CHAPTER 1

Set Theory

Chapter Highlights

Set, Representation of a set, Types of sets, Operations on sets, Algebra of sets, Cartesian product of two sets, Relations, Types of relations on a set, Equivalence relation, Congruence modulo *m*.

SET

A *set* is a well defined collection of objects such that given an object, it is possible to determine whether that object belongs to the given collection or not.

For example, the collection of all students of Delhi University is a set, whereas, collection of all good books on mathematics is not a set, since a mathematics book considered good by one person might be considered bad or average by another.

Notations

The sets are usually denoted by capital letters A, B, C, etc. and the members or elements of the set are denoted by lower-case letters a, b, c etc. If x is a member of the set A, we write $x \in A$ (read as 'x belongs to A') and if x is not a member of the set A, we write $x \notin A$ (read as 'x does not belong to A'). If x and y both belong to A, we write $x, y \in A$.

REPRESENTATION OF A SET

Usually, sets are represented in the following two ways:

1. Roster form or tabular form
2. Set builder form or rule method

Roster Form

In this form, we list all the members of the set within braces (curly brackets) and separate these by commas.

For example, the set A of all odd natural numbers less than 10 in the roster form is written as:

$$A = \{1, 3, 5, 7, 9\}$$

IMPORTANT POINTS

- In roster form, every element of the set is listed only once.
- The order in which the elements are listed is immaterial For example, each of the following sets denotes the same set $\{1, 2, 3\}$, $\{3, 2, 1\}$, $\{1, 3, 2\}$.

Set-builder Form

In this form, we write a variable (say x) representing any member of the set followed by a property satisfied by each member of the set.

For example, the set A of all prime numbers less than 10 in the set-builder form is written as

$$A = \{x \mid x \text{ is a prime number less than } 10\}$$

The symbol '|' stands for the words 'such that'. Sometimes, we use the symbol ':' in place of the symbol '|'.

TYPES OF SETS

Empty Set or Null Set

A set which has no element is called the *null set* or *empty set*. It is denoted by the symbol Φ.

For example, each of the following is a null set:

1. The set of all real numbers whose square is –1.
2. The set of all rational numbers whose square is 2.
3. The set of all those integers that are both even and odd.

A set consisting of atleast one element is called a *non-empty set*.

Singleton Set

A set having only one element is called *singleton set*.

For example, {0} is a singleton set, whose only member is 0.

Finite and Infinite Set

A set which has finite number of elements is called a *finite set*. Otherwise, it is called an *infinite set*.

For example, the set of all days in a week is a finite set whereas, the set of all integers, denoted by {…, – 2, – 1, 0, 1, 2, …} or {x | x is an integer}, is an infinite set.

An empty set Φ which has no element, is a finite set.

The number of distinct elements in a finite set A is called the *cardinal number of the set A* and it is denoted by $n(A)$.

Equal Sets

Two sets A and B are said to be *equal*, written as $A = B$, if every element of A is in B and every element of B is in A.

Equivalent Sets

Two finite sets A and B are said to be *equivalent*, if $n(A) = n(B)$.

CAUTION

Equal sets are equivalent but equivalent sets need not be equal.

For example, the sets $A = \{4, 5, 3, 2\}$ and $B = \{1, 6, 8, 9\}$ are equivalent but are not equal.

Subset

Let A and B be two sets. If every element of A is an element of B, then A is called a *subset* of B and we write $A \subseteq B$ or $B \supseteq A$ (read as 'A is contained in B' or B contains A'). B is called *superset* of A.

IMPORTANT POINTS

- If $A \subseteq B$ and $A \neq B$, we write $A \subset B$ or $B \supset A$ (read as : A is a proper subset of B or B is a proper superset of A).
- Every set is a subset and a superset of itself.
- If A is not a subset of B, we write $A \not\subset B$.
- The empty set is the subset of every set.
- If A is a set with $n(A) = m$, then the number of subsets of A are 2^m and the number of proper subsets of A are $2^m – 1$.

For example, let $A = \{3, 4\}$, then the subsets of A are ϕ, {3}, {4}, {3, 4}. Here, $n(A) = 2$ and number of subsets of $A = 2^2 = 4$.

Also, $\{3\} \subset \{3, 4\}$ and $\{2, 3\} \not\subseteq \{3, 4\}$

Power Set

The set of all subsets of a given set A is called the *power set* of A and is denoted by $P(A)$.

For example, if $A = \{1, 2, 3\}$, then

$$P(A) = [\phi, \{1\}, \{2\}, \{3\}, \{1, 2\} \{1, 3\}, \{2, 3\}, \{1, 2, 3\}]$$

Clearly, if A has n elements, then its power set $P(A)$ contains exactly 2^n elements.

TRICK(S) FOR PROBLEM SOLVING

Number of elements in $P\{P[P(\phi)]\}$ is 4

or Cardinal Number of $P\{P[P(\phi)]\} = 4$

Since, $P(\phi) = \{\phi\}$

Also, $P[P(\phi)] = \{\phi, \{\phi\}\}$

and $P\{P[P(\phi)]\} = \{\phi, \{\phi\}, [\{\phi\}], [\phi, \{\phi\}]\}$

Hence, $n[P\{P[P(\phi)]\}] = 4$

Euler–Venn Diagrams

To express the relationship among sets, we represent them pictorially by means of diagrams, known as Euler–Venn Diagrams or simply Venn diagrams.

In Venn diagrams, the universal set U is represented by the rectangular region and its subsets are represented by closed bounded circles inside this rectangular region.

OPERATIONS ON SETS

Union of Two Sets

The union of two sets A and B, written as $A \cup B$ (read as 'A union B'), is the set consisting of all the elements which are either in A or in B or in both. Thus,

$$A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$$

Clearly, $x \in A \cup B$

$\Rightarrow$ $x \in A$

or $x \in B$,

and $x \notin A \cup B$

$\Rightarrow$ $x \notin A$

and $x \notin B$.

For example, if $A = \{a, b, c\ d\}$ and $B = \{c, d, e, f\}$, then $A \cup B = \{a, b, c, d, e, f\}$.

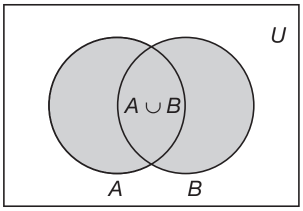


Fig. 1.1

Intersection of Two Sets

The intersection of two sets A and B, written as $A \cap B$ (read as 'A intersection B') is the set consisting of all the common elements of A and B. Thus,

$$A \cap B = \{x: x \in A \text{ and } x \in B\}$$

Clearly, $x \in A \cap B$

$\Rightarrow$ $x \in A$ and $x \in B$,

and $x \notin A \cap B$

$\Rightarrow$ $x \notin A$ or $x \notin B$.

For example, if $A = \{a, b, c, d\}$ and $B = \{c, d, e, f\}$, then $A \cap B = \{c, d\}$.

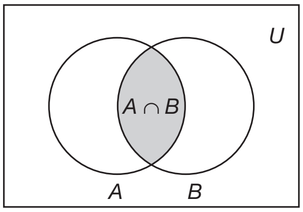


Fig. 1.2

Disjoint Sets

Two sets A and B are said to be *disjoint*, if $A \cap B = \phi$, i.e., A and B have no element in common.

For example, if $A = \{1, 2, 5\}$ and $B = \{2, 4, 6\}$, then $A \cap B = \phi$, so A and B are disjoint sets.

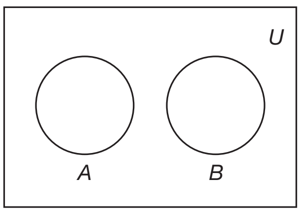


Fig. 1.3

Difference of Two Sets

If A and B are two sets, then their difference $A - B$ is defined as

$$A - B = \{x: x \in A \text{ and } x \notin B\}$$

Similarly,

$$B - A = \{x: x \in B \text{ and } x \notin A\}$$

For example,

if $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$,

then $A - B = \{2, 4\}$ and $B - A = \{7, 9\}$

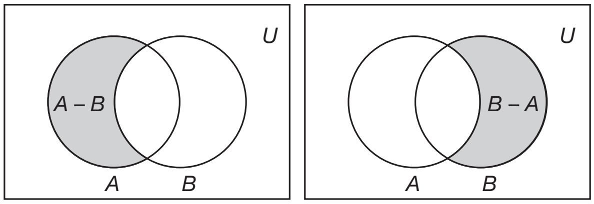


Fig. 1.4(a–b)

NOTE

- $A - B \neq B - A$
- The sets $A - B$, $B - A$ and $A \cap B$ are disjoint sets
- $A - B \subseteq A$ and $B - A \subseteq B$
- $A - \phi = A$ and $A - A = \phi$

Symmetric Difference of Two Sets

The symmetric difference of two sets A and B, denoted by $A \,\Delta\, B$, is defined as

$$A \,\Delta\, B = (A - B) \cup (B - A).$$

For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$ then $A \,\Delta\, B = (A - B) \cup (B - A) = \{2, 4\} \cup \{7, 9\} = \{2, 4, 7, 9\}$.

