = 1$

$\Rightarrow\ 9t^2 + 30t + 25 + 25 + 16t^2 - 40t + 25 = 25t^2$

$\Rightarrow\ 75 - 10t = 0, \therefore\ t = \dfrac{75}{10} = \dfrac{15}{2}$

Hence, $x = -\dfrac{\frac{45}{2}+5}{\frac{75}{2}} = \dfrac{-55}{75} = -\dfrac{11}{15}$

$y = \dfrac{5-30}{\frac{75}{2}} = \dfrac{-50}{75} = \dfrac{-10}{15}$

and, $z = -\dfrac{1}{\frac{15}{2}} = -\dfrac{2}{15}$

$\therefore$ unit vector along c

$= \dfrac{-11i}{15} - \dfrac{10j}{15} - \dfrac{2k}{15}$ or $-\dfrac{11i+10j+2k}{15}$.

37. Since $|a+b+c| = 1 \ \Rightarrow\ (a+b+c)\cdot(a+b+c) = 1$

$\Rightarrow\ 1+1+1+2(a\cdot b + b\cdot c + c\cdot a) = 1$

$\Rightarrow\ a\cdot b + b\cdot c + c\cdot a = -1$

$\Rightarrow\ \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = -1$

So, at least one of $\cos\theta_1$, $\cos\theta_2$ and $\cos\theta_3$ must be negative.

38. Since x and y are linearly independent,

$20a - 15b = 15b - 12c = 12c - 20a = 0 \ \Rightarrow\ \dfrac{a}{3} = \dfrac{b}{4} = \dfrac{c}{5}$

$\Rightarrow\ c^2 = a^2 + b^2 \Rightarrow \Delta ABC$ is right-angled.

39. We have,

$\alpha(a\times b) + \beta(b\times c) + \gamma(c\times a) = 0$

Taking dot product with c, we have

$\alpha[a\ b\ c] + \beta[b\ c\ c] + \gamma[c\ a\ c] = 0$

i.e., $\alpha[a\ b\ c] + 0 + 0 = 0$

i.e., $\alpha[a\ b\ c] = 0$

Similarly, taking dot product with b and c, we have

$\gamma[a\ b\ c] = 0,\ \beta[a\ b\ c] = 0$

Now, even if one of $\alpha, \beta, \gamma \neq 0$, then we have $[a\ b\ c] = 0 \Rightarrow a, b, c$ are coplanar.

40. Since $p\times q = r$

$\therefore\ p\times(p\times q) = p\times r$

$\Rightarrow\ (p\cdot q)p - (p\cdot p)q = p\times r$

$\Rightarrow\ cp - (p)^2 q = p\times r$

$\Rightarrow\ (p)^2 q = cp - p\times r$

$\Rightarrow\ q = \dfrac{cp - p\times r}{|p|^2}$.

41. $r\times a = b\times a \ \Rightarrow\ (r-b)\times a = \vec{0}$

$\Rightarrow\ r-b$ is paralllel to a.

$\therefore\ r - b = \lambda a$ i.e. $r = b + \lambda a$ (1)

Similarly, $r\times b = a\times b$ can be written as

$r = a + \mu b$ (2)

For point of intersection of the two lines (1) and (2),

we get $b + \lambda a = a + \mu b \ \Rightarrow\ \lambda = \mu = 1$

Hence, the required point of intersection is given by

$r = a + b = i + j + 2i - k = 3i + j - k$.

42. Let $a = 2i + 4j - 5k,\ b = i + 2j + 3k$

$\therefore$ diagonals of the parallelogram are

$p = a + b$ and $q = b - a$

i.e., $p = 3i + 6j - 2k$

and, $q = -i - 2j + 8k$

$\therefore$ unit vectors along the diagonals are

$\dfrac{3i+6j-2k}{\sqrt{9+36+4}}$ and $\dfrac{-i-2j+8k}{\sqrt{1+4+64}}$

i.e., $\dfrac{3i+6j-2k}{7}$ and $\dfrac{-i-2j+8k}{\sqrt{69}}$

$\therefore$ unit vector parallel to one of the diagonals is

$\dfrac{1}{7}(3i + 6j - 2k)$

More than One Option Correct Type

43. Given $x + y = a$

$\Rightarrow\ y = a - x$ (1)

$x\times y = b$ (2)

$x\cdot a = 1$ (3)

From (1) and (2), we get

$x\times(a - x) = b$

$\Rightarrow\ x\times a - x\times x = b \Rightarrow x\times a = b$

$\Rightarrow\ a\times(x\times a) = a\times b \Rightarrow (a\cdot a)x - (a\cdot x)a = a\times b$

$\Rightarrow\ |a|^2 x - 1\cdot a = a\times b$ [From (3)]

$\Rightarrow\ x = \dfrac{(a + a\times b)}{a^2}$

and, $y = a - x = \dfrac{(a^2-1)a - a\times b}{a^2}$.

44. We have, $a\times(b\times c) + (a\cdot b)\,b$

$= (4 - 2\beta - \sin\alpha)\,b + (\beta^2 - 1)\,c$ (1)

and, $(c\cdot c)\,a = c$ (2)

where b and c are non-collinear vectors and α, β are scalars

From (2), $(c\cdot c)\,a\cdot c = c\cdot c$

$\therefore\ a\cdot c = 1$ (3)

From (1), we get

$(a\cdot c)\,b - (a\cdot b)\,c + (a\cdot b)\,b$

$= (4 - 2\beta - \sin\alpha)\,b + (\beta^2 - 1)\,c$

or, $\{1 + (a\cdot b)\}\,b - (a\cdot b)\,c$

$= (4 - 2\beta - \sin\alpha)\,b + (\beta^2 - 1)\,c$

$\Rightarrow\ 1 + (a\cdot b) = 4 - 2\beta - \sin\alpha$ (4)

and, $a \cdot b = -(\beta^2 - 1)$ (5)

$\therefore$ $\sin\alpha = 1 + (1-\beta)^2 \Rightarrow \beta = 1, \sin\alpha = 1$

i.e., $\alpha = \dfrac{\pi}{2} + 2n\pi, n \in I.$

45. We have, $x \times b = c \times b$

$\Rightarrow$ $a \times (x \times b) = a \times (c \times b)$

$\Rightarrow$ $(a \cdot b)\, x - (a \cdot x)\, b = a \times (c \times b)$

$\Rightarrow$ $x = \dfrac{a \times (c \times b)}{a \cdot b}$ $(\because\ a \perp x).$

