

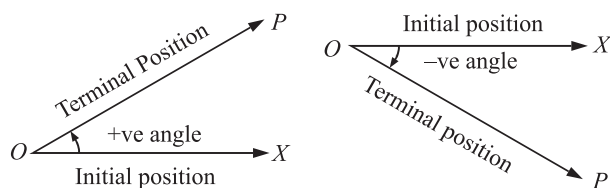
# Trigonometric Ratios and Identities

## Chapter Highlights

Angle 1, Measurement of Angles 1, Relation between Different Systems of Measurement of Angles 2, Relation between sides and interior angles of a regular polygon 2, Fundamental Trigonometric Identities 2, Signs of Trigonometric Ratios in Different Quadrants 2, Increase and Decrease of Trigonometric Functions 3, Domain and Range of Trigonometric Ratios 3, Trigonometric Ratios of Standard Angles 4, Trigonometric Ratios for Some Special Angles 4, Trigonometric ratios of allied angles 5, Table for Trigonometric Ratios of Allied Angles 5, Trigonometric ratios in terms of each other 6, Addition and Subtraction Formulae 6, Transformation Formulae 7, Product into Sum or Difference 7, Sum and Difference into Product 7, Trigonometric Ratios of Multiple Angles 7, Trigonometric Ratios of Submultiple Angles 8, Greatest and Least Values of the Expression 9, Conditional Identities 9, Graphs of Trigonometric Functions 9.

## ANGLE

Let a revolving line starting from  $OX$  revolves about its end point  $O$  on a plane in the direction of arrow and occupy the position  $OP$ . It is said to trace out an angle  $XOP$ . Here  $OX$  is called the *initial position* and  $OP$ , the *terminal position*. The fixed point  $O$  is called the *vertex*.



(a)

(b)

Fig. 25.1

An angle is considered as the figure traced by rotating a given ray about its end point.

## MEASUREMENT OF ANGLES



### IMPORTANT POINTS

If the rotation is in anticlockwise sense, the angle measured is positive and if the rotation is in clockwise sense, the angle measured is negative.

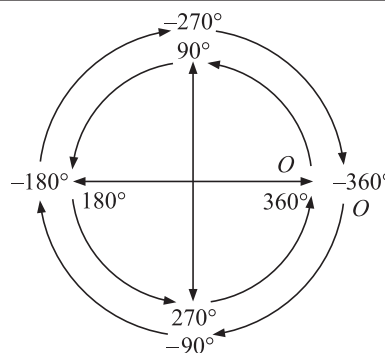


Fig. 25.2

There are three systems for measurement of an angle:

### 1. Sexagesimal System or English System

In this system an angle is measured in degrees, minutes and seconds. A complete rotation describes  $360^\circ$ .

1 right angle =  $90^\circ$ , (read as 90 degrees)

$1^\circ = 60'$  (read as 60 minutes)

$1' = 60''$  (read as 60 seconds)

### 2. Centesimal or French System

In this system an angle is measured in grades, minutes and seconds.

1 right angle =  $100^\circ$  (read as 100 grades)

$$1^g = 100' \text{ (read as 100 minutes)}$$

$$1' = 100'' \text{ (read as 100 seconds)}$$



**CAUTION**

1' of centesimal system  $\neq$  1' of sexagesimal system  
 1'' of centesimal system  $\neq$  1'' of sexagesimal system

**3. Radian or Circular Measure**

A radian is a constant angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle and is denoted by  $1^c$ .

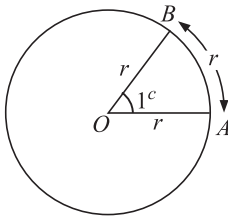


fig. 25.3

$\angle AOB = 1$  radian.

This angle does not depend upon the radius of the circle from which it is derived.



**CAUTION**

Radian is a unit to measure angle and it should not be interpreted that  $\pi$  stands for  $180^\circ$ ,  $\pi$  is a real number whereas  $\pi^c$  stands for  $180^\circ$ .

**REMEMBER**

$$\pi \text{ radians} = 180^\circ = 200^g.$$

**RELATION BETWEEN DIFFERENT SYSTEMS OF MEASUREMENT OF ANGLES**

$$1^\circ = \frac{10}{9} \text{ grades}; 1^g = \frac{9}{10} \text{ degrees}$$

$$1^\circ = \frac{\pi}{180} \text{ radians} = 0.0172 \text{ radians};$$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} = 57^\circ 17' 45''$$

$$1^g = \frac{\pi}{200} \text{ radians}; 1 \text{ radian} = \frac{200}{\pi} \text{ grades}$$

Thus if the measure of an angle in degrees, grades and radians be  $D$ ,  $G$  and  $\theta$  respectively, then

$$\frac{D}{180} = \frac{G}{200} = \frac{\theta}{\pi}$$

**RELATION BETWEEN SIDES AND INTERIOR ANGLES OF A REGULAR POLYGON**

- Sum of interior angles of polygon of  $n$  sides

$$= (2n - 4) \times 90^\circ$$

- Each interior angle of a regular polygon of  $n$  sides

$$= \frac{2n - 4}{n} \times 90^\circ$$

**TRICK(S) FOR PROBLEM SOLVING**

- The angle between two consecutive digits in a clock is  $30^\circ$ .
- The hour hand rotates through an angle of  $30^\circ$  in one hour or  $\left(\frac{1}{2}\right)^\circ$  in one minute
- The minute hand rotates through an angle of  $6^\circ$  in one minute.

**FUNDAMENTAL TRIGONOMETRIC IDENTITIES**

- $\sin^2\theta + \cos^2\theta = 1$  or  $\cos^2\theta = 1 - \sin^2\theta$

$$\text{or } \sin^2\theta = 1 - \cos^2\theta$$

- $1 + \tan^2\theta = \sec^2\theta$  or  $\sec^2\theta - \tan^2\theta = 1$

- $1 + \cot^2\theta = \text{cosec}^2\theta$  or  $\text{cosec}^2\theta - \cot^2\theta = 1$

**TRICK(S) FOR PROBLEM SOLVING**

Since  $\sin^2\theta + \cos^2\theta = 1$ ,  $|\sin\theta| \leq 1$  and  $|\cos\theta| \leq 1$   
 $\Rightarrow -1 \leq \sin\theta \leq 1$  and  $-1 \leq \cos\theta \leq 1$ ;  
 $0 \leq \sin^2\theta \leq 1$ ,  $0 \leq \cos^2\theta \leq 1$ .  
 Since  $\text{cosec}\theta = 1/\sin\theta$ ,  $\text{cosec}\theta \geq 1$  or  $\text{cosec}\theta \leq -1$   
 Also, since  $\sec\theta = 1/\cos\theta$ ,  $\sec\theta \geq 1$  or  $\sec\theta \leq -1$

**SIGNS OF TRIGONOMETRIC RATIOS IN DIFFERENT QUADRANTS**

The following table describes the signs of various  $t$ -ratios in different quadrants. Also, refer to, the figure given below the table.

Quadrant	I	II	III	IV
$MP = y$	+ve	+ve	-ve	-ve
$OM = x$	+ve	-ve	-ve	+ve

Quadrant	I	II	III	IV
$\sin \theta = \frac{y}{r}$	$\frac{+ve}{+ve} = +ve$	$\frac{+ve}{+ve} = +ve$	$\frac{-ve}{+ve} = -ve$	$\frac{-ve}{+ve} = -ve$
$\cos \theta = \frac{x}{r}$	$\frac{+ve}{+ve} = +ve$	$\frac{-ve}{+ve} = -ve$	$\frac{-ve}{+ve} = -ve$	$\frac{+ve}{+ve} = +ve$
$\tan \theta = \frac{y}{x}$ , $x \neq 0$	$\frac{+ve}{+ve} = +ve$	$\frac{+ve}{-ve} = -ve$	$\frac{-ve}{-ve} = +ve$	$\frac{-ve}{+ve} = -ve$

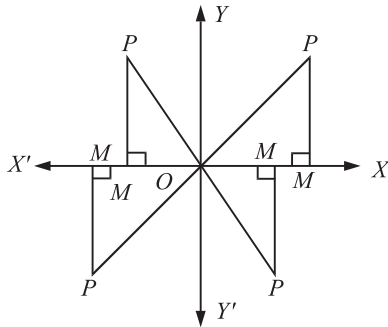


Fig. 25.4

The signs of other  $t$ -ratios can be found by using reciprocal relations, i.e.

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}, \sec \theta = \frac{1}{\cos \theta} \text{ and } \cot \theta = \frac{1}{\tan \theta}. \text{ So, we have}$$

Quadrant: →	I	II	III	IV
t-ratios which are +ve	All	$\sin \theta$ $\operatorname{cosec} \theta$	$\tan \theta$ $\cot \theta$	$\cos \theta$ $\sec \theta$

**REMEMBER**

- In the first quadrant, **all** are **+ve**.
- In the second quadrant, **sin** and **cosec** are **+ve**
- In the third quadrant, **tangent** and **cotangent** are **+ve**
- In the fourth quadrant, **cosine** and **secant** are **+ve**.

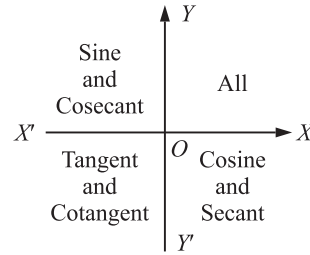


fig.25.5

**Simple rule to remember:**

add – sugar – to – coffee

or

after – school – to – college

In the above, 'a' stands for 'all', 's' stands for 'sine', 't' stands for 'tan' and 'c' stands for 'cos'. The reciprocals of these ratios are also positive in the respective quadrants.

**INCREASE AND DECREASE OF TRIGONOMETRIC FUNCTIONS**

We can discuss the way of increase and decrease of trigonometric functions as described in the following tables:

	I quadrant	II quadrant	III quadrant	IV quadrant
$\sin \theta$	increases from 0 to 1	decreases from 1 to 0	decreases from 0 to -1	increases from -1 to 0
$\cos \theta$	decreases from 1 to 0	decreases from 0 to -1	increases from -1 to 0	increases from 0 to 1
$\tan \theta$	increases from 0 to $\infty$	decreases from $\infty$ to 0	increases from 0 to $\infty$	increases from $-\infty$ to 0
$\cot \theta$	decreases from $\infty$ to 0	increases from 0 to $\infty$	decreases from $\infty$ to 0	decreases from 0 to $-\infty$
$\sec \theta$	increases from 1 to $\infty$	increases from $-\infty$ to -1	decreases from -1 to $-\infty$	decreases from $\infty$ to 1
$\operatorname{cosec} \theta$	decreases from $\infty$ to 1	increases from 1 to $\infty$	increases from $-\infty$ to -1	decreases from -1 to $-\infty$

**DOMAIN AND RANGE OF TRIGONOMETRIC RATIOS**

Functions	Domain	Range
$\sin x, \cos x$	$(-\infty, \infty)$	$[-1, 1]$
$\tan x$	$(-\infty, \infty) - \left\{ (2n+1)\frac{\pi}{2} \mid n \in I \right\}$	$(-\infty, \infty)$
$\cot x$	$(-\infty, \infty) - \{n\pi \mid n \in I\}$	$(-\infty, \infty)$

Functions	Domain	Range
$\sec x$	$(-\infty, \infty) - \left\{ (2n+1)\frac{\pi}{2} \mid n \in I \right\}$	$(-\infty, -1] \cup [1, \infty)$
$\operatorname{cosec} x$	$(-\infty, \infty) - \{n\pi \mid n \in I\}$	$(-\infty, -1] \cup [1, \infty)$

**TRIGONOMETRIC RATIOS OF STANDARD ANGLES**

Angles T-Ratios	0°	30°	45°	60°	90°
sin x	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

Angles T-Ratios	0°	30°	45°	60°	90°
cosec x	Undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec x	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Undefined
cot x	Undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

**TRIGONOMETRIC RATIOS FOR SOME SPECIAL ANGLES**

Angle	sin θ	cos θ	tan θ	cot θ	sec θ	cosec θ
0°/0	0	1	0	∞	1	∞
15° $\frac{\pi}{12}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	$\frac{2\sqrt{2}}{\sqrt{3}+1}$	$\frac{2\sqrt{2}}{\sqrt{3}-1}$
18° $\frac{\pi}{10}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\frac{4}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{5}+1$
22.5° $\frac{\pi}{8}$	$\frac{\sqrt{2}-\sqrt{2}}{2}$	$\frac{\sqrt{2}+\sqrt{2}}{2}$	$\sqrt{2}-1$	$\sqrt{2}+1$	$\sqrt{4-2\sqrt{2}}$	$\sqrt{4+2\sqrt{2}}$
30° $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
36° $\frac{\pi}{5}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	$\sqrt{5}-1$	$\frac{4}{\sqrt{10-2\sqrt{5}}}$
45° $\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
54° $\frac{3\pi}{10}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\sqrt{5}-1$
60° $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
67.5° $\frac{3\pi}{8}$	$\frac{\sqrt{2}+\sqrt{2}}{2}$	$\frac{\sqrt{2}-\sqrt{2}}{2}$	$\sqrt{2}+1$	$\sqrt{2}-1$	$\sqrt{4+2\sqrt{2}}$	$\sqrt{4-2\sqrt{2}}$
72° $\frac{2\pi}{5}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{5}+1$	$\frac{4}{\sqrt{10+2\sqrt{5}}}$
72° $\frac{5\pi}{12}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{2\sqrt{2}}{\sqrt{3}-1}$	$\frac{2\sqrt{2}}{\sqrt{3}+1}$
90° $\frac{\pi}{2}$	1	0	∞	0	∞	1

## TRIGONOMETRIC RATIOS OF ALLIED ANGLES

i.e. the angles  $-\theta$ ,  $90^\circ \pm \theta$ ,  $180^\circ \pm \theta$ ,  $270^\circ \pm \theta$  and  $360^\circ \pm \theta$  are called allied angles.

Two angles are said to be allied when their sum or difference is either zero or a multiple of  $90^\circ$ .

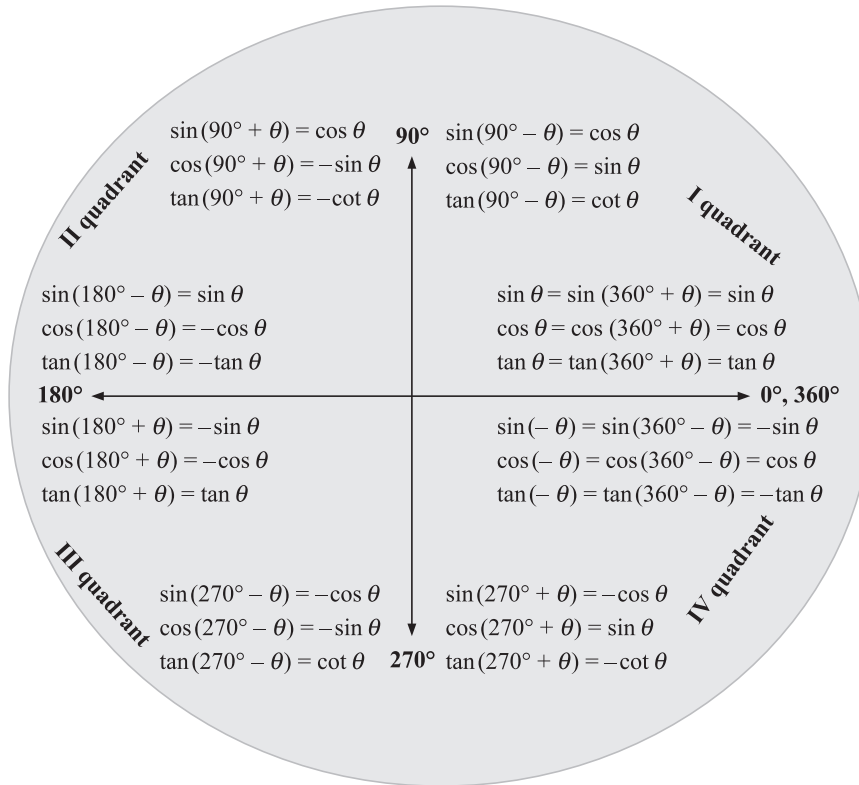


Fig. 25.6

## TABLE FOR TRIGONOMETRIC RATIOS OF ALLIED ANGLES

	$-\theta$	$90^\circ - \theta$	$90^\circ + \theta$	$180^\circ - \theta$	$180^\circ + \theta$	$270^\circ - \theta$	$270^\circ + \theta$	$360^\circ - \theta$	$360^\circ + \theta$
$\sin \theta$	$-\sin \theta$	$\cos \theta$	$\cos \theta$	$\sin \theta$	$-\sin \theta$	$-\cos \theta$	$-\cos \theta$	$-\sin \theta$	$\sin \theta$
$\cos \theta$	$\cos \theta$	$\sin \theta$	$-\sin \theta$	$-\cos \theta$	$-\cos \theta$	$-\sin \theta$	$\sin \theta$	$\cos \theta$	$\cos \theta$
$\tan \theta$	$-\tan \theta$	$\cot \theta$	$-\cot \theta$	$-\tan \theta$	$\tan \theta$	$\cot \theta$	$-\cot \theta$	$-\tan \theta$	$\tan \theta$

### WORKING RULE TO FIND ALLIED ANGLES

**Case I:** When the angle is  $n\pi \pm \theta$ , where  $n \in \mathbb{I}$  and  $\theta$  is acute.

- There is no change in trigonometric function i.e., sin remains sin, cos remains cos and tan remains tan. Angle associated becomes  $\theta$ .
- The sign is affixed according to the quadrant in which the angle lies.

**Case II:** When the angle is  $\frac{n\pi}{2} \pm \theta$ , where  $n$  is an odd integer and  $\theta$  is acute.

- The trigonometric function is replaced by its cofunction i.e. sin changes to cos, tan changes to cot and sec changes to cosec and vice-versa.

Angle associated becomes  $\theta$ .

- The sign is affixed according to the quadrant in which the angle lies.

Note that the sign is always decided on the basis of the operating function.



