

Trigonometric Equations

Chapter Highlights

Trigonometric Equation, Solution or Root of a Trigonometric Equation, Method for Finding Principal Value (Solution), Solution of an Equation of the Form, Working Rule to Solve an Equation of the Form $A\cos\theta + B\sin\theta = C$, Solutions of Basic Trigonometric Inequalities.

TRIGONOMETRIC EQUATION

An equation involving one or more trigonometric ratios of unknown angles is called a trigonometric equation.

For example, $2\cos\theta + 3\cos 2\theta = 0$, $\cos^2\theta + \sin\theta = \frac{1}{3}$, etc. are trigonometric equations in unknown angle θ .

CAUTION

A trigonometric equation is different from a trigonometrical identity. An identity is satisfied for every value of the unknown angle e.g., $\cos^2x = 1 - \sin^2x$ is true $\forall x \in R$, while a trigonometric equation is satisfied for some particular values of the unknown angle.

SOLUTION OR ROOT OF A TRIGONOMETRIC EQUATION

A value of the unknown angle which satisfies the given equation is called a solution of the equation. For example, consider the equation $\sqrt{2}\sin\theta = 1$. Clearly, $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ satisfy this equation. Therefore, $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are solutions of the equation $\sqrt{2}\sin\theta = 1$.

Since all trigonometrical ratios are periodic in nature, generally a trigonometric equation has more than one solution or an infinite number of solutions. There are basically three types of solutions.

- (i) **Particular solution:** A specific value of unknown angle satisfying the equation.
- (ii) **Principal solution:** Smallest numerical value of the unknown angle satisfying equation (Numerically smallest particular solution).
- (iii) **General solution:** Complete set of values of the unknown angle satisfying the equation. It contains all particular solutions as well as principal solutions.

METHOD FOR FINDING PRINCIPAL VALUE (SOLUTION)

- (i) First draw a trigonometrical circle and mark the quadrant in which the angle may lie.
- (ii) Select anticlockwise direction for 1st and 2nd quadrants and clockwise direction for 3rd and 4th quadrants.
- (iii) Find the angle in the first rotation.
- (iv) Select the numerically least angle. The angle thus found will be principle value.
- (v) In case two angles one with positive sign and the other with negative sign qualify for the numerically least angle, then it is the convention to select the angle with positive sign as principal value.

For example, if $\sin\theta = -\frac{1}{2}$, θ lies in 3rd or 4th quadrant. For 3rd and 4th quadrant, we select clockwise direction. So we get two values $-\frac{\pi}{6}$ and $-\frac{5\pi}{6}$. Between these two, $-\frac{\pi}{6}$ has the least numerical value. Hence principal value is $-\frac{\pi}{6}$.

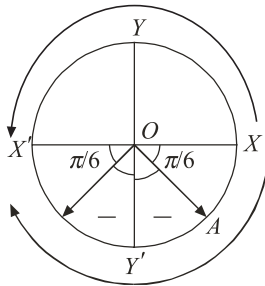


Fig. 26.1

General Solutions of Trigonometric Equations

Trigonometric Equation	General Solution
$\sin\theta = 0$	$\theta = n\pi$
$\cos\theta = 0$	$\theta = n\pi + \pi/2$
$\tan\theta = 0$	$\theta = n\pi$
$\sin\theta = 1$	$\theta = 2n\pi + \pi/2$
$\cos\theta = 1$	$\theta = 2n\pi$
$\tan\theta = 1$	$\theta = n\pi + \pi/4$
$\sin\theta = -1$	$\theta = 2n\pi - \pi/2$
$\cos\theta = -1$	$\theta = (2n + 1)\pi$
$\tan\theta = -1$	$\theta = n\pi - \pi/4$
$\sin\theta = k, -1 \leq k \leq 1$	$\theta = n\pi + (-1)^n \sin^{-1}k$
$\cos\theta = k, -1 \leq k \leq 1$	$\theta = 2n\pi \pm \cos^{-1}k$
$\tan\theta = k, -\infty < k < \infty$	$\theta = n\pi + \tan^{-1}k$
$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n \alpha$
$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha$
$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha$
$\sin^2\theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha$
$\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha$
$\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha$

SOLVED EXAMPLES

1. If $\sin\theta = k$ for exactly one value of θ , $\theta \in \left[0, \frac{7\pi}{3}\right]$, then the value of k is

- (A) 1 (B) -1
(C) $1/\sqrt{2}$ (D) 0

Solution (A, B)

Clearly $-1 \leq k \leq 1$. For any value of k other than 1 and -1, $\sin\theta$ has two values (in quadrant, I II or III, IV).

$\therefore k = 1$ or -1

2. Number of ordered pairs (a, x) satisfying the equation $\sec^2(a + 2)x + a^2 - 1 = 0; -\pi < x < \pi$ is
- (A) 2 (B) 1
(C) 3 (D) infinite

Solution (C)

Given equation is

$$\begin{aligned} \sec^2(a + 2)x + a^2 - 1 &= 0 \\ \Rightarrow \tan^2(a + 2)x + a^2 &= 0 \\ \Rightarrow \tan^2(a + 2)x &= 0 \text{ and } a = 0 \end{aligned}$$

$$\Rightarrow \tan^2 2x = 0 \Rightarrow x = 0, \frac{\pi}{2}, -\frac{\pi}{2}$$

$\therefore (0, 0), (0, \pi/2), (0, -\pi/2)$ are ordered pairs satisfying the equation.

3. If $\frac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} - \frac{\cos\theta}{\sqrt{1 + \cot^2\theta}} - 2\tan\theta \cot\theta = -1$,

$0 \in [0, 2\pi]$, then

- (A) $\theta \in \left(0, \frac{\pi}{2}\right) - \left\{\frac{\pi}{4}\right\}$
(B) $\theta \in \left(\frac{\pi}{2}, \pi\right) - \left\{\frac{3\pi}{4}\right\}$
(C) $\theta \in \left(\pi, \frac{3\pi}{2}\right) - \left\{\frac{5\pi}{4}\right\}$
(D) $\theta \in (0, \pi) - \left\{\frac{\pi}{4}, \frac{\pi}{2}\right\}$

Solution (D)

Since $\sin\theta - \cos\theta \neq 0$

$$\therefore \tan\theta \neq 1 \quad \therefore \theta \neq \frac{\pi}{4}, \frac{5\pi}{4}$$

Now $\sin^2\theta + \cos^2\theta + \sin\theta \cos\theta - |\sin\theta| \cos\theta - 2\tan\theta \cot\theta = -1$

$$\Rightarrow 1 + \cos\theta(\sin\theta - |\sin\theta|) - 2 = -1$$

$$\Rightarrow \cos\theta(\sin\theta - |\sin\theta|) = 0$$

$$\therefore \theta \in (0, \pi) - \left\{\frac{\pi}{4}, \frac{\pi}{2}\right\}$$

4. General solution of the equation

$$(\sqrt{3} - 1)\sin\theta + (\sqrt{3} + 1)\cos\theta = 2 \text{ is}$$

- (A) $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ (B) $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{12}$
(C) $2n\pi \pm \frac{\pi}{4} - \frac{\pi}{12}$ (D) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12}$

Solution (A)

Let $\sqrt{3} + 1 = r \cos\alpha$, and $\sqrt{3} - 1 = r \sin\alpha$

$$\therefore r^2 = (\sqrt{3} + 1)^2 + (\sqrt{3} - 1)^2 = 8,$$

$$\begin{aligned} \text{and } \tan \alpha &= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \tan\left(\frac{\pi}{12}\right), \text{ i.e., } \alpha = \frac{\pi}{12} \end{aligned}$$

From the given equation, we get $r \cos(\theta - \alpha) = 2$

$$\Rightarrow \cos\left(\theta - \frac{\pi}{12}\right) = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$$

$$\begin{aligned} \therefore \theta - \pi/12 &= 2n\pi \pm \pi/4 \\ \text{or } \theta &= 2n\pi \pm \pi/4 + \pi/12 \end{aligned}$$

5. For $m \neq n$, if $\tan m\theta = \tan n\theta$, then the different values of θ are in
- (A) A.P.
 (B) H.P.
 (C) G.P.
 (D) no particular sequence

Solution (A)

We have, $\tan m\theta = \tan n\theta \Rightarrow m\theta = n\theta + k\pi$

$$\therefore \theta = \frac{k\pi}{m-n}, \text{ where } k \in \mathbb{Z}.$$

These values of θ are in **A.P.** with first term $\frac{\pi}{m-n}$ and common difference $\frac{\pi}{m-n}$.

6. The number of the solutions of the equation $\cos(\pi\sqrt{x-4}) \cos(\pi\sqrt{x}) = 1$ is
- (A) >2 (B) 2
 (C) 1 (D) 0

Solution (C)

Clearly, $x \geq 4$ (Since $\sqrt{x-4}$ is real) so that \sqrt{x} is also real.

Again, if $\cos(\pi\sqrt{x}) < 1$ then $\cos(\pi\sqrt{x-4}) > 1$ and if $\cos(\pi\sqrt{x}) > 1$ then $\cos(\pi\sqrt{x-4}) < 1$ (since their product = 1).

But both of these are not possible (since $\cos\theta$ cannot be greater than 1).

$$\therefore \cos(\pi\sqrt{x-4}) = 1 \text{ and } \cos(\pi\sqrt{x}) = 1$$

$$\therefore x - 4 = 0 \text{ and } x = 0$$

But $x = 0$ is not possible, $\therefore x = 4$ is the only solution.

7. The number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all x is
- (A) 0 (B) 1
 (C) 3 (D) infinite

Solution (D)

We have, $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$

$$\Rightarrow a_1 + a_2 \cos 2x + \frac{a_3}{2}(1 - \cos 2x) = 0$$

which is zero for all values of x ,

$$\text{if } a_1 = -\frac{a_3}{2} = -a_2$$

$$\text{or } a_1 = -\frac{k}{2}, a_2 = \frac{k}{2}, a_3 = k$$

for any $k \in \mathbb{R}$.

Hence, the required number of triplets is **infinite**.

8. The least difference between the roots, in the first quadrant $\left(0 \leq x \leq \frac{\pi}{2}\right)$, of the equation $4\cos x (2 - 3\sin^2 x) + (\cos 2x + 1) = 0$, is
- (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$
 (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution (A)

We have,

$$4\cos x (2 - 3\sin^2 x) + (\cos 2x + 1) = 0$$

$$\Rightarrow 4\cos x (3\cos^2 x - 1) + 2\cos^2 x = 0$$

$$\Rightarrow 2\cos x (6\cos^2 x + \cos x - 2) = 0$$

$$\Rightarrow 2\cos x (3\cos x + 2)(2\cos x - 1) = 0$$

$$\Rightarrow \text{either } \cos x = 0 \text{ which gives } x = \pi/2$$

$$\text{or } \cos x = -2/3,$$

which gives no value of x for which $0 \leq x \leq \pi/2$

$$\text{or } \cos x = 1/2, \text{ which gives } x = \pi/3$$

So, the required difference = $\pi/2 - \pi/3 = \pi/6$.