Also, $a \times (c \times b) = -(c \times b) \times a = (b \times c) \times a$

$\therefore$ $x = \dfrac{(b \times c) \times a}{b \cdot a}.$

46. Let $c = c_1 i + c_2 j$. Since $c \perp b$ $\therefore$ $b \cdot c = 0$

$\Rightarrow$ $(4i + 3j)\cdot(c_1 i + c_2 j) = 0 \Rightarrow 4c_1 + 3c_2 = 0$

$\Rightarrow$ $c_2 = -\dfrac{4}{3}c_1$. Thus, $c = \dfrac{c_1}{3}[3i - 4j].$

Now, the projection of $a = \pm(a_1 i + a_2 j)$ on b is 1 and on c is 2

$\therefore$ $1 = \left|\dfrac{4a_1 + 3a_2}{5}\right|$ and $2 = \left|\dfrac{3a_1 - 4a_2}{5}\right|$

Solving, we get $a_1 = 2$ and $a_2 = -1$ $\therefore$ $a = \pm(2i - j).$

47. Let $a = 2i - 2j + k$ and $b = i + 2j - 2k$

$\therefore$ $|a| = 3$ and $|b| = 3.$

$\therefore$ a vector along bisectors

$= \dfrac{a}{|a|} \pm \dfrac{b}{|b|} = \dfrac{2i - 2j + k}{3} \pm \dfrac{i + 2j - 2k}{3}$

$= i - \dfrac{1}{3}k, \dfrac{1}{3}i - \dfrac{4}{3}j + k.$

$\therefore$ The required vector

$$= 2 \cdot \frac{i - \frac{1}{3}k}{\sqrt{1^2 + \left(-\frac{1}{3}\right)^2}}, 2 \cdot \frac{\frac{1}{3}i - \frac{4}{3}j + k}{\left(\frac{1}{3}\right)^2 + \left(-\frac{4}{3}\right)^2 + 1^2}$$

$$= \frac{2}{\sqrt{10}}(3i - k), \frac{2}{\sqrt{26}}(i - 4j + 3k).$$

48. We have,

$(i + j + 3k)\, x + (3i - 3j + k)\, y + (-4i + 5j)\, z$

$= \lambda(xi + yj + xk)$

$\Rightarrow$ $(x + 3y - 4z - \lambda x)i + (x - 3y + 5z - \lambda y)j$

$+ (3x + y + 0z - \lambda z)k = 0.$

Above is a relation of the form

$li + mj + nk = 0$

where i, j and k are non-coplanar and hence each of the coefficients l, m, n is zero.

$\therefore$ $(1 - \lambda)\, x + 3y - 4z = 0$

$x - (3 + \lambda)\, y + 5z = 0$

$3x + y - \lambda z = 0.$

Eliminating x, y, z we get

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -3-\lambda & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (1 - \lambda)(3\lambda + \lambda^2 - 5) - 3(-\lambda - 15) - 4(1 + 9 + 3\lambda) = 0$

$\Rightarrow -\lambda^3 + \lambda^2(1 - 3) + \lambda(5 + 3 + 3 - 12) + (-5 + 45 - 40) = 0$

$\Rightarrow$ $-\lambda^3 - 2\lambda^2 - \lambda = 0$ or $\lambda(\lambda^2 + 2\lambda + 1) = 0$

$\therefore$ $\lambda(\lambda + 1)^2 = 0$ which gives $\lambda = 0, -1, -1.$

49. We have, $a = 12i + 4j + 3k$

$b = 8i - 12j - 9k$

$c = 33i - 4j - 24k$

$\therefore$ $|a| = \sqrt{(12)^2 + 4^2 + 3^2} = \sqrt{144 + 16 + 9}$

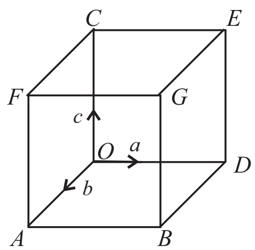
$= \sqrt{169} = 13$

$|b| = \sqrt{(8)^2 + (-12)^2 + (-9)^2}$

$= \sqrt{64 + 144 + 81} = \sqrt{289} = 17$

$|c| = \sqrt{(33)^2 + (-4)^2 + (-24)^2}$

$= \sqrt{1689 + 16 + 576} = \sqrt{1681} = 41.$



Now, area $OABD$ = area $CFGE = |a \times b|$.

$$a \times b = \begin{vmatrix} i & j & k \\ 12 & 4 & 3 \\ 8 & -12 & -9 \end{vmatrix}$$

$= i(-36 + 36) - j(-108 - 24) + k(-144 - 32)$

$= 0.\, i + 132j - 176k$

$\therefore$ $|a \times b| = \sqrt{0 + (132)^2 + (-176)^2} = \sqrt{(484000)}$

$= 220.$

In the same way, area $FABG$ = area $CODE = |a \times c|$

$|a \times c| = \sqrt{(189225)}$ or $|a \times c| = 435$

area $FAOC$ = area $GBDE = |b \times c|$

$\therefore$ $|b \times c| \sqrt{(207025)} = 455$

$$\text{Volume of parallelopiped} = [a\, b\, c] = \begin{vmatrix} 12 & 1 & 3 \\ 8 & -12 & -9 \\ 33 & -4 & -24 \end{vmatrix}$$

$$= (4 \times 3)\begin{vmatrix} 12 & 1 & 1 \\ 8 & -3 & -3 \\ 33 & -1 & -8 \end{vmatrix}$$

$$= 12\begin{vmatrix} 12 & 1 & 1 \\ 8 & -3 & -3 \\ 33 & -1 & -8 \end{vmatrix}$$

$= 12\,[12\,(24 - 3) - 1\,(-64 + 99) + 1\,(-8 + 99)]$

$= 12\,(12 \times 21 - 1 \times 35 + 91)$

$= 12\,(252 + 56) = 12 \times 308 = 3696.$

$\therefore$ Lengths of the edges are 13, 17, 41.

Areas of the faces are 220, 435, 455 and volume 3696.

50. Let the required vector be $\alpha = d_1 i + d_2 j + d_3 k$ where $d_1^2 + d_2^2 + d_3^2 = 51$ (given) (1)

Since, each of the given vectors a, b, c is a unit vector

$$\therefore \cos\theta = \frac{d \cdot a}{|d||a|} = \frac{d \cdot b}{|d||b|} = \frac{d \cdot c}{|d||c|}$$

$\Rightarrow$ $d \cdot a = d \cdot b = d \cdot c$ as $|d| = \sqrt{51}$ cancels out and $|a| = |b| = |c| = 1$

$$\Rightarrow \frac{1}{3}(d_1 - 2d_2 + 2d_3) = \frac{1}{5}(-4d_1 + 0d_2 - 3d_3) = d_2$$

$\therefore$ $d_1 - 5d_2 + 2d_3 = 0$ from 1st and 3rd

and $4d_1 + 5d_2 + 3d_3 = 0$ from 2nd and 3rd. Solving the above two equations, we get

$$\frac{d_1}{-15-10} = \frac{d_2}{8-3} = \frac{d_3}{5+20}$$

$$\Rightarrow \frac{d_1}{5} = \frac{d_2}{-1} = \frac{d_3}{-5} = \lambda \text{ (say)}$$

Putting for d_1, d_2 and d_3 in (1), we get

$(25 + 1 + 25)\,\lambda^2 = 51$, $\therefore$ $\lambda = \pm 1$.