### IMPORTANT POINTS

- $\sin n\pi = 0$ ,  $\cos n\pi = (-1)^n$ .
- $\sin(n\pi + \theta) = (-1)^n \sin\theta$ ,  $\cos(n\pi + \theta) = (-1)^n \cos\theta$
- For odd integer  $n$ ,

$$\sin \frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}, \cos \frac{n\pi}{2} = 0$$

- $\sin\left(\frac{n\pi}{2} + \theta\right) = \begin{cases} (-1)^{\frac{n-1}{2}} \cos\theta, & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} \sin\theta, & \text{if } n \text{ is even.} \end{cases}$

$$\cos\left(\frac{n\pi}{2} + \theta\right) = \begin{cases} (-1)^{\frac{n+1}{2}} \sin\theta, & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} \cos\theta, & \text{if } n \text{ is even.} \end{cases}$$

To find ratios for  $n\pi - \theta$  and  $\frac{n\pi}{2}$ , replace  $\theta$  by  $-\theta$  in all of the above

### TRIGONOMETRIC RATIOS IN TERMS OF EACH OTHER

	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\operatorname{cosec} \theta$
$\sin \theta$	$\sin \theta$	$\sqrt{1 - \cos^2 \theta}$	$\frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}}$	$\frac{1}{\sqrt{1 + \cot^2 \theta}}$	$\frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta}$	$\frac{1}{\operatorname{cosec} \theta}$
$\cos \theta$	$\sqrt{1 - \sin^2 \theta}$	$\cos \theta$	$\frac{1}{\sqrt{1 + \tan^2 \theta}}$	$\frac{\cot \theta}{\sqrt{1 + \cot^2 \theta}}$	$\frac{1}{\sec \theta}$	$\frac{\sqrt{\operatorname{cosec}^2 \theta - 1}}{\operatorname{cosec} \theta}$
$\tan \theta$	$\frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$	$\frac{\sqrt{1 - \cos^2 \theta}}{\cos \theta}$	$\tan \theta$	$\frac{1}{\cot \theta}$	$\sqrt{\sec^2 \theta - 1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2 \theta - 1}}$
$\cot \theta$	$\frac{\sqrt{1 - \sin^2 \theta}}{\sin \theta}$	$\frac{\cos \theta}{\sqrt{1 - \cos^2 \theta}}$	$\frac{1}{\tan \theta}$	$\cot \theta$	$\frac{1}{\sqrt{\sec^2 \theta - 1}}$	$\sqrt{\operatorname{cosec}^2 \theta - 1}$
$\sec \theta$	$\frac{1}{\sqrt{1 - \sin^2 \theta}}$	$\frac{1}{\cos \theta}$	$\sqrt{1 + \tan^2 \theta}$	$\frac{\sqrt{1 + \cot^2 \theta}}{\cot}$	$\sec \theta$	$\frac{\operatorname{cosec} \theta}{\sqrt{\operatorname{cosec}^2 \theta - 1}}$
$\operatorname{cosec} \theta$	$\frac{1}{\sin \theta}$	$\frac{1}{\sqrt{1 - \cos^2 \theta}}$	$\frac{\sqrt{1 + \sin^2 \theta}}{\tan \theta}$	$\sqrt{1 + \cot^2 \theta}$	$\frac{\sec \theta}{\sqrt{\sec^2 \theta - 1}}$	$\operatorname{cosec} \theta$

### ADDITION AND SUBTRACTION FORMULAE

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
- $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$
- $\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$

- $\sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B$
- $\cos(A + B) \cos(A - B) = \cos^2 A - \sin^2 B$
- $\sin(A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$   
( $\Sigma$  (one sine and two cos) – three sines)  
or  
 $= \cos A \cos B \cos C (\tan A + \tan B + \tan C - \tan A \tan B \tan C)$
- $\cos(A + B + C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$  (three cos –  $\Sigma$ (one cos and two sine))  
or  
 $= \cos A \cos B \cos C (1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$

$$13. \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$14. \tan A + \tan B + \tan C - \tan A \tan B \tan C \\ = \frac{\sin(A+B+C)}{\cos A \cos B \cos C}$$

$$15. 1 - \tan A \tan B - \tan B \tan C - \tan C \tan A \\ = \frac{\cos(A+B+C)}{\cos A \cos B \cos C}$$

$$16. \tan\left(\frac{\pi}{4} + A\right) = \frac{1 + \tan A}{1 - \tan A}$$

$$17. \tan\left(\frac{\pi}{4} - A\right) = \frac{1 - \tan A}{1 + \tan A}$$

$$18. \sin(A_1 + A_2 + \dots + An) = \cos A_1 \cos A_2 \dots \cos An (S_1 - S_3 + S_5 - \dots)$$

$$19. \cos(A_1 + A_2 + \dots + An) = \cos A_1 \cos A_2 \dots \cos An (1 - S_2 + S_4 - S_6 + \dots)$$

$$20. \tan(A_1 + A_2 + \dots + A_n) = \frac{S_1 - S_3 + S_5 - \dots}{1 - S_2 + S_4 - S_6 + \dots}$$

where  $S_1 = \sum \tan A_1$ ,  $S_2 = \sum \tan A_1 \tan A_2$ ,  
 $S_3 = \sum \tan A_1 \tan A_2 \tan A_3$  and so on.

$$21. \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin(\alpha + (n-1)\beta)$$

$$= \frac{\sin\left(\alpha + (n-1)\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}} \sin\left(\frac{n\beta}{2}\right)$$

If  $\beta = \alpha$ , then

$$\sin \alpha + \sin 2\alpha + \dots + \sin n\alpha = \frac{\sin\left(\frac{n+1}{2}\alpha\right) \times \sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

$$22. \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (n-1)\beta)$$

$$= \frac{\cos\left(\alpha + (n-1)\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}} \sin\left(\frac{n\beta}{2}\right)$$

If  $\beta = \alpha$ , then

$$\cos \alpha + \cos 2\alpha + \dots + \cos n\alpha = \frac{\cos\left((n+1)\frac{\alpha}{2}\right) \times \sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

## TRANSFORMATION FORMULAE

### PRODUCT INTO SUM OR DIFFERENCE

- $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$ ,  $A > B$
- $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$ ,  $A > B$
- $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$
- $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

### SUM AND DIFFERENCE INTO PRODUCT

- $\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$
- $\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$
- $\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$
- $\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$
- $\tan C + \tan D = \frac{\sin(C+D)}{\cos C \cos D}$
- $\tan C - \tan D = \frac{\sin(C-D)}{\cos C \cos D}$
- $\cot C + \cot D = \frac{\sin(C+D)}{\sin C \sin D}$
- $\cot C - \cot D = \frac{\sin(D-C)}{\sin C \sin D}$

## TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES

(An Angle of the form  $n\theta$ ,  $n \in \mathbb{I}$ )

- $\sin 2\theta = 2 \sin \theta \cos \theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$
- $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1$   
 $= 1 - 2 \sin^2 \theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$
- $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
- $\cot 2\theta = \frac{\cot^2 \theta - 1}{2 \cot \theta}$

5.  $1 + \cos 2\theta = 2 \cos^2 \theta$ ,  $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$

6.  $1 - \cos 2\theta = 2 \sin^2 \theta$ ,  $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$

7.  $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ ,  $\sin^3 \theta = \frac{1}{4}(3 \sin \theta - \sin 3\theta)$   
 $= 4 \sin(60^\circ - \theta) \sin \theta \sin(60^\circ + \theta)$

8.  $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$ ,  $\cos^3 \theta = \frac{1}{4}(\cos 3\theta + 3 \cos \theta)$   
 $= 4 \cos(60^\circ - \theta) \cos \theta \cos(60^\circ + \theta)$   
 $= 4 \cos(120^\circ - \theta) \cos \theta \cos(120^\circ + \theta)$

9.  $\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$   
 $= \tan(60^\circ - \theta) \tan \theta \tan(60^\circ + \theta)$

10.  $\cot 3\theta = \frac{\cot^3 \theta - 3 \cot \theta}{3 \cot^2 \theta - 1}$

11.  $\cos A \cos 2A \cos 2^2 A \dots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$

**TRIGONOMETRIC RATIOS OF SUBMULTIPLE ANGLES**

(An Angle of the form  $\frac{\theta}{n}$ ,  $n \in \mathbb{I}$ )

1.  $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{2 \tan \theta/2}{1 + \tan^2 \theta/2}$

2.  $\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = 2 \cos^2 \frac{\theta}{2} - 1 = 1 - 2 \sin^2 \frac{\theta}{2}$   
 $= \frac{1 - \tan^2 \theta/2}{1 + \tan^2 \theta/2}$

3.  $\tan \theta = \frac{2 \tan \theta/2}{1 - \tan^2 \theta/2}$

4.  $\cot \theta = \frac{\cot^2 \frac{\theta}{2} - 1}{2 \cot \frac{\theta}{2}}$

5.  $\cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2}$

6.  $\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}$

7.  $\tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta}$

8.  $\cot^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{1 - \cos \theta}$

9.  $\frac{1 - \cos \theta}{\sin \theta} = \tan \frac{\theta}{2}$

10.  $\frac{1 + \cos \theta}{\sin \theta} = \cot \frac{\theta}{2}$

11.  $\sin \frac{\theta}{2} \pm \cos \frac{\theta}{2} = \sqrt{2} \sin \left( \frac{\pi}{4} \pm \theta \right) = \sqrt{2} \cos \left( \theta \mp \frac{\pi}{4} \right)$

**TRICK(S) FOR PROBLEM SOLVING**

■  $\left| \sin \frac{A}{2} + \cos \frac{A}{2} \right| = \sqrt{1 + \sin A}$   
 or  $\sin \frac{A}{2} + \cos \frac{A}{2} = \pm \sqrt{1 + \sin A}$   
 i.e.,  $\begin{cases} +, & \text{If } 2n\pi - \pi/4 \leq A/2 \leq 2n\pi + 3\pi/4 \\ -, & \text{other wise} \end{cases}$

■  $\left| \sin \frac{A}{2} - \cos \frac{A}{2} \right| = \sqrt{1 - \sin A}$   
 or  $\left( \sin \frac{A}{2} - \cos \frac{A}{2} \right) = \pm \sqrt{1 - \sin A}$   
 i.e.,  $\begin{cases} +, & \text{If } 2n\pi - \pi/4 \leq A/2 \leq 2n\pi + 5\pi/4 \\ -, & \text{other wise} \end{cases}$

■ (i)  $\tan \frac{A}{2} = \frac{\pm \sqrt{\tan^2 A + 1} - 1}{\tan A}$   
 $= \pm \frac{\sqrt{1 - \cos A}}{\sqrt{1 + \cos A}}$   
 $= \frac{1 - \cos A}{\sin A}$

where  $A \neq (2n + 1)\pi$

(ii)  $\cot \frac{A}{2} = \pm \frac{\sqrt{1 + \cos A}}{\sqrt{1 - \cos A}}$   
 $= \frac{1 + \cos A}{\sin A}$

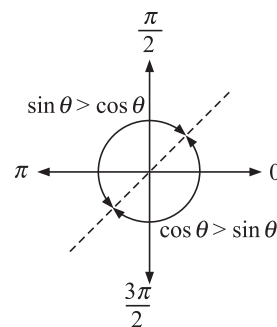
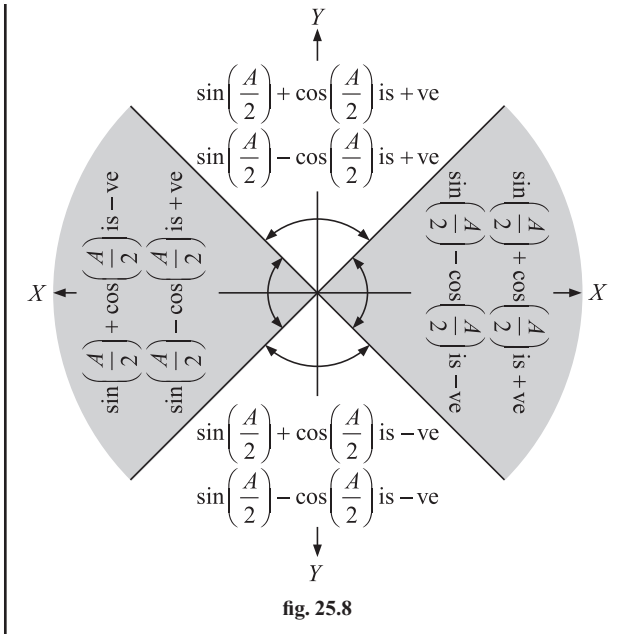


fig. 25.7

where  $A \neq 2n\pi$

The ambiguities of signs are removed by locating the quadrant in which  $\frac{A}{2}$  lies or you can follow the adjoining figure.



## GREATEST AND LEAST VALUES OF THE EXPRESSION

$$a \sin \theta + b \cos \theta$$

Let  $a = r \cos \alpha$ ,  $b = r \sin \alpha$ , then

$$a^2 + b^2 = r^2 \text{ or } r = \sqrt{a^2 + b^2}$$

Then  $a \sin \theta + b \cos \theta = r(\sin \theta \cos \alpha + \cos \theta \sin \alpha)$

$$= r \sin(\theta + \alpha)$$

But  $-1 \leq \sin(\theta + \alpha) \leq 1$ , so

$$-r \leq r \sin(\theta + \alpha) \leq r$$

or  $-\sqrt{a^2 + b^2} \leq a \sin \theta + b \cos \theta \leq \sqrt{a^2 + b^2}$

Thus, the greatest and least values of  $a \sin \theta + b \cos \theta$  are  $\sqrt{a^2 + b^2}$  and  $-\sqrt{a^2 + b^2}$  respectively.

### TRICK(S) FOR PROBLEM SOLVING

$$\sin^2 x + \operatorname{cosec}^2 x \geq 2, \text{ for every real } x$$

$$\cos^2 x + \sec^2 x \geq 2, \text{ for every real } x$$

$$\tan^2 x + \cot^2 x \geq 2, \text{ for every real } x$$

## CONDITIONAL IDENTITIES

When the angles  $A$ ,  $B$  and  $C$  satisfy a given relation, many interesting identities can be established connecting the trigonometric functions of these angles. In providing these identities, we require the properties of complementary and supplementary angles. For example, if  $A + B + C = \pi$ , then

- $\sin(B + C) = \sin A$ ,  $\cos B = -\cos(C + A)$

- $\cos(A + B) = -\cos C$ ,  $\sin C = \sin(A + B)$
- $\tan(C + A) = \tan B$ ,  $\cot A = -\cot(B + C)$
- $\cos \frac{A+B}{2} = \sin \frac{C}{2}$ ,  $\cos \frac{C}{2} = \sin \frac{A+B}{2}$
- $\sin \frac{C+A}{2} = \cos \frac{B}{2}$ ,  $\sin \frac{A}{2} = \cos \frac{B+C}{2}$
- $\tan \frac{B+C}{2} = \cot \frac{A}{2}$ ,  $\tan \frac{B}{2} = \cot \frac{C+A}{2}$

### REMEMBER

If  $A + B + C = \pi$ , then

- $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$
- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
- $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$
- $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$
- $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$
- $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
- $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

## GRAPHS OF TRIGONOMETRIC FUNCTIONS

1. Graph of  $y = \sin x$

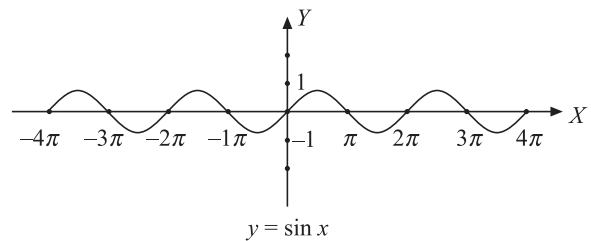


Fig. 25.9

2. Graph of  $y = \cos x$

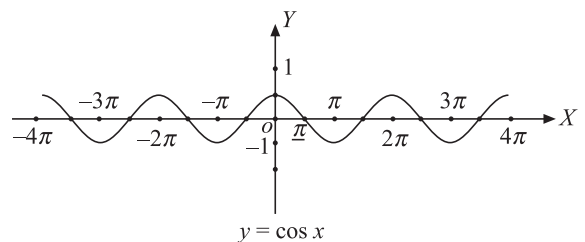


Fig. 25.10

3. Graph of  $y = \tan x$

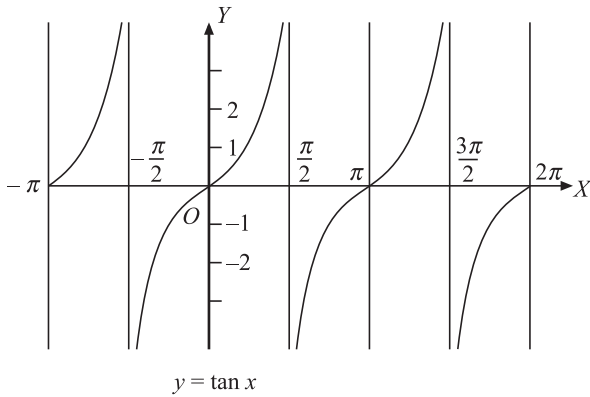


Fig. 25.11

4. Graph of  $y = \cot x$

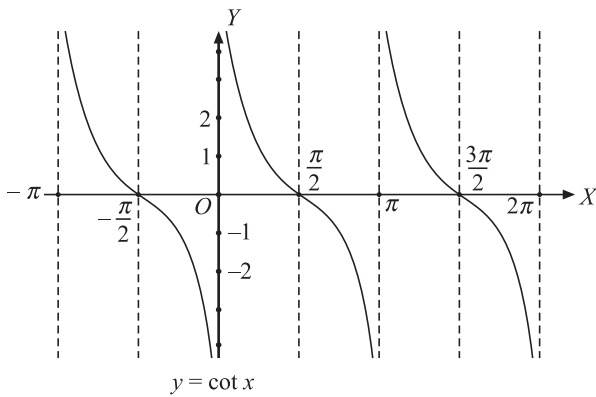


Fig. 25.12

5. Graph of  $y = \sec x$  and  $y = \operatorname{cosec} x$

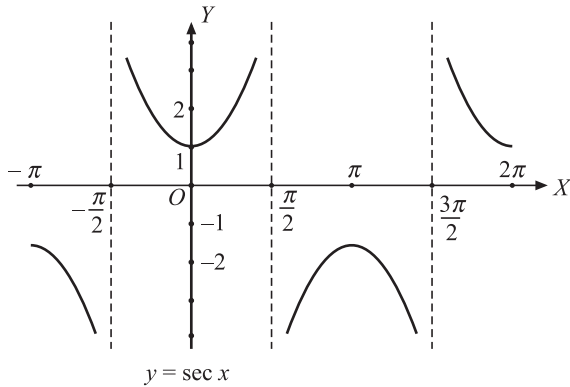


Fig. 25.13

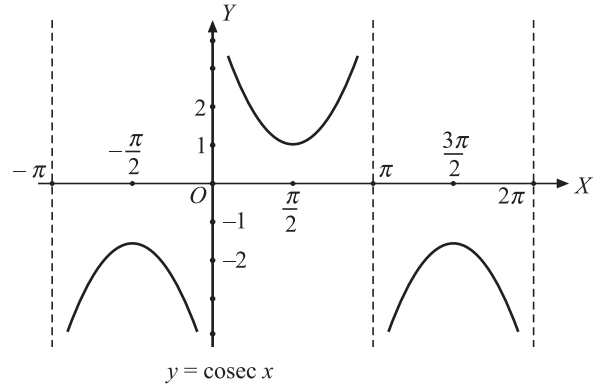


Fig. 25.14

6. Graph of  $y = 3 \sin 2x$

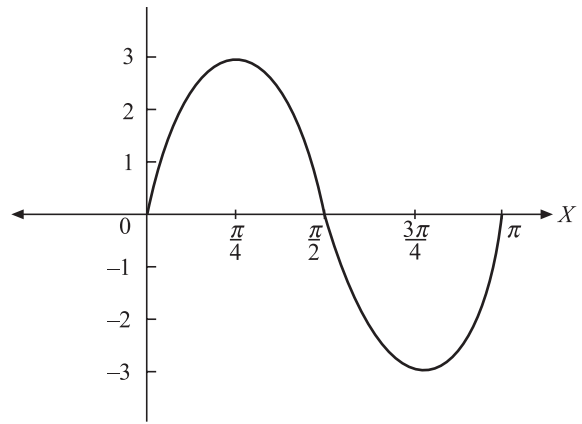


Fig. 25.15

Since the period of  $\sin x$  is  $2\pi$ . Therefore, the period of  $\sin 2x$  is  $\frac{2\pi}{2} = \pi$ . Also,  $-3 \leq 3 \sin 2x \leq 3$ . The graph of  $y = 3 \sin 2x$  is drawn for a period  $0 \leq x \leq \pi$ . The complete graph is simply the repetition of the portion.