9. Solution of the equation $4\cot 2\theta = \cot^2 \theta - \tan^2 \theta$ is
- (A) $\theta = n\pi \pm \frac{\pi}{2}$ (B) $\theta = n\pi \pm \frac{\pi}{3}$
 (C) $\theta = n\pi \pm \frac{\pi}{4}$ (D) none of these

Solution (C)

We have, $4\cot 2\theta = \cot^2 \theta - \tan^2 \theta$

$$\Rightarrow \frac{4}{\tan 2\theta} = \frac{4}{\tan^2 \theta} - \tan^2 \theta$$

$$\text{Put } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\therefore \frac{4(1 - \tan^2 \theta)}{2 \tan \theta} = \frac{1 - \tan^4 \theta}{\tan^2 \theta}$$

$$\Rightarrow (1 - \tan^2 \theta) [2 \tan \theta - (1 + \tan^2 \theta)] = 0$$

$$\Rightarrow (1 - \tan^2 \theta) (\tan^2 \theta - 2 \tan \theta + 1) = 0$$

$$\Rightarrow (1 - \tan^2\theta)(\tan\theta - 1)^2 = 0$$

$$\Rightarrow \tan\theta = 1, -1. \therefore \theta = n\pi \pm \frac{\pi}{4}$$

10. Let α, β be any two positive values of x for which $2\cos x, |\cos x|$ and $1 - 3\cos^2 x$ are in G.P. The minimum value of $|\alpha - \beta|$ is

- (A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$
 (C) $\frac{\pi}{2}$ (D) none of these

Solution (D)

Since

$$2\cos x, |\cos x| \text{ and } 1 - 3\cos^2 x \text{ are in G.P.}$$

$$\therefore \cos^2 x = 2\cos x(1 - 3\cos^2 x)$$

$$\Rightarrow 6\cos^3 x + \cos^2 x - 2\cos x = 0$$

$$\Rightarrow \cos x(2\cos x - 1)(3\cos x + 2) = 0$$

$$\Rightarrow \cos x = 0, \frac{1}{2}, -\frac{2}{3} \therefore x = \frac{\pi}{2}, \frac{\pi}{3}, \cos^{-1}\left(-\frac{2}{3}\right)$$

$$\therefore \text{If } \alpha = \frac{\pi}{2}, \beta = \frac{\pi}{3}, (\because \alpha, \beta \text{ are positive})$$

$$\text{then } |\alpha - \beta| = \frac{\pi}{6}$$

11. The most general values of θ for which $\sin\theta - \cos\theta = a \in R$ ($1, a^2 - 6a + 10$) are given by

- (A) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$ (B) $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{4}$
 (C) $2n\pi + \frac{\pi}{4}$ (D) none of these

Solution (B)

We have,

$$\sin\theta - \cos\theta = \min_{a \in R} \{1, a^2 - 6a + 11\}.$$

$$\text{Since } a^2 - 6a + 11 = (a - 3)^2 + 2 > 2 \text{ for all } a$$

$$\therefore \sin\theta - \cos\theta = 1 \Rightarrow \sin\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}$$

$$\Rightarrow \theta - \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4}$$

$$\therefore \theta = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{4}, \text{ where } n \in Z.$$

12. The least positive non-integral solution of the equation $\sin\pi(x^2 + x) = \sin\pi x^2$ is

- (A) rational
 (B) irrational of the form \sqrt{p}
 (C) irrational of the form $\frac{\sqrt{p-1}}{4}$, where p is an odd integer

- (D) irrational of the form $\frac{\sqrt{p+1}}{4}$, where p is an even integer

Solution (C)

$$\text{We have, } \sin\pi(x^2 + x) = \sin\pi x^2$$

$$\Rightarrow \pi(x^2 + x) = n\pi + (-1)^n \pi x^2$$

$$\therefore \text{Either } x^2 + x = 2m + x^2 \Rightarrow x = 2m \in Z.$$

$$\text{or } x^2 + x = k - x^2, \text{ where } k \text{ is an odd integer}$$

$$\Rightarrow 2x^2 + x - k = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1+8k}}{4}$$

For least positive non-integral solution,

$$x = \frac{-1 + \sqrt{1+8k}}{4} = \frac{\sqrt{p-1}}{4}, \text{ where } p \text{ is an odd integer.}$$

13. In the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the equation $\log_{\sin\theta}(\cos 2\theta) = 2$ has

- (A) no solution
 (B) a unique solution
 (C) two solutions
 (D) infinitely many solutions

Solution (B)

$$\text{We have, } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\therefore -1 \leq \sin\theta \leq 1, \text{ here } 0 < \sin\theta < 1$$

$$\text{Now, } \log_{\sin\theta} \cos 2\theta = 2$$

$$\Rightarrow \cos 2\theta = \sin^2 \theta$$

$$\Rightarrow 1 - 2\sin^2 \theta = \sin^2 \theta$$

$$\Rightarrow 3\sin^2 \theta = 1 \Rightarrow \sin^2 \theta = \frac{1}{3}$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{3}} \quad (\because 0 < \sin\theta < 1)$$

The given equation has a **unique solution**.

14. The number of solutions of the equation $\sin x = \cos 3x$ in $[0, \pi]$ is

- (A) 1 (B) 2
 (C) 3 (D) 4

Solution (C)

The given equation can be written as

$$\sin x = 4\cos^3 x - 3\cos x$$

$$\text{i.e., } \sec^2 x \tan x + 3\sec^2 x - 4 = 0$$

In terms of $\tan x$, this leads to the equation

$$\tan^3 x + 3\tan^2 x + \tan x - 1 = 0$$

$$\Rightarrow (\tan x + 1)(\tan^2 x + 2\tan x - 1) = 0$$

$$\Rightarrow \tan x = -1 \text{ or } \tan 2x = 1$$

$$\text{i.e., } x = \frac{3\pi}{4}, \frac{\pi}{8}, \frac{5\pi}{8}$$

15. The number of values of x in $[0, 2\pi]$ satisfying the equation $|\cos x - \sin x| \geq \sqrt{2}$ is,

- (A) 0 (B) 1
(C) 2 (D) 3

Solution (C)

Given equation is $|\cos x - \sin x| \geq \sqrt{2}$

Since $|\cos x - \sin x| \leq \sqrt{1+1} = \sqrt{2}$

\therefore We must have $|\cos x - \sin x| = \sqrt{2}$

$$\Rightarrow \left| \cos\left(x + \frac{\pi}{4}\right) \right| = 1 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) = 1, -1$$

$$\therefore x + \frac{\pi}{4} = 0, 2\pi, 4\pi, 6\pi, \dots \text{ or } \pi, 3\pi, 5\pi, \dots$$

$$\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

16. If $|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$, then possible values of x are:

(A) $n\pi$ or $n\pi + (-1)^n \frac{\pi}{6}, n \in I$

(B) $n\pi$ or $2n\pi + \frac{\pi}{2}$ or $n\pi + (-1)^n \frac{\pi}{6}, n \in I$

(C) $n\pi + (-1)^n \frac{\pi}{6}, n \in I$

(D) $n\pi, n \in I$

Solution (C, D)

The equation holds if $|\cos x| = 1$

i.e., if $x = n\pi, n \in I$

If $|\cos x| \neq 1$, then $\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2} = 0$

$$\Rightarrow \sin x = 1 \text{ or } \frac{1}{2}$$

$\sin x \neq 1$, as in that case $\cos x = 0$

$$\therefore \sin x = \frac{1}{2} \Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}$$

17. A set of values of x satisfying the equation

$$\cos^2\left(\frac{1}{2}px\right) + \cos^2\left(\frac{1}{2}qx\right) = 1$$

form an arithmetic progression with common difference

- (A) $\frac{2}{p+q}$ (B) $\frac{2}{p-q}$
(C) $\frac{\pi}{p+q}$ (D) none of these

Solution (D)

The given equation can be written as

$$1 + \cos px + 1 + \cos qx = 2$$

$$\Rightarrow \cos\left(\frac{p+q}{2}\right)x \cos\left(\frac{p-q}{2}\right)x = 0$$

$$\cos\left(\frac{p+q}{2}\right)x = 0 \Rightarrow x = \frac{(2n+1)\pi}{p+q}, n = 0, \pm 1, \pm 2, \dots$$

which forms an A.P. with common difference = $\frac{2\pi}{p+q}$

$$\cos\left(\frac{p-q}{2}\right)x = 0 \Rightarrow x = \frac{(2n+1)\pi}{p-q}, n = 0, \pm 1, \pm 2, \dots$$

which forms an A.P. with common difference = $\frac{2\pi}{p-q}$


CAUTION

In each of the above cases, $n \in Z$, where Z is the set of integers

SOLUTION OF AN EQUATION OF THE FORM

$$a \cos \theta + b \sin \theta = c$$

The given equation is $a \cos \theta + b \sin \theta = c$. (1)

Divide throughout by $\sqrt{a^2 + b^2}$

i.e., by $\sqrt{(\text{coeff. of } \cos \theta)^2 + (\text{coeff. of } \sin \theta)^2}$, we get

$$\frac{a}{\sqrt{a^2 + b^2}} \cos \theta + \frac{b}{\sqrt{a^2 + b^2}} \sin \theta = \frac{c}{\sqrt{a^2 + b^2}} \quad (2)$$

Let α be the least +ve angle such that

$$\frac{a}{\sqrt{a^2 + b^2}} = \cos \alpha \text{ and } \frac{b}{\sqrt{a^2 + b^2}} = \sin \alpha$$

\therefore (2) becomes

$$\cos \alpha \cos \theta + \sin \alpha \sin \theta = \frac{c}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \cos(\theta - \alpha) = \cos \beta \text{ (say),}$$

$$\text{where } \cos \beta = \frac{c}{\sqrt{a^2 + b^2}}$$

$$\therefore \theta - \alpha = 2n\pi \pm \beta$$

$$\Rightarrow \theta = 2n\pi \pm \beta + \alpha, \text{ where } n \in Z.$$

WORKING RULE TO SOLVE AN EQUATION OF THE FORM $a \cos \theta + b \sin \theta = c$

1. Divide throughout by $\sqrt{a^2 + b^2}$
i.e., by $\sqrt{(\text{coeff. of } \cos \theta)^2 + (\text{coeff. of } \sin \theta)^2}$
2. Write L.H.S. as a single cosine.
3. Use the formula

$$\theta = 2n\pi \pm \alpha, n \in Z \text{ for } \cos \theta = \cos \alpha$$

**CAUTION**

Check that $|c| \leq \sqrt{a^2 + b^2}$. If it is not satisfied, no real solution exists.

TRICK(S) FOR PROBLEM SOLVING

- Squaring should be avoided as far as possible. If squaring is done, then check for extra solutions.

For example, consider the equation $\sin\theta + \cos\theta = 1$.

On squaring, we get

$$1 + \sin 2\theta = 1 \text{ or } \sin 2\theta = 0$$

$$\Rightarrow \theta = \frac{n\pi}{2}, n = 0, \pm 1, \pm 2, \dots$$

The values of the angle, $\theta = \pi$ and $\theta = \frac{3\pi}{2}$ do not satisfy the given equation. So, we get extra solutions. Thus, if squaring is must, verify each of the solutions.

- Never cancel a common factor containing ' θ ' from the two sides of an equation.

For example, consider the equation $\tan\theta = \sqrt{2} \sin\theta$. If we divide both sides by $\sin\theta$, we get $\cos\theta = \frac{1}{\sqrt{2}}$, which

is clearly not equivalent to the given equation as the solutions obtained by $\sin x = 0$ are lost. Thus, **instead of**

dividing an equation by a common factor, take this factor out as a common factor from all terms of the equation.

- Make sure that the answer should not contain any value of unknown ' θ ' which makes any of the terms undefined.
- If $\tan\theta$ or $\sec\theta$ is involved in the equation, θ should not be an odd multiple of $\pi/2$.
- If $\cot\theta$ or $\operatorname{cosec}\theta$ is involved in the equation, θ should not be a multiple of π or 0.
- The value of $\sqrt{f(\theta)}$ is always positive. For example,

$$\sqrt{\cos^2 \theta} = |\cos \theta| \text{ and not } \pm \cos \theta.$$

- All the solutions should satisfy the given equation and lie in the domain of the variable of the given equation.
- Check the validity of the given equation, e.g., $2\cos\theta + \sin\theta = 3$ can never be true for any θ as the value $(2\cos\theta + \sin\theta)$ can never exceed $\sqrt{2^2 + 1^2} = \sqrt{5}$. So there is no solution for this equation.

**CAUTION**

If the argument in inequalities is not x but $x + a$ or ax or $\frac{x}{a}$ then always write general solution first in terms of the argument and then find x .

