CHAPTER 11

# Limits

## Chapter Highlights

Limit of a function, Indeterminate forms, Algebra of limits, Evaluation of limits, Algebraic limits, Limit of an algebraic function when $x \to \infty$, Trigonometric limits, Exponential and logarithmic limits, Evaluation of limits using L'Hospital's rule

## LIMIT OF A FUNCTION

Let a function $f$ be defined at every point in the neighbourhood of $a$ (an open interval about $a$) except possibly at $a$. If as $x$ approaches closer and closer to $a$, but not equal to $a$, then the value of the function $f(x)$ approaches a real number $l$. The number $l$ is referred to as the limit of $f(x)$ as $x$ tends to $a$ and we write it as

$$\lim_{x \to a} f(x) = l$$

Note that $f(x)$ approaches $l$ means the absolute difference between $f(x)$ and $l$, i.e. $|f(x) - l|$ can be made as small as we please.

When the values of $f(x)$ do not approach a single finite value as $x$ approaches $a$, we say that the limit Does not exist.

**CAUTION**

A number is said to be a limiting value only if it is finite and real, otherwise we say that the limit does not exist.

### Right Hand Limit

We say that right hand limit of $f(x)$ as $x$ tends to '$a$' exists and is equal to $l_1$ if as $x$ approaches '$a$' through values greater than '$a$', the values of $f(x)$ approach a definite unique real number $l_1$ and we write

$$\lim_{x \to a^+} f(x) = l_1 \quad \text{or} \quad f(a+0) = l_1$$

### Method for Finding Right Hand Limit

To evaluate $\lim_{x \to a^+} f(x)$

1. Put $x = a + h$ in $f(x)$ to get
2. Take the limit as $h \to 0$.

### Left Hand Limit

We say that left hand limit of $f(x)$ as $x$ tends to '$a$' exists and is equal to $l_2$ if as $x$ approaches '$a$' through values less than '$a$', the values of $f(x)$ approach a definite unique real number $l_2$ and we write

$$\lim_{x \to a^-} f(x) = l_2$$

or

$$f(a-0) = l_2$$

### Method for Finding Left Hand Limit

To evaluate $\lim_{x \to a^-} f(x)$.

1. Put $x = a - h$ in $f(x)$ to get $\lim_{h \to 0} f(a+h)$.
2. Take the limit as $h \to 0$.

**CAUTION**

For finding $\lim_{x \to a} f(x)$, we study the behaviour of the function $f$ in the neighbourhood of '$a$' and not at '$a$'. Thus, the function $f$ may or may not be defined at $x = a$.

## SOLVED EXAMPLES

**1.** $\lim_{x \to 5} \dfrac{x^2 - 9x + 20}{x - [x]} =$

(A) 1
(B) 0
(C) Does not exist
(D) Cannot be determined

**Solution: (C)**

$$\lim_{x \to 5} \frac{x^2 - 9x + 20}{x - [x]} = \lim_{x \to 5} \frac{(x-4)(x-5)}{x - [x]}$$

$$\text{LHL} = \lim_{x \to 5-} \frac{(x-4)(x-5)}{x - [x]}$$

$$= \lim_{h \to 0} \frac{(5-h-4)(5-h-5)}{(5-h) - [5-h]} \quad (h > 0)$$

$$= \lim_{h \to 0} \frac{(1-h)(-h)}{5-h-4} = \lim_{h \to 0} \frac{(1-h)(-h)}{(1-h)}$$

$\therefore$ LHL = 0

Also, $$\text{RHL} = \lim_{x \to 5} \frac{(x-4)(x-5)}{x - [x]}$$

$$= \lim_{h \to 0} \frac{(5+h-4)(5+h-5)}{(5+h) - [5+h]}$$

$$= \lim_{h \to 0} \frac{(1+h)(h)}{5+h-5} = \lim_{h \to 0} \frac{(1+h)h}{h}$$

$\therefore$ RHL = 1

As LHL $\neq$ RHL

$\therefore$ Limit does not exist.

**2.** If $A_i = \dfrac{x - a_i}{|x - a_i|}$, $i = 1, 2, \ldots, n$ and if $a_1 < a_2 < a_3 < \ldots < a_n$.

Then $\lim_{x \to a_m} (A_1 A_2 \ldots A_n)$, $1 \le m \le n$

(A) is equal to $(-1)^m$
(B) is equal to $(-1)^{m+1}$
(C) is equal to $(-1)^{m-1}$
(D) Does not exist

**Solution: (D)**

We have, $A_i = \dfrac{x - a_i}{|x - a_i|}$, $i = 1, 2, \ldots, n$

and $a_1 < a_2 < \ldots a_{n-1} < a_n$.

Let $x$ be in the left neighbourhood of $a_m$.

Then,

$$x - a_i < 0 \text{ for } i = m, m+1, \ldots n$$

and $x - a_i > 0$ for $i = 1, 2, \ldots, m-1$

$\therefore$ $A_i = \dfrac{x - a_i}{-(x - a_i)} = -1$, for $i = m, m+1, \ldots, n$

and $A_i = \dfrac{x - a_i}{x - a_i} = 1$, for $i = 1, 2, \ldots, m-1$

Similarly, if $x$ is in the right neighbourhood of $a_i$

Then $x - a_i < 0$ for $i = m+1, \ldots., n$ and $x - a_i > 0$ for $i = 1, 2, \ldots, m$

$\therefore$ $A_i = \dfrac{x - a_i}{-(x - a_i)} = -1$ for $i = m+1, \ldots n$

and $A_i = \dfrac{x - a_i}{x - a_i} = 1$ for $i = 1, 2, \ldots, m$

Now, $\lim_{x \to a_m^-} (A_1 A_2 \ldots A_n) = (-1)^{n-m+1}$

and $\lim_{x \to a_m^+} (A_1 A_2 \ldots A_n) = (-1)^{n-m}$

Hence, $\lim_{x \to a_m} (A_1 A_2 \ldots A_n)$ Does not exist.

## INDETERMINATE FORMS

If a unique value cannot be assigned to $f(a)$, then $f(x)$ is said to be inderminate at $x = a$.

Most general of all indeterminate forms is $\dfrac{0}{0}$, others being

**1.** $\infty - \infty = \dfrac{1}{0} - \dfrac{1}{0} = \dfrac{0-0}{0} = \dfrac{0}{0}$ which is indeterminate and hence is $(\infty - \infty)$

**2.** $\dfrac{\infty}{\infty} = \dfrac{1/0}{1/0} = \dfrac{0}{0}$ which is indeterminate and hence is $\left(\dfrac{\infty}{\infty}\right)$

**3.** $0 \times \infty = 0. \dfrac{1}{0} = \dfrac{0}{0}$ which is indeterminate and hence is $(0 \times \infty)$

**4.** $1^\infty$

Let $y = 1^\infty$

$\Rightarrow$ $\log y = \infty \log 1 = \infty \times 0$ which is indeterminate and hence is $1^\infty$

**5.** $0^0$

Let $y = 0^0$

$\Rightarrow$ $\log y = 0 \cdot \log 0 = 0 \times \infty$ which is indeterminate and hence is $0^o$

**6.** $\infty^o$

Let $y = \infty^o$

$\Rightarrow$ $\log y = 0 \cdot \log \infty = 0 \times \infty$ which is indeterminate and hence is $\infty^o$.

## ALGEBRA OF LIMITS

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then following results are true:

1. $\lim_{x\to a} [f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = l + m$.
2. $\lim_{x\to a} [f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = l - m$.
3. $\lim_{x\to a} k\cdot f(x) = k\cdot \lim_{x\to a} f(x) = kl$,

   where $k$ is a constant.
4. $\lim_{x\to a} [f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = lm$.
5. $\lim_{x\to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{l}{m}$ (provided $m \neq 0$).
6. $\lim_{x\to a} (fog)(x) = \lim_{x\to a} f[g(x)] = f\left(\lim_{x\to a} g(x)\right) = f(m)$.

   In particular,

   **(a)** $\lim_{x\to a} \log g(x) = \log\left(\lim_{x\to a} g(x)\right) = \log m$.

   **(b)** $\lim_{x\to a} e^{g(x)} = e^{\lim_{x\to a} g(x)} = e^m$.
7. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = l^n$, for all $n \in N$.
8. Sandwich Theorem (or Squeeze Principle).
   If $f$, $g$ and $h$ are functions such that $f(x) \le g(x) \le h(x)$ for all $x$ in some neighbourhood of the point $a$ (except possibly at $x = a$) and if $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$.
9. If $\lim_{x\to a} f(x) = l$, then $\left|\lim_{x\to a} f(x)\right| = |l|$.

**IMPORTANT POINTS**

Sandwich theorem helps in calculating the limits, when limits cannot be calculated using the usual method

### SOLVED EXAMPLE

3. $\lim_{n\to\infty} \dfrac{\{x\}+\{2x\}+\{3x\}+\ldots+\{nx\}}{n^2}$, where $\{x\} = x - [x]$ denotes the fractional part of $x$, is

(A) 1 (B) 0

(C) $\dfrac{1}{2}$ (D) None of these

**Solution: (B)**

Since, $0 \le \{rx\} < 1$ for $r = 1, 2, 3, \ldots, n$

$$\Rightarrow \quad 0 \le \sum_{r=1}^{n} \{rx\} < \sum_{r=1}^{n} (1)$$

$$\Rightarrow \quad 0 \le \sum_{r=1}^{n} \{rx\} < n$$

Dividing throughout by $n^2$, we have

$$\frac{0}{n^2} \le \frac{\sum_{r=1}^{n} \{rx\}}{n^2} < \frac{1}{n}$$

$$\Rightarrow \quad \lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{\sum_{r=1}^{n} \{rx\}}{n^2} < \lim_{n\to\infty} \frac{1}{n}$$

$$\Rightarrow \quad 0 \le \lim_{n\to\infty} \frac{\sum_{r=1}^{n} \{rx\}}{n^2} < 0$$

$$\therefore \quad 0 \le \lim_{n\to\infty} \frac{\{x\}+\{2x\}+\ldots+\{nx\}}{n^2} < 0$$

According to Sandwich Theorem or Squeeze Principle

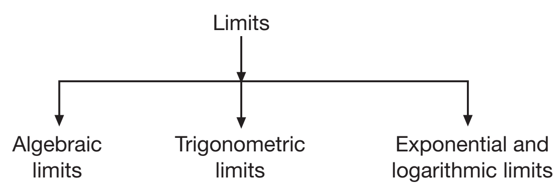
$$\lim_{n\to\infty} \frac{\{x\}+\{2x\}+\ldots+\{nx\}}{n^2} = 0$$

**CAUTION**

The converse of the above result may not be true, i.e., $\lim_{x\to a} |f(x)| = |l| = |l| \Rightarrow \lim_{x\to a} f(x) = l$

## EVALUATION OF LIMITS

The problems on limits can be divided into the following categories:




## ALGEBRAIC LIMITS

The following methods are useful for evaluating limits of algebraic functions:

## Method of Factorization

If $f(x)$ and $g(x)$ are polynomials and $g(a) \neq 0$, then we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)}.$$

Now, if $f(a) = 0 = g(a)$, then $(x - a)$ is a factor of both $f(x)$ and $g(x)$. We cancel this common factor $(x - a)$ from both the numerator and denominator and again put $x = a$ in the given expression. If we get a meaningful number then that number is the limit of the given expression, otherwise we repeat this process till we get a meaningful number.

### SOLVED EXAMPLE

**4.** $\lim_{x\to 1}\frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$ is equal to

(A) $\frac{1}{9}$ (B) $\frac{1}{6}$

(C) $\frac{1}{3}$ (D) None of these

**Solution: (A)**

$$\lim_{x\to 1}\frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2} = \lim_{y\to 1}\frac{y^2-2y+1}{(y^3-1)^2}$$

[Putting $\sqrt[3]{x} = y$; as $x \to 1, y \to 1$]

$$= \lim_{y\to 1}\frac{(y-1)^2}{(y-1)^2(y^2+y+1)^2}$$

$$= \lim_{y\to 1}\frac{1}{(y^2+y+1)^2} = \frac{1}{9}.$$

## Method of Rationalization

This method is useful where radical signs (i.e., expressions of the form $\sqrt{a} \pm \sqrt{b}$ ) are involved either in the numerator or in the denominator or both. The numerator or (and) the denominator (as required) is (are) rationalised and limit taken after cancelling out the common factors.

### SOLVED EXAMPLE

**5.** The value of $\lim_{x\to 3}\left(\log_a \frac{x-3}{\sqrt{x+6}-3}\right)$ is

(A) $\log_a 6$
(B) $\log_a 3$
(C) $\log_a 2$
(D) None of these

**Solution: (A)**

$$\lim_{x\to 3}\left[\log_a \frac{x-3}{\sqrt{x+6}-3}\right]$$

$$= \lim_{x\to 3}\left[\log_a \frac{(x-3)(\sqrt{x+6}+3)}{(x-3)}\right]$$

$$= \lim_{x\to 3}\log_a(\sqrt{x+6}+3) = \log_a 6$$

## Standard Formula

$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1},$$

where $n \in Q$, the set of rational numbers.

### SOLVED EXAMPLE

**6.** $\lim_{x\to 1}\sum_{r=1}^{n}\frac{x^r - 1^r}{x-1} =$

(A) 0 (B) $\frac{n(n+1)}{2}$

(C) 1 (D) None of these

**Solution: (B)**

We have, $\lim_{x\to 1}\sum_{r=1}^{n}\frac{x^r-1^r}{x-1}$

$$= \sum_{r=1}^{n} r\cdot 1^{r-1} = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

## LIMIT OF AN ALGEBRAIC FUNCTION WHEN $X \to \infty$

In order to find the limit of a function of the type $\frac{f(x)}{g(x)}$ as $x \to \infty$, where $f(x)$ and $g(x)$ are algebraic functions of $x$, it is convenient to divide all the terms of $f(x)$ and $g(x)$ by the highest power of $x$ in numerator and denominator both and use the following standard limits:

**1.** $\lim_{x\to\infty}\frac{1}{x} = 0$

**2.** $\lim_{x\to\infty}\frac{1}{x^p} = 0$, if $p > 0$.

### TRICK(S) FOR PROBLEM SOLVING

- $$\lim_{n\to\infty} a^n = \begin{cases} \infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } -1 < a < 1 \\ \text{does not exist}, & \text{if } a \le -1 \end{cases}$$

- $$\lim_{x\to\infty} \frac{a_0x^p + a_1x^{p-1} + \ldots\ldots + a_{p-1}x + a_p}{b_0x^q + b_1x^{q-1} + \ldots\ldots + b_{q-1}x + b_q} = \begin{cases} \dfrac{a_0}{b_0}, & \text{if } p = q \\ 0, & \text{if } p < q \\ \infty, & \text{if } p > q \end{cases}$$

## Some Useful Summations

1. $\Sigma n = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$
2. $\Sigma n^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
3. $\Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$
4. $\Sigma ar^{n-1} = a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$; provided $r < 1$.

## SOLVED EXAMPLES

**7.** The value of

$$\lim_{n\to\infty} \frac{1}{n^4}\left[1\left(\sum_{k=1}^{n} k\right) + 2\left(\sum_{k=1}^{n-1} k\right) + 3\left(\sum_{k=1}^{n-2} k\right) + \ldots + n\cdot 1\right]$$

will be

(A) $\dfrac{1}{24}$ (B) $\dfrac{1}{12}$ (C) $\dfrac{1}{6}$ (D) $\dfrac{1}{3}$

**Solution: (A)**

The $(r + 1)$th term of the series is

$$t_{r+1} = (r+1)\sum_{k=1}^{n-r} k$$

$\Rightarrow$ $t_{r+1} = (r + 1)[1 + 2 + 3 + \ldots (n - r) \text{ terms}]$

$\Rightarrow$ $t_{r+1} = (r+1)\dfrac{1}{2}(n-r)(n-r+1)$

$\Rightarrow$ $t_{r+1} = \dfrac{1}{2}(r+1)(n^2 - rn + n - rn + r^2 - r)$

$\Rightarrow$ $t_{r+1} = \dfrac{1}{2}(r+1)(r^2 - (1+2n)r + n^2)$

$\Rightarrow$ $t_{r+1} = \dfrac{1}{2}(r^3 - 2nr^2 + (n^2 - 2n - 1)r + n^2)$

Now, $S = \displaystyle\sum_{r=0}^{n-1} t_{r+1}$

$\therefore$ $S = \dfrac{1}{2}\displaystyle\sum_{r=1}^{n}[r^3 - 2nr^2 + (n^2 - 2n - 1)r + n^2]$

$$\Rightarrow S = \frac{1}{2}\left[\left\{\frac{n(n+1)}{2}\right\}^2 - 2n\left\{\frac{1}{6}n(n+1)(2n+1)\right\} + (n^2 - 2n - 1)\left\{\frac{1}{2}n(n+1)\right\} + n^2(n)\right]$$

Solving and rearranging, we have

$$S = \frac{1}{24}(n^4 - 11n^3 - 19n^2 + 6n)$$

$$\therefore \lim_{n\to\infty}\frac{S}{n^4} = \lim_{n\to\infty}\frac{1}{24}\left(\frac{n^4 - 11n^3 - 19n^2 + 6n}{n^4}\right)$$

$$= \frac{1}{24}\lim_{n\to\infty}\left(1 - \frac{11}{n} - \frac{19}{n^2} + \frac{6}{n^3}\right)$$

$$\therefore \lim_{n\to\infty}\frac{1}{n^4}\sum_{r=0}^{n-1}\left\{(r+1)\sum_{k=1}^{n-r}k\right\} = \frac{1}{24}$$

**8.** $\displaystyle\lim_{n\to\infty}\prod_{r=3}^{n}\left(\frac{r^3-1}{r^3+1}\right)$

(A) $\dfrac{1}{3}$ (B) $\dfrac{6}{7}$

(C) $-\dfrac{2}{3}$ (D) None of these

**Solution: (B)**

$(n - 2)$th factor of the series is

$$t_n = \frac{n-1}{n+1}\cdot\frac{n^2+n+1}{n^2-n+1}$$

Therefore, required limit $= \displaystyle\lim_{n\to\infty} t_3t_4t_5 \ldots t_{n-2}t_{n-1}t_n$

$$= \lim_{n\to\infty}\left[\left(\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}\cdots\frac{n-3}{n-1}\cdot\frac{n-2}{n}\cdot\frac{n-1}{n+1}\right)\cdot\left(\frac{13}{7}\right)\cdot\frac{21}{13}\cdot\frac{31}{21}\cdots\frac{n^2+n+1}{n^2-n+1}\right]$$

$$= \lim_{n\to\infty}\frac{2\cdot 3}{n(n+1)}\cdot\frac{n^2+n+1}{7} = \frac{6}{7}.$$

**9.** If $[x]$ denotes the integral part of $x$, then

$$\lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{k=1}^{n}[k^2x]\right) =$$

(A) 0 (B) $\dfrac{x}{2}$

(C) $\dfrac{x}{3}$ (D) $\dfrac{x}{6}$

**Solution: (C)**

$$L = \lim_{n \to \infty} \frac{1}{n^3}\left(\sum_{k=1}^{n} [k^2 x]\right)$$

Since $k^2x - 1 \le (k^2x) < k^2x$

$$\Rightarrow \quad \sum_{k=1}^{n} (k^2x - 1) \le \sum_{k=1}^{n} (k^2x) < \sum_{k=1}^{n} k^2 x$$

$$\Rightarrow \quad x\left(\sum_{k=1}^{n} k^2\right) - \sum_{k=1}^{n} (1) \le \sum_{k=1}^{n} [k^2x] < x\left(\sum_{k=1}^{n} k^2\right)$$

$$\Rightarrow \quad \frac{xn(n+1)(2n+1)}{6} - n$$

$$\le \sum_{k=1}^{n} [k^2x] < \frac{xn(n+1)(2n+1)}{6}$$

Dividing throughout by $n^3$, we have

$$\frac{xn(n+1)(2n+1)}{6n^3} - \frac{1}{n^2}$$

$$\le \sum_{k=1}^{n} \frac{[k^2x]}{n^3} < \frac{xn(n+1)(2n+1)}{6n^3}$$

$$\Rightarrow \quad \frac{x}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{1}{n^2} \le \sum_{k=1}^{n} \frac{[k^2x]}{n^3}$$

$$< \frac{x}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Taking limits as $n \to \infty$, we get

$$\lim_{n \to \infty}\left[\frac{x}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{1}{n^2}\right] \le L$$

$$< \lim_{n \to \infty} \frac{x}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Since, as $n \to \infty$, we have $\frac{1}{n} \to 0$

$$\Rightarrow \quad \frac{x}{3} \le L < \frac{x}{3}$$

According to Squeeze Principle or Sandwich Theorem, we have

$$L = \frac{x}{3}.$$

**10.** $\lim_{n \to \infty}\left\{\frac{7}{10} + \frac{29}{10^2} + \frac{133}{10^3} + \ldots + \frac{5^n + 2^n}{10^n}\right\}$ is equal to

(A) $\frac{3}{4}$ (B) 2 (C) $\frac{5}{4}$ (D) $\frac{1}{2}$

**Solution: (C)**

$$\text{Required limit} = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{5^r + 2^r}{10^r}$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \left\{\left(\frac{1}{2}\right)^r + \left(\frac{1}{5}\right)^r\right\}$$

$$= \lim_{n \to \infty} \left\{\frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} + \frac{1}{5} \frac{1 - \left(\frac{1}{5}\right)^n}{1 - \frac{1}{5}}\right\}$$

$$= 1 + \frac{1}{4} = \frac{5}{4}$$

**11.** $\lim_{x \to \infty} f(x)$, where $\frac{2x-3}{x} < f(x) < \frac{2x^2+5x}{x^2}$, is

(A) 1 (B) 2 (C) –1 (D) –2

**Solution: (B)**

$$\lim_{x \to \infty} \frac{2x-3}{x} = \lim_{x \to \infty}\left(2 - \frac{3}{x}\right) = 2$$

and $$\lim_{x \to \infty} \frac{2x^2+5x}{x^2} = \lim_{x \to \infty}\left(2 + \frac{5}{x}\right) = 2,$$

$\therefore$ Using Sandwich theorem, $\lim_{x \to \infty} f(x) = 2$.

## TRIGONOMETRIC LIMITS

For finding the limits of trigonometric functions, we use trigonometric transformations and simplify. The following results are quite useful:

**1.** **(a)** $\lim_{x \to 0} \frac{\sin x}{x} = 1$

**(b)** $\lim_{x \to 0} \cos x = 1$

**(c)** $\lim_{x \to 0} \frac{\tan x}{x} = 1$

**(d)** $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

**(e)** $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

**(f)** $\lim_{x \to 0} \frac{\sin x^0}{x} = \frac{\pi}{180}$.

**2.** $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$, where $a \neq 0$, on taking $x = a + h$.

## Some Useful Expansions

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \infty; x \in \mathbb{R}$

2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \infty; x \in \mathbb{R}$

3. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^3 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots \infty;$ $|x| < \frac{\pi}{2}$

4. $\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \ldots \infty;\ 0 < |x| < \frac{\pi}{2}$

5. $\operatorname{cosec} x = \frac{1}{x} + \frac{x}{6} + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \ldots \infty; 0 < |x| < \pi$

6. $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2}{945}x^5 - \ldots \infty; 0 < |x| < \pi$

7. $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5x^7}{2.4.6.7} + \ldots \infty; |x| < 1$

8. $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$

$$= \frac{\pi}{2} - \left\{x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \ldots \infty\right\}; |x| < 1$$

9. $\tan^{-1} x =$

$$\begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \infty; & |x| < 1 \\ \pm\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \ldots \infty; & \begin{cases} + \text{ if } x \ge 1 \\ - \text{ if } x \le -1 \end{cases} \end{cases}$$

## SOLVED EXAMPLES

**12.** $\lim_{x\to 0} \frac{1 - \cos^3 x}{x \sin x \cos x} =$

(A) $\frac{3}{2}$ $\quad$ $\frac{1}{2}$

(C) 1 $\quad$ (D) None of these

**Solution: (A)**

$$\lim_{x\to 0} \frac{1 - \cos^3 x}{x \sin x \cos x}$$

$$= \lim_{x\to 0} \frac{(1 - \cos x)(1 + \cos^3 x + \cos x)}{x \sin x \cos x}$$

$$= \lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}(1 + \cos^2 x + \cos x)}{2x \sin\frac{x}{2} \cos\frac{x}{2} \cos x}$$

$$= \lim_{x\to 0} \frac{1}{2}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{1 + \cos^2 x + \cos x}{\cos\frac{x}{2}\cdot\cos x}$$

$$= \frac{1}{2}\cdot 1\cdot\frac{3}{1} = \frac{3}{2}$$

**13.** $\lim_{n\to\infty}\left(\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\ldots\cos\frac{x}{2^n}\right) =$

(A) $\frac{x}{\sin x}$ $\quad$ (B) $\frac{\sin x}{x}$

(C) 0 $\quad$ (D) None of these

**Solution: (B)**

Required Limit $= \lim_{n\to\infty} \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\cos\frac{x}{2^n}$

$$= \lim_{n\to\infty}\frac{1}{2\sin\frac{x}{2^n}}\left[\cos\frac{x}{2}\ldots\cos\frac{x}{2^{n-1}}\left(2\sin\frac{x}{2^n}\cos\frac{x}{2^n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{2^2\sin\frac{x}{2^n}}\left[\cos\frac{x}{2}\ldots\left(2\cos\frac{x}{2^{n-1}}\sin\frac{x}{2^{n-1}}\right)\right]$$

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$$= \lim_{n\to\infty}\frac{1}{2^n\sin\frac{x}{2^n}}\left(2\cos\frac{x}{2}\sin\frac{x}{2}\right) = \lim_{n\to\infty}\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$$

$$= \frac{\sin x}{x}\lim_{n\to\infty}\left\{\frac{\left(\frac{x}{2^n}\right)}{\sin\left(\frac{x}{2^n}\right)}\right\} = \frac{\sin x}{x}.$$

**14.** $\lim_{n\to\infty}\left(\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \ldots + \frac{1}{2^n}\tan\frac{\theta}{2^n}\right) =$

(A) $\frac{1}{\theta}$ $\quad$ (B) $\frac{1}{\theta} - 2\cot 2\theta$

(C) $2\cot 2\theta$ $\quad$ (D) None of these

**Solution: (B)**

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$$

$$\frac{1}{\cot 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$\Rightarrow \quad \cot 2\theta = \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$$

$$\Rightarrow \quad 2\cot 2\theta = \frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad 2\cot 2\theta = \cot\theta - \tan\theta$$

$$\Rightarrow \quad \tan\theta = \cot\theta - 2\cot 2\theta \qquad (1)$$

Now, $\tan\theta = \cot\theta - 2\cot 2\theta$

$$\Rightarrow \quad \frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2} - \cot\theta$$

$$\Rightarrow \quad \frac{1}{2^2}\tan\frac{\theta}{2^2} = \frac{1}{2^2}\cot\frac{\theta}{2} - \frac{1}{2}\cot\theta$$

$$\Rightarrow \quad \frac{1}{2^n}\tan\frac{\theta}{2^n} = \frac{1}{2^n}\cot\frac{\theta}{2^n} - \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}}$$

$$\Rightarrow \quad S = -2\cot 2\theta + \frac{1}{2^n}\cot\frac{\theta}{2^n}$$

Therefore,

$$\lim_{n\to\infty}\left(\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \ldots + \frac{1}{2^n}\tan\frac{\theta}{2^n}\right)$$

$$= \lim_{n\to\infty} S = \lim_{n\to\infty}\left(-2\cot 2\theta + \frac{1}{2^n}\cot\frac{\theta}{2^n}\right)$$

$$= -2\cot 2\theta + \lim_{n\to\infty}\frac{1}{\theta}\left(\frac{\frac{\theta}{2^n}}{\tan\frac{\theta}{2^n}}\right)$$

$$= -2\cot 2\theta + \frac{1}{\theta}$$

**15.** $\lim\limits_{x\to 0}\dfrac{\tan([-\pi^2]x) - x^2\tan([-\pi^2])}{\sin^2 x}$ equals, where [ ] denotes the greatest integer function

(A) 0 (B) 1
(C) tan 10 – 10 (D) ∞

**Solution: (C)**

Since, $[-\pi^2] = [-9.87] = -10$, therefore

$$\lim_{x\to 0}\frac{\tan[-\pi^2]x^2 - x^2\tan[-\pi^2]}{\sin^2 x}$$

$$= \lim_{x\to 0}\frac{\tan(-10x^2) - x^2\tan(-10)}{x^2}\frac{x^2}{\sin^2 x}$$

$$= -\lim_{x\to 0}\frac{\tan(10x^2)}{x^2} + \lim_{x\to 0}\frac{x^2\tan 10}{x^2}$$

$$= \lim_{x\to 0}(-10)\frac{\tan(10x^2)}{10x^2} + \tan 10$$

$$= \tan 10 - 10$$

**16.** $\lim\limits_{h\to 0}\dfrac{2\left[\sqrt{3}\sin\left(\frac{\pi}{6}+h\right) - \cos\left(\frac{\pi}{6}+h\right)\right]}{\sqrt{3}h\,(\sqrt{3}\cos h - \sin h)}$ is equal to

(A) $\frac{4}{3}$ (B) $-\frac{4}{3}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

**Solution: (A)**

$$\lim_{h\to 0}\frac{2\left[\sqrt{3}\sin\left(\frac{\pi}{6}+h\right) - \cos\left(\frac{\pi}{6}+h\right)\right]}{\sqrt{3}h\,(\sqrt{3}\cos h - \sin h)}$$

$$= \lim_{h\to 0}\frac{2\left[\sqrt{3}\left(\frac{1}{2}\cos h + \frac{\sqrt{3}}{2}\sin h\right) - \left(\frac{\sqrt{3}}{2}\cos h - \frac{1}{2}\sin h\right)\right]}{\sqrt{3}h(\sqrt{3}\cos h - \sin h)}$$

$$= \lim_{h\to 0}\frac{2[2\sin h]}{\sqrt{3}h(\sqrt{3}\cos h - \sin h)}$$

$$= \lim_{h\to 0}\frac{4\cdot\frac{\sin h}{h}}{\sqrt{3}(\sqrt{3}\cos h - \sin h)}$$

$$= \frac{4}{\sqrt{3}(\sqrt{3}-0)} = \frac{4}{3}$$

**17.** If $\lim\limits_{x\to 0}\dfrac{x^n - \sin x^n}{x - \sin^n x}$ is non-zero finite, then $n$ may be equal to

(A) 1 (B) 2
(C) 3 (D) None of these

**Solution: (A)**

Clearly, for $n = 1$, $\lim\limits_{x\to 0}\dfrac{x - \sin x}{x - \sin x} = 1$.