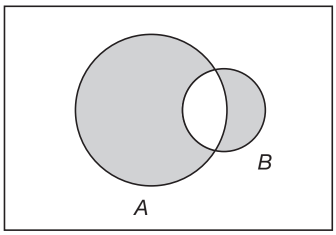


Fig. 1.5

Complement of a Set

If U is a universal set and A is a subset of U, then the complement of A is the set which contains those elements of U, which are not contained in A and is denoted by A' or A^c. Thus,

$$A' = \{x: x \in U \text{ and } x \notin A\}$$

For example,

if $U = \{1, 2, 3, 4, \ldots\}$ and $A = \{2, 4, 6, 8, \ldots\}$,

then, $A' = \{1, 3, 5, 7, \ldots\}$

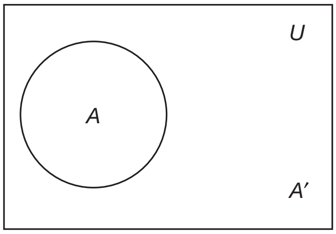


Fig. 1.6

NOTE

- $U' = \phi$
- $\phi' = U$
- $A \cup A' = U$
- $A \cap A' = \phi$

ALGEBRA OF SETS

1. **Idempotent Laws:** For any set A, we have
   (a) $A \cup A = A$
   (b) $A \cap A = A$
2. **Identity Laws:** For any set A, we have:
   (a) $A \cup \phi = A$
   (b) $A \cap \phi = \phi$
   (c) $A \cup U = U$
   (d) $A \cap U = A$
3. **Commutative Laws:** For any two sets A and B, we have
   (a) $A \cup B = B \cup A$
   (b) $A \cap B = B \cap A$
4. **Associative Laws:** For any three sets A, B and C, we have
   (a) $A \cup (B \cup C) = (A \cup B) \cup C$
   (b) $A \cap (B \cap C) = (A \cap B) \cap C$
5. **Distributive Laws:** For any three sets A, B and C, we have
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. For any two sets A and B, we have
   (a) $P(A) \cap P(B) = P(A \cap B)$
   (b) $P(A) \cup P(B) \subseteq P(A \cup B)$, where $P(A)$ is the power set of A.
7. If A is any set, we have $(A')' = A$.
8. **Demorgan's Laws:** For any three sets A, B and C, we have
   (a) $(A \cup B)' = A' \cap B'$
   (b) $(A \cap B)' = A' \cup B'$
   (c) $A - (B \cup C) = (A - B) \cap (A - C)$
   (d) $A - (B \cap C) = (A - B) \cup (A - C)$

Key Results on Operations on Sets

1. $A \subseteq A \cup B$, $B \subseteq A \cup B$, $A \cap B \subseteq A$, $A \cap B \subseteq B$
2. $A - B = A \cap B'$
3. $(A - B) \cup B = A \cup B$
4. $(A - B) \cap B = \phi$
5. $A \subseteq B \Leftrightarrow B' \subseteq A'$
6. $A - B = B' - A'$
7. $(A \cup B) \cap (A \cup B') = A$
8. $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$
9. $A - (A - B) = A \cap B$
10. $A - B = B - A \Leftrightarrow A = B$
11. $A \cup B = A \cap B \Leftrightarrow A = B$
12. $A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)$

Some Results about Cardinal Number

If A, B and C are finite sets and U be the finite universal set, then

1. $n(A') = n(U) - n(A)$
2. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
3. $n(A \cup B) = n(A) + n(B)$,
   where A and B are disjoint non-empty sets
4. $n(A \cap B') = n(A) - n(A \cap B)$
5. $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
6. $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
7. $n(A - B) = n(A) - n(A \cap B)$
8. $n(A \cap B) = n(A \cup B) - n(A \cap B') - n(A' \cap B)$
9. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
10. If $A_1, A_2, A_3, \ldots A_n$ are disjoint sets, then
    $n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = n(A_1) + n(A_2) + n(A_3) + \ldots + n(A_n)$
11. $n(A \,\Delta\, B)$ = number of elements which belong to exactly one of A or B

CARTESIAN PRODUCT OF TWO SETS

If A and B are any two non-empty sets, then *cartesian product* of A and B is defined as

$$A \times B = [(a, b) : a \in A \text{ and } b \in B]$$

CAUTION

$A \times B \neq B \times A$

TRICK(S) FOR PROBLEM SOLVING

- If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.
- If A has n elements and B has m elements then $A \times B$ has mn elements.
- If $A_1, A_2, \ldots, A_p$ are p non-empty sets, then their cartesian product, is defined as $\prod_{i=1}^{p} A_i = [(a_1, a_2, a_3, \ldots, a_p);\ a_i \in A_i$ for all $i]$

Key Results on Cartesian Product

If A, B, C are three sets, then

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. $A \times (B - C) = (A \times B) - (A \times C)$
4. $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$, where S and T are two sets.
5. If $A \subseteq B$, then $(A \times C) \subseteq (B \times C)$
6. If $A \subseteq B$, then $(A \times B) \cap (B \times A) = A^2$
7. If $A \subseteq B$ and $C \subseteq D$ then $A \times C \subseteq B \times D$
8. If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$
9. If A and B are two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.
10. $A \times B = B \times A$ if and only if $A = B$
11. $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
12. $A \times (B' \cap C')\,' = (A \times B) \cup (A \times C)$

SOLVED EXAMPLES

1. If $n(U) = 60$, $n(A) = 35$, $n(B) = 24$ and $n(A \cup B)' = 10$ then $n(A \cap B)$ is

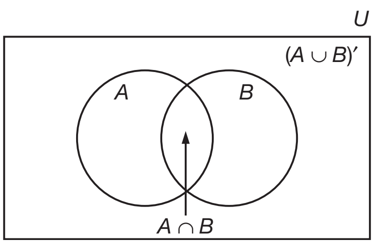
(A) 9 (B) 8
(C) 6 (D) None of these

Solution: (A)

We have,

$$n(A \cup B) = n(U) - n(A \cup B)' = 60 - 10 = 50$$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$



$$\Rightarrow \quad 50 = 35 + 24 - n(A \cap B)$$

$$\Rightarrow \quad n(A \cap B) = 59 - 50 = 9.$$

2. Let $A = \{2, 3, 4\}$ and $X = \{0, 1, 2, 3, 4\}$, then which of the following statements is correct?

(A) $\{0\} \in A'$ in X
(B) $\phi \in A'$ with respect to. X
(C) $\{0\} \subset A'$ with respect to X
(D) $0 \subset A'$ with respect to X.

Solution: (C)

We have, A' in X = The set of elements in X which are not in $A = \{0, 1\}$

$\{0\} \in A'$ in X is false, because $\{0\}$ is not an element of A' in X.

$\phi \subset A'$ in X is false, because ϕ is not an element of A' in X

$\{0\} \subset A'$ in X is correct, because the only element of $\{0\}$ namely 0 also belongs to A' in X.

$0 \subset A'$ in X is false, because 0 is not a set.

3. If $X = \{8^n - 7n - 1 / n \in N\}$ and $Y = \{49(n-1) / n \in N\}$, then

(A) $X \subset Y$ (B) $Y \subset X$
(C) $X = Y$ (D) None of these

Solution: (A)

We have, $8^n - 7n - 1$

$$= (7+1)^n - 7n - 1 = ({}^nC_2 7^2 + {}^nC_3 7^3 + \ldots + {}^nC_n 7^n)$$

$$= 49({}^nC_2 + {}^nC_3 7 + \ldots + {}^nC_n 7^{n-2}) \text{ for } n \geq 2$$

For $n = 1$, $8^n - 7n - 1 = 0$

Thus, $8^n - 7n - 1$ is a multiple of 49 for $n \geq 2$ and 0 for $n = 1$. Hence, X consists of all positive integral multiples of 49 of the form $49\,K_n$. where $K_n = {}^nC_2 + {}^nC_3 7 + \ldots + {}^nC_n 7^{n-2}$ together with zero. Also, Y consists of all positive integral multiples of 49 including zero. Therefore, $X \subset Y$.

4. The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to

(A) $A \cap B$ (B) $A \cap C'$
(C) $B \cap C'$ (D) $B' \cap C'$

Solution: (C)

$$(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$$

$$= (A \cup B \cup C) \cap (A' \cup B \cup C) \cap C'$$

$$= [(A \cap A') \cup (B \cup C)] \cap C'$$

$$= (\phi \cup B \cup C) \cap C' = (B \cup C) \cap C'$$

$$= (B \cap C') \cup (C \cap C')$$

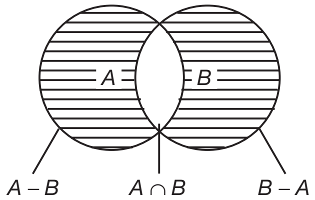
$$= (B \cap C') \cup \phi = B \cap C'$$

5. If A, B and C are non-empty subsets of a set, then $(A - B) \cup (B - A)$ equals

(A) $(A \cap B) \cup (A \cup B)$ (B) $(A \cup B) - (A \cap B)$
(C) $A - (A \cap B)$ (D) $(A \cup B) - B$

Solution: (B)

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$



6. Let A and B two non-empty subsets of a set X such that A is not a subset of B then

(A) A is subset of the complement of B
(C) B is a subset of A
(C) A and B are disjoint
(D) A and the complement of B are non-disjoint

Solution: (D)

Since $A \not\subseteq B$, $\exists\, x \in A$ such that $x \notin B$

Then $x \in B'$.