In a similar way the graphs of other trigonometric functions can be drawn.

## EXERCISES

## Single Option Correct Type

1. The value of  $\cos a \cos 2a \cos 3a \dots \cos 999 a$ , where  $a = \frac{2\pi}{1999}$ , is
- (A)  $\frac{1}{2^{99}}$  (B)  $\frac{1}{2^{999}}$   
 (C)  $\frac{1}{2^{9999}}$  (D)  $\frac{1}{2^{1999}}$
2. Let  $a_1 = \left(\tan \frac{\pi}{8}\right)^{\tan \frac{\pi}{8}}$ ,  $a_2 = \left(\tan \frac{\pi}{8}\right)^{\cot \frac{\pi}{8}}$ ,  
 $a_3 = \left(\cot \frac{\pi}{8}\right)^{\tan \frac{\pi}{8}}$ ,  $a_4 = \left(\cot \frac{\pi}{8}\right)^{\cot \frac{\pi}{8}}$ . Then,
- (A)  $a_4 > a_3 > a_2 > a_1$  (B)  $a_3 > a_4 > a_2 > a_1$   
 (C)  $a_4 > a_3 > a_1 > a_2$  (D)  $a_3 > a_1 > a_2 > a_4$
3. If  $x \cos^2 3\theta + y \cos^4 \theta = 16 \cos^6 \theta + 9 \cos^2 \theta$  be an identity, then
- (A)  $x = -1, y = 24$  (B)  $x = 1, y = 24$   
 (C)  $x = 24, y = 1$  (D) none of these
4.  $|\tan \theta + \sec \theta| = |\tan \theta| + |\sec \theta|$ ,  $0 \leq \theta \leq 2\pi$  is possible only if
- (A)  $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$  (B)  $\theta \in [0, \pi]$   
 (C)  $\theta \in \left[0, \frac{\pi}{2}\right)$  (D) none of these
5. If  $\sin \theta, \sin \phi$  and  $\cos \theta$  are in G.P., then the roots of the equation  $x^2 + 2x \cot \phi + 1 = 0$  are always
- (A) real (B) imaginary  
 (C) equal (D) greater than 1
6. If  $\cos 25^\circ + \sin 25^\circ = k$ , then  $\cos 50^\circ$  is equal to
- (A)  $k\sqrt{2-k^2}$  (B)  $-\sqrt{2-k^2}$   
 (C)  $\sqrt{2-k^2}$  (D)  $-k\sqrt{2-k^2}$
7. If  $\frac{2 \sin \alpha}{1 + \cos \alpha + \sin \alpha} = x$  then  $\frac{1 - \cos \alpha + \sin \alpha}{1 + \sin \alpha} = \dots$
- (A)  $\frac{1}{x}$  (B)  $x$   
 (C)  $1 - x$  (D)  $1 + x$
8. If  $e^{-\pi/2} < \theta < \pi/2$ , then
- (A)  $\cos \log \theta < \log \cos \theta$   
 (B)  $\cos \log \theta > \log \cos \theta$   
 (C)  $\cos \log \theta \leq \log \cos \theta$   
 (D) none of these
9.  $\sin \theta = \frac{1}{2} \left( \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \right)$  necessarily implies
- (A)  $x > y$   
 (B)  $x < y$   
 (C)  $x = y$   
 (D) both  $x$  and  $y$  are purely imaginary
10. If  $xy + yz + zx = 1$ , then  $\sum \frac{x+y}{1-xy} =$
- (A)  $\frac{4}{xyz}$  (B)  $\frac{1}{xyz}$   
 (C)  $xyz$  (D) none of these
11.  $\cos 12^\circ \cos 24^\circ \cos 36^\circ \cos 48^\circ \cos 72^\circ \cos 96^\circ$  equals
- (A)  $-\frac{1}{2^6}$  (B)  $\frac{1}{2^8}$   
 (C)  $\frac{1}{2^7}$  (D)  $-\frac{1}{2^7}$
12. If  $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2}\right)$ , then  $\frac{\sin(\alpha + \beta + \gamma)}{\sin \alpha + \sin \beta + \sin \gamma}$  is
- (A)  $< 1$  (B)  $> 1$   
 (C)  $= 1$  (D) none of these
13. If  $\left| \cos \theta \left\{ \sin \theta + \sqrt{\sin^2 \theta + \sin^2 \alpha} \right\} \right| \leq k$ , then the value of  $k$  is
- (A)  $\sqrt{1 + \cos^2 \alpha}$  (B)  $\sqrt{1 + \sin^2 \alpha}$   
 (C)  $\sqrt{2 + \sin^2 \alpha}$  (D)  $\sqrt{2 + \cos^2 \alpha}$
14. The maximum value of  $(\cos \alpha_1) (\cos \alpha_2) \dots (\cos \alpha_n)$  under the restrictions  $0 \leq \alpha_1, \alpha_2, \dots, \alpha_n \leq \frac{\pi}{2}$  and  $(\cot \alpha_1) (\cot \alpha_2) \dots (\cot \alpha_n) = 1$  is
- (A)  $\frac{1}{2^{n/2}}$  (B)  $\frac{1}{2^n}$   
 (C)  $\frac{1}{2n}$  (D) 1

15. The inequality  $2^{\sin\theta} + 2^{\cos\theta} \geq 2^{\left(\frac{1-\frac{1}{\sqrt{2}}}{\sqrt{2}}\right)}$  holds for  
 (A)  $0 \leq \theta < \pi$  (B)  $\pi \leq \theta < 2\pi$   
 (C) for all real  $\theta$  (D) none of these
16. The expression  $2^{\sin\theta} + 2^{-\cos\theta}$  is minimum when  $\theta$  is equal to  
 (A)  $2n\pi + \frac{\pi}{4}, n \in I$  (B)  $2n\pi + \frac{7\pi}{4}, n \in I$   
 (C)  $n\pi \pm \frac{\pi}{4}, n \in I$  (D) none of these
17. If  $x_{n+1} = \sqrt{\frac{1+x_n}{2}}$ , then  $\cos\left(\frac{\sqrt{1-x_0^2}}{x_1 x_2 x_3 \dots \text{to } \infty}\right)$  ( $-1 < x_0 < 1$ ) is equal to  
 (A)  $x_0$  (B)  $1/x_0$   
 (C) 1 (D) -1
18. If  $0 < \theta < \pi$ , then  
 (A)  $1 + \cot\theta \leq \cot\frac{\theta}{2}$  (B)  $1 + \cot\theta \geq \cot\frac{\theta}{2}$   
 (C)  $1 + \cot\frac{\theta}{2} \geq \cot\theta$  (D)  $1 + \cot\frac{\theta}{2} \leq \cot\theta$
19. If  $\cos(\theta - \alpha) = a$  and  $\sin(\theta - \beta) = b$  ( $0 < \theta - \alpha, \theta - \beta < \pi/2$ ), then  $\cos^2(\alpha - \beta) + 2ab \sin(\alpha - \beta)$  is equal to  
 (A)  $a^2 - b^2$  (B)  $a^2 + b^2$   
 (C)  $2a^2 b^2$  (D)  $a^2 b^2$
20. If in the triangle  $ABC$ ,  $\tan\frac{A}{2}, \tan\frac{B}{2}$  and  $\tan\frac{C}{2}$  are in harmonic progression, then the least value of  $\cot\frac{B}{2}$  is  
 (A)  $\sqrt{2}$  (B)  $\sqrt{3}$   
 (C) 2 (D) none of these
21. If  $x \sin a + y \sin 2a + z \sin 3a = \sin 4a$   $x \sin b + y \sin 2b + z \sin 3b = \sin 4b$   $x \sin c + y \sin 2c + z \sin 3c = \sin 4c$ , then the roots of the equation  $t^3 - \frac{z}{2}t^2 - \frac{y+2}{4}t + \frac{z-x}{8} = 0$ ;  $a, b, c \neq n\pi$ , are  
 (A)  $\sin a, \sin b, \sin c$   
 (B)  $\cos a, \cos b, \cos c$   
 (C)  $\sin 2a, \sin 2b, \sin 2c$   
 (D)  $\cos 2a, \cos 2b, \cos 2c$
22. If  $a \sin x + b \cos(x + \theta) + b \cos(x - \theta) = d$ , then the minimum value of  $|\cos\theta|$  is  
 (A)  $\frac{1}{2|a|}\sqrt{d^2 - a^2}$  (B)  $\frac{1}{2|b|}\sqrt{d^2 - a^2}$   
 (C)  $\frac{1}{2|b|}\sqrt{a^2 - d^2}$  (D) none of these
23. If  $\sin\theta + \cos\theta = \frac{\sqrt{7}}{2}$  and  $0 < \theta < \pi/6$ , then  $\tan\left(\frac{\theta}{2}\right)$  equals  
 (A)  $\sqrt{7} - 2$  (B)  $\frac{1}{3}(\sqrt{7} - 2)$   
 (C)  $2 - \sqrt{7}$  (D)  $\frac{1}{3}(2 - \sqrt{7})$
24. If  $\sin(\theta + \alpha) = a$  and  $\sin(\theta + \beta) = b$  ( $0 < \alpha, \beta, \theta < \frac{\pi}{2}$ ) then  $\cos^2(\alpha - \beta) - 4ab \cos(\alpha - \beta)$  is equal to  
 (A)  $1 - 2a^2 - 2b^2$  (B)  $1 + 2a^2 + 2b^2$   
 (C)  $1 - a^2 - b^2$  (D) none of these
25. If  $\sin x + \operatorname{cosec} x + \tan y + \cot y = 4$ , where  $x$  and  $y \in \left[0, \frac{\pi}{2}\right]$ , then  $\tan\frac{y}{2}$  is a root of the equation  
 (A)  $\alpha^2 + 2\alpha + 1 = 0$  (B)  $\alpha^2 + 2\alpha - 1 = 0$   
 (C)  $2\alpha^2 - 2\alpha - 1 = 0$  (D) none of these
26. The value of  $2 \sin^2\theta + 4 \cos(\theta + \alpha) \sin\alpha \sin\theta + \cos 2(\alpha + \theta)$  is  
 (A)  $\cos\theta + \cos\alpha$  (B) independent of  $\theta$   
 (C) independent of  $\alpha$  (D) none of these
27. The value of  $\cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \dots \cos 2^{n-1}\theta$  for  $\theta = \frac{\pi}{2^n + 1}$  is  
 (A) 1 (B)  $\frac{1}{2^n}$   
 (C)  $2^n$  (D) none of these
28. The sum of the series  $\sin\theta \cdot \sec 3\theta + \sin 3\theta \cdot \sec 3^2\theta + \sin 3^2\theta \sec 3^3\theta + \dots$  up to  $n$  terms is  
 (A)  $\frac{1}{2}(\tan 3^n\theta - \tan\theta)$  (B)  $(\tan 3^n\theta - \tan\theta)$   
 (C)  $\tan 3^n\theta - \tan 3^{n-1}\theta$  (D) none of these
29. If  $x_1, x_2, x_3, \dots, x_n$  are in A.P. whose common difference is  $\alpha$ , then the value of  $\sin\alpha [\sec x_1 \sec x_2 + \sec x_2 \sec x_3 + \dots + \sec x_{n-1} \sec x_n]$  is equal to  
 (A)  $\frac{\sin n\alpha}{\cos x_1 \cos x_n}$  (B)  $\frac{\sin(n-1)\alpha}{\cos x_1 \cos x_n}$   
 (C)  $\frac{\sin(n+1)\alpha}{\cos x_1 \cos x_n}$  (D) none of these

**More than One Option Correct Type**

30. If  $\cos 5\theta = a \cos \theta + b \cos 3\theta + c \cos 8\theta + d$ , then  
 (A)  $a = 20$  (B)  $b = -20$   
 (C)  $c = 16$  (D)  $d = 5$
31. If  $\frac{\sin \alpha}{\sin \beta} = \frac{\sqrt{3}}{2}$  and  $\frac{\cos \alpha}{\cos \beta} = \frac{\sqrt{5}}{2}$ ,  $0 < \alpha < \beta < \frac{\pi}{2}$ , then  
 (A)  $\tan \alpha = 1$  (B)  $\tan \alpha = \frac{\sqrt{3}}{\sqrt{5}}$   
 (C)  $\tan \beta = \frac{\sqrt{3}}{\sqrt{5}}$  (D)  $\tan \beta = 1$
32. If  $\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x = 7$  and  $\sin 2x = a - b\sqrt{7}$ , then  
 (A)  $a = 8$  (B)  $b = 22$   
 (C)  $a = 22$  (D)  $b = 8$
33. Let  $n$  be an odd integer. If  $\sin n\theta = \sum_{r=0}^n b_r \sin^r \theta$ , for every value of  $\theta$ , then  
 (A)  $b_0 = 0$  (B)  $b_0 = n$   
 (C)  $b_1 = 0$  (D)  $b_1 = n$
34. If  $\alpha$  and  $\beta$  be the solutions of  $a \cos \theta + b \sin \theta = c$ , then  
 (A)  $\sin \alpha + \sin \beta = \frac{2bc}{a^2 + b^2}$   
 (B)  $\sin \alpha \sin \beta = \frac{c^2 - a^2}{a^2 + b^2}$   
 (C)  $\sin \alpha + \sin \beta = \frac{2ac}{b^2 + c^2}$   
 (D)  $\sin \alpha \cdot \sin \beta = \frac{a^2 - b^2}{b^2 + c^2}$
35. Let  $n$  be a fixed positive integer such that  $\sin\left(\frac{\pi}{2n}\right) + \cos\left(\frac{\pi}{2n}\right) = \frac{\sqrt{n}}{2}$ , then  
 (A)  $n = 4$  (B)  $n = 5$   
 (C)  $n = 6$  (D) none of these
36. If  $a \cos^2 3\alpha + b \cos^4 \alpha = 16 \cos^6 \alpha + 9 \cos^2 \alpha$  is identity, then  
 (A)  $a = 1$  (B)  $a = 24$   
 (C)  $b = 1$  (D)  $b = 24$
37. If  $A$  and  $B$  are acute angle such that  $A + B$  and  $A - B$  satisfy the equation  $\tan^2 \theta - 4 \tan \theta + 1 = 0$ , then  
 (A)  $A = \frac{\pi}{4}$  (B)  $B = \frac{\pi}{6}$   
 (C)  $A = \frac{\pi}{6}$  (D)  $B = \frac{\pi}{4}$
38. For  $0 < \phi < \pi/2$ , if  $x = \sum_{n=0}^{\infty} \cos^{2n} \phi$ ,  $y = \sum_{n=0}^{\infty} \sin^{2n} \phi$ , and  $z = \sum_{n=0}^{\infty} \cos^{2n} \phi \sin^{2n} \phi$ , then  $xyz =$   
 (A)  $xy + z$  (B)  $xz + y$   
 (C)  $x + y + z$  (D)  $yz + x$
39. Let  $f_n(\theta) = \tan \frac{\theta}{2} (1 + \sec \theta)(1 + \sec 2\theta)(1 + \sec 4\theta) \dots (1 + \sec 2^n \theta)$ , then  
 (A)  $f_2\left(\frac{\pi}{16}\right) = 1$  (B)  $f_3\left(\frac{\pi}{32}\right) = 1$   
 (C)  $f_4\left(\frac{\pi}{64}\right) = 1$  (D)  $f_5\left(\frac{\pi}{128}\right) = 1$
40. If  $(a - b) \sin(\theta + \phi) = (a + b) \sin(\theta - \phi)$  and  $a \tan \frac{\theta}{2} - b \tan \frac{\phi}{2} = c$ , then  
 (A)  $b \tan \phi = a \tan \theta$   
 (B)  $a \tan \phi = b \tan \theta$   
 (C)  $\sin \phi = \frac{2bc}{a^2 - b^2 - c^2}$   
 (D)  $\sin \theta = \frac{2ac}{a^2 - b^2 + c^2}$
41. If  $\alpha$ ,  $\beta$  and  $\gamma$  are connected by the relation  $2 \tan^2 \alpha \tan^2 \beta \tan^2 \gamma + \tan^2 \alpha \tan^2 \beta + \tan^2 \beta \tan^2 \gamma + \tan^2 \gamma \tan^2 \alpha = 1$ , then  
 (A)  $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 1$   
 (B)  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 2$   
 (C)  $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 1$   
 (D)  $\cos(\alpha + \beta) \cos(\alpha - \beta) = -\cos^2 \gamma$

### Match the Column Type

42.