Hence, the required vectors are $\pm(5i - j - 5k)$.

51. Since they are unit vectors, we have $|a| = |b| = 1 = |c|$. It is also given that the angle θ between a and c equals that between b and c, i.e.,

$$\frac{a \cdot c}{|a||c|} = a \cdot c = \cos\theta = \frac{b \cdot c}{|b||c|} = b \cdot c.$$

Since $a \cdot b = 0$, we get from the given value of c,

$a \cdot c = \alpha a \cdot a + \beta a \cdot b + \gamma a \cdot (a \times b) = \alpha$,

i.e., $\alpha = \cos\theta$, and similarly, $b \cdot c = \cos\theta = \beta$.

That is , $\alpha = \beta = \cos\theta$, so that answer (a) is correct. Next, we have

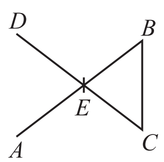
$1 = c \cdot c = 2\alpha^2 + \gamma^2 |a \times b|^2$

$= 2\alpha^2 + \gamma^2[\,|a|^2 |b|^2 - (a \cdot b)^2] = 2\alpha^2 + \gamma^2$

$\Rightarrow$ $\gamma^2 = 1 - 2\alpha^2 = 1 - 2\cos^2\theta = -\cos 2\theta$

$$\Rightarrow \alpha^2 = \beta^2 = \frac{1-\gamma^2}{2} = \frac{1+\cos 2\theta}{2}$$

52. Let $EB = pAB$, $CE = qCD$



then, $0 < p, q \le 1$

Since $EB + BC + CE = 0$

$\therefore$ $pm\,(2i - 6j + 2k) + (i - 2j) + qn\,(-6i + 15j - 3k) = 0$

$\Rightarrow$ $(2pm + 1 - 6qn)i + (-6pm - 2 + 15qn)j + (2pm - 6qn)k = 0$

$\Rightarrow$ $2pm - 6qn + 1 = 0$, $-6pm - 2 + 15qn = 0$, $2pm - 6qn = 0$.

Solving these, we get

$p = 1/(2m)$ and $q = 1/(3n)$.

$\therefore$ $0 < 1/(2m) \le 1$ and $0 < 1/(3n) \le 1$

$\Rightarrow$ $m \ge 1/2$ and $n \ge 1/3$.

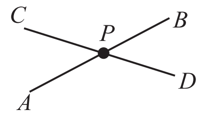
The area of $\Delta BCE = \frac{1}{2}|EB \times BC| = \frac{1}{2}\sqrt{6}$.

53. Given A $= (9i - j + 7k)$

$C = (7i - 2j + 7k)$

$AB = 4i - j + 3k$

and, $CD = 2i - j + 2k$



Let the position vector of P, the point of intersection of AB and CD be r.

Then, $AP = s\,AB$

$\Rightarrow$ $r - (9i - j + 7k) = s\,(4i - j + 3k)$

and, $CP = t\,CD$

$\Rightarrow$ $r - (7i - 2j + 7k) = t\,(2i - j + 2k)$

$\therefore$ $(4s + 9)\,i - (s + 1)\,j + (3s + 7)\,k$

$= (2t + 7)\,i - (t + 2)\,j + (2t + 7)\,k$

$\therefore$ Equating the coefficients and solving, we get

$s = -2$, $t = -3$, $\therefore$ $r = i + j + k$

$\therefore$ Position vector of $P = i + j + k$ (1)

$$AB \times CD = \begin{vmatrix} i & j & k \\ 4 & -1 & 3 \\ 2 & -1 & 2 \end{vmatrix} = I - 2j - 2k.$$

$\therefore$ $PQ \,||\, (i - 2j - 2k)$. As $|PQ| = 15$,

$$\therefore PQ = \pm 15\frac{(i - 2j - 2k)}{3} = \pm 5(i - 2j - 2k)$$

$\Rightarrow$ p.v. of Q – p.v. of $P = \pm\,5\,(i - 2j - 2k)$.

$\Rightarrow$ p.v. of Q $= (i + j + k) \pm 5\,(i - 2j - 2k)$

$= (6i - 9j - 9k)$ or $(-4i + 11j + 11k)$.

54. We have, $DA = a$, $AB = b$ and $CB = ka$

Since X, Y are mid-points of DB and AC.

$$\therefore OX = \frac{OB + OD}{2},\ OY = \frac{OA + OC}{2}$$

$$\therefore XY = OY - OX = \frac{OA + OC - OB - OD}{2} = \frac{DA + BC}{2} = \frac{a - ka}{2}$$

$$\therefore XY = \frac{(1-k)a}{2}$$

$$\therefore |XY| = \pm\left(\frac{1-k}{2}\right)|a| = 4 \text{ (given)}.$$

$\Rightarrow \left(\frac{1-k}{2}\right)17 = 4$ (taking positive sign)

$\Rightarrow 1-k = \frac{8}{17} \Rightarrow k = 1 - \frac{8}{17} = \frac{9}{17}$

and taking negative sign, $-\left(\frac{1-k}{2}\right)17 = 4$

$\Rightarrow k-1 = \frac{8}{17} \Rightarrow k = 1 + \frac{8}{17} = \frac{25}{17}.$

Hence, $k = \frac{9}{17}$ or $\frac{25}{17}$.

55. Let θ be the angle between unit vectors a and b. As a and b are non-collinear, $\theta \neq 0$ and $\theta \neq \pi$.

We have $a \cdot b = |a||b|\cos\theta = \cos\theta$.

Now, $|u|^2 = |a - (a \cdot b)\, b|^2 = |a - \cos\theta\, b|^2$

$= |a|^2 + \cos^2\theta\, |b|^2 - 2\cos\theta (a \cdot b)$

$= 1 + \cos^2\theta - 2\cos^2\theta = 1 - \cos^2\theta$

$= \sin^2\theta.$

Also, $|v|^2 = |a \times b|^2 = |a|^2\, |b|^2 \sin^2\theta$

$\therefore$ $|v|^2 = |u|^2 \Rightarrow |v| = |u|$.

