CHAPTER

5

Matrices

Chapter Highlights

Matrix, Types of matrices, Algebra of matrices, Symmetric matrix, Orthogonal matrix, Idempotent matrix, Involutory matrix, Nilpotent matrix, Singular matrix

MATRIX

A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n, written as $m \times n$ matrix.

Such an array is enclosed by [ ] or ( ) or || ||. An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

In compact form the above matrix is represented by $A = [a_{ij}]_{\times n}$. The numbers $a_{11}, a_{12}, \ldots$ etc. are known as elements of the matrix A, a_{ij} belongs to the ith row and jth column and is called the (i, j)th element of the matrix $A = [a_{ij}]$.

For example, $A = \begin{bmatrix} 3 & 7 & 2 \\ 0 & -1 & 9 \end{bmatrix}$ is a matrix having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements:

$$a_{11} = 3, a_{12} = 7,$$

$$a_{13} = 2, a_{21} = 0,$$

$$a_{22} = -1, a_{23} = 9.$$

CAUTION

$m \times n$ does not indicate multiplication

TYPES OF MATRICES

Row Matrix

A matrix having only one row is called a row matrix or a row vector. For example, $A = [2\ 3\ {-4}\ 1]$ is a row matrix of order 1×4.

Column Matrix

A matrix having only one column is called a column matrix or a column vector.

For example, $A = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a column matrix of order 3×1.

Zero Matrix or Null Matrix

A matrix each of whose elements is zero, is called a zero matrix or a null matrix.

For example, the matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, are null matrices of order 2×2 and 2×3 respectively.

Square Matrix

A matrix in which number of rows is equal to the number of columns, say n, is called a square matrix of order n.

For example, the matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & -3 & 4 \\ 3 & 4 & 2 \\ 5 & 3 & 6 \end{bmatrix}$ are square matrices of order 2 and 3 respectively.

Diagonal Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called a diagonal matrix if all the elements except those in the leading diagonal are zero, i.e., $a_{ij} = 0$ for $i \neq j$. In other words

$$A = \text{diag. } [a_{11}\ a_{22}\ a_{33}\ \dots\ a_{nn}]$$

For example, the matrix $A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. is a diagonal matrix, and is denoted by $A = \text{diag. } [6\ 4\ -2]$.

Scalar Matrix

A square matrix in which every non-diagonal element is zero and all diagonal elements are equal, is known as scalar matrix.

For example, the matrices $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are scalar matrices of order 2 and 3 respectively.

Unit Matrix

A square matrix in which every non-diagonal element is zero and every diagonal element is 1, is called a unit matrix or an identity matrix. Thus, a square matrix $A = [a_{ij}]_{n \times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 0 \text{ when } i \neq j \\ 1 \text{ when } i = j \end{cases}$$

A unit matrix of order n is denoted by I_n or I. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ are unit matrices}$$

of order 2 and 3 respectively.

Upper Triangular Matrix

A square matrix $A = [a_{ij}]$ is called upper triangular matrix if $a_{ij} = 0$ for all $i > j$. For example, the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix} \text{ is an upper triangular matrix.}$$

Lower Triangular Matrix

A square matrix $A = [a_{ij}]$ is called a lower triangular matrix if $a_{ij} = 0$ for all $i < j$. For example, the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 2 & 4 & 6 & 7 \end{bmatrix} \text{ is a lower triangular matrix.}$$

Trace of a Matrix

The sum of the diagonal elements of a square matrix A is called the trace of A and is denoted by tr (A). For example, if

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 6 \\ 3 & 1 & 4 \end{bmatrix}, \text{ then tr } (A) = 1 - 1 + 4 = 4.$$

Sub Matrix

A matrix which is obtained from a given matrix by deleting any number of rows or columns or both is called a sub-matrix of the given matrix. For example,

$$\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \text{ is a sub-matrix of the matrix } \begin{bmatrix} 5 & 3 & 1 \\ 4 & 2 & -1 \\ 6 & 3 & 5 \end{bmatrix}.$$

Equality of Matrices

Two matrices A and B are said to be equal if they are of same order and all the corresponding elements are equal. It is written as $A = B$. For example,

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1+2 & 1+3 \\ 2+3 & 1 & 0 \end{bmatrix}$$

are equal matrices, whereas

$$C = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 5 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

are not equal, because their orders are not same.

ALGEBRA OF MATRICES

Addition of Matrices

Let A and B be two matrices each of order $m \times n$. Then the sum matrix $A + B$ is defined only if matrices A and B are of same order. The new matrix, say $C = A + B$ is of order $m \times n$ and is obtained by adding the corresponding elements of A and B. Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of same order then the sum $A + B$ is defined to be the matrix of order $m \times n$ such that $A + B = C = [a_{ij} + b_{ij}]$ for all i and j.

For example, if $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$

Then $C = A + B = \begin{bmatrix} 2+4 & 3-2 & 4+3 \\ 1+1 & 0+1 & 5+4 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 7 \\ 2 & 1 & 9 \end{bmatrix}$,

whereas the addition of

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

is not defined since the two matrices are not of same order. If A is any matrix, the negative of A, denoted by $-A$, is the matrix obtained by replacing each entry in A by its negative. For example, if

$$A = \begin{bmatrix} 2 & -1 \\ 5 & 4 \\ -6 & 0 \end{bmatrix}, \text{ then } -A = \begin{bmatrix} -2 & 1 \\ -5 & -4 \\ 6 & 0 \end{bmatrix}$$

CAUTION

Two matrices cannot be added if they are of different order

Properties of Addition of Matrices

1. *Matrix addition is commutative*. If A and B are two matrices of the same order, then $A + B = B + A$.
2. *Matrix addition is associative*. If A, B and C are three matrices of the same order, then $(A + B) + C = A + (B + C)$.
3. *Existence of additive identity*. If O is the zero matrix of the same order as that of the matrix A, then $A + O = A = O + A$.
4. *Existence of additive inverse*. If A is any matrix, then $A + (-A) = O = (-A) + A$.
5. *Cancellation laws hold good in case of addition of matrices*. If A, B, C are matrices of the same order, then

$$A + B = A + C \Rightarrow B = C \quad \text{(left cancellation law)}$$

and $$B + A = C + A \Rightarrow B = C \quad \text{(right cancellation law)}$$

NOTE

The zero matrix plays the same role in matrix addition as the number zero does in addition of numbers.

Subtraction of Matrices

Let A and B be two matrices of the same order. Then by $A - B$, we mean $A + (-B)$. In other words, to find $A - B$ we subtract each element of B from the corresponding element of A.

For example, if

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 1 \\ 7 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 4 \\ 2 & 5 \\ 6 & 3 \end{bmatrix}$$

then $$A - B = \begin{bmatrix} 2+3 & 3-4 \\ 6-2 & 1-5 \\ 7-6 & -2-3 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 4 & -4 \\ 1 & -5 \end{bmatrix}.$$

Multiplication of a Matrix by a Scalar

Let $A = [a_{ij}]$ be an $m \times n$ matrix and k be any scalar. Then the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k and is denoted by kA. Thus, if $A = [a_{ij}]_{m \times n}$, then $kA = [ka_{ij}]_{m \times n}$. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -6 & 8 \\ 0 & 2 & 5 \end{bmatrix}$$

then

$$3A = \begin{bmatrix} 3 & 6 & 9 \\ 12 & -18 & 24 \\ 0 & 6 & 15 \end{bmatrix} \text{ and } \frac{1}{2}A = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 2 & -3 & 4 \\ 0 & 1 & \frac{5}{2} \end{bmatrix}$$

Properties of Scalar Multiplication

1. If A and B are two matrices of the same order and k be a scalar, then
$$k(A + B) = kA + kB$$
2. If k_1 and k_2 are two scalars and A is a matrix, then
$$(k_1 + k_2)A = k_1A + k_2A$$
3. If k_1 and k_2 are two scalars and A is a matrix, then
$$(k_1k_2)A = k_1(k_2A) = k_2(k_1A)$$
4. If A is any matrix, then $1A = A$.

Multiplication of Matrices

Two matrices A and B can be multiplied only if the number of columns in A (pre-multiplier) is same as the number of rows in B (post multiplier). For example, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$ are two matrices of order $m \times n$ and $n \times p$ respectively, then their product AB is of order $m \times p$ and is defined as

$$(AB)_{i,k} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + \ldots + a_{in}\, b_{nk}$$

$$= [a_{i1}\; a_{i2}\; \ldots\; a_{in}] \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$$

$$= (i\text{th row of } A)\ (k\text{th column of } B).$$

$(AB)_{ik}$ = Sum of the product of elements of ith row of A with the corresponding elements of kth column of B.

CAUTION

If the number of columns in A is not equal to the number of rows in B, we cannot find the product AB of the two matrices A and B.

REMARKS

- If A and B are square matrices of the same order, say n, then both the products AB and BA are defined and each is a sqaure matrix of order n.
- In the matrix product AB, the matrix A is called pre-multiplier (prefactor) and B is called post-multiplier (post-factor).
- The rule of multiplication of matrices is row column wise (or → ↓ wise), viz.., the first row of AB is obtained by multiplying the first row of A with first, second, third, … columns of B respectively. Similarly second row of A with first, second, third, … columns of B respectively and so on.

Let $A = \begin{bmatrix} 1 & -4 \\ 5 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 & 1 & 6 \\ 2 & 7 & 3 & 8 \end{bmatrix}$

be two matrices.

Since the number of columns in A are equal to the number of rows in B, the product AB is defined. As order of matrix A is 3×2 and B is 2×4, the product AB will be of order 3×4.

$$AB = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}$$

The entry c_{11} is obtained by summing the products of each entry in row 1 of A by the corresponding entry in column 1 of B, i.e.,

$$c_{11} = (1)\ (-2) + (-4)\ (2) = -10$$

Similarly, for c_{21}, we use the entries in row 2 of A and those in column 1 of B:

$$c_{21} = (5)\ (-2) + (3)\ (2) = -4.$$

Also, $c_{12} = (1)\ (4) + (-4)\ (7) = -24$

$$c_{13} = (1)\ (1) + (-4)\ (3) = -11$$

$$c_{14} = (1)\ (6) + (-4)\ (8) = -26$$

$$c_{22} = (5)\ (4) + (3)\ (7) = 41$$

$$c_{23} = (5)\ (1) + (3)\ (3) = 14$$

$$c_{24} = (5)\ (6) + (3)\ (8) = 54$$

$$c_{31} = (0)\ (-2) + (2)\ (2) = 4$$

$$c_{32} = (0)\ (4) + (2)\ (7) = 14$$

$$c_{33} = (0)\ (1) + (2)\ (3) = 6$$

$$c_{34} = (0)\ (6) + (2)\ (8) = 16.$$

Thus, $AB = \begin{bmatrix} -10 & -24 & -11 & -26 \\ -4 & 41 & 14 & 54 \\ 4 & 14 & 6 & 16 \end{bmatrix}$

The product BA is not defined since the number of columns of B is not equal to the number of rows of A. This shows that matrix multiplication is not commutative. That is, for any two matrices A and B, it is usually the case that $AB \neq BA$ (even if both products are defined).

Properties of Matrix Multiplication

1. *Multiplication is distributive over matrix addition*. If A, B, C are $m \times n$, $n \times p$ and $n \times p$ matrices respectively, then
$$A\ (B + C) = AB + AC$$
2. *Multiplication is associative*. If A, B, C are matrices of order $m \times n$, $n \times p$ and $p \times r$ respectively, then
$$(AB)C = A\ (BC)$$
3. *Multiplicative identity*. If A is an $m \times n$ matrix and I_n the identity matrix of order $n \times n$ and I_m the identity matrix of order $m \times m$, then
$$I_m A = A \text{ and } AI_n = A$$
In particular if A is a square matrix of order n, then
$$AI_n = I_n A = A$$
4. $AB = 0$ (null matrix) does not necessarily imply that $A = 0$ or $B = 0$ or both $= 0$. For example, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \neq 0$ and
$B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
5. If A is a square matrix of order n, then A^2 is defined as AA. In general $A^m = AA \ldots A$ (m times), where m is any positive integer.
6. If I be a unit matrix, then $I = I^2 = I^3 = \ldots = I^n$.