Column-I	Column-II
I. If $8\sin^3\theta \sin 3\theta = \sum_{k=0}^n \alpha_k \cos k\theta$ is an identity in $\theta$ , where $\alpha_k$ 's are constants, then $n =$	(A) 0
II. If $\alpha \cos A = \beta \cos\left(A + \frac{2\pi}{3}\right) = \gamma \cos\left(A + \frac{4\pi}{3}\right)$ , then $\alpha\beta + \beta\gamma + \gamma\alpha =$	(B) 1
III. The number of all possible 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ such that $a_1 + a_2 \sin x + a_3 \cos x + a_4 \sin 2x + a_5 \cos 2x = 0$ holds for all $x$ is	(C) 6
IV. The value of $\prod_{p=1}^{10} \left( \sin \frac{2p\pi}{11} - i \cos \frac{2p\pi}{11} \right)$ is	(D) -1

43.

Column-I	Column-II
I. If $\sin\theta = 3 \sin(\theta + 2\alpha)$ , then the value of $\tan(\theta + \alpha) + 2 \tan \alpha$ is	(A) 0
II. If $p \sin\theta + q \cos\theta = a$ and $p \cos\theta - q \sin\theta = b$ then $\frac{p+a}{q+b} + \frac{q-b}{p-a} + 1$ is equal to	(B) 1
III. The value of the expression $\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{10\pi}{7} - \sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14}$ is	(C) $\sec \theta$
IV. If $\sec\theta + \tan\theta = 1$ , then one root of the equation $(a - 2b + c)x^2 + (b - 2c + a)x + (c - 2a + b) = 0$ is	(D) $-\frac{1}{4}$

44.

Column-I	Column-II
I. Least ratio of the sides of regular $n$ -sided polygon inside and outside the unit circle is	(A) 1
II. The tangents of two acute angles are 3 and 2. The sine of twice their difference is	(B) 0

- III. If  $n = \frac{\pi}{4\alpha}$ , then  $\tan \alpha \tan 2\alpha \tan 3\alpha \dots \tan (2n-1)\alpha$  is equal to (C)  $\frac{1}{2}$
- IV. If  $x = y \cos \frac{2\pi}{3} = z \cos \frac{4\pi}{3}$ , then  $xy + yz + zx =$  (D)  $\frac{7}{25}$
- V. The ratio of the greatest value of  $2 - \cos x + \sin^2 x$  to its least value is (E)  $\frac{13}{4}$

45.

Column-I	Column-II
I. The value of $\frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15}$ is	(A) $\frac{2}{1}$
II. If $A$ and $B$ be acute positive angles satisfying $3\sin^2 A + 2\sin^2 B = 1$ and $3\sin 2A - 2\sin 2B = 0$ , then $\cos(A + 2B) =$	(B) $\frac{1}{16}$
III. The number of integral values of $k$ for which the equation $7 \cos x + 5 \sin x = 2k + 1$ has a solution is	(C) 0
IV. If $A = \tan 27^\circ - \tan \theta$ and $B = \frac{\sin \theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta}$ then $\frac{A}{B} =$	(D) 8

46.

Column-I	Column-II
I. If $\alpha, \beta, \gamma$ and $\delta$ are four solutions of the equation $\tan\left(\theta + \frac{\pi}{4}\right) = 3 \tan 3\theta$ , no two of which have equal tangents, then the value of $\tan \alpha + \tan \beta + \tan \gamma + \tan \delta$ is	(A) $\sqrt{2}$
II. If $\frac{\cos(\theta_1 - \theta_2)}{\cos(\theta_1 + \theta_2)} + \frac{\cos(\theta_3 + \theta_4)}{\cos(\theta_3 - \theta_4)} = 0$ , then $\tan \theta_1 \tan \theta_2 \tan \theta_3 \tan \theta_4 =$	(B) $\sqrt{3}$
III. If $\sec(\alpha - \beta)$ , $\sec \alpha$ and $\sec(\alpha + \beta)$ are in A. P. (with $\beta \neq 0$ ), then $\cos \alpha \sec \frac{\beta}{2} =$	(C) -1

IV. If  $\alpha = \frac{2 \cos \beta - 1}{2 - \cos \beta}$  ( $0 < \alpha < \beta < \pi$ ), (D) 0

then  $\frac{\tan \alpha/2}{\tan \beta/2}$  is equal to

II.  $\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ$  is equal to (B)  $\frac{1}{8}$

III.  $\frac{1}{\cos 290^\circ} + \frac{1}{\sqrt{3} \sin 250^\circ}$  is equal to (C) 4

IV.  $\sin \frac{\pi}{18} \sin \frac{5\pi}{18} \sin \frac{7\pi}{18}$  is equal to (D)  $\frac{4}{\sqrt{3}}$

47.

**Column-I**

**Column-II**

I.  $\tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha - \cot \alpha + 1$  (A) 1  
is equal to

### Previous Year's Questions

48. In a triangle  $ABC$ ,  $2ca \sin \frac{A-B+C}{2}$  is equal to: [2002]

- (A)  $a^2 + b^2 - c^2$  (B)  $c^2 + a^2 - b^2$   
(C)  $b^2 - c^2 - a^2$  (D)  $c^2 - a^2 - b^2$

49.  $\sin^2 \theta = \frac{4xy}{(x+y)^2}$  is true if and only if: [2002]

- (A)  $x + y \neq 0$  (B)  $x = y, x \neq 0, y \neq 0$   
(C)  $x = y$  (D)  $x \neq 0, y \neq 0$

50. The value of  $\frac{1 - \tan^2 15^\circ}{1 + \tan^2 15^\circ}$  is: [2002]

- (A) 1 (B)  $\sqrt{3}$   
(C)  $\frac{\sqrt{3}}{2}$  (D) 2

51. If  $\tan \theta = -\frac{4}{3}$ , then  $\sin \theta$  is: [2002]

- (A)  $-\frac{4}{5}$  but not  $\frac{4}{5}$  (B)  $-\frac{4}{5}$  or  $\frac{4}{5}$   
(C)  $\frac{4}{5}$  but not  $-\frac{4}{5}$  (D) none of these

52. If  $\sin(\alpha + \beta) = 1, \sin(\alpha - \beta) = \frac{1}{2}$ , then  $\tan(\alpha + 2\beta) \tan(2\alpha + \beta)$  is equal to: [2002]

- (A) 1 (B) -1  
(C) zero (D) none of these

53. If  $y = \sin^2 \theta + \operatorname{cosec}^2 \theta, \theta \neq 0$  then: [2002]

- (A)  $y = 0$  (B)  $y \leq 2$   
(C)  $y \geq -2$  (D)  $y \geq 2$

54. In a triangle  $ABC, a = 4, b = 3, \angle A = 60^\circ$ , then  $c$  is the root of the equation: [2002]

- (A)  $c^2 - 3c - 7 = 0$  (B)  $c^2 + 3c + 7 = 0$   
(C)  $c^2 - 3c + 7 = 0$  (D)  $c^2 + 3c - 7 = 0$

55. In a  $\triangle ABC, \tan \frac{A}{2} = \frac{5}{6}, \tan \frac{C}{2} = \frac{2}{5}$ , then: [2002]

- (A)  $a, c, b$  are in AP (B)  $a, b, c$  are in AP  
(C)  $b, a, c$  are in AP (D)  $a, b, c$  are in GP

56. The equation  $a \sin x + b \cos x = c$  where  $|c| > \sqrt{a^2 + b^2}$  has: [2002]

- (A) a unique solution  
(B) infinite number of solutions  
(C) no solution  
(D) none of the above

57. If  $\alpha$  is a root of  $25 \cos^2 \theta + 5 \cos \theta - 12 = 0, \frac{\pi}{2} < \alpha < \pi$  then  $\sin 2\alpha$  is equal to: [2002]

- (A)  $\frac{24}{25}$  (B)  $-\frac{24}{25}$   
(C)  $\frac{13}{18}$  (D)  $-\frac{13}{18}$

58. If in a triangle  $ABC, a \cos^2 \left(\frac{C}{2}\right) + c \cos^2 \left(\frac{A}{2}\right) = \frac{3b}{2}$ , then the sides  $a, b$  and  $c$  [2003]

- (A) are in A.P. (B) are in G.P.  
(C) are in H.P. (D) satisfy  $a + b = c$

59. Let  $\alpha, \beta$  be such that  $\pi < \alpha - \beta < 3\pi$ . If  $\sin \alpha + \sin \beta = -\frac{21}{65}$  and  $\cos \alpha + \cos \beta = -\frac{27}{65}$ , then the value of  $\cos \frac{\alpha - \beta}{2}$  is [2004]

- (A)  $-\frac{3}{\sqrt{130}}$  (B)  $\frac{3}{\sqrt{130}}$   
(C)  $\frac{6}{65}$  (D)  $-\frac{6}{65}$

60. If  $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ , then

the difference between the maximum and minimum values of  $u^2$  is given by [2004]

- (A)  $2(a^2 + b^2)$  (B)  $2\sqrt{a^2 + b^2}$   
 (C)  $(a + b)^2$  (D)  $(a - b)^2$

61. The sides of a triangle are  $\sin\alpha$ ,  $\cos\alpha$  and  $\sqrt{1 + \sin\alpha \cos\alpha}$  for some  $0 < \alpha < \frac{\pi}{2}$ . Then the greatest angle of the triangle is [2004]

- (A)  $60^\circ$  (B)  $90^\circ$   
 (C)  $120^\circ$  (D)  $150^\circ$

62. In a triangle  $PQR$ ,  $\angle R = \frac{\pi}{2}$ . If  $\tan\left(\frac{P}{2}\right)$  and  $\tan\left(\frac{Q}{2}\right)$  are the roots of  $ax^2 + bx + c = 0$ ,  $a \neq 0$  then [2005]

- (A)  $a = b + c$  (B)  $c = a + b$   
 (C)  $b = c$  (D)  $b = a + c$

63. In a triangle  $ABC$ ,  $\angle C = \frac{\pi}{2}$ . If  $r$  is the inradius and  $R$  is the circumradius of the triangle  $ABC$ , then  $2(r + R)$  equals [2005]

- (A)  $b + c$  (B)  $a + b$   
 (C)  $a + b + c$  (D)  $c + a$

64. If the roots of the quadratic equation  $x^2 + px + q = 0$  are  $\tan 30^\circ$  and  $\tan 15^\circ$ , respectively then the value of  $2 + q - p$  is [2006]

- (A) 2 (B) 3  
 (C) 0 (D) 1

65. The number of values of  $x$  in the interval  $[0, 3\pi]$  satisfying the equation  $2 \sin^2 x + 5 \sin x - 3 = 0$  is [2006]

- (A) 4 (B) 6  
 (C) 1 (D) 2

66. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length  $x$ . The maximum area enclosed by the park is [2006]

- (A)  $\frac{3}{2}x^2$  (B)  $\sqrt{\frac{x^3}{8}}$   
 (C)  $\frac{1}{2}x^2$  (D)  $\pi x^2$

67. If  $0 < x < \pi$  and  $\cos x + \sin x = \frac{1}{2}$ , then  $\tan x$  is [2006]

- (A)  $\frac{(1 - \sqrt{7})}{4}$  (B)  $\frac{(4 - \sqrt{7})}{3}$   
 (C)  $-\frac{(4 + \sqrt{7})}{3}$  (D)  $\frac{(1 + \sqrt{7})}{4}$

68. Let A and B denote the statements [2009]  
 A:  $\cos\alpha + \cos\beta + \cos\lambda = 0$

B:  $\sin\alpha + \sin\beta + \sin\lambda = 0$

If  $\cos(\beta - \lambda) + \cos(\beta - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$ , then

- (A) A is true and B is false  
 (B) A is false and B is true  
 (C) both A and B are true  
 (D) both A and B are false

69. Let

$\cos(\alpha + \beta) = \frac{4}{5}$  and  $\sin(\alpha - \beta) = \frac{5}{13}$ , where  $0 \leq \alpha, \beta \leq \frac{\pi}{4}$ ,

then  $\tan 2\alpha =$  [2010]

- (A)  $\frac{56}{33}$  (B)  $\frac{19}{12}$   
 (C)  $\frac{20}{7}$  (D)  $\frac{25}{16}$

70. For a regular polygon, let  $r$  and  $R$  be the respective radii of the inscribed and the circumscribed circles. A false statement among the following is [2010]

- (A) There is a regular polygon with  $\frac{r}{R} = \frac{1}{\sqrt{2}}$   
 (B) There is a regular polygon with  $\frac{r}{R} = \frac{2}{3}$   
 (C) There is a regular polygon with  $\frac{r}{R} = \frac{\sqrt{3}}{2}$   
 (D) There is a regular polygon with  $\frac{r}{R} = \frac{1}{2}$

71. If  $A = \sin^2 x + \cos^4 x$  then, for all real values of  $x$ , [2011]

- (A)  $\frac{13}{16} \leq A \leq 1$  (B)  $1 \leq A \leq 2$   
 (C)  $\frac{3}{4} \leq A \leq \frac{13}{16}$  (D)  $\frac{3}{4} \leq A \leq 1$

72. In a  $\Delta PQR$ , if  $3 \sin P + 4 \cos Q = 6$  and  $4 \sin Q + 3 \cos P = 1$ , then the angle  $R$  is equal to [2012]

- (A)  $\frac{5\pi}{6}$  (B)  $\frac{\pi}{6}$   
 (C)  $\frac{\pi}{4}$  (D)  $\frac{3\pi}{4}$

73.  $ABCD$  is a trapezium such that  $AB$  and  $CD$  are parallel and  $BC \perp CD$ . If angle  $ADB = \theta$ ,  $BC = p$  and  $CD = q$ , then  $AB$  is equal to [2013]

- (A)  $\frac{p^2 + q^2 \cos\theta}{p \cos\theta + q \sin\theta}$  (B)  $\frac{p^2 + q^2}{p^2 \cos\theta + q^2 \sin\theta}$   
 (C)  $\frac{(p^2 + q^2) \sin\theta}{(p \cos\theta + q \sin\theta)^2}$  (D)  $\frac{(p^2 + q^2) \sin\theta}{p \cos\theta + q \sin\theta}$

74. The expression  $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$  can be written as [2013]
- (A)  $\sec A \operatorname{cosec} A + 1$  (B)  $\tan A + \cot A$  (C)  $\frac{1}{6}$  (D)  $\frac{1}{3}$   
 (C)  $\sec A + \operatorname{cosec} A$  (D)  $\sin A \cos A + 1$  (C)  $\frac{1}{4}$  (D)  $\frac{1}{12}$
75. Let  $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$  where  $x \in R$  and  $k \geq 1$  then, the value of  $f_4(x) - f_6(x)$  equals [2014]

## ANSWER KEYS

### Single Option Correct Type

1. (B) 2. (C) 3. (B) 4. (A) 5. (A) 6. (A) 7. (B) 8. (B) 9. (C) 10. (B)  
 11. (A) 12. (A) 13. (B) 14. (A) 15. (C) 16. (B) 17. (A) 18. (A) 19. (B) 20. (B)  
 21. (B) 22. (B) 23. (B) 24. (A) 25. (B) 26. (B) 27. (B) 28. (A) 29. (B)

### More than One Option Correct Type

30. (b, c) 31. (b, d) 32. (c, d) 33. (a, d) 34. (a, b) 35. (b, c) 36. (a, d) 37. (a, b) 38. (a, c)  
 39. (a, b, c, d) 40. (b, c, d) 41. (a, b, c)

### Match the Column Type

42. I  $\leftrightarrow$  (C), II  $\leftrightarrow$  (A), III  $\leftrightarrow$  (B), IV  $\leftrightarrow$  (E) 43. I  $\leftrightarrow$  (B), II  $\leftrightarrow$  (B), III  $\leftrightarrow$  (D), IV  $\leftrightarrow$  (C)  
 44. I  $\leftrightarrow$  (D), II  $\leftrightarrow$  (A), III  $\leftrightarrow$  (B), IV  $\leftrightarrow$  (E) 45. I  $\leftrightarrow$  (B), II  $\leftrightarrow$  (C), III  $\leftrightarrow$  (D), IV  $\leftrightarrow$  (A)  
 46. I  $\leftrightarrow$  (D), II  $\leftrightarrow$  (C), III  $\leftrightarrow$  (A), IV  $\leftrightarrow$  (B) 47. I  $\leftrightarrow$  (A), II  $\leftrightarrow$  (C), III  $\leftrightarrow$  (D), IV  $\leftrightarrow$  (B)

### Previous Year's Questions

48. (B) 49. (B) 50. (C) 51. (B) 52. (A) 53. (D) 54. (A) 55. (A) 56. (C) 57. (B)  
 58. (A) 59. (A) 60. (D) 61. (C) 62. (B) 63. (B) 64. (B) 65. (A) 66. (C) 67. (C)  
 68. (C) 69. (A) 70. (B) 71. (D) 72. (B) 73. (D) 74. (A) 75. (D)

## HINTS AND SOLUTIONS

### Single Option Correct Type

1. Let  $P = \cos a \cos 2a \cos 3a \dots \cos (999a)$

$$\text{Let } Q = \sin a \sin 2a \sin 3a \dots \sin (999a)$$

$$\text{Thus, } 2^{999} PQ = 2 \sin a \cos a (2 \sin 2a \cos 2a)$$

$$(2 \sin 3a \cos 3a) \dots$$

$$(2 \sin (999a) \cos (999a))$$

$$= \sin 2a \sin 4a \sin 6a \dots \sin (1998a)$$

$$= (\sin 2a \sin 4a \sin 6a \dots \sin 998a)$$

$$[-\sin (2\pi - 1000a)] [-\sin (2\pi - 1002a)] \dots$$

$$[-\sin (2\pi - 1998a)]$$

$$= \sin 2a \sin 4a \dots \sin 998a \sin 999a$$

$$\sin 997a \dots \sin a$$

$$= Q \quad [\because 2\pi = 1999a]$$

$$\text{Clearly, } Q \neq 0$$

$$\therefore 2^{999} PQ = Q \Rightarrow P = \frac{1}{2^{999}}$$

2. For  $\alpha > 1$ , the function  $y = \alpha^x$  is an increasing function

$$\text{Since } 0 < \frac{\pi}{8} < \frac{\pi}{4}$$

$$\therefore \cot \frac{\pi}{8} > \cot \frac{\pi}{4} > \tan \frac{\pi}{8}$$

$$\Rightarrow a_4 > a_3 \quad (1)$$

For  $\alpha < 1$ , the function  $y = \alpha^x$  is a decreasing function.

$$\therefore a_2 < a_1 \quad (2)$$

$$\text{Again, } \cot \frac{\pi}{8} > 1 > \tan \frac{\pi}{8} > 0$$

$$\therefore a_1 < 1 \text{ and } a_3 > 1 \quad \therefore a_1 < a_3 \quad (3)$$

From (1), (2) and (3), we get

$$a_2 < a_1 < a_3 < a_4$$

3. We have,  $x \cos^2 3\theta + y \cos^4 \theta = 16 \cos^6 \theta + 9 \cos^2 \theta$

$$\Rightarrow x(4 \cos^3 \theta - 3 \cos \theta)^2 + y \cos^4 \theta$$

$$= 9 \cos^2 \theta + 16 \cos^6 \theta$$

$$\Rightarrow x(16 \cos^6 \theta + 9 \cos^2 \theta - 24 \cos^4 \theta) + y \cos^4 \theta$$

$$= 9 \cos^2 \theta + 16 \cos^6 \theta$$

$$\Rightarrow 9x \cos^2 \theta + (y - 24x) \cos^4 \theta + 16x \cos^6 \theta$$

$$= 9 \cos^2 \theta + 16 \cos^6 \theta$$

On comparing, we get

$$9x = 9 \text{ and } y - 24x = 0$$

$$\therefore x = 1 \text{ and } y = 24.$$

4.  $|\tan \theta + \sec \theta| = |\tan \theta| + |\sec \theta|$  only if  $\sec \theta$  and  $\tan \theta$  both have same sign

$$\text{i.e., } \sec \theta \cdot \tan \theta \geq 0 \text{ or } \frac{\sin \theta}{\cos^2 \theta} \geq 0$$

$$\Rightarrow \sin \theta \geq 0 \text{ and } \cos \theta \neq 0$$

$$\Rightarrow \theta \in [0, \pi] - \left\{ \frac{\pi}{2} \right\}$$

5. We have,  $\sin \theta, \sin \phi$  and  $\cos \theta$  are in G.P.

$$\Rightarrow \sin^2 \phi = \sin \theta \cos \theta \Rightarrow \frac{1 - \cos 2\phi}{2} = \frac{\sin 2\theta}{2}$$

$$\Rightarrow \cos 2\phi = 1 - \sin 2\theta \geq 0. \quad (i)$$

Now, the discriminant of the given equation is

$$4 \cot^2 \phi - 4 = 4 \left( \frac{\cos^2 \phi - \sin^2 \phi}{\sin^2 \phi} \right)$$

$$= 4 \cos 2\phi \times \operatorname{cosec}^2 \phi \geq 0 \quad (\because \text{of (i)})$$

$\therefore$  Roots of the given equation are always **real**.