Also, $u \cdot a = (a - (a \cdot b)\, b) \cdot a = a \cdot a - (a \cdot b)(b \cdot a)$

$= |a|^2 - \cos^2\theta = 1 - \cos^2\theta = \sin^2\theta$

$\therefore$ $|u| + |u \cdot a| = \sin\theta + \sin^2\theta \neq |u|$.

Next, $u \cdot b = a \cdot b - (a \cdot b)(b \cdot b) = a \cdot b - (a \cdot b)$

$= 0$ (1)

$\therefore$ $|u| + |u \cdot b| = |u| + 0 = |u| = |v|$.

Also, $u \cdot (a+b) = u \cdot a + u \cdot b = u \cdot a$

$\Rightarrow$ $|u| + u \cdot (a+b) = |u| + u \cdot a \neq |v|$.

56. Equation of the plane containing i and $i+j$ is $[r - iii + j] = 0$

or, $(r-i) \cdot [i \times (i+j)] = 0$

or, $\{(x-1)\, i + yj + zk\} \cdot k = 0$

or, $z = 0$ (1)

Equation of the plane containing $i - j$ and $i + k$ is

$[r - (i-j)i - ji + k] = 0$

or, $[(\bar{r} - i + j) \cdot [(i-j) \times (i+k)] = 0$

or, $[(x-1)i + (y+1)j + zk] \cdot (-i - j + k) = 0$

or, $-(x-1) - (y+1) + z = 0$

or, $x + y - z = 0.$ (2)

Let $a = a_1 i + a_2 j + a_3 k.$

Since a is parallel to (1) and (2)

$a_3 = 0$ and $a_1 + a_2 - a_3 = 0$

$\Rightarrow$ $a_1 = -a_2,\, a_3 = 0.$

Thus, a vector in the direction of a is $u = i - j$.

If θ is the angle between a and $i - 2j + 2k$, then

$$\cos\theta = \pm\frac{(1)(1) + (-1)(-2)}{\sqrt{1+1}\,\sqrt{1+4+4}} = \pm\frac{3}{(\sqrt{2})(3)}$$

$$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

57. Any vector coplanar with vectors $i + j + 2k$ and $i + 2j + k$ is

$x(i + j + 2k) + y(i + 2j + k)$

i.e., $(x+y)i + (x+2y)j + (2x+y)k.$ (1)

If this is perpendicular to $i + j + k$, then

$(x+y) + (x+2y) + (2x+y) = 0$

$\Rightarrow$ $4x + 4y = 0 \Rightarrow y = -x.$

Substituting $y = -x$ in (1), the vector coplanar with the given vectors is $x(-j+k)$.

So, the required unit vector $= \frac{x(-j+k)}{\sqrt{x^2(1^2+1^2)}}$

$= \pm\frac{(-j+k)}{\sqrt{2}}$

58. Let $c = xi + yj + zk$.

Given, $|c| = |a| = |b|$ [equal magnitude]

$\Rightarrow$ $x^2 + y^2 + z^2 = 2$ (1)

Also, $\frac{a \cdot b}{|a||b|} = \frac{b \cdot c}{|b||c|} = \frac{c \cdot a}{|c||a|}$ [equally inclined]

$$\Rightarrow \frac{0+1+0}{\sqrt{2}\,\sqrt{2}} = \frac{y+z}{\sqrt{2}\,\sqrt{2}} = \frac{x+y}{\sqrt{2}\,\sqrt{2}}$$

$\Rightarrow$ $1 = y + z = x + y$ (2)

Solving equations (1) and (2), we have

$(1-y)^2 + y^2 + (1-y)^2 = 2$

i.e., $3y^2 - 4y = 0$

$\Rightarrow y = 0, 4/3$ and corresponding $x = 1, -1/3$ and $z = 1, -1/3$.

Hence, $c = (1, 0, 1)$ or $\left(\frac{-1}{3}, \frac{4}{3}, \frac{-1}{3}\right)$.

59. We have $a = -i + 2j + 1k$

Let the new vector be $b = xi + yj + zk$

We have, $|a| = |b|$

i.e., $x^2 + y^2 + z^2 = 1^2 + 2^2 + 1^2 = 6$ (1)

and, $a \cdot b = 0$

i.e., $-x + 2y + z = 0$

Also a, b and y-axis lie in one plane (2)

$\Rightarrow$ a, b, j are coplanar

$\Rightarrow$ $[a\, b\, j] = 0$

i.e., $\begin{vmatrix} -1 & 2 & 1 \\ x & y & z \\ 0 & 1 & 0 \end{vmatrix} = 0$

i.e., $z + x = 0$ (3)

From equations (2) and (3), we have

$z = -x, y = x$

Putting in equation (1), we get $x = \pm\sqrt{2}$

Hence, $b = \pm\sqrt{2}\,(i + j - k)$.

60. Given $|a| = 1, |c| = 1, |b| = 4$

$a \times b = 2a \times c$ and angle between a and c is $\cos^{-1}\left(\frac{1}{4}\right)$

Now, $a \cdot c = |a||c| \cos\left[\cos^{-1}\frac{1}{4}\right] = (1)\,(1)\left(\frac{1}{4}\right) = \frac{1}{4}$

$\therefore \quad a \cdot c = \frac{1}{4}$ (1)

Again, $b - 2c = \lambda a$ (2)

$\Rightarrow \quad a \cdot b - 2a \cdot c = \lambda a \cdot a$

$\Rightarrow \quad a \cdot b - \frac{1}{2} = 1 \qquad [\because \quad a \cdot a = a^2 = a^2 = 1]$

$\Rightarrow \quad a \cdot b = \lambda + \frac{1}{2}$ (3)

Again, from (2)

$b \cdot b - 2b \cdot c = \lambda\, a \cdot b$

$\Rightarrow \quad 16 - 2b \cdot c = \lambda\left(\lambda + \frac{1}{2}\right)$ (by (3))

$\Rightarrow \quad b \cdot c = 8 - \frac{\lambda^2}{2} - \frac{\lambda}{4}$ (4)

Also, from (2)

$b \cdot c - 2c \cdot c = \lambda\,\vec{a} \cdot \vec{c}$

$\therefore \quad 8 - \frac{\lambda^2}{2} - \frac{\lambda}{4} - 2(1) = \lambda \cdot \left(\frac{1}{4}\right)$

$\Rightarrow \quad 32 - 2\lambda^2 - \lambda - 8 = \lambda$

$\Rightarrow \quad 2\lambda^2 + 2\lambda - 24 = 0 \Rightarrow \lambda^2 + \lambda - 12 = 0$

$\Rightarrow \quad (\lambda + 4)\,(\lambda - 3) = 0, \therefore \quad \lambda = 3, -4.$

61. Let a_0 and b_0 be unit vectors along a and b respectively, then, $a_0 = (1/9)(7i - 4i - 4k)$

and, $b_0 = \frac{1}{3}(-2i - j + 2k)$

The required vector $c = \lambda(a_0 + b_0)$ (where λ is a scalar)