SOLVED EXAMPLES

1. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $n \in N$, then A^{4n} equals

(A) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

Solution: (C)

We have,

$$A^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{4n} = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{n \text{ times}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. If $AB = A$ and $BA = B$, then B^2 is equal to

(A) B (B) A (C) 1 (D) 0

Solution: (A)

Since $\quad BA = B,$

$\therefore \quad (BA)\,B = BB = B^2$

$\Rightarrow \quad B\,(AB) = B^2 \;\Rightarrow\; BA = B^2 \qquad (\because\ AB = A)$

$\Rightarrow \quad B = B^2 \qquad (\because\ BA = B)$

3. If $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, the value of X^n is

(A) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (B) $\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$

(C) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (D) None of these

Solution: (D)

We have,

$$X^2 = X \times X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

For $n = 2$, matrices in (A), (B) and (C) do not match with

$$\begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

4. If A and B are two matrices such that $AB = B$ and $BA = A$, then $A^2 + B^2 =$

(A) $2AB$ (B) $2BA$

(C) $A + B$ (D) AB

Solution: (C)

We have,

$$A^2 + B^2 = AA + BB = A\,(BA) + B\,(AB) \qquad (\because\ AB = B \text{ and } BA = A)$$

$$= (AB)\,A + (BA)\,B$$

$$= BA + AB = A + B \qquad (\because\ AB = B \text{ and } BA = A)$$

5. If A and B are square matrices of same order such that $(A + B)^2 = A^2 + B^2 + 2AB$, then

(A) $AB = BA$ (B) $A = B$

(C) $A = B'$ (D) $A = -B$

Solution: (A)

We have,

$$(A + B)^2 = A^2 + B^2 + 2AB$$

$$\Rightarrow \quad (A + B)\,(A + B) = A^2 + B^2 + 2AB$$

$$\Rightarrow \quad A^2 + AB + BA + B^2 = A^2 + B^2 + 2AB$$

$$\Rightarrow \quad BA = AB$$

6. If $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ the A^8 equals

(A) $64B$ (B) $-64B$

(C) $-128B$ (D) $128B$

Solution: (D)

We have,

$$A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix} = i\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = i\,B$$

$$\Rightarrow \quad A^2 = (iB)^2 = i^2 B^2 = -B^2 = -\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = -\,2B$$

$$\Rightarrow \quad A^4 = (-2B)^2 = 4B^2 = 4(2B) = 8B$$

$$\Rightarrow \quad A^8 = (A^4)^2 = (8B)^2 = 64B^2 = 128B$$

7. If A and B are two matrices such that $A + B$ and AB are both defined, then

(A) A and B are two matrices not necessarily of same order

(B) A and B are square matrices of same order

(C) number of columns of A = number of rows of B
(D) None of the above.

Solution: (B)

Both matrices are of same order for $A+B$ to be defined. Now let both are of order $m \times n$. But AB is defined only when number of columns in A is same as number of rows in B. So both are square matrices of same order.

8. If $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ then A^{64} is

(A) $\begin{bmatrix} 1 & 32 \\ 32 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ 32 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 32 \\ 0 & 1 \end{bmatrix}$ (D) None of these

Solution: (C)

$$A^2 = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

Similarly, $A^{64} = \begin{bmatrix} 1 & 32 \\ 0 & 1 \end{bmatrix}$

9. If A = $\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$, then A^{100} is equal to

(A) $\begin{bmatrix} 1 & 0 \\ 110 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ 25 & 1 \end{bmatrix}$ (D) None of these

Solution: (B)

$$A^2 = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2(1/2) & 1 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 1 & 0 \\ 2(1/2) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3(1/2) & 1 \end{bmatrix}$$

Continuing in this way, we get

$$A^{100} = \begin{bmatrix} 1 & 0 \\ 100(1/2) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$$

Transpose of a Matrix

Let A be an $m \times n$ matrix. Then, the $n \times m$ matrix obtained by interchanging the rows and columns of A is called the transpose of A, and is denoted by A' or A^t. Thus,

1. if order of A is $m \times n$, then, the order of A' is $n \times m$.
2. (i, j)th element of A = (j, i)th element of A'.

For example,

if $A = \begin{bmatrix} 2 & -3 & -1 \\ 4 & 2 & 3 \end{bmatrix}$,

then $A' = \begin{bmatrix} 2 & 4 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}_{3\times 2}$

Properties of the Transpose of a Matrix

1. Let A and B be two matrices of order $m \times n$, then $(A \pm B)' = A' \pm B'$.
2. Let A be a matrix of order $m \times n$ and k be a scalar, then $(kA)' = kA'$.
3. Let A and B be two matrices of order $m \times n$ and $n \times p$ respectively. Then, $(AB)' = B'A'$.
4. The double transpose of any matrix is the original matrix. For example, if A is any matrix, then $(A')' = A$.

SOLVED EXAMPLES

10. If A is 3×4 matrix and B is a matrix such that $A'B$ and BA' are both defined. Then B is of the type

(A) 3×4 (B) 3×3 (C) 4×4 (D) 4×3

Solution: (A)

Let the order of B be $m \times n$.

Since A is 3×4 matrix,

∴ A' is 4×3 matrix.

Since $A'B$ is defined,

∴ number of columns of A' must be equal to number of rows of B,

∴ $m = 3$.

Also, since BA' is defined,

∴ number of columns of B must be equal to number of rows of A',

∴ $n = 4$.

∴ B is **3** × **4** matrix.

11. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^T(Q^{2005})P$ is equal to

(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} \sqrt{3}/2 & 2005 \\ 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 2005 \\ \sqrt{3}/2 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & \sqrt{3}/2 \\ 0 & 2005 \end{bmatrix}$

Solution: (A)

We have,

$$Q = P\,A\,P^T$$

$$\Rightarrow \quad P^T Q = AP^T \qquad (\because \text{ as } P^TP = I = PP^T)$$

$$\therefore \quad P^T Q^{2005} P = A\,P^T Q^{2004} P$$

$$= A^2 P^T Q^{2003} P = A^3 P^T Q^{2002} P$$

$$= A^{2004} P^T (QP)$$

$$= A^{2004} P^T (PA)$$

$$(Q = P\,A\,P^T \quad \Rightarrow \quad Q\,P = P\,A) = A^{2005}$$

$$= \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

SYMMETRIC MATRIX

A square matrix A is said to be symmetric if $A' = A$. That is, the matrix $A = [a_{ij}]_{n \times n}$ is said to be symmetric provided $a_{ij} = a_{ji}$ for all i and j.

For example,

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & -3 \\ 5 & -3 & 6 \end{bmatrix} \text{ is symmetric,}$$

Since

$$A' = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & -3 \\ 5 & -3 & 6 \end{bmatrix} = A$$

Skew Symmetric Matrix

A square matrix A is said to be skew symmetric, if $A' = -A$. That is, the matrix $A = [a_{ij}]_{n \times n}$ is skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j.

For example,

$$A = \begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & 3 \\ -7 & -3 & 0 \end{bmatrix} = -A$$

IMPORTANT POINTS

Elements of main diagonal of a skew-symmetric matrix are all zero, because by definition,

$$a_{ii} = -a_{ii} \quad \Rightarrow \quad 2a_{ii} = 0$$

or $\qquad a_{ii} = 0$ for all values of i.

Properties of Symmetric and Skew-symmetric Matrices

1. If A is a square matrix, then
   (a) $A + A'$ is symmetric
   (b) $A - A'$ is skew-symmetric.
2. If A and B are two symmetric (or skew-symmetric) matrices of the same order, then so is $A + B$.
3. If A is symmetric (or skew-symmetric) matrix and k is a scalar, then kA is also symmetric (or skew-symmetric).
4. If A and B are symmetric matrices of the same order, then the product AB is symmetric if and only if $AB = BA$.
5. Every square matrix can be expressed uniquely as the sum of a symmetric and a skew-symmetric matrix.
6. The matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.
7. All positive integral powers of a symmetric matrix are symmetric.
8. All positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.
9. If A and B are symmetric matrices of the same order, then
   (a) $AB - BA$ is a skew-symmetric matrix.
   (b) $AB + BA$ is a symmetric matrix.
10. If A is any square matrix, then AA' and $A'A$ are both symmetric matrices.

SOLVED EXAMPLES

12. Which of the following is correct?

(A) $B'AB$ is symmetric if A is symmetric
(B) $B'AB$ is skew-symmetric if A is symmetric
(C) $B'AB$ is symmetric if A is skew-symmetric
(D) $B'AB$ is skew-symmetric if A is skew-symmetric

Solution: (A, D)

Let A be a symmetric matrix.

Then $\qquad A' = A.$

Now, $\qquad (B'AB)' = B'A'(B')' \qquad [\because (AB)' = B'A']$

$$= B'A'B \qquad [\because (B')' = B]$$

$$= B'AB \qquad [\because\ A' = A]$$

$\Rightarrow$ $B'AB$ is a symmetric matrix.

Now, let A be a skew-symmetric matrix.

Then $\qquad A' = -A.$

$$\begin{aligned} \therefore \quad (B'AB)' &= B'A'(B')' && [\because\ (AB)' = B'A'] \\ &= B'A'B && [\because\ (B')' = B] \\ &= B'(-A)B && [\because\ A' = -A] \\ &= -B'AB \end{aligned}$$

$\therefore$ $B'AB$ is a skew-symmetric matrix.

13. If A is symmetric as well as skew symmetric matrix, then A is

(A) diagonal
(B) null
(C) triangular
(D) None of these

Solution: (B)

Let $\qquad A = [a_{ij}]$

Since A is skew-symmetric,

$\therefore \qquad a_{ii} = 0$ and $a_{ij} = -a_{ji}$ $(i \neq j)$

A is symmetric as well, so $a_{ij} = a_{ji}$ for all i and j.

$\therefore \qquad a_{ij} = 0$ for all $i \neq j$

Hence, $a_{ij} = 0$ for all i and j i.e. A is a null matrix.

14. If A is a 3 × 3 skew-symmetric matrix, then trace of A is

(A) $|A|$
(B) 1
(C) –1
(D) None of these

Soluiton: (A)

As A is a skew symmetric matrix

$$\Rightarrow \quad a_{ii} = 0\ \forall\ i \quad \Rightarrow \quad \text{trace}(A) = 0$$

Also, $$|A| = |A'| = |-A| = (-1)^3|A|$$

$$\Rightarrow \quad 2|A| = 0 \quad \Rightarrow \quad |A| = 0$$

$$\therefore \quad \text{trace}(A) = |A|.$$

ORTHOGONAL MATRIX

A square matrix of order $n \times n$ is said to be orthogonal if $AA' = I_n = A'A$. For example, if

$$A = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix},$$

then $$A' = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$$

Also, $$AA' = \frac{1}{8}\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly, $\quad A'A = I.$

Hence A is orthogonal.

SOLVED EXAMPLES

15. If A is an orthogonal matrix, then $|A|$ is

(A) 1
(B) –1
(C) 0
(D) None of these

Solution: (A, B)

Since the matrix A is orthogonal

$$A'A = AA' = I_n$$

$$\Rightarrow \quad |A'A| = |AA'| = |I_n|$$

$$\Rightarrow \quad |A'|\,|A| = 1$$

$$\Rightarrow \quad |A|\,|A| = 1$$

or $$|A|^2 = 1$$

$$\therefore \quad |A| = \pm 1$$

16. Let $A = \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$ be an orthogonal matrix then the values of a, b, c are

(A) $b = \pm\dfrac{1}{\sqrt{6}}$, $c = \pm\dfrac{1}{\sqrt{3}}$

(B) $a = \pm\dfrac{1}{\sqrt{2}}$, $c = \pm\dfrac{1}{\sqrt{6}}$

(C) $a = \pm\dfrac{1}{\sqrt{2}}$, $b = \pm\dfrac{1}{\sqrt{6}}$

(D) All of these

Solution: (D)

$$A = \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix} \text{ and } A' = \begin{pmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{pmatrix}$$

As A is orthogonal

$$\therefore \quad AA' = I$$

$$\Rightarrow \quad \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}\begin{pmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore$ Using definition of equality of two matrices

$$4b^2 + c^2 = 1,\ 2b^2 - c^2 = 0,\ a^2 + b^2 + c^2 = 1$$

On solving them,

$$a = \pm\frac{1}{\sqrt{2}},\ b = \pm\frac{1}{\sqrt{6}},\ c = \pm\frac{1}{\sqrt{3}}$$

IDEMPOTENT MATRIX

A square matrix A is said to be idempotent if $A^2 = A$.

For example, if

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix},$$

then $$A^2 = AA = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence, A is idempotent.

INVOLUTORY MATRIX

A square matrix A is said to be involutory matrix if $A^2 = I$.

For example, if

$$A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix},$$

then $$A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A is involutory.

NILPOTENT MATRIX

A square matrix A is said to be nilpotent matrix if there exists a positive integer m such that $A^m = 0$. If m is the least positive integer such that $A^m = 0$, then m is called the index of the nilpotent matrix A.

For example, if

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix},$$

then $$A^2 = AA = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Hence, the matrix A is nilpotent of the index 2.

SOLVED EXAMPLE

17. The matrix $A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$ is

(A) Nilpotent
(B) Idempotent
(C) Orthogonal
(D) Involutary

Solution: (B)

In order to get the result, we go to the definition of these matrices and check

(A) $AA' = 1$ for othogonal matrix.

(B) If A^2, A^3 … or A^k is zero *i.e.* $A^2 = 0$ or $A^3 = 0$ or $A^k = 0$ then A is said to be nilpotent.

(C) If $A^2 = I$ then A is involutary matrix.

(D) If $A^2 = A$ then A is said to be idempotent and by checking, we get $A^2 = A$.

$\therefore$ A is idempotent matrix.