6. We have,  $\cos 50^\circ = \cos^2 25^\circ - \sin^2 25^\circ$

$$= (\cos 25^\circ + \sin 25^\circ)(\cos 25^\circ - \sin 25^\circ)$$

$$= k(\cos 25^\circ - \sin 25^\circ).$$

$$\text{But } (\cos 25^\circ + \sin 25^\circ)^2 + (\cos 25^\circ - \sin 25^\circ)^2 = 2$$

$$\Rightarrow \cos 25^\circ - \sin 25^\circ = \sqrt{2 - k^2}$$

(As  $\cos 25^\circ - \sin 25^\circ$  is positive)

$$\therefore \cos 50^\circ = k\sqrt{2 - k^2}.$$

7. Given,  $x = \frac{2 \sin \alpha}{1 + \cos \alpha + \sin \alpha}$

$$\Rightarrow x(1 + \cos \alpha + \sin \alpha) = 2 \sin \alpha$$

$$\Rightarrow x(1 + \cos \alpha) = (2 - x) \sin \alpha$$

$$\text{Let } y = \frac{1 - \cos \alpha + \sin \alpha}{1 + \sin \alpha}. \text{ Now } \frac{x}{y} = \frac{2 \sin \alpha (1 + \sin \alpha)}{(1 + \sin \alpha)^2 - \cos^2 \alpha}$$

$$\Rightarrow \frac{x}{y} = \frac{2 \sin \alpha (1 + \sin \alpha)}{1 + \sin^2 \alpha + 2 \sin \alpha - 1 + \sin^2 \alpha} = 1$$

$$\Rightarrow x = y$$

8. We have,  $e^{-\pi/2} < \theta < \pi/2$

$$\Rightarrow -\frac{\pi}{2} \log e < \log \theta < \log \frac{\pi}{2}$$

$$\text{Now, } \log_e e = 1 \text{ and } \frac{\pi}{2} < e, \quad \therefore \log \frac{\pi}{2} < \log e = 1$$

$$\therefore -\frac{\pi}{2} < \log \theta < 1 < \frac{\pi}{2}$$

$\Rightarrow \log \theta$  lies in 1st and 4th quadrant

$\therefore \cos(\log \theta)$  is positive

Now,  $0 < \cos \theta < 1 \Rightarrow \log \cos \theta < \log 1 = 0$

$\Rightarrow \log \cos \theta$  is negative

From (1) and (2), we conclude  $\cos \log \theta > \log \cos \theta$ .

9. Put  $t = \sqrt{\frac{x}{y}} > 0$

$$t + \frac{1}{t} = \left( \sqrt{t} - \frac{1}{\sqrt{t}} \right)^2 + 2 \geq 2$$

equality holding iff  $t = 1$

Also,  $t + \frac{1}{t} = 2 \sin \theta \leq 2$ , so that  $t$  should necessarily be 1.

i.e.,  $x = y$ .

10. Put  $x = \tan A, y = \tan B, z = \tan C$

$\therefore \tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

$\Rightarrow \tan C[\tan A + \tan B] = 1 - \tan A \tan B$

$$\begin{aligned} \text{Now, } \sum \frac{x+y}{1-xy} &= \sum \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B} + \frac{\tan B + \tan C}{1 - \tan B \tan C} + \frac{\tan C + \tan A}{1 - \tan C \tan A} \\ &= \frac{1}{\tan C} + \frac{1}{\tan A} + \frac{1}{\tan B} \\ &= \frac{\tan A \tan B + \tan B \tan C + \tan C \tan A}{\tan A \tan B \tan C} \\ &= \frac{1}{\tan A \tan B \tan C} = \frac{1}{xyz} \end{aligned}$$

11.  $\cos \frac{\pi}{15} \cos \frac{2\pi}{15} \cos \frac{3\pi}{15} \cos \frac{4\pi}{15} \cos \frac{6\pi}{15} \cos \frac{8\pi}{15} \left( \frac{\pi}{15} = 12^\circ \right)$

$$\begin{aligned} &= \left( \cos \frac{\pi}{15} \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \right) \cos \frac{3\pi}{15} \cos 2 \left( \frac{3\pi}{15} \right) \\ &= \frac{\sin 2^4 \left( \frac{\pi}{15} \right)}{2^4 \sin \frac{\pi}{15}} \times \frac{1}{2^2} \frac{\sin 2^2 \left( \frac{3\pi}{15} \right)}{\sin \left( \frac{3\pi}{15} \right)} \\ &= \frac{\sin \frac{16\pi}{15}}{16 \sin \frac{\pi}{15}} \times \frac{1}{2^2} \frac{\sin \left( \frac{12\pi}{15} \right)}{\sin \frac{3\pi}{15}} \\ &= \frac{1}{64} \frac{\sin \left( \pi + \frac{\pi}{15} \right)}{\sin \frac{\pi}{15}} \times \frac{\sin \left( \pi - \frac{3\pi}{15} \right)}{\sin \frac{3\pi}{15}} = -\frac{1}{2^6} \end{aligned}$$

12. We have,  $\sin \alpha + \sin \beta + \sin \gamma - \sin(\alpha + \beta + \gamma)$   
 $= \sin \alpha + \sin \beta + \sin \gamma - \sin \alpha \cos \beta \cos \gamma$

(1)  $-\cos \alpha \sin \beta \cos \gamma - \cos \alpha \cos \beta \sin \gamma + \sin \alpha \sin \beta \sin \gamma$   
 $= \sin \alpha [1 - \cos \beta \cos \gamma] + \sin \beta [1 - \cos \alpha \cos \gamma]$

(2)  $+ \sin \gamma [1 - \cos \alpha \cos \beta] + \sin \alpha \sin \beta \sin \gamma > 0$   
 $\therefore \sin \alpha + \sin \beta + \sin \gamma > \sin(\alpha + \beta + \gamma)$

$$\Rightarrow \frac{\sin(\alpha + \beta + \gamma)}{\sin \alpha + \sin \beta + \sin \gamma} < 1$$

13. Let  $u = \cos \theta \left\{ \sin \theta + \sqrt{\sin^2 \theta + \sin^2 \alpha} \right\}$

$$\Rightarrow (u - \sin \theta \cos \theta)^2 = \cos^2 \theta (\sin^2 \theta + \sin^2 \alpha)$$

$$\Rightarrow u^2 \tan^2 \theta - 2u \tan \theta + u^2 - \sin^2 \alpha = 0$$

Since  $\tan \theta$  is real, therefore

$$4u^2 - 4u^2 (u^2 - \sin^2 \alpha) \geq 0$$

$$\Rightarrow u^2 - (1 + \sin^2 \alpha) \leq 0$$

$$\Rightarrow |u| \leq \sqrt{1 + \sin^2 \alpha}$$

14.  $\cos \alpha_1 \cos \alpha_2 \dots \cos \alpha_n$

$= \sin \alpha_1 \sin \alpha_2 \dots \sin \alpha_n$  (given)

$$\Rightarrow \cos^2 \alpha_1 \dots \cos^2 \alpha_n = \frac{2(\sin \alpha_1 \cos \alpha_1) \dots 2(\sin \alpha_n \cos \alpha_n)}{2^n}$$

$$\Rightarrow \cos^2 \alpha_1 \dots \cos^2 \alpha_n = \frac{\sin 2\alpha_1 \dots \sin 2\alpha_n}{2^n}$$

$$\Rightarrow \cos \alpha_1 \dots \cos \alpha_n = \frac{(\sin 2\alpha_1 \dots \sin 2\alpha_n)^{1/2}}{2^{n/2}}$$

Since, maximum value of  $\sin \alpha = 1$

So, maximum value of  $\cos \alpha_1 \dots \cos \alpha_n = \frac{1}{2^{n/2}}$

15. We have,

$$\frac{1}{2} [2^{\sin \theta} + 2^{\cos \theta}] \geq \sqrt{2^{\sin \theta} \cdot 2^{\cos \theta}}$$

[A.M.  $\geq$  G.M.]

$$\Rightarrow 2^{\sin \theta} + 2^{\cos \theta} \geq 2 \cdot 2^{(\sin \theta + \cos \theta)/2}$$

(1)

Now,  $(\sin \theta + \cos \theta) = \sqrt{2} \sin \left( \theta + \frac{\pi}{4} \right)$

$\geq -\sqrt{2}$  for all real  $\theta$ .

$$\therefore 2^{\sin \theta} + 2^{\cos \theta} \geq 2 \cdot 2^{(\sin \theta + \cos \theta)/2}$$

$$> 2 \cdot 2^{-\sqrt{2}/2} = 2^{1-(1/\sqrt{2})}$$

Thus,  $2^{\sin \theta} + 2^{\cos \theta} \geq 2^{1-(1/\sqrt{2})}$ , for all real  $\theta$ .

16. Since A.M.  $\geq$  G.M.

$$\therefore \frac{2^{\sin \theta} + 2^{-\cos \theta}}{2} \geq \sqrt{2^{\sin \theta} \cdot 2^{-\cos \theta}}$$

$$\Rightarrow 2^{\sin \theta} + 2^{-\cos \theta} \geq 2^{1+\frac{1}{2}(\sin \theta - \cos \theta)}$$

$$= 2^{1+\frac{1}{\sqrt{2}} \sin \left( \theta - \frac{\pi}{4} \right)}$$

Thus,  $2^{\sin \theta} + 2^{-\cos \theta}$  is minimum

when  $\sin\left(\theta - \frac{\pi}{4}\right) = -1$  i.e.,  $\theta - \frac{\pi}{4} = 2n\pi + \frac{3\pi}{4}$

or,  $\theta = 2n\pi + \frac{7\pi}{4}$ ,  $n \in I$ .

17. Let  $x_0 = \cos\theta$ , then  $x_1 = \sqrt{\frac{1}{2}(1 + \cos\theta)}$   
 $= \cos \theta/2$ ,  $x_2 = \cos(\theta/2^2)$ ,  $x_3 = \cos(\theta/2^3)$ ,...and so on.

$$\therefore \left[ \begin{array}{c} \sqrt{1-x_0^2} \\ x_1, x_2, x_3, \dots \text{to } \infty \end{array} \right]$$

$$= \frac{\sin\theta}{\cos\frac{\theta}{2} \cos\frac{\theta}{2^2} \dots \cos\frac{\theta}{2^n} \dots \infty}$$

$$= \frac{2\sin\frac{\theta}{2} \cos\frac{\theta}{2}}{\cos\frac{\theta}{2} \cos\frac{\theta}{2^2} \dots \cos\frac{\theta}{2^n} \dots \infty}$$

$$= \frac{2^2 \sin\frac{\theta}{2^2} \cos\frac{\theta}{2^2}}{\cos\frac{\theta}{2^2} \cos\frac{\theta}{2^3} \dots \cos\frac{\theta}{2^n} \dots \infty}$$

$$= \lim_{n \rightarrow \infty} \frac{2^n \sin\frac{\theta}{2^n}}{\cos\frac{\theta}{2^{n+1}}}$$

$$= \lim_{n \rightarrow \infty} \theta \left( \frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}} \right) \frac{1}{\cos\frac{\theta}{2^{n+1}}} = \theta$$

$$\therefore \cos \left[ \begin{array}{c} \sqrt{1-x_0^2} \\ x_1, x_2, \dots \infty \end{array} \right] = \cos\theta = x_0.$$

18.  $1 + \cot\theta - \cot\frac{\theta}{2} = 1 + \frac{\cot^2\frac{\theta}{2} - 1}{2\cot\frac{\theta}{2}} - \cot\frac{\theta}{2}$

$$= \frac{2\cot\frac{\theta}{2} + \cot^2\frac{\theta}{2} - 1 - 2\cot^2\frac{\theta}{2}}{2\cot\frac{\theta}{2}} = \frac{\left(\cot\frac{\theta}{2} - 1\right)^2}{2\cot\frac{\theta}{2}} \leq 0$$

$$\left[ \because 0 < \frac{\theta}{2} < \frac{\pi}{2} \right]$$

$$\Rightarrow 1 + \cot\theta \leq \cot\frac{\theta}{2}$$

19. We have,

$$\sin(\alpha - \beta) = \sin(\alpha - \theta + \theta - \beta)$$

$$= \sin[(\theta - \beta) - (\theta - \alpha)]$$

$$= \sin(\theta - \beta) \cos(\theta - \alpha) - \cos(\theta - \beta) \sin(\theta - \alpha)$$

$$= ba - \sqrt{1-b^2} \sqrt{1-a^2}$$

$$\text{and } \cos(\alpha - \beta) = \cos[\theta - \beta] - (\theta - \alpha)]$$

$$= \cos(\theta - \beta) \cos(\theta - \alpha) + \sin(\theta - \beta)$$

$$\sin(\theta - \alpha)$$

$$= a\sqrt{1-b^2} + b\sqrt{1-a^2}$$

Substituting these values in the given expression, we get

$$\cos^2(\alpha - \beta) + 2ab \sin(\alpha - \beta)$$

$$= (a\sqrt{1-b^2} + b\sqrt{1-a^2})^2 + 2ab[ab - \sqrt{(1-a^2)}\sqrt{(1-b^2)}]$$

$$= a^2(1-b^2) + b^2(1-a^2) + 2ab\sqrt{(1-a^2)}\sqrt{(1-b^2)}$$

$$+ 2a^2b^2 - 2ab\sqrt{(1-a^2)}\sqrt{(1-b^2)} = a^2 + b^2.$$

20. Given,  $A + B + C = \pi$

$$\Rightarrow \cot\frac{A}{2} \cot\frac{B}{2} \cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} \quad \text{(i)}$$

But  $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$  are in H.P.  $\Rightarrow \cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$

are in A.P.  $\Rightarrow \cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2} \quad \text{(ii)}$

From (i) and (ii), we get  $\cot\frac{A}{2} \cdot \cot\frac{B}{2} \cdot \cot\frac{C}{2} = 3\cot\frac{B}{2}$

$$\therefore \cot\frac{A}{2} \cdot \cot\frac{C}{2} = 3 \quad \text{(iii)}$$

Now,  $\frac{\cot\frac{A}{2} + \cot\frac{C}{2}}{2} \geq \sqrt{\cot\frac{A}{2} \cot\frac{C}{2}}$

$$\Rightarrow \frac{2\cot\frac{B}{2}}{2} \geq \sqrt{3} \quad \text{[From (ii) and (iii)]}$$

$$\therefore \cot\frac{B}{2} \geq \sqrt{3}$$

21. The first equation can be written as  $x \sin a + y \times 2 \sin a \cos a + z \sin a(3 - 4 \sin^2 a)$

$$= 2 \times 2 \sin a \cos a \cos 2a$$

$$\Rightarrow x + 2y \cos a + z(3 + 4 \cos^2 a - 4)$$

$$= 4 \cos a(2 \cos^2 a - 1) \text{ as } \sin a \neq 0$$

$$\Rightarrow 8 \cos^3 a - 4z \cos^2 a - (2y + 4) \cos a + (z - x) = 0$$

$$\Rightarrow \cos^3 a - \frac{z}{2} \cos^2 a - \frac{y+2}{4} \cos a + \frac{z-x}{8} = 0$$

which shows that  $\cos a$  is root of the equation

$$t^3 - \frac{z}{2}t^2 - \frac{y+2}{4}t + \frac{z-x}{8} = 0$$

Similarly, from second and third equations we can verify that  $\cos b$  and  $\cos c$  are the roots of the above equation

22.  $a \sin x + b \cos(x + \theta) + b \cos(x - \theta) = d$

$$\Rightarrow a \sin x + 2b \cdot \cos x \cdot \cos \theta = d$$

$$\Rightarrow |d| \leq \sqrt{a^2 + 4b^2 \cdot \cos^2 \theta} \Rightarrow \frac{d^2 - a^2}{4b^2} \leq \cos^2 \theta$$

$$\Rightarrow |\cos \theta| \geq \frac{\sqrt{d^2 - a^2}}{2|b|}$$

23. From the given condition, we have

$$\frac{2 \tan(\alpha/2)}{1 + \tan^2(\alpha/2)} + \frac{1 - \tan^2(\alpha/2)}{1 + \tan^2(\alpha/2)} = \frac{\sqrt{7}}{2}$$

$$\Rightarrow 2 \left( 2 \tan \frac{\alpha}{2} + 1 - \tan^2 \frac{\alpha}{2} \right) = \sqrt{7} \left( 1 + \tan^2 \frac{\alpha}{2} \right)$$

$$\Rightarrow (\sqrt{7} + 2) \tan^2 \frac{\alpha}{2} - 4 \tan \frac{\alpha}{2} + (\sqrt{7} - 2) = 0.$$