$= \lambda(1/9)i - (7/9)j + (2/9)k$

$\Rightarrow \quad |c|^2 = \lambda^2 \times \left(\frac{1 + 49 + 4}{81}\right)$

$\Rightarrow \quad (3\sqrt{6})^2 = (54/81)\lambda^2 \quad \Rightarrow \quad \lambda = \pm 9.$

$\therefore \quad c = \pm(i - 7j + 2k).$

Passage Based Questions

62. $r = xi + yi + zk$. then $|r| = \sqrt{x^2 + y^2 + z^2}$

and, $\phi = \ln|r| = \frac{1}{2}\ln(x^2 + y^2 + z^2).$

$$\nabla\phi = \frac{1}{2}\nabla\ln(x^2 + y^2 + z^2)$$

$$= \frac{1}{2}\left\{i\frac{\partial}{\partial x}\ln(x^2 + y^2 + z^2) + j\frac{\partial}{\partial y}\ln(x^2 + y^2 + z^2) + k\frac{\partial}{\partial z}\ln(x^2 + y^2 + z^2)\right\}$$

$$= \frac{1}{2}\left\{i\frac{2x}{x^2 + y^2 + z^2} + j\frac{2y}{x^2 + y^2 + z^2} + k\frac{2z}{x^2 + y^2 + z^2}\right\}$$

$$= \frac{xi + yj + zk}{x^2 + y^2 + z^2} = \frac{r}{r^2}.$$

63. $\nabla\phi = \nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right)$

$$= \nabla\{(x^2 + y^2 + z^2)^{-1/2}\}$$

$$= i\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-1/2} + j\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{-1/2} + k\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{-1/2}$$

$$= i\left\{-\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}2x\right\} + j\left\{-\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}2y\right\} + k\left\{-\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}2z\right\}$$

$$= \frac{-xi - yi - zk}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{r}{r^3}.$$

64. $\nabla r^n = \nabla(\sqrt{x^2 + y^2 + z^2})^n = \nabla(x^2 + y^2 + z^2)^{n\backslash 2}$

$$= i\frac{\partial}{\partial x}\{(x^2 + y^2 + z^2)^{n/2}\} + j\frac{\partial}{\partial y}\{(x^2 + y^2 + z^2)^{n/2}\} + k\frac{\partial}{\partial z}\{(x^2 + y^2 + z^2)^{n/2}\}$$

$$= i\left\{\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1}2x\right\} + j\left\{\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1}2y\right\} + k\left\{\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1}2z\right\}$$

$$= n(x^2 + y^2 + z^2)^{\frac{n}{2}-1}(xi + yj + zk)$$

$$= n(r^2)^{\frac{n}{2}-1}r = nr^{n-2}r$$

65. $\nabla\phi = i\frac{\partial}{\partial x}(2x^3y^2z^4) + j\frac{\partial}{\partial y}(2x^3y^2z^4) + k\frac{\partial}{\partial z}(2x^3y^2z^4)$

$= 6x^2y^2z^4\,i + 4x^3\,y\,z^4j + 8x^3y^2z^3\,k$

Then,

$$\nabla \cdot \nabla\phi = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \cdot (6x^2y^2z^4i + 4x^3yz^4j + 8x^3y^2z^3k)$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

$\therefore \quad k = 4.$

66. $\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}[-x(x^2+y^2+z^2)^{-3/2}]$$

$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$

$$= \frac{2x^2 - y^2 - x^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2 - z^2 - x^2}{(x^2+y^2+z^2)^{5/2}}$$

and, $\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}}$

Then, by addition,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = 0.$$

67. $\nabla \cdot \left(\frac{r}{r^3}\right) = \nabla \cdot (r^{-3}r)$

$= (\nabla r^{-3}) \cdot r + (r^{-3})\nabla \cdot r \quad [\because \ \nabla \cdot (\phi A) = (\nabla \phi).A + \phi(\nabla \cdot A)]$

$= -3r^{-5}r \cdot r + 3r^{-3} = 0 \quad [\because \ \nabla \cdot r = 3]$

68. $\nabla \times (\nabla \phi) = \nabla \times \left(\frac{\partial \phi}{\partial x}i + \frac{\partial \phi}{\partial y}j + \frac{\partial \phi}{\partial z}k\right)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)\right]i + \left[\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)\right]j + \left[\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right]k$$

$$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right)i + \left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right)j + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)k = 0.$$

69. $\nabla \cdot (\nabla \times A) = \nabla \cdot \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$= \nabla \cdot \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)i + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)j + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)k\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} = 0.$$

Match the Column Type

70 I. $BD \cdot AC = |BD||AC|\cos\theta$, **where** θ **is the angle between** BD **and** AC.

$\Rightarrow \quad (OD - OB) \cdot (OC - OA)$

$= |OD - OB||OC - OA|\cos\theta$

$\Rightarrow \quad (a - 2b - b) \cdot (2a + 3b - a) = |a - 3b||a + 3b|\cos\theta$

$\Rightarrow \quad a^2 - 9b^2 = |a - 3b||a + 3b|\cos\theta$

$\Rightarrow \quad 0 = |a - 3b||a + 3b|\cos\theta \, [\because |a| = 3|b|, \ \therefore \ a^2 = 9b^2]$

$\Rightarrow \quad \cos\theta = 0, \therefore \quad \theta = \frac{\pi}{2}.$

II. Let angles between a and b and between a and c be α and β, respectively.

We have, $a \times (b \times c) = \frac{1}{2}(b + c)$

or, $\quad (a \cdot c)b - (a \cdot b)c = \frac{1}{2}b + \frac{1}{2}c$

or, $\quad (a \cdot c - 1/2)\, b - (a \cdot b + 1/2)\, c = 0$

or, $\quad a \cdot c = \frac{1}{2}, a \cdot b = -\frac{1}{2}$ (As b and c are non-parallel)

or, $\quad \cos\beta = \frac{1}{2}, \cos\alpha = -\frac{1}{2}$ [a, b, c are unit vectors]

or, $\quad \beta = \frac{\pi}{3}, \alpha = \frac{2\pi}{3}.$

Hence, angle between a and b is $\frac{2\pi}{3}$.

III. A normal to the plane P_1 is given by $N_1 = a \times b$ and a normal to the plane P_2 is given by $N_2 = c \times d$.

Since $(a \times b) \times (c \times d) = 0$, we get $N_1 \times N_2 = 0$.