NOTE

- Minimum number of zeros in a triangular matrix is given by, $\frac{n(n-1)}{2}$, where n is order of matrix.
- A triangular matrix $A = [a_{ij}]_{n \times n}$ is called strictly triangular if $a_{ij} = 0$ for $1 \le i \le n$.
- The multiplication of two triangular matrices is a triangular matrix.
- If A is involutary matrix, then $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $(I + A)\ (I - A) = 0$.
- Trace of a skew symmetric matrix is always 0.

SINGULAR MATRIX

A square matrix A is said to be singular matrix if determinant of A denoted by det A or $|A|$ is zero, i.e., $|A| = 0$, otherwise, it is a non-singular matrix.

For example, the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 4 & -3 & 4 \end{bmatrix}$

is singular as $\begin{vmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 4 & -3 & 4 \end{vmatrix} = 0$

CAUTION

See next chapter for expansion of the determinant.

SOLVED EXAMPLES

18. Let A and B be two non-null square matrices. If the product AB is a null matrix, then

(A) A is singular (B) B is singular
(C) A is non-singular (D) B is non-singular

Solution: (A, B)

Let B be non-singular, then B^{-1} exists.

Now, $AB = O$ (given)

$\Rightarrow \quad (AB)\, B^{-1} = OB^{-1}$

(post-multiplying both sides by B^{-1})

$\Rightarrow \quad A\,(BB^{-1}) = O$ (by associativity)

$\Rightarrow \quad AI_n = O$ $(\because\ BB^{-1} = I_n)$

$\Rightarrow \quad A = O$

But A is a non-null matrix. Hence, B is a singular matrix. Similarly it can be shown that A is a singular matrix.

19. Which of the following is correct?

(A) Skew symmetric matrix of even order is always singular.
(B) Skew symmetric matrix of odd order is non-singular.
(C) Skew symmetric matrix of odd order is singular.
(D) None of the above.

Solution: (C)

Since the determinant of a skew symmetric matrix of odd order is zero,

$\therefore$ The matrix is singular.

Adjoint of a Square Matrix

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be the cofactor of a_{ij} in the determinant $|A|$. Then the adjoint of A, denoted by adj A, is defined as the transpose of the cofactor matrix.

The adjoint of a square matrix A is obtained on replacing each (i, j)th element of A by the cofactor of the (j, i)th element in $|A|$.

For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 4 & 3 & 2 \end{bmatrix},$$

then we have

$C_{11} = -3$ $C_{12} = 6$ $C_{13} = -3$
$C_{21} = 5$ $C_{22} = -10$ $C_{23} = 5$
$C_{31} = 2$ $C_{32} = -4$ $C_{33} = 2$

Thus, $$\text{adj } A = \begin{bmatrix} -3 & 5 & 2 \\ 6 & -10 & -4 \\ -3 & 5 & 2 \end{bmatrix}$$

Properties of the Adjoint of a Matrix

1. If A is a square matrix of order n, then
$$A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A,$$
where I_n is a square matrix of order n.
2. If A is a square matrix of order n, then adj $(A') = (\text{adj } A)'$.
3. If A and B are two square matrices of the same order, then adj $(AB) = (\text{adj } B)\,(\text{adj } A)$.
4. adj (adj A) $= |A|^{n-2} A$, where A is a non-singular matrix.
5. $|\text{adj } A| = |A|^{n-1}$ and $|\text{adj (adj } A)| = |A|^{(n-1)^2}$, where A is a non-singular matrix.
6. Adjoint of a diagonal matrix is a diagonal matrix.
7. $adj(A^m) = (adj\, A)^m,\ m \in N$
8. $adj(kA) = k^{n-1}(adj\, A),\ k \in R$
9. $adj(I_n) = I_n$
10. $adj(O) = O$
11. A is symmetric $\Rightarrow$ *adj A* is also symmetric.
12. A is diagonal $\Rightarrow$ *adj A* is also diagonal.
13. A is triangular $\Rightarrow$ *adj A* is also triangular.
14. A is singular $\Rightarrow |adj\, A| = 0$

SOLVED EXAMPLES

20. If $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, then det. [adj (adj A)] is

(A) $(14)^4$ (B) $(14)^3$ (C) $(14)^2$ (D) $(14)^1$

Solution: (A)

We know that adj (adj A) = $|A|^{n-2} A$ if $|A| \neq 0$, provided order of A is n.

$$\therefore \quad \text{adj}\,(\text{adj}\,A) = |A|\,A \qquad (\text{as } n = 3)$$

$$\therefore \quad \det[\text{adj}\,(\text{adj}\,A)] = |A|^3 \det A = |A|^4.$$

But
$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = 14$$

$$\therefore \quad \det[\text{adj}\,(\text{adj}\,A)] = (14)^4.$$

21. If A is a singular matrix, then adj A is

(A) non-singular (B) singular

(C) symmetric (D) not defined

Solution: (B)

Let A be of order $n \times n$.

Since $\quad A \times \text{adj}\,A = |A|\,I$

$$\therefore \quad |A \times \text{adj}\,A| = |A|^n$$

$$\Rightarrow \quad |A|\,|\text{adj}\,A| = |A|^n$$

$$\therefore \quad |\text{adj}\,A| = |A|^{n-1}$$

Since A is singular,

$$\therefore \quad |A| = 0.$$

$$\therefore \quad |\text{adj}\,A| = 0.$$

Hence, adj A is singular.

Inverse of a Square Matrix

Let A be any n-rowed square matrix. The inverse of A denoted by A^{-1} is determined by the formula:

$$A^{-1} = \frac{1}{|A|}\,(\text{adj}\,A)$$

It may be noted that

$$AA^{-1} = A^{-1}A = I$$

Properties of the Inverse of a Matrix

1. A square matrix is invertible if and only if it is non-singular.
2. The inverse of the inverse is the original matrix itself, i.e., $(A^{-1})^{-1} = A$.
3. The inverse of the transpose of a matrix is the transpose of its inverse, i.e. $(A')^{-1} = (A^{-1})'$.
4. If A and B are two invertieble matrices of the same order, then AB is also invertible and moreover
$$(AB)^{-1} = B^{-1}A^{-1}$$
5. Let A, B, C be square matrices of the same order n. If A is a non-singular matrix, then
   (a) $AB = AC \Rightarrow B = C$ (Left cancellation law)
   (b) $BA = CA \Rightarrow B = C$ (Right cancellation law)
   Note that these cancellation laws hold only if the matrix A is non-singular.
6. If A is a non-singular matrix such that A is symmetric then A^{-1} is also symmetric.
7. If A is a non-singular matrix, then $|A^{-1}| = |A|^{-1}$.
8. If A, B, C are three invertible matrices of the same order then ABC is invertible and more over
$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$
9. $A = \text{diag}\,(a_1 a_2 \ldots a_n) \Rightarrow A^{-1} = \text{diag}\,(a_1^{-1}\; a_2^{-1} \ldots a_n^{-1})$
10. A is a scalar matrix $\Rightarrow A^{-1}$ is also a scalar matrix.
11. A is triangular, $|A| \neq 0 \Rightarrow A^{-1}$ is also triangular.
12. Every invertible matrix possesses a unique inverse.

SOLVED EXAMPLES

22. Let A be an invertible matrix, which of the following is not true?

(A) $(A')^{-1} = (A^{-1})'$

(B) $A^{-1} = |A|^{-1}$

(C) $(A^2)^{-1} = (A^{-1})^2$

(D) None of these

Solution: (B)

$A^{-1} = |A|^{-1}$ is not true, as L.H.S. is a matrix and R.H.S. is a number.

23. If B is a non-singular matrix and A is a square matrix, then det $(B^{-1}AB)$ is equal to

(A) det (A^{-1}) (B) det (B^{-1})

(C) det (A) (D) det (B)

Solution: (C)

$$\det(B^{-1}AB) = \det(B^{-1}) \det A \det B$$
$$= \det(B^{-1}) \times \det B \times \det A = \det(B^{-1}B) \times \det A$$
$$= \det(I) \times \det A = 1 \times \det A = \det A.$$

24. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{6}\left[A^2 + cA + dI\right]$$

where $c, d \in R$, the pair of values (c, d) are

(A) (6, 11) (B) (6, –11)

(C) (–6, 11) (D) (–6, –11)

Solution: (C)

Given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}, A^{-1} = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$$

$$cA = \begin{bmatrix} c & 0 & 0 \\ 0 & c & c \\ 0 & -2c & 4c \end{bmatrix}, dI = \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$$

$$\therefore \; A^{-1} = \frac{1}{6}\left[A^2 + cA + dI\right]$$

$$\Rightarrow \quad 6 = 1 + c + d, \quad \text{(By equality of matrices)}$$

$\therefore$ (–6, 11) satisfy the relation

Solution of a System of Linear Equations by Matrix Method

Consider a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots \quad\quad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n. \end{aligned}$$

We can express these equations as a single matrix equation

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{B}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}$$

Let $|A| \neq 0$, so that A^{-1} exists uniquely. Pre-multiplying both sides of $AX = B$ by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B$$

or $$(A^{-1}A)X = A^{-1}B$$

or $$IX = A^{-1}B$$

or $$X = A^{-1}B$$

Hence $X = A^{-1}B$ is the unique solution of $AX = B$, $|A| \neq 0$.

Criterion of Consistency

Let $AX = B$ be a system of n linear equations in n variables.

1. If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.
2. If $|A| = 0$ and (adj A) $B = 0$, then the system of equations is consistent and has infinitely many solutions.
3. If $|A| = 0$ and (adj A) $B \neq 0$, then the system of equations is inconsistent, i.e., it has no solution.

Homogeneous Equations

The system of equations $AX = B$ is said to be homogeneous if the constants $b_1, b_2, \ldots b_n$ are all zero. That is, if the matrix B is a zero matrix and the system is of the form

$$AX = O$$

where O is a null matrix of order $n \times 1$.

1. If $|A| \neq 0$, then its only solution $X = 0$, is called the trivial solution.
2. If $|A| = 0$, then $AX = O$ has a non-trivial solution. It will have infinitely many solutions.

SOLVED EXAMPLE

25. The value of a for which the system of equations $ax + y + z = 0$, $x + ay + z = 0$, $x + y + z = 0$, possess non-zero solutions are given by,

(A) 1, 2
(B) 1, –1
(C) 1
(D) None of these

Solution: (C)

The set of homogeneous equations will have a non-zero solution if $\Delta = 0$

i.e., $$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad a = 1$$

EXERCISES

Single Option Correct Type

1. If A and B are symmetric matrices and $AB = BA$, then $A^{-1}B$ is a

(A) symmetric matrix
(B) skew-symmetric matrix
(C) identity matrix
(D) None of these

2. If the product of the matrix $B = \begin{bmatrix} 2 & 6 & 4 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ with a matrix A has inverse $C = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix}$, then A^{-1} equals

(A) $\begin{bmatrix} -3 & -5 & 5 \\ 0 & 9 & 14 \\ 2 & 2 & 6 \end{bmatrix}$ (B) $\begin{bmatrix} -3 & 5 & 5 \\ 0 & 0 & 9 \\ 2 & 14 & 16 \end{bmatrix}$

(C) $\begin{bmatrix} -3 & -5 & -5 \\ 0 & 9 & 2 \\ 2 & 14 & 6 \end{bmatrix}$ (D) $\begin{bmatrix} -3 & -3 & -5 \\ 0 & 9 & 2 \\ 2 & 14 & 6 \end{bmatrix}$

3. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$ then the value of α is

(A) ±1 (B) ±2 (C) ±3 (D) ±5

4. If A is an involutory matrix and I is unit matrix of the same order then, $(I - A)(I + A) =$

(A) 0 (B) A (C) I (D) $2A$

5. Matrix A is such that $A^2 = 2A - I$, where I is unit matrix then for $n \geq 2$, $A^n =$

(A) $nA - (n - 1)I$ (B) $nA - I$
(C) $2^{n-1} A - (n - 1)I$ (D) $2^{n-1}A - I$

6. If $AB = A$ and $BA = B$, where A and B are square matrices, then

(A) $B^2 = B$ and $A^2 = A$
(B) $B^2 = A$ and $A^2 = B$
(C) $AB = BA$
(D) None of these

7. If A is a square matrix, B is a singular matrix of same order, then for a positive integer n, $(A^{-1}BA)^n$ equals

(A) $A^{-n} B^n A^n$ (B) $A^n B^n A^{-n}$
(C) $A^{-1} B^n A$ (D) $n\,(A^{-1} B A)$

8. If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ then A^n equals

(A) $\begin{pmatrix} 3n & -4n \\ n & -n \end{pmatrix}$ (B) $\begin{pmatrix} 3^n & 4^n (-1)^n \\ 1^n & (-1)^n \end{pmatrix}$

(C) $\begin{pmatrix} 3+n & -(4+n) \\ n & -n \end{pmatrix}$ (D) None of these

9. If $A = [a_{ij}]$ is a scalar matrix of order $n \times n$ such that $a_{ij} = k$ for all i, then trace of A is equal to