$\therefore$ $A \cap B' \neq \phi$

7. Two finite sets have m and n elements, then total number of subsets of the first set is 56 more that the total number of subsets of the second. The values of m and n are,

(A) 7, 6 (B) 6, 3 (D) 5, 1 (D) 8, 7

Solution: (B)

Since the two finite sets have m and n elements, so number of subsets of these sets will be 2^m and 2^n respectively. According to the question

$$2^m - 2^n = 56$$

putting $m = 6, n = 3$, we get

$$2^6 - 2^3 = 56 \quad \text{or} \quad 64 - 8 = 56$$

8. Let $U = R$ (the set of all real numbers) If $A = \{x : x \in R, 0 < x < 2\}$, $B = \{x : x \in R, 1 < x \leq 3\}$, then

(A) $A \cup B = \{x : x \in R \text{ and } 0 < x \leq 3\}$
(B) $A \cap B = \{x : x \in R \text{ and } 1 < x < 2\}$
(C) $A - B = \{x : x \in R \text{ and } 0 < x \leq 1\}$
(D) All of these

Solution: (D)

We have

$$A' = R - A = \{x : \in R \text{ and } x \notin A\}$$

$$= \{x : (x \in R \text{ and } x \geq 2) \text{ or } (x \in R \text{ and } x \leq 0)\}$$

$$= \{x : x \in R \text{ and } x \geq 2\} \cup \{x : x \in R \text{ and } x \leq 0\}$$

Similarly,

$B' = \{x : x \in R \text{ and } x \leq 1\} \cup \{x : x \in R \text{ and } x > 3\}$

$\therefore$ $A \cup B = \{x : x \in R \text{ and } 0 < x \leq 3\}$,

$A \cap B = \{x : x \in R \text{ and } 1 < x < 2\}$

$A - B = \{x : x \in R \text{ and } 0 < x \leq 1\}$

9. If $Y \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$, then

(A) The smallest set of Y is $\{3, 5, 9\}$
(B) The smallest set of Y is $\{2, 3, 5, 9\}$
(C) The largest set of Y is $\{1, 2, 3, 4, 9\}$
(D) The largest set of Y is $\{2, 3, 4, 9\}$

Solution: (A and C)

Since the set on the right hand side has 5 elements,

$\therefore$ smallest set of Y has three elements and largest set of Y has five elements,

$\therefore$ smallest set of Y is $\{3, 5, 9\}$

and largest set of Y is $\{1, 2, 3, 4, 9\}$.

10. If A has 3 elements and B has 6 elements, then the minimum number of elements in the set $A \cup B$ is

(A) 6 (B) 3
(C) ϕ (D) None of these

Solution: (A)

Clearly the number of elements in $A \cup B$ will be minimum when $A \subset B$. Hence the minimum number of elements in $A \cup B$ is the same as the number of elements in B, that is, 6.

11. Suppose $A_1, A_2, \ldots A_{30}$ are thirty sets, each with five elements and $B_1, B_2, \ldots, B_n$ are n sets each with three elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$

If each element of S belongs to exactly ten of the $A_i's$ and exactly nine of the $B_j's$ then $n =$

(A) 45 (B) 35
(C) 40 (D) None of these

Solution: (A)

Given A_i's are thirty sets with five elements each, so

$$\sum_{i=1}^{30} n(A_i) = 5 \times 30 = 150 \qquad (1)$$

If there are m distinct elements in S and each element of S belongs to exactly 10 of the A_i's, we have

$$\sum_{i=1}^{30} n(A_i) = 10 \text{ m} \qquad (2)$$

$\therefore$ From Eq. (1) and (2), we get

$$10\,m = 150$$

$$\therefore \quad m = 15 \quad (3)$$

Similarly $\sum_{j=1}^{30} n(B_j) = 3n$ and $\sum_{j=1}^{30} n(B_j) = 9\,m$

$$\therefore \quad 3\,n = 9\,m$$

$$\Rightarrow \quad n = \frac{9m}{3} = 3\,m = 3 \times 15 = 45 \quad \text{[from (3)]}$$

Hence, $n = 45$.

12. If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$, $B = \{2, 4, \ldots, 18\}$ and N is the universal set, then $A' \cup ((A \cup B) \cap B')$ is

(A) A (B) N
(C) B (D) None of these

Solution: (B)

We have,

$$(A \cup B) \cap B' = A$$

$$[(A \cup B) \cap B'] \cup A' = A \cup A' = N.$$

13. If X and Y are two sets and X' denotes the complement of X, then $X \cap (X \cup Y)'$ equals

(A) X (B) Y
(C) ϕ (D) None of these

Solution: (C)

$$X \cap (X \cup Y)' = X \cap (X' \cap Y')$$

[$\because$ By De-Morgan's Law $(A \cup B)' = (A' \cap B')$]

$$= (X \cap X') \cap Y' = \phi \cap Y' = \phi$$

14. In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. The number of persons liking tennis only and not cricket is

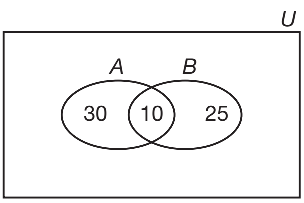
(A) 21 (B) 25
(C) 15 (D) None of these

Solution: (B)

Let A be the set of people who like cricket and B the set of people who like tennis.

Then $n(A \cup B) = 65$

$$n(A) = 40,\ n(A \cap B) = 10$$



$\because$ $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$$65 = 40 + n(B) - 10$$

$$n(B) = 65 - 40 + 10 = 35$$

Number of people who like only tennis

$$= n(B) - n(A \cap B) = 35 - 10 = 25$$

$\therefore$ Number of people who like tennis only and not cricket = 25.

15. In a group of 1000 people, there are 750 people who can speak Hindi and 400 who can speak English. Then number of persons who can speak Hindi only is

(A) 300 (B) 400
(C) 600 (D) None of these

Solution: (C)

Here

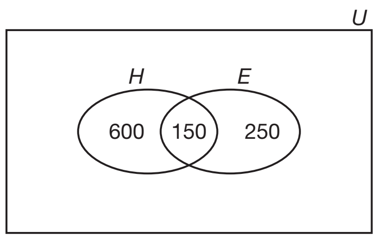
$$n(H \cup E) = 1000,\ n(H) = 750,$$

$$n(E) = 400$$

Using $n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$$1000 = 750 + 400 - n(H \cap E)$$

$$\Rightarrow \quad n(H \cap E) = 1150 - 1000 = 150.$$



Number of people who can speak Hindi only

$$= n(H \cap E') = n(H) - n(H \cap E)$$

$$= 750 - 150 = 600.$$

16. If $f: R \to R$, defined by $f(x) = x^2 + 1$, then the values of $f^{-1}(17)$ and $f^{-1}(-3)$ respectively are

(A) ϕ, $\{4, -4\}$ (B) $\{3, -3\}$, ϕ
(C) ϕ, $\{3, -3\}$ (D) $\{4, -4\}$, ϕ

Solution: (D)

Let $y = x^2 + 1$. Then for $y = 17$,

we have $x = \pm 4$ and for $y = -3$, x becomes imaginary that is, there is no value of x.

Hence, $f(17) = \{-4, 4\}$

and $f^{-1}(-3) = \phi$

17. In a statistical investigation of 1,003 families of Kolkata, it was found that 63 families had neither a radio nor a TV, 794 families had a radio and 187 had a TV. The number of families in that group having both a radio and a TV is

(A) 36 (B) 41
(C) 32 (D) None of these

Solution: (B)

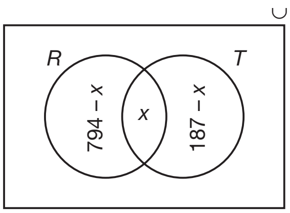
Let R be the set of families having a radio and T, the set of families having a TV, then

$n(R \cup T)$ = The no. of families having at least one of radio and TV

$$= 1003 - 63 = 940$$

$$n(R) = 794 \text{ and } n(T) = 187$$

Let x families had both a radio and a TV i.e.,



$$n(R \cap T) = x$$

The number of families who have only radio $= 794 - x$ and the number of families who have only TV $= 187 - x$

From Venn diagram,

$$794 - x + x - 187 - x = 940$$

$$\Rightarrow \quad 981 - x = 940 \text{ or } x = 981 - 940 = 41$$

Hence, the required no. of families having both a radio and a TV = 41.

18. In a city, three daily newspapers A, B, C are published. 42% of the people in that city read A, 51% read B and 68% read C. 30% read A and B; 28% read B and C; 36% read A and C; 8% do not read any of the three newspapers. The percentage of persons who read all the three papers is

(A) 25% (B) 18%
(C) 20% (D) None of these

Solution: (A)

Let the no. of persons in the city be 100.