SOLUTIONS OF BASIC TRIGONOMETRIC INEQUALITIES

Inequality	Solution ($n \in \mathbb{Z}$)
$\sin x > \alpha$, where $ \alpha < 1$	$\sin^{-1}\alpha + 2n\pi < x < \pi - \sin^{-1}\alpha + 2n\pi$
$\sin x < \alpha$, where $ \alpha < 1$	$-\pi - \sin^{-1}\alpha + 2n\pi < x < \sin^{-1}\alpha + 2n\pi$
$\cos x > \alpha$, where $ \alpha < 1$	$-\cos^{-1}\alpha + 2n\pi < x < \cos^{-1}\alpha + 2n\pi$
$\cos x < \alpha$, where $ \alpha < 1$	$\cos^{-1}\alpha + 2n\pi < x < 2\pi - \cos^{-1}\alpha + 2n\pi$
$\tan x > \alpha$, where $-\infty < \alpha < \infty$	$\tan^{-1}\alpha + n\pi < x < \frac{\pi}{2} + n\pi$
$\tan x < \alpha$, where $-\infty < \alpha < \infty$	$-\frac{\pi}{2} + n\pi < x < \tan^{-1}\alpha + n\pi$
$\cot x > \alpha$, where $-\infty < \alpha < \infty$	$n\pi < x < \cot^{-1}\alpha + n\pi$
$\cot x < \alpha$, where $-\infty < \alpha < \infty$	$\cot^{-1}\alpha + n\pi < x < \pi + n\pi$

SOLVED EXAMPLES

18. If $0 \leq x \leq 2\pi$ and $|\cos x| \leq \sin x$, then

- (A) $x \in \left[0, \frac{\pi}{4}\right]$ (B) $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
- (C) $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$ (D) none of these

Solution (C)

We have, $|\cos x| \leq \sin x \Rightarrow \sin x \geq 0$

($\because |\cos x| \geq 0$)

$$\Rightarrow x \notin (\pi, 2\pi)$$

If $x = 2\pi$, $|\cos 2\pi| \leq \sin 2\pi$ which is not possible,

$\therefore x \in [0, \pi]$

If $x \in \left[0, \frac{\pi}{2}\right]$, then $|\cos x| \leq \sin x \Rightarrow \cos x \leq \sin x$

$$\Rightarrow x \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right]$$

If $x \in \left(\frac{\pi}{2}, \pi \right)$, then

$$|\cos x| \leq \sin x \Rightarrow -\cos x \leq \sin x \Rightarrow \tan x \leq -1$$

($\because \cos x < 0$)

$$\Rightarrow x \in \left(\frac{\pi}{2}, \frac{3\pi}{4} \right]. \therefore x \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right] \cup \left(\frac{\pi}{2}, \frac{3\pi}{4} \right]$$

19. The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$, in the variable x has a real root. then p can take any value in the interval

- (A) $(0, \pi)$ (B) $\left(\frac{-\pi}{2}, \frac{\pi}{2} \right)$
 (C) $(-\pi, 0)$ (D) $(0, 2\pi)$

Solution (A)

$$\text{Discriminant} = \cos^2 p - 4(\cos p - 1)\sin p$$

For equation to have real roots,

$$\cos^2 p - 4(\cos p - 1)\sin p \geq 0$$

$$\Rightarrow \cos^2 p \geq 4(\cos p - 1)\sin p$$

$$\Rightarrow \cos^2 p \geq -8 \sin^2 \frac{p}{2}$$

For $p = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$, R.H.S. > 1 but L.H.S. < 1 ,

\therefore the choices (B), (C) and (D) are ruled out. The correct alternative is (A)

20. The equation $\sin^4 x + \cos^4 x = a$ has a solution for

- (A) all of values of a (B) $a = 1$
 (C) $a = \frac{1}{2}$ (D) $\frac{1}{2} < a < 1$

Solution (B, C, D)

The given equation can be written as

$$\Rightarrow 1 - 2 \sin^2 x \cos^2 x = a$$

$$\Rightarrow \sin^2 2x = 2(1 - a) \Rightarrow 2(1 - a) \leq 1$$

$$\text{and } 2(1 - a) \geq 0 \Rightarrow 1/2 \leq a \leq 1$$

21. The equation $\sin^4 x - (k + 2)\sin^2 x - (k + 3) = 0$ possesses a solution if

- (A) $k > -3$
 (B) $k < -2$
 (C) $-3 \leq k \leq -2$
 (D) k is any positive integer

Solution (C)

$$\text{We have, } \sin^4 x - (k + 2)\sin^2 x - (k + 3) = 0$$

$$\Rightarrow \sin^2 x = \frac{(k + 2) \pm \sqrt{(k + 2)^2 + 4(k + 3)}}{2}$$

$$= \frac{(k + 2) \pm (k + 4)}{2}$$

$$\Rightarrow \sin^2 x = k + 3 \quad (\because \sin^2 x = -1 \text{ is not possible})$$

Since $0 \leq \sin^2 x \leq 1$, $\therefore 0 \leq k + 3 \leq 1$ or $-3 \leq k \leq -2$.

22. The equation $\cos 2x + a \sin x = 2a - 7$ has a solution for
 (A) all a (B) $a > 6$
 (C) $a < 2$ (D) $a \in [2, 6]$

Solution (D)

$$\text{Given equation is } \cos 2x + a \sin x = 2a - 7$$

$$\Rightarrow 1 - 2\sin^2 x + a \sin x = 2a - 7$$

$$\Rightarrow 2\sin^2 x - a \sin x + 2a - 8 = 0$$

$$\Rightarrow \sin x = \frac{a \pm \sqrt{a^2 - 16(a - 4)}}{4} = \frac{a \pm (a - 8)}{4}$$

$$\Rightarrow \sin x = \frac{a - 4}{2} \text{ or } \sin x = 2 \text{ (not possible)}$$

$$\therefore \sin x = \frac{a - 4}{2}$$

$$\because -1 \leq \sin x \leq 1, \therefore -1 \leq \frac{a - 4}{2} \leq 1 \Rightarrow 2 \leq a \leq 6$$

23. The set of all x in $(-\pi, \pi)$ satisfying $|4 \sin x - 1| < \sqrt{5}$ is given by

- (A) $\left(-\frac{\pi}{10}, \frac{3\pi}{10} \right)$ (B) $\left(-\frac{\pi}{10}, \pi \right)$
 (C) $(-\pi, \pi)$ (D) $\left(-\pi, \frac{3\pi}{10} \right)$

Solution (A)

$$\text{We have, } |4 \sin x - 1| < \sqrt{5}$$

$$\Rightarrow -\sqrt{5} < 4 \sin x - 1 < \sqrt{5}$$

$$\Rightarrow -\left(\frac{\sqrt{5} - 1}{4} \right) < \sin x < \frac{\sqrt{5} + 1}{4}$$

$$\Rightarrow -\sin \frac{\pi}{10} < \sin x < \cos \frac{\pi}{5}$$

$$\Rightarrow \sin \left(-\frac{\pi}{10} \right) < \sin x < \sin \left(\frac{\pi}{2} - \frac{\pi}{5} \right)$$

$$\Rightarrow \sin \left(-\frac{\pi}{10} \right) < \sin x < \sin \left(\frac{3\pi}{10} \right)$$

$$\Rightarrow x \in \left(-\frac{\pi}{10}, \frac{3\pi}{10} \right) \quad [\because x \in (-\pi, \pi)]$$

24. The number of values of x in $[0, 2\pi]$ satisfying the equation $|\cos x - \sin x| \geq \sqrt{2}$, is

- (A) 0 (B) 1
 (C) 2 (D) 3

Solution (C)

$$\text{Given equation is } |\cos x - \sin x| \geq \sqrt{2}.$$

Since $|\cos x - \sin x| \leq \sqrt{1+1} = \sqrt{2}$,

\therefore we must have $|\cos x - \sin x| = \sqrt{2}$

$$\Rightarrow \left| \cos\left(x + \frac{\pi}{4}\right) \right| = 1 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) = 1, -1.$$

$$\therefore x + \frac{\pi}{4} = 0, 2\pi, 4\pi, 6\pi, \dots, \pi, 3\pi, 5\pi, \dots$$

$$\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

25. The set of all x in $(-\pi, \pi)$ satisfying $|4\sin x - 1| < \sqrt{5}$ is given by

(A) $\left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$

(B) $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$

(C) $\left(\frac{\pi}{10}, -\frac{3\pi}{10}\right)$

(D) none of these

Solution (A)

Since $|4\sin x - 1| < \sqrt{5}$

$$\Rightarrow -\sqrt{5} < 4\sin x - 1 < \sqrt{5}$$

$$\Rightarrow -\left(\frac{\sqrt{5}-1}{4}\right) < \sin x < \frac{\sqrt{5}+1}{4}$$

$$\Rightarrow -\sin\frac{\pi}{10} < \sin x < \cos\frac{\pi}{5}$$

$$\Rightarrow \sin\left(-\frac{\pi}{10}\right) < \sin x < \sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right)$$

$$\Rightarrow \sin\left(-\frac{\pi}{10}\right) < \sin x < \sin\frac{3\pi}{10}$$

$$\therefore x \in \left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$$

EXERCISES

Single Option Correct Type

- The equation $2\cos^2\left(\frac{x}{2}\right) \cdot \sin^2x = x^2 + \frac{1}{x^2}$, $0 \leq x \leq \frac{\pi}{2}$ has
 - one real solution
 - no solution
 - more than one real solution
 - none of these
- The general solution of the equation $\sin^{50}x - \cos^{50}x = 1$ is
 - $2n\pi + \frac{\pi}{2}$
 - $2n\pi + \frac{\pi}{3}$
 - $n\pi + \frac{\pi}{2}$
 - $n\pi + \frac{\pi}{3}$
- General solution of the equation $(\sqrt{3}-1)\sin\theta + (\sqrt{3}+1)\cos\theta = 2$ is
 - $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$
 - $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{12}$
 - $2n\pi \pm \frac{\pi}{4} - \frac{\pi}{12}$
 - $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12}$
- The number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2 \cos 2x + a_3 \sin^2x = 0$ for all x is
 - 0
 - 1
 - 3
 - infinite
- The equation $\sin^4x - (k+2)\sin^2x - (k+3) = 0$ possesses a solution if
 - $k > -3$
 - $k < -2$
 - $-3 \leq k \leq -2$
 - k is any positive integer
- The least positive non-integral solution of the equation $\sin \pi(x^2 + x) = \sin \pi x^2$ is
 - rational
 - irrational of the form \sqrt{p}
 - irrational of the form $\frac{\sqrt{p}-1}{4}$, where p is an odd integer
 - irrational of the form $\frac{\sqrt{p}+1}{4}$, where p is an even integer
- If $\sin^2x - 2\sin x - 1 = 0$ has exactly four different solutions in $x \in [0, n\pi]$, then minimum value of n can be ($n \in \mathbb{N}$)

- (A) 4 (B) 3
(C) 2 (D) 1
8. A set of values of x satisfying the equation $\cos^2\left(\frac{1}{2}px\right) + \cos^2\left(\frac{1}{2}qx\right) = 1$ form an arithmetic progression with common difference
- (A) $\frac{2}{p+q}$ (B) $\frac{2}{p-q}$
(C) $\frac{\pi}{p+q}$ (D) none of these
9. If $0 \leq x \leq 2\pi$ and $|\cos x| \leq \sin x$, then
- (A) $x \in \left[0, \frac{\pi}{4}\right]$ (B) $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
(C) $x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (D) none of these
10. The general solution of the equation $\frac{1 - \sin x + \dots + (-1)^n \sin^n x + \dots}{1 + \sin x + \dots + \sin^n x + \dots} = \frac{1 - \cos 2x}{1 + \cos 2x}$ $x \neq (2n+1)\frac{\pi}{2}$, $n \in Z$ is
- (A) $(-1)^n \left(\frac{\pi}{3}\right) + n\pi$ (B) $(-1)^n \left(\frac{\pi}{6}\right) + n\pi$
(C) $(-1)^{n+1} \left(\frac{\pi}{6}\right) + n\pi$ (D) $(-1)^{n-1} \left(\frac{\pi}{3}\right) + n\pi$
11. The general solution of the equation $\sum_{r=1}^n \cos r^2 \theta \sin r \theta = \frac{1}{2}$ is
- (A) $\frac{4k-1}{n(n+1)} \frac{\pi}{2}$, $k \in Z$ (B) $\frac{2k+1}{n(n+1)} \frac{\pi}{2}$, $k \in Z$
(C) $\frac{4k+1}{n(n+1)} \frac{\pi}{2}$, $k \in Z$ (D) none of these
12. The solution of $\sin^8 x + \cos^8 x = \frac{17}{32}$ is
- (A) $\frac{n\pi}{2} \pm \frac{\pi}{8}$ (B) $n\pi \pm \frac{\pi}{4}$
(C) $n\pi \pm \frac{\pi}{8}$ (D) no solution
13. The general solution of the equation $2^{\cos^2 \theta} + 1 = 3 \cdot 2^{-\sin^2 \theta}$ is
- (A) $2n\pi \pm \frac{\pi}{2}$, $n\pi$, $n \in Z$
(B) $n\pi \pm \frac{\pi}{2}$, $2n\pi$, $n \in Z$
(C) $n\pi \pm \frac{\pi}{2}$, $n\pi$, $n \in Z$
(D) none of these
14. The solution of the inequality $\log_{1/2} \sin \theta > \log_{1/2} \cos \theta$ in $[0, 2\pi]$ is
- (A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(C) $\left(0, \frac{\pi}{4}\right)$ (D) none of these
15. If $\cos 3x + \sin\left(2x - \frac{7\pi}{6}\right) = -2$ then $x =$
- (A) $\frac{\pi}{3}(6k+1)$ (B) $\frac{\pi}{3}(6k-1)$
(C) $\frac{\pi}{3}(2k+1)$ (D) none of these
where $k \in Z$.
16. If $\tan^2[\pi(x+y)] + \cot^2[\pi(x+y)] = 1 + \sqrt{\frac{2x}{1+x^2}}$, where $x, y \in R$, then least positive value of y is
- (A) $\frac{5}{4}$ (B) $\frac{1}{4}$
(C) $\frac{3}{4}$ (D) 2
17. The general value of y satisfying the equation $1 - 2x - x^2 = \tan^2(x+y) + \cot^2(x+y)$ is
- (A) $2n\pi \pm \frac{\pi}{4}$ (B) $n\pi \pm \frac{\pi}{4}$
(C) $\frac{n\pi}{2} \pm \frac{\pi}{4}$ (D) none of these
18. If $[\sin x] + [\sqrt{2} \cos x] = -3$, $x \in [0, 2\pi]$ ($[\cdot]$ denotes the greatest integer function) then x belongs to
- (A) $\left[\frac{5\pi}{4}, 2\pi\right]$ (B) $\left(\frac{5\pi}{4}, 2\pi\right)$
(C) $\left(\pi, \frac{5\pi}{4}\right)$ (D) $\left[\pi, \frac{5\pi}{4}\right]$
19. The number of solutions of the equation $\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}x + 4$