**18.** $\lim\limits_{x\to 0}\left(\operatorname{cosec}^3 x\cdot\cot x - 2\cot^3 x\cdot\operatorname{cosec} x + \dfrac{\cot^4 x}{\sec x}\right)$ is equal to

(A) 1 (B) – 1
(C) 0 (D) None of these

**Solution: (A)**

$$\lim_{x\to 0}\left[\operatorname{cosec}^3 x\cdot\cot x - 2\cot^3 x\cdot\operatorname{cosec} x + \frac{\cot^4 x}{\sec x}\right]$$

$$= \lim_{x\to 0}\left(\frac{\cos x}{\sin^4 x} - \frac{2\cos^3 x}{\sin^4 x} + \frac{\cos^5 x}{\sin^4 x}\right)$$

$$= \lim_{x\to 0}\frac{\cos x(1-\cos^2 x)^2}{\sin^4 x} = \lim_{x\to 0}\cos x = 1$$

## EXPONENTIAL AND LOGARITHMIC LIMITS

For finding the limits of exponential and logarithmic functions, the following results are useful:

1. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

2. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, a > 0$

3. $\lim_{x\to 0} \frac{a^x - b^x}{x} = \log_e \left(\frac{a}{b}\right); \ a, b > 0$

4. $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

5. $\lim_{x\to 0} (1+x)^{1/x} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

6. $\lim_{h\to 0} (1+ah)^{1/h} = e^a$

7. $\lim_{x\to\infty} \frac{\log x}{x^m} = 0, (m > 0)$

8. $\lim_{x\to 0} \frac{\log_a (1+x)}{x} = \log_a e, (a > 0, a \neq 1)$

9. $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a$

10. $\lim_{x\to\infty} \left(1+\frac{1}{f(x)}\right)^{f(x)} = e$, where $f(x) \to \infty$ as $x \to \infty$.

11. $\lim_{x\to a} (1+f(x))^{1/f(x)} = e.$

## Some Useful Expansions

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to $\infty$

2. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$ to $\infty$

3. $\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ to $\infty$, $-1 < x \leq 1$

4. $\log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$ to $\infty$, $-1 \leq x < 1$

5. $a^x = e^{x \log a} = 1 + x \log a + \frac{(x \log a)^2}{2!} + \ldots$ to $\infty$

6. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$ to $\infty$, $-1 < x < 1$, $n$ being any negative integer or fraction.

The expansion formulae mentioned above can be used with advantage in simplification and evaluation of limits.

For example, $\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

$$= \lim_{x\to 0} \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) - 1 + \frac{x^2}{2}}{x^4}$$

$$= \lim_{x\to 0} \left(\frac{1}{4!} + \text{terms containing } x \text{ and its powers}\right)$$

$$= \frac{1}{4!} = \frac{1}{24}$$

**CAUTION**

If [·] denotes the greatest integer function, then

$$\lim_{x\to 0} [-x] = [0] = 0$$

Is the above statement true?

No. If fact, $\lim_{x\to 0} [-x] = \lim_{x\to 0} [0 - x] = \lim_{x\to 0} -1 = -1$

Thus, limit must be applied only after removing [·] sign.

## SOLVED EXAMPLES

19. $\lim_{x\to 1} \frac{x \sin\{x\}}{x-1}$, where $\{x\}$ denotes the fractional part of $x$, is equal to

(A) –1 (B) 0
(C) 1 (D) Does not exist

**Solution**: (D)

$$\text{LHL} = \lim_{x\to 1-} \frac{x \sin\{x\}}{x-1}$$

Let $x = 1 - h$, as $x \to 1$, $h \to 0$

$\Rightarrow$ $\text{LHL} = \lim_{h\to 0} \frac{(1-h)\sin\{1-h\}}{h}$

$\Rightarrow$ $\text{LHL} = \lim_{h\to 0} \frac{(1-h)\sin(1-h)}{h}$

$\therefore$ $\text{LHL} = \lim_{h\to 0} \frac{(1-h)}{h}\sin(1) = \infty$

Now, $\text{RHL} = \lim_{x\to 1+} \frac{x \sin(x)}{x-1}$

Let $x = 1 + h$, as $x \to 1$, $h \to 0$

$\Rightarrow$ $\text{RHL} = \lim_{h\to 0} \frac{(1+h)\sin(1+h)}{h}$

$\Rightarrow$ $\text{RHL} = \lim_{h\to 0} \frac{(1+h)\sin h}{h} = \lim_{h\to 0}(1+h)$

$\therefore$ $\text{RHL} = (1 + 0) = 1$

Since LHL $\neq$ RHL,

$\therefore$ the limit of the function Does not exist at $x = 1$.

**20.** $\lim_{x \to 0^+} \left(\frac{b}{x}\right)\left[\frac{x}{a}\right]$ where $a > 0$, $b > 0$ and $[x]$ denotes greatest integer less than or equal to $x$ is

(A) $\frac{1}{a}$ (B) $b$ (C) $\frac{b}{a}$ (D) 0

**Solution: (D)**

$$\lim_{x \to 0^+} \left(\frac{b}{x}\right)\left[\frac{x}{a}\right] = \lim_{h \to 0} \left(\frac{b}{0+h}\right)\left[\frac{0+h}{a}\right]$$

$$= \lim_{h \to 0} \left(\frac{b}{h}\right)\left[\frac{h}{a}\right] = 0$$

**21.** If $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0 & , [x] = 0 \end{cases}$, where $[x]$ denotes the greatest integer $\leq x$, then $\lim_{x \to 0} f(x)$ equals

(A) 0 (B) –1
(C) 1 (D) None of these

**Solution: (A)**

$$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{\sin[-h]}{[-h]} = \lim_{h \to 0} \frac{\sin(-1)}{(-1)} = \sin 1.$$

$$\lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{\sin[h]}{[h]}$$

$$= 1 \qquad [\because h \to 0 \Rightarrow (h) \to 0]$$

$\therefore \lim_{x \to 0} f(x)$ does not exist.

**22.** Let $f(x) = x - [x]$, where $[x]$ denotes the greatest integer $\leq x$ and $g(x) = \lim_{n \to \infty} \frac{[f(x)]^{2n} - 1}{[f(x)]^{2n} + 1}$, then $g(x) =$

(A) 0 (B) 1
(C) –1 (D) None of these

**Solution: (C)**

As $0 \leq x - [x] < 1 \ \forall x \in R, 0 \leq f(x) < 1$.

$$\therefore \quad \lim_{n \to \infty} [f(x)]^{2n} = 0$$

Thus, for $x \in R$, $g(x) = \lim_{n \to \infty} \frac{[f(x)]^{2n} - 1}{[f(x)]^{2n} + 1}$

$$= \frac{0-1}{0+1} = -1$$

**23.** If $f(x) = \begin{cases} \frac{\tan^{-1}([x]+x)}{[x]-2x}, & [x] \neq 0 \\ 0 & , [x] = 0 \end{cases}$

where $[x]$ denotes the greatest integer less than or equal to $x$, then $\lim_{x \to 0} f(x)$ is equal to

(A) $-\frac{1}{2}$ (B) 1
(C) $\frac{\pi}{4}$ (D) Does not exist

**Solution: (D)**

$$\text{LHL} = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{\tan^{-1}([-h]-h)}{[-h]+2h}$$

$$= \lim_{h \to 0} \frac{\tan^{-1}(-1-h)}{(2h-1)}$$

$$= \lim_{h \to 0} \frac{\tan^{-1}(1+h)}{(1-2h)}$$

$$= \frac{\pi/4}{1} = \frac{\pi}{4}.$$

$$\text{RHL} = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{\tan^{-1}([h]+h)}{[h]-2h}$$

$$= \lim_{h \to 0} \frac{\tan^{-1}(h)}{-2h} = -\frac{1}{2}$$

Since LHL $\neq$ RHL

$\therefore \lim_{x \to 0} f(x)$ Does not exist.

## TRICK(S) FOR PROBLEM SOLVING

- If $\lim_{x \to a} f(x) = A > 0$ and $\lim_{x \to a} g(x) = B$, then

$$\lim_{x \to a} [f(x)]^{g(x)} = A^B$$

- If $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$, then

$$\lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)[f(x)-1]}$$

## SOLVED EXAMPLES

**24.** If $\lim_{x \to 0} \frac{729^x - 243^x - 81^x + 9^x + 3^x - 1}{x^3} = k(\log 3)^3$, then $k =$

(A) 4 (B) 5
(C) 6 (D) None of these

**Solution: (C)**

Required limit

$$= \lim_{x \to 0} \frac{243^x(3^x - 1) - 9^x(3^{2x} - 1) + (3^x - 1)}{x^3}$$

$$= \lim_{x \to 0} \frac{(3^x - 1)\{(243)^x - (27)^x - 9^x + 1\}}{x^3}$$

$$= \lim_{x \to 0} \frac{(3^x - 1)\{(243)^x - (27)^x - 9^x + 1\}}{x^3}$$

$$= \lim_{x \to 0} \frac{(3^x - 1)}{x} \frac{(9^x - 1)}{x} \frac{(27^x - 1)}{x}$$

$= \log 3 \cdot \log 9 \cdot \log 27$

$= \log 3 \cdot 2\log 3 \cdot 3\log 3$

$= 6(\log 3)^3 = k(\log 3)^3$ (given)

$\therefore \quad k = 6$

**25.** If $\alpha$ and $\beta$ be the roots of $ax^2 + bx + c = 0$, then $\lim_{x \to \alpha} (1 + ax^2 + bx + c)^{1/(x-\alpha)}$ is

(A) $\log |a(\alpha - \beta)|$ (B) $e^{a(\alpha - \beta)}$

(C) $e^{a(\beta - \alpha)}$ (D) None of these

**Solution: (B)**

$$\lim_{x \to \alpha} (1 + ax^2 + bx + c)^{1/(x-\alpha)} = e^{\lim_{x \to \alpha} \frac{1}{(x-\alpha)}[(1+ax^2+bx+c)-1]}$$

[Using $\lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)[f(x)-1]}$ provided $f(x) \to 1$ and $g(x) \to \infty$ as $x \to a$]

$$= e^{\lim_{x \to \alpha} \frac{(ax^2+bx+c)}{(x-\alpha)}} = e^{\lim_{x \to \alpha} \frac{a(x-\alpha)(x-\beta)}{(x-\alpha)}}$$

[$\because$ $\alpha, \beta$ are roots of $ax^2 + bx + c = 0$]

$= e^{a(\alpha - \beta)}$

**26.** $\lim_{x \to a} \left(\frac{\sin x}{\sin a}\right)^{\frac{1}{x-a}}$, $a \neq n\pi$, $n$ is an integer, equals

(A) $e^{\cot a}$ (B) $e^{\tan a}$ (C) $e^{\sin a}$ (D) $e^{\cos a}$

**Solution: (A)**

$$\lim_{x \to a} \left(\frac{\sin x}{\sin a}\right)^{\frac{1}{x-a}} = \lim_{x \to a} \left(1 + \frac{\sin x - \sin a}{\sin a}\right)^{\frac{1}{x-a}}$$

$$= \lim_{x \to a} \left[\left\{1 + \left(\frac{\sin x - \sin a}{\sin a}\right)\right\}^{\frac{\sin a}{\sin x - \sin a}}\right]^{\frac{\sin x - \sin a}{(x-a)\sin a}}$$

$$= e^{\lim_{x \to a} \frac{\sin x - \sin a}{(x-a)\sin a}}$$

$$= e^{\lim_{x \to a} \frac{2}{x-a} \cos\left(\frac{x+a}{2}\right) \sin\left(\frac{x-a}{2}\right) \cdot \frac{1}{\sin a}}$$

$$= e^{\lim_{x \to a} \cos\left(\frac{x+a}{2}\right)\left[\sin\left(\frac{x-a}{2}\right) \Big/ \left(\frac{x-a}{2}\right)\right] \frac{1}{\sin a}}$$

$$= e^{\frac{\cos a}{\sin a}} = e^{\cot a}$$

**27.** $\lim_{x \to 0} \left(\frac{1 + \tan x}{1 + \sin x}\right)^{1/\sin x}$ is equal to

(A) 0 (B) 1

(C) –1 (D) None of these

**Solution: (B)**

Let $f(x) = \dfrac{1 + \tan x}{1 + \sin x}$

and $g(x) = \dfrac{1}{\sin x}$.

Clearly $f(x) \to 1$ and $g(x) \to \infty$ as $x \to 0$.

$$\therefore \quad \lim_{x \to 0} \left(\frac{1 + \tan x}{1 + \sin x}\right)^{1/\sin x} = e^{\lim_{x \to 0} \frac{1}{\sin x}\left(\frac{1+\tan x}{1+\sin x} - 1\right)}$$

$\left\{\text{Using } \lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)[f(x)-1]}\right\}$

$$= e^{\lim_{x \to 0} \frac{1}{\sin x}\left(\frac{\tan x - \sin x}{1 + \sin x}\right)} = e^{\lim_{x \to 0} \frac{1 - \cos x}{\cos x(1 + \sin x)}} = e^0 = 1$$

**28.** $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}}$ is equal to

(A) $e^{\pi/2}$ (B) $e^{2/\pi}$ (C) $e^{-2/\pi}$ (D) $e^{-\pi/2}$

**Solution: (B)**

$$\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}} = e^{\lim_{x \to a} \tan\left(\frac{\pi x}{2a}\right) \cdot \left(2 - \frac{x}{a} - 1\right)}$$

$\left\{\text{Using } \lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)[f(x)-1]}\right.$ as $f(x) \to 1$ and $g(x) \to \infty$ as $x \to a\left.\right\}$

$$= e^{\lim_{x \to a} \left(1 - \frac{x}{a}\right) \tan\left(\frac{\pi x}{2a}\right)} = e^{\lim_{x \to a} \frac{(1 - x/a)}{\cot(\pi x/2a)}}$$

$$= e^{\lim_{x \to a} \frac{-1/a}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right) \cdot \frac{\pi}{2a}}} = e^{\lim_{x \to a} \frac{2}{\pi} \sin^2\left(\frac{\pi x}{2a}\right)} = e^{2/\pi}$$

**29.** $\lim_{x \to 0} (\cos x + a \sin bx)^{\frac{a}{x}}$ is equal to

(A) $e^{-a^2 b}$ (B) $e^{ab^2}$ (C) $e^{a^2 b}$ (D) $e^{-b^2 a}$

**Solution: (C)**

$$\lim_{x \to 0} (\cos x + a \sin bx)^{\frac{a}{x}} = e^{\lim_{x \to 0} \frac{a}{x}(\cos x + a \sin bx - 1)}$$

$\left[\text{Using } \lim_{x \to a} [f(x)]^{\phi(x)} = e^{\lim_{x \to a} \phi(x)[f(x)-1]}\right.$ as $f(x) \to 1$ and $\phi(x) \to \infty$ as $x \to a\left.\right]$

$$= e^{\lim_{x \to 0} \frac{a(-\sin x + ab\cos bx)}{1}} = e^{a^2 b}$$

**30.** The value of $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$ is

(A) 1 (B) –1
(C) 0 (D) None of these

**Solution: (B)**

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}} = e^{\lim_{x\to 0}\frac{\sin x}{x-\sin x}\left(\frac{\sin x}{x}-1\right)}$$

$$\left[\text{Using } \lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}\right.$$

as $f(x)=\frac{\sin x}{x}\to 1$ and $g(x)$

$$\left.= \frac{\sin x}{x-\sin x} = \frac{\frac{\sin x}{x}}{1-\frac{\sin x}{x}} \to \infty \qquad \text{as } x\to 0\right]$$

$$= e^{\lim_{x\to 0} -\frac{\sin x}{x}} = e^{-1}$$

**31.** $\lim_{x\to -1}\left(\frac{x^4+x^2+x+1}{x^2-x+1}\right)^{\frac{1-\cos(x+1)}{(x+1)^2}}$ is equal to

(A) 1 (B) $\left(\frac{2}{3}\right)^{1/2}$ (C) $\left(\frac{3}{2}\right)^{1/2}$ (D) $e^{1/2}$

**Solution: (B)**

$$\lim_{x\to -1}\left(\frac{x^4+x^2+x+1}{x^2-x+1}\right)^{\frac{1-\cos(x+1)}{(x+1)^2}}$$

$$= \lim_{x\to -1}\left(\frac{x^4+x^2+x+1}{x^2-x+1}\right)^{\frac{2\sin^2\left(\frac{x+1}{2}\right)}{(x+1)^2}}$$

$$= \lim_{x\to -1}\left(\frac{x^4+x^2+x+1}{x^2-x+1}\right)^{\frac{1}{2}\left(\frac{\sin\left(\frac{x+1}{2}\right)}{\left(\frac{x+1}{2}\right)}\right)^2} = \left(\frac{2}{3}\right)^{1/2}$$

**32.** $\lim_{n\to\infty}\left(\cos\frac{x}{n}\right)^n$ is equal to

(A) $e^1$ (B) $e^{-1}$
(C) 1 (D) None of these

**Solution: (C)**

$$\lim_{n\to\infty}\left(\cos\frac{x}{n}\right)^n = e^{\lim_{n\to\infty} n\left(\cos\frac{x}{n}-1\right)}$$

$$= e^{\lim_{n\to\infty} -n\cdot 2\sin^2\left(\frac{x}{2n}\right)}$$

$$= e^{-2\lim_{n\to\infty}\left(\frac{\sin\left(\frac{x}{2n}\right)}{\frac{x}{2n}}\right)^2\cdot\frac{x^2}{4n^2}\cdot n}$$

$$= e^{-2\times\lim_{n\to\infty}\frac{x^2}{4n}} = e^0 = 1$$

## EVALUATION OF LIMITS USING L'HOSPITAL'S RULE

Besides the methods given above to evaluate limits, there is yet another method for finding limits, usually known as L'Hospital's Rule as given below for indeterminate forms:

1. $\left(\frac{0}{0}\right)$ form: If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, provided the limit on the R.H.S. exists.
   Here, $f'$ is derivative of $f$.
2. $\left(\frac{\infty}{\infty}\right)$ form: If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, provided the limit on the R.H.S. exists.

Note that sometimes we have to repeat the process if the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again.

### TRICK(S) FOR PROBLEM SOLVING

- L'Hospital's Rule is applicable only when $\frac{f(x)}{g(x)}$ becomes of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
- If the form is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, simplify the given expression till it reduces to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L'Hospital's rule.
- For applying L'Hospital's rule differentiate the numerator and denominator separately.

### CAUTION

L' Hospital's rule cannot be applied in every problem.

Consider the example, $\lim_{x\to 0}\frac{3x+\sin 2x}{3x-\sin 2x}\left(\text{form }\frac{\infty}{\infty}\right)$.

Here, if we apply L' Hospital's rule, we get

$$\lim_{x\to\infty}\frac{3x+\sin 2x}{3x-\sin 2x} = \lim_{x\to\infty}\frac{3+2\cos 2x}{3-2\cos 2x}$$

Now, both the numerator and denominator are undefined because $\lim_{x\to\infty} \cos 2x$ Does not exist.

We can find the above limit as:

$$\lim_{x\to\infty}\frac{3x+\sin 2x}{3x-\sin 2x}=\lim_{x\to\infty}\frac{3+2\left(\frac{\sin 2x}{2x}\right)}{3-2\left(\frac{\sin 2x}{2x}\right)}=\frac{3+2(0)}{3-2(0)}$$

$$= 1, \text{ since } \lim_{x\to\infty}\frac{\sin 2x}{2x}=0$$

## SOLVED EXAMPLES

**33.** If $\alpha$ is a repeated root of $ax^2 + bx + c = 0$, then

$$\lim_{x\to\alpha}\frac{\tan(ax^2+bx+c)}{(x-\alpha)^2} \text{ is}$$

(A) $a$ (B) $b$ (C) $c$ (D) 0

**Solution: (A)**

$$\lim_{x\to\alpha}\frac{\tan(ax^2+bx+c)}{(x-\alpha)^2}$$

$$\left(\frac{0}{0} \text{ form as } a\alpha^2+b\alpha+c=0\right)$$

$$= \lim_{x\to\alpha}\frac{(2ax+b)\sec^2(ax^2+bx+c)}{2(x-\alpha)}$$

$$\left(\frac{0}{0} \text{ form as } \alpha \text{ being a repeated root of } ax^2+bx+c=0,\ 2a\alpha+b=0\right)$$

$$= \lim_{x\to\alpha}\frac{2a\sec^2(ax^2+bx+c)+(2ax+b)^2\times 2\sec^2(ax^2+bx+c)\tan(ax^2+bx+c)}{2}$$

$$= \frac{2a}{2} = a.$$

**34.** $\lim_{x\to 0}\frac{e^x+e^{-x}+2\cos x-4}{x^4}$ is equal to

(A) 0 (B) 1 (C) $\frac{1}{6}$ (D) $-\frac{1}{6}$

**Solution: (C)**

$$\lim_{x\to 0}\frac{e^x+e^{-x}+2\cos x-4}{x^4} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{e^x-e^{-x}-2\sin x}{4x^3} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{12x^2} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{e^x-e^{-x}+2\sin x}{24x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{e^x+e^{-x}+2\cos x}{24} = \frac{4}{24} = \frac{1}{6}.$$

**35.** If $f(2)$, $g(x)$ be differentiable functions and $f(1) = g(1) = 2$ then $\lim_{x\to 1}\frac{f(1)g(x)-f(x)g(1)-f(1)+g(1)}{g(x)-f(x)}$ is equal to

(A) 0 (B) 1
(C) 2 (D) None of these

**Solution: (C)**

$$\lim_{x\to 1}\frac{f(1)g(x)-f(x)g(1)-f(1)+g(1)}{g(x)-f(x)} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 1}\frac{f(1)g'(x)-f'(x)g(1)}{g'(x)-f'(x)}$$

$$= 2\lim_{x\to 1}\frac{g'(x)-f'(x)}{g'(x)-f'(x)}$$

$$= 2.$$

**36.** Let $f(x)$ be a twice differentiable function and $f''(0) = 5$, then $\lim_{x\to 0}\frac{3f(x)-4f(3x)+f(9x)}{x^2}$ is equal to

(A) 30 (B) 120
(C) 40 (D) None of these

**Solution: (B)**

$$\lim_{x\to 0}\frac{3f(x)-4f(3x)+f(9x)}{x^2} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{3f'(x)-12f'(3x)+9f'(9x)}{2x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{3f''(x)-36f''(3x)+81f''(9x)}{2}$$

$$= \frac{3f''(0)-36f''(0)+81f''(0)}{2} = 24f''(0)$$

$$= 24\cdot 5 = 120.$$

**37.** If $f(9) = 9$ and $f'(9) = 1$, then $\lim_{x\to 9}\frac{3-\sqrt{f(x)}}{3-\sqrt{x}}$ is equal to

(A) 0 (B) 1
(C) –1 (D) None of these

**Solution: (B)**

$$\lim_{x\to 9}\frac{3-\sqrt{f(x)}}{3-\sqrt{x}} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 9} \frac{0 - \dfrac{1}{2\sqrt{f(x)}}\cdot f'(x)}{0 - \dfrac{1}{2\sqrt{x}}}$$

[Using L'Hospital's Rule]

$$= \lim_{x\to 9} \frac{\sqrt{x}}{\sqrt{f(x)}}\cdot f'(x) = \frac{3}{3} \times f'(9) = 1.$$

**38.** If $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$ be finite, then the value of $a$ and the limit are given by

(A) – 2, 1 (B) – 2, –1
(C) 2, 1 (D) 2, –1

**Solution: (B)**

Let $k = \lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$

$$= \lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2}$$

[Using L'Hospital's Rule]

We require 2 cos2$x$ + $a$ cos $x$ = 0 for $x$ = 0 as denominator is zero.

$\therefore$ $a = -2$.

Hence, $k = \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6} = \frac{-8+2}{6} = -1.$$

**39.** $\lim_{x\to 0} x^x$ is equal to

(A) 0 (B) 1
(C) – 1 (D) None of these

**Solution: (B)**

Let $y = \lim_{x\to 0} x^x$

$\Rightarrow$ $\log y = \lim_{x\to 0} x\log x = \lim_{x\to 0} \frac{\log x}{1/x}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{1/x}{-1/x^2} = -\lim_{x\to 0} x = 0$$

$\Rightarrow$ $y = e^0 = 1$.