If θ is an angle between P_1 and P_2, we get

$|N_1| \times |N_2| \sin\theta = 0$ or $\sin\theta = 0 \Rightarrow \theta = 0.$

IV. We have, $|a \cdot b| = |a \times b|$

$\Rightarrow \ |a||b||\cos\theta| = |a||b||\sin\theta|$

(where 'θ' is the angle between a and b)

$\Rightarrow \ |\cos\theta| = |\sin\theta| \ \Rightarrow \ \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ (as $0 \le \theta \le \pi$)

But $a \cdot b < 0$, $\therefore \ \theta = \frac{3\pi}{4}$.

Assertion-Reason Type

71. We have, $|a \times r|^2 + |a \cdot r|^2 = |a|^2 |r|^2$

$\Rightarrow \ |J|^2 + (a \cdot r)^2 = |a|^2 |r|^2$

$\Rightarrow \ (a \cdot r) = \pm\sqrt{|a|^2 |r|^2 - 1}.$

This shows that $a \cdot r$ depends on $|r|$ for given a.

Hence, $a \cdot r$ is an arbitrary constant.

72. As a, b, c are non-coplanar, $b \times c$, $c \times a$, $a \times b$ are also non-coplanar.

So, any vector can be expressed as a linear combination of these vectors.

Let $a = \lambda b \times c + \mu c \times a + \nu a \times b$

$\therefore \ a \cdot a = \lambda\,[b\ c\ a],\ a \cdot b = \mu\,[c\ a\ b],\ a \cdot c = \nu\,[a\ b\ c]$

$\therefore \ a = \frac{(a \cdot a) b \times c}{[bca]} + \frac{(a \cdot b) c \times a}{[cab]} + \frac{(a \cdot c) a \times b}{[abc]}$

$\therefore \ (a \cdot a)\, b \times c + (a \cdot b)\, c \times a + (a \cdot c)\, a \times b = [b\ c\ a]\, a.$

73. We have,

$i \times [(a - j) \times i] = (i \cdot i)\,(a - j) - [i\,.\,(a - j)]\, i$

$= a - j - (i \cdot a) i$

$j \times [(a - k) \times j] = (j \cdot j)\,(a - k) - [(j \cdot (a - k)]\, j$

$= a - k - (j \cdot a)\, j$

$k \times [(a - i) \times k] = (k \cdot k)\,(a - i) - [(k \cdot (a - i)] k$

$= a - i - (k \cdot a)\, k$

According to the given condition, we have

$i \times [(a - j) \times i] + j \times [(a - k) \times j] + k \times [(a - i) \times k] = 0$

i.e., $3a - (i + j + k) - [(a \cdot i)\, i + (a \cdot j)\, j + (a \cdot k)\, k] = 0$

i.e., $3a - a = i + j + k$

i.e., $a = \frac{1}{2}(i + j + k).$

Previous Year's Questions

74. Given two vectors lie in xy–plane. Hence a vector coplanar with them is

$\vec{a} = x\vec{i} + y\vec{j}$

$\because \ \vec{a} \perp (\hat{i} - \hat{j}) \ \Rightarrow \ \vec{a} \cdot (\hat{i} - \hat{j}) = 0$

$\Rightarrow \ (x\hat{i} + y\hat{j}) \cdot (\hat{i} - \hat{j}) = 0 \ \Rightarrow \ x - y = 0$

$\Rightarrow$ x = y

$\therefore \ \vec{a} = x\hat{i} + x\hat{j}$ and $|\vec{a}| - \sqrt{x^2 + x^2} = x\sqrt{2}$

Required unit vector $= \frac{\vec{a}}{|\vec{a}|} = \frac{x(\hat{i} + \hat{j})}{x\sqrt{2}}$

$\therefore \ = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

75. The vector, $\hat{i} + x\hat{j} + z\hat{k}$ if it is doubled in magnitude becomes

$4\hat{i} + (4x - 2)\hat{j} + 2\hat{k}$

$\therefore \ 2|\hat{i} + x\hat{j} + 3\hat{k}| = |4\hat{i} + (4x - 2)\hat{j} + 2\hat{k}|$

$\Rightarrow \ 2\sqrt{1 + x^2 + 9} = \sqrt{16 + (4x - 2)^2 + 4}$

$\Rightarrow \ 40 + 4x^2 = 20 + (4x - 2)^2$

$\Rightarrow \ 3x^2 - 4x - 4 = 0$

$\Rightarrow \ 3x^2 - 6x + 2x - 4 = 0$

$\Rightarrow \ 3x\,(x - 2) + 2(x - 2) = 0$

$\Rightarrow \ (x - 2)\,(3x + 2) = 0$

$\Rightarrow \ x = 2, -\frac{2}{3}$

76. Key Idea: If the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ represent the sides of a triangle, then $\vec{a} + \vec{b} + \vec{c} = 0$

$\because \ \vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \ \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow \ (\vec{a} + \vec{b}) \times \vec{c} = -\vec{c} \times \vec{c}$

$\Rightarrow \ \vec{a} \times \vec{c} + \vec{b} \times \vec{c} = 0$

$\Rightarrow \ \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Similarly, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

Hence $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

77. Given that $\vec{a} = xi + y\hat{j} + z\hat{k}$ and $\vec{b} = \hat{j}$ are such that $\vec{a}, \vec{c}$ and $\vec{b}$ form a right handed system.

$$\therefore \ \vec{c} = \vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix}$$

$= \hat{i}z - x\hat{k}$

78. The equation of a line through the center represented by vector $\hat{j} + 2\hat{k}$ and normal to the given plane is

$\vec{r} = \hat{j} + 2\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ (1)

This meets the plane at a point for which we must have

$[(\hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})] \cdot (\hat{i} + 2\hat{j} + 2\hat{k}) = 15$

$\Rightarrow \ 6 + 9\lambda = 15$

$\Rightarrow \quad \lambda = 1$

On putting $\lambda = 1$ in Eq. (i), we get

$\vec{r} = \hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore$ Centre of the circle is (1, 3, 4).

79. $\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$\Rightarrow \quad (1 + abc)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$

$\Rightarrow \quad abc = -1.$

Hence, (8) is the correct answer

80. $(\vec{a} + 2\vec{b}) = t_1\vec{c}$ (1)

And $\vec{b} + 3\vec{c} = t_2\vec{a}$ (2)

$(1) - 2\times(2) \Rightarrow \vec{a}(1 + 2t_2) + \vec{c}(-t_1 - 6) = 0 \Rightarrow 1 + 2t_2 = 0$

$\Rightarrow t_2 = -1/2 \ \& \ t_1 = -6.$

(since $\vec{a}$ and $\vec{c}$ are non–collinear)

Putting the value of t_1 and t_2 in (1) and (2), we get $\vec{a} + 2\vec{b} + 6\vec{c} = \vec{0}$.