- (A) forms an empty set
 (B) is only one
 (C) is only two
 (D) is more than 2
20. The number of solutions of the equation $|\cos x| = 2[x]$, where $[\cdot]$ is the greatest integer, is
 (A) one (B) two
 (C) infinite (D) nil
21. The general solution of $\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$ is
 (A) $n\pi + \frac{\pi}{8}$ (B) $\frac{n\pi}{2} + \frac{\pi}{8}$
 (C) $(-1)^n \frac{n\pi}{2} + \frac{\pi}{8}$ (D) $2n\pi + \cos^{-1} \frac{3}{2}$
22. $\sin x + 2 \sin 2x = 3 + \sin 3x$, $0 \leq x \leq 2\pi$ has
 (A) 2 solutions in I quadrant
 (B) one solution in II quadrant
 (C) no solution in any quadrant
 (D) one solution in each quadrant
23. The solution of the equation $1 + \sin^2 ax = \cos x$, where a is irrational, is
 (A) $x = 0$ (B) $x = \frac{n\pi}{a}$
 (C) $x = 2n\pi$ (D) none of these
24. The values of α for which the equation $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ may be valid, are
 (A) $-\frac{3}{2} \leq \alpha \leq 1$ (B) $0 \leq \alpha \leq \frac{1}{2}$
 (C) $-\frac{3}{2} \leq \alpha \leq \frac{1}{2}$ (D) none of these
25. If α and β be two distinct values of θ lying between 0 and 2π , satisfying the equation $3 \cos \theta + 4 \sin \theta = 2$, then the value of $\sin(\alpha + \beta)$ is
 (A) $\frac{12}{25}$ (B) $\frac{24}{25}$
 (C) $\frac{13}{25}$ (D) none of these
26. $|\tan x + \sec x| = |\tan x| + |\sec x|$, $x \in [0, 2\pi]$, if and only if x belongs to the interval
 (A) $(\pi, 2\pi]$
 (B) $[0, \pi]$
 (C) $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
 (D) $\left[\pi, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$
27. $|\cos x| = \cos x - 2 \sin x$ if
 (A) $x = n\pi$
 (B) $x = 2n\pi$ or $(2n+1)\pi + \frac{\pi}{4}$
 (C) $x = n\pi + \frac{\pi}{4}$
 (D) $x = n\pi$ or $n\pi + \frac{\pi}{4}$
28. A solution of the equation $(1 - \tan \theta)(1 + \tan \theta) \sec^2 \theta + 2 \tan^2 \theta = 0$, where θ lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is given by
 (A) $\theta = 0$ (B) $\theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$
 (C) $\theta = \frac{\pi}{6}$ (D) $\theta = -\frac{\pi}{6}$
29. If $\sqrt{p} \cos x - 2 \sin x = \sqrt{2} + \sqrt{2-p}$ has a solution, then $p \in$
 (A) $[\sqrt{5}+1, 4]$ (B) $[\sqrt{5}-1, 2]$
 (C) $[\sqrt{3}+1, 3]$ (D) none of these
30. The value of 'b' such that the equation $\frac{b \cos x}{2 \cos 2x - 1} = \frac{b + \sin x}{(\cos^2 x - 3 \sin^2 x) \tan x}$ possess solutions, belongs to the set
 (A) $\left(-\infty, \frac{1}{2}\right)$ (B) $\left(\frac{1}{2}, \infty\right)$
 (C) $(-\infty, \infty)$ (D) $\left(-\infty, \frac{1}{2}\right) \cup (1, \infty)$

More than One Option Correct Type

31. If $\sin \theta = k$ for exactly one value of θ , $\theta \in \left[0, \frac{7\pi}{3}\right]$, then the value of k is
 (A) 1 (B) -1
 (C) $1/\sqrt{2}$ (D) 0
32. The equation $\sin^4 x + \cos^4 x = a$ has a solution for
 (A) all of values of a (B) $a = 1$
 (C) $a = \frac{1}{2}$ (D) $\frac{1}{2} < a < 1$
33. If $|\cos x|^{\sin^2 x - \frac{3}{2} \sin x + \frac{1}{2}} = 1$, then possible values of x are
 (A) $n\pi$ or $n\pi + (-1)^n \frac{\pi}{6}$, $n \in I$
 (B) $n\pi$ or $2n\pi + \frac{\pi}{2}$ or $n\pi + (-1)^n \frac{\pi}{6}$, $n \in I$
 (C) $n\pi + (-1)^n \frac{\pi}{6}$, $n \in I$
 (D) $n\pi$, $n \in I$

34. The equation $3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2\sin^2 x} = 28$ is satisfied for the values of x given by
 (A) $\cos x = 0$ (B) $\tan x = -1$
 (C) $\tan x = 1$ (D) none of these
35. The general value of θ , satisfying the equation $2\cos 2\theta + \sqrt{2}\sin \theta = 2$, is
 (A) $n\pi$ (B) $n\pi + (-1)^n \frac{\pi}{3}$
 (C) $n\pi + (-1)^n \frac{\pi}{6}$ (D) none of these.
36. The value of θ , lying between $\theta = 0$ and $\theta = \frac{\pi}{2}$ and satisfying the equation
- $$\begin{vmatrix} 1 + \cos^2 \theta & \sin^2 \theta & 4 \sin 4\theta \\ \cos^2 \theta & 1 + \sin^2 \theta & 4 \sin 4\theta \\ \cos^2 \theta & \sin^2 \theta & 1 + 4 \sin 4\theta \end{vmatrix} = 0, \text{ is}$$
- (A) $\frac{11\pi}{24}$ (B) $\frac{7\pi}{24}$
 (C) $\frac{5\pi}{24}$ (D) none of these
37. Solution of the system of equations $2^{\sin x} + \cos y = 1$, $16^{\sin^2 x + \cos^2 y} = 4$ is
 (A) $x = n\pi + (-1)^n \frac{\pi}{6}, y = 2n\pi \pm \frac{2\pi}{3}$
 (B) $x = n\pi + (-1)^n \frac{\pi}{6}, y = 2n\pi \pm \frac{\pi}{3}$
 (C) $x = n\pi - (-1)^n \frac{\pi}{6}, y = 2n\pi \pm \frac{2\pi}{3}$
 (D) $x = n\pi - (-1)^n \frac{\pi}{6}, y = 2n\pi \pm \frac{\pi}{3}$
38. Solution of the equation $4\sin^4 x + \cos^4 x = 1$ is
 (A) $x = n\pi$
 (B) $x = 2n\pi \pm \cos^{-1}\left(\sqrt{\frac{3}{5}}\right)$
 (C) $x = (2n + 1) \frac{\pi}{2}$
 (D) none of these
39. The solution of the equation $\frac{\sqrt{3}}{2} \sin x - \cos x = \cos^2 x$ is
 (A) $x = (2n + 1)\pi$ (B) $x = 2n\pi \pm \frac{\pi}{3}$
 (C) $x = 2n\pi \pm \frac{\pi}{6}$ (D) none of these
40. Solution of the equation $\sin^3 \theta + \sin \theta \cos \theta + \cos^3 \theta = 1$ is
 (A) $\theta = 2n\pi + \frac{\pi}{4}$ (B) $\theta = 2n\pi - \frac{\pi}{4}$
 (C) $\theta = 2n\pi + \frac{\pi}{2}$ (D) $\theta = 2n\pi$
41. Solution of the equation $\sqrt{3}(\cos \theta - \sqrt{3}\sin \theta) = 4\sin 2\theta \cos 3\theta$ is
 (A) $\theta = \frac{n\pi}{6} + \frac{\pi}{18}$, if n is even
 (B) $\theta = \frac{n\pi}{4} + \frac{\pi}{12}$ if n is odd
 (C) $\theta = \frac{n\pi}{4} - \frac{\pi}{12}$, if n is odd
 (D) none of these
42. The values of x in $(-\pi, \pi)$ which satisfy the equation $8^{1 + |\cos x| + \cos^2 x + |\cos^3 x| + \dots \text{to infinity}} = 4^3$ are
 (A) $\pm \frac{\pi}{4}$ (B) $\pm \frac{\pi}{3}$
 (C) $\pm \frac{2\pi}{3}$ (D) none of these
43. Solution of the equation $\sin 6x + \cos 4x + 2 = 0; 0 < x < 2\pi$ is
 (A) $x = \frac{\pi}{3}$ (B) $x = \frac{\pi}{4}$
 (C) $x = \frac{4\pi}{3}$ (D) $x = \frac{5\pi}{4}$
44. $\sqrt{\cos 2x} + \sqrt{1 + \sin 2x} = \sqrt{\sin x + \cos x}$ if
 (A) $x = 2n\pi$ (B) $x = n\pi - \frac{\pi}{4}$
 (C) $\sin x + \cos x = 0$ (D) $x = n\pi$
45. If $[x]$ denotes the greatest integer less than or equal to x , then the equation $\sin x = [1 + \sin x] + [1 - \cos x]$ has no solution in
 (A) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (B) $\left[\frac{\pi}{2}, \pi\right]$
 (C) $\left[\pi, \frac{3\pi}{2}\right]$ (D) R

Match the Column Type

46.

Column-I	Column-I
I. If $2\sin^2x + 3\sin x - 2 > 0$ and $x^2 - x - 2 < 0$ (x is measured in radians), then $x \in$	(A) $(0, \pi) - \left\{ \frac{\pi}{4}, \frac{\pi}{2} \right\}$
II. If $\frac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} - \frac{\cos\theta}{\sqrt{1 + \cos^2\theta}} - 2 \tan\theta \cot\theta = -1$, $\theta \in [0, 2\pi]$, then $\theta \in$	(B) $(0, \pi)$
	(C) $\left(-\frac{\pi}{10}, \frac{3\pi}{10} \right)$
III. The set of all x in $(-\pi, \pi)$ satisfying $ 4\sin x - 1 < \sqrt{5}$ is given by	(D) $\left(\frac{\pi}{6}, 2 \right)$

47.

Column-I	Column-I
I. The least difference between the roots, in the first quadrant $\left(0 \leq x \leq \frac{\pi}{2} \right)$, of the equation $4\cos x(2 - 3\sin^2x) + (\cos 2x + 1) = 0$, is	(A) $\frac{\pi}{2}$
II. If α, β be any two positive values of x for which $2\cos x, \cos x $ and $1 - 3\cos^2x$ are in G.P., then the minimum value of $ \alpha - \beta $ is	(B) $\frac{\pi}{6}$
III. The values of x between 0 and 2π which satisfy the equation $\sin x \sqrt{8\cos^2x} = 1$ are in A.P. with common difference	(C) $\frac{\pi}{3}$
IV. The solution of the equation $(\sin\theta + \cos\theta)[(\sin\theta + \cos\theta)^{1 + \sin 2\theta}] = 2$, $-\pi \leq \theta \leq \pi$, is	(D) $\frac{\pi}{4}$

Assertion-Reason Type

Instructions: In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:

- (A) Assertion(A) is True and Reason(R) is True; Reason(R) is a correct explanation for Assertion(A)
 (B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)
 (C) Assertion(A) is True, Reason(R) is False
 (D) Assertion(A) is False, Reason(R) is True

48. Assertion: The general solution of the equation $2^{\sin\theta} +$

$$2^{\cos\theta} = 2^{1 - \frac{1}{\sqrt{2}}} \text{ is } \theta = n\pi + \frac{\pi}{4}$$

Reason: For any two numbers a and b , A.M. \geq G.M.

49. Assertion: Solution of the equation $4\cot^2\theta = \cot^2\theta - \tan^2\theta$ is $\theta = n\pi \pm \frac{\pi}{4}$

Reason: $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi \pm \alpha$.