**40.** $\lim_{x\to 0}\left(\frac{\ln\cos x}{\sqrt[4]{1+x^2}-1}\right)$ is equal to

(A) 2 (B) –2 (C) 1 (D) –1

**Solution: (B)**

$$\lim_{x\to 0}\left(\frac{\ln\cos x}{\sqrt[4]{1+x^2}-1}\right) = \lim_{x\to 0} \frac{\ln[1+(\cos x - 1)]}{\sqrt[4]{1+x^2}-1}$$

$$= 4\lim_{x\to 0} \frac{\cos x - 1}{x^2}$$

$$\left[\because \ln[1+(\cos x-1)] \sim (\cos x - 1) \text{ and } \left(\sqrt[4]{1+x^2}-1\right) \sim \frac{x^2}{4}\right]$$

$$= -4\lim_{x\to 0} \frac{x^2/2}{x^2} \quad \left[\because (1-\cos x) \sim \frac{x^2}{2}\right]$$

$= -2$

**41.** $\lim_{x\to 2} \frac{2^x - x^2}{x^x - 2^2}$ is equal to

(A) $\frac{\log 2 - 1}{\log 2 + 1}$ (B) $\frac{\log 2 + 1}{\log 2 - 1}$
(C) 1 (D) –1

**Solution: (A)**

$$\lim_{x\to 2} \frac{2^x - x^2}{x^x - 2^2} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 2} \frac{2^x\log 2 - 2x}{x^x(1+\log x)} = \frac{4\log 2 - 4}{4(1+\log 2)} = \frac{\log 2 - 1}{\log 2 + 1}.$$

**42.** $\lim_{x\to \frac{\pi}{2}} \frac{\sin x - (\sin x)^{\sin x}}{1 - \sin x + \ln \sin x}$ equals

(A) 1 (B) 2 (C) 3 (D) 4

**Solution: (B)**

Let $\sin x = h$, then as $x \to \pi/2$, $h \to 1$

$\therefore$ given limit

$$= \lim_{h\to 1} \frac{h - h^h}{1 - h + \ln h} = \lim_{h\to 1} \frac{1 - h^h - h^h\ln h}{-1 + 1/h}$$

(Using L' Hospital Rule)

$$= \lim_{h\to 1} \frac{-h^h - 2h^h\ln h - h^{h-1} - h^h(\ln h)^2}{-1/h^2}$$

$$= \frac{-1-1}{-1} = 2$$

**43.** The value of $\lim_{x\to a} \sqrt{a^2 - x^2}\cot\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}$ is

(A) $\frac{2a}{\pi}$ (B) $-\frac{2a}{\pi}$
(C) $\frac{4a}{\pi}$ (D) $-\frac{4a}{\pi}$

**Solution: (C)**

$$\lim_{x\to a}\sqrt{a^2-x^2}\cot\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}} \quad (0\times\infty \text{ form})$$

$$=\lim_{x\to a}\frac{\sqrt{a^2-x^2}}{\tan\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to a}\frac{\frac{-2x}{2\sqrt{a^2-x^2}}}{-\sec^2\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}\times\frac{\pi}{2}\times\frac{2a}{2(a+x)\sqrt{a^2-x^2}}}$$

$$=\frac{4a}{\pi}$$

# EXERCISES

## Single Option Correct Type

**1.** The value of $\lim_{x\to 0}\left(\left[\frac{100x}{\sin x}\right]+\left[\frac{99\sin x}{x}\right]\right)$, where $[\cdot]$ represents greatest integer function, is

(A) 199 (B) 198
(C) 0 (D) None of these

**2.** If $f(x)=\sin x, \quad x\neq n\pi,$
$\quad = 2, \quad x = n\pi$
where $n\in Z$ and
$g(x)=x^2+1, \quad x\neq 2,$
$\quad =3, \quad x=2.$
then $\lim_{x\to 0} g[f(x)]$ is

(A) 1 (B) 0
(C) 3 (D) Does not exist

**3.** The value of $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$ is

(A) $\frac{1}{2}$ 1
(C) 0 (D) None of these

**4.** The value of $\lim_{x\to\infty} x\left[\tan^{-1}\frac{x+1}{x+2}-\frac{\pi}{4}\right]$ is

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 1 (D) $-1$

**5.** $\lim_{n\to\infty}\cos\left(\pi\sqrt{n^2+n}\right)$, $n\in$ Z is equal to

(A) 0 (B) 1
(C) $-1$ (D) None of these

**6.** $\lim_{n\to\infty}\frac{n^k\sin^2(n!)}{n+2}$ $0<k<1$, is equal to

(A) $\infty$ (B) 1
(C) 0 (D) None of these

**7.** $\lim_{x\to 1}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}$

(A) exists and it equals $\sqrt{2}$
(B) exists and it equals $-\sqrt{2}$
(C) Does not exist because $(x-1)\to 0$
(D) Does not exist because left hand limit is not equal to right hand limit

**8.** The value of $\lim_{x\to\infty}\frac{x^5}{5^x}$ is

(A) 1 (B) $-1$
(C) 0 (D) None of these

**9.** $\lim_{x\to 0}(\cos x+\sin x)^{\frac{1}{x}}$ is equal to

(A) $e$ (B) $e^2$ (C) $e^{-1}$ (D) 1

**10.** The value of $\lim_{x\to\frac{\pi}{4}}\frac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin 2x}$ is

(A) $\frac{3}{\sqrt{2}}$ (B) $\frac{\sqrt{2}}{3}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2}$

**11.** The value of $\lim_{h\to 0}\frac{\ln(1+2h)-2\ln(1+h)}{h^2}$ is

(A) 1 (B) $-1$
(C) 0 (D) None of these

**12.** The value of $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{e^{1/n}}{n}+\frac{e^{2/n}}{n}+\ldots+\frac{e^{(n-1)/n}}{n}\right)$ is

(A) 1 (B) 0 (C) $e-1$ (D) $e+1$

**13.** $\lim_{x\to 1}\frac{x\sin(x-[x])}{x-1}$, where $[\cdot]$ denotes the greatest integer function, is equal to

(A) 1 (B) –1
(C) $\infty$ (D) Does not exist

**14.** If $f(x) = \int \frac{2\sin x - \sin 2x}{x^3}\,dx$, $x \neq 0$, then $\lim_{x\to 0} f'(x)$ is

(A) 0 (B) $\infty$ (C) –1 (D) 1

**15.** $\lim_{x\to \pi/2} \frac{\left[\frac{x}{2}\right]}{\ln(\sin x)}$ (where [·] denotes the greatest integer function)

(A) Does not exist (B) equals 1
(C) equals 0 (D) equals –1

**16.** $\lim_{m\to\infty} \lim_{n\to\infty} (1+\cos^{2m} n!\pi x)$ is equal to

(A) 2 (B) 1
(C) 0 (D) None of these

**17.** $\lim_{x\to 0}\left[\frac{\sin([x-3])}{[x-3]}\right]$, where [·] represents greatest integer function, is

(A) 0 (B) 1
(C) Does not exist (D) sin 1

**18.** The values of constants $a$ and $b$ so that

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x+1} - ax - b\right) = 0 \text{ are}$$

(A) $a = 1, b = -1$ (B) $a = -1, b = 1$
(C) $a = 0, b = 0$ (D) $a = 2, b = -1$

**19.** $\lim_{n\to\infty}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}\right)$ is equal to

(A) 1 (B) –1
(C) 0 (D) None of these

**20.** $\lim_{x\to\infty}\frac{(\log x)^2}{x^n}$, $n > 0$ is equal to

(A) 1 (B) 0 (C) – 1 (D) $\infty$

**21.** If the $r$th term, $t_r$, of a series is given by

$t_r = \frac{r}{r^4+r^2+1}$, then $\lim_{n\to\infty}\sum_{r=1}^{n} t_r$ is

(A) 1 (B) $\frac{1}{2}$
(C) $\frac{1}{3}$ (D) None of these

**22.** $\lim_{x\to n}(-1)^{[x]}$, where $[x]$ denotes the greatest integer less than or equal to $x$, is equal to

(A) $(-1)^n$ (B) $(-1)^{n-1}$
(C) 0 (D) Does not exist

**·23.** $\lim_{\substack{x\to 1\\ y\to 0}}\frac{y^3}{x^3-y^2-1}$ as $(x, y) \to (1, 0)$ along the line $y = x - 1$ is given by

(A) 1 (B) $\infty$
(C) 0 (D) None of these

**24.** $\lim_{n\to\infty}\frac{1-2+3-4+5-6+\ldots-2n}{\sqrt{n^2+1}+\sqrt{4n^2-1}}$ is equal to

(A) $\frac{1}{3}$ (B) $-\frac{1}{3}$
(C) $-\frac{1}{5}$ (D) None of these

**25.** The value of $\lim_{x\to-\infty}\left[\frac{x^4\sin(1/x)+x^2}{1+|x|^3}\right]$ is

(A) 1 (B) –1 (C) 0 (D) $\infty$

**26.** $\lim_{x\to 2}\frac{2^x+2^{3-x}-6}{2^{-x/2}-2^{1-x}}$ is equal to

(A) 8 (B) 4
(C) 2 (D) None of these

**27.** $\lim_{x\to 0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^4}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$ is equal to

(A) $\frac{1}{16}$ (B) $-\frac{1}{16}$ (C) $\frac{1}{32}$ (D) $-\frac{1}{32}$

**28.** $\lim_{n\to\infty}[\log_{n-1}(n)\cdot\log_n(n+1)\cdot\log_{n+1}(n+2)\ldots\log_{n^k-1}(n^k)]$ is equal to

(A) $\infty$ (B) $n$
(C) $k$ (D) None of these

**29.** $\lim_{n\to\infty}\left[\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n+1)(2n+3)}\right]$ is equal to

(A) 1 (B) $\frac{1}{2}$
(C) $-\frac{1}{2}$ (D) None of these

**30.** The value of $\lim_{x\to\infty}\left[\frac{1^{1/x}+2^{1/x}+3^{1/x}+\ldots+n^{1/x}}{n}\right]^{nx}$ is

(A) $n!$ (B) $n$ (C) $(n-1)!$ (D) 0

**31.** $\lim_{n\to\infty}(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2n})$, $|x| < 1$, is equal to

(A) $\frac{1}{x-1}$ (B) $\frac{1}{x-1}$ (C) $1-x$ (D) $x-1$

**32.** $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$, ($n$ integer), for

(A) no value of $n$
(B) all values of $n$
(C) only negative values of $n$
(D) only positive values of $n$

**33.** The value of $\lim_{x\to 1} \frac{x^n + x^{n-1} + x^{n-2} + \ldots + x^2 + x - n}{x-1}$ is

(A) $\frac{n(n+1)}{2}$ (B) 0
(C) 1 (D) $n$

**34.** If $t_r = \frac{1^2+2^2+3^2+\ldots+r^2}{1^3+2^3+3^3+\ldots+r^3}$ and $S_n = \sum_{r=1}^{n} (-1)^r \cdot t_r$, then $\lim_{n\to\infty} S_n$ is given by

(A) $\frac{2}{3}$ (B) $-\frac{2}{3}$ (C) $\frac{1}{3}$ (D) $-\frac{1}{3}$

**35.** If $\lim_{x\to 0} \frac{(1+a^3)+8e^{1/x}}{1+(1-b^3)e^{1/x}} = 2$, then

(A) $a = 1, b = (-3)^{1/3}$ (B) $a = 1, b = 3^{1/3}$
(C) $a = -1, b = -(3)^{1/3}$ (D) None of these

**36.** If $a = \min \{x^2 + 4x + 5, x \in R\}$ and $b = \lim_{\theta\to 0} \frac{1-\cos 2\theta}{\theta^2}$, then the value of $\sum_{r=0}^{n} a^r \cdot b^{n-r}$ is

(A) $\frac{2^{n+1}-1}{4\cdot 2^n}$ (B) $2^{n+1} - 1$
(C) $\frac{2^{n+1}-1}{3\cdot 2^n}$ (D) None of these

**37.** $\lim_{n\to\infty} \frac{1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)}{n^3}$ is equal to

(A) 1 (B) –1
(C) $\frac{1}{3}$ (D) None of these

**38.** $\lim_{x\to 0} \frac{\log(1+x+x^2)+\log(1-x+x^2)}{\sec x - \cos x}$ is equal to

(A) 1 (B) –1 (C) 0 (D) $\infty$

**39.** $\lim_{x\to e} \frac{\ln x - 1}{|x-e|}$ is equal to

(A) $\frac{1}{e}$ (B) $-\frac{1}{e}$
(C) $e$ (D) Does not exist

**40.** If $x_1 = 3$ and $x_{n+1} = \sqrt{2+x_n}$, $n \geq 1$, then $\lim_{n\to\infty} x_n$ is equal to

(A) –1 (B) 2 (C) $\sqrt{5}$ (D) 3

**41.** The value of $\lim_{x\to\infty} \frac{3^{x+1}-5^{x+1}}{3^x - 5^x}$ is

(A) 5 (B) $\frac{1}{5}$
(C) –5 (D) None of these

**42.** $\lim_{n\to\infty} \frac{1}{n}\left(1+e^{1/n}+e^{2/n}+\ldots+e^{\frac{n-1}{n}}\right)$ is equal to

(A) $e$ (B) $-e$
(C) $e-1$ (D) $1-e$

**43.** $\lim_{x\to\infty} \sqrt{\frac{x+\sin x}{x-\cos x}} =$

(A) 0 (B) 1
(C) – 1 (D) None of these

**44.** If $S_n = \sum_{i=1}^{n} a_i$ and $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} \frac{S_{n+1}-S_n}{\sqrt{\sum_{i=1}^{n} i}}$ is equal to

(A) 0 (B) $a$
(C) $\sqrt{2}a$ (D) None of these

**45.** The value of $\lim_{n\to\infty} \left[\sqrt[3]{n^2-n^3} + n\right]$ is

(A) $\frac{1}{3}$ (B) $-\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $-\frac{2}{3}$

**46.** The value of $\lim_{n\to\infty} \frac{\sqrt[4]{n^5+2} - \sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2} - \sqrt[2]{n^3+1}}$ is

(A) 1 (B) 0 (C) –1 (D) $\infty$

**47.** The integer $n$ for which $\lim_{x\to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number, is

(A) 1 (B) 2 (C) 3 (D) 4

**48.** The value of $\lim_{x\to\infty} \frac{2\sqrt{x}+3\sqrt[3]{x}+5\sqrt[5]{x}}{\sqrt{3x-2}+\sqrt[3]{2x-3}}$ is

(A) $\frac{2}{\sqrt{3}}$ (B) $\sqrt{3}$
(C) $\frac{1}{\sqrt{3}}$ (D) None of these

**49.** $\lim_{x\to 0} \frac{x\sqrt[3]{z^2-(z-x)^2}}{\left(\sqrt[3]{8xz-4x^2}+\sqrt[3]{8xz}\right)^4}$ is equal to

(A) $\frac{z}{2^{11/3}}$ (B) $\frac{1}{2^{23/3}\cdot z}$

(C) $2^{21/3}z$ (D) None of these

**50.** In a circle of radius $r$, an isosceles triangle $ABC$ is inscribed with $AB = AC$. If the $\Delta ABC$ has perimeter $P = 2\left[\sqrt{2hr - h^2} + \sqrt{2hr}\right]$ and area $A = h\sqrt{2hr - h^2}$, where $h$ is the altitude from $A$ to $BC$, then $\lim_{h\to 0^+} \frac{A}{P^3}$ is equal to

(A) $128r$ (B) $\frac{1}{128r}$

(C) $\frac{1}{64r}$ (D) None of these

**51.** $\lim_{x\to 2}\left(\frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$

(A) equals $\frac{1}{\sqrt{2}}$ (B) Does not exist

(C) equals $\frac{1}{\sqrt{2}}$ (D) equals $-\sqrt{2}$

**52.** $\lim_{n\to\infty}\left(\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\ldots\cos\frac{x}{2^n}\right) =$

(A) $\frac{x}{\sin x}$ (B) $\frac{\sin x}{x}$

(C) 0 (D) None of these

**53.** The value of

$$\lim_{n\to\infty}\frac{1}{n^4}\left[1\left(\sum_{k=1}^{n}k\right)+2\left(\sum_{k=1}^{n-1}k\right)+3\left(\sum_{k=1}^{n-2}k\right)+\ldots+n\cdot 1\right]$$

will be

(A) $\frac{1}{24}$ (B) $\frac{1}{12}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$

**54.** If $[x]$ denotes the integral part of $x$, then

$$\lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{k=1}^{n}[k^2x]\right) =$$

(A) 0 (B) $\frac{x}{2}$ (C) $\frac{x}{3}$ (D) $\frac{x}{6}$

**55.** $\lim_{n\to\infty}\left(\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \ldots + \frac{1}{2^n}\tan\frac{\theta}{2^n}\right) =$

(A) $\frac{1}{\theta}$ (B) $\frac{1}{\theta} - 2\cot 2\theta$

(C) $2\cot 2\theta$ (D) None of these

**56.** $\lim_{n\to 0}\frac{4^{3n-2} - 9^{n+1}}{8^{2n-1} - 9^{n-1}} =$

(A) $\frac{1}{2}$ (B) 81

(C) Does not exist (D) None of these

**57.** If $A_i = \frac{x - a_i}{|x - a_i|}$, $i = 1, 2, \ldots, n$ and if $a_1 < a_2 < a_3 \ldots < a_n$. Then, $\lim_{x\to a_m}(A_1A_2\ldots A_n)$, $1 \le m \le n$

(A) is equal to $(-1)^m$

(B) is equal to $(-1)^{m+1}$

(C) is equal to $(-1)^{m-1}$

(D) Does not exist

**58.** The value of $\lim_{x\to 0}\left(\left[\frac{100x}{\sin x}\right] + \left[\frac{99\sin x}{x}\right]\right)$, where $[\cdot]$ represents greatest integer function, is

(A) 199 (B) 198

(C) 0 (D) None of these

**59.** The value of $\lim_{x\to\infty}\frac{x^5}{5^x}$ is

(A) 1 (B) –1

(C) 0 (D) None of these

**60.** If the $r$th term, $t_r$, of a series is given by

$t_r = \frac{r}{r^4 + r^2 + 1}$, then $\lim_{n\to\infty}\sum_{r=1}^{n}t_r$ is

(A) 1 (B) $\frac{1}{2}$

(C) $\frac{1}{3}$ (D) None of these

**61.** The value of $\lim_{x\to\infty}\left[\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \ldots + n^{1/x}}{n}\right]^{nx}$ is

(A) $n!$ (B) $n$ (C) $(n-1)!$ (D) 0

**62.** $\lim_{n\to\infty}(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2n})$, $|x| < 1$, is equal to

(A) $\frac{1}{x-1}$ (B) $\frac{1}{1-x}$

(C) $1-x$ (D) $x-1$

**63.** $\lim_{x\to\infty}\frac{x^n}{e^x} = 0$, ($n$ integer), for

(A) no value of $n$

(B) all values of $n$

(C) only negative values of $n$

(D) only positive values of $n$

**64.** If $a = \min\{x^2 + 4x + 5, x \in R\}$ and $b = \lim_{\theta\to 0}\frac{1-\cos 2\theta}{\theta^2}$ then the value of $\sum_{r=0}^{n}a^r \cdot b^{n-r}$ is

(A) $\dfrac{2^{n+1}-1}{4\cdot 2^n}$ (B) $2^{n+1}-1$

(C) $\dfrac{2^{n+1}-1}{3\cdot 2^n}$ (D) None of these

**65.** $\lim\limits_{x\to 0} \dfrac{\log(1+x+x^2)+\log(1-x+x^2)}{\sec x - \cos x}$ is equal to

(A) 1 (B) –1 (C) 0 (D) $\infty$

**66.** The value of $\lim\limits_{n\to\infty} \dfrac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2}-\sqrt[2]{n^3+1}}$ is

(A) 1 (B) 0 (C) –1 (D) $\infty$

**67.** $\lim\limits_{x\to 0} \dfrac{x\sqrt[3]{z^2-(z-x)^2}}{\left(\sqrt[3]{8xz-4x^2}+\sqrt[3]{8xz}\right)^4}$ is equal to

(A) $\dfrac{z}{2^{11/3}}$ (B) $\dfrac{1}{2^{23/3}\cdot z}$

(C) $2^{21/3}z$ (D) None of these

**68.** In a circle of radius $r$, an isosceles triangle $ABC$ is inscribed with $AB = AC$. If the $\Delta ABC$ has perimeter $P = 2\left[\sqrt{2hr-h^2}+\sqrt{2hr}\right]$ and area $A = h\sqrt{2hr-h^2}$, where $h$ is the altitude from $A$ to $BC$, then $\lim\limits_{h\to 0^+}\dfrac{A}{P^3}$ is equal to

(A) $128r$ (B) $\dfrac{1}{128r}$

(C) $\dfrac{1}{64r}$ (D) None of these

**69.** $\lim\limits_{x\to\pi/3} \dfrac{\cos\left(x+\dfrac{\pi}{6}\right)}{(1-2\cos x)^{2/3}} =$

(A) 1 (B) –1

(C) 0 (D) None of these

**70.** $\lim\limits_{x\to 0} \dfrac{\ln(2-\cos 2x)}{\ln^2(\sin 3x+1)}$ is equal to

(A) $\dfrac{2}{9}$ (B) $-\dfrac{2}{9}$

(C) 0 (D) None of these

**71.** $\lim\limits_{x\to 1/\alpha} \dfrac{1-\cos(cx^2+bx+a)}{(1-x\alpha)^2}$, where $\alpha$ is a root of $ax^2+bx+c=0$, is equal to

(A) $\dfrac{b^2-4ac}{2\alpha^2}$ (B) $\dfrac{b^2-4ac}{\alpha^2}$

(C) $\dfrac{4ac-b^2}{2\alpha^2}$ (D) None of these

**72.** $\lim\limits_{x\to 0} \dfrac{\sqrt{1-\sqrt{\cos x}}}{x} =$

(A) $\dfrac{1}{2}$ (B) $-\dfrac{1}{2}$

(C) Does not exist (D) None of these

**73.** $\lim\limits_{x\to 2} \dfrac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}} =$

(A) $\dfrac{17}{23}$ (B) $\dfrac{34}{23}$

(C) 1 (D) None of these

**74.** $\lim\limits_{x\to 0} \dfrac{(2^m+x)^{1/m}-(2^n+x)^{1/n}}{x}$ is equal to

(A) $\dfrac{1}{m2^m}-\dfrac{1}{n2^n}$ (B) $\dfrac{1}{m2^m}+\dfrac{1}{n2^n}$

(C) $\dfrac{1}{m2^{m-1}}-\dfrac{1}{n2^{n-1}}$ (D) None of these

**75.** $\lim\limits_{x\to 4} \dfrac{(\cos\theta)^x-(\sin\theta)^x-\cos 2\theta}{x-4} =$

(A) $\cos^4\theta\, ln\cos\theta - \sin^4\theta\, ln\sin\theta$

(B) $\cos^4\theta\, ln\cos\theta + \sin^4\theta\, ln\sin\theta$

(C) $\cos^4\theta\, ln\sin\theta - \sin^4\theta\, ln\cos\theta$

(D) None of these

**76.** $\lim\limits_{x\to 0} \left(\dfrac{x-1+\cos x}{x}\right)^{1/x} =$

(A) $e^{1/2}$ (B) $e^{-1/2}$

(C) $e^{1/4}$ (D) None of these

**77.** $\lim\limits_{x\to\infty} \left[\dfrac{e}{(1+1/x)^x}\right]^x =$

(A) $e$ (B) $e^{-1}$ (C) $e^{1/2}$ (D) $e^{-1/2}$

**78.** $\lim\limits_{x\to 0} \left[\dfrac{a\sin x}{x}\right]+\left[\dfrac{b\tan x}{x}\right]$, where $a$, $b$ are integers and [ ] denotes integral part, is equal to

(A) $a+b$ (B) $a+b-1$

(C) $a-b$ (D) $a-b-1$

**79.** $\lim\limits_{n\to\infty} \dfrac{[x]+[2x]+[3x]+\ldots+[nx]}{1+2+3+\ldots+n} =$

(A) $x$ (B) $2x$

(C) 0 (D) None of these

**80.** $\lim\limits_{n\to\infty} n^2\left(x^{1/n}-x^{1/n+1}\right)$, $x>0$ is equal to

(A) 0 (B) $e^x$

(C) $lnx$ (D) None of these

**81.** If $\lim_{x\to 0}\left[1+x+\frac{f(x)}{x}\right]^{1/x}=e^3$, then $\lim_{x\to 0}\left[1+\frac{f(x)}{x}\right]^{1/x}=$

(A) $e$ (B) $e^2$
(C) $e^3$ (D) None of these

**82.** If $y = x+\cfrac{\sqrt{x}}{x+\cfrac{\sqrt{x}}{x+\cfrac{\sqrt{x}}{\dots\infty}}}$, then $\lim_{x\to\infty}\frac{x}{y}$ is equal to

(A) 1 (B) –1
(C) 0 (D) None of these

**83.** $\lim_{x\to 0}\frac{\cos x-(\cos x)^{\cos x}}{1-\cos x+ln(\cos x)}=$

(A) 0 (B) 1
(C) 2 (D) None of these

**84.** The value of $\lim_{x\to \pi/4}\frac{(\tan x)^{\tan x}-\tan x}{ln(\tan x)-\tan x+1}$ is

(A) –2 (B) 1
(C) 0 (D) None of these

**85.** $\lim_{x\to\pi/2}\left(1^{1/\cos^2 x}+2^{1/\cos^2 x}+\dots+n^{1/\cos^2 x}\right)^{\cos^2 x}=$

(A) $n$ (B) $\frac{n(n+1)}{2}$
(C) $n!$ (D) None of these

**86.** $\lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(r^2+\frac{3}{4}\right)=$

(A) 0 (B) $\tan^{-1}2$
(C) $\frac{\pi}{4}$ (D) None of these

**87.** The value of $\lim_{x\to 5\pi/4}[\sin x+\cos x]$, where [·] denotes the greatest integer function, is

(A) 2 (B) –2 (C) 1 (D) –1

**88.** $\lim_{m\to\infty}\lim_{n\to\infty}$

$$\left(\frac{1+\sqrt[n]{1^n+2^n}+\sqrt[n]{2^n+3^n}+\sqrt[n]{3^n+4^n}+\dots+\sqrt[n]{(m-1)^n+m^n}}{m^2}\right)$$

(A) 0 (B) 1 (C) –1 (D) $\frac{1}{2}$

**89.** The value of $\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{2^r}\right]$, where [·] denotes the greatest integer, is

(A) 0 (B) 1 (C) –1 (D) $\frac{1}{2}$

**90.** The value of $\lim_{x\to\infty}|x|^{[\cos x]}$, where [·] denotes the greatest integer, is

(A) 0 (B) 1
(C) –1 (D) Does not exist

**91.** If $a_1 = 1$ and $a_n = n(1 + a_{n-1})$, $\forall\, n \ge 2$, then

$\lim_{n\to\infty}\left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)\dots\left(1+\frac{1}{a_n}\right)=$

(A) 0 (B) $e$
(C) $e^2$ (D) Does not exist

**92.** $\lim_{n\to\infty}n^{-n^2}\left[(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)\dots\left(n+\frac{1}{2^{n-1}}\right)\right]^n$

(A) $e$ (B) $e^2$
(C) $e^4$ (D) None of these

**93.** If $\lim_{x\to y}\frac{x^y-y^x}{x^x-y^y}=\frac{1-k}{1+k}$, then $k=$

(A) $\log y$ (B) $e^y$
(C) $y$ (D) None of these

**94.** $\lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(\frac{r^3-r+\frac{1}{r}}{2}\right)$ is equal to

(A) 0 (B) $\pi$
(C) $\frac{\pi}{2}$ (D) None of these

**95.** The value of $\lim_{n\to\infty}\left(\left[\frac{n\sin x}{x}\right]+\left[\frac{n\tan x}{x}\right]\right)$, where [·] denotes the greatest integer function, is

(A) $n$ (B) $2n+1$
(C) $2n-1$ (D) None of these

**96.** $\lim_{x\to 0}\left[\frac{x^2}{\sin x\tan x}\right]$, where [·] denotes the greatest integer function, is

(A) 0 (B) 1
(C) 2 (D) Does not exist

**97.** $\lim_{\theta\to 0}\frac{\cos^2(1-\cos^2(1-\cos^2(1\dots\cos^2\theta))}{\sin\left(\frac{\pi(\sqrt{\theta+4}-2)}{\theta}\right)}=$

(A) 1 (B) 0 (C) $\sqrt{2}$ (D) $-\sqrt{2}$

**98.** Let $f(x)=\sqrt{\frac{\tan x-\sin\{\tan^{-1}(\tan x)\}}{\tan x+\cos^2(\tan x)}}$, then $\lim_{x\to\frac{\pi}{2}}f(x)=$

(A) 1
(B) –1
(C) 0
(D) Does not exist

**99.** $\lim_{x \to a} \left\{ \left[ \left( \frac{a^{1/2} + x^{1/2}}{a^{1/2} - x^{1/4}} \right)^{-1} - \frac{2(ax)^{1/4}}{x^{3/4} - a^{1/4}x^{1/2} + a^{1/2}x^{1/4} - a^{3/4}} \right]^{-1} - 2^{\log_a^4} \right\} =$

(A) $a^{3/4}$ (B) $a$

(C) $a^2$ (D) None of these

**100.** $\lim_{x \to 1} \frac{(\log(1+x) - \log 2)(3.4^{x-1} - 3x)}{\{(7+x)^{1/3} - (1+3x)^{1/2}\} \sin \pi x} =$

(A) $\frac{9}{\pi} \log \frac{4}{e}$ (B) $\frac{3}{\pi} \log \frac{4}{e}$

(C) $\frac{9}{\pi} \log \frac{2}{e}$ (D) None of these

**101.** $\lim_{x \to 1} \frac{(1-x)(1-x^2)\ldots(1-x^{2n})}{[(1-x)(1-x^2)\ldots(1-x^n)]^2} =$

(A) $n!$ (B) $\frac{(2n)!}{n!}$

(C) $\frac{(2n)!}{(n!)^2}$ (D) None of these

**102.** If $\sum_{r=1}^{k} \cos^{-1} \alpha r = \frac{k\pi}{2}$ for any $k \geq 1$

and $\theta = \sum_{r=1}^{k} (\alpha r)^r$, then $\lim_{x \to \theta} \frac{(1+x^2)^{1/3} - (1-2x)^{1/4}}{x + x^2}$ is

(A) 1 (B) –1 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

**103.** If $\alpha$ and $\beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\lim_{x \to \frac{1}{\alpha}} \sqrt{\frac{1 - \cos(cx^2 + bx + a)}{2(1 - \alpha x)^2}} =$

(A) $\left| \frac{c}{2\alpha} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \right|$ (B) $\left| \frac{c}{2\beta} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \right|$

(C) $\left| \frac{c}{\alpha\beta} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \right|$ (D) None of these

**104.** Given a real valued function $f$ such that

$$f(x) = \begin{cases} \frac{\tan^2\{x\}}{x^2 - [x]^2}, & x > 0 \\ 1, & x = 0 \\ \sqrt{\{x\} \cot \{x\}}, & x < 0 \end{cases}$$

The value of $\cot^{-1} \left( \lim_{x \to 0} f(x) \right)^2$ is

(A) 0 (B) 1

(C) –1 (D) None of these

**105.** If $\lim_{x \to 0} \frac{x^a \sin^b x}{\sin(x^c)}$, $a, b, c \in R - \{0\}$ exists and has non-zero value, then

(A) $a, b, c$ are in A.P.