Then we have

$$n(A) = 42, n(B) = 51, n(C) = 68;$$

$$n(A \cap B) = 30, n(B \cap C) = 28, n(A \cap C) = 36$$

$$n(A \cup B \cup C) = 100 - 8 = 92$$

Using

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$$
$$- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Substituting the above values, we have

$$92 = 42 + 51 + 68 - 30 - 28 - 36 + n(A \cap B \cap C)$$

$$\Rightarrow \quad n(A \cap B \cap C) = 92 - 161 + 94$$

$$\Rightarrow \quad n(A \cap B \cap C) = 92 - 67 = 25$$

Hence, 25% of the people read all the three papers.

19. Out of 800 boys in a school, 224 played cricket, 240 played hockey and 336 played basketball. Of the total, 64 played both basketball and hockey; 80 played cricket and basketball and 40 played cricket and hockey; 24 played all the three games. The number of boys who did not play any game is

(A) 160 (B) 240 (C) 216 (D) 128

Solution: (A)

$$n(C) = 224, n(H) = 240, n(B) = 336$$

$$n(H \cap B) = 64, n(B \cap C) = 80$$

$$n(H \cap C) = 0, n(C \cap H \cap B) = 24$$

$$n(C^c \cap H^c \cap B^c) = n[(C \cup H \cup B)^c]$$

$$= n(U) - n(C \cup H \cup B)$$

$$= 800 - [n(C) + n(H) + n(B) - n(H \cap C)$$
$$- n(H \cap B) - n(C \cap B) + n(C \cap H \cap B)]$$

$$= 800 - [224 + 240 + 336 - 64 - 80 - 40 + 24]$$

$$= 800 - [824 - 184] = 984 - 824 = 160.$$

20. In a certain town 25% families own a phone and 15% own a car, 65% families own neither a phone nor a car. 2000 families own both a car and a phone. Consider the following statements in this regard:

1. 10% families own both a car and a phone.
2. 35% families own either a car or a phone.
3. 40,000 families live in the town.

Which of the above statements are correct?

(A) 1 and 2 (B) 1 and 3
(C) 2 and 3 (D) 1, 2 and 3

Solution: (C)

$$n(P) = 25\%, n(C) = 15\%,$$

$$n(P' \cap C') = 65\%, n(P \cap C) = 2000$$

Since, $n(P' \cap C') = 65\%$

$\therefore \quad n(P \cup C)' = 65\%$

$\therefore \quad n(P \cup C) = 35\%$

Now, $\quad n(P \cup C) = n(P) + n(C) - n(P \cap C)$

$\therefore \quad 35 = 25 + 15 - n(P \cap C)$

$\therefore \quad n(P \cap C) = 40 - 35 = 5$

Thus, $\quad n(P \cap C) = 5\%$

But, $\quad n(P \cap C) = 2000$

$\therefore$ 5% of the total = 2000

$\therefore$ total no. of families $= \dfrac{2000 \times 100}{5} = 40000$

$\therefore \quad n(P \cup C) = 35\%,$

Total no. of families = 40,000 and $n(P \cap C) = 5\%$.

21. If P, Q and R are subsets of a set A, then $R \times (P' \cup Q')'$ equals

(A) $(R \times P) \cap (R \times Q)$ (B) $(R \times Q) \cap (R \times P)$
(C) $(R \times P) \cup (R \times Q)$ (D) None of these

Solution: (A)

$$R \times (P' \cup Q')' = R \times [(P')' \cap (Q')']$$
$$= R \times (P \cap Q)$$
$$= (R \times P) \cap (R \times Q)$$

22. If sets A and B are defined as
$A = \{(x, y) : y = e^x, x \in R\}$
$B = \{(x, y) : y = x, x \in R\}$ then

(A) $B \subset A$ (B) $A \subset B$
(C) $A \cap B = \phi$ (D) $A \cup B = A$

Solution: (C)

Since $\quad y = e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

$\therefore e^x > x \; \forall \; x \in R$ so that the two curves given by $y = e^x$ and $y = x$ do not intersect in any point Hence, there is no common point, so that $A \cap B = \phi$.

23. The solution of $3x^2 - 12x = 0$ when

(A) $x \in N$ is $\{4\}$
(B) $x \in I$ is $\{0, 4\}$
(C) $x \in S = \{a + ib : b \neq 0, a, b \in R\}$ is ϕ
(D) All of these

Solution: (D)

We have,

$$3x^2 - 12x = 0$$

$$\Rightarrow \quad 3x(x - 4) = 0$$

$$\Rightarrow \quad x = 0, 4$$

Now, if $x \in N$, then the solution set is $\{4\}$.

Also, if $x \in I$, then the solution set is $\{0, 4\}$.

Further, since there is no root of the form $a + ib$, where a, b are real and $b \neq 0$,

$\therefore$ if $x \in S = \{a + ib : b \neq 0, a, b \in R\}$ then the solution set is ϕ.

RELATIONS

Let A, B be any two non-empty sets, then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.

If R is a relation from A to B and if $(a, b) \in R$, then we write $a \, R \, b$ and say that 'a is related to b' and if $(a, b) \notin R$, then we write $a \not R \, b$ and say that a is not related to b.

Key Results on Relations

1. Every subset of $A \times A$ is said to be a relation on A.
2. If A has m elements and B has n elements, then $A \times B$ has mn elements and total number of different relations from A to B is 2^{mn}.
3. Let R be a relation from A to B, *i.e.* $R \subseteq A \times B$, then
   Domain of $R = \{a : a \in A, (a, b) \in R$ for some $b \in B\}$
   Range of $R = \{b : b \in B, (a, b) \in R$ for some $a \in A\}$
   For example, let $A = \{1, 3, 4, 5, 7\}$, $B = \{2, 4, 6, 8\}$ and R be the relation 'is one less than' from A to B, then $R = [(1, 2), (3, 4), (5, 6), (7, 8)]$.
   Here, domain of $R = \{1, 3, 5, 7\}$ and range of $R = \{2, 4, 6, 8\}$.

NOTE

Domain of a relation from A to B is a subset of A and its range is a subset of B

Identity Relation

R is an *identity relation* if $(a, b) \in R$ if $a = b$, $a \in A$, $b \in A$. In other words, every element of A is related to only itself.

Universal Relation

Let A be any set and R be the set $A \times A$, then R is called the *universal relation* in A.

Void Relation

ϕ is called void relation in a set.

TRICK(S) FOR PROBLEM SOLVING

The void and the universal relations on a set A are respectively the smallest and the largest relations on A

Inverse Relation

Let $R \subseteq A \times B$ be a relation from A to B. Then $R^{-1} \subseteq B \times A$ is defined by

$$R^{-1} = [(b, a): (a, b) \in R]$$

Thus,

$$(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}, \forall a \in A, b \in B$$

IMPORTANT POINTS

- dom (R^{-1}) = range (R) and range (R^{-1}) = dom (R)
- $(R^{-1})^{-1} = R$. For example, if $R = [(1, 2), (3, 4), (5, 6)]$
  then $R^{-1} = [(2, 1), (4, 3), (6, 5)]$
  and $(R^{-1})^{-1} = [(1, 2), (3, 4), (5, 6)] = R$.
  dom (R) = (1, 3, 5), range (R) = (2, 4, 6)
  and dom (R^{-1}) = (2, 4, 6), range (R^{-1}) = (1, 3, 5)
  So, dom (R^{-1}) = range (R) and range (R^{-1}) = dom (R)

TYPES OF RELATIONS ON A SET

Let A be a non-empty set, then a relation R on A is said to be:

1. **Reflexive:** If $a R a, \forall a \in A$, i.e., if $(a, a) \in R, \forall a \in A$
2. **Symmetric:** If $a R b \Rightarrow b R a, \forall a, b \in A$, i.e., if $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$
3. **Anti-symmetric:** If $a R b$ and $b R a \Rightarrow a = b, \forall a, b \in A$
4. **Transitive:** If $a R b$ and $b R c \Rightarrow a R c, \forall a, b, c \in A$ i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$

EQUIVALENCE RELATION

A relation R on a non-empty set A is called an *equivalence relation* if and only if it is

1. reflexive
2. symmetric and
3. transitive

That is, R satisfies following properties:

1. $a R a, \forall a \in A$
2. $a R b \Rightarrow b R a, \forall a, b \in A$
3. $a R b, b R c \Rightarrow a R c, \forall a, b, c \in A$

For example, let I be the set of all integers, m be a positive integer. Then the relation, R on I is defined by,

$$R = [(x, y): x, y \in I, x - y \text{ is divisible by } m]$$

Consider any $x, y, z \in I$.