(A) k^n (B) $\dfrac{n}{k}$
(C) nk (D) None of these

10. The matrix $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ is

(A) unitary (B) orthogonal
(C) nilpotent (D) involutory

11. The value of θ in $[0, 2\pi]$ such that the matrix

$$\begin{bmatrix} 2\sin\theta - 1 & \sin\theta & \cos\theta \\ \sin(\theta + \pi) & 2\cos\theta - \sqrt{3} & \tan\theta \\ \cos(\theta - \pi) & \tan(\pi \boxtimes \theta) & 0 \end{bmatrix}$$

is skew-symmetric, is

(A) $\pi/2$ (B) $\pi/3$ (C) $\pi/4$ (D) $\pi/6$

12. If $E(\theta) = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and θ and ϕ differ by an odd multiple of $\dfrac{\pi}{2}$, then $E(\theta)\,E(\phi)$ is a

(A) null matrix (B) unit matrix
(C) diagonal matrix (D) None of these

13. If A and B are two square matrices such that $B = -A^{-1}BA$, then $(A + B)^2 =$

(A) 0 (B) $A^2 + B^2$
(C) $A^2 + 2AB + B^2$ (D) $A + B$

14. If $A = \begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}$ and $AB = I$, then $B =$

(A) $\cos^2\dfrac{\theta}{2}\,A$ (B) $\cos^2\dfrac{\theta}{2}\,A^T$

(C) $\cos^2\dfrac{\theta}{2}\,I$ (D) None of these

15. For each real number x such that $-1 < x < 1$, let $A(x)$ be the matrix $(1-x)^{-1}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$ and $z = \dfrac{x+y}{1+xy}$. Then

(A) $A(z) = A(x) + A(y)$
(B) $A(z) = A(x)[A(y)]^{-1}$
(C) $A(z) = A(x)\,A(y)$
(D) $A(z) = A(x) - A(y)$

16. The inverse of a skew symmetric matrix of odd order is

(A) a symmetric matrix
(B) a skew symmetric matrix
(C) diagonal matrix
(D) does not exist

17. The number of solutions of equations $x_2 - x_3 = 1, -x_1 + 2x_3 = 2, x_1 - 2x_2 = 3$ is

(A) zero (B) one
(C) two (D) infinite

18. If $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is to be the square root of two-rowed unit matrix, then α, β and γ should satisfy the relation

(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 - \beta\gamma = 0$
(C) $1 - \alpha^2 + \beta\gamma = 0$ (D) $\alpha^2 + \beta\gamma - 1 = 0$

19. Let A and B be two symmetric matrices of order 3.

Statement 1: $A(BA)$ and $(AB)A$ are symmetric matrices.
Statement 2: AB is symmetric matrix if matrix multiplication of A with B is commutative.

(A) Statement 1 is false, Statement 2 is true
(B) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1
(C) Statement 1 is true, Statement 2 is true; Statement 2 is **not** a correct explanation for Statement 1
(D) Statement 1 is true, Statement 2 if false

20. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to:

(A) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (B) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

(C) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ (D) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

21. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, be a 2×2 matrix where a, b, c, d take the values 0 or 1 only. The number of such matrices which have inverses is:

(A) 8 (B) 7 (C) 6 (D) 5

22. Let A be a 2×2 matrix with real entries. Let I be the 2×2 identity matrix. Denote by tr (A), the sum of diagonal entries of A. Assume that $A^2 = I$.

Statement 1: If $A \neq I$ and $A \neq -I$, then $\det A = -1$.
Statement 2: If $A \neq I$ and $A \neq -I$, then tr $(A) \neq 0$.

(A) Statement 1 is false, Statement 2 is true
(B) Statement 1 is true, Statement 2 is true, Statement 2 is a correct explanation for Statement 1
(C) Statement 1 is true, Statement 2 is true; Statement 2 is **not** a correct explanation for Statement 1
(D) Statement 1 is true, Statement 2 is false

23. Let A be a square matrix all of whose entries are integers. Then which one of the following is true?

(A) If $\det A = \pm 1$, then A^{-1} exists but all its entries are not necessarily integers
(B) If $\det A \neq \pm 1$, then A^{-1} exists and all its entries are non-integers
(C) If $\det A = \pm 1$, then A^{-1} exists and all its entries are integers
(D) If $\det A = \pm 1$, then A^{-1} need not exist

24. If B, C are square matrices of order n and if $A = B + C$, $BC = CB$, $C^2 = 0$, then for any positive integer p, $A^{p+1} = B^k[B + (p+1)C]$, where $k =$

(A) p (B) $p + 1$ (C) $p + 2$ (D) $p - 1$

25. If $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, then det. (adj (adj A)) is

(A) $(14)^4$ (B) $(14)^3$ (C) $(14)^2$ (D) $(14)^1$

26. If $abc = p$ and $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$ such that $AA' = I$, then a, b, c are the roots of the equation

(A) $x^3 + p = 0$ (B) $x^3 \pm x^2 + p = 0$
(C) $x^3 \pm 3x^2 + p = 0$ (D) $x^3 \pm 2x^2 + p = 0$

27. If A is a singular matrix, then adj A is

(A) non-singular (B) singular
(C) symmetric (D) not defined

28. Matrix A is such that $A^2 = 2A - I$, where I is unit matrix then for $n \geq 2$, $A^n =$

(A) $nA - (n-1)I$ (B) $nA - I$
(C) $2^{n-1}A - (n-1)I$ (D) $2^{n-1}A - I$

29. For each real number x such that $-1 < x < 1$, let $A(x)$ be the matrix $(1-x)^{-1}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$ and $z = \dfrac{x+y}{1+xy}$. Then,

(A) $A(z) = A(x) + A(y)$ (B) $A(z) = A(x)[A(y)]^{-1}$
(C) $A(z) = A(x)\,A(y)$ (D) $A(z) = A(x) - A(y)$

30. The inverse of a skew-symmetric matrix of odd order is

(A) a symmetric matrix
(B) a skew-symmetric matrix
(C) diagonal matrix
(D) does not exist

31. If A is a square matrix, B is a singular matrix of same order, then for a positive integer n, $(A^{-1}BA)^n$ equals

(A) $A^{-n}B^nA^n$ (B) $A^nB^nA^{-n}$
(C) $A^{-1}B^nA$ (D) $n(A^{-1}BA)$

32. If A is an invertible matrix, then

(A) adj $A' = (\text{adj } A)'$ (B) adj $A' =$ adj A
(C) adj $A' = A'$ (D) None of these

33. If A is a non-singular square matrix of order n, then adj (adj A) is equal to

(A) $|A|^n A$ (B) $|A|^{n-1}A$
(C) $|A|^{n-2}A$ (D) None of these

34. If x, y, z are in A.P. with common differences d and the rank of the matrix $\begin{vmatrix} 4 & 5 & x \\ 5 & 6 & y \\ 6 & k & z \end{vmatrix}$ is 2 then the values of d and k are

(A) $\dfrac{x}{4}$; arbitrary number (B) arbitrary number, 7
(C) x, 5 (D) $\dfrac{x}{2}$, 6.

35. If $D = \text{diag}(a_1\ a_2\ a_3 \ldots a_n)$, where $a_i \neq 0$ for all $i = 1, 2, \ldots, n$, then D^{-1} is equal to

(A) I_n
(B) D
(C) diag $(a_1^{-1}\ a_2^{-1}\ a_3^{-1} \ldots a_n^{-1})$
(D) None of these

36. If A is a non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then BB' is

(A) I (B) B^{-1}
(C) $(B^{-1})'$ (D) None of these

37. If $A^3 = 0$ and $A^n \neq I$ for $n = 1, 2$ then $(I - A)^{-1}$ is

(A) $I + A$ (B) $I + A + A^2$
(C) $I - A + A^2$ (D) None of these

38. A and B are two non-singular matrices of the same order such that $A^n = I$ for some positive integer $n > 1$. Then, $BA^{n-1}B^{-1} - BA^{-1}B^{-1}$

(A) is a null matrix
(B) is an identity matrix
(C) a singular matrix
(D) None of these

39. The number of different matrices which can be formed using 12 different real numbers is

(A) 6 (12)! (B) 12 (12)!
(C) 4 (12)! (D) None of these

40. A skew-symmetric matrix A satisfies the relation $A^2 + I = 0$, where I is a unit matrix. Then, A is

(A) Idempotent matrix (B) Orthogonal matrix
(C) Nilpotent matrix (D) None of these

41. Let A be an $n \times n$ matrix such that $A^n = \alpha A$, where α is a real number different from 1 and -1. Then, the matrix $A + I_n$ is

(A) singular
(B) non-singular, i.e., invertible
(C) scalar matrix
(D) None of these

42. If adj $B = A$ and P, Q are two unimodular matrices, i.e., $|P| = 1 = |Q|$, then $(Q^{-1}BP^{-1})^{-1}$ is equal to

(A) PAQ (B) PBQ (C) QAP (D) QBP

43. If $A = \begin{bmatrix} 1 & \dfrac{\alpha}{n} \\ -\dfrac{\alpha}{n} & 1 \end{bmatrix}$, then

(A) $\lim\limits_{n\to\infty} A^n = 0$ (B) $\lim\limits_{n\to\infty} \dfrac{1}{n} A^n = 0$
(C) $\lim\limits_{n\to\infty} \dfrac{1}{n^2} A^n = 0$ (D) None of these

44. If $A^k = 0$ for some value of k and $(I - A)^p = I + A + A^2 + \ldots + A^{k-1}$, then p is

(A) -1 (B) -2
(C) -3 (D) None of these

45. If A satisfies the equation $x^3 - 5x^2 + 4x + kI = 0$, then A^{-1} exists if

(A) $k \neq -1$ (B) $k \neq 0$
(C) $k \neq 1$ (D) None of these

46. If M is a 3×3 matrix, where $M'M = I$ and det $M = 1$, then det $(M - I) =$

(A) 0 (B) 1
(C) -1 (D) None of these

More than One Option Correct Type

47. Let A and B be two non-null square matrices. If the product AB is a null matrix, then

(A) A is singular
(B) B is singular
(C) A is non-singular
(D) B is non-singular

48. The rank of the matrix $\begin{bmatrix} -1 & 2 & 5 \\ 2 & -4 & a-4 \\ 1 & -2 & a+1 \end{bmatrix}$ is

(A) 1 if $a = 6$ (B) 2 if $a = 1$
(C) 3 if $a = 2$ (D) 1 if $a = -6$

49. The system of equations $2x - 3y + 6z - 5t = 3$, $y - 4z + t = 1$, $4x - 5y + 8z - 9t = k$ has

(A) no solution if $k \neq 7$
(B) no solution if $k = 7$
(C) infinite solutions if $k \neq 7$
(D) infinite solutions if $k = 7$

50. Which of the following is correct?

(A) If A is a symmetric matrix, then A^n is symmetric, $n \in N$
(B) If A is a skew-symmetric matrix then A^n is symmetric if n is even, $n \in N$
(C) If A is a skew-symmetric matrix then A^n is skew-symmetric if n is odd, $n \in N$
(D) All of these

51. If A is a non-singular matrix, then

(A) A^{-1} is symmetric if A is symmetric
(B) A^{-1} is skew-symmetric if A is symmetric
(C) $|A^{-1}| = |A|$
(D) $|A^{-1}| = |A|^{-1}$

52. Which of the following is true?

(A) Transpose of an orthogonal matrix is also orthogonal
(B) Every orthogonal matrix is non-singular
(C) Product of the two orthogonal matrices is also orthogonal
(D) Inverse of an orthogonal matrix is also orthogonal

53. Suppose, a, b, c are real numbers such that $abc = 1$. If the matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ is such that $A'A = I$, then the value of $a^3 + b^3 + c^3$ is

(A) 1 (B) 2 (C) 3 (D) 4

54. Let A, B, C be 2×2 matrices with entries from the set of real numbers. Define operation '*' as follows

$$A * B = \frac{1}{2}(AB + BA), \text{ then}$$

(A) $A * I = A$
(B) $A * A = A^2$
(C) $A * B = B * A$
(D) $A * (B + C) = A * B + A * C$

55. If A and B are two matrices such that $AB = BA$, then $\forall\, n \in N$

(A) $A^n B = BA^n$
(B) $(AB)^n = A^n B^n$
(C) $(A + B)^n = {}^nC_0 A^n + {}^nC_1 A^{n-1} B + {}^nC_2 A^{n-2} B^2 + \ldots + {}^nC_n B^n$
(D) $A^{2n} - B^{2n} = (A^n - B^n)(A^n + B^n)$

56. If $A^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$, then

(A) $|A| = 2$
(B) $adj.\ A = \begin{bmatrix} \frac{1}{2} & 0 & -1 \\ -1 & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$
(C) $|adj.\ A| = 4$
(D) $|A'| = \dfrac{1}{2}$

57. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then

(A) $A^3 = I$ (B) $A^{-1} = A^2$
(C) $A^n = A$, $\forall\, n \neq 4$ (D) None of these

58. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, the values of α, β such that $(\alpha I + \beta A)^2 = A^2$ arc

(A) $\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}$ (B) $\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}$
(C) $\pm\frac{i}{\sqrt{2}}, \pm\frac{i}{\sqrt{2}}$ (D) $\pm\frac{i}{\sqrt{2}}, \mp\frac{i}{\sqrt{2}}$

Passage Based Questions

Passage 1

A matrix is said to be of rank r when it contains at least one non-zero minor of order r and no such minor of order $r + 1$. The rank of a matrix is denoted by $\rho(A)$.