(B) $a, c, b$ are in A.P.

(C) $a, c, b$ are in G.P.

(D) None of these

## More than One Option Correct Type

**106.** $\lim_{m \to \infty} \lim_{n \to \infty} (1 + \cos^{2m} n! \pi x)$ is equal to

(A) 2 (B) 1

(C) 0 (D) None of these

**107.** If $\lim_{x \to 0} \frac{(1+a^3) + 8e^{1/x}}{1 + (1-b^3)e^{1/x}} = 2$, then

(A) $a = -1$ (B) $a = 1$

(C) $b = (-3)^{1/3}$ (D) $3^{1/3}$

**108.** $\lim_{x \to \infty} \left( \frac{a^{1/x} + b^{1/x} + c^{1/x}}{3} \right)^{3x} =$

(A) $(a + b + c)$ (B) $e^{\log(a+b+c)}$

(C) $abc$ (D) $e^{\log(abc)}$

**109.** If $\lim_{x \to 0} \frac{axe^x - b\log(1+x) + cxe^{-x}}{x^2 \sin x} = 2$, then

(A) $a = 3$ (B) $b = 12$

(C) $c = 9$ (C) $a = -3$

## Passage Based Questions

***Passage 1***

We know that if $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m (\neq 0)$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

However, if $\lim_{x \to a} g(x) = 0 = \lim_{x \to a} f(x)$, we cannot say anything definite about the existence of $\lim_{x \to a} \frac{f(x)}{g(x)}$. Though in some cases this limit exists. Any expression of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ is termed as an indeterminate form. Many other expressions like $\infty - \infty$, $1^\infty$, $\infty^0$, $0^0$, $0 \times \infty$ which can be reduced to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are also called indeterminate forms.

If $\frac{f(x)}{g(x)}$ is indeterminate at $x = a$ of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

where $f'$ is derivative of $f$.

If $\frac{f'(x)}{g'(x)}$, too, is indeterminate at $x = a$ of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$

This can be continued till we finally arrive at a determinate result.

**110.** If $\lim_{x\to 0} \frac{\sin 2x + a \sin x}{x^3}$ be finite, then the value of $a$ and the limit are given by

(A) –2, 1 (B) –2, –1 (C) 2, 1 (D) 2, –1

**111.** The value of $\lim_{x\to 0} \sqrt{a^2 - x^2} \cot \frac{\pi}{2} \sqrt{\frac{a-x}{a+x}}$ is

(A) $\frac{2a}{\pi}$ (B) $-\frac{2a}{\pi}$ (C) $\frac{4a}{\pi}$ (D) $-\frac{4a}{\pi}$

### *Passage 2*

For a function $f$, let $\lim_{x\to a} f(x) \neq 1$ but $f(x)$ is

$$\lim_{x\to a} \{f(x)\}^{g(x)},$$

we write $\{f(x)\}^{g(x)} = e^{\log_e \{f(x)\}^{g(x)}}$

$$\Rightarrow \quad \lim_{x\to a} \{f(x)\}^{g(x)} = e^{\lim_{x\to a} g(x) \log_e f(x)}$$

In case, $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} \{f(x)\}^{g(x)} = \lim_{x\to a} (1 + f(x) - 1)^{g(x)}$$

$$= e^{\lim_{x\to a} (f(x)-1)\, g(x)}$$

**112.** $\lim_{x\to 0} |x|^{\sin x}$ equals

(A) 0 (B) 1
(C) –1 (D) None of these

**113.** If $\alpha$ and $\beta$ be the roots of $ax^2 + bx + c = 0$, then

$$\lim_{x\to\alpha} (1 + ax^2 + bx + c)^{1/(x-\alpha)} \text{ is}$$

(A) $\log |a(\alpha - \beta)|$ (B) $e^{a(\alpha-\beta)}$
(C) $e^{a(\beta-\alpha)}$ (D) None of these

### *Passage 3*

Let $f$, $g$ and $h$ be real valued functions defined on an interval $I \subseteq R$ except possibly for some point $c$ such that

$$\lim_{x\to c} f(x) = l = \lim_{x\to c} h(x)$$

and, $f(x) \le g(x) \le h(x)$, $\forall\, x \in I$.

Then, $\lim_{x\to c} g(x) = l$.

**114.** $\lim_{n\to\delta+\infty} \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n} =$

(A) 1 (B) –1
(C) 0 (D) None of these

**115.** $\lim_{n\to\infty} \frac{\{x\} + \{2x\} + \{3x\} + \ldots + \{nx\}}{n^2}$,

where $\{x\} = x - [x]$ denotes the fractional part of $x$, is

(A) 1 (B) 0
(C) $\frac{1}{2}$ (D) None of these

**116.** $\lim_{x\to 0^+} \left( \lim_{n\to\infty} \frac{[1^2 x^x] + [2^2 x^x] + \ldots + [n^2 x^x]}{n^3} \right)$, where $[\cdot]$ denotes the greatest integer function, is equal to

(A) $-\frac{1}{3}$ (B) $\frac{1}{3}$
(C) 0 (D) None of these

## Match the Column Type

**117.**

| Column-I | Column-II |
|---|---|
| (I) $\lim_{n\to\infty} \left[\sqrt[3]{n^2 - n^3} + n\right]$ | (A) $\frac{1}{9}$ |
| (II) $\lim_{x\to 1} \frac{\sqrt[3]{x^2} - 2\sqrt[3]{x} + 1}{(x-1)^2}$ | (B) $\frac{6}{7}$ |
| (III) $\lim_{n\to\infty} \prod_{r=3}^{n} \left(\frac{r^3 - 1}{r^3 + 1}\right)$ | (C) $\frac{1}{3}$ |
| (IV) $\lim_{n\to\infty} \left(\cos \frac{x}{n}\right)^n$ | (D) 1 |

**118.**

| Column-I | Column-II |
|---|---|
| (I) $\lim_{x\to\frac{\pi}{2}} \dfrac{\sin x-(\sin x)^{\sin x}}{1-\sin x+ln \sin x}$ | (A) 2 |
| (II) $\lim_{n\to\infty}\left\{\dfrac{7}{10}+\dfrac{29}{10^2}+\dfrac{133}{10^3}+\ldots+\dfrac{5^n+2^n}{10^n}\right\}$ | (B) $-\dfrac{1}{2}$ |
| (III) $\lim_{x\to\infty} x\left[\tan^{-1}\dfrac{x+1}{x+2}-\dfrac{\pi}{4}\right]$ | (C) 0 |
| (IV) $\lim_{n\to\infty}\dfrac{n^k\sin^2(n!)}{n+2}, 0<k<1$ | (D) $\dfrac{5}{4}$ |

## Assertion-Reason Type

**Instructions:** *In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:*

(A) Assertion(A) is True and Reason(R) is True; Reason(R) is a correct explanation for Assertion(A)
(B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)
(C) Assertion(A) is True, Reason(R) is False
(D) Assertion(A) is False, Reason(R) is True

**119. Assertion:** If $t_r = \dfrac{1^2+2^2+3^2+\ldots+r^2}{1^3+2^3+3^3+\ldots+r^3}$ and $S_n = \sum_{r=1}^{n}(-1)^r.t_r$, then $\lim_{n\to\infty} S_n = \dfrac{2}{3}$

**Reason:** $1^2+2^2+3^2+\ldots+r^2 = \dfrac{r(r+1)(2r+1)}{6}$

and $1^3+2^3+3^3+\ldots+r^3 = \left(\dfrac{r(r+1)}{2}\right)^2$

**120. Assertion:** If $x_1 = 3$ and $x_{n+1} = \sqrt{2+x_n}, n \geq 1$, then $\lim_{n\to\infty} x_n = 2$

**Reason:** A monotonically decreasing sequence which is bounded below is convergent

**121. Assertion:** $\lim_{n\to\infty}\dfrac{1}{n}\left(1+e^{1/n}+e^{2/n}+\ldots+e^{(n-1)/n}\right) = e-1$

**Reason:** $1+r+r^2+\ldots+r^{n-1}$

$$= \begin{cases} \dfrac{1-r^n}{1-r} & \text{if } r<1 \\ \dfrac{r^n-1}{r-1} & \text{if } r>1 \end{cases}.$$

**122. Assertion:** $\lim_{x\to0}\dfrac{e-(1+x)^{1/x}}{x} = \dfrac{e}{2}$

**Reason:** $\lim_{x\to0}\dfrac{ln(1+x)-x}{x} = 0$

and $\lim_{x\to0}\dfrac{ln(1+x)-x}{x^2} = -\dfrac{1}{2}$

**123. Assertion:** $\lim_{x\to0}\left([f(x)]+x^2\right)^{\frac{1}{\{f(x)\}}} = e$, where $f(x) = \dfrac{\tan x}{x}$ and $[\cdot]$, $\{\ \}$ denote integral and fractional parts, respectively

**Reason:** $\lim_{x\to0}\dfrac{\left[\dfrac{\tan x}{x}\right]+x^2-1}{\left\{\dfrac{\tan x}{x}\right\}} = 3$

**124. Assertion:** $\lim_{n\to\infty}\dfrac{1}{2}\tan\dfrac{x}{2}+\dfrac{1}{2^2}\tan\dfrac{x}{2^2}+\ldots+\dfrac{1}{2^n}\tan\dfrac{x}{2^n} = -\cot x+\dfrac{1}{x}$

**Reason:** $\cot x+\dfrac{1}{2}\tan\dfrac{x}{2} = \dfrac{1}{2}\cot\dfrac{x}{2}$

**125. Assertion:** $\lim_{\theta\to0}\dfrac{\cot\theta\tan^{-1}(m\tan\theta)-m\cos^2(\theta/2)}{\sin^2(\theta/2)} = m-\dfrac{4}{3}m^3$

**Reason:** $\lim_{x\to0}\dfrac{x-\tan x}{x^3} = \dfrac{1}{3}$

## Previous Year's Questions

**126.** $\lim_{x\to\infty}\frac{\sqrt{1-\cos 2x}}{\sqrt{2}x}$ is **[2002]**

(A) 1 (B) − 1
(C) Zero (D) Does not exist

**127.** $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$ is equal to **[2002]**

(A) $e^4$ (B) $e^2$
(C) $e^3$ (D) $e$

**128.** For $x \in$ R, $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^2$ is equal to **[2002]**

(A) $e$ (B) $e^{-1}$
(B) $e^{-5}$ (D) $e^5$

**129.** Let $f(2) = 4$ and $f'(2) = 4$. Then $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}$ is given by **[2002]**

(A) 2 (B) − 2
(C) − 4 (D) 3

**130.** $\lim_{x\to\pi/2}\frac{\left[1-\tan\left(\frac{x}{2}\right)\right][1-\sin x]}{\left[1+\tan\left(\frac{x}{2}\right)\right][\pi-2x]^3}$ is **[2003]**

(A) $\frac{1}{8}$ (B) 0

(C) $\frac{1}{32}$ (D) $\infty$

**131.** If $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x}=k$, the value of $k$ is **[2003]**

(A) 0 (B) $-\frac{1}{3}$

(C) $\frac{2}{3}$ (D) $-\frac{2}{3}$

**132.** Let $f(a) = g(a) = k$ and their $n^{\text{th}}$ derivatives $f^n(a)$, $g^n(a)$ exist and are not equal for some $n$. Further if $\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4$, then the value of $k$ is **[2003]**

(A) 4 (B) 2
(C) 1 (D) 0

**133.** If $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$, then the values of $a$ and $b$, are **[2004]**

(A) $a \in \underline{\underline{R}}, b \in \underline{\underline{R}}$ (B) $a = 1, b \in \underline{\underline{R}}$

(C) $a \in \underline{\underline{R}}, b = \underline{\underline{R}}$ (D) $a = 1$ and $b = 2$

**134.** Let $\alpha$ and $\beta$ be the distinct roots of $ax^2 + bx + c = 0$, then $\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to **[2005]**

(A) $\frac{a^2}{2}(\alpha-\beta)^2$ (B) 0

(C) $-\frac{a^2}{2}(\alpha-\beta)^2$ (D) $\frac{1}{2}(\alpha-\beta)^2$

**135.** Let $f : R \to R$ be a positive increasing function such that $\lim_{x\to\infty}\frac{f(3x)}{f(x)}=1$. Then, $\lim_{x\to\infty}\frac{f(2x)}{f(x)}=$ **[2010]**

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$

(C) 3 (D) 1

**136.** Limit of $\left(\frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$ as $x$ tends to 2 **[2011]**

(A) equals $\sqrt{2}$ (B) equals $-\sqrt{2}$

(C) equals $\frac{1}{\sqrt{2}}$ (D) does not exist

**137.** The value of $\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to **[2013]**

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) $-\frac{1}{4}$

**138.** Let $f(x)$ be a forth degree polynomial having extreme values at $x = 1$ and $x = 2$. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]=3$, then $f(2)$ is equal to **[2015]**

(A) −4 (B) 0
(C) 4 (D) −8

**139.** The value of $\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to **[2015]**

(A) 3
(B) 2
(C) $\frac{1}{2}$
(D) 4

**140.** Let $p=\lim_{x\to 0+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then log $p$ is equal to **[2016]**

(A) $\frac{1}{4}$
(B) 2
(C) 1
(D) $\frac{1}{2}$

## ANSWER KEYS

### Single Option Correct Type

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1.** (B) | **2.** (A) | **3.** (A) | **4.** (B) | **5.** (A) | **6.** (C) | **7.** (D) | **8.** (C) | **9.** (A) | **10.** (A) |
| **11.** (B) | **12.** (C) | **13.** (D) | **14.** (D) | **15.** (C) | **16.** (a, b) | **17.** (C) | **18.** (A) | **19.** (A) | **20.** (B) |
| **21.** (B) | **22.** (D) | **23.** (C) | **24.** (B) | **25.** (B) | **26.** (A) | **27.** (C) | **28.** (C) | **29.** (B) | **30.** (A) |
| **31.** (B) | **32.** (B) | **33.** (A) | **34.** (B) | **35.** (A) | **36.** (B) | **37.** (C) | **38.** (A) | **39.** (D) | **40.** (B) |
| **41.** (A) | **42.** (C) | **43.** (B) | **44.** (A) | **45.** (A) | **46.** (B) | **47.** (C) | **48.** (A) | **49.** (B) | **50.** (B) |
| **51.** (B) | **52.** (B) | **53.** (A) | **54.** (C) | **55.** (B) | **56.** (C) | **57.** (D) | **58.** (B) | **59.** (C) | **60.** (B) |
| **61.** (A) | **62.** (B) | **63.** (B) | **64.** (B) | **65.** (A) | **66.** (B) | **67.** (B) | **68.** (B) | **69.** (C) | **70.** (A) |
| **71.** (A) | **72.** (C) | **73.** (B) | **74.** (C) | **75.** (A) | **76.** (B) | **77.** (C) | **78.** (B) | **79.** (A) | **80.** (C) |
| **81.** (B) | **82.** (A) | **83.** (C) | **84.** (A) | **85.** (A) | **86.** (B) | **87.** (B) | **88.** (D) | **89.** (A) | **90.** (B) |
| **91.** (B) | **92.** (B) | **93.** (A) | **94.** (C) | **95.** (C) | **96.** (A) | **97.** (C) | **98.** (A) | **99.** (C) | **100.** (A) |
| **101.** (C) | **102.** (C) | **103.** (A) | **104.** (D) | **105.** (D) | | | | | |

### More than One Option Correct Type

**106.** (A) and (B) **107.** (B) and (C) **108.** (C) and (D) **109.** (A), (B) and (C)

### Passage Based Questions

**110.** (B) **111.** (C) **112.** (B) **113.** (B) **114.** (C) **115.** (B) **116.** (B)

### Match the Column Type

**117.** (I) → (C); (II) → (A); (III) → (B); (IV) → (D)
**118.** (I) → (A); (II) → (D); (III) → (B); (IV) → (C)

### Assertion-Reason Type

**119.** (D) **120.** (A) **121.** (A) **122.** (A) **123.** (D) **124.** (A) **125.** (C)

### Previous Year's Questions

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **126.** (D) | **127.** (A) | **128.** (C) | **129.** (C) | **130.** (C) | **131.** (C) | **132.** (A) | **133.** (B) | **134.** (A) | **135.** (D) |
| **136.** (D) | **137.** (C) | **138.** (A) | **139.** (B) | **140.** (D) | | | | | |

# HINTS AND SOLUTIONS

## Single Option Correct Type

**1.** We know that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}\to 1^-$ and $\lim\limits_{x\to 0}\dfrac{x}{\sin x}\to 1^+$

$$\therefore\quad \lim_{x\to 0}\left[100\,\frac{x}{\sin x}\right]+\lim_{x\to 0}\left[99\,\frac{\sin x}{x}\right]=100+98=198.$$

The correct option is (B)

**2.** $g\,[f(x)]=[f(x)]^2+1,\quad f(x)\neq 2$

$3,\quad f(x)=2$

$\therefore\quad g\,[f(x)]=\sin^2 x+1,\quad x\neq n\pi$

$3,\quad x=n\pi$

$$\text{RHL}=\lim_{h\to 0}g\left[f(0+h)\right]=\lim_{h\to 0}(\sin^2 h+1)=1.$$

$$\text{LHL}=\lim_{h\to 0}g\left[f(0-h)\right]=\lim_{h\to 0}(\sin^2 h+1)=1.$$

$$\therefore\quad \lim_{x\to 0}g\left[f(x)\right]=1$$

The correct option is (A)

**3.** $$\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]$$

$$=\lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{x}\left(1+\frac{1}{\sqrt{x}}\right)^{1/2}}{\sqrt{x}\left[\left(1+\frac{1}{\sqrt{x}}\sqrt{1+\frac{1}{\sqrt{x}}}\right)^{1/2}+1\right]}$$

$$=\frac{1}{1+1}=\frac{1}{2}$$

The correct option is (A)

**4.** $$\lim_{x\to\infty}x\left[\tan^{-1}\frac{x+1}{x+2}-\frac{\pi}{4}\right]$$

$$=\lim_{x\to\infty}x\left[\tan^{-1}\frac{x+1}{x+2}-\tan^{-1}1\right]=\lim_{x\to\infty}x\tan^{-1}\left(\frac{\frac{x+1}{x+2}-1}{1+\frac{x+1}{x+2}}\right)$$

$$=\lim_{x\to\infty}x\tan^{-1}\left(\frac{-1}{2x+3}\right)=-\lim_{x\to\infty}\frac{\tan^{-1}\left(\frac{1}{2x+3}\right)}{\left(\frac{1}{2x+3}\right)}\cdot\frac{1}{\left(2+\frac{3}{x}\right)}$$

$$=-1\times\frac{1}{2}=-\frac{1}{2}$$

The correct option is (B)

**5.** $$\lim_{n\to\infty}\cos\left[\pi\sqrt{n^2+n}\right]=\lim_{n\to\infty}\cos\left[n\pi\left(1+\frac{1}{n}\right)^{1/2}\right]$$

$$=\lim_{n\to\infty}\cos\left[n\pi\left(1+\frac{1}{2n}-\frac{1}{8n^2}+\ldots\right)\right]$$

$$=\lim_{n\to\infty}\cos\left(n\pi+\frac{\pi}{2}-\frac{\pi}{8n}+\ldots\right)$$

$$=-\lim_{n\to\infty}\sin\left(n\pi-\frac{\pi}{8n}+\ldots\right)$$

$$=-\lim_{n\to\infty}(-1)^{n-1}\sin\left(\frac{\pi}{8n}-\ldots\right)$$

$$=0\qquad\left(\because\ \frac{\pi}{8n}-\ldots\to 0\text{ as }n\to\infty\right)$$

The correct option is (A)

**6.** $$\lim_{n\to\infty}\frac{n^k\sin^2(n!)}{n+2}=\lim_{n\to\infty}\frac{n^k\sin^2(n!)}{n\left(1+\frac{2}{n}\right)}$$

$$=\lim_{n\to\infty}\frac{\sin^2(n!)}{n^{1-k}\left(1+\frac{2}{n}\right)}$$

$$=\frac{\text{a finite quantity}}{\infty}$$

[$\because$ $\sin^2(n\,!)$ always lies between 0 and 1. Also, since $1-k>0$, $\therefore$ $n^{1-k}\to\infty$ as $n\to\infty$]

$=0.$

The correct option is (C)

**7.** $$\lim_{x\to 1}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}=\lim_{x\to 1}\frac{\sqrt{2\sin^2(x-1)}}{x-1}$$

$$=\lim_{x\to 1}\frac{\sqrt{2}\,|\sin(x-1)|}{(x-1)}$$

$$\text{LHL}=\lim_{x\to 1^-}\frac{\sqrt{2}\,|\sin(x-1)|}{(x-1)}=\lim_{h\to 0}-\frac{\sqrt{2}\,|\sin(1-h-1)|}{(1-h-1)}$$

$$=\lim_{h\to 0}\frac{\sqrt{2}\,|-\sin h|}{-h}=-\sqrt{2}\lim_{h\to 0}\frac{\sin h}{h}$$

$$=-\sqrt{2}\cdot 1=-\sqrt{2}$$

$$\text{RHL}=\lim_{x\to 1^+}\frac{\sqrt{2}\,|\sin(x-1)|}{(x-1)}=\lim_{h\to 0}\frac{\sqrt{2}\,|\sin(1+h-1)|}{(1+h-1)}$$

$$=\lim_{h\to 0}\frac{\sqrt{2}\,|\sin h|}{h}=\sqrt{2}\lim_{h\to 0}\frac{\sin h}{h}=\sqrt{2}\cdot 1=\sqrt{2}$$

Since LHL ≠ RHL,

$$\therefore \lim_{x\to 1}\frac{\sqrt{1-\cos 2(x-1)}}{x-1} \text{ does not exist.}$$

The correct option is (D)

**8.** $\lim_{x\to\infty}\frac{x^5}{5^x} = \lim_{x\to\infty}\frac{x^5}{e^{x\log 5}} = \lim_{x\to\infty}\frac{x^5}{e^{kx}}$,

where $k = \log 5$

$$= \lim_{x\to\infty}\frac{x^5}{\left(1+kx+\frac{k^2x^2}{2!}+\frac{k^3x^3}{3!}+\frac{k^4x^4}{4!}+\frac{x^5k^5}{5!}+\frac{k^6x^6}{6!}+...\right)}$$

$$= \lim_{x\to\infty}\frac{1}{\left[\left(\frac{1}{x^5}+k\cdot\frac{1}{x^4}+\frac{k^2}{2!}\cdot\frac{1}{x^3}+\frac{k^3}{3!}\cdot\frac{1}{x^2}+\frac{k^4}{4!}\cdot\frac{1}{x}\right)+\frac{k^5}{5!}+\left(\frac{k^6}{6!}x+...\right)\right]}$$

$$= \frac{1}{\infty} = 0$$

The correct option is (C)

**9.** $\lim_{x\to 0}(\cos x+\sin x)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}(\cos x+\sin x-1)}$

$$= e^{\lim_{x\to 0}\frac{(-\sin x+\cos x)}{1}}$$

(Using L'Hospital's Rule)

$$= e^1 = e$$

The correct option is (A)

**10.** $\lim_{x\to\frac{\pi}{4}}\frac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin 2x}$ $\quad\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to\frac{\pi}{4}}\frac{-3(\cos x+\sin x)^2(-\sin x+\cos x)}{-2\cos 2x}$$

(Using L'Hospital's Rule)

$$= \lim_{x\to\frac{\pi}{4}}\frac{-3(\cos x+\sin x)(\cos^2 x-\sin^2 x)}{-2\cos 2x}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{-3(\cos x+\sin x)\cos 2x}{-2\cos 2x}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{3(\cos x+\sin x)}{2} = \frac{3}{2}\cdot\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}$$

The correct option is (A)

**11.** $\lim_{h\to 0}\frac{\ln(1+2h)-2\ln(1+h)}{h^2}$

$$= -\lim_{h\to 0}\frac{\ln(1+h)^2-\ln(1+2h)}{h^2}$$

$$= -\lim_{h\to 0}\frac{\ln\left(\frac{(1+h)^2}{1+2h}\right)}{h^2}$$

$$= -\lim_{h\to 0}\frac{\ln\left(1+\frac{h^2}{1+2h}\right)}{\left(\frac{h^2}{1+2h}\right)(1+2h)}$$

$$= -\lim_{h\to 0}\frac{\ln\left(1+\frac{h^2}{1+2h}\right)}{\left(\frac{h^2}{1+2h}\right)}\cdot\frac{1}{1+2h} = -1$$

$$\left[\text{Using } \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$$

The correct option is (B)

**12.** $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{e^{1/n}}{n}+\frac{e^{2/n}}{n}+...+\frac{e^{(n-1)/n}}{n}\right]$

$$= \lim_{n\to\infty}\left[\frac{1+e^{1/n}+(e^{1/n})^2+...+(e^{1/n})^{n-1}}{n}\right]$$

$$= \lim_{n\to\infty}\frac{1\cdot\left[(e^{1/n})^n-1\right]}{n(e^{1/n}-1)} = (e-1)\lim_{n\to\infty}\frac{1}{\left(\frac{e^{1/n}-1}{1/n}\right)}$$

$$= (e-1)\times 1 = (e-1).$$

The correct option is (C)

**13.** RHL $= \lim_{h\to 0}\frac{(1+h)\sin(1+h-[1+h])}{1+h-1}$

$$= \lim_{h\to 0}\frac{(1+h)\sin(1+h-1)}{h}$$

$$= \lim_{h\to 0}(1+h)\frac{\sin h}{h} = 1$$

LHL $= \lim_{h\to 0}\frac{(1-h)\sin(1-h-[1-h])}{1-h-1}$

$$= \lim_{h\to 0}\frac{(1-h)\sin(1-h)}{-h} = -\infty$$

Since LHL ≠ RHL,

$$\therefore \lim_{x\to 1}\frac{x\sin(x-[x])}{x-1} \text{ does not exist.}$$

The correct option is (D)

**14.** $f(x) = \int\frac{2\sin x-\sin 2x}{x^3}dx$

$$\Rightarrow f'(x) = \frac{d}{dx}\int\frac{2\sin x-\sin 2x}{x^3}dx = \frac{2\sin x-\sin 2x}{x^3}$$

$$\therefore \lim_{x\to 0}f'(x) = \lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3} \quad\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{2\cos x-2\cos 2x}{3x^2} \quad\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{-2\sin x+4\sin 2x}{6x} \quad\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{6}$$

$$= \frac{6}{6} = 1$$

The correct option is (D)

**15.** $\because \quad \frac{\pi}{4} < 1,$

$\therefore \quad \left(\frac{\pi}{4}\right) = 0$

$\therefore \quad \lim_{x\to\pi/2} \frac{\left(\frac{x}{2}\right)}{\ln(\sin x)} = 0$

The correct option is (C)

**16.** We know that $|\cos\theta| \le 1$ for all $\theta$.

So, if $|\cos n!\, px| < 1$,

$$\lim_{m\to\infty}\lim_{n\to\infty}(1+\cos^{2m} n!\pi x) = (1+0) = 1$$

and if $\quad |\cos n!\, px| = 1$,

$$\lim_{m\to\infty}\lim_{n\to\infty}(1+\cos^{2m} n!\pi x) = \lim_{m\to\infty}\lim_{n\to\infty}(1+1^{2m})$$

$$= \lim_{m\to\infty}\lim_{n\to\infty}(1+1) = 2$$

The correct option is (A) and (B)

**17.** LHL $= \lim_{x\to 0^-}\left[\frac{\sin([x-3])}{[x-3]}\right] = \left[\frac{\sin(-4)}{-4}\right]$

$= \left[\frac{\sin 4}{4}\right] = -1 \qquad \because \quad \pi < 4 < \frac{3\pi}{2}$

RHL $= \lim_{x\to 0^+}\left[\frac{\sin[x-3]}{[x-3]}\right] = \left[\frac{\sin(-3)}{-3}\right]$

$= \left[\frac{\sin 3}{3}\right] = 0 \qquad \because \quad \frac{\pi}{2} < 3 < \pi.$

The correct option is (C)

**18.** We have, $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1} - ax - b\right) = 0$

$\Rightarrow \quad \lim_{x\to\infty}\frac{(x^2+1)-(ax+b)(x+1)}{x+1} = 0$

$\Rightarrow \quad \lim_{x\to\infty}\frac{x^2(1-a)-(a+b)x-b+1}{x+1} = 0$

$\Rightarrow \quad 1 - a = 0$ and $a + b = 0$

$\Rightarrow \quad a = 1$ and $b = -1$.