1. Since $x - x = 0 = 0 \cdot m \Rightarrow x - x$ is divisible by m
   $\Rightarrow (x, x) \in R \Rightarrow R$ is reflexive.
2. Let $(x, y) \in R \Rightarrow x - y$ is divisible by m
   $\Rightarrow x - y = mq$, for some $q \in I$
   $\Rightarrow y - x = m(-q)$
   $\Rightarrow y - x$ is divisible by m
   $\Rightarrow (y, x) \in R$
   Thus, $(x, y) \in R \Rightarrow (y, x) \in R \Rightarrow R$ is symmetric.
3. Let $(x, y) \in R$ and $(y, z) \in R$
   $\Rightarrow x - y$ is divisible by m and $y - z$ is divisible by m
   $\Rightarrow x - y = mq$ and $y - z = mq'$ for some $q, q' \in I$
   $\Rightarrow (x - y) + (y - z) = m(q + q')$
   $\Rightarrow x - z = m(q + q'), q + q' \in I$
   $\Rightarrow (x, z) \in R$
   Thus, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$, so R is transitive.

Hence the relation R is reflexive, symmetric and transitive and therefore it is an equivalence relation.

CAUTION

Every identity relation is reflexive but every reflexive relation need not be an identity relation. Also, identity relation is reflexive, symmetric and transitive.

CONGRUENCE MODULO *m*

Let m be a positive integer and $x, y \in I$, then x is said to be congruent to y modulo m, written as $x \equiv y \pmod{m}$, if $x - y$ is divisible by m.

For example, $155 \equiv 7 \pmod{4}$ as

$$\frac{155 - 7}{4} = \frac{148}{4} = 37 \text{ (integer)}$$

But $27 \not\equiv 5 \pmod{4}$ as $\frac{27 - 5}{4} = \frac{22}{4}$ (Not an integer)

IMPORTANT POINTS

- The universal relation on a non-void set is reflexive
- The identity and the universal relations on a non-void set are symmetric and transitive
- The identity relation on a set is an anti-symmetric relation
- The relation R on a set A is symmetric if and only if $R = R^{-1}$
- If R and S are two equalvalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.
- The union of two equivalence equivalence relations on a set is not necessarily an equivalence realtion on the set.
- The inverse of an equivalence relation is an equivalence relation.

CAUTION

A reflexive relation on a set is not necessarily symmetric.

Equivalence Classes of an Equivalence Relation

Let R be an equivalence relation in A ($\neq \phi$), Let $a \in A$. Then the equivalence class of a, denoted by $[a]$ or $\{ \overline{a} \}$ is defined as the set of all those points of A which are related to a under the relation R. Thus, $[a] = \{x \in A : x\, Ra\}$

It is easy to see that

1. $b \in [a] \Rightarrow a \in [b]$
2. Two equivalence classes are either disjoint or identical

SOLVED EXAMPLES

24. Let n be a fixed positive integer. Let a relation R be defined on I (the set of all integers) as follows: $a\, R\, b$ if $n/(a - b)$, that is, if $a - b$ is divisible by n, then, the relation R is

(A) reflexive only
(B) symmetric only
(C) transitive only
(D) an equivalence relation

Solution: (D)

R is reflexive since for any integer a we have $a - a = 0$ and 0 is divisible by n. Hence, $a\, R\, a\ \forall\ a \in I$.

R is symmetric, let $a\, R\, b$. Then by definition of R, $a - b = nk$ where $k \in I$. Hence, $b - a = (-k)\, n$ where $-k \in I$ and so $b\, R\, a$. Thus we have shown that

$$a\, R\, b \quad \Rightarrow \quad b\, R\, a$$

R is transitive, let $a\, R\, b$ and $b\, R\, c$. Then by definition of R, we have $a - b = k_1 n$ and $b - c = k_2 n$, where k_1, $k_2 \in I$. It then follows that

$$a - c = (a - b) + (b - c) = k_1 n + k_2 n = (k_1 + k_2)n$$

where $k_1 + k_2 \in I$

25. Let R be a relation defined as $a\, R\, b$ if $| a - b | > 0$. Then, the relation R is

(A) Reflexive (B) Symmetric
(C) Transitive (D) None of these

Solution: (B)

R is not reflexive since $| a - a | = 0$ and so $| a - a |\ 0$.

Thus $a \not R\, a$ for any real number a.

R is symmetric since if $| a - b | > 0$, then

$$| b - a | = | a - b | > 0.$$

Thus $a\, R\, b \Rightarrow b$ R a

R is not transitive. For example,

consider the numbers 3, 7, 3. Then we have 3 R 7 since $| 3 - 7 | = 4 > 0$ and 7 R 3 since $| 7 - 3 | = 4 > 0$.

But $3 \not R\ 3$ since $| 3 - 3 | = 0$ so that $| 3 - 3 | \ngtr 0$.

26. Let R be a relation defined as $a\, R\, b$ if $1 + ab > 0$. Then, the relation R is

(A) Reflexive (B) Symmetric
(C) Transitive (D) None of these

Solution: (A) and (B)

Here relation R is reflexive since $1 + a \times a > 0\ \forall$ real numbers a. It is symmetric since $1 + ab > 0 \Rightarrow 1 + ba > 0$. However R is not transitive: consider three real numbers $2, -\dfrac{1}{6}$ and -2. We have

$$1 + 2 \times \left(-\frac{1}{6}\right) = \frac{2}{3} > 0$$

and $$1 + \left(-\frac{1}{6}\right)(-2) = \frac{4}{3} > 0$$

Hence, $$2\, R \left(-\frac{1}{6}\right) \text{ and } \left(-\frac{1}{6}\right) R\, (-2)$$

But $2 \not R -2$ since $1 + 2\,(-2) = -3 \ngtr 0$

27. Let R be a relation defined as $a\, R\, b$ if $| a | \leq b$. Then, relation R is

(A) Reflexive (B) Symmetric
(C) Transitive (D) None of these

Solution: (C)

R is not reflexive, if $-a$ is any negative real number, then $| -a | > -a$ so that $-a \not R -a$. R is not symmetric consider the real numbers $a = -2$ and $b = 3$. Then $a\, R\, b$ since $|-2| < 3$. But $b \not R\, a$ since $| 3 | > -2$.

R is transitive: let a, b, c be three real numbers such that

$$| a | \leq b \text{ and } | b | \leq c.$$

But $| a | \leq b \Rightarrow b \geq 0$, and so $| b | \leq c \Rightarrow b \leq c$.

It follows $| a | \leq c$.

Thus $a\, R\, b$ and $b\, R\, c \Rightarrow a\, R\, c$.

28. N is the set of natural numbers. The relation R is defined on $N \times N$ as follows $(a, b)\, R\, (c, d) \Leftrightarrow a + d = b + c$. Then, R is

(A) Reflexive only (B) Symmetric only
(C) Transitive only (D) an equivalence relation

Solution: (D)

We have, $(a, b)\ R\ (a, b)$ for all $(a, b) \in N \times N$ since $a + b = b + a$.

Hence, R is reflexive.

R is symmetric: we have

$$(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \Rightarrow d + a = c + b$$

$$\Rightarrow \quad c + b = d + a \Rightarrow (c, d)\ R\ (a, b)$$

R is transitive: let

$$(a, b)\ R\ (c, d) \text{ and } (c, d)\ R\ (e, f).$$

Then by definition of R, we have

$$a + d = b + c \text{ and } c + f = d + e,$$

$$\Rightarrow \quad a + d + c + f = b + c + d + e$$

or $\quad a + f = b + e.$

Hence, $\quad (a, b)\ R\ (e, f)$

Thus, $\quad (a, b)\ R\ (c, d) \quad \text{and} \quad (c, d)\ R\ (e, f)$

$$\Rightarrow \quad (a, b)\ R\ (e, f)$$

29. Let $S = \{1, 2, 3, 4, 5\}$ and let $A = S \times S$. Define the relation R on A as follows $(a, b)\ R\ (c, d)$ if and only if $ad = cb$. Then, R is

(A) reflexive only
(B) symmetric only
(C) transitive only
(D) equivalence relation

Solution: (D)

Given that $S = \{1, 2, 3, 4, 5\}$ and $A = S \times S$

A relation R on A is defined as follows: "$(a, b)\ R\ (c, d)$" if and only if $ad = cb$

(A) R is reflexive, since $ab = ba$
$\Rightarrow \quad ba = ab$, therefore,
$(a, b)\ R\ (b, a)\ \forall\ a, b \in S$

(B) R is symmetric,
since $(a, b)\ R\ (c, d)$
$\Rightarrow \quad ad = cb \Rightarrow cd = da$
$\Rightarrow \quad (c, d)\ R\ (a, b)\ \forall\ a, b \in S$

(C) R is transitive $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$
$\Rightarrow \quad ad = cb$ and $cf = ed \Rightarrow adcf = cb\ ed$
$\Rightarrow \quad cd\ (af) = cd\ (be) \Rightarrow af = eb$
$\Rightarrow \quad (a, b)\ R\ (e, f)\ \forall\ a, b, c, d, e, f \in S$

30. The relation 'less than' in the set of natural numbers is

(A) only symmetric
(B) only transitive
(C) only reflexive
(D) equivalence relation

Solution: (B)

The relation 'less than' is only transitive because

$$x < y, y < z \Rightarrow x < z, x, y, z \in N$$

$$\therefore \quad x\ R\ y, y\ R\ z \Rightarrow x\ R\ z$$

31. If R and R' are symmetric relations (not disjoint) on a set A, then the relation $R \cap R'$ is

(A) reflexive (B) symmetric
(C) transitive (D) None of these

Solution: (B)

Since $R \cap R'$ are not disjoint, there is at least one ordered pair, say, (a, b) in $R \cap R'$.