Previous Year's Questions

50. If $0 \leq x < 2\pi$, then the number of real values of x , which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is: [2016]

(A) 9

(B) 3

(C) 5

(D) 7

ANSWER KEYS

Single Option Correct Type

1. (B) 2. (C) 3. (A) 4. (D) 5. (C) 6. (C) 7. (B) 8. (D) 9. (C) 10. (B)
 11. (C) 12. (A) 13. (C) 14. (C) 15. (A) 16. (B) 17. (D) 18. (C) 19. (B) 20. (D)
 21. (B) 22. (C) 23. (A) 24. (C) 25. (B) 26. (C) 27. (B) 28. (B) 29. (B) 30. (A)

More than One Option Correct Type

31. (A, B) 32. (B, C, D) 33. (C, D) 34. (A, B) 35. (A, C) 36. (A, B) 37. (A, D) 38. (A, B) 39. (A, B) 40. (C, D)
 41. (A, C) 42. (B, C) 43. (B, D) 44. (A, B, C) 45. (A, B, C, D)

Match the Column Type

46. I ↔ (D), II ↔ (A), III ↔ (C), 47. I ↔ (B), II ↔ (B), III ↔ (D), IV ↔ (D)

Assertion-Reason Type

48. (A) 49. (A)

Previous Year's Questions

50. (D)

HINTS AND SOLUTIONS

Single Option Correct Type

1. Since $x^2 + x^{-2} = (x - x^{-1})^2 + 2 \geq 2$
 and, $2\cos^2 \frac{x}{2} \sin^2 x \leq 2$,
 \therefore the given equation is valid only if
 $2\cos^2 \frac{x}{2} \sin^2 x = 2$
 $\Leftrightarrow \cos \frac{x}{2} = \operatorname{cosec} x = 1$, which cannot be true.
2. We have, $\sin^{50} x - \cos^{50} x = 1$
 $\Rightarrow \sin^{50} x = 1 + \cos^{50} x$
 Since $\sin^{50} x \leq 1$ and $1 + \cos^{50} x \geq 1$, therefore, the two sides are equal only if
 $\sin^{50} x = 1 = 1 + \cos^{50} x$ i.e., $\sin^{50} x = 1$ and $\cos^{50} x = 0$
 $\therefore x = n\pi + \frac{\pi}{2}$, $n \in \mathbb{Z}$.
3. Let $\sqrt{3} + 1 = r \cos \alpha$, and $\sqrt{3} - 1 = r \sin \alpha$
 $\therefore r^2 = (\sqrt{3} + 1)^2 + (\sqrt{3} - 1)^2 = 8$,
 and, $\tan \alpha = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{1 - 1/\sqrt{3}}{1 + 1/\sqrt{3}} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$
 $= \tan(\pi/12)$, i.e., $\alpha = \pi/12$.
 From the given equation, we get $r \cos(\theta - \alpha) = 2$
 $\Rightarrow \cos(\theta - \pi/12) = 1/\sqrt{2} = \cos(\pi/4)$
 $\therefore \theta - \pi/12 = 2n\pi \pm \pi/4$
 or $\theta = 2n\pi \pm \pi/4 + \pi/12$.
4. We have, $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$
 $\Rightarrow a_1 + a_2 \cos 2x + \frac{a_3}{2} (1 - \cos 2x) = 0$
 which is zero for all values of x ,
 if $a_1 = -\frac{a_3}{2} = -a_2$
 or $a_1 = -\frac{k}{2}$, $a_2 = \frac{k}{2}$, $a_3 = k$
 for any $k \in \mathbb{R}$.
 Hence, the required number of triplets is **infinite**.
5. We have, $\sin^4 x - (k+2) \sin^2 x - (k+3) = 0$
 $\Rightarrow \sin^2 x = \frac{(k+2) \pm \sqrt{(k+2)^2 + 4(k+3)}}{2}$
 $= \frac{(k+2) \pm (k+4)}{2}$
 $\Rightarrow \sin^2 x = k+3$ ($\because \sin^2 x = -1$ is not possible)
 Since $0 \leq \sin^2 x \leq 1$, $\therefore 0 \leq k+3 \leq 1$ or $-3 \leq k \leq -2$.
6. We have, $\sin \pi(x^2 + x) = \sin \pi x^2$
 $\Rightarrow \pi(x^2 + x) = n\pi + (-1)^n \pi x^2$
 \therefore Either $x^2 + x = 2m + x^2 \Rightarrow x = 2m \in \mathbb{Z}$.
 or, $x^2 + x = k - x^2$, where k is an odd integer
 $\Rightarrow 2x^2 + x - k = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1+8k}}{4}$.

For least positive non-integral solution,

$$x = \frac{-1 + \sqrt{1+8k}}{4} = \frac{\sqrt{p-1}}{4}, \text{ where } p \text{ is an odd integer.}$$

7. We have, $\sin^2 x - 2\sin x - 1 = 0$

$$\Rightarrow (\sin x - 1)^2 = 2 \Rightarrow \sin x - 1 = \pm \sqrt{2}$$

$$\Rightarrow \sin x = 1 - \sqrt{2} \text{ as } \sin x \not\geq 1.$$

There are 2 solutions in $[0, 2\pi]$ and two more in $[2\pi, 4\pi]$.

Thus, minimum value of n is 4.

8. The given equation can be written as

$$1 + \cos px + 1 + \cos qx = 2$$

$$\Rightarrow \cos\left(\frac{p+q}{2}\right)x \cos\left(\frac{p-q}{2}\right)x = 0$$

$$\cos\left(\frac{p+q}{2}\right)x = 0 \Rightarrow x = \frac{(2n+1)\pi}{p+q}, n = 0, \pm 1, \pm 2 \dots \text{ which}$$

forms an A. P. with common difference = $\frac{2\pi}{p+q}$

$$\cos\left(\frac{p-q}{2}\right)x = 0 \Rightarrow x = \frac{(2n+1)\pi}{p-q}, n = 0, \pm 1, \pm 2, \dots$$

which forms an A. P. with common difference = $\frac{2\pi}{p-q}$

9. We have, $|\cos x| \leq \sin x$

$$\Rightarrow \sin x \geq 0 \quad [\because |\cos x| \geq 0]$$

$$\Rightarrow x \notin (\pi, 2\pi)$$

If $x = 2\pi$, $|\cos 2\pi| \leq \sin 2\pi$, which is not possible

$$\therefore x \in [0, \pi]$$

If $x \in \left[0, \frac{\pi}{2}\right]$, then $|\cos x| \leq \sin x$

$$\Rightarrow \cos x \leq \sin x \Rightarrow x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

If $x \in \left[\frac{\pi}{2}, \pi\right]$, then $|\cos x| \leq \sin x$

$$\Rightarrow -\cos x \leq \sin x \Rightarrow \tan x \leq -1 \quad (\because \cos x < 0)$$

$$\Rightarrow x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$$

Combining two results, we get

$$x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$$

$$\text{or, } x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

10.
$$\frac{1 - \sin x + \dots + (-1)^n \sin^n x + \dots}{1 + \sin x + \dots + \sin^n x + \dots} = \frac{1 - \cos 2x}{1 + \cos 2x}$$

$$\Rightarrow \frac{1}{1 + \sin x} \times \frac{1 - \sin x}{1} = \frac{2 \sin^2 x}{2 \cos^2 x} \text{ as } -1 < \sin x < 1$$

$$\Rightarrow 1 - \sin x = \frac{\sin^2 x (1 + \sin x)}{1 - \sin^2 x}$$

$$\Rightarrow (1 - \sin x)^2 = \sin^2 x \Rightarrow 1 - 2 \sin x = 0$$

$$\Rightarrow \sin x = \frac{1}{2} = \sin \frac{\pi}{6} \Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}$$

11. We have,
$$\sum_{r=1}^n \cos r^2 \theta \sin r \theta = \frac{1}{2}$$

$$\Rightarrow \sum_{r=1}^n 2 \cos r^2 \theta \sin r \theta = 1$$

$$\Rightarrow \sum_{r=1}^n [\sin r(r+1)\theta - \sin r(r-1)\theta] = 1$$

$$\Rightarrow \sin n(n+1)\theta = 1 \Rightarrow \theta = \frac{4k+1}{n(n+1)} \cdot \frac{\pi}{2}, k \in \mathbb{Z}$$

12. Given,
$$\sin^8 x + \cos^8 x = \frac{17}{32}$$

$$\Rightarrow (\sin^4 x + \cos^4 x)^2 - 2 \sin^4 x \cos^4 x = \frac{17}{32}$$

$$\Rightarrow (1 - 2 \sin^2 x \cos^2 x)^2 - 2 \sin^4 x \cos^4 x = \frac{17}{32}$$

$$\Rightarrow 1 - 4 \sin^2 x \cos^2 x + 2 \sin^4 x \cos^4 x = \frac{17}{32}$$

$$\Rightarrow 1 - (2 \sin x \cos x)^2 + \frac{1}{8} (2 \sin x \cos x)^4 = \frac{17}{32}$$

$$\Rightarrow 1 - \sin^2 2x + \frac{1}{8} \sin^4 2x - \frac{17}{32} = 0$$

$$\Rightarrow 4 \sin^4 2x - 32 \sin^2 2x + 15 = 0$$

$$\Rightarrow 2 \sin^2 2x - 1 = 0 \quad \left[\because \sin^2 2x \neq \frac{15}{2} \right]$$

$$\Rightarrow \sin^2 2x = \frac{1}{2} \Rightarrow \sin^2 2x = \sin^2 \frac{\pi}{4}$$

$$\Rightarrow 2x = n\pi \pm \frac{\pi}{4}, n \in \mathbb{I}$$

$$\Rightarrow x = \frac{n\pi}{2} \pm \frac{\pi}{8}, n \in \mathbb{I}$$

13. We have,
$$2^{\cos^2 \theta} + 1 = 3 \cdot 2^{-\sin^2 \theta}$$

$$\Rightarrow 2^{1-2\sin^2 \theta} + 1 = 3 \cdot 2^{-\sin^2 \theta}$$

$$\Rightarrow 2z^2 - 3z + 1 = 0, \text{ where } z = 2^{-\sin^2 \theta} \Rightarrow z = \frac{1}{2}, 1.$$

$$\text{If } z = \frac{1}{2}, \text{ then } 2^{-\sin^2 \theta} = 2^{-1} \Rightarrow \sin^2 \theta = 1 \Rightarrow \theta = n\pi \pm \frac{\pi}{2}$$

$$\text{If } z = 1, \text{ then } 2^{-\sin^2 \theta} = 2^0 \Rightarrow \sin^2 \theta = 0 \Rightarrow \sin \theta = 0$$

$$\Rightarrow \theta = n\pi$$

$$\text{Thus, } \theta = n\pi, n\pi \pm \frac{\pi}{2}$$

14.
$$\sin \theta > 0 \Rightarrow \theta \in (0, \pi) \tag{1}$$

$$\cos \theta > 0 \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) \tag{2}$$

From (1) and (2), $\theta \in \left(0, \frac{\pi}{2}\right)$.