By means of elementary transformations every non-zero matrix of rank r can be reduced to one of the following forms

(A) $\begin{bmatrix} I_r & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & & 0 \end{bmatrix}$ (B) $\begin{bmatrix} I_r \\ \cdots \\ 0 \end{bmatrix}$

(C) $\begin{bmatrix} I_r & \vdots & 0 \end{bmatrix}$ (D) $[I_r]$

Where I_r is a r-rowed unit matrix. These are called the normal forms of the given matrix and the value of r is the rank of the matrix. In order to find the rank of a given matrix, reduce the matrix to its normal form. This process of reducing a matrix of rank r to its normal form is known as 'The sweep-out process'.

59. The rank of the matrix $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$ is

(A) 1 (B) 2 (C) 3 (D) 4

60. Rank of the matrix $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ is

(A) 1 (B) 2 (C) 3 (D) 4

Passage 2

A non-zero matrix A is said to be in Echelon from if it satisfies the following conditions

1. All non-zero rows of A, if any, precede the zero rows
2. The number of zeroes preceding the first non-zero element in a row, is less than the number of such zeros in the succeeding row.
3. The first non-zero element in a row, is unity.
   For example, is in Echelon form.

The number of non-zero rows of a matrix in the Echelon form, is its rank. For example, the rank of the above matrix is 3. By elementary transformations, we reduce the matrix to Echelon form and then find its rank as the rank of a matrix remains unaltered by elementary transformations.

61. The rank of the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ is

(A) 1 (B) 2
(C) 3 (D) can't determine

62. The rank of the matrix $A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$ is

(A) 1 (B) 2 (C) 3 (D) 0

Assertion-Reason Type

Instructions: *In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:*

(A) Assertion(A) is True and Reason(R) is True; Reason(R) is a correct explanation for Assertion(A)
(B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)
(C) Assertion(A) is True, Reason(R) is False
(D) Assertion(A) is False, Reason(R) is True

63. **Assertion:** A is the $n \times n$ matrix whose elements are all '1' and B is the $n \times n$ matrix whose diagonal elements are all 'n' and other elements are '$n - r$'. Then, $(B - rI)[B - (n^2 - nr + r)I] = 0$ because

Reason: A^2 is a scalar multiple of A.

Previous Year's Questions

64. If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then **[2003]**

(A) $\alpha = a^2 + b^2, \beta = ab$
(B) $\alpha = a^2 + b^2, \beta = 2ab$
(C) $\alpha = a^2 + b^2, \beta = a^2 - b^2$
(D) $\alpha = 2ab, \beta = a^2 + b^2$

65. Let $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. The only correct statement about the matrix A is **[2004]**

(A) A is a zero matrix
(B) $A^2 = I$
(C) A^{-1} does not exist
(D) $A = (-1)I$, where I is a unit matrix

66. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}(10)$ $B = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix}$. If B is the inverse of matrix A, then α is **[2004]**

(A) -2 (B) 5 (C) 2 (D) -1

67. If $A^2 - A + I = 0$, then the inverse of A is **[2005]**

(A) $A + I$ (B) A
(C) $A - I$ (D) $I - A$

68. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then which one of the following holds for all $n \geq 1$, by the principle of mathematical induction **[2005]**

(A) $A^n = nA - (n-1)I$
(B) $A^n = 2^{n-1}A - (n-1)I$
(C) $A^n = nA + (n-1)I$
(D) $A^n = 2^{n-1}A + (n-1)I$

69. If A and B are square matrices of order $n \times n$ such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following will always be true? **[2006]**

(A) $A = B$
(B) $AB = BA$
(C) either of A or B is a zero matrix
(D) either of A or B is an identity matrix

70. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in N$. Then **[2006]**

(A) there cannot exist any B such that $AB = BA$
(B) there exist more than one but finite number of B's such that $AB = BA$
(C) there exists exactly one B such that $AB = BA$
(D) there exist infinitely many B's such that $AB = BA$

71. Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$, If $|A^2| = 25$ then $|\alpha|$ equals **[2007]**

(A) 5 (B) 1
(C) 1/5 (D) 5

72. The number of 3×3 non-singular matrices, with four entries as 1 and all other entries as 0, is **[2010]**

(A) 5 (B) 6
(C) at least 7 (D) less than 4

73. Let A and B be two symmetric matrices of order 3. **[2011]**

Statement 1: $A(BA)$ and $(AB)A$ are symmetric matrices.
Statement 2: AB is symmetric matrix if matrix multiplication of A and B is commutative.

(A) Statement 1 is true, Statement 2 is true; Statement 2 is **not** a correct explanation for Statement 1
(B) Statement 1 is true, Statement 2 is false.
(C) Statement 1 is false, Statement 2 is true.
(D) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1

74. Let A = $\begin{pmatrix} 100 \\ 210 \\ 321 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to **[2012]**

(A) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (B) $\begin{pmatrix} -1 \\ 1 \\ -0 \end{pmatrix}$ (C) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ (D) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

75. The number of values of k, for which the system of equations **[2013]**

$(k + 1)x + 8y = 4k$
$kx + (k + 3)y = 3k - 1$

has no solution, is

(A) 1 (B) 2 (C) 3 (D) infinite

76. If A is a 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then BB' equals **[2014]**

(A) $I + B$ (B) I (C) B^{-1} (D) $(B^{-1})'$

77. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then ordered pair (a, b) is equal to **[2015]**

(A) $(-2, 1)$ (B) $(2, 1)$
(C) $(-2, -1)$ (D) $(2, -1)$

78. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and A adj $A = A A^T$, then $5a + b$ is equal to **[2016]**

(A) 13 (B) -1 (C) 5 (D) 4

ANSWER KEYS

Single Option Correct Type

1. (A) **2.** (C) **3.** (C) **4.** (A) **5.** (A) **6.** (A) **7.** (C) **8.** (D) **9.** (C) **10.** (C)
11. (D) **12.** (A) **13.** (B) **14.** (B) **15.** (C) **16.** (D) **17.** (A) **18.** (B) **19.** (C) **20.** (D)
21. (C) **22.** (D) **23.** (C) **24.** (A) **25.** (A) **26.** (B) **27.** (B) **28.** (A) **29.** (C) **30.** (D)
31. (C) **32.** (A) **33.** (C) **34.** (B) **35.** (C) **36.** (A) **37.** (B) **38.** (A) **39.** (C) **40.** (B)
41. (B) **42.** (A) **43.** (B) **44.** (A) **45.** (B) **46.** (A)

More than One Option Correct Type

47. (A) and (B) **48.** (B) and (D) **49.** (A) and (D) **50.** (A), (B), (C) and (D)
51. (A) and (D) **52.** (A), (B), (C) and (D) **53.** (B) and (D) **54.** (A), (B), (C) and (D)
55. (A), (B), (C) and (D) **56.** (B) and (D) **57.** (A) and (B) **58.** (A) and (D)

Passage Based Questions

59. (C) **60.** (D) **61.** (B) **62.** (A)

Assertion-Reason Type

63. (A)

Previous Year's Questions

64. (B) **65.** (B) **66.** (B) **67.** (D) **68.** (A) **69.** (B) **70.** (D) **71.** (C) **72.** (C) **73.** (A)
74. (D) **75.** (A) **76.** (B) **77.** (C) **78.** (C)

HINTS AND SOLUTIONS

Single Option Correct Type

1. We have, $AB = BA = B'A' = (AB)'$

$\Rightarrow$ AB is symmetric.

Also, $ABA^{-1} = BAA^{-1} = B$ $(\because AB = BA)$

$\Rightarrow A^{-1}ABA^{-1} = A^{-1}B \Rightarrow BA^{-1} = A^{-1}B$

Therefore, $(A^{-1}B)' = (BA^{-1}) = (A^{-1})'B' = \text{A}'B$

$(\because A^{-1}$ and B are symmetric)

Thus, the matrix $A^{-1}B$ is symmetric.

The correct option is (A)

2. We have $(BA)^{-1} = C \Rightarrow A^{-1}B^{-1} = C \Rightarrow A^{-1} = CB$

$$\therefore \quad A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -5 & -5 \\ 0 & 9 & 2 \\ 2 & 14 & 6 \end{bmatrix}$$

The correct option is (C)

3. Given: $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = |A|^3 = 125.$

Now $|A| = \alpha^2 - 4$

$\Rightarrow (\alpha^2 - 4)^3 = 125 = 5^3$

$\Rightarrow \alpha^2 - 4 = 5$

$\Rightarrow \alpha = \pm 3$

The correct option is (C)

4. Since A is an involutory matrix

$\therefore A^2 = I$

$\Rightarrow I - A^2 = 0$

$\therefore (I - A)(I + A) = 0$

The correct option is (A)

5. $A^2 = 2A - 1$ (1)

Multiplying by A, we have

$A^3 = 2A^2 - A = 2(2A - I) - A$ [Using (1)]

$\Rightarrow A^3 = 3A - 2I$

Again, multiplying by A, we get

$A^4 = 3A^2 - 2AI \Rightarrow A^4 = 3(2A - I) - 2A$ [Using (1)]

$\Rightarrow \quad A^4 = 6A - 3I - 2A$

$\Rightarrow \quad A^4 = 4A - 3I$

Hence, by induction, we have

$A^n = nA - (n-1)I$

The correct option is (A)

6. We have $AB = A$

$\Rightarrow \quad A(BA) = A \qquad (\because BA = B)$

$\Rightarrow \quad (AB)A = A$

$\Rightarrow \quad AA = A \qquad (\therefore AB = A)$

$\Rightarrow \quad A^2 = A$

Again $BA = B$

$\Rightarrow \quad B(AB) = B \qquad (\because AB = A)$

$\Rightarrow \quad (BA)B = B$

$\Rightarrow \quad BB = B \Rightarrow B^2 = B$

The correct option is (A)

7. Consider $n = 2$

$$\therefore \quad (A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA) = (A^{-1}B)(A^{-1})(BA) = A^{-1}B^2A$$

Again for $n = 3$,

we have

$(A^{-1}BA)^3 = (A^{-1}B^2A)(A^{-1}BA) = A^{-1}B^3A$

$\therefore$ Generalizing the case $(A^{-1}BA)^n = A^{-1}B^nA$

The correct option is (C)

8. Consider

$$A^2 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

But when $n = 2$ no choice among (A), (B), (C) are match with A^2

The correct option is (D)

9. Trace of $A = a_{11} + a_{22} + \ldots a_{nn} = k + k + \ldots + k = nk$.

The correct option is (C)

10. $$A^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\therefore \quad A$ is nilpotent

The correct option is (C)

11. The matrix can be written as

$$\begin{bmatrix} 2\sin\theta - 1 & \sin\theta & \cos\theta \\ -\sin\theta & 2\cos\theta - \sqrt{3} & \tan\theta \\ -\cos\theta & -\tan\theta & 0 \end{bmatrix}$$

The above matrix is skew symmetric if

$2\sin\theta - 1 = 0$ and $2\cos\theta - \sqrt{3} = 0$

The simultaneous equations hold in $[0, 2\pi]$ if $\theta = \frac{\pi}{6}$

The correct option is (D)

12. Let $S = E(\theta)\,E(\phi)$

$$\Rightarrow \quad S = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$\Rightarrow \quad S = \begin{bmatrix} \cos\theta\cos\phi\cos(\theta \boxtimes \phi) & \cos\theta\sin\phi\cos(\theta \boxtimes \phi) \\ \cos\phi\sin\theta\cos(\theta \boxtimes \phi) & \sin\theta\sin\phi\cos(\theta \boxtimes \phi) \end{bmatrix}$$

$$\Rightarrow \quad S = \begin{bmatrix} \cos\theta\cos\phi\cos(2n+1)\frac{\pi}{2} & \cos\theta\sin\phi\cos(2n+1)\frac{\pi}{2} \\ \cos\phi\sin\theta\cos(2n+1)\frac{\pi}{2} & \sin\theta\sin\phi\cos(2n+1)\frac{\pi}{2} \end{bmatrix}$$

$$\therefore \quad S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \left[\because \theta - \phi = (2n+1)\,\frac{\pi}{2}\ \text{(given)}\right]$$

The correct option is (A)

13. Given, $B = -A^{-1}BA$

$\therefore \quad AB = -AA^{-1}BA = -IBA = -BA \qquad (\therefore AB = -BA)$

Now $\quad (A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$

$\qquad = A^2 + B^2 \qquad (\therefore BA = -AB)$

Thus, $\quad (A+B)^2 = A^2 + B^2$

The correct option is (B)

14. $|A| = 1 + \tan^2\frac{\theta}{2} = \sec^2\frac{\theta}{2}$

$AB = I \quad \Rightarrow \quad B = IA^{-1}$

$$= \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}}{\sec^2\frac{\theta}{2}} = \cos^2.\frac{\theta}{2}\ A^T$$

The correct option is (B)

15. $$A(z) = A\left(\frac{x+y}{1+xy}\right) = \left[\frac{1+xy}{(1-x)(1-y)}\right]$$

$$\begin{bmatrix} 1 & -\left(\frac{x+y}{1+xy}\right) \\ -\left(\frac{x+y}{1+xy}\right) & 1 \end{bmatrix}$$

$\therefore \quad A(x)\cdot A(y) = A(z)$

The correct option is (C)

16. Let A be a skew symmetric matrix of order n. By definition

$A' = -A$

$\Rightarrow \quad |A'| = |-A| \quad \Rightarrow \quad |A| = (-1)^n\,|A|$

$\Rightarrow \quad |A| = -|A| \qquad (\because n \text{ is odd})$

$\Rightarrow \quad 2|A| = 0 \quad \Rightarrow \quad |A| = 0$

$\therefore \quad A^{-1}$ does not exist.