81. Work done by the forces $\vec{F_1}$ and $\vec{F_2}$ is $(\vec{F_1} + \vec{F_2})\cdot\vec{d}$ where $\vec{d}$ is the displacement.

Now, according to question

$\vec{F_1} + \vec{F_2} = (4\hat{i} + \hat{j} - 3\hat{k}) + (3\hat{i} + \hat{j} - \hat{k}) = 7\hat{i} + 2\hat{j} - 4\hat{k}$ and

$\vec{d} = (5\hat{i} + 4\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} + 2\hat{j} - 2\hat{k}$. Hence $(\vec{F_1} + \vec{F_2})\cdot\vec{d}$ is 40.

82. Condition for given three vectors to be coplanar is $\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & 2\lambda - 1 \end{vmatrix} = 0 \Rightarrow \lambda = 0, 1/2.$

Hence given vectors will be non coplanar for all real values of λ. except for 0 and 1/2.

83. Projection of $\vec{v}$ along $\vec{u}$ and $\vec{w}$ along $\vec{u}$ is $\dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|}$ and $\dfrac{\vec{w}\cdot\vec{u}}{|\vec{u}|}$ respectively

Now, according to question

$\dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|} = \dfrac{\vec{w}\cdot\vec{u}}{|\vec{u}|} \Rightarrow \vec{v}\cdot\vec{u} = \vec{w}\cdot\vec{u}\cdot$ and $\vec{v}\cdot\vec{w} = 0$

$|\vec{u} - \vec{v} + \vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{u}\cdot\vec{w} = 14$

$\Rightarrow |\vec{u} - \vec{v} + \vec{w}| = \sqrt{14}$

84. $(\vec{a}\times\vec{b})\times\vec{c} = \dfrac{1}{3}|\vec{b}||\vec{c}|\vec{a} \Rightarrow (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a} = \dfrac{1}{3}|\vec{b}||\vec{c}|\vec{a}$

85. By triangle law,

$\overrightarrow{PA} + \overrightarrow{AC} + \overrightarrow{CP} = 0$

$\overrightarrow{PB} + \overrightarrow{BC} + \overrightarrow{CP} = 0$

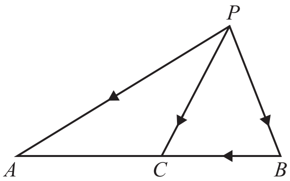
Adding, we get

$\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{AC} + \overrightarrow{BC} + 2\overrightarrow{CP} = 0$

Since $\overrightarrow{AC} = -\overrightarrow{BC}$

& $\overrightarrow{CP} = -\overrightarrow{PC}$

$\Rightarrow \quad \overrightarrow{PA} + \overrightarrow{PB} - 2\overrightarrow{PC} = 0.$



86. Distance between the line with equation $\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and the plane $\vec{r}\cdot(\hat{i} + 5\hat{j} + \hat{k}) = 5$ equivalently $x + 5y + z = 5$ is equal to the perpendicular distance of point (2, –2, 3) from the plane

i.e., $\left|\dfrac{2 - 10 + 3 - 5}{\sqrt{1 + 5^2 + 1}}\right| = \dfrac{10}{3\sqrt{3}}.$

87. Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$\vec{a}\times\hat{i} = z\hat{j} - y\hat{k}$

$\Rightarrow \quad (\vec{a}\times\hat{i})^2 = y^2 + z^2$

Similarly $(\vec{a}\times\hat{j})^2 = x^2 + z^2$

and $(\vec{a}\times\hat{k})^2 = x^2 + y^2 \Rightarrow (\vec{a}\times\hat{i})^2 = y^2 + z^2$

Similarly $(\vec{a}\times\hat{k})^2 = x^2 + y^2$

$\Rightarrow \quad (\vec{a}\times\hat{i})^2 + (\vec{a} + \hat{j})^2 + (\vec{a}\times\hat{k})^2 = 2(x^2 + y^2 + z^2) = 2\vec{a}^2.$

88. Given that a, b, c are in H.P.

$\Rightarrow \quad \dfrac{2}{b} - \dfrac{1}{a} - \dfrac{1}{c} = 0$

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{1}{c} = 0$

$\Rightarrow \quad \dfrac{x}{-1} = \dfrac{y}{2} = \dfrac{1}{-1}$

$\therefore \quad x = 1, y = -2$

89. Vectors $a\hat{i} + a\hat{j} + c\hat{k}, \hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ are coplanar

Therefore, $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0 \Rightarrow c^2 = ab$

$\therefore$ a, b, c are in G.P.

90. $[\lambda(\vec{a} + \vec{b})\ \lambda^2\vec{b}\ \lambda\vec{c}] = [\vec{a}\ \vec{b} + \vec{c}\ \vec{b}]$

$\Rightarrow \quad \begin{vmatrix} \lambda & \lambda & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix}$

$\Rightarrow \quad \lambda^4 = -1$

Hence no real value of λ.

91. Given $\vec{a}=\hat{i}-\hat{k}, \vec{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$ and

$\vec{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$.

$[\vec{a}\vec{b}\vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})$

$$\Rightarrow \quad \vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & 1 & 1-x\\ y & x & 1+x-y\end{vmatrix}=\hat{i}(1+x-x-y-x^2)$$

$$-\hat{j}(x+x^2-xy-y+xy)+\hat{k}(x^2-y)$$

$\hat{a}\cdot(\vec{b}\times\vec{c})=1$

Which does not depend on x and y.

92. Given condition $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c}), \vec{a}\cdot\vec{b}\neq 0, \vec{b}\cdot\vec{c}\neq 0$

$\Rightarrow \quad (\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$\Rightarrow \quad (\vec{a}\cdot\vec{b})\vec{c}=(\vec{b}\cdot\vec{c})\vec{a}$

$\Rightarrow \quad \vec{a}\,||\,\vec{c}$

93. The vectors

$\overrightarrow{BA}=\hat{i}-2\hat{j}+6\hat{k}$

$\overrightarrow{CA}=(2-a)\hat{i}+2\hat{j}$

$\overrightarrow{CB}=(1-a)\hat{i}-6\hat{j}$

imply

$\overrightarrow{CA}\cdot\overrightarrow{CB}=0$

$\Rightarrow \quad (2-a)(1-a)=0$

$\Rightarrow \quad a=2,1$

94. Given that $|2\hat{u}\times 3\hat{v}|=1$

$\Rightarrow \quad 6|\hat{u}||\hat{v}||\sin\theta|=1$

$\Rightarrow \quad \sin\theta=\frac{1}{6}$

Hence there is exactly one value of θ for which $2\hat{u}\times 3\hat{v}$ is a unit vector.