Also, $\log_{1/2} \sin \theta > \log_{1/2} \cos \theta$

$$\Rightarrow \sin \theta < \cos \theta \text{ in } \left(0, \frac{\pi}{2}\right) \Rightarrow \theta \in \left(0, \frac{\pi}{4}\right).$$

15. We have $\cos 3x + \sin\left(2x - \frac{7\pi}{6}\right) = -2$

$$\Rightarrow 1 + \cos 3x + 1 + \sin\left(2x - \frac{7\pi}{6}\right) = 0$$

$$\Rightarrow (1 + \cos 3x) + 1 - \cos\left(2x - \frac{2\pi}{3}\right) = 0$$

$$\Rightarrow 2\cos^2 \frac{3x}{2} + 2\sin^2\left(x - \frac{\pi}{3}\right) = 0$$

$$\Rightarrow \cos \frac{3x}{2} = 0 \text{ and } \sin\left(x - \frac{\pi}{3}\right) = 0$$

$$\Rightarrow \frac{3x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \dots \text{ and } x - \frac{\pi}{3} = 0, \pi, 2\pi, \dots$$

$$\Rightarrow x = \frac{\pi}{3}$$

Therefore, the general solution of

$$\begin{aligned} \cos \frac{3x}{2} = 0 \text{ and } \sin\left(x - \frac{\pi}{3}\right) = 0 \text{ is } x &= 2k\pi + \frac{\pi}{3} \\ &= \frac{\pi}{3}(6k+1), \text{ where } k \in Z \end{aligned}$$

16. Minimum value of L.H.S. is

$$2\left(x + \frac{1}{x} \text{ for } x > 0\right) \text{ and } 1 + \sqrt{\frac{2}{x + \frac{1}{x}}} \leq 2$$

Equality is possible only when both sides are 2 at $x = 1$

$$\text{i.e., } \tan^2 [\pi(x+y)] = 1 \Rightarrow \pi(x+y) = n\pi \pm \frac{\pi}{4}$$

$$\Rightarrow \pi(x+y) = \frac{5\pi}{4} \text{ (for least value of } y)$$

$$\therefore y = \frac{1}{4}$$

17. The given equation can be written as

$$3 - 2x - x^2 = 1 + \tan^2(x+y) + 1 + \cot^2(x+y)$$

$$\Rightarrow 4 - (x+1)^2 = \sec^2(x+y) + \operatorname{cosec}^2(x+y)$$

$$\Rightarrow \cos^2(x+y) \sin^2(x+y) [4 - (x+1)^2] = 1$$

$$\Rightarrow \sin^2(2x+2y) [4 - (x+1)^2] = 4$$

$$\text{Since } \sin^2(2x+2y) \leq 1 \text{ and } 4 - (x+1)^2 \leq 4$$

$$\therefore (1) \text{ holds only if } \sin^2(2x+2y) = 1$$

$$\text{and, } 4 - (x+1)^2 = 4$$

From (3), we get $x = -1$

Putting in (2), we get $\sin(2y-2) = \pm 1$

$$\Rightarrow 2y-2 = n\pi \pm \frac{\pi}{2}, n \in Z$$

$$\Rightarrow y = 1 + (2n \pm 1)\frac{\pi}{4}, n \in Z$$

$$\therefore x = -1, y = 1 + (2n \pm 1)\frac{\pi}{4}, n \in Z$$

18. We have, $[\sin x] + [\sqrt{2} \cos x] = -3$

$$\Rightarrow [\sin x] = -1 \text{ and } [\sqrt{2} \cos x] = -2$$

$$\Rightarrow -1 \leq \sin x < 0 \text{ or } x \in (\pi, 2\pi)$$

$$\text{and, } -2 \leq \sqrt{2} \cos x < -1$$

$$\text{or, } \cos x < \frac{-1}{\sqrt{2}} \text{ or } x \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$$

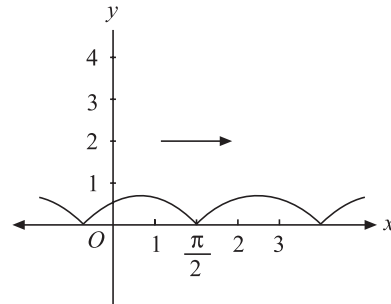
$$\text{So } x \in \left(\pi, \frac{5\pi}{4}\right)$$

19. $\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}x + 4$

$$= (x - \sqrt{3})^2 + 1$$

\therefore R.H.S > 1 . So, the solution exists if and only if $x - \sqrt{3} = 0 \Rightarrow x = \sqrt{3}$ and then equation is obviously satisfied.

20. From the graph, $|\cos x|$ and $2[x]$ do not cut each other for any real value of x . Hence, number of solutions is nil.



21. $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$$\Rightarrow 2\sin 2x \cos x - 3\sin 2x - 2\cos 2x \cos x$$

$$+ 3\cos 2x = 0$$

$$\Rightarrow \sin 2x(2\cos x - 3) - \cos 2x(2\cos x - 3) = 0$$

$$\Rightarrow (\sin 2x - \cos 2x)(2\cos x - 3) = 0$$

$$\Rightarrow \sin 2x = \cos 2x \quad (\because \cos x \neq 3/2)$$

$$\Rightarrow 2x = 2n\pi \pm \left(\frac{\pi}{2} - 2x\right) \text{ i.e., } x = \frac{n\pi}{2} + \frac{\pi}{8}$$

22. From the given equation we have

$$\sin x - \sin 3x + 2\sin 2x = 3$$

$$\Rightarrow 2\sin x \cos 2x - 2\sin 2x + 3 = 0$$

$$\Rightarrow (\sin x + \cos 2x)^2 + (\sin 2x - 1)^2 + 3$$

$$= \sin^2 x + \cos^2 2x + \sin^2 2x + 1$$

$$\Rightarrow (\sin x + \cos 2x)^2 + (\sin 2x - 1)^2 + \cos^2 x = 0$$

which is possible only if

$$\sin x + \cos 2x = 0, \sin 2x = 1 \text{ and } \cos x = 0$$

which is not possible for any value of x .

23. The given equation is
 $1 + \sin^2 ax = \cos x$ (1)

We have, $1 + \sin^2 ax \geq 1$ and $\cos x \leq 1$

\therefore The equation (1) has a solution only if $\cos x = 1$

and, $1 + \sin^2 ax = 1$ i.e., $\sin^2 ax = 0 \Rightarrow \sin ax = 0$

Now, $\cos x = 1$ gives $x = 2n\pi, n \in I$

and, $\sin ax = 0$ gives $ax = m\pi \Rightarrow x = \frac{m\pi}{a}, m \in I$

The equation (1) has a solution only if for some values of m and n , we have

$$m\pi/a = 2n\pi \text{ or } 2na = m. \quad (2)$$

But a is irrational and m and n are integers.

\therefore (2) is possible only if $m = 0 = n$. But then we have $x = 0$.

Hence, $x = 0$ is the only solution satisfying the given equation.

24. Given equation is

$$\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$$

$$\Rightarrow (\sin^2 x)^2 + (\cos^2 x)^2 + 2\sin^2 x \cos^2 x - 2\sin^2 x \cos^2 x$$

$$+ \sin 2x + \alpha = 0$$

$$\Rightarrow (\sin^2 x + \cos^2 x)^2 - \frac{1}{2} (4\sin^2 x \cos^2 x) + \sin 2x + \alpha = 0$$

$$\Rightarrow 1 - \frac{1}{2} (\sin 2x)^2 + \sin 2x + \alpha = 0$$

$$\Rightarrow 2 - \sin^2 2x + 2\sin 2x + 2\alpha = 0$$

$$\Rightarrow \sin^2 2x - 2\sin 2x - 2 - 2\alpha = 0$$

$$\Rightarrow \sin 2x = \frac{2 \pm \sqrt{[(2)^2 - 4 \times 1(-2 - 2\alpha)]}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{(4 + 8 + 8\alpha)}}{2} = \frac{2 \pm \sqrt{(12 + 8\alpha)}}{2}$$

$$= \frac{2\{1 \pm \sqrt{(3 + 2\alpha)}\}}{2} = 1 \pm \sqrt{(3 + 2\alpha)}$$

If $\sin 2x = 1 + \sqrt{(3 + 2\alpha)} > 1$ which is not possible

$$\text{If } \sin 2x = 1 - \sqrt{(3 + 2\alpha)} \quad (i)$$

$$\Rightarrow -1 \leq 1 - \sqrt{(3 + 2\alpha)} \leq 1$$

$$\Rightarrow -2 \leq -\sqrt{(3 + 2\alpha)} \leq 0 \Rightarrow 0 \leq (3 + 2\alpha) \leq 4$$

$$\Rightarrow -3 \leq 2\alpha \leq 1 \Rightarrow -\frac{3}{2} \leq \alpha \leq \frac{1}{2}.$$

25. Since α, β satisfy the given equation,

$$\therefore 3\cos\alpha + 4\sin\alpha = 2 \text{ and } 3\cos\beta + 4\sin\beta = 2$$

On subtracting, we get $3(\cos\alpha - \cos\beta) + 4(\sin\alpha - \sin\beta) = 0$

$$\Rightarrow 3 \cdot 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta - \alpha}{2} + 4 \cdot 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} = 0$$

$$\Rightarrow 2 \sin \left(\frac{\beta - \alpha}{2} \right) \left[3 \sin \frac{\alpha + \beta}{2} - 4 \cos \frac{\alpha + \beta}{2} \right] = 0$$

Here, $\sin \frac{\beta - \alpha}{2} \neq 0$ because $\beta \neq \alpha$ and

$$\left(\frac{\beta - \alpha}{2} \right)_{\max} < \pi \text{ and } \left(\frac{\beta - \alpha}{2} \right)_{\min} > -\pi$$

$$\therefore 3 \sin \frac{\alpha + \beta}{2} - 4 \cos \frac{\alpha + \beta}{2} = 0$$

$$\text{or, } \tan \frac{\alpha + \beta}{2} = \frac{4}{3}$$

$$\therefore \sin(\alpha + \beta) = \frac{2 \tan \frac{\alpha + \beta}{2}}{1 + \tan^2 \frac{\alpha + \beta}{2}} = \frac{2 \times \frac{4}{3}}{1 + \frac{16}{9}} = \frac{24}{25}.$$

26. $|\tan x + \sec x| = |\tan x| + |\sec x|$ iff $\sec x$ and $\tan x$ both have same sign. $\Rightarrow \sec x \cdot \tan x \geq 0 \Rightarrow \frac{\sin x}{\cos^2 x} \geq 0$, but $\cos x \neq 0 \Rightarrow$

$$x \in \left[0, \frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \pi \right]$$

27. $|\cos x| = \cos x - 2 \sin x$

$$\Rightarrow \cos x = \cos x - 2 \sin x \text{ if } \cos x \geq 0$$

$$\Rightarrow \sin x = 0 \Rightarrow x = 2n\pi \text{ (as } \cos x \geq 0), n \in I$$

Also, $|\cos x| = \cos x - 2 \sin x$

$$\Rightarrow -\cos x = \cos x - 2 \sin x \text{ if } \cos x < 0$$

$$\Rightarrow \cos x - \sin x = 0 \Rightarrow \tan x = 1$$

$$\text{Now, } \cos x < 0 \text{ and } \tan x = 1 \Rightarrow \tan x = \tan \frac{5\pi}{4}$$

$$\Rightarrow x = 2n\pi + \left(\frac{5\pi}{4} \right) = (2n + 1)\pi + \frac{\pi}{4}$$

28. We have, $(1 - \tan\theta)(1 + \tan\theta) \sec^2\theta + 2 \tan^2\theta = 0$

$$\Rightarrow (1 - \tan^2\theta) + 2 \tan^2\theta = 0$$

$$\text{Put } \tan^2\theta = x, \therefore (1 - x^2) + 2x = 0$$

$$\Rightarrow 2x = x^2 - 1 = y \text{ (say)}$$

$$\therefore y = 2^x \text{ and } y = x^2 - 1$$

By inspection, $x = 3, \therefore \tan^2\theta = 3$

$$\Rightarrow \tan \theta = \pm \sqrt{3} \Rightarrow \theta = \pm \frac{\pi}{3}$$

29. The given equation has a solution if $p \geq 0, 2 - p \geq 0$

$$\text{and, } \left| \sqrt{p} \cos x - 2 \sin x \right| \leq \sqrt{p + 4}$$

$$\Rightarrow p \geq 0, 2 - p \geq 0$$

$$\text{and, } \left| \sqrt{2} + \sqrt{2 - p} \right| \leq \sqrt{p + 4} \Rightarrow p \geq 0, 2 - p \geq 0$$

$$\text{and, } \left(\sqrt{2} + \sqrt{2 - p} \right)^2 \leq p + 4 \Rightarrow p \geq 0, p \leq 2$$

$$\text{and, } 2 + 2 - p + 2 \sqrt{4 - 2p} \leq p + 4$$

$$\Rightarrow 0 \leq p \leq 2 \text{ and } 2\sqrt{4 - 2p} \leq 2p$$

$$\Rightarrow p^2 + 2p - 4 \geq 0$$

$$\Rightarrow 0 \leq p \leq 2 \text{ and } (p + 1)^2 \leq 5$$

$$\Rightarrow 0 \leq p \leq 2 \text{ and } p \in (-\infty, -\sqrt{5} - 1] \cup [\sqrt{5} - 1, \infty)$$

$$\Rightarrow p \in [\sqrt{5} - 1, 2]$$

30. For the domain of definition of the given equation, we have,

$$(i) 2 \cos 2x - 1 \neq 0 \Rightarrow x \neq n\pi \pm \frac{\pi}{6}$$

$$(ii) \tan x \neq 0 \Rightarrow x \neq \pm \frac{n\pi}{2} \text{ [For odd multiples of } \frac{\pi}{2}, \tan x \text{ is not defined]}$$

$$(iii) \cos^2 x - 3 \sin^2 x \neq 0 \Rightarrow x \neq n\pi \pm \frac{\pi}{6},$$

$$\text{Also, } 2\cos 2x - 1 = 2(\cos^2 x - \sin^2 x) - (\cos^2 x + \sin^2 x) = \cos^2 x - 3\sin^2 x$$