The correct option is (D)

17. The system is $0x_1 + x_2 - x_3 = 1$

$-x_1 + 0x_2 + 2x_3 = 2$

$x_1 - 2x_2 + 0x_3 = 3$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ or } AX = B$$

Clearly, $|A| = 0$

$$\text{Now Adj } A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$\therefore$ (Adj A) $B \neq 0 \Rightarrow$ system is inconsistent

The correct option is (A)

18. Since $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is a square root of I_2 i.e., two rowed unit matrix

$$\therefore \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 + \beta\gamma & \alpha\beta - \alpha\beta \\ \gamma\alpha - \gamma\alpha & \gamma\beta + \alpha^2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \alpha^2 + \beta\gamma \end{bmatrix}$$

$\therefore \quad \alpha^2 + \beta\gamma = 1 \Rightarrow 1 - \alpha^2 - \beta\gamma = 0$

The correct option is (B)

19. Clearly both statements are true but statement 2 is not a correct explanation of statement 1.

The correct option is (C)

20. $A(u_1 + u_2) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, |A| = 1$

$$A^{-1} = \frac{1}{|A|}\text{adj} A$$

$$u_1 + u_2 = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

The correct option is (D)

21. $|A| = ad - bc$

If A is invertible then $|A| \neq 0$

$\Rightarrow \quad ad \neq bc$

$\Rightarrow$ (i) $ad = 0$ and $bc = 1$

or (ii) $ad = 1$ and $bc = 0$

$\Rightarrow$ $(a, d, b, c) = (0, 0, 1, 1), (0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 0, 0),$

$(1, 1, 0, 1), (1, 1, 1, 0)$

$\Rightarrow$ 6 matrices

The correct option is (C)

22. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad a^2 + bc = 1 = bc + d^2$ and $(a + d)c = 0 = (a + d)b$

Since $A \neq 1$, $A \neq 1$, $a = -d$ and hence

$$\det A = \begin{vmatrix} \sqrt{1 - bc} & b \\ c & -\sqrt{1 - bc} \end{vmatrix}$$

$$= -1 + bc - bc = -1$$

Statement 1 is true.

But tr. $A = 0$, and hence statement-2 is false.

The correct option is (D)

23. Each entry of A is integer, so the cofactor of every entry is an integer and hence each entry in the adjoint of matrix A is integer.

Now det $A = \pm 1$ and $A^{-1} = \dfrac{1}{\det(A)}$(adj A)

$\Rightarrow$ all entries in A^{-1} are integers.

The correct option is (C)

24. $A^{p+1} = (B + C)^{p+1}$

$$= {}^{p+1}C_0\, B^{p+1} + {}^{p+1}C_1\, B^pC + {}^{p+1}C_2\, B^{p-1}\, C^2 + \ldots + {}^{p+1}C_r\, B^{p+1-r}C^r + \ldots$$

[using $BC = CB$]

But $C^2 = 0$

$\Rightarrow \quad C^3 = C^4 = \ldots = C^r = 0.$

Thus, $A^{p+1} = {}^{p+1}C_0\, B^{p+1} + {}^{p+1}C_1\, B^pC$

$$= B^{p+1} + (p + 1)B^pC = B^p[B + (p + 1)C].$$

Therefore, $k = p$.

The correct option is (A)

25. We know that adj (adj A) $= |A|^{n-2}A$ if $|A| \neq 0$, provided order of A is n.

$\therefore$ adj (adj A) $= |A|\, A$ (as $n = 3$)

$\therefore$ det (adj (adj A)) $= |A|^3 \det A = |A|^4$.

$$\text{But } |A| = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix} = 14.$$

$\therefore$ det (adj (adj A)) $= (14)^4$.

The correct option is (A)

26. We have,

$$AA' = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b^2+c^2 & ac+ab+bc & ab+bc+ca \\ ca+ab+bc & a^2+b^2+c^2 & cb+ba+ac \\ ab+cb+ac & bc+ca+ab & a^2+b^2+c^2 \end{bmatrix} = I.$$

$\therefore \quad a^2+b^2+c^2 = 1$ and $ab+bc+ca = 0$.

$\Rightarrow \quad a+b+c = \pm 1$

Also, $abc = p$. Therefore, a, b, c are the roots of the equation $x^3 \pm x^2 + p = 0$.

The correct option is (B)

27. Let A be of order $n \times n$.

Since $A \times \text{adj } A = |A|\, I$

$\therefore \quad |A \times \text{adj } A| = |A|^n$

$\Rightarrow \quad |A|\, |\text{adj } A| = |A|^n$

$\therefore \quad |\text{adj } A| = |A|^{n-1}$

Since A is singular,

$\therefore \quad |A| = 0.$

$\therefore \quad |\text{adj } A| = 0.$

Hence, adj A is singular.

The correct option is (B)

28. $A^2 = 2A - I \qquad (1)$

Multiplying by A, we have

$A^3 = 2A^2 - A = 2(2A - I) - A \qquad$ {Using (1)}

$\Rightarrow \quad A^3 = 3A - 2I$

Again, multiplying by A, we get

$A^4 = 3A^2 - 2AI \Rightarrow A^4 = 3(2A - I) - 2A \qquad$ {Using (1)}

$\Rightarrow \quad A^4 = 6A - 3I - 2A$

$\Rightarrow \quad A^4 = 4A - 3I$

Hence, by induction, we have $A^n = nA - (n-1)I$

The correct option is (A)

29. $$A(z) = A\left(\frac{x+y}{1+xy}\right) = \left[\frac{1+xy}{(1-x)(1-y)}\right]$$

$$\begin{bmatrix} 1 & -\left(\dfrac{x+y}{1+xy}\right) \\ -\left(\dfrac{x+y}{1+xy}\right) & 1 \end{bmatrix}$$

$\therefore \quad A(x) \cdot A(y) = A(z).$

The correct option is (C)

30. Let A be a skew-symmetric matrix of order n. By definition $A' = -A$

$\Rightarrow \quad |A'| = |-A| \Rightarrow |A| = (-1)^n\, |A|$

$\Rightarrow \quad |A| = -|A| \qquad [\because n \text{ is odd}]$

$\Rightarrow \quad 2\,|A| = 0 \Rightarrow |A| = 0.$

$\therefore \quad A^{-1}$ does not exist.

The correct option is (D)

31. Consider $n = 2$

$$\therefore \quad (A^{-1} BA)^2 = (A^{-1}BA)(A^{-1}BA)$$
$$= (A^{-1}B)(AA^{-1})(BA) = A^{-1}B^2A$$

Again, for $n = 3$, we have

$(A^{-1}BA)^3 = (A^{-1}B^2A)(A^{-1}BA) = A^{-1}B^3A$

$\therefore \quad$ Generalizing the case $(A^{-1}BA)^n = A^{-1}B^nA$

The correct option is (C)

32. Since A is invertible matrix, therefore $|A| \neq 0$.

Since $|A| = |A'|$

$\therefore \quad |A'| \neq 0$ i.e., A' is invertible.

Now, A adj $A = |A|\, I_n$

$\Rightarrow \quad (A \text{ adj } A)' = (|A|\, I_n)' \Rightarrow (\text{adj } A)'\, A' = |A|\, I_n \qquad (1)$

Also, $(\text{adj } A')\, A' = |A'|\, I_n$

$\Rightarrow \quad (\text{adj } A')\, A' = |A|\, I_n \qquad (\because |A| = |A'|) \qquad (2)$

From (1) and (2),

we have $(\text{adj } A')\, A' = (\text{adj } A)'\, A'$

$\Rightarrow \quad \text{adj } A' = (\text{adj } A)' \qquad$ (by cancellation law)

The correct option is (A)

33. We know that B (adj B) = $|B|\, I_n$ for every square matrix B of order n.

Replacing B by adj A,

we get (adj A) (adj (adj A)) = $|\text{adj } A|\, I_n$

$\Rightarrow \quad \text{adj } A\, (\text{adj } (\text{adj } A)) = |A|^{n-1}\, I_n$

$(\because |\text{adj } A| = |A|^{n-1})$

$\Rightarrow \quad A\, \{\text{adj } A\, (\text{adj } (\text{adj } A))\} = A\, \{|A|^{n-1}\, I_n\}$

(Pre-multiplying both sides by A)

$\Rightarrow \quad (A \text{ adj } A)\, (\text{adj } (\text{adj } A)) = |A|^{n-1}\, (AI_n)$

(by associativity)

$\Rightarrow \quad |A|\, I_n\, (\text{adj adj } A) = |A|^{n-1}\, A$

$(\because A \text{ adj } A = |A|\, I_n)$

$\Rightarrow \quad |A|\, (\text{adj adj } A) = |A|^{n-1}\, A$

$\Rightarrow \quad \text{adj adj } A = |A|^{n-2}\, A$

$(\because |A| \neq 0 \therefore$ dividing both sides by $|A|)$

The correct option is (C)

34. $$\text{Let } A = \begin{bmatrix} 4 & 5 & x \\ 5 & 6 & y \\ 6 & k & z \end{bmatrix} \sim \begin{bmatrix} 4 & 5 & x \\ 5 & 6 & x+d \\ 6 & k & x+2d \end{bmatrix}$$

$[\because x, y, z$ are in A.P.$]$

$$\sim \begin{bmatrix} 4 & 5 & x \\ 5 & 6 & x+d \\ 0 & k-7 & 0 \end{bmatrix}$$

[Applying $R_3 \to R_3 + R_1 - 2R_2$]

Rank of A is 2 if $k - 7 = 0$ i.e. $k = 7$ and d is any arbitrary number.

The correct option is (B)

35. $$\text{We have, } D = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

$$|D| = \begin{vmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} = a_1\, a_2\, a_3\, \ldots\, a_n$$

$$\text{adj } D = \begin{bmatrix} a_2a_3\ldots a_n & 0 & . \; . & 0 \\ 0 & a_1a_3\ldots a_n & . \; . & 0 \\ \vdots & \vdots & \vdots \; \vdots & \vdots \\ 0 & 0 & . \; . & a_1a_2\ldots a_{n-1} \end{bmatrix}$$

$$\therefore \quad D^{-1} = \frac{1}{|D|} \text{ adj } D = \frac{1}{a_1a_2a_3\ldots a_n}$$

$$= \begin{bmatrix} a_2\, a_3\ldots\, a_n & 0 & \cdot & 0 \\ 0 & a_1\, a_3\, \ldots\, a_n & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdot & a_1\, a_2\ldots a_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{a_1} & 0 & . & 0 \\ 0 & \frac{1}{a_2} & . & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & . & \frac{1}{a_n} \end{bmatrix} = \text{diag.}\,(a_1^{-1}\; a_2^{-1}\; a_3^{-1}\; \ldots\; a_n^{-1})$$

The correct option is (C)

36. We have, $B = A^{-1}A'$

Then, $BB' = (A^{-1}A')\,(A^{-1}A')' = (A^{-1}A')\,A(A^{-1})'$

$= A^{-1}A'\,A(A')^{-1}$ [by property of inverse]

$= A^{-1}\,(AA')\,(A')^{-1} = I.$ [$A'A = AA'$ given]

The correct option is (A)

37. We have,

$(I - A)\,(I + A + A^2) = I - A^3 = 1.$

This proves that $I - A$ and $I + A + A^2$ are inverse of each other.

The correct option is (B)

38. We have,

$A^n = I \;\Rightarrow\; A(A^{n-1}) = I \;\Rightarrow\; A^{n-1} = A^{-1}$

Thus, $BA^{n-1}\,B^{-1} - BA^{-1}\,B^{-1} = BA^{-1}\,B^{-1} - BA^{-1}\,B^{-1} = 0.$

The correct option is (A)

39. We can form matrices of order 12 × 1, 1 × 12, 6 × 2, 2 × 6, 4 × 3, 3 × 4, i.e., matrices of 6 different orders.

Again, 12 different real numbers can be arranged in 12! ways.

$\therefore$ Total number of different matrices = 6 (12)!