This quadratic has the roots

$$\tan \frac{\alpha}{2} = \frac{4 \pm \sqrt{16 - 4(\sqrt{7} + 2)(\sqrt{7} - 2)}}{2(\sqrt{7} + 2)} = \frac{4 \pm 2}{2(\sqrt{7} + 2)}$$

That is,  $\tan(\alpha/2) = 3/(\sqrt{7} + 2)$  or  $1/(\sqrt{7} + 2)$ . The given condition on  $\alpha$  implies  $0 < \alpha/2 < \pi/12$ . Therefore, we get

$$0 < \tan \frac{\alpha}{2} < \tan \frac{\pi}{12} \Rightarrow 0 < \tan \frac{\alpha}{2} < 2 - \sqrt{3}.$$

Only the second root satisfies this condition, because

$$\frac{3}{\sqrt{7} + 2} = \sqrt{7} - 2 > 2 - \sqrt{3}$$

$$\text{and, } \frac{1}{\sqrt{7} + 2} = \frac{1}{3}(\sqrt{7} - 2) < 2 - \sqrt{3}.$$

Hence, the required value of  $\tan(\alpha/2)$  is  $\frac{1}{3}(\sqrt{7} - 2)$

24. Using,

$$\begin{aligned} \cos(\alpha - \beta) &= \cos[(\alpha + \theta) - (\theta + \beta)] \\ &= \cos(\theta + \alpha) \cos(\theta + \beta) + \sin(\theta + \alpha) \sin(\theta + \beta) \\ &= \sqrt{1 - a^2} \sqrt{1 - b^2} + ab, \end{aligned}$$

the given expression can be written as

$$\begin{aligned} 2 \cos^2(\alpha - \beta) - 1 - 4ab \cos(\alpha - \beta) &= 2(\sqrt{1 - a^2} \sqrt{1 - b^2} + ab)^2 \\ &\quad - 1 - 4ab(\sqrt{1 - a^2} \sqrt{1 - b^2} + ab) \\ &= 2[(1 - a^2)(1 - b^2) - a^2 b^2] - 1 = 1 - 2a^2 - 2b^2. \end{aligned}$$

25. Given that  $\sin x + \operatorname{cosec} x + \tan y + \cot y = 4$

$$\Rightarrow x = \frac{\pi}{2} \text{ and } y = \frac{\pi}{4}$$

$$\Rightarrow \tan y = 1$$

$$\Rightarrow \frac{2 \tan y/2}{1 - \tan^2 y/2} = 1 \Rightarrow \tan^2 \frac{y}{2} + 2 \tan \frac{y}{2} - 1 = 0.$$

26. We have,

$$\begin{aligned} 2 \sin^2 \theta + 4 \cos(\theta + \alpha) \sin \alpha \sin \theta + \cos 2(\alpha + \theta) &= 2 \sin^2 \theta + 2 \cos(\theta + \alpha) 2 \sin \alpha \sin \theta + \cos 2(\alpha + \theta) \\ &= 2 \sin^2 \theta + 2 \cos(\theta + \alpha) [\cos(\theta - \alpha) - \cos(\theta + \alpha)] \end{aligned}$$

$$\begin{aligned} &+ \cos 2(\alpha + \theta) \\ &= 2 \sin^2 \theta + 2 \cos(\theta + \alpha) \cos(\theta - \alpha) \\ &\quad - 2 \cos^2(\theta + \alpha) + \cos 2(\alpha + \theta) \\ &= 2 \sin^2 \theta + 2(\cos^2 \theta - \sin^2 \alpha) - 2 \cos^2(\theta + \alpha) \\ &\quad + [2 \cos^2(\alpha + \theta) - 1] \\ &= 2 \sin^2 \theta + 2 \cos^2 \theta - 2 \sin^2 \alpha - 1 \\ &= 2(\sin^2 \theta + \cos^2 \theta) - 2 \sin^2 \alpha - 1 \\ &= 2 - 2 \sin^2 \alpha - 1 = 1 - 2 \sin^2 \alpha = \cos 2\alpha, \end{aligned}$$

which is independent of  $\theta$ .

27. We have,  $\cos \theta \cdot \cos 2\theta \cdot \cos 2^2 \theta \dots \cos 2^{n-1} \theta$

$$= \frac{1}{2 \sin \theta} (2 \sin \theta \cdot \cos \theta \cdot \cos 2\theta \cdot \cos 2^2 \theta \dots \cos 2^{n-1} \theta)$$

$$= 2 \sin \theta (\sin 2\theta \cdot \cos 2\theta \cdot \cos 2^2 \theta \dots \cos 2^{n-1} \theta)$$

$$= \frac{1}{2^2 \sin \theta} (2 \sin 2\theta \cdot \cos 2\theta \cdot \cos 2^2 \theta \dots \cos 2^{n-1} \theta)$$

$$= \frac{1}{2^2 \sin \theta} (\sin 2^2 \theta \cos 2^2 \theta \dots \cos 2^{n-1} \theta)$$

.....  
.....

$$= \frac{1}{2^{n-1} \sin \theta} \sin 2^{n-1} \theta \cdot \cos 2^{n-1} \theta$$

$$= \frac{1}{2^n \sin \theta} (2 \sin 2^{n-1} \theta \cos^{n-1} \theta) = \frac{1}{2^n \sin \theta} \sin 2^n \theta$$

$$= \frac{1}{2^n \sin \theta} [\sin(\pi - \theta)] \left[ \because \theta = \frac{\pi}{2^n + 1}, \therefore 2^n \theta + \theta = \pi \right]$$

$$= \frac{1}{2^n \sin \theta} \cdot \sin \theta = \frac{1}{2^n}.$$

28. We have,  $\tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$

$$\Rightarrow \frac{1}{\cot \theta} = \frac{\frac{2}{\cot \frac{\theta}{2}}}{1 - \frac{1}{\cot^2 \frac{\theta}{2}}} = \frac{2 \cot \frac{\theta}{2}}{\cot^2 \frac{\theta}{2} - 1}$$

$$\Rightarrow \cot \theta = \frac{\cot^2 \frac{\theta}{2} - 1}{2 \cot \frac{\theta}{2}}$$

$$\text{Now, } \cot \frac{\theta}{2} - 1 - \cot \theta = \cot \frac{\theta}{2} - 1 - \frac{\cot^2 \frac{\theta}{2} - 1}{2 \cot \frac{\theta}{2}}$$

$$= \frac{2 \cot^2 \frac{\theta}{2} - 2 \cot \frac{\theta}{2} - \cot^2 \frac{\theta}{2} + 1}{2 \cot \frac{\theta}{2}}$$

$$= \frac{\cot^2 \frac{\theta}{2} - 2\cot \frac{\theta}{2} + 1}{2\cot \frac{\theta}{2}} = \frac{\left(\cot \frac{\theta}{2} - 1\right)^2}{2\cot \frac{\theta}{2}}$$

Now, since  $0 < \theta < \pi$ ,  $\therefore 0 < \frac{\theta}{2} < \frac{\pi}{2} \Rightarrow \cot \frac{\theta}{2}$  is positive.

Also, square of a real number  $\geq 0$

$$\therefore \cot \frac{\theta}{2} - 1 - \cot \theta \geq 0 \Rightarrow \cot \frac{\theta}{2} \geq 1 + \cot \theta.$$

29. We have,  $\frac{\sin \alpha}{\cos x_1 \cos x_2} = \frac{\sin(x_2 - x_1)}{\cos x_1 \cos x_2}$

$$= \frac{\sin x_2 \cos x_1 - \cos x_2 \sin x_1}{\cos x_1 \cos x_2}$$

$$= \tan x_2 - \tan x_1.$$

$\therefore \sin \alpha [\sec x_1 \sec x_2 + \sec x_2 \sec x_3 + \dots \text{ to } n \text{ terms}]$

$$= (\tan x_2 - \tan x_1) + (\tan x_3 - \tan x_2) +$$

$$\dots + (\tan x_n - \tan x_{n-1})$$

$$= \tan x_n - \tan x_1$$

$$= \frac{\sin x_n \cos x_1 - \cos x_n \sin x_1}{\cos x_n \cos x_1}$$

$$= \frac{\sin(x_n - x_1)}{\cos x_1 \cos x_n} = \frac{\sin(n-1)\alpha}{\cos x_1 \cos x_n}$$

### More than One Option Correct Type

30.  $\cos 5\theta = \cos(4\theta + \theta)$

$$= \cos 4\theta \cos \theta - \sin 4\theta \sin \theta$$

$$= (2\cos^2 2\theta - 1) \cos \theta - 2\sin 2\theta \cos 2\theta \sin \theta$$

$$= (2[2\cos^2 \theta - 1]^2 - 1) \cos \theta - 2.2 \sin \theta \cos \theta$$

$$[2\cos^2 \theta - 1] \sin \theta$$

$$= \cos \theta [2(4\cos^4 \theta + 1 - 4\cos^2 \theta) - 1]$$

$$- 4\sin^2 \theta \cos \theta [2\cos^2 \theta - 1]$$

$$= 8\cos^5 \theta - 8\cos^3 \theta + 2\cos \theta - \cos \theta$$

$$- 4(1 - \cos^2 \theta) \cos \theta (2\cos^2 \theta - 1)$$

$$= 8\cos^5 \theta - 8\cos^3 \theta + \cos \theta - 4\cos \theta$$

$$[2\cos^2 \theta - 1 - 2\cos^4 \theta + \cos^2 \theta]$$

$$= 16\cos^5 \theta - 20\cos^3 \theta + 5\cos \theta$$

$$\therefore a = 5, b = -20, c = 16, d = 0$$

31. We have,  $\frac{\sin \alpha}{\sin \beta} \cdot \frac{\cos \beta}{\cos \alpha} = \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}}$

$$\Rightarrow \frac{\tan \alpha}{\tan \beta} = \frac{\sqrt{3}}{\sqrt{5}} \Rightarrow \frac{\tan \alpha}{\sqrt{3}} = \frac{\tan \beta}{\sqrt{5}} = k \text{ (say)}$$

Also,  $2\sin \alpha = \sqrt{3} \sin \beta$

$$\Rightarrow \frac{2 \tan \alpha}{\sqrt{1 + \tan^2 \alpha}} = \frac{\sqrt{3} \tan \beta}{\sqrt{1 + \tan^2 \beta}} \Rightarrow \frac{2\sqrt{3}k}{\sqrt{1 + 3k^2}} = \frac{\sqrt{3} \cdot \sqrt{5}k}{\sqrt{1 + 5k^2}}$$

$$\Rightarrow 4(1 + 5k^2) = 5(1 + 3k^2) \Rightarrow 5k^2 = 1 \Rightarrow k = \frac{1}{\sqrt{5}}$$

$$\therefore \tan \alpha = \frac{\sqrt{3}}{\sqrt{5}} \text{ and } \tan \beta = 1.$$

32.  $\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x = 7$

$$\Rightarrow (\sin x + \cos x) + \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) + \left(\frac{1}{\cos x} + \frac{1}{\sin x}\right) = 7$$

$$\Rightarrow (\sin x + \cos x) + \left(\frac{1}{\sin x \cos x}\right) + \left(\frac{\sin x + \cos x}{\sin x \cos x}\right) = 7$$

$$\Rightarrow (\sin x + \cos x) \left(1 + \frac{2}{\sin 2x}\right) = 7 - \frac{2}{\sin 2x}$$

Squaring, we get

$$(1 + \sin 2x) \left(1 + \frac{4}{\sin^2 2x} + \frac{4}{\sin 2x}\right)$$

$$= 49 + \frac{4}{\sin^2 2x} - \frac{28}{\sin 2x}$$

$$\Rightarrow \sin^3 2x - 44 \sin^2 2x + 36 \sin 2x = 0$$

$$\Rightarrow \sin^2 2x - 44 \sin 2x + 36 = 0 \quad [\because \sin 2x \neq 0]$$

$$\Rightarrow \sin^2 2x = 22 - 8\sqrt{7} \Rightarrow a = 22, b = 8.$$

Hence,  $(a, b) = (22, 8)$

33. Putting  $\theta = 0$ ,

we get  $0 = b_0, \therefore \sin \theta = \sum_{r=0}^n b_r \sin^r \theta$

$$\Rightarrow \frac{\sin n\theta}{\sin \theta} = \sum_{r=1}^n b_r (\sin \theta)^{r-1}$$

$$= b_1 + b_2 \sin \theta + b_3 \sin^2 \theta + \dots + b_n \sin^{n-1} \theta$$

Taking limit as  $\theta \rightarrow 0$ , we obtain  $\lim_{\theta \rightarrow 0} \frac{\sin n\theta}{\sin \theta} = b_1$

$$\Rightarrow b_1 = n \left[ \because \lim_{\theta \rightarrow 0} \frac{\sin n\theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{n \cos n\theta}{\cos \theta} = \frac{n \times 1}{1} = n \right]$$

34. We have,  $a \cos \theta = c - b \sin \theta$

$$\Rightarrow a^2 (1 - \sin^2 \theta) = c^2 - 2bc \sin \theta + b^2 \sin^2 \theta$$

[Squaring both sides]

$$\Rightarrow (a^2 + b^2) \sin^2 \theta - 2bc \sin \theta + (c^2 - a^2) = 0.$$

Since  $\alpha$  and  $\beta$  are the values of  $\theta$  as given,

$\therefore$  roots of the above equation are  $\sin \alpha$  and  $\sin \beta$ .

$$\therefore \sin \alpha + \sin \beta = \text{sum of roots} = \frac{2bc}{a^2 + b^2}$$

and,  $\sin \alpha \sin \beta = \text{product of roots} = \frac{c^2 - a^2}{a^2 + b^2}$ .

35. We have,

$$\begin{aligned}\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} &= \sqrt{2} \sin \left( \frac{\pi}{4} + \frac{\pi}{2n} \right) \\ \Rightarrow \frac{\sqrt{n}}{2} &= \sqrt{2} \sin \left( \frac{\pi}{4} + \frac{\pi}{2n} \right)\end{aligned}$$

So, for  $n > 1$ ,  $\frac{\sqrt{n}}{2\sqrt{2}} = \sin \left( \frac{\pi}{4} + \frac{\pi}{2n} \right) > \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ .

Thus,  $n > 4$ .

Since  $\sin \left( \frac{\pi}{4} + \frac{\pi}{2n} \right) < 1$  for all  $n > 2$ ,

we get  $\frac{\sqrt{n}}{2\sqrt{2}} < 1$  or  $n < 8$ .

$\therefore 4 < n < 8$ .

36. Given,  $a \cos^2 3\alpha + b \cos^4 \alpha = 16 \cos^6 \alpha + 9 \cos^2 \alpha$   
 $\Rightarrow 9 \cos^2 \alpha + 16 \cos^6 \alpha = a (4 \cos^3 \alpha - 3 \cos \alpha)^2 + b \cos^4 \alpha$   
 $= a (16 \cos^6 \alpha + 9 \cos^2 \alpha - 24 \cos^4 \alpha) + b \cos^4 \alpha$   
 $= 9a \cos^2 \alpha + (b - 24a) \cos^4 \alpha + 16a \cos^6 \alpha$   
 Comparing, we get  $9a = 9 \Rightarrow a = 1, b - 24a = 0$   
 $\therefore b = 24, a = 24$

37. Since  $A + B$  and  $A - B$  satisfy the given equation,  
 therefore,  $\tan(A + B) + \tan(A - B) = 4$  (1)  
 $\tan(A + B) \cdot \tan(A - B) = 1$  (2)  
 From (1) and (2), we have

$$\begin{aligned}\tan(A + B + A - B) &= \frac{\tan(A + B) + \tan(A - B)}{1 - \tan(A + B)\tan(A - B)} = \infty \\ \Rightarrow 2A &= \frac{\pi}{2} \text{ or } A = \frac{\pi}{4}.\end{aligned}$$

Therefore, from (1), we have

$$\begin{aligned}\frac{1 + \tan B}{1 - \tan B} + \frac{1 - \tan B}{1 + \tan B} &= 4 \\ \Rightarrow \frac{(1 + \tan B)^2 + (1 - \tan B)^2}{1 - \tan^2 B} &= 4 \\ \Rightarrow \frac{2(1 + \tan^2 B)}{1 - \tan^2 B} &= 4 \Rightarrow \frac{1 - \tan^2 B}{1 + \tan^2 B} = \frac{1}{2} \\ \Rightarrow \cos 2B = \frac{1}{2} &\Rightarrow 2B = \frac{\pi}{3} \text{ or } B = \frac{\pi}{6}.\end{aligned}$$

38. We have,

$$\begin{aligned}x &= \sum_{n=0}^{\infty} \cos^{2n} \phi = \frac{1}{1 - \cos^2 \phi} = \operatorname{cosec}^2 \phi \\ y &= \sum_{n=0}^{\infty} \sin^{2n} \phi = \frac{1}{1 - \sin^2 \phi} = \sec^2 \phi \\ z &= \sum_{n=0}^{\infty} \cos^{2n} \phi \sin^{2n} \phi\end{aligned}$$

$$= \frac{1}{1 - \cos^2 \phi \sin^2 \phi} = \frac{1}{1 - 1/xy} = \frac{xy}{xy - 1}$$

$$\Rightarrow xyz - z = xy \Rightarrow xyz = xy + z$$

$$\text{Also, } xy = \frac{1}{\sin^2 \phi \cos^2 \phi} = \frac{\sin^2 \phi + \cos^2 \phi}{\sin^2 \phi \cos^2 \phi}$$

$$= \frac{1}{\cos^2 \phi} + \frac{1}{\sin^2 \phi} = x + y,$$

So that, we can write  $xyz = x + y + z$ .