95. $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{c}=x\hat{i}+(x-2)\hat{j}-\hat{k}$

Therefore, $\begin{vmatrix}x & x-2 & -1\\ 1 & 1 & 1\\ 1 & -1 & 2\end{vmatrix}=0$

$\Rightarrow \quad 3x+2-x+2=0$

$\Rightarrow \quad 2x=-4$

$\Rightarrow \quad x=-2.$

96. Let for some $\lambda, \vec{a}=\lambda(\hat{b}+\hat{c})$, then

$\alpha\hat{i}+2\hat{j}+\beta\hat{k}=\lambda(\frac{\hat{i}+2\hat{j}+\hat{k}}{\sqrt{2}})$

$\Rightarrow \quad \lambda=\sqrt{2}\alpha$ and $\lambda=\sqrt{2}$ and $\lambda=\sqrt{2}\beta$

$\Rightarrow \quad \alpha=1$ and $\beta=1$.

97. Since $\vec{a}=8\vec{b}, \vec{c}=-7\vec{b}$.

$\therefore \quad \vec{a}$ and $\vec{b}$ are like vectors and $\vec{b}$ and $\vec{c}$ are unlike.

$\Rightarrow \quad \vec{a}$ and $\vec{c}$ will be unlike

Hence, the angle between $\vec{a}$ *and* $\vec{c}=\pi$.

98. $(3p^2-pq+2q^2)[\vec{u}\quad\vec{v}\quad\vec{w}]=0$

But since $\vec{u},\vec{v},\vec{w}$ are non–coplanar, $[\vec{u}\quad\vec{v}\quad\vec{w}]\neq 0$ which implies that

$3p^2-pq+2q^2=0$

$\Rightarrow \quad 2p^2+p^2-pq+(\frac{q}{2})^2+\frac{7q^2}{4}=0$

$\Rightarrow \quad 2p^2+(p-\frac{q}{2})^2+\frac{7}{4}q^2=0=0$

$\Rightarrow \quad p=0, q=0, p=\frac{q}{2}$

This is possible only when $p=0$, $q=0$ and therefore exactly one value of (p, q).

99. Projection of a vector on coordinate axes are $x_2-x_1, y_2-y_1, z_2-z_1$

So, $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}=\sqrt{36+9+4}=7$

The D.C's of the vector are $\frac{6}{7},-\frac{3}{7},\frac{2}{7}$ respectively.

100. $\vec{c}=\vec{b}\times\vec{a}$

$\Rightarrow \quad \vec{b}\cdot\vec{c}=0$

$\Rightarrow \quad (b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\cdot(\hat{i}-\hat{j}-\hat{k})=0$

$\Rightarrow \quad b_1-b_2-b_3=0$

and $\vec{a}\cdot\vec{b}=3$

$\Rightarrow \quad b_2-b_3=3$

$\Rightarrow \quad b_1=b_2+b_3=3+2b_3$

$\Rightarrow \quad \vec{b}=(3+2b_3)\hat{i}+(3+b_3)\hat{j}+b_3\hat{k}$

101. $\vec{a}\cdot\vec{b}=0$,

$\vec{b}\cdot\vec{c}=0$,

$\vec{c}\cdot\vec{a}=0$

$\Rightarrow \quad 2\lambda+4+\mu=0\lambda-1+2\mu=0$

Solving we get:

$\lambda=-3$, $\mu=2$.

102. $(2\bar{a}-\bar{b}).\{(\bar{a}\times\bar{b})\times(\bar{a}+2\bar{b})\}$

$=(2\bar{a}-\bar{b}).\{2\bar{a}-\bar{b})\{[\bar{a}.(\bar{a}+2\bar{b})]\bar{b}-[\bar{b}.(\bar{a}+2\bar{b})\bar{a}]\}$

$=-5(\bar{a})2(\bar{b})2+5(\bar{a}\bar{b})2=-5$

103. $\bar{b}\times c=\bar{b}\times\bar{d}$

$\Rightarrow \quad \bar{a}\times(\bar{b}\times\bar{c})=\bar{a}\times(\bar{b}\times\bar{d})$

$\Rightarrow \quad (\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}=(\bar{a}.\bar{d})\bar{b}-(\bar{a}.\bar{b})\bar{d}$

$\Rightarrow \quad (\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})c=-(\bar{a}.\bar{b})\bar{d}$

$\therefore \quad \bar{d}=\bar{c}-(\frac{\bar{a}.\bar{c}}{\bar{a}.\bar{b}})\bar{b}$

104. $\vec{c}\cdot\vec{d}=\vec{0} \Rightarrow 5|\vec{a}|^2+6\vec{a}\cdot\vec{b}-8|\vec{b}|^2=0$

$\Rightarrow 6\vec{a}\cdot\vec{b}=3 \Rightarrow \vec{a}\cdot\vec{b}=\frac{1}{2} \Rightarrow (\vec{a}\cdot\vec{b})=\frac{\pi}{3}$

105. $\overline{AE}=$ vector component of $\vec{q}$ on $\vec{p}$

$\overline{AE}=\frac{(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{q})}\vec{p}$

$\therefore$ From $\Delta ABE; \overrightarrow{AB}+\overrightarrow{BE}=\overrightarrow{AE}$

$\Rightarrow \vec{q}+\vec{r}=\frac{(\vec{p}\cdot\vec{q})\vec{p}}{(\vec{p}\cdot\vec{q})}$

$\Rightarrow \vec{r}=-\vec{q}+\frac{(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{q})}\vec{p}$

106. Let M be the median, then

$\overrightarrow{AM}=\frac{\overrightarrow{AB}+\overrightarrow{AC}}{2}$

$\overrightarrow{AM}=4\hat{i}=-\hat{j}+4\hat{k}$

$\therefore |\overrightarrow{AM}|=\sqrt{16+16+1}=\sqrt{33}$

107. $[\vec{a}\times\vec{b}\vec{b}\times\vec{c}\vec{c}\times\vec{a}]=\lambda[\vec{a}\vec{b}\vec{c}]^2$

$\lambda=1$

108. Since $(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$

$\Rightarrow -\vec{b}\cdot\vec{c}=\frac{1}{3}|\vec{b}||\vec{c}|$

$\Rightarrow -|\vec{b}||\vec{c}|\cos\theta=\frac{1}{3}|\vec{b}||\vec{c}|$

$\Rightarrow \cos\theta=-\frac{1}{3}$

$\Rightarrow \sin\theta=\frac{2\sqrt{2}}{3}.$

109. Given expression is $\left(\vec{a}\cdot\vec{c}-\frac{\sqrt{3}}{2}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}+\frac{\sqrt{3}}{2}\right)\vec{c}=0$

$\Rightarrow \vec{a}\cdot\vec{b}=\cos\theta=-\sqrt{3}/2 \Rightarrow \theta=5\pi/6$