The correct option is (C)

40. Since A is skew-symmetric, therefore $A' = -A$

Again, $A^2 + I = 0 \;\Rightarrow\; A^2 = 0 - I = -I$

$\therefore \quad A.A = -I$

$\Rightarrow \quad A.A.A' = -IA' = I(-A')$

$= -I(-A) = IA = A.$

$\Rightarrow \quad A^{-1}\cdot A.A.A' = A^{-1}A$

$\Rightarrow \quad IAA' = I \Rightarrow AA' = I$

$\therefore \quad A$ is orthogonal matrix.

The correct option is (B)

41. Let $B = A + I_n$. Since $A = B - I_n$, the condition $A^n = \alpha A$ can be written in the form

$(B - I_n)^n = \alpha\,(B - I_n)$

$\Rightarrow \quad B^{n - n}C_1\,B^{n-1} + {}^nC_2\,B^{n-2} + \ldots + (-1)^n I_n$

$= \alpha B - \alpha I_n$

$\Rightarrow \quad B^{n - n}C_1\,B^{n-1} + {}^nC_2\,B^{n-2} + \ldots + (-1)^{n-1}\,B - \alpha B$

$= -\alpha I_n - (-1)^n\, I_n$

$\Rightarrow \quad B\,(B^{n-1-n}C_1\,B^{n-2} + {}^nC_2\,B^{n-3} + \ldots + (-1)^{n-1}\,I_n - \alpha I_n)$

$= [(-1)^{n+1} - \alpha]\, I_n$

Since $(-1)^{n+1} - \alpha \neq 0$, $\therefore$ B is invertible.

$\therefore \quad A + I_n$ is invertible i.e., non-singular.

The correct option is (B)

42. Writing $Q^{-1}BP^{-1}$ as $(Q^{-1}B)P^{-1}$, we have

adj $[(Q^{-1}B)P^{-1}] =$ (adj. P^{-1}) (adj Q^{-1} B)

[$\because$ adj $(AB) =$ adj B. adj. A]

$=$ (adj. P^{-1}) (adj B) (adj Q^{-1})

$= (|P^{-1}|\,(P^{-1})^{-1}\,A\,(|Q^{-1}|\,(Q^{-1})^{-1})$ $\left[\because A^{-1} = \dfrac{adj\ A}{|\,A\,|}\right]$

$= |P^{-1}|\,PA\,|Q^{-1}|\,Q$

$= \dfrac{1}{|\,P\,|}\,PA\,|\,Q^{-1}\,|\,Q$ $[\because (P^{-1})^{-1} = P, (Q^{-1})^{-1} = Q]$

$= PAQ$ $(\because |P| = |Q| = 1)$

The correct option is (A)

43. $A = \begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix} = \frac{1}{n}\begin{pmatrix} n & \alpha \\ -\alpha & n \end{pmatrix} = \frac{1}{n}\, B$

where, $B = \begin{pmatrix} n & \alpha \\ -\alpha & n \end{pmatrix}$

Put $n = r\cos\theta$, $\alpha = r\sin\theta$,

$\therefore \quad n^2 + \alpha^2 = r^2,\ \theta = \tan^{-1}\dfrac{\alpha}{n}$

$$\therefore \quad A = \frac{1}{n}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$= \frac{r}{n}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\therefore \quad A^n = \frac{r^n}{n^n}\begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

$$= \left(\frac{n^2 + \alpha^2}{n^2}\right)^{n/2}\begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

$$\therefore \quad \frac{A^n}{n} = \left(1 + \frac{\alpha^2}{n^2}\right)^{n/2}\begin{pmatrix} \frac{\cos n\theta}{n} & \frac{\sin n\theta}{n} \\ -\frac{\sin n\theta}{n} & \frac{\cos n\theta}{n} \end{pmatrix}$$

Now, $$\underset{n\to\infty}{Lt}\left(1+\frac{\alpha^2}{n^2}\right)^{n/2} = \underset{n\to\infty}{Lt}\left[\left(1+\frac{\alpha^2}{n^2}\right)^{\frac{n^2}{\alpha^2}}\right]^{\frac{\alpha^2}{2n}} = e^0 = 1$$

$$\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\cos n\theta = \underset{n\to\infty}{\text{Lt}}\ \frac{\sin n\theta}{n} = 0$$

[∵ cos $(n\theta)$, sin $(n\theta)$ lie between -1 and 1]

$$\therefore\quad \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}A^n = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0.$$

The correct option is (B)

44. Let $B = I + A + A^2 + \ldots + A^{k-1}$

$\therefore\quad B(I-A) = (I + A + A^2 + \ldots + A^{k-1})\,(I - A)$

$= I - A + A + A^2 - A^3 + \ldots + A^{k-1} - A^k$

$= I - A^k = I \qquad (\because A^k = 0)$

$\therefore\quad B = (I - A)^{-1}$

Hence, $(I - A)^{-1} = I + A + A^2 + \ldots + A^{k-1}$

Thus, $p = -1$.

The correct option is (A)

45. Since A satisfies the equation $x^3 - 5x^2 + 4x + kI = 0$, therefore, $A^3 - 5A^2 + 4A + kI = 0$

A^{-1} exists if $k \neq 0$ since if $k = 0$, then the above equation gives $A = 0$ and in that case A^{-1} will not exist.

The correct option is (B)

46. $\det M = 1 \Rightarrow \det M' = 1$

Now, $\det (M - I) = \det (M - I) \det M'$

$= \det (MM' - IM') \qquad (\because \det A \det B = \det (AB))$

$= \det (I - M') = -\det (M' - I) = -\det (M - I)$

Thus, $\det (M - I) = -\det (M - I) \Rightarrow \det (M - I) = 0$.

The correct option is (A)

More than One Option Correct Type

47. Let B be non-singular, then B^{-1} exists.

Now, $AB = 0$ (given) $\Rightarrow (AB)\,B^{-1} = 0B^{-1}$

(post-multiplying both sides by B^{-1})

$\Rightarrow\quad A\,(BB^{-1}) = 0 \qquad$ (by associativity)

$\Rightarrow\quad AI_n = 0 \qquad (\because BB^{-1} = I_n)$

$\Rightarrow\quad A = 0.$

But A is a non-null matrix. Hence, B is a singular matrix. Similarly, it can be shown that A is a singular matrix.

The correct option is (A) and (B)

48. $$A = \begin{bmatrix}-1 & 2 & 5\\ 2 & -4 & a-4\\ 1 & -2 & a+1\end{bmatrix} = \begin{bmatrix}0 & 0 & a+6\\ 0 & 0 & -a-6\\ 1 & -2 & a+1\end{bmatrix}$$

(Operating $R_1 \to R_1 + R_3$ and $R_2 \to R_2 - 2R_3$)

$$= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -a-6\\ 1 & -2 & a+1\end{bmatrix} \qquad (\text{Operating } R_1 \to R_1 + R_2)$$

When $a = -6$, $A = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & -2 & -5\end{bmatrix}$, $\quad\therefore \rho(A) = 1$

When $a = 6$, $A = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -12\\ 1 & -2 & -5\end{bmatrix}$, $\quad\therefore \rho(A) = 2$

When $a = 1$, $A = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -7\\ 1 & -2 & -2\end{bmatrix}$, $\quad\therefore \rho(A) = 2$

When $a = 2$, $A = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -8\\ 1 & -2 & 3\end{bmatrix}$, $\quad\therefore \rho(A) = 2.$

The correct option is (B) and (D)

49. The augmented matrix $C = [A\ B]$

$$= \begin{bmatrix}2 & -3 & 6 & -5 & 3\\ 0 & 1 & -4 & 1 & 1\\ 4 & -5 & 8 & -9 & k\end{bmatrix}$$

$$\sim \begin{bmatrix}2 & -3 & 6 & -5 & 3\\ 0 & 1 & -4 & 1 & 1\\ 0 & 1 & -4 & 1 & k-6\end{bmatrix}\ [R_3 \to R_3 - 2R_1]$$

$$\sim \begin{bmatrix}2 & -3 & 6 & -5 & 3\\ 0 & 1 & -4 & 1 & 1\\ 0 & 0 & 0 & 0 & k-7\end{bmatrix}\ [R_3 \to R_3 - R_2]$$

(i) There is no solution if $\rho(A) \neq \rho(C)$

Here, $\rho(A) = 2$. $\rho(C) = 3$ if $k - 7 \neq 0$ or $k \neq 7$.

(ii) There are infinite solutions if $\rho(A) = \rho(C) < 4$.

Already, $\rho(A) = 2$. $\rho(C) = 2$ if $k - 7 = 0$ or $k = 7$.

The correct option is (A) and (D)

50. Let A be a symmetric matrix.

Then, $(A^n)' = (AAA \ldots n \text{ times})'$

$= (A'\,A'\,A' \ldots A'\ n \text{ times})$

$= (A')^n = A^n \qquad (\because A' = A)$

$\therefore\quad A^n$ is also a symmetric matrix.

Now, let A be a skew-symmetric matrix. Then $A' = -A$.

We have, $(A^n)' = (AAA \ldots n \text{ times})$

$= (A'\,A'\,A' \ldots A'\ n \text{ times}) \qquad [\because (AB)' = B'\,A']$

$= (A')^n = (-A)^n \qquad (\because A' = -A)$

$= (-1)^n A^n$

$$= \begin{Bmatrix}A^n & \text{if } n \text{ is even}\\ -A^n & \text{if } n \text{ is odd}\end{Bmatrix}$$

Hence, A^n is symmetric if n is even and skew-symmetric if n is odd.

The correct option is (A), (B), (C) and (D)

51. Since $|A| \neq 0$, therefore A^{-1} exists.

Now, $AA^{-1} = I = A^{-1}A$

$\Rightarrow \quad (AA^{-1})' = I' = (A^{-1}A)'$

$\Rightarrow \quad (A^{-1})'\,A' = I = A'\,(A^{-1})'$

$\Rightarrow \quad (A^{-1})'\,A = I = A\,(A^{-1})' \qquad (\because A' = A)$

$\Rightarrow \quad A^{-1} = (A^{-1})' \Rightarrow A^{-1}$ is symmetric.

Also, since $|A| \neq 0$,

$\therefore \quad A^{-1}$ exists such that

$AA^{-1} = I = A^{-1}A \quad \Rightarrow \quad |AA^{-1}| = |I|$

$\Rightarrow \quad |A|\,|A^{-1}| = 1 \qquad (\because |AB| = |A|\,|B|)$

$\Rightarrow \quad |A^{-1}| = \dfrac{1}{|A|}$

The correct option is (A) and (D)

52. (A) For any orthogonal matrix A, we have

$A'A = I$

Let B be a matrix such that $AB = I$.

Now we have

$A' = A'I$ [by property of unit matrix]

$= A'(AB) = (A'A)B = IB = B$

Therefore, $(A')'(A) = AA' = AB = I$

$\Rightarrow \quad A'$ is orthogonal.

(B) For any orthogonal matrix A, we have

$A'A = I$

$\Rightarrow \quad |A'A| = |I| \quad \Rightarrow \quad |A'|\,|A| = 1$

$\Rightarrow \quad |A| \neq 0$, i.e., A is non-singular.

(C) Let A and B be two orthogonal matrices, therefore,

$(AB)'\,(AB) = B'A'AB$ [by property of transpose]

$= B'(A'A)B$ [by asssociative law]

$= B'(IB) = B'B = I$

$\Rightarrow \quad AB$ is orthogonal.

(D) Let A be orthogonal matrix and B be its inverse matrix. Then, we have,

$A'A = I$ (1)

and, $\quad AB = I = BA$ (2)

Now, we have,

$(AB)'\,(AB) = B'A'AB$

$= B'(A'A)B = B'(IB)B = B'B$ (3)

Also, from equation (2), we have

$(AB)' = I' = I$

i.e., $\quad B'B = I$ [Using equation (3)]

$\Rightarrow \quad B$ is orthogonal.

The correct option is (A), (B), (C) and (D)

53. We can see that $A' = A$.

Therefore, $A'A = I$

$\Rightarrow \quad A^2 = I \quad \Rightarrow \quad |A^2| = |I|$

$\Rightarrow \quad |A^2| = 1 \quad \Rightarrow \quad |A| = \pm 1$

Therefore, $a^3 + b^3 + c^3 - 3abc = \pm 1$

$\Rightarrow \quad a^3 + b^3 + c^3 = 2, 4.$

The correct option is (B) and (D)

54. (A) $A * I = \dfrac{1}{2}(AI + IA) = \dfrac{1}{2}(A + A) = A$

$\therefore \quad A * I = A,$

$\therefore \quad$ (A) holds.

(B) $A * A = \dfrac{1}{2}(AA + AA) = \dfrac{1}{2}(A^2 + A^2) = A^2$

$\therefore \quad$ (B) holds.

(C) $A * B = \dfrac{1}{2}(AB + BA)$

$B * A = \dfrac{1}{2}(BA + AB) = \dfrac{1}{2}(AB + BA)$

[$\because$ addition is commutative]

$\therefore \quad A * B = B * A,$

$\therefore \quad$ (C) holds

(D) $A * (B + C) = \dfrac{1}{2}(A(B + C) + (B + C)A)$

$= \dfrac{1}{2}(AB + AC + BA + CA)$

$= \dfrac{1}{2}(AB + BA) + \dfrac{1}{2}(AC + CA)$

$= A * B + A * C$

$\therefore \quad$ (D) holds.