The correct option is (A)

**19.** $\lim_{n\to\infty}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}\right]$

$= \lim_{n\to\infty}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$

$= \lim_{n\to\infty}\left[1-\frac{1}{n+1}\right] = 1 - 0 = 1.$

The correct option is (A)

**20.** $\lim_{x\to\infty}\frac{(\log x)^2}{x^n} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$

$\lim_{x\to\infty}\frac{2\log x\cdot\frac{1}{x}}{nx^{n-1}} = \lim_{x\to\infty}\frac{2\log x}{nx^n} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$

$= \lim_{x\to\infty}\frac{2}{n^2x^n} = 0$

The correct option is (B)

**21.** $t_r = \frac{r}{r^4+r^2+1} = \frac{r}{(r^2+1)^2-r^2}$

$= \frac{1}{2}\left(\frac{1}{r^2-r+1}-\frac{1}{r^2+r+1}\right)$

$= \frac{1}{2}\left[\frac{1}{r(r-1)+1}-\frac{1}{(r+1)r+1}\right]$

$\therefore \quad \sum_{r=1}^{n} t_r = \sum_{r=1}^{n}\frac{1}{2}[f(r)-f(r+1)],$

where $f(r) = \frac{1}{r(r-1)+1} = \frac{1}{2}[f(1)-f(n+1)]$

$= \frac{1}{2}\left[1-\frac{1}{(n+1)n+1}\right] \to \frac{1}{2}$ as $n\to\infty$

The correct option is (B)

**22.** LHL $= \lim_{h\to 0}(-1)^{[n-h]} = \lim_{h\to 0}(-1)^{n-1} = (-1)^{n-1}$

RHL $= \lim_{h\to 0}(-1)^{[n+h]} = \lim_{h\to 0}(-1)^{n} = (-1)^n$

Since LHL $\ne$ RHL

$\therefore \quad \lim_{x\to n}(-1)^{[x]}$ does not exist.

The correct option is (D)

**23.** Since $y = x - 1$,

$\therefore \quad x = y + 1.$

As $(x, y) \to (1, 0)$ along the line $y = x - 1$, $x = y + 1$ holds throughout.

$\therefore \quad \lim_{\substack{x\to 1\\ y\to 0}}\frac{y^3}{x^3-y^2-1} = \lim_{y\to 0}\frac{y^3}{(y+1)^3-y^2-1}$

$\lim_{y\to 0}\frac{y^3}{y^3+2y^2+3y} = \lim_{y\to 0}\frac{y^2}{y^2+2y+3} = \frac{0}{3} = 0$

The correct option is (C)

**24.** $\lim_{n\to\infty}\frac{1-2+3-4+5-6+\ldots-2n}{\sqrt{n^2+1}+\sqrt{4n^2-1}}$

$= \lim_{n\to\infty}\frac{[1+3+5+7+\ldots+(2n-1)]-(2+4+6+\ldots+2n)}{n\sqrt{1+\frac{1}{n^2}}+n\sqrt{4-\frac{1}{n^2}}}$

$= \lim_{n\to\infty}\frac{\frac{n}{2}[2\cdot 1+(n-1)\cdot 2]-\frac{n}{2}[2\cdot 2+(n-1)\cdot 2]}{n\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{4-\frac{1}{n^2}}\right)}$

$$= \lim_{n\to\infty} \frac{\frac{n}{2}\cdot 2n - \frac{n}{2}2(n+1)}{n\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{4-\frac{1}{n^2}}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2-n^2-n}{n\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{4-\frac{1}{n^2}}\right)}$$

$$= \lim_{n\to\infty} \frac{-n}{n\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{4-\frac{1}{n^2}}\right)}$$

$$= \lim_{n\to\infty} \frac{-1}{\sqrt{1+\frac{1}{n^2}}+\sqrt{4-\frac{1}{n^2}}} = \frac{-1}{1+2} = \frac{-1}{3}$$

The correct option is (B)

**25.** $\lim_{x\to-\infty}\left(\frac{x^4\sin(1/x)\;+x^2}{1+|x|^3}\right)$

$$= \lim_{y\to\infty}\frac{-y^4\sin\frac{1}{y}+y^2}{1+y^3}$$

(Putting $x = -y$; as $x\to-\infty$, $y\to\infty$)

$$= \lim_{y\to\infty}\frac{-\left(\frac{\sin\frac{1}{y}}{\frac{1}{y}}\right)+\frac{1}{y}}{1+\frac{1}{y^3}} = \frac{-1+0}{1+0} = -1$$

The correct option is (B)

**26.** $\lim_{x\to2}\frac{2^x+2^{3-x}-6}{2^{-x/2}-2^{1-x}}$

$$= \lim_{x\to2}\frac{(2^{2x}+2^3-6\cdot2^x)/2^x}{\frac{1}{2^{x/2}}-\frac{2}{2^x}}$$

$$= \lim_{x\to2}\frac{2^{2x}-6\cdot2^x+8}{2^{x/2}-2} = \lim_{x\to2}\frac{(2^x-4)(2^x-2)}{(2^{x/2}-2)}$$

$$= \lim_{x\to2}\frac{(2^{x/2}+2)(2^{x/2}-2)(2^x-2)}{(2^{x/2}-2)}$$

$$= \lim_{x\to2}(2^{x/2}+2)(2^x-2) = (2+2)\cdot(4-2) = 8$$

The correct option is (A)

**27.** $\lim_{x\to0}\frac{8}{x^8}\left[1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right]$

$$= \lim_{x\to0}\frac{8}{x^8}\left[\left(1-\cos\frac{x^2}{2}\right)-\cos\frac{x^4}{4}\left(1-\cos\frac{x^2}{2}\right)\right]$$

$$= \lim_{x\to0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)$$

$$= \lim_{x\to0}\frac{8}{x^8}\cdot2\sin^2\frac{x^2}{4}\cdot2\sin^2\frac{x^2}{8}$$

$$= \lim_{x\to0}\frac{32}{x^8}\left(\frac{\sin\frac{x^2}{4}}{\frac{x^2}{4}}\right)^2\left(\frac{x^2}{4}\right)^2\left(\frac{\sin\frac{x^2}{8}}{\frac{x^2}{8}}\right)^2\left(\frac{x^2}{8}\right)^2$$

$$= \frac{1}{32}$$

The correct option is (C)

**28.** $\lim_{n\to\infty}\left[\log_{n-1}(n)\cdot\log_n(n+1)\cdot\log_{n+1}(n+2)\ldots\log_{n^k-1}(n^k)\right]$

$$= \lim_{n\to\infty}\left[\frac{\log n}{\log(n-1)}\cdot\frac{\log(n+1)}{\log n}\cdot\frac{\log(n+2)}{\log(n+1)}\cdots\frac{\log(n^k)}{\log(n^k-1)}\right]$$

$$\left(\text{Using } \log_n m = \frac{\log m}{\log n}\right)$$

$$= \lim_{n\to\infty}\frac{\log n^k}{\log(n-1)} = k\lim_{n\to\infty}\frac{\log n}{\log(n-1)} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= k\lim_{n\to\infty}\frac{1/n}{1/n-1} \qquad \text{(Using L'Hospital's Rule)}$$

$$= k\lim_{n\to\infty}\left(1-\frac{1}{n}\right) = k$$

The correct option is (C)

**29.** $\lim_{n\to\infty}\left[\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\ldots+\frac{1}{(2n+1)(2n+3)}\right]$

$$= \lim_{n\to\infty}\left[\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)\ldots+\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{2n+3}\right) = \frac{1}{2}\;(1-0) = \frac{1}{2}$$

The correct option is (B)

**30.** $\lim_{x\to\infty}\left(\frac{1^{1/x}+2^{1/x}+3^{1/x}+\ldots+n^{1/x}}{n}\right)^{nx}$

$$= \lim_{y\to0}\left(\frac{1^y+2^y+3^y+\ldots+n^y}{n}\right)^{\frac{n}{y}}$$

$$= e^{\lim_{y\to0}\frac{n}{y}\left(\frac{1^y+2^y+3^y+\ldots+n^y}{n}-1\right)}$$

$$= e^{\lim_{y\to0}\left(\frac{1^y+2^y+3^y+\ldots+n^y-n}{y}\right)}$$

$$= e^{\lim_{y\to0}\left[\frac{(1^y-1)}{y}+\frac{(2^y-1)}{y}+\frac{(3^y-1)}{y}+\ldots+\frac{(n^y-1)}{y}\right]}$$

$$= e^{(\log1+\log2+\log3+\ldots+\log n)}$$

$$= e^{\log(1\cdot2\cdot3\ldots n)} = n!$$

The correct option is (A)

**31.** $\lim_{n\to\infty} (1+x)(1+x^2)(1+x^4)\ldots(1+x^{2n})$

$= \lim_{n\to\infty} \frac{(1-x)(1+x)(1+x^2)(1+x^4)...(1+x^{2n})}{1-x}$

$= \lim_{n\to\infty} \frac{(1-x^2)(1+x^2)(1+x^4)...(1+x^{2n})}{1-x}$

$\vdots \quad \vdots \quad \vdots$

$= \lim_{n\to\infty} \frac{1-x^{4n+2}}{1-x} = \frac{1}{1-x}$ for $|x| < 1$

The correct option is (B)

**32.** **Case I:** $n$ is a positive integer

$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x}$

$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \ldots = \lim_{x\to\infty} \frac{n!}{e^x}$

(Using L'Hospital's Rule repeatedly)

$= 0$

**Case II:** $n$ is a negative integer.

$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{x^{-m}}{e^x}$

(Putting $n = -m$, where $m$ is a positive integer)

$= \lim_{x\to\infty} \frac{1}{x^m e^x} = \frac{1}{\infty} = 0.$

**Case III:** $n = 0$

$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0.$

Hence, $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for all values of $n$.

The correct option is (B)

**33.** $\lim_{x\to 1} \frac{x^n + x^{n-1} + x^{n-2} + ... + x^2 + x - n}{x-1}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to 1} \frac{nx^{n-1} + (n-1)x^{n-2} + ... + 2x + 1}{1}$

(Using L'Hospital's Rule)

$= n + (n-1) + \ldots + 2 + 1 = \frac{n(n+1)}{2}$

The correct option is (A)

**34.** $t_r = \frac{1^2 + 2^2 + 3^2 + ... + r^2}{1^3 + 2^3 + 3^3 + ... + r^3}$

$= \frac{r(r+1)(2r+1)}{6} \cdot \left(\frac{2}{r(r+1)}\right)^2$

$= \frac{2}{3}\left(\frac{1}{r} + \frac{1}{r+1}\right)$

$\therefore \quad S_n = \frac{2}{3}\left[-\left(1+\frac{1}{2}\right) + \left(\frac{1}{2}+\frac{1}{3}\right) - \left(\frac{1}{3}+\frac{1}{4}\right) + \ldots \pm \left(\frac{1}{n}+\frac{1}{n+1}\right)\right]$

$= \frac{2}{3}\left(-1 \pm \frac{1}{n+1}\right)$

$\therefore \quad \lim_{n\to\infty} S_n = -\frac{2}{3}$

The correct option is (B)

**35.** We have $2 = \lim_{x\to 0} \frac{(1+a^3) + 8e^{1/x}}{1 + (1-b^3)e^{1/x}}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$ (1)

$\Rightarrow \quad 2 = \lim_{x\to 0} \frac{0 + 8e^{1/x}(-1/x^2)}{0 + (1-b^3)e^{1/x}(-1/x^2)}$

(Using L'Hospital's Rule)

$\Rightarrow \quad 1 - b^3 = 4$

$\Rightarrow \quad b^3 = -3$

$\Rightarrow \quad b = (-3)^{1/3}$

$\therefore$ From Eq. (1), $2 = \lim_{x\to 0} \frac{(1+a^3) + 8e^{1/x}}{1 + 4e^{1/x}}$

$\Rightarrow \quad 1 + a^3 = 2$ i.e., $a = 1$

Hence $a = 1$ and $b = (-3)^{1/3}$.

The correct option is (A)

**36.** $x^2 + 4x + 5 = (x+2)^2 + 1 \geq 1$. So, $a = 1$.

Also, $b = \lim_{\theta\to 0} \frac{1-\cos 2\theta}{\theta^2} = \lim_{\theta\to 0} \frac{2\sin^2\theta}{\theta^2} = 2.$

$\therefore \quad \sum_{r=0}^{n} a^r \cdot b^{n-r} = b^n + ab^{n-1} + a^2 b^{n-2} + \ldots + a^n$

$= \frac{b^n\left[1 - \left(\frac{a}{b}\right)^{n+1}\right]}{1 - \frac{a}{b}} = \frac{2^n\left[1 - \left(\frac{1}{2}\right)^{n+1}\right]}{1 - \frac{1}{2}}$

$= \frac{2^{n+1}(2^{n+1} - 1)}{2^{n+1}} = (2^{n+1} - 1)$

The correct option is (B)

**37.** $\lim_{n\to\infty} \frac{1\cdot 2 + 2\cdot 3 + 3\cdot 4 + ... + n(n+1)}{n^3}$

$= \lim_{n\to\infty} \frac{\Sigma n(n+1)}{n^3} = \lim_{n\to\infty} \frac{\Sigma n^2 + \Sigma n}{n^3}$

$= \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

$= \lim_{n\to\infty} \left[\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{1}{2}\cdot\left(\frac{1}{n} + \frac{1}{n^2}\right)\right]$

$= \frac{1}{6} \times 1 \times 2 = \frac{1}{3}.$

The correct option is (C)

**38.** $\lim_{x\to 0} \frac{\log(1+x+x^2) + \log(1-x+x^2)}{\sec x - \cos x}$

$= \lim_{x\to 0} \frac{\log\left[(1+x^2)^2 - x^2\right]}{(1-\cos^2 x)/\cos x}$

$= \lim_{x\to 0} \frac{\log(1+x^2+x^4)}{\sin x \tan x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0}\frac{\log[1+x^2(1+x^2)]}{x^2(1+x^2)}\cdot x^2(1+x^2)\cdot\frac{1}{\frac{\sin x}{x}\cdot\frac{\tan x}{x}\cdot x^2}$$

$$= 1.\left[\text{as } \lim_{x\to 0}\frac{\log(1+x)}{x}=1\right]$$

The correct option is (A)

**39.** $$\text{LHL} = \lim_{h\to 0}\frac{\ln(e-h)-1}{|e-h-e|} = \lim_{h\to 0}\frac{\log_e e\left(1-\frac{h}{e}\right)-1}{|-h|}$$

$$= \lim_{h\to 0}\frac{\log e+\log\left(1-\frac{h}{e}\right)-1}{h}$$

$$= \lim_{h\to 0}\frac{-\frac{h}{e}-\frac{h^2}{2e^2}-\ldots}{|e+h-e|} = -\frac{1}{e}$$

$$\text{RHL} = \lim_{h\to 0}\frac{\ln(e+h)-1}{|e+h-e|}$$

$$= \lim_{h\to 0}\frac{\log e\left(1+\frac{h}{e}\right)-1}{|h|}$$

$$= \lim_{h\to 0}\frac{\log e+\log\left(1+\frac{h}{e}\right)-1}{h}$$

$$= \lim_{h\to 0}\frac{\frac{h}{e}-\frac{h^2}{2e^2}+\ldots}{h} = \frac{1}{e}$$

Since LHL $\neq$ RHL

$\therefore \quad \lim_{x\to e}\frac{\ln x-1}{|x-e|}$ does not exist.

The correct option is (D)

**40.** We have

$x_1 = 3, x_{n+1} = \sqrt{2+x_n}$

$x_2 = \sqrt{2+x_1} = \sqrt{2+3} = \sqrt{5}$

$x_3 = \sqrt{2+x_2} = \sqrt{2+\sqrt{5}}$

$\therefore \quad x_1 > x_2 > x_3$

It can be easily shown by mathematical induction that the sequence $x_1, x_2, \ldots x_n, \ldots$ is a monotonically decreasing sequence bounded below by 2. So it is convergent.

Let $\lim_{n\to\infty} x_n = x$. Then

$x_{n+1} = \sqrt{2+x_n}$

$\Rightarrow \quad \lim_{n\to\infty} x_{n+1} = \sqrt{2+\lim_{n\to\infty} x_n}$

$\Rightarrow \quad x = \sqrt{2+x}$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x-2)(x+1) = 0$

$\Rightarrow \quad x = 2 \qquad (\because x_n > 0\ \forall n, \therefore x > 0)$

The correct option is (B)

**41.** $$\lim_{x\to\infty}\frac{3^{x+1}-5^{x+1}}{3^x-5^x} = \lim_{x\to\infty}\frac{3\cdot 3^x-5\cdot 5^x}{3^x-5^x}$$

$$= \lim_{x\to\infty}\frac{3\cdot\left(\frac{3}{5}\right)^x-5}{\left(\frac{3}{5}\right)^x-1} = \frac{-5}{-1}$$

$$= 5. \qquad \left(\because \lim_{n\to\infty} a^n = 0, \text{ if } -1<a<1\right)$$

The correct option is (A)

**42.** $$\lim_{n\to\infty}\frac{1}{n}\left(1+e^{1/n}+e^{2/n}+\ldots+e^{\frac{n-1}{n}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\cdot\frac{1-\left(e^{1/n}\right)^n}{1-e^{1/n}} = \lim_{n\to\infty}\frac{1-e}{n\left(1-1-\frac{1}{n}-\frac{1}{2!}\cdot\frac{1}{n^2}\ldots\right)}$$

$$= \lim_{n\to\infty}\frac{1-e}{-1-\frac{1}{2!}\cdot\frac{1}{n}\ldots} = \frac{1-e}{-1} = e-1$$

The correct option is (C)

**43.** $$\lim_{x\to\infty}\sqrt{\frac{x+\sin x}{x-\cos x}} = \lim_{x\to\infty}\sqrt{\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}} = \sqrt{\frac{1+0}{1-0}} = 1.$$

$$\left[\because \lim_{x\to\infty}\frac{\sin x}{x} = \lim_{y\to 0} y\sin\left(\frac{1}{y}\right) = \text{O}\times\text{ (a finite quantity)} = 0.\text{ Similarly } \lim_{x\to\infty}\frac{\cos x}{x} = 0\right]$$

The correct option is (B)

**44.** $S_n = \sum_{i=1}^{n} a_i,\ \lim_{n\to\infty} a_n = a$

$S_{n+1} - S_n = a_{n+1}$

So, $$\lim_{n\to\infty}\frac{a_{n+1}}{\sqrt{\frac{n(n+1)}{2}}} = \lim_{n\to\infty}\frac{\sqrt{2}a}{\sqrt{n(n+1)}} = 0$$

The correct option is (A)

**45.** $$\lim_{n\to\infty}\left[\sqrt[3]{n^2-n^3}+n\right] = \lim_{n\to\infty} n\left[\left(-1+\frac{1}{n}\right)^{1/3}+1^{1/3}\right]$$

$$= \lim_{n\to\infty} n\cdot\frac{\left(\frac{1}{n}-1\right)+1}{\left(\frac{1}{n}-1\right)^{2/3}+1-\left(\frac{1}{n}-1\right)^{1/3}}$$

$$\left(\text{Using } a+b = \frac{a^3+b^3}{a^2-ab+b^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{\left(\frac{1}{n}-1\right)^{2/3}+1-\left(\frac{1}{n}-1\right)^{1/3}} = \frac{1}{1+1+1} = \frac{1}{3}$$

The correct option is (A)

**46.** $\lim_{n\to\infty} \dfrac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2}-\sqrt[2]{n^3+1}}$

$= \lim_{n\to\infty} \dfrac{n^{5/4}\sqrt[4]{1+\frac{2}{n^5}} - n^{2/3}\sqrt[3]{1+\frac{1}{n^2}}}{n^{4/5}\sqrt[5]{1+\frac{2}{n^4}} - n^{3/2}\sqrt[2]{1+\frac{1}{n^3}}}$

$= \lim_{n\to\infty} \dfrac{\frac{n^{5/4}}{n^{3/2}}\sqrt[4]{1+\frac{2}{n^5}} - \frac{n^{2/3}}{n^{3/2}}\sqrt[3]{1+\frac{1}{n^2}}}{\frac{n^{4/5}}{n^{3/2}}\sqrt[5]{1+\frac{2}{n^4}} - \frac{n^{3/2}}{n^{3/2}}\sqrt[2]{1+\frac{1}{n^3}}}$

(Dividing the numerator and denominator by the highest power $n^{3/2}$)

$= \lim_{n\to\infty} \dfrac{\frac{1}{n^{1/4}}\sqrt[4]{1+\frac{2}{n^5}} - \frac{1}{n^{5/6}}\sqrt[3]{1+\frac{1}{n^2}}}{\frac{1}{n^{7/10}}\sqrt[5]{1+\frac{2}{n^4}} - \sqrt[2]{1+\frac{1}{n^3}}} = \dfrac{0-0}{0-1} = 0.$

The correct option is (B)

**47.** Minimum power in numerator on $x$ is 3. So $n = 3$.

The correct option is (C)

**48.** $\lim_{x\to\infty} \dfrac{2\sqrt{x}+3\sqrt[3]{x}+5\sqrt[5]{x}}{\sqrt{3x-2}+\sqrt[3]{2x-3}}$

$= \lim_{x\to\infty} \dfrac{2\sqrt{x}+3\sqrt[3]{x}+5\sqrt[5]{x}}{\sqrt{x}\sqrt{3-\frac{2}{x}}+\sqrt[3]{x}\sqrt[3]{2-\frac{3}{x}}}$

$= \lim_{x\to\infty} \dfrac{2+\frac{3}{x^{1/6}}+\frac{5}{x^{3/10}}}{\sqrt{3-\frac{2}{x}}+\frac{1}{x^{1/6}}\sqrt[3]{2-\frac{3}{x}}}$

(Dividing the numerator and denominator by the highest power $x^{1/2}$)

$= \dfrac{2}{\sqrt{3}}$

The correct option is (A)

**49.** $\lim_{x\to 0} \dfrac{x\sqrt[3]{z^2-(z-x)^2}}{(\sqrt[3]{8xz-4x^2}+\sqrt[3]{8xz})^4}$

$= \lim_{x\to 0} \dfrac{x\sqrt[3]{2xz-x^2}}{(\sqrt[3]{x}\sqrt[3]{8z-4x}+\sqrt[3]{8z}\sqrt[3]{x})^4}$

$= \lim_{x\to 0} \dfrac{x^{4/3}\sqrt[3]{2z-x}}{x^{4/3}\left(\sqrt[3]{8z-4x}+\sqrt[3]{8z}\right)^4}$

$= \dfrac{\sqrt[3]{2z}}{\left(2\sqrt[3]{8z}\right)^4}$

$= \dfrac{1}{2^{23/3}\cdot z}$

The correct option is (B)

**50.** $\lim_{h\to 0^+} \dfrac{A}{P^3} = \lim_{h\to 0^+} \dfrac{h\sqrt{2hr-h^2}}{8\left(\sqrt{2hr-h^2}+\sqrt{2hr}\right)^3}$

$= \lim_{h\to 0} \dfrac{h\cdot\sqrt{h}\sqrt{2r-h}}{8\cdot\sqrt{h}\cdot h\left(\sqrt{2r-h}+\sqrt{2r}\right)^3}$

$= \lim_{h\to 0} \dfrac{\sqrt{2r-h}}{8\left(\sqrt{2r-h}+\sqrt{2r}\right)^3}$

$= \dfrac{\sqrt{2r}}{8\left(2\sqrt{2r}\right)^3} = \dfrac{1}{128r}$

The correct option is (B)

**51.** $\lim_{x\to 2}\left(\dfrac{\sqrt{1-\cos 2(x-2)}}{x-2}\right)$

$= \lim_{x\to 2} \dfrac{\sqrt{2}\,|\sin(x-2)|}{x-2}$

which doesn't exist as LHL $= -\sqrt{2}$ whereas RHL $= \sqrt{2}$

The correct option is (B)

**52.** Required limit$= \lim_{n\to\infty} \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\cos\frac{x}{2^n}$

$= \lim_{n\to\infty} \dfrac{1}{2\sin\frac{x}{2^n}}\left\{\cos\frac{x}{2}\ldots\cos\frac{x}{2^{n-1}}\left(2\sin\frac{x}{2^n}\cos\frac{x}{2^n}\right)\right\}$

$= \lim_{n\to\infty} \dfrac{1}{2^2\sin\frac{x}{2^n}}\left\{\cos\frac{x}{2}\ldots\left(2\cos\frac{x}{2^{n-1}}\sin\frac{x}{2^{n-1}}\right)\right\}$

……………………………………………………………..