But $(a, b) \in R \cap R' \Rightarrow (a, b) \in R$ and $(a, b) \in R'$ since R and R' are symmetric relations, we get

$$(b, a) \in R \quad \text{and} \quad (b, a) \in R'$$

and consequently $(b, a) \in R \cap R'$

similarly if any other ordered pair $(c, d) \in R \cap R'$, then we must also have, $(d, c) \in R \cap R'$

Hence, $R \cap R'$ is symmetric

32. With reference to a universal set, the inclusion of a subset in another, is relation which is

(A) symmetric only (B) equivalence
(C) reflexive only (D) None of these

Solution: (C)

Let the universal set be

$$U = \{x_1, x_2, x_3 \ldots x_n\}$$

We know every set is a subset of itself. Therefore, inclusion of a subset is reflexive Now the elements of the set $\{x_1\}$ are included in the set $\{x_1, x_2\}$ but converse is not true i.e.,

$$\{x_1\} \subset \{x_1, x_2\} \text{ but } \{x_1, x_2\} \not\subset \{x_1\}$$

Hence, the inclusion of a subset is not symmetric.

Thus, the inclusion of a subset is not an equivalence relation.

33. Let R be the relation on the set R of all real numbers defined by $a\ R\ b$ if $|a - b| \leq 1$. Then R is

(A) reflexive (B) symmetric
(C) transitive (D) anti-symmetric

Solution: (A and B)

$$|a - a| = 0 < 1 \text{ so } a\ R\ a\ \forall\ a \in R.$$

$\therefore \quad R$ is reflexive

$$a\ R\ b \Rightarrow |a - b| \leq 1 \Rightarrow |b - a| \leq 1 \Rightarrow b\ R\ a.$$

$\therefore \quad R$ is symmetric

$2\,R\,3/2$, $3/2\,R\,2$ but $2 \neq 3/2$ so R is not anti-symmetric. Finally, $1\,R2$, $2\,R\,3$ but $1 \not R\ 3$ as

$$|1-3| = 2 > 1.$$

34. If R is a relation '<' from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$ i.e., $(a, b) \in R$ if $a < b$, then $R \text{ o } R^{-1}$ is

(A) $[(1, 2), (1, 5), (2, 3) (2, 5) (3, 5), (4, 5)]$
(B) $[(3, 1), (5, 1), (5, 2), (5, 3), (5, 4)]$
(C) $[(3, 3), (3, 5), (5, 3), (5, 5)]$
(D) $[(3, 3), (3, 4), (4, 5)]$

Solution: (C)

Here

$$R = [(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)]$$

$$\therefore \quad R^{-1} = [(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)]$$

Hence,

$$R\,0\,R^{-1} = [(3, 3), (3, 5), (5, 3), (5, 5)]$$

35. A relation R on the set of complex numbers is defined by, $z_1\,R\,z_2 \Leftrightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real, then the relation R is

(A) equivalence
(B) only reflexive
(C) only transitive
(D) None of these

Solution: (A)

Since $\dfrac{z_1 - z_1}{z_1 + z_1} = 0$, which is real $\forall\ z_1 \in C$, therefore R is reflexive.

For $\quad z_1, z_2 \in C, z_1\,R\,z_2 \Rightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

$\Rightarrow \quad -\left(\dfrac{z_1 - z_2}{z_1 + z_2}\right)$ is real $\Rightarrow \left(\dfrac{z_2 - z_1}{z_2 + z_1}\right)$ is real

$\Rightarrow \quad z_2\,R\,z_1.$

For

$$z_1, z_2, z_3 \in C,$$

let $\quad z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$ and $z_3 = a_3 + ib_3$.

Now, $\quad z_1\,R\,z_2 \Rightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

$$\Rightarrow \frac{(a_1 - a_2) + i(b_1 - b_2)}{(a_1 + a_2) + i(b_1 + b_2)} \text{ is real}$$

$$\Rightarrow \frac{(a_1 - a_2) + i(b_1 - b_2)}{(a_1 + a_2) + i(b_1 + b_2)} \times \frac{(a_1 + a_2) - i(b_1 + b_2)}{(a_1 + a_2) - i(b_1 + b_2)} \text{ is real}$$

$$\Rightarrow \frac{(a_1 - a_2)(a_1 + a_2) + (b_1 - b_2)(b_1 + b_2) + i[(b_1 - b_2)(a_1 + a_2) - (b_1 + b_2)(a_1 - a_2)]}{(a_1 + a_2)^2 + (b_1 + b_2)^2} \text{ is real}$$

$$\Rightarrow (a_1 + a_2)(b_1 - b_2) - (a_1 - a_2)(b_1 + b_2) = 0$$

$$\Rightarrow 2a_2b_1 - 2b_2a_1 = 0$$

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \text{ or } \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

and $\quad \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

Similarly,

$$z_2\,R\,z_3 \Rightarrow \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Therefore,

$$z_1\,R\,z_2 \text{ and } z_2\,R\,z_3$$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

and $\quad \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_3}{b_3}$$

$$\Rightarrow z_1\,R\,z_3$$

$\therefore \quad R$ is transitive.
Hence R is an equivalence relation.

EXERCISES

Single Option Correct Type

1. Let F_1 be the set of all parallelograms, F_2 the set of rectangles, F_3 the set of rhombuses, F_4 the set of squares and F_5 the set of trapeziums in a plane then F_1 is equal to

(A) $F_2 \cap F_3$
(B) $F_2 \cup F_3 \cup F_4 \cup F_1$
(C) $F_3 \cap F_4$
(D) None of these

2. (*i*) Let R be the relation on the set R of all real numbers defined by setting $a\,R\,b$ if $|a - b| \leq \dfrac{1}{2}$. Then R is

(A) Reflexive and symmetric but not transitive
(B) Symmetric and transitive but not reflexive
(C) Transitive but neither reflexive nor symmetric
(D) None of these

3. n/m means that n is a factor of m, then the relation '/' is
(A) reflexive and symmetric.
(B) transitive and reflexive.
(C) reflexive, transitive and symmetric.
(D) reflexive, transitive and not symmetric.

4. Set A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$?
(A) 18 (B) 9 (C) 6 (D) 3

5. Let R be a relation defined on the set of natural numbers N as
$R = [(x, y): x \in N, y \in N, 2x + y = 41]$. Then
(A) Domain of $R = \{1, 2, 3, \ldots, 19, 20\}$
(B) Range of $R = \{39, 37, 35, 9, 7, 5, 3, 1\}$
(C) R is reflexive
(D) R is symmetric

6. Let $A = \{x: x \in R, |x| < 1\}$
$B = \{x: x \in R, |x - 1| \geq 1\}$
and $A \cup B = R - D$, then the set D is
(A) $\{x: 1 < x \leq 2\}$ (B) $\{x: 1 \leq x < 2\}$
(C) $\{x: 1 \leq x \leq 2\}$ (D) None of these

7. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be subset of A consisting of all determinants with value 1. Let C be the subset A of consisting of all determinants with value –1. Then
(A) C is empty.
(B) B has as many elements as C.
(C) $A = B \cup C$.
(D) B has twice as many elements as C.

8. Let A and B be two sets then $(A \cup B)' \cup (A' \cap B)$ is equal to
(A) B' (B) B (C) A (D) A'

9. If A is the set of even natural numbers less than 8 and B is the set of prime numbers less then 7, then the number of relations from A to B is
(A) 2^9 (b) 9^2 (C) 3^2 (D) $2^9 - 1$

10. If P, Q and R are subsets of a set A, then $R \times (P' \cup Q')'$ equals
(A) $(R \times P) \cap (R \times Q)$ (B) $(R \times Q) \cap (R \times P)$
(C) $(R \times P) \cup (R \times Q)$ (D) None of these

11. For real numbers x and y, define a relation R, $x R y$ if and only if $x - y + \sqrt{2}$ is an irrational number. Then the relation R is
(A) reflexive
(B) symmetric
(C) transitive
(D) an equivalence relagtion

12. If A = $\{ x : x^2 = 1\}$ and B $\{ x : x^4 = 1\}$, then $A \Delta B$ is equal to:
(A) $\{i, -i\}$ (B) $\{-1, 1\}$
(C) $\{-1, 1, i, -i\}$ (D) None of these

13. Which of the following is a singleton set?
(A) $\{x : |x| < 1, x \in Z\}$
(B) $\{x : |x| = 5, x \in Z\}$
(C) $\{x : x^2 = 1, x \in Z\}$
(D) $\{x : x^2 + x + 1 = 0, x \in R\}$

14. Consider the following relations:
(1) $A - B = A - (A \cap B)$
(2) $A = (A \cap B) \cup (A - B)$
(3) $A - (B \cup C) = (A - B) \cup (A - C)$
Which of these is/are correct?
(A) 1 and 3 (B) 2 only
(C) 2 and 3 (D) 1 and 2