The correct option is (A), (B), (C) and (D)

55. Given $AB = BA$

(A) The result $A^nB = BA^n$ is true for $n = 2$, since

$A^2B = A(AB) = A(BA) = (AB)A = (BA)A = BA^2$

Let the result be true for $n = k$

i.e., $A^kB = BA^k$

For $n = k + 1$, we have

$A^{k+1}B = A(A^kB) = A(BA^k) = (AB)A^k = (BA)A^k$

$= BAA^{k+1}$

Thus, the result is true for $n = k + 1$

Hence, by induction the result is true $\forall\, n \in N$

(B), (C) Since A and B commute, both these options hold.

(D) $(A^n - B^n)\,(A^n + B^n) = A^{2n} + A^n B^n - B^n A^n - B^{2n}$

$= A^{2n} - B^{2n}.$ [A and B commute]

The correct option is (A), (B), (C) and (D)

56. We have,

$$|A^{-1}| = \begin{vmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} = -2\,(-2 + 1) = 2$$

(A) $|A| = \dfrac{1}{|A^{-1}|} = \dfrac{1}{2}$

(B) $$adj\,A = A^{-1}\,|A| = \frac{1}{2}\begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & -1 \\ -1 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{bmatrix}$$

(C) $|adj\, A| = -1\left(\dfrac{-1}{2} + \dfrac{1}{4}\right) = \dfrac{1}{4}$

(D) $|A'| = |A| = \dfrac{1}{2}$

The correct option is (B) and (D)

57. $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Then, $A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

and, $A^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

i.e., $A(A^2) = A^2(A) = I$

$\Rightarrow$ A^2 is the inverse of A $[|A| \neq 0 \Rightarrow A^{-1}$ exists$]$

Also, $A^4 = A^3A = IA = A$

But $A^5 = A^4A = A^2 \neq A$.

The correct option is (A) and (B)

58. $A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1$

Also, $(\alpha I + \beta A)^2 = (\alpha I + \beta A)\,(\alpha I + \beta A)$

$= \alpha^2 I^2 + \alpha\beta IA + \beta\alpha AI + \beta^2 A^2$

$= \alpha^2 I + 2\alpha\beta A + \beta^2 A^2 \quad (\because I^2 = I, IA = AI)$

$= \alpha^2 I + 2\alpha\beta A - \beta^2 I \quad (\because A^2 = -I)$

$= (\alpha^2 - \beta^2)I + 2\alpha\beta A$

$= \begin{bmatrix} a^2 - \beta^2 & 0 \\ 0 & a^2 - \beta^2 \end{bmatrix} + 2\alpha\beta\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} a^2 - \beta^2 & 2\alpha\beta \\ -2\alpha\beta & a^2 - \beta^2 \end{bmatrix}$

On equating corresponding elements in $(\alpha I + \beta A)^2$ and A^2 we get

$\alpha^2 - \beta^2 = 0$ (1)

$2\alpha\beta = 1$ (2)

From (1),

$\alpha = \pm\beta$. If $\alpha = \beta$, then from (2),

$2\beta^2 = 1$

$\Rightarrow$ $\beta = \pm\dfrac{1}{\sqrt{2}}$ whence $\alpha = \pm\dfrac{1}{\sqrt{2}}$.

Again, if $\alpha = -\beta$, then from (2),

$2\beta^2 = -1$

$\Rightarrow$ $\beta = \mp\dfrac{i}{\sqrt{2}}$ whence $\alpha = \pm\dfrac{i}{\sqrt{2}}$.

The correct option is (A) and (D)

Passage Based Questions

59. We have,

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix} \quad C_1 \leftrightarrow C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 3 & -1 & 2 \end{bmatrix} \quad \begin{matrix} C_3 \to C_3 - 2C_1 \\ C_4 \to C_4 - C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -1 & 2 \end{bmatrix} \quad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 2 \end{bmatrix} \quad (C_3 \to C_3 + C_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \quad (R_3 \to R_3 - 3R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \left(R_3 \to \frac{1}{2}R_3\right)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (C_4 \to C_4 - C_3)$$

Hence, $[I_3 : 0]$ is the normal form of A and, therefore, the rank of the matrix A is 3.

The correct option is (C)

60. We have,

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \quad \begin{pmatrix} C_2 \to C_2 + C_1 \\ C_3 \to C_3 - 2C_1 \\ C_4 \to C_4 + 3C_1 \end{pmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 2\\ 0 & 3 & 1 & 4\\ 0 & 5 & -8 & 4\end{bmatrix} \quad (R_2 \leftrightarrow R_4)$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 3 & 1 & -2\\ 0 & 5 & -8 & 4\end{bmatrix} \quad \left(C_4 \to C_4 - 2C_2\right)$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -2\\ 0 & 5 & -8 & 4\end{bmatrix} \quad \left(R_3 \to R_3 - 3R_2\right)$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 5 & -8 & -12\end{bmatrix} \quad \left(C_4 \to C_4 + 2C_3\right)$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -12\end{bmatrix} \quad \begin{bmatrix}R_4 \to R_4 - 5R_1,\\ R_4 \to R_4 + 8R_3\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = I_4 \left[C_4 \to \frac{-1}{12} C_4\right]$$

Hence, $\rho(A) = 4$.

The correct option is (D)

61. $A = \begin{bmatrix}2 & 3 & 4\\ 3 & 1 & 2\\ -1 & 2 & 2\end{bmatrix} \sim \begin{bmatrix}-1 & 2 & 2\\ 3 & 1 & 2\\ 2 & 3 & 4\end{bmatrix} R_1 \leftrightarrow R_3$

$$\sim\begin{bmatrix}-1 & 2 & 2\\ 0 & 7 & 8\\ 0 & 7 & 8\end{bmatrix} \quad \begin{matrix}R_2 \to R_2 + 3R_1\\ R_3 \to R_3 - 2R_1\end{matrix}$$

$$\sim\begin{bmatrix}-1 & 2 & 2\\ 0 & 7 & 8\\ 0 & 0 & 0\end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim\begin{bmatrix}1 & -2 & -2\\ 0 & 1 & \frac{8}{7}\\ 0 & 0 & 0\end{bmatrix} \quad \begin{matrix}R_1 \to -R_1\\ R_2 \to \frac{1}{7} R_2\end{matrix}$$

The last equivalent matrix is in Echelon form. The number of non-zero rows in this matrix = 2

Rank of A = Number of non-zero rows = 2.

The correct option is (B)

62. $A = \begin{bmatrix}1 & 3 & 4 & 3\\ 3 & 9 & 12 & 9\\ -1 & -3 & -4 & -3\end{bmatrix}$

$$\sim\begin{bmatrix}1 & 3 & 4 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \quad \begin{matrix}R_2 \to R_2 - 3R_1\\ R_3 \to R_3 + R_1\end{matrix}$$

The equivalent matrix is in Echelon form. The number of non-zero rows in this matrix is 1. Therefore, the rank of $A = 1$.

Assertion-Reason Type

63. Here, $A = \begin{bmatrix}1 & 1 & ....1\\ 1 & 1 & ....1\\ 1 & 1 & ....1\end{bmatrix}$

$$\therefore \quad A^2 = A.A = \begin{bmatrix}1 & 1 & ....1\\ 1 & 1 & ....1\\ 1 & 1 & ....1\end{bmatrix}\begin{bmatrix}1 & 1 & ....1\\ 1 & 1 & ....1\\ 1 & 1 & ....1\end{bmatrix}$$

$$= \begin{bmatrix}n & n & ....n\\ n & n & ....n\\ . & . & ....\\ n & n & ....n\end{bmatrix} = nA$$

$$B = \begin{bmatrix}n & n-r & ....n-r\\ n-r & n & ....n-r\\ .... & .... & ....\\ n-r & n-r & ....n\end{bmatrix}$$

$$\therefore \quad B - rI = \begin{bmatrix}n-r & n-r & ....n-r\\ n-r & n-r & ....n-r\\ .... & .... & ....\\ n-r & n-r & ....n-r\end{bmatrix}$$

$$= (n-r)\,A$$

Hence, $(B - rI)\,[B - (n^2 - nr + r)\,I]$

$$= (B - rI)\,[(B - rI) - n(n-r)\,I]$$

$$= (n-r)\,A[(n-r)\,A - n(n-r)\,I]$$

$$= (n-r)^2\,A\,(A - nI)$$

$$= (n-r)^2\,A^2 - n(n-r)^2\,AI$$

$$= (n-r)^2\,]\,[A^2 - nA]$$

$$= (n-r)^2\,[nA - nA] \qquad [\because A^2 = nA]$$

$$= (n-r)^2\,(0) = 0.$$

$$= B\,A$$

The correct option is (A)

Previous Year's Questions

64. We have $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ so that

$$A^2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix}$$

$\Rightarrow \quad \alpha = a^2 + b^2, \beta = 2ab.$

The correct option is (B)

65. For given A, $AA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$.

The correct option is (B)

66. For $AB = I$, we have

$A(10\,B) = 10\,I$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 & 5-\alpha \\ 0 & 10 & \alpha-5 \\ 0 & 0 & 5+\alpha \end{bmatrix} = 10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ if } \alpha = 5\,.$$

The correct option is (B)

67. Given $A^2 - A + I = 0$

Multiplying $A - I$ on both sides

$A^{-1}A^2 - A^{-1}A + A^{-1} - I = A^{-1} \cdot 0$

$\Rightarrow \quad A - I + A^{-1} = 0$

or $\quad A^{-1} = I - A.$

The correct option is (D)

68. By the principle of mathematical induction (1) is true.

The correct option is (A)

69. Given condition $A^2 - B^2 = (A - B)\,(A + B)$

$\Rightarrow \quad A^2 - B^2 = A^2 + AB - BA - B^2$

$\Rightarrow \quad AB = BA.$

The correct option is (B)

70. Given that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$\Rightarrow AB = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}$, and

$$BA = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3b & 4b \end{bmatrix}$$

Now, $AB = BA$ only when $a = b$

The correct option is (D)

71. The product

$$A^2 = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 25 & 25\alpha + 5\alpha^2 & 5\alpha + 25\alpha^2 + 5\alpha \\ 0 & \alpha^2 & 5\alpha^2 + 25\alpha \\ 0 & 0 & 25 \end{bmatrix}$$

$\Rightarrow \quad 625\alpha^2 = 25$

$\Rightarrow \quad |\alpha| = \dfrac{1}{5}$

The correct option is (C)

72. First row with exactly one zero; total number of cases = 6

First row 2 zeros we get more cases

Total we get more than 7.

Directions: Questions Number 72 to 76 are Assertion-Reason type questions. Each of these questions contains two statements.

Statement 1: (Assertion) and **Statement 2: (Reason)**

Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice.

The correct option is (C)

73. $A^T = A, B^T = B:$

$(A(BA))^T = (BA)^T A^T = (A^T B^T)A = (AB)A = A(BA)$

$((AB)A)^T = A^T (AB)^T = A(B^T A^T) = A(BA) = (AB)A$

$\therefore \quad$ Statement 1 is correct

Statement 2

$(AB)^T = B^T A^T = BA = AB \qquad (\because$ AB is commutative)

Statement 2 is also correct but it is not correct explanation of Statement 1

The correct option is (A)

74. $A = \begin{pmatrix} 100 \\ 210 \\ 321 \end{pmatrix}$

Let $u_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}; u_2 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$

$$Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \quad u_1 + u_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

The correct option is (D)

75. For no solution

$$\frac{K+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1} \qquad (1)$$

$\Rightarrow \quad (k+1)(k+3) - 8k = 0$

or, $\quad k^2 - 4k + 3 = 0 \Rightarrow k = 1, 3$

But for $k = 1$, equation (1) is not satisfied

Hence $k = 3$.

The correct option is (A)

76. $B = A^{-1}A' \Rightarrow AB = A'$

$ABB' = A'B' = (BA)' = (A^{-1}A'A)' = (A^{-1}AA')' = A$.

$\Rightarrow \quad BB' = I$.

The correct option is (B)

77. Given that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$

Let $AA^T = \left[b_{ij}\right]_{3\times 3}$

Then, $\quad b_{23} = 0 \Rightarrow 0 = 2a + 2 - 2b$,

$\qquad b_{13} = 0 \Rightarrow 0 = a + 4 + 2b$

$\Rightarrow \quad 3a + 6 = 0 \Rightarrow a = -2, b = -1$.

The correct option is (C)

75. $\quad A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A^T = \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix}$$

Now, A adj $A = |A| I_2 = \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix}$

Given $AA^T = A$. adj A

$15a - 2b = 0 \qquad (1)$

$10a + 3b = 13 \qquad (2)$

Solving we get

$5a = 2$ and $b = 3$

$\therefore \quad 5a + b = 5$

The correct option is (C)