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$= \lim_{n\to\infty} \dfrac{1}{2^n\sin\frac{x}{2^n}}\left(2\cos\frac{x}{2}\sin\frac{x}{2}\right)$

$= \lim_{n\to\infty} \dfrac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$

$= \dfrac{\sin x}{x}\lim_{n\to\infty}\left\{\dfrac{\left(\frac{x}{2^n}\right)}{\sin\left(\frac{x}{2^n}\right)}\right\} = \dfrac{\sin x}{x}$

The correct option is (B)

**53.** The $(r+1)$th term of the series is

$t_{r+1} = (r+1)\sum_{k=1}^{n-r} k$

$\Rightarrow\ t_{r+1} = (r+1)\{1+2+3+\ldots(n-r)\text{ terms}\}$

$\Rightarrow\ t_{r+1} = (r+1)\frac{1}{2}(n-r)(n-r+1)$

$\Rightarrow\ t_{r+1} = \frac{1}{2}(r+1)(n^2-rn+n-rn+r^2-r)$

$$\Rightarrow \quad t_{r+1} = \frac{1}{2}(r+1)(r^2 - (1+2n)r + n^2)$$

$$\Rightarrow \quad t_{r+1} = \frac{1}{2}(r^3 - 2nr^2 + (n^2 - 2n - 1)r + n^2)$$

Now, $\quad S = \sum_{r=0}^{n-1} t_{r+1}$

$$\therefore \quad S = \frac{1}{2}\sum_{r=1}^{n} \{r^3 - 2nr^2 + (n^2 - 2n - 1)r + n^2\}$$

$$\Rightarrow \quad S = \frac{1}{2}\left[\left\{\frac{n(n+1)}{2}\right\}^2 - 2n\left\{\frac{1}{6}n(n+1)(2n+1)\right\} + (n^2 - 2n - 1)\left\{\frac{1}{2}n(n+1)\right\} + n^2(n)\right]$$

Solving and rearranging, we have,

$$S = \frac{1}{24}\{n^4 - 11n^3 - 19n^2 + 6n\}$$

$$\therefore \quad \lim_{n\to\infty} \frac{S}{n^4} = \lim_{n\to\infty} \frac{1}{24}\left(\frac{n^4 - 11n^3 - 19n^2 + 6n}{n^4}\right)$$

$$= \frac{1}{24}\lim_{n\to\infty}\left(1 - \frac{11}{n} - \frac{19}{n^2} + \frac{6}{n^3}\right)$$

$$\therefore \quad \lim_{n\to\infty} \frac{1}{n^4}\sum_{r=0}^{n-1}\left\{(r+1)\sum_{k=1}^{n-r} k\right\} = \frac{1}{24}$$

The correct option is (A)

**54.** Let $L = \lim_{n\to\infty} \frac{1}{n^3}\left(\sum_{k=1}^{n} [k^2 x]\right)$

Since $\quad k^2x - 1 \le [k^2x] < k^2x$

$$\Rightarrow \quad \sum_{k=1}^{n}(k^2x - 1) \le \sum_{k=1}^{n}[k^2x] < \sum_{k=1}^{n} k^2xa$$

$$\Rightarrow \quad x\left(\sum_{k=1}^{n} k^2\right) - \sum_{k=1}^{n}(1) \le \sum_{k=1}^{n}[k^2x] < x\left(\sum_{k=1}^{n} k^2\right)$$

$$\Rightarrow \quad \frac{xn(n+1)(2n+1)}{6} - n \le \sum_{k=1}^{n}[k^2x] < \frac{xn(n+1)(2n+1)}{6}$$

Dividing throughout by $n^3$, we have

$$\frac{xn(n+1)(2n+1)}{6n^3} - \frac{1}{n^2} \le \sum_{k=1}^{n}\frac{[k^2x]}{n^3} < \frac{xn(n+1)(2n+1)}{6n^3}$$

$$\Rightarrow \quad \frac{x}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{1}{n^2} \le \sum_{k=1}^{n}\frac{[k^2x]}{n^3} < \frac{x}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Taking limits as $n \to \infty$, we get

$$\lim_{n\to\infty}\left\{\frac{x}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{1}{n^2}\right\} \le L < \lim_{n\to\infty}\frac{x}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Since, as $n \to \infty$, we have $\frac{1}{n} \to 0$

$$\Rightarrow \quad \frac{x}{3} \le L < \frac{x}{3}$$

According to Squeeze Principle or Sandwich Theorem, we have

$$L = \frac{x}{3}.$$

The correct option is (C)

**55.** $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$

$$\Rightarrow \quad \frac{1}{\cot 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$\Rightarrow \quad \cot 2\theta = \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$$

$$\Rightarrow \quad 2\cot 2\theta = \frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad 2\cot 2\theta = \cot\theta - \tan\theta$$

$$\Rightarrow \quad \tan\theta = \cot\theta - 2\cot 2\theta \qquad (1)$$

Now, $\tan\theta = \cot\theta - 2\cot 2\theta$

$$\Rightarrow \quad \frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2} - \cot\theta$$

$$\Rightarrow \quad \frac{1}{2^2}\tan\frac{\theta}{2^2} = \frac{1}{2^2}\cot\frac{\theta}{2} - \frac{1}{2}\cot\theta$$

… … …

… … …

$$\Rightarrow \quad \frac{1}{2^n}\tan\frac{\theta}{2^n} = \frac{1}{2^n}\cot\frac{\theta}{2^n} - \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}}$$

$$\Rightarrow \quad S = -2\cot 2\theta + \frac{1}{2^n}\cot\frac{\theta}{2^n}$$

$$= \lim_{n\to\infty}\left(\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \ldots + \frac{1}{2^n}\tan\frac{\theta}{2^n}\right)$$

$$\therefore \quad \lim_{n\to\infty} S = \lim_{n\to\infty}\left(-2\cot 2\theta + \frac{1}{2^n}\cot\frac{\theta}{2^n}\right)$$

$$= -2\cot 2\theta + \lim_{n\to\infty}\frac{1}{\theta}\left[\frac{\frac{\theta}{2^n}}{\tan\frac{\theta}{2^n}}\right]$$

$$= -2\cot 2\theta + \frac{1}{\theta}$$

The correct option is (B)

**56.** $\lim_{n\to\infty} \frac{4^{3n-2} - 9^{n+1}}{8^{2n-1} - 9^{n-1}}$

$$= \lim_{n\to+\infty} \frac{4^{-2}\cdot 64^n - 9\cdot 9^n}{8^{-1}\cdot 64^n - 9^{-1}\cdot 9^n}$$

$$= \lim_{n\to+\infty} \frac{4^{-2} - 9\left(\frac{9}{64}\right)^n}{8^{-1} - 9^{-1}\left(\frac{9}{64}\right)^n}$$

$$= \frac{4^{-2} - 0}{8^{-1} - 0} = \frac{1}{2}$$

$$= \lim_{n\to-\infty} \frac{4^{-2}\left(\frac{64}{9}\right)^n - 9}{8^{-1}\left(\frac{64}{9}\right)^n - 9^{-1}}$$

$$= \frac{0-9}{0-9^{-1}} = 81$$

Hence, limit does not exist.

The correct option is (C)

**57.** We have, $A_i = \frac{x - a_i}{|x - a_i|}$, $i = 1, 2, \ldots, n$

and, $a_1 < a_2 < \ldots a_{n-1} < a_n$

Let $x$ be in the left neighbourhood of $a_m$

Then, $x - a_i < 0$ for $i = m, m + 1, \ldots n$

and $x - a_i > 0$ for $i = 1, 2, \ldots, m - 1$

$\therefore \quad A_i = \frac{x - a_i}{-(x - a_i)} = -1$, for $i = m, m + 1, \ldots, n$

and, $A_i = \frac{x - a_i}{x - a_i} = 1$, for $i = 1, 2, \ldots, m - 1$

Similarly, if $x$ is in the right neighbourhood of $a_i$

Then, $x - a_i < 0$ for $i = m + 1, \ldots, n$

and $x - a_i > 0$ for $i = 1, 2, \ldots, m$

$\therefore \quad A_i = \frac{x - a_i}{-(x - a_i)} = -1$ for $i = m + 1, \ldots n$

and, $A_i = \frac{x - a_i}{x - a_i} = 1$ for $i = 1, 2, \ldots, m$

Now, $\lim_{x\to a_m^-} (A_1 A_2 \ldots A_n) = (-1)^{n-m+1}$

and, $\lim_{x\to a_m^+} (A_1 A_2 \ldots A_n) = (-1)^{n-m}$

Hence, $\lim_{x\to a_m} (A_1 A_2 \ldots A_n)$ does not exist.

The correct option is (D)

**58.** We know that $\lim_{x\to 0} \frac{\sin x}{x} \to 1^-$

$\lim_{x\to 0} \frac{x}{\sin x} \to 1^+$

$$\therefore \quad \lim_{x\to 0}\left[100 \frac{x}{\sin x}\right] + \lim_{x\to 0}\left[99 \frac{\sin x}{x}\right] = 100 + 98 = 198.$$

The correct option is (B)

**59.** $\lim_{x\to\infty} \frac{x^5}{5^x}$

$$= \lim_{x\to\infty} \frac{x^5}{e^{x\log 5}}$$

$$= \lim_{x\to\infty} \frac{x^5}{e^{kx}}, \text{ where } k = \log 5$$

$$= \lim_{x\to\infty} \frac{x^5}{\left(1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + \frac{k^4x^4}{4!} + \frac{x^5k^5}{5!} + \frac{k^6x^6}{6!} + \ldots\right)}$$

$$= \lim_{x\to\infty} \frac{1}{\left[\left(\frac{1}{x^5} + k\cdot\frac{1}{x^4} + \frac{k^2}{2!}\cdot\frac{1}{x^3} + \frac{k^3}{3!}\cdot\frac{1}{x^2} + \frac{k^4}{4!}\cdot\frac{1}{x}\right) + \frac{k^5}{5!} + \left(\frac{k^6}{6!}x + \ldots\right)\right]}$$

$$= \frac{1}{\infty} = 0$$

The correct option is (C)

**60.** $t_r = \frac{r}{r^4 + r^2 + 1} = \frac{r}{(r^2+1)^2 - r^2}$

$$= \frac{1}{2}\left[\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{r(r-1)+1} - \frac{1}{(r+1)r+1}\right]$$

$$\therefore \quad \sum_{r=1}^{n} t_r = \sum_{r=1}^{n} \frac{1}{2}[f(r) - f(r+1)],$$

where, $f(r) = \frac{1}{r(r-1)+1}$

$$= \frac{1}{2}[f(1) - f(n+1)]$$

$$= \frac{1}{2}\left[1 - \frac{1}{(n+1)n+1}\right] \to \frac{1}{2} \text{ as } n \to \infty$$

The correct option is (B)

**61.** $\lim_{x\to\infty}\left[\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \ldots + n^{1/x}}{n}\right]^{nx}$

$$= \lim_{y\to 0}\left[\frac{1^y + 2^y + 3^y + \ldots + n^y}{n}\right]^{\frac{n}{y}}$$

$$= e^{\lim_{y\to 0}\frac{n}{y}\left[\frac{1^y + 2^y + 3^y + \ldots + n^y}{n} - 1\right]}$$

$$= e^{\lim_{y\to 0}\left[\frac{1^y + 2^y + 3^y + \ldots + n^y - n}{y}\right]}$$

$$= e^{\lim_{y\to 0}\left[\frac{(1^y - 1)}{y} + \frac{(2^y - 1)}{y} + \frac{(3^y - 1)}{y} + \ldots + \frac{(n^y - 1)}{y}\right]}$$

$$= e^{(\log 1 + \log 2 + \log 3 + \ldots + \log n)}$$

$$= e^{\log(1\cdot 2\cdot 3 \ldots . n)} = n!$$

The correct option is (A)

**62.** $\lim_{n\to\infty} (1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2n})$

$$= \lim_{n\to\infty} \frac{(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2n})}{1-x}$$

$$= \lim_{n\to\infty} \frac{(1-x^2)(1+x^2)(1+x^4)\ldots(1+x^{2n})}{1-x}$$

. . .

. . .

. . .

$= \lim_{n\to\infty} \frac{1-x^{4n+2}}{1-x} = \frac{1}{1-x}$ for $|x| < 1$

The correct option is (B)

**63.** **Case I:** $n$ is a positive integer

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x}$$

$$= \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \ldots = \lim_{x\to\infty} \frac{n!}{e^x}$$

[Using L'Hospital's rule repeatedly]

$= 0$

**Case II:** $n$ is a negative integer.

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{x^{-m}}{e^x}$$

[Putting $n = -m$, where $m$ is a positive integer]

$$= \lim_{x\to\infty} \frac{1}{x^m e^x} = \frac{1}{\infty} = 0$$

**Case III:** $n = 0$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

Hence, $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for all values of $n$.

The correct option is (B)

**64.** $x^2 + 4x + 5 = (x+2)^2 + 1 \geq 1.$

So, $a = 1$

Also, $b = \lim_{\theta\to 0} \frac{1-\cos 2\theta}{\theta^2} = \lim_{\theta\to 0} \frac{2\sin^2\theta}{\theta^2} = 2$

$$\therefore \sum_{r=0}^{n} a^r \cdot b^{n-r} = b^n + ab^{n-1} + a^2 b^{n-2} + \ldots + a^n$$

$$= \frac{b^n\left[1-\left(\frac{a}{b}\right)^{n+1}\right]}{1-\frac{a}{b}} = \frac{2^n\left[1-\left(\frac{1}{2}\right)^{n+1}\right]}{1-\frac{1}{2}}$$

$$= \frac{2^{n+1}(2^{n+1}-1)}{2^{n+1}} = (2^{n+1}-1)$$

The correct option is (B)

**65.** $\lim_{x\to 0} \frac{\log(1+x+x^2)+\log(1-x+x^2)}{\sec x - \cos x}$

$$= \lim_{x\to 0} \frac{\log\left[(1+x^2)^2 - x^2\right]}{(1-\cos^2 x)/\cos x}$$

$$= \lim_{x\to 0} \frac{\log(1+x^2+x^4)}{\sin x \tan x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\log(1+x^2(1+x^2))}{x^2(1+x^2)} \cdot x^2(1+x^2) \cdot \frac{1}{\frac{\sin x}{x} \cdot \frac{\tan x}{x} \cdot x^2}$$

$$= 1.\left(\text{as } \lim_{x\to 0} \frac{\log(1+x)}{x} = 1\right)$$

The correct option is (A)

**66.** $\lim_{n\to\infty} \frac{\sqrt[4]{n^5+2} - \sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2} - \sqrt[2]{n^3+1}}$

$$= \lim_{n\to\infty} \frac{n^{5/4}\sqrt[4]{1+\frac{2}{n^5}} - n^{2/3}\sqrt[3]{1+\frac{1}{n^2}}}{n^{4/5}\sqrt[5]{1+\frac{2}{n^4}} - n^{3/2}\sqrt[2]{1+\frac{1}{n^3}}}$$

$$= \lim_{n\to\infty} \frac{\frac{n^{5/4}}{n^{3/2}}\sqrt[4]{1+\frac{2}{n^5}} - \frac{n^{2/3}}{n^{3/2}}\sqrt[3]{1+\frac{1}{n^2}}}{\frac{n^{4/5}}{n^{3/2}}\sqrt[5]{1+\frac{2}{n^4}} - \frac{n^{3/2}}{n^{3/2}}\sqrt[2]{1+\frac{1}{n^3}}}$$

[Dividing the numerator and denominator by the highest power $n^{3/2}$]

$$= \lim_{n\to\infty} \frac{\frac{1}{n^{1/4}}\sqrt[4]{1+\frac{2}{n^5}} - \frac{1}{n^{5/6}}\sqrt[3]{1+\frac{1}{n^2}}}{\frac{1}{n^{7/10}}\sqrt[5]{1+\frac{2}{n^4}} - \sqrt[2]{1+\frac{1}{n^3}}} = \frac{0-0}{0-1} = 0.$$

The correct option is (B)

**67.** $\lim_{x\to 0} \frac{x\sqrt[3]{z^2-(z-x)^2}}{(\sqrt[3]{8xz-4x^2}+\sqrt[3]{8xz})^4}$

$$= \lim_{x\to 0} \frac{x\sqrt[3]{2xz-x^2}}{(\sqrt[3]{x}\sqrt[3]{8z-4x}+\sqrt[3]{8z}\sqrt[3]{x})^4}$$

$$= \lim_{x\to 0} \frac{x^{4/3}\sqrt[3]{2z-x}}{x^{4/3}\left[\sqrt[3]{8z-4x}+\sqrt[3]{8z}\right]^4} = \frac{\sqrt[3]{2z}}{\left[2\sqrt[3]{8z}\right]^4}$$

$$= \frac{1}{2^{23/3} \cdot z}$$

The correct option is (B)

**68.** $\lim_{h\to 0^+} \frac{A}{P^3} = \lim_{h\to 0^+} \frac{h\sqrt{2hr-h^2}}{8\left[\sqrt{2hr-h^2}+\sqrt{2hr}\right]^3}$

$$= \lim_{h\to 0} \frac{h\cdot\sqrt{h}\sqrt{2r-h}}{8\cdot\sqrt{h}\cdot h\left[\sqrt{2r-h}+\sqrt{2r}\right]^3}$$

$$= \lim_{h\to 0} \frac{\sqrt{2r-h}}{8\left[\sqrt{2r-h}+\sqrt{2r}\right]^3}$$

$$= \frac{\sqrt{2r}}{8\left(2\sqrt{2r}\right)^3} = \frac{1}{128r}$$

The correct option is (B)

**69.** $\lim_{x\to\pi/3} \frac{\cos\left(x+\pi/6\right)}{(1-2\cos x)^{2/3}} = \lim_{z\to 0} \frac{\cos\left(\pi/2+z\right)}{\left[1-2\cos\left(\pi/3+z\right)\right]^{2/3}}$

[putting $x - \pi/3 = z$]

$$= \lim_{z\to 0} \frac{-\sin z}{\left(1-\cos z+\sqrt{3}\sin z\right)^{2/3}}$$

$$= \lim_{z\to 0} \frac{-2\sin\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right)}{\left[2\sin\left(\frac{z}{2}\right)\right]^{2/3}\left[\sin\left(\frac{z}{2}\right)+\sqrt{3}\cos\left(\frac{z}{2}\right)\right]^{2/3}}$$

$$= \lim_{z\to 0} \frac{-2^{1/3}\left[\sin\left(\frac{z}{2}\right)\right]^{1/3}\cos\left(\frac{z}{2}\right)}{\left[\sin\left(\frac{z}{2}\right)+\sqrt{3}\cos\left(\frac{z}{2}\right)\right]^{2/3}} = \frac{-2^{1/3}\cdot 0\cdot 1}{(\sqrt{3})^{2/3}} = 0$$

The correct option is (C)

**70.** $$\lim_{x\to 0}\frac{ln\,(2-\cos 2x)}{ln^2(\sin 3x+1)} = \lim_{x\to 0}\frac{ln\,\{1+(1-\cos 2x)\}}{ln^2(1+\sin 3x)}$$

$$= \lim_{x\to 0}\frac{1-\cos 2x}{(\sin 3x)^2} = \lim_{x\to 0}\frac{2x^2}{(3x)^2} = \frac{2}{9}$$

The correct option is (A)

**71.** $$\lim_{x\to 1/\alpha}\frac{1-\cos(cx^2+bx+a)}{(1-x\alpha)^2}$$

$$= \lim_{x\to 1/\alpha}\frac{1-\cos(cx^2+bx+a)}{(cx^2+bx+a)^2}\cdot\frac{(cx^2+bx+a)^2}{(1-x\alpha)^2}$$

$$= \lim_{z\to 0}\frac{1-\cos z}{z^2}\cdot\lim_{y\to\alpha}\frac{(ay^2+by+c)^2}{y^2(y-\alpha)^2}$$

[putting $cx^2+bx+a=z$ and $x=1/y$]

$$= \frac{1}{2}\lim_{y\to\alpha}\frac{a^2(y-\alpha)^2(y-\beta)^2}{y^2(y-\alpha)^2}$$

[If $\alpha$, $\beta$ are roots of $ax^2+bx+c=0$ then $ax^2+bx+c=a(x-\alpha)(x-\beta)$]

$$= \frac{a^2(\alpha-\beta)^2}{2\alpha^2} = \frac{a^2}{2\alpha^2}\left[(\alpha+\beta)^2-4\alpha\beta\right]$$

$$= \frac{a^2}{2\alpha^2}\left(\frac{b^2}{a^2}-\frac{4c}{a}\right) = \frac{b^2-4ac}{2\alpha^2}$$

The correct option is (A)

**72.** $$\lim_{x\to 0}\frac{\sqrt{1-\sqrt{\cos x}}}{x}$$

$$= \lim_{x\to 0}\frac{\sqrt{1-\cos x}}{x}\cdot\frac{1}{\sqrt{1+\sqrt{\cos x}}}$$

$$= \lim_{x\to 0}\frac{\sqrt{2}\left|\sin\left(x/2\right)\right|}{2\left(x/2\right)}\cdot\frac{1}{\sqrt{1+\sqrt{\cos x}}}$$

Now, we have,

$$\text{LHL} = \lim_{x\to 0^-}\frac{-1}{\sqrt{2}}\cdot\frac{\sin\left(x/2\right)}{x/2}\cdot\frac{1}{\sqrt{1+\sqrt{\cos x}}}$$

$$= -\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{-1}{2}$$

$$\text{RHL} = \lim_{x\to 0^+}\frac{1}{\sqrt{2}}\cdot\frac{\sin\left(x/2\right)}{x/2}\cdot\frac{1}{\sqrt{1+\sqrt{\cos x}}}$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2}$$

Hence, limit does not exist.

The correct option is (C)

**73.** $$\lim_{x\to 2}\frac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$$

$$= \lim_{x\to 2}\frac{(x+7)-9(2x-3)}{\sqrt{x+7}+3\sqrt{2x-3}}\times\left[\text{using } a-b=\frac{a^2-b^2}{a+b}\right]$$

$$\frac{(x+6)^{2/3}+2(x+6)^{1/3}(3x-5)^{1/3}+4(3x-5)^{2/3}}{(x+6)-8\,(3x-5)}$$

$$\left[\text{using } a-b=\frac{a^3-b^3}{a^2+ab+b^2}\right]$$

$$= \lim_{x\to z}\frac{-17\,(x-2)}{\sqrt{x+7}+\sqrt[3]{2x-3}}$$

$$\times\frac{(x+6)^{2/3}+2(x+6)^{1/3}(3x-5)^{1/3}+4(3x-5)^{2/3}}{-23\,(x-2)}$$

$$= \frac{-17}{\sqrt{9}+3\sqrt{1}}\cdot\frac{8^{2/3}+2.8^{1/3}+4}{-23} = \frac{17}{6}\cdot\frac{12}{23} = \frac{34}{23}$$

The correct option is (B)

**74.** $$\lim_{x\to 0}\frac{(2^m+x)^{1/m}-(2^n+x)^{1/n}}{x}$$

$$= \lim_{x\to 0}\frac{(2^m+x)^{1/m}-2}{x} - \lim_{x\to 0}\frac{(2^n+x)^{1/n}-2}{x}$$

$$= \lim_{a\to 2}\frac{a-2}{a^m-2^m} - \lim_{b\to 2}\frac{b-2}{b^n-2^n}$$

[Putting $2^m+x=a^m$ and $2^n+x=b^n$]

$$= \frac{1}{m\,2^{m-1}} - \frac{1}{n\,2^{n-1}}$$

The correct option is (C)

**75.** $$\lim_{x\to 4}\frac{(\cos\theta)^x-(\sin\theta)^x-\cos 2\theta}{x-4}$$

$$= \lim_{y\to 0}\frac{(\cos\theta)^{y+4}-(\sin\theta)^{y+4}-(\cos^4\theta-\sin^4\theta)}{y}$$

[Putting $x-4=y$ and $\cos 2\theta=\cos^4\theta-\sin^4\theta$]

$$= \lim_{y\to 0}\cos^4\theta\left[\frac{(\cos\theta)^y-1}{y}\right]-\sin^4\theta\left[\frac{(\sin\theta)^y-1}{y}\right]$$

$= \cos^4\theta\, ln\cos\theta - \sin^4\theta\, ln\sin\theta$

The correct option is (A)

**76.** $\lim_{x\to 0}\left(\frac{x-1+\cos x}{x}\right)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln\left(\frac{x-1+\cos x}{x}\right)}$

Now, $\lim_{x\to 0}\frac{1}{x}\, ln\left(1-\frac{1-\cos x}{x}\right)$

$$= \lim_{x\to 0}\frac{\cos x-1}{x^2}\cdot\frac{ln\left(1+\frac{\cos x-1}{x}\right)}{\frac{\cos x-1}{x}}$$

$$= \lim_{x\to 0}\frac{-2\sin^2(x/2)}{4(x/2)^2}\cdot\lim_{z\to 0}\frac{\ln(1+z)}{z} = \frac{-1}{2}$$

$\left[\text{Putting } z = \frac{\cos x-1}{x}\text{, we can see that}\right.$

$\left.\lim_{x\to 0}\frac{\cos x-1}{x} = \lim_{n\to 0} x\left(\frac{\cos x-1}{x^2}\right) = 0\cdot\frac{-1}{2} = 0\right]$

Hence, the required limit is $e^{-1/2}$.

The correct option is (B)

**77.** $\lim_{x\to\infty}\left[\frac{e}{(1+1/x)^x}\right]^x = \lim_{y\to 0}\left[\frac{e}{(1+y)^{1/y}}\right]^{1/y}$

$$= \lim_{y\to 0} e^{\frac{1}{y}ln\left[\frac{e}{(1+y)^{1/y}}\right]}$$

Now, we have,

$$\lim_{y\to 0}\frac{1}{y}\, ln\left[\frac{e}{(1+y)^{1/y}}\right] = \lim_{y\to 0}\frac{ln\, e-\frac{1}{y}ln(1+y)}{y}$$

$$= \lim_{y\to 0}\frac{y-ln(1+y)}{y^2}$$

$$= \lim_{y\to 0}\frac{y-\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\ldots\right)}{y^2}$$

$$= \lim_{y\to 0}\frac{1}{2}-\frac{y}{3}+\frac{y^2}{4}-\ldots = \frac{1}{2}$$

Hence, the required limit is $e^{1/2}$.