39. We have,  $\left( \tan \frac{\theta}{2} \right) (1 + \sec \theta) = \tan \left( \frac{\theta}{2} \right) \left( \frac{1 + \cos \theta}{\cos \theta} \right)$

$$= \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \cdot \frac{2 \cos^2 \frac{\theta}{2}}{\cos \theta} = \frac{2 \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{\cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

Therefore, by repeated use of (1), we have

$$\begin{aligned}f_n(\theta) &= \tan \theta (1 + \sec 2\theta) (1 + \sec 4\theta) && (1 + \sec 2^n \theta) \\ &= \tan 2\theta (1 + \sec 4\theta) && (1 + \sec 2^n \theta) \\ &= \tan 4\theta (1 + \sec 8\theta) && (1 + \sec 2^n \theta) \\ &= \tan 2^n \theta\end{aligned}$$

$$\text{Now, } f_2 \left( \frac{\pi}{16} \right) = \tan \left( 2^2 \frac{\pi}{16} \right) = \tan \frac{\pi}{4} = 1$$

40.  $(a - b) \sin(\theta + \phi) = (a + b) \sin(\theta - \phi)$

$$\Rightarrow \frac{\sin(\theta + \phi)}{\sin(\theta - \phi)} = \frac{a + b}{a - b}$$

$$\Rightarrow \frac{\sin(\theta + \phi) + \sin(\theta - \phi)}{\sin(\theta + \phi) - \sin(\theta - \phi)} = \frac{2a}{2b}$$

$$\Rightarrow \frac{2 \sin \theta \cos \phi}{2 \cos \theta \sin \phi} = \frac{a}{b} = \frac{\tan \theta}{\tan \phi} = \frac{a}{b}$$

$$\Rightarrow b \tan \theta = a \tan \phi$$

$$\Rightarrow \frac{2a \tan \frac{\phi}{2}}{1 - \tan^2 \frac{\phi}{2}} = \frac{2b \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

$$\Rightarrow \frac{a \left( \frac{a \tan \frac{\theta}{2} - c}{b} \right)}{1 - \left( \frac{a \tan \frac{\theta}{2} - c}{b} \right)^2} = \frac{b \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

$$\Rightarrow \frac{\left( a^2 \tan \frac{\theta}{2} - ac \right)}{b^2 - \left( \tan^2 \frac{\theta}{2} + c^2 - 2ac \tan \frac{\theta}{2} \right)} = \frac{\tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

$$\Rightarrow \left( a^2 \tan \frac{\theta}{2} - ac \right) \left( 1 - \tan^2 \frac{\theta}{2} \right)$$

$$= b^2 \tan \frac{\theta}{2} - a^2 \tan^3 \frac{\theta}{2} - c^2 \tan \frac{\theta}{2} + 2ac \tan^2 \frac{\theta}{2}$$

$$\Rightarrow a^2 \tan \frac{\theta}{2} - ac - a^2 \tan^3 \frac{\theta}{2} + ac \tan^2 \frac{\theta}{2}$$

$$= b^2 \tan \frac{\theta}{2} - a^2 \tan^3 \frac{\theta}{2} - c^2 \tan \frac{\theta}{2} + 2ac \tan^2 \frac{\theta}{2}$$

$$\Rightarrow (a^2 + c^2 - b^2) \tan \frac{\theta}{2} = ac \left[ 1 + \tan^2 \frac{\theta}{2} \right]$$

$$\Rightarrow \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} = \frac{2ac}{c^2 + a^2 - b^2}$$

$$\Rightarrow \sin \theta = \frac{2ac}{c^2 + a^2 - b^2}$$

Similarly,  $\sin \phi = \frac{2bc}{a^2 - b^2 - c^2}$

41. The given relation can be written

$$\sum \tan^2 \beta \tan^2 \gamma = 1 - 2 \tan^2 \alpha \tan^2 \beta \tan^2 \gamma.$$

Using this, we have

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$$

$$= \frac{\tan^2 \alpha}{1 + \tan^2 \alpha} + \frac{\tan^2 \beta}{1 + \tan^2 \beta} + \frac{\tan^2 \gamma}{1 + \tan^2 \gamma}$$

$$= \frac{\sum \tan^2 \alpha (1 + \tan^2 \beta)(1 + \tan^2 \gamma)}{(1 + \tan^2 \alpha)(1 + \tan^2 \beta)(1 + \tan^2 \gamma)}$$

$$\frac{\sum \tan^2 \alpha + 2(1 - 2 \tan^2 \alpha \tan^2 \beta \tan^2 \gamma) + 3 \tan^2 \alpha \tan^2 \beta \tan^2 \gamma}{1 + \sum \tan^2 \alpha + (1 - 2 \tan^2 \alpha \tan^2 \beta \tan^2 \gamma) + \tan^2 \alpha \tan^2 \beta \tan^2 \gamma}$$

$$= \frac{2 + \sum \tan^2 \alpha - \tan^2 \alpha \tan^2 \beta \tan^2 \gamma}{2 + \sum \tan^2 \alpha - \tan^2 \alpha \tan^2 \beta \tan^2 \gamma} = 1.$$

This proves (a). Answer (b) follows because

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 3 - (\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma) = 2.$$

Using these results, answer (c) follows:

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sum (\cos^2 \alpha - \sin^2 \alpha)$$

$$= \sum \cos^2 \alpha - \sum \sin^2 \alpha$$

$$= 2 - 1 = 1$$

However, answer (d) implies

$$\cos^2 \alpha - \sin^2 \beta = -\cos^2 \gamma$$

$$\Rightarrow \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

which is not correct, from (b).

### Match the Column Type

42. I We have,  $\sum_{k=0}^n \alpha_k \cos k\theta = 8 \sin^3 \theta \sin 3\theta$

$$= 8 \left( \frac{3 \sin \theta - \sin 3\theta}{4} \right) \sin 3\theta$$

$$= 6 \sin \theta \sin 3\theta - 2 \sin^2 3\theta$$

$$= 3(\cos 2\theta - \cos 4\theta) - (1 - \cos 6\theta)$$

$$= -1 + 3 \cos 2\theta - 3 \cos 4\theta + \cos 6\theta$$

$$\therefore n = 6.$$

II. We have,

$$\alpha \cos A = \beta \cos \left( A + \frac{2\pi}{3} \right) = \gamma \cos \left( A + \frac{4\pi}{3} \right)$$

Dividing by  $\alpha\beta\gamma$ , we get  $\alpha\beta + \beta\gamma + \gamma\alpha$

$$= \frac{1}{k} \left[ \cos A + \cos \left( A + \frac{2\pi}{3} \right) + \cos \left( A + \frac{4\pi}{3} \right) \right]$$

$$= \frac{1}{k} \left[ \cos A + 2 \cos \left( \pi + A \right) \cos \frac{\pi}{3} \right]$$

$$= \frac{1}{k} [\cos A - \cos A] = 0.$$

III. Since the equation  $a_1 + a_2 \sin x + a_3 \cos x + a_4 \sin 2x + a_5 \cos 2x = 0$  holds for all values of  $x$ ,

$$a_1 + a_3 + a_5 = 0 \quad (\text{on putting } x = 0)$$

$$a_1 - a_3 + a_5 = 0 \quad (\text{on putting } x = \pi)$$

$$\Rightarrow a_3 = 0 \text{ and } a_1 + a_5 = 0 \tag{1}$$

Putting  $x = \pi/2$  and  $3\pi/2$ , we get

$$a_1 + a_2 - a_5 = 0 \text{ and } a_1 - a_2 - a_5 = 0$$

$$\Rightarrow a_2 = 0 \text{ and } a_1 - a_5 = 0 \tag{2}$$

(1) and (2) give

$$a_1 = a_2 = a_3 = a_5 = 0$$

The given equation reduces to  $a_4 \sin 2x = 0$ . This is true for all values of  $x$ , therefore,  $a_4 = 0$

$$\text{Hence, } a_1 = a_2 = a_3 = a_4 = a_5 = 0$$

Thus, the number of 5-tuples is one

$$\text{IV. } \prod_{p=1}^{10} (\sin 2p\pi - i \cos 2p\pi)$$

$$= (i)^{10} \prod_{p=1}^{10} (\cos 2p\pi + i \sin 2p\pi)$$

$$= - \prod_{p=1}^{10} e^{\frac{2p\pi i}{11}} = -e^{\frac{2\pi i(1+2+\dots+10)}{11}}$$

$$= -e^{10\pi i}$$

$$= -1(\cos 10\pi + i \sin 10\pi) = -1$$

43. I Given,  $\sin \theta = 3 \sin(\theta + 2\alpha)$

$$\Rightarrow \sin(\theta + \alpha - \alpha) = 3 \sin(\theta + \alpha + \alpha)$$

$$\Rightarrow \sin(\theta + \alpha) \cos \alpha - \cos(\theta + \alpha) \sin \alpha$$

$$= 3 \sin(\theta + \alpha) \cos \alpha + 3 \cos(\theta + \alpha) \sin \alpha$$

$$\Rightarrow -2 \sin(\theta + \alpha) \cos \alpha = 4 \cos(\theta + \alpha) \sin \alpha$$

$$\Rightarrow -\frac{\sin(\theta + \alpha)}{\cos(\theta + \alpha)} = \frac{2 \sin \alpha}{\cos \alpha}$$

$$\Rightarrow -\tan(\theta + \alpha) = 2 \tan \alpha$$

$$\Rightarrow \tan(\theta + \alpha) + 2 \tan \alpha = 0.$$

II. We have,  $p \sin \theta + q \cos \theta = a$

$$\text{and, } p \cos \theta - q \sin \theta = b$$

Squaring (1) and (2), and then adding, we get

$$(p \sin \theta + q \cos \theta)^2 + (p \cos \theta - q \sin \theta)^2 = a^2 + b^2$$

$$\Rightarrow p^2(1) + q^2(1) - a^2 - b^2 = 0$$

$$\Rightarrow (p^2 - a^2) + (q^2 - b^2) = 0$$

$$\Rightarrow (p + a)(p - a) + (q + b)(q - b) = 0$$

$$\Rightarrow \frac{p + a}{q + b} + \frac{q - b}{p - a} = 0$$

III.  $\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14}$

$$= \cos\left(\frac{\pi}{2} - \frac{\pi}{14}\right) \cos\left(\frac{\pi}{2} - \frac{3\pi}{14}\right) \cos\left(\frac{\pi}{2} - \frac{5\pi}{14}\right)$$

$$= \cos\left(\frac{3\pi}{7}\right) \cos\left(\frac{2\pi}{7}\right) \cos\left(\frac{\pi}{7}\right)$$

$$= -\cos\left(\frac{\pi}{7}\right) \cos\left(2\frac{\pi}{7}\right) \cos\left(\frac{4\pi}{7}\right)$$

$$\text{Also, } \cos \frac{10\pi}{7} = \cos \frac{4\pi}{7}$$

$$\text{so, } \cos \frac{\pi}{7} \cos 2\frac{\pi}{7} \cos 10\frac{\pi}{7} - \sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14}$$

$$= 2 \cos\left(\frac{\pi}{7}\right) \cos\left(2\frac{\pi}{7}\right) \cos\left(4\frac{\pi}{7}\right)$$

$$= 2 \cdot \frac{\sin\left(8\frac{\pi}{7}\right)}{2^3 \cdot \sin\left(\frac{\pi}{7}\right)} = -\frac{1}{4}$$

IV.  $\sum (a - 2b + c) = 0$ , so one root is 1

$$\text{Now, } \sec \theta + \tan \theta = 1$$

$$\text{Also, } \sec^2 \theta - \tan^2 \theta = 1$$

$$\Rightarrow \sec \theta - \tan \theta = 1 \Rightarrow \sec \theta = 1$$

so, one root is  $\sec \theta$

44. I Given  $\tan \alpha = 3$  and  $\tan \beta = 2$

$$\therefore \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{3 - 2}{1 + 3 \times 2} = \frac{1}{7}$$

$$\Rightarrow \sin(\alpha - \beta) = \frac{1}{\sqrt{50}} \text{ and } \cos(\alpha - \beta) = \frac{7}{\sqrt{50}}$$

$$\therefore \sin 2(\alpha - \beta) = 2 \sin(\alpha - \beta) \cos(\alpha - \beta)$$

$$= 2 \times \frac{1}{\sqrt{50}} \times \frac{7}{\sqrt{50}} = \frac{7}{25}.$$

II. We have,

$$\tan \alpha \cdot \tan(2n - 1)\alpha = \tan \alpha \cdot \tan\left(\frac{\pi}{2\alpha} - 1\right)\alpha$$

$$= \tan \alpha \cdot \tan\left(\frac{\pi}{2} - \alpha\right)$$

$$= \tan \alpha \cot \alpha = 1.$$

$\therefore$  The given expression = 1.

III. We have,  $x = y \cos \frac{2\pi}{3} = z \cos \frac{4\pi}{3} = k$  (say)

$$\Rightarrow \frac{1}{x} = \frac{1}{k}, \frac{1}{y} = \frac{\cos \frac{2\pi}{3}}{k}, \frac{1}{z} = \frac{\cos \frac{4\pi}{3}}{k}$$

$$\therefore \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{k} \left(1 + \cos \frac{2\pi}{3} + \cos \frac{4\pi}{3}\right) \\ = \frac{1}{k} \left(1 - \frac{1}{2} - \frac{1}{2}\right) = 0$$

$$\Rightarrow xy + xz + yz = 0.$$

IV. We have,

$$2 - \cos x + \sin^2 x = 2 - \cos x + 1 - \cos^2 x$$

$$= -(\cos^2 x + \cos x) + 3$$

$$= -\left[\left(\cos x + \frac{1}{2}\right)^2 - \frac{1}{4}\right] + 3$$

$$= \frac{13}{4} - \left(\cos x + \frac{1}{2}\right)^2.$$

$\therefore$  Maximum value occurs at  $\cos x = -\frac{1}{2}$  and it is  $\frac{13}{4}$  and minimum value occurs at  $\cos x = 1$  and it is 1

$\therefore$  The required ratio is  $\frac{13}{4}$ .

45. I The given expression

$$= \cos \frac{2\pi}{15} \cdot \cos \frac{4\pi}{15} \cdot \cos \frac{8\pi}{15} \cos \frac{16\pi}{15}$$

$$\left[ \because \cos\left(\frac{16\pi}{15}\right) = \cos\left(2\pi - \frac{14\pi}{15}\right) = \cos \frac{14\pi}{15} \right]$$

$$= \cos A \cdot \cos 2A \cdot \cos 2^2 A \cdot \cos 2^3 A, \text{ where } A = \frac{2\pi}{15}$$

$$= \frac{\sin 2^4 A}{2^4 \sin A} = \frac{\sin \frac{32\pi}{15}}{16 \sin \frac{2\pi}{15}} = \frac{\sin 384^\circ}{16 \sin 24^\circ}$$

$$= \frac{1}{16} [\sin 384^\circ = \sin (360^\circ + 24^\circ) = \sin 24^\circ]$$

II. From the given relations we have

$$\sin 2B = \left(\frac{3}{2}\right) \sin 2A$$

$$\text{and, } 3 \sin^2 A = 1 - 2 \sin^2 B = \cos 2B.$$

$$\text{so that } \tan 2B = \frac{(3/2) \cdot \sin 2A}{3 \sin^2 A} = \cot A$$

$$\text{or, } 1 - \tan 2B \tan A = 0$$

$$\Rightarrow A + 2B = \frac{\pi}{2}.$$

III. The given equation can be written as  $r \cos(x - \alpha) = 2k + 1$  where  $r \cos \alpha = 7$ ,  $r \sin \alpha = 5$

$$\Rightarrow \cos(x - \alpha) = \frac{2k + 1}{\sqrt{74}} \text{ as } r^2 = 7^2 + 5^2 = 74$$

$$\Rightarrow -1 \leq \frac{2k + 1}{\sqrt{74}} \leq 1$$

$$\Rightarrow -\sqrt{74} \leq 2k + 1 \leq \sqrt{74}$$

$$\Rightarrow -8 \leq 2k + 1 \leq 8 \text{ (For integral values of } k)$$

$$\Rightarrow -4 \leq k \leq 3$$

$$k = -4, -3, -2, -1, 0, 1, 2, 3$$

which gives 8 integral values of  $k$ .

IV. We can write

$$A = \tan 27\theta - \tan 9\theta + \tan 9\theta - \tan 3\theta + \tan 3\theta - \tan \theta$$

But,

$$\tan 3\theta - \tan \theta = \frac{\sin 3\theta \cos \theta - \cos 3\theta \sin \theta}{\cos 3\theta \cos \theta}$$

$$= \frac{\sin 2\theta}{\cos 3\theta \cos \theta} = \frac{2 \sin \theta}{\cos 3\theta}$$

$$\therefore A = 2 \left[ \frac{\sin 9\theta}{\cos 27\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin \theta}{\cos 3\theta} \right] = 2B$$

$$\therefore \frac{A}{B} = \frac{2}{1}$$

46. I Using,

$$\tan\left(\theta + \frac{\pi}{4}\right) = \frac{1 + \tan \theta}{1 - \tan \theta}$$

$$\text{and } 3 \tan 3\theta = \frac{3(3 \tan \theta - \tan^3 \theta)}{1 - 3 \tan^2 \theta},$$

the given equation becomes

$$3 \tan^4 \theta - 6 \tan^2 \theta + 8 \tan \theta - 1 = 0.$$

If  $\tan \alpha$ ,  $\tan \beta$ ,  $\tan \gamma$  and  $\tan \delta$  are the roots of this equation, then the sum of these roots,  $\tan \alpha + \tan \beta + \tan \gamma + \tan \delta$  equals zero, since the coefficient of  $\tan^3 \theta$  is zero.