Now, the given equation reduces to

$$b\sin x = b + \sin x$$

$$\Rightarrow \sin x = \frac{b}{b-1},$$

$$\because -1 \leq \sin x \leq 1 \therefore -1 \leq \frac{b}{b-1} \leq 1$$

$$\Rightarrow \frac{b}{b-1} + 1 \leq 0 \text{ and } \frac{b}{b-1} - 1 \leq 0$$

$$\Rightarrow \frac{2b-1}{b-1} \geq 0 \text{ and } \frac{1}{b-1} \leq 0$$

$$\Rightarrow b \leq \frac{1}{2} \text{ or } b > 1 \text{ and } b < 1 \Rightarrow b \leq \frac{1}{2}$$

When $b = \frac{1}{2}$, $\sin x = 1$, which is not possible

$$\therefore b < \frac{1}{2}$$

More than One Option Correct Type

31. Clearly, $-1 \leq k \leq 1$. For any value of k other than 1 and -1 , $\sin \theta$ has two values (in quadrant, I, II or III, IV).

$$\therefore k = 1 \text{ or } -1.$$

32. The given equation can be written as

$$1 - 2\sin^2 x \cos^2 x = a$$

$$\Rightarrow \sin^2 2x = 2(1-a) \Rightarrow 2(1-a) \leq 1$$

$$\text{and, } 2(1-a) \geq 0 \Rightarrow 1/2 \leq a \leq 1.$$

33. The equation holds if $|\cos x| = 1$

i.e., if $x = n\pi$, $n \in I$

$$\text{If } |\cos x| \neq 1, \text{ then } \sin^2 x - \frac{3}{2}\sin x + \frac{1}{2} = 0$$

$$\Rightarrow \sin x = 1 \text{ or } \frac{1}{2}$$

$\sin x \neq 1$, as in that case $\cos x = 0$

$$\therefore \sin x = \frac{1}{2} \Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}$$

34. We have,

$$3^{\sin 2x} + 2\cos^2 x + 3^{1-\sin 2x} + 2\sin^2 x = 28$$

$$\Rightarrow 3^{\sin 2x} + 2\cos^2 x + 3^{3-(\sin 2x + 2\cos^2 x)} = 28$$

$$\text{Let } y = 3^{\sin 2x + 2\cos^2 x}$$

$$\therefore y + \frac{27}{y} = 28$$

$$\Rightarrow y^2 - 28y + 27 = 0 \Rightarrow y = 27 \text{ or } 1$$

$$\text{If } y = 27, \text{ then } 3^{\sin 2x + 2\cos^2 x} = 3^3$$

$$\Rightarrow \sin 2x + (2\cos^2 x - 1) = 2$$

$$\Rightarrow \sin 2x + \cos 2x = 2$$

Dividing both sides by $\sqrt{2}$, we have

$$\sin\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$$

which is not possible

Now, if $y = 1$

$$\Rightarrow 3^{\sin 2x + 2\cos^2 x} = 1 = 3^0 \Rightarrow \sin 2x + 2\cos^2 x = 0$$

$$\Rightarrow 2\cos x (\sin x + \cos x) = 0$$

$$\therefore \cos x = 0 \text{ or } \tan x = -1$$

35. We have, $2\cos 2\theta + \sqrt{2}\sin \theta = 2$

$$\Rightarrow \sqrt{2}\sin \theta = 2(1 - \cos 2\theta)$$

$$\Rightarrow \sqrt{2}\sin \theta = 4\sin^2 \theta \Rightarrow \sqrt{2}\sin \theta - 4\sin^2 \theta = 0$$

$$\Rightarrow \sqrt{2}\sin \theta [1 - 2\sqrt{2}\sin^2 \theta] = 0$$

If $\sqrt{2}\sin \theta = 0$, $\sin \theta = 0$ [$\because \sqrt{z} = 0 \Rightarrow z = 0$]

$$\Rightarrow \theta = n\pi, n \in I.$$

$$\text{If } 1 - 2\sqrt{2}\sin^2 \theta = 0, \sin^2 \theta = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow (\sin \theta)^2 = \left(\frac{1}{2}\right)^{\frac{3}{2}} \Rightarrow \sin \theta = \frac{1}{2} = \sin \frac{\pi}{6}$$

$$\Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I.$$

$$\therefore \theta = n\pi, n\pi + (-1)^n \frac{\pi}{6}, n \in I.$$

36. The given equation can be written as

$$\begin{vmatrix} 1 + \cos^2 \theta & \sin^2 \theta & 4\sin 4\theta \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0$$

[Applying $R_3 \rightarrow R_3 - R_2$ and $R_2 \rightarrow R_2 - R_1$]

$$\Rightarrow \begin{vmatrix} 2 & \sin^2 \theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 0$$

[Applying $C_1 \rightarrow C_1 + C_2$]

$$\Rightarrow 2 + 4\sin 4\theta = 0 \Rightarrow \sin 4\theta = -\frac{1}{2}$$

$$\Rightarrow 4\theta = n\pi + (-1)^n \left(-\frac{\pi}{6}\right) \Rightarrow \theta = \frac{n\pi}{4} + (-1)^{n+1} \frac{\pi}{24}.$$

We have to choose values of θ s.t. $0 < \theta < \frac{\pi}{2}$

$$\therefore \theta = \frac{7\pi}{24}, \frac{11\pi}{24}$$

37. We have, $2^{\sin x + \cos y} = 1 = 2^0$

$$\Rightarrow \sin x + \cos y = 0$$

$$\text{and, } 16^{\sin^2 x + \cos^2 y} = 4 = 16^{1/2}$$

(1)

$$\Rightarrow \sin^2 x + \cos^2 y = 1/2 \quad (2)$$

Eliminating $\cos y$ from (1) and (2), we get

$$2\sin^2 x \cdot \frac{1}{2}, \text{ or } \sin x = \pm \frac{1}{2}.$$

When $\sin x = \frac{1}{2}$.

$$\cos y = -\frac{1}{2} = -\cos(\pi/3) = \cos(\pi - \pi/3)$$

$$\therefore x = n\pi + (-1)^n (\pi/6), y = 2n\pi \pm (2\pi/3)$$

When $\sin x = -\frac{1}{2}, \cos x = \frac{1}{2}$

$$\therefore x = n\pi - (-1)^n (\pi/6) \text{ and } y = 2n\pi \pm (\pi/3).$$

38. We have, $4\sin^4 x + \cos^4 x = 1$

$$\Rightarrow 4(1 - \cos^2 x)^2 + \cos^4 x = 1$$

$$\Rightarrow 4(1 + \cos^4 x - 2\cos^2 x) + \cos^4 x - 1 = 0$$

$$\Rightarrow 4 + 4\cos^4 x - 8\cos^2 x + \cos^4 x - 1 = 0$$

$$\Rightarrow 5\cos^4 x - 8\cos^2 x + 3 = 0$$

$$\Rightarrow 5\cos^4 x - 5\cos^2 x - 3\cos^2 x + 3 = 0$$

$$\Rightarrow 5\cos^2 x (\cos^2 x - 1) - 3(\cos^2 x - 1) = 0$$

$$\Rightarrow (\cos^2 x - 1)(5\cos^2 x - 3) = 0$$

If $\cos^2 x = 1, x = n\pi, n \in I$.

If $5\cos^2 x - 3 = 0 \Rightarrow \cos^2 x = (3/5)$

$$\Rightarrow \cos x = \pm \sqrt{(3/5)}$$

Therefore, $x = 2n\pi \pm \cos^{-1} \sqrt{(3/5)}$.

39. We have, $\frac{\sqrt{3}}{2} \sin x = \cos x + \cos^2 x$.

Squaring both sides, we get

$$3(1 - \cos^2 x) = 4(\cos^2 x + 2\cos^3 x + \cos^4 x)$$

$$\Rightarrow 4\cos^4 x + 8\cos^3 x + 7\cos^2 x - 3 = 0$$

$$\Rightarrow (\cos x + 1)(2\cos x - 1)(2\cos^2 x + 3\cos x + 3) = 0$$

When $\cos x = -1 = \cos \pi, x = 2n\pi + \pi = (2n + 1)\pi$.

When $\cos x = \frac{1}{2} = \cos \frac{\pi}{3}, x = 2n\pi \pm \frac{\pi}{3}$.

When $2\cos^2 x + 3\cos x + 3 = 0,$

the discriminant $= 9 - 4 \cdot 2 \cdot 3 < 0$.

\therefore This factor does not give any real values of $\cos x$.

Hence, $x = (2n + 1)\pi, 2n\pi \pm \frac{\pi}{3}, n \in I$.

40. We have, $(\sin^3 \theta + \cos^3 \theta) - (1 - \sin \theta \cos \theta) = 0$

$$\Rightarrow (\sin \theta + \cos \theta)(\sin^2 \theta + \cos^2 \theta - \sin \theta \cos \theta)$$

$$- (1 - \sin \theta \cos \theta) = 0$$

$$\Rightarrow (\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta) - (1 - \sin \theta \cos \theta) = 0$$

$$\Rightarrow (1 - \sin \theta \cos \theta)(\sin \theta + \cos \theta - 1) = 0$$

If $1 - \sin \theta \cos \theta = 0,$

then, $2\sin \theta \cos \theta = 2$ or $\sin 2\theta = 2$

which is not possible because $\sin 2\theta$ can never be greater than 1.

If, $\sin \theta + \cos \theta = 1$

then, $\frac{1}{\sqrt{2}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta = \frac{1}{\sqrt{2}}$

(Divide by $\sqrt{2}$)

$$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4}$$

$$\Rightarrow \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}.$$

$$\therefore \theta = 2n\pi + \frac{\pi}{4} + \frac{\pi}{4}, 2n\pi - \frac{\pi}{4} + \frac{\pi}{4}$$

i.e., $\theta = 2n\pi + \pi/2$, or $2n\pi$.

41. We have,

$$\sqrt{3}(\cos \theta - \sqrt{3} \sin \theta) = 4\sin 2\theta \cos 3\theta$$

$$\Rightarrow \sqrt{3} \cos \theta - 3\sin \theta = 2(\sin 5\theta - \sin \theta)$$

$$\Rightarrow \sqrt{3} \cos \theta - \sin \theta = 2\sin 5\theta$$

Dividing by $\sqrt{3+1} = 2$, we get

$$\frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta = \sin 5\theta$$

$$\Rightarrow \sin \frac{\pi}{3} \cos \theta - \cos \frac{\pi}{3} \sin \theta = \sin 5\theta$$

$$\Rightarrow \sin\left(\frac{\pi}{3} - \theta\right) = \sin 5\theta$$

$$\Rightarrow 5\theta = n\pi + (-1)^n \left(\frac{\pi}{3} - \theta\right)$$

$$\Rightarrow 5\theta = n\pi + \frac{\pi}{3} - \theta \text{ if } n \text{ is even}$$

and, $5\theta = n\pi - \left(\frac{\pi}{3} - \theta\right)$ if n is odd.

$$\therefore \theta = \frac{n\pi}{6} + \frac{\pi}{18} \text{ if } n \text{ is even}$$

and, $\theta = \frac{n\pi}{4} - \frac{\pi}{12}$ if n is odd.

42. We have, $8^{1 + |\cos x| + \cos 2x + |\cos 3x| + \dots \text{to infinity}} = 8^2$.

$$\Rightarrow 1 + |\cos x| + \cos^2 x + |\cos^3 x| + \dots \text{to infinity} = 2 \quad (1)$$

This is an infinite geometric series with first term 1 and common ratio $|\cos x|$.

We know that $|\cos x| \leq 1$.

Here, $|\cos x| \neq 1$, because when $|\cos x| = 1$, equation (1) becomes meaningless ($\infty = 2$)

$$\therefore |\cos x| < 1.$$

Hence, the equation (1) reduces to

$$\frac{1}{1 - |\cos x|} = 2 \Rightarrow 1 - |\cos x| = \frac{1}{2} \Rightarrow |\cos x| = \frac{1}{2}$$

$$\Rightarrow \cos x = \pm \frac{1}{2}.$$

$$\text{When } \cos x = \frac{1}{2} = \cos \frac{\pi}{3}, x = 2n\pi \pm \frac{\pi}{3}$$

$$\text{When } \cos x = -\frac{1}{2} = \cos \frac{2\pi}{3},$$

$$x = 2n\pi \pm \frac{2\pi}{3} \text{ where } n = 0, \pm 1, \dots$$

The values of x in the given interval $(-\pi, \pi)$ are

$$= \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}$$

43. Given: $\sin 6x + \cos 4x + 2 = 0$

or, $\sin 6x + \cos 4x = -2$

$$\Rightarrow \sin 6x = -1$$

and, $\cos 4x = -1$ both should be satisfied simultaneously.