15. Let R be a reflexive relation on a finite set A having n elements, and let there be m ordered pairs in R. Then
(A) $m \geq n$ (B) $m \leq n$
(C) $m = n$ (D) None of these

16. If two sets A and B are having 99 elements in common then the number of elements common to each of the sets $A \times B$ and $B \times A$ are
(A) 2^{99} (B) 99^2 (C) 100
(D) 18 (E) 9

17. The relation R defined on the set $A = [1, 2, 3, 4, 5]$ by $R = [(x, y): |x^2 - y^2| < 16]$ is given by,
(A) [(1, 1), (2, 1) (3, 1), (4, 1), (2, 3)]
(B) [(2, 2), (3, 2) (4, 2), (2, 4)]
(C) [(3, 3), (3, 4) (5, 4), (4, 3), (3, 1)]
(D) None of these

18. Let L denotes the set of all straight lines in a plane. Let a relation R be defined by $\alpha R \beta \Leftrightarrow \alpha \perp \beta$, $\alpha, \beta \in L$. Then R is
(A) reflexive
(B) symmetric
(C) transitive
(D) None of these

19. Let R = [(2,3), (3,4)] be a relation defined on the set of natural numbers. The minimum number of ordered pairs required to be added in R so that enlarged relation be comes an equivalence relation is
(A) 3 (B) 5 (C) 7 (D) 9

20. The solution of $8x \equiv 6 \pmod{14}$ is
(A) [8], [6] (B) [6], [14]
(C) [6], [13] (D) [8], [14], [16]

21. Let R be the set of real numbers.

Statement 1: $A = \{(x, y) \in R \times R : y - x$ is an integer$\}$ is an equivalence relation of R.

Statement 2: $B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation of R.

(A) Statement 1 is false, Statement 2 is true
(B) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1
(C) Statement 1 is true, Statement 2 is true; Statement 2 is **not** a correct explanation for Statement 1
(D) Statement 1 is true, Statement 2 is false

22. Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can formed such that $Y \subseteq X$, $Z \subseteq X$ and $Y \cap Z$ is empty, is

(A) 5^2 (B) 3^5 (C) 2^5 (D) 5^3

23. Let R be the real line. Consider the following subsets of the plane $R \times R$.

$S = [(x, y) : y = x + 1$ and $0 < x < 2]$, $T = [(x, y): x - y$ is an integer$]$. Which one of the following is true?

(A) Neither S nor T is an equivalence relation on R
(B) Both S and T are equivalence relations on R
(C) S is an equivalence relation on R but T is not
(D) T is an equivalence relation on R but S is not

Previous Year's Questions

24. Let $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$. The relation R is **[2004]**

(A) a function (B) reflexive
(C) not symmetric (D) transitive

25. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$ be a relation the set $A = \{3, 6, 9, 12\}$. The relation is **[2005]**

(A) reflexive and transitive only
(B) reflexive only
(C) an equivalence relation
(D) reflexive and symmetric only

26. Let W denote the words in the English dictionary. Define the relation R by: **[2006]**

$R = \{(x, y) \in W \times W \mid$ the words x and y have at least one letter in common$\}$. Then R is

(A) not reflexive, symmetric and transitive
(B) reflexive, symmetric and not transitive
(C) reflexive, symmetric and transitive
(D) reflexive, not symmetric and transitive

27. The set $S = \{1, 2, 3, \ldots, 12)$ is to be partitioned into three sets A, B, C of equal size. Thus, $A \cup B \cup C = S$, $A \cap B = B \cap C = A \cap C = \phi$. The number of ways to partition S is **[2007]**

(A) $\dfrac{12!}{3!(4!)^3}$ (B) $\dfrac{12!}{3!(3!)^4}$

(C) $\dfrac{12!}{(4!)^3}$ (D) $\dfrac{12!}{(3!)^4}$

28. Let R be the real line. Consider the following subsets of the plane $R \times R$.

$S = \{(x, y) : y = x + 1$ and $0 < x < 2\}$, $T = \{(x, y) : x - y$ is an integer$\}$. Which one of the following is true? **[2008]**

(A) neither S nor T is an equivalence relation on R
(B) both S and T are equivalence relations on R
(C) S is an equivalence relation on R but T is not
(D) T is an equivalence relation on R but S is not

29. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then **[2009]**

(A) $A = B$ (B) $A = C$
(C) $B = C$ (D) $A \cap B = \phi$

30. Let S be a non-empty subset of R. Consider the following statement: **[2010]**

P: There is a rational number $x \in S$ such that $x > 0$.

Which of the following statements is the negation of the statement P ?

(A) There is no rational number $x \in S$ such that $x \leq 0$
(B) Every rational number $x \in S$ satisfies $x \leq 0$
(C) $x \in S$ and $x \leq 0 \Rightarrow x$ is not rational
(D) There is a rational number $x \in S$ such that $x \leq 0$

31. Let R be the set of all real numbers. **[2011]**

Statement 1: $A = \{(x, y) \in R \times R : y - x$ is an integer$\}$ is an equivalence relation on R.

Statement 2: $B = \{(x, y) \in R \times R : x = \alpha y$ for some rational number $\alpha\}$ is an equivalence relation on R.

(A) Statement 1 is true, Statement 2 is true; Statement 2 is **not** a correct explanation for Statement 1
(B) Statement 1 is true, Statement 2 is false.
(C) Statement 1 is false, Statement 2 is true.
(D) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1

32. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is **[2013]**

(A) 220 (B) 219
(C) 211 (D) 256

33. Let A and B be two sets containing four and two elements respectively. Then the number of subsets of the set $A \times B$, each having at least three elements is: **[2015]**

(A) 256 (B) 275 (C) 510 (D) 219

ANSWER KEYS

Single Option Correct Type

1. (B) **2.** (A) **3.** (B) **4.** (C) **5.** (A and B) **6.** (B) **7.** (B) **8.** (D) **9.** (A)
10. (A) **11.** (A) **12.** (A) **13.** (A) **14.** (D) **15.** (A) **16.** (B) **17.** (D) **18.** (B) **19.** (C)
20. (C) **21.** (D) **22.** (B) **23.** (D)

Previous Year's Questions

24. (C) **25.** (A) **26.** (B) **27.** (C) **28.** (D) **29.** (C) **30.** (B) **31.** (B) **32.** (B) **33.** (D)

HINTS AND SOLUTIONS

Single Option Correct Type

1. Since every rectangle, rhombus and square is a parallelogram so, $F_1 = F_2 \cup F_3 \cup F_4 \cup F_1$

The correct option is (B)

2. R is reflexive since $|a - a| = 0 < \frac{1}{2}$ for all $a \in R$.

R is symmetric since $|a - b| < \frac{1}{2} \Rightarrow |b - a| < \frac{1}{2}$

R is not transitive: we take three numbers

$\frac{3}{4}, \frac{1}{3}, \frac{1}{8}$, then

$$\left|\frac{3}{4} - \frac{1}{3}\right| = \frac{5}{12} < \frac{1}{2} \text{ and } \left|\frac{1}{3} - \frac{1}{8}\right| = \frac{5}{24} < \frac{1}{2}$$

But $\left|\frac{3}{4} - \frac{1}{8}\right| = \frac{5}{8} > \frac{1}{2}$

Thus $\frac{3}{4} R \frac{1}{3}$ and $\frac{1}{3} R \frac{1}{8}$ but $\frac{3}{4} \not{R} \frac{1}{8}$

The correct option is (A)

3. '/' is reflexive since every natural number is a factor of itself, that is n/n for $n \in N$. '/' is transitive: if n is a factor of m and m is a factor of p, then n is surely a factor of p. Thus 'n/m' and 'm/p' $\Rightarrow$ 'n/p'. However '/' is not symmetric: for example, 2 is factor of 4 but 4 is not a factor of 2.

The correct option is (B)

4. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 3 + 6 - n(A \cap B)$

Since maximum number of elements in $A \cap B = 3$,

$\therefore$ minimum no. of elements in $A \cup B = 9 - 3 = 6$.

The correct option is (C)

5. The relation R can be written as

$R = [(1, 39), (2, 37), (3, 35), \ldots(10, 21), (11, 19), \ldots (19, 3), (20, 1)]$

$\therefore$ Domain of $R = \{1, 2, 3, \ldots, 19, 20\}$

Range of $R = \{39, 37, 35, 9, 7, 5, 3, 1\}$

R is not reflexive since $(x, x) \notin R \ \forall \ x \in N$.

For example, $(1, 1) \notin R$

R is not symmetric since $(1, 39) \in R$ but $(39, 1) \notin R$.