II. The given equation can be written as

$$\Rightarrow \frac{\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2}{\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2}$$

$$+ \frac{\cos \theta_3 \cos \theta_4 - \sin \theta_3 \sin \theta_4}{\cos \theta_3 \cos \theta_4 + \sin \theta_3 \sin \theta_4}$$

$$\Rightarrow \frac{1 + \tan \theta_1 \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2} + \frac{1 - \tan \theta_3 \tan \theta_4}{1 + \tan \theta_3 \tan \theta_4} = 0$$

$$\Rightarrow \frac{2 + 2 \tan \theta_1 \tan \theta_2 \tan \theta_3 \tan \theta_4}{(1 - \tan \theta_1 \tan \theta_2)(1 + \tan \theta_3 \tan \theta_4)} = 0$$

$$\text{showing that } \tan \theta_1 \tan \theta_2 \tan \theta_3 \tan \theta_4 = -1$$

III. For the given A. P., we have  $2 \sec \alpha = \sec(\alpha - \beta) + \sec(\alpha + \beta)$ , which gives

$$\frac{2}{\cos \alpha} = \frac{1}{\cos(\alpha - \beta)} + \frac{1}{\cos(\alpha + \beta)} = \frac{2 \cos \alpha \cos \beta}{\cos^2 \alpha - \sin^2 \beta}$$

$$\Rightarrow \cos^2 \alpha - \sin^2 \beta = \cos^2 \alpha \cos \beta$$

$$\Rightarrow \cos^2 \alpha (1 - \cos \beta) = \sin^2 \beta$$

$$\Rightarrow \cos^2 \alpha \left(2 \sin^2 \frac{\beta}{2}\right) = 4 \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2}$$

$$\Rightarrow \cos^2 \alpha \sec^2 \frac{\beta}{2} = 2$$

$$\text{IV. } 1 + \cos \alpha = 1 + \frac{2 \cos \beta - 1}{2 - \cos \beta} = \frac{2 - \cos \beta + 2 \cos \beta - 1}{2 - \cos \beta}$$

$$= \frac{1 + \cos \beta}{2 - \cos \beta}$$

$$\Rightarrow \cos^2 \frac{\alpha}{2} = \frac{\cos^2 \beta / 2}{1 + 2 \sin^2 \beta / 2} \tag{1}$$

$$\Rightarrow 1 - \cos^2 \frac{\alpha}{2} = 1 - \frac{\cos^2 \beta / 2}{1 + 2 \sin^2 \beta / 2}$$

$$= \frac{1 + 2 \sin^2 \beta / 2 - \cos^2 \beta / 2}{1 + 2 \sin^2 \beta / 2}$$

$$= \frac{1 + 2 \sin^2 \beta / 2 - [1 - \sin^2 \beta / 2]}{1 + 2 \sin^2 \beta / 2}$$

$$\Rightarrow \sin^2 \alpha / 2 = \frac{3 \sin^2 \beta / 2}{1 + 2 \sin^2 \beta / 2} \tag{2}$$

Dividing equation (2) by (1), we get

$$\tan^2 \frac{\alpha}{2} = 3 \tan^2 \frac{\beta}{2} \Rightarrow \frac{\tan \alpha / 2}{\tan \beta / 2} = \sqrt{3}$$

47. I Using the identity  $\cot x - \tan x = 2 \cot 2x$ , we have

$$\tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha - \cot \alpha$$

$$\begin{aligned}
 &= \tan \alpha - \cot \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha \\
 &= -2 \cot 2\alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha \\
 &= -4 \cot 4\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha \\
 &= -8 \cot 8\alpha + 8 \cot 8\alpha = 0.
 \end{aligned}$$

II. We have,

$$\begin{aligned}
 &\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ \\
 &= \tan 9^\circ - \tan 27^\circ - \cot 27^\circ + \cot 9^\circ \\
 &= \tan 9^\circ + \cot 9^\circ - (\tan 27^\circ + \cot 27^\circ) \\
 &= \frac{1}{\sin 9^\circ \cos 9^\circ} - \frac{1}{\sin 27^\circ \cos 27^\circ} = \frac{2}{\sin 18^\circ} - \frac{2}{\sin 54^\circ} \\
 &= \frac{8}{\sqrt{5}-1} - \frac{8}{\sqrt{5}+1} = \frac{8(\sqrt{5}+1-\sqrt{5}+1)}{5-1} = 4.
 \end{aligned}$$

III. The given expression is equal to

$$\begin{aligned}
 &= \frac{1}{\cos(270^\circ + 20^\circ)} + \frac{1}{\sqrt{3} \sin(270^\circ - 20^\circ)} \\
 &= \frac{1}{\sin 20^\circ} + \frac{1}{\sqrt{3}(-\cos 20^\circ)} \\
 &= \frac{\sqrt{3} \cos 20^\circ - \sin 20^\circ}{\sqrt{3} \sin 20^\circ \cos 20^\circ} = \frac{\frac{\sqrt{3}}{2} \cos 20^\circ - \frac{1}{2} \sin 20^\circ}{\frac{\sqrt{3}}{4} \sin 40^\circ}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ}{\frac{\sqrt{3}}{4} \sin 40^\circ} \\
 &= \frac{\sin 40^\circ}{\frac{\sqrt{3}}{4} \sin 40^\circ} = \frac{4}{\sqrt{3}}.
 \end{aligned}$$

IV.  $\sin \pi/8 \sin 5\pi/18 \sin 7\pi/18$

$$\begin{aligned}
 &= \sin 10^\circ \sin 50^\circ \sin 70^\circ \\
 &= \frac{1}{2} [\cos 40^\circ - \cos 60^\circ] \sin 70^\circ \\
 &= \frac{1}{2} \cos 40^\circ \sin 70^\circ - \frac{1}{4} \sin 70^\circ \\
 &= \frac{1}{4} [\sin 110^\circ + \sin 30^\circ] - \frac{1}{4} \sin 70^\circ \\
 &= \frac{1}{4} \sin(180^\circ - 70^\circ) + \frac{1}{4} \times \frac{1}{2} - \frac{1}{4} \sin 70^\circ = \frac{1}{8}.
 \end{aligned}$$

## Previous Year's Questions

48. We know that:

The sum of angles  $A + B + C = \pi$

$$\Rightarrow A + C = \pi - B$$

$$\Rightarrow \frac{A - B + C}{2} = \frac{\pi - B}{2}$$

$$\therefore 2ca \sin\left(\frac{A - B + C}{2}\right) = 2ca \sin\left(\frac{\pi - B}{2}\right)$$

$$= 2ac \cos B = 2ac \left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

$$= a^2 + c^2 - b^2$$

49.  $\because \sin \theta < 1$ ,

$$\sin^2 \theta < 1$$

$$\Rightarrow \frac{4xy}{(x+y)^2} \leq 1$$

$$\Rightarrow 0 \leq (x+y)^2 - 4xy$$

$$\Rightarrow x^2 + y^2 + 2xy - 4xy \geq 0$$

$$\Rightarrow (x-y)^2 \geq 0$$

which is true for all real  $x$  and  $y$  provided  $x+y \neq 0$ , otherwise

$\frac{4xy}{(x+y)^2}$  will be meaningless.



### NOTE

$\sec^2 \theta = \frac{4xy}{(x+y)^2}$  is possible only, if  $x = y$ .

50. Key Idea: Cosine in terms of tangent

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$\frac{1 - \tan^2 15^\circ}{1 + \tan^2 15^\circ} = \cos 30^\circ$$

$$\therefore = \frac{\sqrt{3}}{2}$$

51. Key Idea:  $\tan \theta$  is negative in second and fourth quadrants.

$$\therefore \tan \theta = -\frac{4}{3}$$

Here  $p = 4$  and  $b = 3$

$$\therefore h = \sqrt{p^2 + b^2} = \sqrt{16 + 9} = \sqrt{25}$$

$$\Rightarrow h = 5$$

$$\therefore \sin \theta = \frac{p}{h} = \frac{4}{5}$$

But  $\tan \theta$  is negative which is possible only if  $\theta$  lies in second or fourth quadrant.

$$\therefore \sin \theta \text{ may be } \frac{4}{5} \text{ or } -\frac{4}{5}.$$

52.  $\therefore \sin(\alpha + \beta) = 1$

$$\Rightarrow \sin(\alpha + \beta) = \sin \frac{\pi}{2}$$

$$\Rightarrow \alpha + \beta = \frac{\pi}{2}$$

and  $\sin(\alpha - \beta) = \frac{1}{2}$

$$\Rightarrow \alpha - \beta = \frac{\pi}{6}$$

Solving eqs. (i) and (ii), we find

$$\therefore \tan(\alpha + 2\beta) \tan(2\alpha + \beta)$$

$$= \tan\left(\frac{2\pi}{3}\right) \tan\left(\frac{5\pi}{6}\right)$$

$$= \left(-\cot \frac{\pi}{6}\right) \left(-\cot \frac{\pi}{3}\right)$$

$$= \sqrt{3} \times \frac{2}{\sqrt{3}} = 1$$

53.  $\therefore y = \sin^2 \theta + \cos^2 \theta$

$$= (\sin \theta - \cos \theta)^2 + 2$$

$$\Rightarrow y \geq 2$$

But at  $\theta = 0$ ,  $y$  becomes meaningless

$$\therefore y \geq 2$$

54. Using  $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

We write for  $a = 4$ ,  $b = 3$  and  $\angle A = 60^\circ$

$$\therefore \cos 60^\circ = \frac{c^2 + 9 - 16}{2 \times 3 \times c}$$

$$\Rightarrow \frac{1}{2} = \frac{c^2 - 7}{2 \times 3c}$$

$$\Rightarrow c^2 - 7 = 3c$$

$$\Rightarrow c^2 - 3c - 7 = 0$$

Thus,  $c$  is the root of above equation.

55. Key Idea:  $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$$\text{Now } \tan \frac{A}{2} \tan \frac{C}{2} = \frac{5}{6} \times \frac{2}{5}$$

$$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \cdot \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{1}{3}$$

$$\Rightarrow \frac{s-b}{s} = \frac{1}{3}$$

$$\Rightarrow 3s - 3b = s$$

$$\Rightarrow 2s = 3b$$

$$\Rightarrow a + b + c = 3b$$

$$\Rightarrow a + c = 2b$$

$a, b, c$  are in AP.

(i) 56. Using  $-\sqrt{a^2 + b^2} \leq a \sin x + b \cos x \leq \sqrt{a^2 + b^2}$ .

Given  $|c| > \sqrt{a^2 + b^2}$  which implies

$$c = a \sin x + b \cos x > \sqrt{a^2 + b^2}, \text{ or}$$

$$c = a \sin x + b \cos x < -\sqrt{a^2 + b^2}$$

(ii) which are reverse inequalities. Then no solution exists for  $a \sin x + b \cos x = c$ .

57.  $\therefore \theta = \alpha$  satisfies the equation  $25 \cos^2 \theta + 5 \cos \theta - 12 = 0$

$$\therefore 25 \cos^2 \alpha + 5 \cos \alpha - 12 = 0$$

$$\Rightarrow 25 \cos^2 \alpha + 20 \cos \alpha - 15 \cos \alpha - 12 = 0$$

$$\Rightarrow 5 \cos \alpha (5 \cos \alpha + 4) - 3(5 \cos \alpha + 4) = 0$$

$$\Rightarrow \cos \alpha = -\frac{4}{3}, \frac{3}{5}$$

But  $\frac{\pi}{2} < \alpha < \pi$

$$\therefore \cos \alpha = -\frac{4}{5} (\because \cos \alpha < 0)$$

$$\Rightarrow \sin \alpha = \frac{3}{5}$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha = -2 \times \frac{3}{5} \times \frac{4}{5}$$

$$\therefore = -\frac{24}{25}$$

58.  $a \left( \frac{1 + \cos C}{2} \right) + c \left( \frac{1 + \cos A}{2} \right) = \frac{3b}{2}$

$$\Rightarrow a + c + b = 3b$$

$$\Rightarrow a + c = 2b$$

Hence, (A) is the correct answer

59. Given that  $\sin \alpha + \sin \beta = \frac{-21}{65}$  and  $\cos \alpha + \cos \beta = \frac{-27}{65}$ .

Squaring and adding, we get

$$2 + 2 \cos(\alpha - \beta) = \frac{1170}{(65)^2}$$

$$\Rightarrow \cos^2 \left( \frac{\alpha - \beta}{2} \right) = \frac{9}{130} \Rightarrow \cos \left( \frac{\alpha - \beta}{2} \right)$$

$$= \frac{-3}{\sqrt{130}} \left( \because \frac{\pi}{2} < \frac{\alpha - \beta}{2} < \frac{3\pi}{2} \right).$$

60. We have  $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$

$$= \sqrt{\frac{a^2+b^2}{2} + \frac{a^2-b^2}{2} \cos 2\theta} + \sqrt{\frac{a^2+b^2}{2} + \frac{b^2-a^2}{2} \cos 2\theta}$$

$$\Rightarrow u^2 = a^2 + b^2 + 2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2 - \left(\frac{a^2-b^2}{2}\right)^2} \cos^2 2\theta$$

Now, min value of  $u^2 = a^2 + b^2 + 2ab$

And the max. value of  $u^2 = 2(a^2 + b^2)$

$$\Rightarrow u_{\max}^2 - u_{\min}^2 = (a-b)^2.$$

61. Greatest side is  $\sqrt{1 + \sin \alpha \cos \alpha}$ , by applying cosine rule we get the greatest angle  $= 120^\circ$ .

62.  $\tan\left(\frac{P}{2}\right), \tan\left(\frac{Q}{2}\right)$  are the roots of  $ax^2 + bx + c = 0$

$$\tan\left(\frac{P}{2}\right) + \tan\left(\frac{Q}{2}\right) = -\frac{b}{a}$$

$$\tan\left(\frac{P}{2}\right) \tan\left(\frac{Q}{2}\right) = \frac{c}{a}$$

$$\frac{\tan\left(\frac{P}{2}\right) + \tan\left(\frac{Q}{2}\right)}{1 - \tan\left(\frac{P}{2}\right) \tan\left(\frac{Q}{2}\right)} = \tan\left(\frac{P}{2} + \frac{Q}{2}\right) = \tan\left(\frac{\pi - R}{2}\right) = 1$$

$$\Rightarrow \frac{-\frac{b}{a}}{1 - \frac{c}{b}} = 1 \Rightarrow -\frac{b}{a} = \frac{a-c}{a} \Rightarrow -b = a - c$$

$$\Rightarrow c = a + b$$

63. We have

$$2r + 2R = c + \frac{2ab}{(a+b+c)} = \frac{(a+b)^2 + c(a+b)}{(a+b+c)} = a + b \quad (\text{since } c^2 = a^2 + b^2)$$

64.  $x^2 + px + q = 0$

Sum and product of the roots respectively are

$$\tan 30^\circ + \tan 15^\circ = -p$$

$$\tan 30^\circ \cdot \tan 15^\circ = q$$

$$\therefore \tan 45^\circ = \frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ} = \frac{-p}{1-q} = 1$$

$$\Rightarrow -p = 1 - q$$

$$\Rightarrow q - p = 1 \therefore 2 + q - p = 3.$$

65.  $2 \sin^2 x + 5 \sin x - 3 = 0$

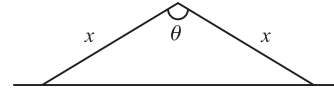
$$\Rightarrow (\sin x + 3)(2 \sin x - 1) = 0$$

$$\Rightarrow \sin x = \frac{1}{2}$$

Now, in the interval  $[0, 3\pi]$   $x$  has 4 values

66.  $\text{Area} = \frac{1}{2} x^2 \sin \theta$

$$\text{Now, } A_{\max} = \frac{1}{2} x^2 \left( \text{at } \sin \theta = 1, \theta = \frac{\pi}{2} \right)$$



67.

$$\cos x + \sin x = \frac{1}{2} \Rightarrow 1 + \sin 2x = \frac{1}{4} \Rightarrow \sin 2x = -\frac{3}{4},$$

so  $x$  is obtuse.

$$\text{Now, } \frac{2 \tan x}{1 + \tan^2 x} = -\frac{3}{4} \Rightarrow 3 \tan^2 x + 8 \tan x + 3 = 0$$

$$\therefore \tan x = \frac{-8 \pm \sqrt{64 - 36}}{6} = \frac{-4 \pm \sqrt{7}}{3}$$

$$\therefore \tan < 0 \therefore \tan x = \frac{-4 - \sqrt{7}}{3}$$

68. Given that  $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$  which is equivalent to

$$2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + 3 = 0$$

$$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + \sin^2 \alpha + \cos^2 \alpha + \sin^2 \beta + \cos^2 \beta + \sin^2 \gamma + \cos^2 \gamma = 0$$

$$\Rightarrow (\sin \alpha + \sin \beta + \sin \gamma)^2 + (\cos \alpha + \cos \beta + \cos \gamma)^2 = 0$$

69.  $\cos(\alpha + \beta) = \frac{4}{5}$

$$\Rightarrow \tan(\alpha + \beta) = \frac{3}{4}$$

$$\sin(\alpha - \beta) = \frac{5}{13}$$

$$\Rightarrow \tan(\alpha - \beta) = \frac{5}{12}$$

70.  $r = \frac{a}{2} \cot \frac{\pi}{n}$

' $a$ ' is side of polygon.

$$R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$$

$$\frac{r}{R} = \frac{\cot \frac{\pi}{n}}{\operatorname{cosec} \frac{\pi}{n}} = \cos \frac{\pi}{n}$$

$$\cos \frac{\pi}{n} \neq \frac{2}{3} \text{ for any } n \in \mathbb{N}.$$

71.  $A = \sin^2 x + \cos^4 x = \frac{7 + \cos 4x}{8} \Rightarrow \frac{3}{4} \leq A \leq 1$

72.  $3 \sin P + 4 \cos Q = 6$  (1)

$4 \sin Q + 3 \cos P = 1$  (2)

From (1) and (2)  $P$  is obtuse.

$$(3 \sin P + 4 \cos Q)^2 + (4 \sin Q + 3 \cos P)^2 = 37$$

$$\Rightarrow 9 + 16 + 24 (\sin P \cos Q + \cos P \sin Q) = 37$$

$$\Rightarrow 24 \sin(P + Q) = 12$$

$$\Rightarrow \sin(P + Q) = \frac{1}{2}$$

$$\Rightarrow P + Q = \frac{5\pi}{6}$$

$$\Rightarrow R = \frac{\pi}{6}$$

73. Applying sine rule in triangle  $ABD$ ,

$$\frac{AB}{\sin \theta} = \frac{BD}{\sin(\theta + \alpha)}$$

$$\Rightarrow AB = \frac{\sqrt{p^2 + q^2} \sin \theta}{\sin \theta \cos \alpha + \cos \theta \sin \alpha} = \frac{\sqrt{p^2 + q^2} \sin \theta}{\frac{\sin \theta \cdot q}{\sqrt{p^2 + q^2}} + \frac{\cos \theta \cdot p}{\sqrt{p^2 + q^2}}}$$

$$\Rightarrow AB = \frac{(p^2 + q^2) \sin \theta}{(p \cos \theta + q \sin \theta)}$$

$$\begin{aligned} 74. \quad & \frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} \\ &= \frac{1}{\cot A(1 - \cot A)} - \frac{\cot^2 A}{(1 - \cot A)} = \frac{1 - \cot^3 A}{\cot A(1 - \cot A)} \\ &= \frac{\operatorname{cosec}^2 A + \cot A}{\cot A} = 1 + \sec A \operatorname{cosec} A \end{aligned}$$

$$\begin{aligned} 75. \quad & \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\sin^6 x + \cos^6 x) \\ &= \frac{3(\sin^4 x + \cos^4 x) - 2(\sin^6 x + \cos^6 x)}{12} \\ &= \frac{3(1 - 2\sin^2 x + \cos^2 x) - 2(1 - 3\sin^2 x + \cos^2 x)}{12} \\ &= \frac{1}{12}. \end{aligned}$$