[\because minimum value of $\sin 6x$ and $\cos 4x$ is -1]

$$\text{Now, } \sin 6x = -1 = \frac{3\pi}{2} \Rightarrow 6x = 2n\pi + \frac{3\pi}{2}$$

$$\therefore x = \frac{3\pi}{2} + \frac{\pi}{4}, \text{ where } n \in I.$$

\Rightarrow Values of x between 0 and 2π are

$$\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{15\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

$$\text{Also, } \cos 4x = -1 = \cos \pi \Rightarrow 4x = 2n\pi + \pi$$

$$\therefore x = \frac{n\pi}{2} + \frac{\pi}{4}, \text{ where } n \in Z$$

$$\Rightarrow \text{Values of } x \text{ between 0 and } 2\pi \text{ are } \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Hence, values of x lying between 0 and 2π and satisfying both the equations are

$$\frac{\pi}{4}, \frac{5\pi}{4}$$

44. The given equation can be written as

$$\sqrt{\cos^2 x - \sin^2 x} + \sqrt{(\cos x + \sin x)^2} = 2\sqrt{\cos x + \sin x}$$

$$\Rightarrow \sqrt{\cos x + \sin x} [\sqrt{\cos x - \sin x} + \sqrt{\cos x + \sin x}]$$

$$= 2\sqrt{\cos x + \sin x}$$

$$\Rightarrow \text{Either } \cos x + \sin x = 0 \Rightarrow \tan x = -1$$

$$\Rightarrow x = n\pi - \frac{\pi}{4} (n \in I)$$

$$\text{or, } \sqrt{\cos x - \sin x} + \sqrt{\cos x + \sin x} = 2$$

$$\Rightarrow 2\cos x + 2\sqrt{\cos^2 x - \sin^2 x} = 4$$

$$\Rightarrow \cos^2 x - \sin^2 x = (2 - \cos x)^2$$

$$\Rightarrow \cos^2 x + 4\cos x - 5 = 0$$

$$\Rightarrow \cos x = \frac{-4 \pm \sqrt{16 + 20}}{2} = -5 \text{ or } 1$$

$$\text{But } \cos x \neq -5 \text{ so } \cos x = 1 \Rightarrow x = 2n\pi$$

(1) 45. At $x = -\frac{\pi}{2}, \frac{3\pi}{2}$

$$[1 + \sin x] = 0, [1 - \cos x] = 1$$

$$\therefore \sin x = 0 + 1 \Rightarrow -1 = 1 \text{ (Absurd)}$$

$$\text{At } x = 0$$

$$(1 + \sin x) = 1, (1 - \cos x) = 0$$

$$\therefore \sin x = 1 + 0 \Rightarrow 0 = 1 \text{ (Absurd)}$$

$$\text{At } x = \frac{\pi}{2}$$

$$[1 + \sin x] = 2, [1 - \cos x] = 1$$

$$\sin x = 2 + 1 = 3 \text{ (Absurd)}$$

$$\text{At } x = \pi$$

$$[1 + \sin x] = 1, [1 - \cos x] = 2$$

$$\sin x = 1 + 2 = 3 \text{ (Absurd)}$$

$$\text{In } \left(-\frac{\pi}{2}, 0\right) [1 + \sin x] = 0, [1 - \cos x] = 0$$

$$\therefore \sin x = 0 + 0 = 0 \text{ (Absurd)}$$

$$\text{In } \left(0, \frac{\pi}{2}\right), [1 + \sin x] = 1, [1 - \cos x] = 0$$

$$\therefore \sin x = 1 + 0 = 1 \text{ (Absurd)}$$

$$\text{In } \left(\frac{\pi}{2}, \pi\right), [1 + \sin x] = 1, [1 - \cos x] = 1$$

$$\therefore \sin x = 1 + 1 = 2 \text{ (Absurd)}$$

$$\text{In } \left(\pi, \frac{3\pi}{2}\right), [1 + \sin x] = 0, [1 - \cos x] = 1$$

$$\therefore \sin x = 0 + 1 = 1 \text{ (Absurd)}$$

$$\therefore \text{all the four results hold.}$$

Match the Column Type

46. I We have, $2\sin^2 x + 3\sin x - 2 > 0$

$$\Rightarrow (2\sin x - 1)(\sin x + 2) > 0$$

$$\Rightarrow 2\sin x - 1 > 0 [\because \sin x + 2 > 0 \forall x \in R]$$

$$\Rightarrow \sin x > 1/2 \Rightarrow x \in (\pi/6, 5\pi/6)$$

$$\text{Also, } x^2 - x - 2 < 0$$

$$\Rightarrow (x - 2)(x + 1) < 0 \Rightarrow -1 < x < 2.$$

$$\text{As, } 2 < \frac{5\pi}{6}, \text{ we obtain that } x \text{ must lie in } \left(\frac{\pi}{6}, 2\right).$$

II. Since $\sin \theta - \cos \theta \neq 0$

$$\therefore \tan \theta \neq 1 \therefore \theta \neq \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\text{Now, } \sin^2 \theta + \cos^2 \theta + \sin \theta \cos \theta$$

$$= |\sin \theta| \cos \theta - 2 \tan \theta \cot \theta = -1$$

$$\Rightarrow 1 + \cos \theta (\sin \theta - |\sin \theta|) - 2 = -1$$

$$\Rightarrow \cos \theta (\sin \theta - |\sin \theta|) = 0$$

$$\therefore \theta \in (0, \pi) - \left\{\frac{\pi}{4}, \frac{\pi}{2}\right\}.$$

III. We have, $|4\sin x - 1| < \sqrt{5}$

$$\Rightarrow -\sqrt{5} < 4\sin x - 1 < \sqrt{5}$$

$$\Rightarrow -\left(\frac{\sqrt{5}-1}{4}\right) < \sin x < \frac{\sqrt{5}+1}{4}$$

$$\Rightarrow -\sin \frac{\pi}{10} < \sin x < \cos \frac{\pi}{5}$$

$$\Rightarrow \sin \left(\frac{-\pi}{10} \right) < \sin x < \sin \left(\frac{\pi}{2} - \frac{\pi}{5} \right)$$

$$\Rightarrow \sin \left(\frac{-\pi}{10} \right) < \sin x < \sin \left(\frac{3\pi}{10} \right)$$

$$\Rightarrow x \in \left(-\frac{\pi}{10}, \frac{3\pi}{10} \right) [\because x \in (-\pi, \pi)].$$

47. I. We have,

$$4\cos x (2 - 3 \sin^2 x) + (\cos 2x + 1) = 0$$

$$\Rightarrow 4\cos x (3\cos^2 x - 1) + 2\cos^2 x = 0$$

$$\Rightarrow 2\cos x (6\cos^2 x + \cos x - 2) = 0$$

$$\Rightarrow 2\cos x (3\cos x + 2)(2\cos x - 1) = 0$$

$$\Rightarrow \text{either } \cos x = 0 \text{ which gives } x = \pi/2$$

$$\text{or, } \cos x = -2/3,$$

which gives no value of x for which $0 \leq x \leq \pi/2$

or, $\cos x = 1/2$, which gives $x = \pi/3$.

So, the required difference = $\pi/2 - \pi/3 = \pi/6$.

II. Since

$2\cos x$, $|\cos x|$ and $1 - 3\cos^2 x$ are in G. P.

$$\therefore \cos^2 x = 2\cos x (1 - 3\cos^2 x)$$

$$\Rightarrow 6\cos^3 x + \cos^2 x - 2\cos x = 0$$

$$\Rightarrow \cos x (2\cos x - 1)(3\cos x + 2) = 0$$

$$\Rightarrow \cos x = 0, \frac{1}{2}, -\frac{2}{3} \therefore x = \frac{\pi}{2}, \frac{\pi}{3}, \cos^{-1} \left(-\frac{2}{3} \right)$$

$$\therefore \text{If } \alpha = \frac{\pi}{2}, \beta = \frac{\pi}{3}, (\because \alpha, \beta \text{ are positive})$$

$$\text{then, } |\alpha - \beta| = \frac{\pi}{6}.$$

III. We have, $\sin x \sqrt{8\cos^2 x} = 1$

$$\Rightarrow \sin x |\cos x| = \frac{1}{2\sqrt{2}}$$

Case I: When $\cos x > 0$

$$\text{In this case, } \sin x \cos x = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \sin 2x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow 2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

$$\Rightarrow x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

As x lies between 0 and 2π and $\cos x > 0$

$$\therefore x = \frac{\pi}{8}, \frac{3\pi}{8}$$

Case II. When $\cos x < 0$

$$\text{In this case } \sin x \cos x = \frac{-1}{2\sqrt{2}}$$

$$\Rightarrow \sin 2x = \frac{-1}{2\sqrt{2}}$$

$$\Rightarrow x = \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$$

$$\Rightarrow x = \frac{5\pi}{8}, \frac{7\pi}{8} \text{ as } \cos x < 0$$

Thus, the values of x satisfying the given equation which lie between 0 and 2π are

$$\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$

These are in A. P. with common difference $\frac{\pi}{4}$.

IV. We know that the maximum values of $\sin \theta + \cos \theta$ and $1 + \sin 2\theta$ are $\sqrt{2}$ and 2, respectively. Also, $(\sqrt{2})^2 = 2$.

\therefore The given equation can hold only if

$$\sin \theta + \cos \theta = \sqrt{2} \text{ and } 1 + \sin 2\theta = 2.$$

$$\text{Now, } \sin \theta + \cos \theta = \sqrt{2} \Rightarrow \cos \left(\theta - \frac{\pi}{4} \right) = 1$$

$$\Rightarrow \theta = 2n\pi + \frac{\pi}{4}$$

$$\Rightarrow \theta = 2n\pi + \frac{\pi}{4}$$

$$\text{Also, } 1 + \sin 2\theta = 2 \Rightarrow \sin 2\theta = 1 \Rightarrow \theta = \frac{n\pi}{2} + (-1)^n \frac{n\pi}{4}$$

Thus, the value of θ in $[-\pi, \pi]$, satisfying both the equations

$$\text{is } \frac{\pi}{4}.$$

Assertion-Reason Type

48. Since A.M. \geq G.M.

$$\therefore \frac{2^{\sin \theta} + 2^{\cos \theta}}{2} \geq \sqrt{2^{\sin \theta} \cdot 2^{\cos \theta}}$$

[The equality holds only if $2^{\sin \theta} = 2^{\cos \theta}$]

$$\Rightarrow 2^{\sin \theta} + 2^{\cos \theta} \geq 2\sqrt{2^{\sin \theta + \cos \theta}}$$

But the minimum value of $\sin \theta + \cos \theta$ is $-\sqrt{2}$.

$$\therefore 2^{\sin \theta} + 2^{\cos \theta} \geq 2\sqrt{2^{-\sqrt{2}}} = 2^{1 - \frac{1}{\sqrt{2}}}$$

The equation holds when the equality holds.

$$\text{i.e., } 2^{\sin \theta} = 2^{\cos \theta} \Rightarrow \sin \theta = \cos \theta$$

$$\Rightarrow \tan \theta = 1 = \tan \frac{\pi}{4} \Rightarrow \theta = n\pi + \frac{\pi}{4}$$

49. We have, $4\cot 2\theta = \cot^2 \theta - \tan^2 \theta$

$$\Rightarrow \frac{4}{\tan 2\theta} = \frac{1}{\tan^2 \theta} - \tan^2 \theta$$

$$\text{Put } \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\therefore \frac{4(1 - \tan^2 \theta)}{2 \tan \theta} = \frac{1 - \tan^4 \theta}{\tan^2 \theta}$$

$$\Rightarrow (1 - \tan^2 \theta) [2 \tan \theta - (1 + \tan^2 \theta)] = 0$$

$$\Rightarrow (1 - \tan^2 \theta) (\tan^2 \theta - 2 \tan \theta + 1) = 0$$

$$\Rightarrow (1 - \tan^2 \theta) (\tan \theta - 1)^2 = 0$$

$$\Rightarrow \tan \theta = 1, -1. \therefore \theta = n\pi \pm \frac{\pi}{4}$$

Previous Year's Questions

50. $2\cos 2x \cos x + 2 \cos 3x \cos x = 0$

$$\Rightarrow 2\cos x (\cos 2x + \cos 3x) = 0$$

$$\Rightarrow 2\cos x 2\cos 5x/2 \cos x/2 = 0$$

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}, \pi, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$$

\therefore 7 Solutions