The correct option is (C)

**78.** $\lim_{x\to 0}\left[\frac{a\sin x}{x}\right]+\left[\frac{b\tan x}{x}\right]$

$\left[\text{as } x\to 0,\ \frac{\sin x}{x}\to 1\text{ but }\frac{\sin x}{x}<1\right.$

$\left.\text{while }\frac{\tan x}{x}\to 1\text{ but }\frac{\tan x}{x}>1\right]$

$= (a-1)+b$

$= a+b-1$

The correct option is (B)

**79.** Let $f(x) = \frac{[x]+[2x]+[3x]+\ldots+[nx]}{1+2+3+\ldots+n}$

Now, we have,

$$f(x) \le \frac{x+2x+3x+\ldots+nx}{1+2+3+\ldots+n} = x$$

and, $f(x) > \frac{(x-1)+(2x-1)+(3x-1)+\ldots+(nx-1)}{1+2+3+\ldots+n}$

$$= \frac{x\,\Sigma n-n}{\Sigma n} = x-\frac{2}{n+1}$$

$[\because x-1\le[x]<x\ \forall x\in R]$

Thus, we have,

$$x-\frac{2}{n+1} < f(x) \le x$$

Now, we have,

$$\lim_{n\to\infty} x-\frac{2}{n+1} = x \text{ and } \lim_{n\to\infty} x = x$$

Hence, by Sandwich Theorem, we have

$$\lim_{n\to\infty} f(x) = x$$

The correct option is (A)

**80.** $\lim_{n\to\infty} n^2\left(x^{1/n}-x^{\frac{1}{n+1}}\right) = \lim_{n\to\infty} n^2\cdot x^{\frac{1}{n+1}}\left(x^{\frac{1}{n}-\frac{1}{n+1}}-1\right)$

$$= \lim_{n\to\infty} x^{\frac{1}{n+1}}\left(x^{\frac{1}{n(n+1)}}-1\right)n^2$$

$$= \lim_{n\to\infty} x^{\frac{1}{n+1}}\cdot\frac{x^{\frac{1}{n(n+1)}}-1}{\frac{1}{n(n+1)}}\cdot\frac{n^2}{n(n+1)}$$

$$= 1\cdot lnx\cdot 1 = lnx$$

The correct option is (C)

**81.** We have,

$$\lim_{x\to 0}\left[1+x+\frac{f(x)}{x}\right]^{1/x} = e^3$$

$$\Rightarrow \lim_{x\to 0} e^{\frac{ln[1+x+g(x)]}{x}} = e^3 \qquad \left[\text{Putting }\frac{f(x)}{x} = g(x)\right]$$

$$\Rightarrow \lim_{x\to 0}\frac{ln[1+x+g(x)]}{x} = 3$$

Since, the denominator approaches zero, the numerator should also approach zero for a finite limit to exist.

Thus, we have,

$$\lim_{x\to 0} g(x) = 0$$

Now, using L'Hospital's rule, the above equation reduces to

$$\lim_{x\to 0}\frac{1+g'(x)}{1+x+g(x)} = 3$$

i.e., $1+g'(0) = 3 \Rightarrow g'(0) = 2$

Hence, we have,

$$\lim_{x\to 0}\left[1+\frac{f(x)}{x}\right]^{1/x} = \lim_{x\to 0} e^{\frac{ln[1+g(x)]}{x}}$$

$$= \lim_{x\to 0} e^{\frac{g'(x)}{1+g(x)}} = e^{g'(0)}$$

$= e^2$

The correct option is (B)

**82.** Let $y = x + \cfrac{\sqrt{x}}{x + \cfrac{\sqrt{x}}{x + \cfrac{\sqrt{x}}{\dots\infty}}} = x + \frac{\sqrt{x}}{y}$

i.e., $y^2 - xy - \sqrt{x} = 0$

i.e., $y = \dfrac{x \pm \sqrt{x^2 + 4\sqrt{x}}}{2}$

We can see $y$ is a positive quantity for positive $x$, therefore

$y = \dfrac{x + \sqrt{x^2 + 4\sqrt{x}}}{2}$

Hence, the required limit is

$$= \lim_{x\to\infty} \frac{x}{y} = \lim_{x\to\infty} \frac{2x}{x + \sqrt{x^2 + 4\sqrt{x}}}$$

$$= \lim_{x\to\infty} \frac{2}{1 + \sqrt{1 + 4x^{-3/2}}} = \frac{2}{1+1} = 1$$

The correct option is (A)

**83.** We have,

$$\lim_{x\to 0} \frac{\cos x - (\cos x)^{\cos x}}{1 - \cos x + \ln(\cos x)}$$

$$= \lim_{x\to 0} \cos x \left[\frac{1 - (\cos x)^{\cos x - 1}}{1 - \cos x + \ln(\cos x)}\right]$$

$$= \lim_{t\to 0} \frac{1 - (1+t)^t}{\ln(1+t) - t} \qquad [\text{Putting } \cos x - 1 = t]$$

$$= \lim_{t\to 0} \frac{t^2 + \frac{t^3(t-1)}{2!} + \frac{t^4(t-1)(t-2)}{3!} + \dots}{\frac{t^2}{2} - \frac{t^3}{3} + \frac{t^4}{4} - \dots} = 2$$

The correct option is (C)

**84.** We have,

$$\lim_{x\to \pi/4} \frac{(\tan x)^{\tan x} - \tan x}{\ln(\tan x) - \tan x + 1}$$

$$= \lim_{t\to 1} \frac{t^t - t}{\ln t - t + 1} \left(\frac{0}{0}\right) \qquad [\text{Putting } \tan x = t]$$

$$= \lim_{t\to 1} \frac{t^t(1 + \ln t) - 1}{\frac{1}{t} - 1} \left(\frac{0}{0}\right)$$

$$= \lim_{t\to 1} \frac{t^t(1+\ln t)^2 + t^t\left(\frac{1}{t}\right)}{\frac{-1}{t^2}} = \frac{1+1}{-1} = -2$$

The correct option is (A)

**85.** We have,

$$\lim_{x\to \pi/2} \left(1^{1/\cos^2 x} + 2^{1/\cos^2 x} + \dots + n^{1/\cos^2 x}\right)^{\cos^2 x}$$

$$= \lim_{y\to\infty} \left(1^y + 2^y + \dots + n^y\right)^{1/n} \qquad \left[\text{Putting } \frac{1}{\cos^2 x} = y\right]$$

$$= \lim_{y\to\infty} n\left[\left(\frac{1}{n}\right)^y + \left(\frac{2}{n}\right)^y + \dots + \left(\frac{n}{n}\right)^y\right]^{1/y}$$

$= n\,(0 + 0 + \dots + 1) = n$

The correct option is (A)

**86.** $\displaystyle\lim_{n\to\infty} \sum_{r=1}^{n} \cot^{-1}\left(r^2 + \frac{3}{4}\right)$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \cot^{-1}\left[\frac{(r - 1/2)(r + 1/2) + 1}{(r + 1/2) - (r - 1/2)}\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \cot^{-1}\left(r - \frac{1}{2}\right) - \cot^{-1}\left(r + \frac{1}{2}\right)$$

$$\left[\because \cot^{-1}\left(\frac{ab+1}{b-a}\right) = \cot^{-1} a - \cot^{-1} b\right]$$

$$= \lim_{n\to\infty} \left[\cot^{-1}\left(\frac{1}{2}\right) - \cot^{-1}\left(\frac{3}{2}\right) + \cot^{-1}\left(\frac{3}{2}\right) - \cot^{-1}\left(\frac{5}{2}\right) + \dots + \cot^{-1}\left(n - \frac{1}{2}\right) - \cot^{-1}\left(n + \frac{1}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\cot^{-1}\left(\frac{1}{2}\right) - \cot^{-1}\left(n + \frac{1}{2}\right)\right]$$

$$= \cot^{-1}\left(\frac{1}{2}\right) - 0 = \tan^{-1} 2$$

The correct option is (B)

**87.** Here, $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

For $x \to \frac{5\pi}{4} + h$, $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \to -\sqrt{2}$,

But greater than $-\sqrt{2}$

$$\therefore \left[\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right] = -2 \qquad (1)$$

Also, for $x \to \frac{5\pi}{4} - h$, $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \to -\sqrt{2}$, but greater than $-\sqrt{2}$

$$\therefore \left[\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right] = -2 \qquad (2)$$

From (1) and (2), we get

$$\lim_{x\to 5\pi/4} [\sin x + \cos x]$$

$$= \lim_{x\to 5\pi/4} -2 = -2$$

The correct option is (B)

**88.** $\lim_{m\to\infty}\lim_{n\to\infty}\left\{\frac{1+\sqrt[n]{1^n+2^n}+\sqrt[n]{2^n+3^n}+\ldots+\sqrt[n]{(m-1)^n+m^n}}{m^2}\right\}$

$= \lim_{m\to\infty}$

$$\left\{\lim_{n\to\infty}\frac{1+2\sqrt[n]{\left(\frac{1}{2}\right)^n+1}+3\sqrt[n]{\left(\frac{2}{3}\right)^n+1}+\ldots+m\sqrt[n]{\left(\frac{(m-1)}{m}\right)^n+1}}{m^2}\right\}$$

$= \lim_{m\to\infty}\frac{1+2+3+\ldots+m}{m^2}$

$\left(\because \left(\frac{1}{2}\right)^n \to 0 \text{ as } n\to\infty; \left(\frac{2}{3}\right)^n \to 0 \text{ as } n\to\infty; \ldots;\right.$

$\left.\left(\frac{m-1}{m}\right)^n \to 0 \text{ as } n\to\infty\right)$

$= \lim_{m\to\infty}\frac{m(m+1)}{2m^2} = \lim_{m\to\infty}\frac{1}{2}\left(1+\frac{1}{m}\right)=\frac{1}{2}$

The correct option is (D)

**89.** We know,

$\sum_{r=1}^{n}\frac{1}{2^r} = \frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}$ i.e., sum of $n$ terms of G.P.

which tends to one as $n\to\infty$ but always remains less than one.

Thus, $\left[\sum_{r=1}^{n}\frac{1}{2^r}\right]\to 0$ as $n\to\infty$

$\therefore \quad \lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{2^r}\right]=0$

The correct option is (A)

**90.** Here, $0<\cos x<1$; if $0-h<x<0+h$

$\therefore \quad [\cos x]=0$

Hence, $\lim_{x\to 0}|x|^{[\cos x]}$

$= \lim_{x\to 0}|x|^0 = \lim_{x\to 0} 1 = 1$

The correct option is (B)

**91.** We have, $a_{n-1}+1 = \frac{a_n}{n}$ (1)

$\therefore \quad \lim_{n\to\infty}\left(\frac{a_1+1}{a_1}\right)\left(\frac{a_2+1}{a_2}\right)\ldots\left(\frac{a_n+1}{a_n}\right)$

$= \lim_{n\to\infty}\left(\frac{a_2}{2}\right)\left(\frac{a_3}{3}\right)\left(\frac{a_4}{4}\right)\ldots\left(\frac{a_{n+1}}{n+1}\right)\cdot\frac{1}{a_1\cdot a_2\ldots a_n}$

$= \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)!} = \lim_{n\to\infty}\frac{1+a_n}{n!}$ [using (1)]

$= \lim_{n\to\infty}\left(\frac{1}{n!}+\frac{a_n}{n!}\right)$

$= \lim_{n\to\infty}\left[\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{a_{n-1}}{(n-1)!}\right]$ [using (1)]

$= \lim_{n\to\infty}\left[\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{1}{(n-2)!}+\ldots+\frac{1}{2!}+\frac{1}{1!}+\frac{a_1}{1!}\right]$

$= \lim_{n\to\infty}\left[\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{1}{(n-2)!}+\ldots+\frac{1}{(2)!}+\frac{1}{1!}+\frac{1}{1}\right]$

[$\because \quad a_1=1$; given]

$= e\left[\text{as, } e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\infty\right]$

The correct option is (B)

**92.** $\lim_{n\to\infty} n^{-n^2}\left[(n+1)\left(n+\frac{1}{2}\right)\ldots\left(n+\frac{1}{2^{n-1}}\right)\right]^n$

$= \lim_{n\to\infty}\left[\frac{(n+1)\left(n+\frac{1}{2}\right)\ldots\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right]^n$

$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{n+\frac{1}{2}}{n}\right)^n\ldots\left(\frac{n+\frac{1}{2^{n-1}}}{n}\right)^n$

$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{2n}\right)^n\ldots\left(1+\frac{1}{2^{n-1}n}\right)^n$ ($1^\infty$ form)

$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{2n}\right)^{\frac{2n}{2}}\ldots\left(1+\frac{1}{2^{n-1}n}\right)^{\frac{2^{n-1}.n}{2^{n-1}}}$

$= e^1\cdot e^{1/2}\cdot e^{1/4}\ldots e^{1/2n-1}$

$\ldots\left\{\text{using; } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{an} = e^a\right\}$

$= e^{(1+1/2+1/4+\ldots)} = e^{\frac{1}{1-\frac{1}{2}}} = e^2$

The correct option is (B)

**93.** $\lim_{x\to y}\frac{x^y-y^x}{x^x-y^y}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to y}\frac{yx^{y-1}-y^x\log y}{x^x(1+x\log x)-0}$ (applying L-Hospital's rule)

$= \lim_{x\to y}\frac{yx^{y-1}-y^x\log y}{x^x(1+x\log x)}$

$= \frac{y\cdot y^{y-1}-y^y\log y}{y^y(\log y+1)}$

$= \frac{1-\log y}{1+\log y}$

The correct option is (A)

**94.** $\lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(\dfrac{r^3-r+\frac{1}{r}}{2}\right)$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\dfrac{2r}{1-r^2+r^4}\right)$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\dfrac{2r}{1-(r^2-r)(r^2+r)}\right)$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\dfrac{(r^2+r)-(r^2-r)}{1-(r^2+r)(r^2-r)}\right)$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\left[\tan^{-1}(r^2+r)-\tan^{-1}(r^2-r)\right]$

$= \lim_{n\to\infty}$ [$\tan^{-1} 2 - \tan^{-1} 0) + (\tan^{-1} 6 - \tan^{-1} 2) + (\tan^{-1} 12 - \tan^{-1} 6) + \ldots + \{\tan^{-1}(n^2+n) - \tan^{-1}(n^2-n)\}$]

$= \lim_{n\to\infty}\{\tan^{-1}(n^2+n) - \tan^{-1}(0)\}$

$= \tan^{-1}(\infty) - \tan^{-1}(0) = \dfrac{\pi}{2}$

$\therefore \quad \lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(\dfrac{r^3-r+\frac{1}{r}}{2}\right) = \dfrac{\pi}{2}$

The correct option is (C)

**95.** We know $n \le [x] < n+1 \quad \Rightarrow \quad [x] = n$

Here, $\dfrac{n\sin x}{x} \to n$ as $x \to 0$ but less than $n$

Also, $\dfrac{n\tan x}{x} \to n$ as $x \to 0$ but more than $n$

Thus, $\quad n-1 \le \left[\dfrac{n\sin x}{x}\right] < n$ as $x \to 0$

$\Rightarrow \quad \left[\dfrac{n\sin x}{x}\right] = n-1$

Again, $\quad n \le \left[\dfrac{n\tan x}{x}\right] < n+1$ as $x \to 0$

$\Rightarrow \quad \left[\dfrac{n\tan x}{x}\right] = n$

Thus, $\quad \lim_{x\to\infty}\left(\left[\dfrac{n\sin x}{x}\right] + \left[\dfrac{n\tan x}{x}\right]\right)$

$= (n-1) + (n) = (2n-1)$

The correct option is (C)

**96.** We know,

$\dfrac{x}{\sin x} \to 1$, as $x \to 0$, but less than 1

Also, $\dfrac{x}{\tan x} \to 1$, as $x \to 0$ but less than 1

Thus, $\dfrac{x^2}{\sin x\tan x} \to 1$ as $x \to 0$, but less than 1.

Hence, $\lim_{x\to 0}\left[\dfrac{x^2}{\sin x\tan x}\right] = 0$ as $0 \le \dfrac{x^2}{\sin x\tan x} < 1$ as $x \to 0$

The correct option is (A)

**97.** $\lim_{\theta\to 0}\dfrac{\cos^2(1-\cos^2(1-\cos^2(1\ldots\cos^2\theta))}{\sin\left(\dfrac{\pi(\sqrt{\theta+4}-2}{\theta}\right)}$

$= \lim_{\theta\to 0}\dfrac{\cos^2(\sin^2(\sin^2\ldots(\sin^2\theta))}{\sin\left(\dfrac{\pi(\sqrt{\theta+4}-2}{\theta}\right)}$

$= \lim_{\theta\to 0}\dfrac{\cos^2(\sin^2(\sin^2\ldots(\sin^2\theta))}{\sin\left(\pi\lim_{\theta\to 0}\dfrac{\theta}{\theta\left(\sqrt{\theta+4}+2\right)}\right)} = \dfrac{\cos^2 0}{\sin\dfrac{\pi}{4}} = \sqrt{2}$

The correct option is (C)

**98.** $\lim_{x\to\frac{\pi}{2}-} f(x) = \lim_{x\to\frac{\pi}{2}-}\sqrt{\dfrac{\tan x - \sin\{\tan^{-1}(\tan x)\}}{\tan x + \cos^2(\tan x)}}$

$= \lim_{x\to\frac{\pi}{2}-}\sqrt{\dfrac{\tan x - \sin x}{\tan x + \cos^2(\tan x)}}$

[$\because \tan^{-1}(\tan x) = x$, where $x < \pi/2$]

$= \lim_{x\to\frac{\pi}{2}}\sqrt{\dfrac{1-\dfrac{\sin x}{\tan x}}{1+\dfrac{\cos^2(\tan x)}{\tan x}}}$

$= \sqrt{\dfrac{1-0}{1+0}} = 1$

$\left[\because \lim_{x\to\frac{\pi}{2}-}\dfrac{\cos^2(\tan x)}{\tan x} = \dfrac{\text{finite quantity}}{\infty} = 0\right]$

Similarly, $\quad \lim_{x\to\frac{\pi}{2}+} f(x) = \lim_{x\to\frac{\pi}{2}+}\sqrt{\dfrac{1+\dfrac{\sin x}{\tan x}}{1+\dfrac{\cos^2(\tan x)}{\tan x}}}$

[$\because \tan^{-1}(\tan x) = x - \pi$, if $x > \pi/2$]

$= \sqrt{\dfrac{1+0}{1+0}} = 1$

$\therefore \quad \lim_{x\to\frac{\pi}{2}} f(x) = 1$

The correct option is (A)

**99.** Put $c = a^{1/4}$ and $z = x^{1/4}$, we get the function whose limit is required, as

$$\left\{\left[\left(\frac{c^2+z^2}{c-z}\right)^{-1} - \frac{2cz}{z^3-cz^2+c^2z-c^3}\right]^{-1} - 2^{\frac{1}{2}\log_4 c^4}\right\}^8$$

$$= \left\{\left[\frac{c^2+z^2}{c-z} - \frac{2cz}{(z-c)(z^2+c^2)}\right]^{-1} - c\right\}^8$$

$= (c-z-c)^8 = z^8 = x^2$

Hence, required limit as $x \to a = a^2$

The correct option is (C)

**100.** $\lim_{x\to 1}\frac{(\log(1+x)-\log 2)(3.4^{x-1}-3x)}{\{(7+x)^{1/3}-(1+3x)^{1/2}\}\sin\pi x}$

$= \lim_{t\to 0}\frac{[\log(2+t)-\log 2][3.4^{t}-3(t+1)}{\{(8+t)^{1/3}-(4+3t)^{1/2}\}\sin\pi(t+1)}$

[By putting $x = 1 + t$]

$= -\lim_{t\to 0}\frac{\log\left(1+\frac{t}{2}\right)(3(4^t-1)-3t)}{\left[2\left(1+\frac{t}{8}\right)^{1/3}-2\left(1+\frac{3t}{4}\right)^{1/2}\right]\sin\pi t}$

$= -\lim_{t\to 0}\frac{1}{\pi}\cdot\frac{\log\left(1+\frac{t}{2}\right)}{\frac{t}{2}}\cdot\left[\frac{3(4^t-1)}{t}-3\right]$

$\frac{\pi}{\sin\pi t}\cdot\frac{t}{\left[1+\frac{t}{24}-1-\frac{3t}{8}+\text{terms containing } t^2, t^3, \text{etc.}\right]}$

$=\frac{3}{\pi}\cdot(3\log 4-3) = \frac{9}{\pi}\log\frac{4}{e}$

The correct option is (A)

**101.** $\lim_{x\to 1}\frac{(1-x)(1-x^2)\ldots(1-x^{2n})}{[(1-x)(1-x^2)\ldots(1-x^n)]^2}$

$\lim_{x\to 1}\frac{(1-x)(1-x^2)\ldots(1-x^n)(1-x^{n+1})(1-x^{n+2})\ldots(1-x^{2n})}{[(1-x)(1-x^2)\ldots(1-x^n)]^2}$

$= \lim_{x\to 1}\frac{(1-x^{n+1})(1-x^{n+2})\ldots(1-x^{2n})}{[(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)]}$

$= \lim_{x\to 1}\frac{(1-x^{n+1})(1-x^{n+2})\ldots(1-x^{2n})}{(1-x)(1-x)\ldots(1-x)}$

$\times\frac{(1-x)(1-x)\ldots(1-x)}{(1-x)(1-x^2)\ldots(1-x^n)}$

$= (n+1)(n+2)\ldots 2n\cdot\frac{1}{1}\cdot\frac{1}{2}\ldots\frac{1}{n} = \frac{(2n)!}{(n!)^2}$

The correct option is (C)

**102.** Since maximum value of $\cos^{-1}x = \frac{\pi}{2}$

$\sum_{r=1}^{k}\cos^{-1}\alpha r = \frac{k\pi}{2}$ is possible if and only if each

$\cos^{-1}\alpha r = \frac{\pi}{2} \Rightarrow \alpha r = 0$

$\therefore\ \theta = \sum_{r=1}^{k}(\alpha r)^r = 0$

$\therefore\ \lim_{x\to\theta}\frac{(1+x^2)^{1/3}-(1-2x)^{1/4}}{x+x^2}$

$= \lim_{x\to 0}\frac{(1+x^2)^{1/3}-(1-2x)^{1/4}}{x+x^2}$

$= \lim_{x\to 0}\frac{\left(1+\frac{1}{3}x^2+O(x^4)\right)-\left(1-\frac{x}{2}+O(x^2)\right)}{x(1+x)}$

$= \lim_{x\to 0}\frac{\frac{1}{2}+\frac{1}{3}x+O(x^2)}{1+x} = \frac{1}{2}$

[O ($x^2$) means terms containing $x^2, x^3, x^4, \ldots$]

The correct option is (C)

**103.** $\because\ ax^2+bx+c=0$ has roots $\alpha$ and $\beta$, therefore

$\frac{a}{x^2}+\frac{b}{x}+c = 0$ i.e., $cx^2+bx+a=0$ has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

$\Rightarrow c\left(x^2+\frac{b}{cx}+\frac{a}{c}\right) = c\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)$

Now, $\lim_{x\to\frac{1}{\alpha}}\sqrt{\left[\frac{1-\cos(cx^2+bx+a)}{2(1-\alpha x)^2}\right]}$

$= \lim_{x\to\frac{1}{\alpha}}\left\{\frac{\sin^2\left(\frac{cx^2+bx+a}{2}\right)}{(1-\alpha x)^2}\right\}^{1/2}$

$= \lim_{x\to\frac{1}{\alpha}}\left|\frac{\sin\left(\frac{cx^2+bx+a}{2}\right)}{1-\alpha x}\right|$

$= \lim_{x\to\frac{1}{\alpha}}\left|\frac{\sin\left(\frac{c}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\right)}{-\alpha\left(x-\frac{1}{\alpha}\right)}\right|$

$= \left|\lim_{x\to\frac{1}{\alpha}}\frac{\sin\left(\frac{c}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\right)}{\frac{c}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}\cdot\lim_{x\to\frac{1}{\alpha}}\frac{\frac{c}{2}\left(x-\frac{1}{\beta}\right)}{-\alpha}\right|$

$= \left|\frac{1\cdot\frac{c}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)}{-\alpha}\right| = \left|\frac{c}{2\alpha}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right|$

The correct option is (A)

**104.** $\lim_{x\to 0^-}f(x) = \lim_{h\to 0}\sqrt{\{-h\}\cot\{-h\}}$

$= \lim_{h\to 0}\sqrt{(1-h)\cot(1-h)} = \sqrt{\cot 1}$

$\lim_{x\to 0^+}f(x) = \lim_{h\to 0}\frac{\tan^2\{h\}}{h^2-[h]^2}$

$= \lim_{h\to 0}\frac{\tan^2 h^2}{h^2} = 1$

$\therefore\ \lim_{x\to 0}f(x)$ does not exist,

The correct option is (D)

**105.** $\lim_{x\to 0}\frac{x^a\sin^b x}{\sin(x^c)} = \lim_{x\to 0}x^{a+b-c}\left(\frac{\sin x}{x}\right)^b\left(\frac{x^c}{\sin(x^c)}\right)$

The above limit is non-zero if $a+b-c=0$

The correct option is (D)

## More than One Option Correct Type

**106.** We know that $|\cos\theta| \le 1$ for all $\theta$.

So, if $|\cos n!\, px| < 1$,

$$\lim_{m\to\infty}\lim_{n\to\infty}(1+\cos^{2m} n!\pi x) = (1+0) = 1$$

and if $|\cos n!\, px| = 1$,

$$\lim_{m\to\infty}\lim_{n\to\infty}(1+\cos^{2m} n!\pi x) = \lim_{m\to\infty}\lim_{n\to\infty}(1+1^{2m})$$

$$= \lim_{m\to\infty}\lim_{n\to\infty}(1+1) = 2$$

The correct option is (A) and (B)

**107.** We have $2 = \lim_{x\to 0}\dfrac{(1+a^3)+8e^{1/x}}{1+(1-b^3)e^{1/x}}\left(\dfrac{\infty}{\infty}\text{ form}\right)$ (1)

$$\Rightarrow\ 2 = \lim_{x\to 0}\frac{0+8e^{1/x}(-1/x^2)}{0+(1-b^3)e^{1/x}(-1/x^2)}$$

[Using L'Hospital's rule]

$\Rightarrow\ 1-b^3 = 4\ \Rightarrow\ b^3 = -3\ \Rightarrow\ b = (-3)^{1/3}$

$\therefore$ From (1),

$$2 = \lim_{x\to 0}\frac{(1+a^3)+8e^{1/x}}{1+4e^{1/x}}$$

$\Rightarrow\ 1 + a^3 = 2$ i.e., $a = 1$

Hence $a = 1$ and $b = (-3)^{1/3}$

The correct option is (B) and (C)

**108.** $\lim_{x\to\infty}\left(\dfrac{a^{1/x}+b^{1/x}+c^{1/x}}{3}\right)^{3x}$

$= \lim_{y\to 0}\left(\dfrac{a^y+b^y+c^y}{3}\right)^{3/y}$ $\left[\text{Putting } \dfrac{1}{x} = y\right]$

$= \lim_{y\to 0} e^{\frac{3}{y}ln\left(\frac{a^y+b^y+c^y}{3}\right)}$

Now, we have,

$$\lim_{y\to 0}\frac{3}{y}ln\left(\frac{a^y+b^y+c^y}{3}\right)$$

$$= \lim_{y\to 0}\frac{3}{y}ln\left(1+\frac{a^y+b^y+c^y-3}{3}\right)$$

$$= \lim_{y\to 0}\frac{a^y+b^y+c^y-3}{y}\cdot\frac{ln\left(1+\frac{a^y+b^y+c^y-3}{3}\right)}{\frac{a^y+b^y+c^y-3}{3}}$$

$$= \lim_{y\to 0}\left(\frac{a^y-1}{y}+\frac{b^y-1}{y}+\frac{c^y-1}{y}\right)\cdot\lim_{t\to 0}\frac{\ln(1+t)}{t}$$

$\left[\text{Putting } \dfrac{a^y+b^y+c^y-3}{3} = t\right]$

$= ln\, a + ln\, b + ln\, c = ln(abc)$

Hence, the required limit is $e^{ln\,(abc)} = abc$

The correct option is (C) and (D)

**109.** Using the expansion, we have,

$$\lim_{x\to 0}\frac{ax\left(1+x+\frac{x^2}{2!}+\dots\right)-b\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right)+cx\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots,\right)}{x^2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\dots\right)}$$

$$\Rightarrow\ \lim_{x\to 0}\frac{x(a-b+c)-x^2\left(a+\frac{b}{2}-c\right)+x^3\left(\frac{a}{2}-\frac{b}{3}+\frac{c}{2}\right)+\dots}{x^2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)}$$

$= 2$

Now, above limit would exist if least power in numerator is greater than or equal to least power in denominator

i.e., coefficient of $x$ and $x^2$ must be zero and coefficient of $x^3$ should be 2.

i.e., $a-b+c = 0,\ a+\dfrac{b}{2}-c = 0,\ \dfrac{a}{2}-\dfrac{b}{3}+\dfrac{c}{2} = 2$

On solving, we get $a = 3$, $b = 12$, $c = 9$

The correct option is (A), (B) and (C)

## Passage Based Questions

**110.** Let $k = \lim_{x\to 0}\dfrac{\sin 2x + a\sin x}{x^3}$ $\left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{2\cos 2x + a\cos x}{3x^2}$$

[Using L'Hospital's rule]

We require $2\cos 2x + a\cos x = 0$ for $x = 0$ as denominator is zero.