The correct option is (A and B)

6. We have

$A = \{x : x \in R, -1 < x < 1\}$

and $B = \{x : x \in R, x - 1 \leq -1 \text{ or } x - 1 \geq 1\}$

$= \{x : x \in R, x \leq 0 \text{ or } x \geq 2\}$

$\therefore$ $A \cup B = R - D$,

where $D = \{x : x \in R, 1 \leq x < 2\}$

The correct option is (B)

7. We know that the interchange of two adjacent rows (or columns) changes the value of a determinant only in sign and not in magnitude. Hence corresponding to every element Δ of B there is an element Δ' in C obtained by interchanging two adjacent rows (or columns) in Δ. It follows that

$n(B) \leq n(C)$

that is, the number of elements in B is less than or equal to the number of elements in C.

Similarly, $n(C) \leq n(B)$

Hence, $n(B) = n(C)$

That is, B has as many elements as C.

The correct option is (B)

8. $(A \cup B)' \cup (A' \cap B) = (A' \cap B') \cup (A' \cap B)$

$= (A' \cup A') \cap (A' \cup B) \cap (B' \cup A') \cap (B' \cup B)$

$= A' \cap [A' \cup (B \cap B')] \cap U$

$= A' \cap (A' \cup \phi) \cap U$

$= A' \cap A' \cap U = A' \cap U = A'$

The correct option is (C)

9. $A = \{2,4,6\}$, $B = \{2,3,5\}$

Number of relations from A to $B = 2^{3\times3} = 2^9$

The correct option is (A)

10. $R \times (P' \cup Q')' = R \times [(P')' \cap (Q')']$

$= R \times (P \cap Q) = (R \times P) \cap (R \times Q)$

The correct option is (A)

11. Clearly $x\ R\ x$ as $x - x + \sqrt{2} = \sqrt{2}$ is an irrational number. Thus, R is reflexive. Also $(\sqrt{2}, 1) \in R$ as $\sqrt{2} - 1 + \sqrt{2} = 2\sqrt{2} - 1$ is an irrational number but $(1, \sqrt{2}) \notin R$ as $1 - \sqrt{2} + \sqrt{2} = 1$ is a rational number. So, R is not symmetric,

$1\ R\ \sqrt{2}$ and $2\sqrt{2}\ R\ \sqrt{2}$

Since, $1\ R\ 2$ and $2\ R\ \sqrt{2}$ but 1 is not related to $\sqrt{2}$. So R is not transitive.

The correct option is (A)

12. $A = \{-1, 1\}$, $B = \{-1, 1, -i, i\}$

$A - B = \phi$, $B - A = \{-i, i\}$

$(A - B) \cup (B - A) = \{-i, i\}$

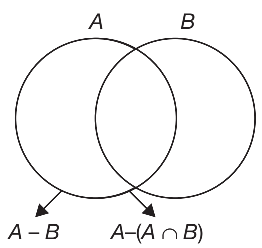
The correct option is (A)

13. $|x| < 1 \Rightarrow -1 < x < 1$ ($x = 0$ integer satisfies it)

The correct option is (A)

14. $A - B = A - (A \cap B)$ is correct

$A = (A \cap B) \cup (A - B)$ is correct



(3) is false.

$\therefore$ (1) and (2) are true.

The correct option is (D)

15. The set consists of n elements and for relation to be reflexive it must have at least n ordered pairs. It has m ordered pairs therefore $m \geq n$.

The correct option is (A)

16. $n\,[(A \times B) \cap (B \times A)]$

$= n\,[(A \cap B) \times (B \cap A)] = n\,(A \cap B) \cdot n\,(B \cap A)$

$= n\,(A \cap B) \cdot n\,(A \cap B) = (99)(99) = 99^2$

The correct option is (B)

17. Here, $R = \{(x, y) : |x^2 - y^2| < 16\}$ and $A = \{1, 2, 3, 4, 5\}$

$\therefore$ R = [(1,2), (1,3), (1,4); (2,1), (2,2), (2,3), (2,4); (3,1), (3,2) (3,3), (3,4); (4,1), (4,2), (4,3); (4,4), (4,5); (5,4), (5,5)]

The correct option is (D)

18. Here $\alpha \perp \beta \Rightarrow \beta \perp \alpha$. Hence, R is symmetric.

The correct option is (B)

19. To make it reflexive, we need to add (2, 2), (3, 3), (4,4). To make symmetric, it requires (3, 2), (4, 3) to be added. To make transitive, (2, 4) and (4, 2) must be added, so, the relation.

R = [(2,2), (3,3), (4,4) (2,3),(3, 2), (3, 4), (4,3), (2,4),(4,2)]

The correct option is (C)

20. Since $8x \equiv 6 \pmod{14}$ i.e, $8x - 6 = 14k$ for $k \in I$.

The values 6 and 13 satisfy this equation, while 8, 14, and 16 do not.

The correct option is (C)

21. Statement 1 is true

We observe that

Reflexivity: xRx as $x - x = 0$ is an integer, $\forall\, x \in A$

Symmetric

Let $(x, y) \in A$

$\Rightarrow$ $y - x$ is an integer

$\Rightarrow$ $x - y$ is also an integer

Transitivity: Let $(x, y) \in A$ and $(y, z) \in A$

$\Rightarrow$ $y - x$ is an integer and $z - y$ is an integer

$\Rightarrow$ $y - x + z - y$ is also an integer

$\Rightarrow$ $z - x$ is an integer

$\Rightarrow$ $(x, z) \in A$

Because of the above properties A is an equivalence relation over R

Statement 2 is false as 'B' is not symmetric on $\mathbb{R}$

We observe that

$0Bx$ as $0 = 0 \cdot x \forall\, x \in \mathbb{R}$ but $(x, 0) \notin B$

The correct option is (D)

22. Every element has 3 options. Either set Y or set Z or none, so number of ordered pairs $= 3^5$.

The correct option is (B)

23. $T = \{(x, y) : x - y \in I\}$

as $0 \in I$ T is a reflexive relation.

If $x - y \in I \Rightarrow y - x \in I$

$\therefore$ T is symmetrical also

If $x - y = I_1$ and $y - z = I_2$

Then $x - z = (x - y) + (y - z) = I_1 + I_2 \in I$

$\therefore$ T is also transitive.

Hence, T is an equivalence relation.

Clearly $x \neq x + 1 \Rightarrow (x, x) \notin S$

$\therefore$ S is not reflexive.

The correct option is (D)

Previous Year's Questions

24. (2, 3) belongs to R but (3, 2) is not.

Hence R is not symmetric.

The correct option is (C)

25. R is Reflexive and transitive only so not an equivalence relation.

e.g. (3, 3), (6, 6), (9, 9), (12, 12) [Reflexive]

(3, 6), (6, 12), (3, 12) [Transitive].

The correct option is (A)

26. Clearly $(x, x) \in R\ \forall\ x \in W$. So, R is reflexive.

Let $(x, y) \in R$, then $(y, x) \in R$ as x and y have at least one letter in common. So, R is symmetric.

But R is not transitive for example

Particularly if x = DELHI, y = DWARKA and z = PARK, then

$(x, y) \in R$ and $(y, z) \in R$ but $(x, z) \notin R$.

The correct option is (B)

27. The required number of ways is ${}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4 = \dfrac{12!}{(4!)^3}$

The correct option is (C)

28. Given $T = \{(x, y) : x - y \in I\}$

As $0 \in I$, T is a reflexive relation.

If $x - y \in I \Rightarrow y - x \in I$

$\therefore$ T is symmetrical also

If $x - y = I_1$ and $y - z = I_2$

Then $x - z = (x - y) + (y - z) = I_1 + I_2 \in I$

$\therefore$ T is also transitive.

Hence T is an equivalence relation.

Clearly $x \neq x + 1 \Rightarrow (x, x) \notin S$

$\therefore$ S is not reflexive.

The correct option is (D)

29. The correct option is (C)

30. P: there is a rational number $x \in S$ such that $x > 0$

$\sim P$: Every rational number $x \in S$ satisfies $x \leq 0$

The correct option is (B)

31. $x - y$ is an integer

$x - x = 0$ is an integer $\Rightarrow A$ is Reflexive

$x - y$ is an integer $\Rightarrow y - x$ is an integer $\Rightarrow A$ is symmetric

$x - y$, $y - z$ are integers

As sum of two integers is an integer.

$\Rightarrow$ $(x - y) + (y - z) = x - z$ is an integer

$\Rightarrow$ A is transitive. Hence statement 1 is true.

Also $\dfrac{x}{x} = 1$ is a rational number $\Rightarrow B$ is reflexive

$\dfrac{x}{x} = \alpha$ is rational $\Rightarrow \dfrac{y}{x}$ need not be rational

i.e., $\dfrac{0}{1}$ is rational $\Rightarrow \dfrac{1}{0}$ is not rational

Hence B is not symmetric

$\Rightarrow$ B is not an equivalence relation.

The correct option is (B)

32. $A \times B$ will have 8 elements.

$\therefore$ The number of subsets of $A \times B$ having 3 or more elements is $2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2 = 256 - 1 - 8 - 28 = 219$

The correct option is (B)

33. $n(A) = 4$, $n(B) = 2$

$n(A \times B) = 8$

$\therefore$ Number of subsets having atleast 3 elements

$$= 2^8 - \left(1 + {}^8C_1 + {}^8C_2\right) = 219$$

The correct option is (D)