$\therefore\ a = -2$

Hence, $k = \lim_{x\to 0}\dfrac{2\cos 2x - 2\cos x}{3x^2}$ $\left(\dfrac{0}{0}\right)$

$= \lim_{x\to 0}\dfrac{-4\sin 2x + 2\sin x}{6x}$ $\left(\dfrac{0}{0}\right)$

$= \lim_{x\to 0}\dfrac{-8\cos 2x + 2\cos x}{6} = \dfrac{-8+2}{6} = -1$

The correct option is (B)

**111.** $\lim_{x\to a}\sqrt{a^2-x^2}\cot\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}$ $(0\cdot\infty \text{ form})$

$= \lim_{x\to a}\frac{\sqrt{a^2-x^2}}{\tan\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to a}\frac{\frac{-2x}{2\sqrt{a^2-x^2}}}{-\sec^2\frac{\pi}{2}\sqrt{\frac{a-x}{a+x}}\times\frac{\pi}{2}\times\frac{2a}{2(a+x)\sqrt{a^2-x^2}}}$

$= \frac{4a}{\pi}$

The correct option is (C)

**112.** $\lim_{x\to 0}|x|^{\sin x} = \lim_{x\to 0}e^{\sin x\log_e|x|} = e^{\lim_{x\to 0}\frac{\log_e|x|}{\operatorname{cosec} x}}$

$= e^{\lim_{x\to 0}\frac{1/x}{\operatorname{cosec} x\cot x}} = e^{\lim_{x\to 0}\frac{\sin^2 x}{x\cos x}}$

$= e^{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\left(\frac{x}{\cos x}\right)} = e^{-(1)2\cdot(0)} = e^0 = 1$

The correct option is (B)

**113.** $\lim_{x\to\alpha}(1+ax^2+bx+c)^{1/(x-\alpha)}$

$= e^{\lim_{x\to\alpha}\frac{1}{(x-\alpha)}[(1+ax^2+bx+c)-1]}$

$= e^{\lim_{x\to\alpha}\frac{(ax^2+bx+c)}{(x-\alpha)}} = e^{\lim_{x\to\alpha}\frac{a(x-\alpha)(x-\beta)}{(x-\alpha)}}$

$[\because\ \alpha, \beta \text{ are roots of } ax^2+bx+c=0]$

$= e^{\alpha(\alpha-\beta)}$.

The correct option is (B)

**114.** We can see that

$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n} > \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\ldots \text{ to } n \text{ terms} = \frac{1}{2^n}$

and, $\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n} = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{2n}\right)$

$< \left(1-\frac{1}{2n}\right)\left(1-\frac{1}{2n}\right) \ldots \text{ to } n \text{ terms}$

$= \left(1-\frac{1}{2n}\right)^n$

Thus, we have,

$\frac{1}{2^n} < \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n} < \left(1-\frac{1}{2n}\right)^n$

Now, we have,

$\lim_{n\to+\infty}\frac{1}{2^n} = 0$ and $\lim_{n\to+\infty}\left(\frac{2n-1}{2n}\right)^n = 0$

Hence, by Sandwich Theorem, we have

$\lim_{n\to+\infty}\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n} = 0$

The correct option is (C)

**115.** Since, $0 \le \{rx\} < 1$ for $r = 1, 2, 3, \ldots, n$

$\Rightarrow\ 0\le\sum_{r=1}^{n}\{rx\} < \sum_{r=1}^{n}(1) \quad\Rightarrow\quad 0\le\sum_{r=1}^{n}\{rx\} < n$

Dividing throughout by $n^2$, we have

$\frac{0}{n^2} \le \frac{\sum_{r=1}^{n}\{rx\}}{n^2} < \frac{1}{n}$

$\Rightarrow\ \lim_{n\to\infty} 0 \le \lim_{n\to\infty}\frac{\sum_{r=1}^{n}\{rx\}}{n^2} < \lim_{n\to\infty}\frac{1}{n}$

$\Rightarrow\ 0 \le \lim_{n\to\infty}\frac{\sum_{r=1}^{n}\{rx\}}{n^2} < 0$

$\therefore\ 0 \le \lim_{n\to\infty}\frac{\{x\}+\{2x\}+\ldots+\{nx\}}{n^2} < 0$

According to Sandwich Theorem or Squeeze Principle

$\lim_{n\to\infty}\frac{\{x\}+\{2x\}+\ldots+\{nx\}}{n^2} = 0$

The correct option is (B)

**116.** We have,

$1^2\cdot x^x - 1 \le [1^2x^x] < 1^2x^x$ $[\because\ x-1 \le [x] < x]$

$2^2\cdot x^x - 1 \le [2^2x^x] < 2^2x^x$

…

…

$n^2\cdot x^x - 1 \le [n^2x^x] < n^2\cdot x^x$

Adding the above inequations,

$\frac{x^x\Sigma n^2 - n}{n^3} \le \frac{\Sigma[n^2x^x]}{n^3} \le \frac{x^x\Sigma n^2}{n^3}$

$\Rightarrow\ x^x\frac{n(n+1)(2n+1)}{6n^3} - \frac{1}{n^2} \le \frac{\Sigma[n^2x^x]}{n^3}$

$\le x^x\frac{n(n+1)(2n+1)}{6n^3}$

Now, applying $\lim_{n\to\infty}$, we have $\frac{x^x}{3} \le \frac{\Sigma[n^2x^x]}{n^3} \le \frac{x^x}{3}$

Hence, by Sandwich Theorem, we have

$\lim_{n\to\infty}\frac{\Sigma[n^2x^x]}{n^3} = \frac{x^x}{3}$

Now, the required unit

$\lim_{n\to 0^+}\left(\lim_{n\to\infty}\frac{\Sigma[n^2x^x]}{n^3}\right) = \frac{1}{3}\lim_{x\to 0+}x^x = \frac{1}{3}$

The correct option is (B)

## Match the Column Type

**117.** **(I)** $\lim_{n\to\infty}\left[\sqrt[3]{n^2-n^3}+n\right] = \lim_{n\to\infty} n\left[\left(-1+\frac{1}{n}\right)^{1/3}+1^{1/3}\right]$

$$= \lim_{n\to\infty} n\cdot\frac{\left(\frac{1}{n}-1\right)+1}{\left(\frac{1}{n}-1\right)^{2/3}+1-\left(\frac{1}{n}-1\right)^{1/3}}$$

$$\left[\text{Using } a+b=\frac{a^3+b^3}{a^2-ab+b^2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{\left(\frac{1}{n}-1\right)^{2/3}+1-\left(\frac{1}{n}-1\right)^{1/3}} = \frac{1}{1+1+1} = \frac{1}{3}$$

The correct option is (C)

**(II)** $\lim_{x\to1}\frac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$

$$= \lim_{y\to1}\frac{y^2-2y+1}{(y^3-1)^2} \quad [\text{Putting } \sqrt[3]{x}=y;\text{ as } x\to1,\ y\to1]$$

$$= \lim_{y\to1}\frac{(y-1)^2}{(y-1)^2(y^2+y+1)^2} = \lim_{y\to1}\frac{1}{(y^2+y+1)^2} = \frac{1}{9}$$

The correct option is (A)

**(III)** $(n-2)$th factor of the series is

$$t_n = \frac{n-1}{n+1}\cdot\frac{n^2+n+1}{n^2-n+1}$$

Therefore, required limit $= \lim_{n\to\infty} t_3t_4t_5 \ldots t_{n-2}t_{n-1}t_n$

$$= \lim_{n\to\infty}\left[\left(\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}\cdots\frac{n-3}{n-1}\cdot\frac{n-2}{n}\cdot\frac{n-1}{n+1}\right)\cdot\left(\frac{13}{7}\right)\cdot\frac{21}{13}\cdot\frac{31}{21}\cdots\frac{n^2+n+1}{n^2-n+1}\right]$$

$$= \lim_{n\to\infty}\frac{2\cdot3}{n(n+1)}\cdot\frac{n^2+n+1}{7} = \frac{6}{7}$$

The correct option is (B)

**(IV)** $\lim_{n\to\infty}\left(\cos\frac{x}{n}\right)^n = e^{\lim_{n\to\infty} n\left(\cos\frac{x}{n}-1\right)}$

$$= e^{\lim_{n\to\infty} -n\cdot2\sin^2\left(\frac{x}{2n}\right)} = e^{-2\lim_{n\to\infty}\left(\frac{\sin\left(\frac{x}{2n}\right)}{\frac{x}{2n}}\right)^2\cdot\frac{x^2}{4n^2}\cdot n}$$

$$= e^{-2\times\lim_{n\to\infty}\frac{x^2}{4n}} = e^0 = 1$$

The correct option is (D)

**118.** **(I)** Let $\sin x = h$, then as $x\to\pi/2$, $h\to1$

$\therefore$ given limit

$$= \lim_{h\to1}\frac{h-h^h}{1-h+\ln h} = \lim_{h\to1}\frac{1-h^h-h^h\ln h}{-1+1/h}$$

[Using L' Hospital rule]

$$= \lim_{h\to1}\frac{-h^h-2h^h\ln h-h^{h-1}-h^h(\ln h)^2}{-1/h^2}$$

$$= \frac{-1-1}{-1} = 2$$

The correct option is (A)

**(II)** Required limit =

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{5^r+2^r}{10^r} = \lim_{n\to\infty}\sum_{r=1}^{n}\left\{\left(\frac{1}{2}\right)^r+\left(\frac{1}{5}\right)^r\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{2}\cdot\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}+\frac{1}{5}\cdot\frac{1-\left(\frac{1}{5}\right)^n}{1-\frac{1}{5}}\right\}$$

$$= 1+\frac{1}{4} = \frac{5}{4}$$

The correct option is (D)

**(III)** $\lim_{x\to\infty} x\left[\tan^{-1}\frac{x+1}{x+2}-\frac{\pi}{4}\right]$

$$= \lim_{x\to\infty} x\left[\tan^{-1}\frac{x+1}{x+2}-\tan^{-1}1\right]$$

$$= \lim_{x\to\infty} x\tan^{-1}\left(\frac{\frac{x+1}{x+2}-1}{1+\frac{x+1}{x+2}}\right)$$

$$= \lim_{x\to\infty} x\tan^{-1}\left(\frac{-1}{2x+3}\right)$$

$$= -\lim_{x\to\infty}\frac{\tan^{-1}\left(\frac{1}{2x+3}\right)}{\left(\frac{1}{2x+3}\right)}\cdot\frac{1}{\left(2+\frac{3}{x}\right)}$$

$$= -1\times\frac{1}{2} = -\frac{1}{2}$$

The correct option is (B)

**(IV)** $\lim_{n\to\infty}\frac{n^k\sin^2(n!)}{n+2} = \lim_{n\to\infty}\frac{n^k\sin^2(n!)}{n\left(1+\frac{2}{n}\right)}$

$$= \lim_{n\to\infty}\frac{\sin^2(n!)}{n^{1-k}\left(1+\frac{2}{n}\right)}$$

$$= \frac{\text{a finite quantity}}{\infty}$$

[$\because$ $\sin^2(n!)$ always lies between 0 and 1. Also, since $1-k>0$,

$\therefore$ $n^{1-k}\to\infty$ as $n\to\infty$]

$= 0$

The correct option is (C)

## Assertion-Reasoning Type

**119.** $t_r = \dfrac{1^2+2^2+3^2+...+r^2}{1^3+2^3+3^3+...+r^3}$

$= \dfrac{r(r+1)(2r+1)}{6}\cdot\left(\dfrac{2}{r(r+1)}\right)^2 = \dfrac{2}{3}\left(\dfrac{1}{r}+\dfrac{1}{r+1}\right)$

$\therefore\ S_n = \dfrac{2}{3}\left[-\left(1+\dfrac{1}{2}\right)+\left(\dfrac{1}{2}+\dfrac{1}{3}\right)-\left(\dfrac{1}{3}+\dfrac{1}{4}\right)+\ldots\pm\left(\dfrac{1}{n}+\dfrac{1}{n+1}\right)\right]$

$= \dfrac{2}{3}\left(-1\pm\dfrac{1}{n+1}\right)$

$\therefore\ \lim\limits_{n\to\infty} S_n = -\dfrac{2}{3}$

The correct option is (D)

**120.** We have,

$x_1 = 3,\ x_{n+1} = \sqrt{2+x_n}$

$x_2 = \sqrt{2+x_1} = \sqrt{2+3} = \sqrt{5}$

$x_3 = \sqrt{2+x_2} = \sqrt{2+\sqrt{5}}$

$\therefore\ x_1 > x_2 > x_3$

It can be easily shown by mathematical induction that the sequence $x_1, x_2, \ldots x_n, \ldots$ is a monotonically decreasing sequence bounded below by 2. So, it is convergent.

Let $\lim\limits_{n\to\infty} x_n = x$. Then,

$x_{n+1} = \sqrt{2+x_n}$

$\Rightarrow\ \lim\limits_{n\to\infty} x_{n+1} = \sqrt{2+\lim\limits_{n\to\infty} x_n}$

$\Rightarrow\ x = \sqrt{2+x}$

$\Rightarrow\ x^2 - x - 2 = 0$

$\Rightarrow\ (x-2)(x+1) = 0$

$\Rightarrow\ x = 2 \qquad (\because x_n > 0\ \forall\, n,\quad \therefore x > 0)$

The correct option is (A)

**121.** $\lim\limits_{n\to\infty}\dfrac{1}{n}\left(1+e^{1/n}+e^{2/n}+...+e^{\frac{n-1}{n}}\right)$

$= \lim\limits_{n\to\infty}\dfrac{1}{n}\cdot\dfrac{1-\left(e^{1/n}\right)^n}{1-e^{1/n}} = \lim\limits_{n\to\infty}\dfrac{1-e}{n\left[1-1-\dfrac{1}{n}-\dfrac{1}{2!}\cdot\dfrac{1}{n^2}...\right]}$

$= \lim\limits_{n\to\infty}\dfrac{1-e}{-1-\dfrac{1}{2!}\cdot\dfrac{1}{n}...} = \dfrac{1-e}{-1} = e-1$

The correct option is (A)

**122.** $\lim\limits_{x\to 0}\dfrac{e-(1+x)^{1/x}}{x} = \lim\limits_{x\to 0}\dfrac{e-e^{\frac{\ln(1+x)}{x}}}{x}$

$= \lim\limits_{x\to 0} -e\cdot\dfrac{e^{\frac{\ln(1+x)-x}{x}}-1}{x}$

$= \lim\limits_{x\to 0} -e\cdot\dfrac{e^y-1}{y}\cdot\dfrac{\ln(1+x)-x}{x^2}$

$\left[\text{Putting } \dfrac{\ln(1+x)-x}{x} = y\right]$

Now, we have,

$\lim\limits_{x\to 0}\dfrac{\ln(1+x)-x}{x}\left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0}\dfrac{\dfrac{1}{1+x}-1}{1} = 0$

Also, $\lim\limits_{x\to 0}\dfrac{\ln(1+x)-x}{x^2}\left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0}\dfrac{\dfrac{1}{1+x}-1}{2x} = \lim\limits_{x\to 0}\dfrac{-1}{2(1+x)} = -\dfrac{1}{2}$

Hence, the required limit is

$= -e\lim\limits_{y\to 0}\dfrac{e^y-1}{y}\cdot\lim\limits_{x\to 0}\dfrac{\ln(1+x)-x}{x^2}$

$= -e\cdot 1\cdot -\dfrac{1}{2} = \dfrac{e}{2}$

The correct option is (A)

**123.** $\lim\limits_{x\to 0}\left([f(x)]+x^2\right)^{\frac{1}{\{f(x)\}}}$

$= \lim\limits_{x\to 0}\left[\left(\left(1+[f(x)]+x^2-1\right)^{\frac{1}{[f(x)]+x^2-1}}\right)\right]^{\frac{[f(x)]+x^2-1}{\{f(x)\}}}$

Now, we have,

$\lim\limits_{x\to 0}\left[\dfrac{\tan x}{x}\right]+x^2-1 = 1+0-1 = 0$

and, $\lim\limits_{x\to 0}\dfrac{[f(x)]+x^2-1}{\{f(x)\}}$

$= \lim\limits_{x\to 0}\dfrac{\left[1+\dfrac{x^2}{3}+\dfrac{2x^4}{15}+...\right]+x^2-1}{\left\{1+\dfrac{x^2}{3}+\dfrac{2x^4}{15}+...\right\}}$

$\left[\tan x = x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+...\right]$

$= \lim\limits_{x\to 0}\dfrac{1+x^2-1}{\dfrac{x^2}{3}+\dfrac{2x^5}{15}+...} = \lim\limits_{x\to 0}\dfrac{1}{\dfrac{1}{3}+\dfrac{2x}{15}+...}$

$= 3$

$\therefore$ Required limit $= e^3$

The correct option is (D)

**124.** $\lim_{n\to\infty} \frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^n} + \ldots + \frac{1}{2^n}\tan\frac{x}{2^n}$

$= \lim_{n\to\infty} -\cot x + \left(\cot x + \frac{1}{2}\tan\frac{x}{2}\right) + \frac{1}{2^2}\tan\frac{x}{2^2} +$

$\ldots + \frac{1}{2^n}\tan\frac{x}{2^n}$

$= \lim_{n\to\infty} -\cot x + \left(\frac{1}{2}\cot\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2}\right) + \ldots + \frac{1}{2^n}\tan\frac{x}{2^n}$

$= \lim_{n\to\infty} -\cot x + \left(\frac{1}{2^2}\cot\frac{x}{2^2} + \frac{1}{2^3}\tan\frac{x}{2^3}\right) + \ldots + \frac{1}{2^n}\tan\frac{x}{2^n}$

$\left[\text{repeatedly using } \cot\theta + \frac{1}{2}\tan\left|\frac{\theta}{2} = \frac{1}{2}\cot\right|\frac{\theta}{2}\right]$

$= \lim_{n\to\infty} -\cot x + \frac{1}{2^n}\cot\frac{x}{2^n}$

$= \lim_{n\to\infty} -\cot x + \frac{1}{x}\left(\frac{x/2^n}{\tan x/2^n}\right)$

$= -\cot x + \frac{1}{x}\lim_{\theta\to 0}\frac{\theta}{\tan\theta} = -\cot x + \frac{1}{x}$

The correct option is (A)

**125.** $\lim_{\theta\to 0}\frac{\cot\theta\tan^{-1}(m\tan\theta) - m\cos^2(\theta/2)}{\sin^2(\theta/2)}$

$= \lim_{\theta\to 0}\frac{\cot\theta\tan^{-1}(m\tan\theta) - m[1-\sin^2(\theta/2)]}{\sin^2(\theta/2)}$

$= \lim_{\theta\to 0}\frac{\cot\theta\tan^{-1}(m\tan\theta) - m}{\sin^2(\theta/2)} + m$

$= \lim_{\theta\to 0}\frac{\tan^{-1}(m\tan\theta) - m\tan\theta}{\tan\theta\sin^2(\theta/2)} + m$

$= \lim_{\theta\to 0}\frac{\tan^{-1}(m\tan\theta) - m\tan\theta}{\theta^3/4} + m$

$= \lim_{x\to 0}\frac{x - \tan x}{x^3 / 4m^3} + m$

[Putting $m\tan\theta = \tan x$; as $q \to 0$, $x \to 0$]

$= m + 4m^3\lim_{x\to 0}\frac{x-\tan x}{x^3}$

Now, we have,

$\lim_{x\to 0}\frac{x-\tan x}{x^3}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{1-\sec^2 x}{3x^2}\left(\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{-2\sec^2 x\tan x}{6x} = \frac{-1}{3}$

Hence, the required limit is $m - (4/3)\,m^3$

The correct option is (C)

## Previous Year's Questions

**126.** Key Idea: Limit of a function exists only, if LHL = RHL.

Now, $\lim_{x\to 0}\frac{\sqrt{1-\cos 2x}}{\sqrt{2}x} = \lim_{x\to 0}\frac{\sqrt{1-1+2\sin^2 x}}{\sqrt{2}x}$

$= \lim_{x\to 0}\frac{\sqrt{2}|\sin x|}{\sqrt{2}x} = \lim_{x\to 0}\frac{|\sin x|}{x}$

Let $f(x) = \frac{|\sin x|}{x}$

Now, $\text{LHL} = \lim_{h\to 0}\frac{|\sin(0-h)|}{0-h}$

$= \lim_{h\to 0}\frac{|\sin h|}{-h} = -1$

and $\text{RHL} = \lim_{h\to 0}\frac{|\sin(0+h)|}{0+h}$

$= \lim_{h\to 0}\frac{|\sin h|}{h} = 1$

$\because$ LHL $\neq$ RHL

$\therefore \lim_{x\to 0}\frac{|\sin x|}{x}$ does not exist.

The correct option is (D)

**127.** Key Idea : $\lim_{x\to\infty}(1+\lambda x)^{1/x} = e^{\lambda}$

Now, the limit $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$

$= \lim_{x\to\infty}\left(1 + \frac{4x+1}{x^2+x+2}\right)^x$

$= \lim_{x\to\infty}\left[\left(1+\frac{4x+1}{x^2+x+2}\right)^{1/\frac{(4x+1)}{x^2+x+2}}\right]^{\frac{(4x+1)x}{x^2+x+2}}$

$= e^{\lim_{x\to\infty}\frac{\left(4+\frac{1}{x}\right)}{1+\frac{1}{x}-\frac{2}{x^2}}}$

$= e^4$

The correct option is (A)

**128.** The limit $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$

$= \lim_{x\to\infty}\left[1 - \frac{5}{x+2}\right]^x$

$$= \lim_{x\to\infty}\left[\left(1+\left(\frac{-5}{x+2}\right)\right)^{1/\left(\frac{5}{x+2}\right)}\right]^{\left(\frac{-5x}{x+2}\right)}$$

$$= e^{\lim_{x\to\infty}\left(-\frac{5}{1+2/x}\right)}$$

$$= e^{-5}$$

**Alternative Method:**

$$\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$$

$$= \lim_{x\to\infty}\frac{\left(1-\frac{3}{x}\right)^x}{\left(1+\frac{2}{x}\right)^x}$$

$$= \frac{e^{-3}}{e^2} = e^{-5}$$

The correct option is (C)

**129.** The limit $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}$

$$= \lim_{x\to 2}\frac{xf(2)-2f(x)+2f(2)-2f(x)}{x-2}$$

$$= \lim_{x\to 2}\frac{f(2)(x-2)-2\{f(x)-f(2)\}}{x-2}$$

$$= f(2)-2\lim_{x\to 2}\frac{f(x)-f(2)\}}{x-2}$$

$$= f(2)-2f'(2) = 4-2\times 4 = -4$$

**Alternative Solution:**

$$\lim_{x\to 2}\left\{\frac{xf(2)-2f(x)}{x-2}\right\}$$

$$= \lim_{x\to 2}\{f(2)-2f'(x)\} \quad \text{(by L' Hopital's Rule)}$$

$$= f(2) - 2f'(2)$$

$$= 4 - 2 \times 4 = -4$$

The correct option is (C)

**130.** The limit, by applying L'Hopital rule,

$$\lim_{x\to\pi/2}\frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)(1-\sin x)}{4\left(\frac{\pi}{4}-\frac{x}{2}\right)(\pi-2x)^2}$$

$$= \lim_{x\to\pi/2}\frac{(-\cos x)}{8(-2)(\pi-2x)}$$

$$= \lim_{x\to\pi/2}\frac{-\sin x}{16(-2)}$$

$$= \frac{1}{32}.$$

The correct option is (C)

**131.** By applying L'Hopital Rule, the given limit equals

$$\lim_{x\to 0}\frac{\frac{1}{3+x}+\frac{1}{3-x}}{1} = \frac{2}{3}.$$

The correct option is (C)

**132.** Applying *L*. Hospital's Rule

$$\lim_{x\to 2a}\frac{f(a)g'(a)-g(a)f'(a)}{g'(a)-f'(a)} = 4$$

$$\frac{k(g'(a)-ff'(a))}{(g'(a)-f'(a))} = 4$$

$k = 4.$

The correct option is (A)

**133.** The limit $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} =$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{\left(\frac{1}{\frac{a}{x}+\frac{b}{x^2}}\right)\times 2x\times\left(\frac{a}{x}+\frac{b}{x^2}\right)} = e^{2a}$$

$$\Rightarrow a = 1, b \in R$$

The correct option is (B)

**134.** Let $L =$

$$\lim_{x\to\alpha}\frac{1-\cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2} = \lim_{x\to\alpha}\frac{2\sin^2\left(a\frac{(x-\alpha)(x-\beta)}{2}\right)}{(x-\alpha)^2}$$

$$= \lim_{x\to\alpha}\frac{2}{(x-\alpha)^2}\times\frac{\sin^2\left(a\frac{(x-\alpha)(x-\beta)}{2}\right)}{\frac{a^2(x-\alpha)^2(x-\beta)^2}{4}}\times\frac{a^2(x-\alpha)^2(x-\beta)^2}{4}$$

Then, the limit $L = \frac{a^2(\alpha-\beta)^2}{2}.$

The correct option is (A)

**135.** $f(x)$ is a positive increasing function

$$\Rightarrow 0 < f(x) < f(2x) < f(3x)$$

$$\Rightarrow 0 < 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$$

$$\Rightarrow \lim_{x\to\infty} 1 \le \lim_{x\to\infty}\frac{f(2x)}{f(x)} \le \lim_{x\to\infty}\frac{f(3x)}{f(x)}$$

By sandwich theorem.

$$\Rightarrow \lim_{x\to\infty}\frac{f(2x)}{f(x)} = 1$$

The correct option is (D)

**136.** $\lim_{x\to 2}\frac{\sqrt{2\sin^2(x-2)}}{x-2}$

$$\lim_{x\to 2}\frac{\sqrt{2}\,|\sin(x-2)|}{x-2}$$

R.H.L. $= \sqrt{2}$, L.H.L. $= -\sqrt{2}$

Limit does not exist.

The correct option is (D)

**137.** $\lim\limits_{x\to 0} \dfrac{(1-\cos 2x)}{x(\tan 4x)}(3+\cos x)$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{4}\left(\frac{4x}{\tan 4x}\right)(3+\cos x)$$

$$= 2\times 1\times \frac{1}{4}\times 1\times (3+1) = 2.$$

The correct option is (C)

**138.** Given that $\lim\limits_{x\to 0}\left(\dfrac{x^2+f(x)}{x^2}\right) = 3$.

Since limit exits, the expression $x^2 + f(x) = ax^4 + bx^3 + 3x^2$

$\Rightarrow f(x) = ax^4 + bx^3 + 2x^2$

$\Rightarrow f'(x) = 4ax^3 + 3bx^2 + 4x$

Also, $f'(x) = 0$ at $x = 1, 2$

$$\Rightarrow a = \frac{1}{2}, b = -2$$

$$\Rightarrow f(x) = \frac{x^4}{2} - 2x^3 + 2x^2$$

$$\Rightarrow f(x) = 8 - 16 + 8 = 0.$$

The correct option is (A)

**139.** The value of the limit

$$\lim_{x\to 0} \frac{2\sin^2 x\times(3+\cos x)}{x\times\left(\dfrac{\tan 4x}{rx}\right)\times 4x} = \frac{2\times 4}{4} = 2.$$

The correct option is (B)

**140.** We have $p = e^{\lim\limits_{x\to 0^+}\frac{1}{2}\left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)^2} = \sqrt{e}$

$\therefore \log p = \dfrac{1}{2}$

The correct option is (D)