CHAPTER 27

Inverse Trigonometric Functions

Chapter Highlights

Inverse Functions, Inverse Trigonometric Functions, Domain and Range of Inverse Trigonometric Functions, Graphs of Inverse Trigonometric Functions, Principal Values for Inverse Trigonometric Functions, Properties of Inverse Trigonometric Functions, Solutions of Basic Inverse Trigonometric Inequalities, Some useful Substitutions.

INVERSE FUNCTIONS

If $f : X \to Y$ is a function which is both one-one and onto, then its inverse function $f^{-1} : Y \to X$ is defined as:

$$y = f(x) \Leftrightarrow f^{-1}(y) = x, \forall x \in X, \forall y \in Y$$

INVERSE TRIGONOMETRIC FUNCTIONS

Consider the sine function with domain R and range $[-1, 1]$. This function is many-one and onto. So, its inverse does not exist. If we restrict its domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then the function

$\sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ given by $\sin\theta = x$

is one-one and onto and therefore it is invertible.

The inverse of sine function is defined as

$\sin^{-1}: [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

such that $\sin^{-1}x = \theta \Leftrightarrow \sin\theta = x$.

Thus, if x is a real number between -1 and 1, then $\sin^{-1}x$ is an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x, i.e.,

$\sin^{-1}x = \theta \Leftrightarrow x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $-1 \le x \le 1$.

The least numerical value among all the values of the angle whose sine is x, is called the principal value of sin–1x.

Similar definitions for $\cos^{-1}x$, $\tan^{-1}x$ and so on can be given.

DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

Table 27.1

Function	Domain	Range (Principal Values)
1. $y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2. $y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
3. $y = \tan^{-1}x$	$(-\infty, \infty)$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
4. $y = \operatorname{cosec}^{-1}x$	$x \ge 1$ or $x \le -1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$
5. $y = \sec^{-1}x$	$x \ge 1$ or $x \le -1$	$0 \le y \le \pi, y \ne \frac{\pi}{2}$
6. $y = \cot^{-1}x$	$(-\infty, \infty)$	$0 < y < \pi$

SOLVED EXAMPLE

1. The solution set of the equation $\tan^{-1}x - \cot^{-1}x = \cos^{-1}(2 - x)$ is

(A) [0, 1] (b) [– 1, 1]
(C) [1, 3] (d) none of these

Solution: (C)

$\tan^{-1}x$ and $\cot^{-1}x$ exist for all $x \in R$.

$\cos^{-1}(2 - x)$ exists if $-1 \le 2 - x \le 1$ *i.e.* $1 \le x \le 3$

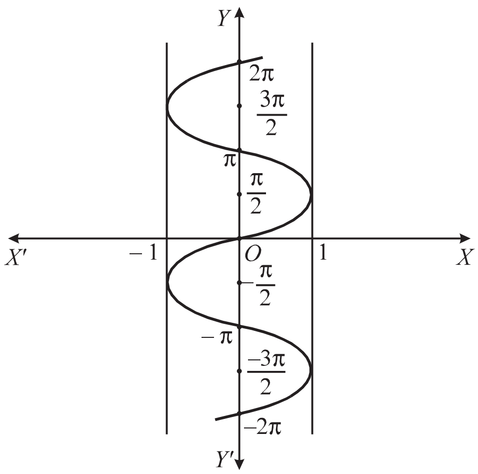
So, the given equation holds for $\mathbf{1 \le x \le 3}$.

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

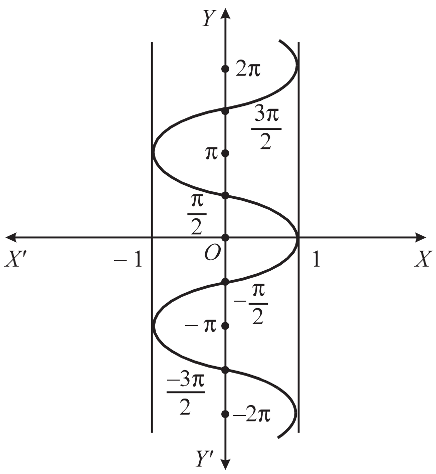
The graphs of inverse trigonometric functions can be drawn from the knowledge of the graphs of corresponding trigonometric functions. These graphs can be obtained by interchanging x and y axis.

The graphs of inverse trigonometric functions are given as follows:

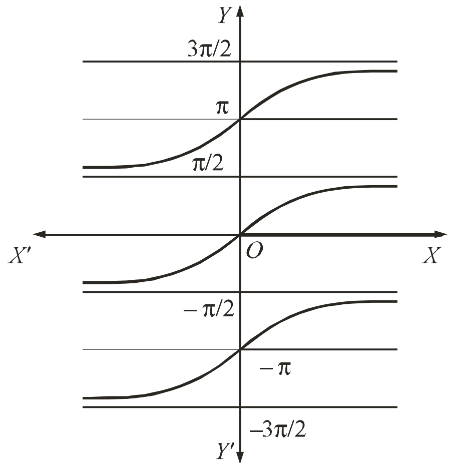
(i) Graph of $\sin^{-1}x$



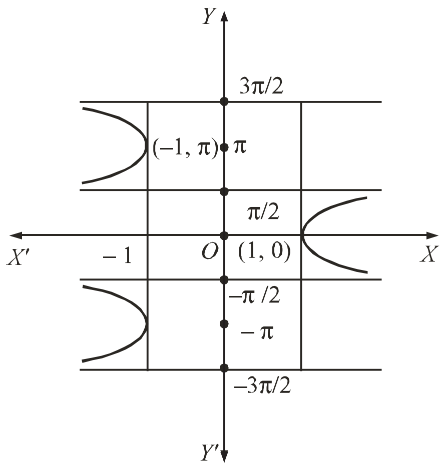
(ii) Graph of $\cos^{-1}x$



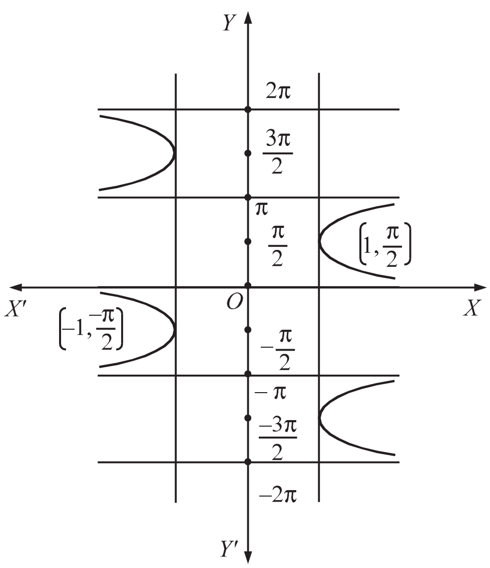
(iii) Graph of $\tan^{-1}x$



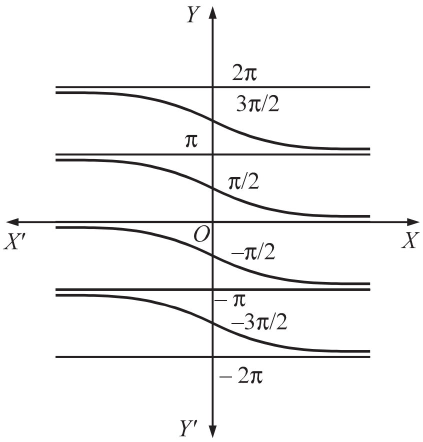
(iv) Graph of $\sec^{-1}x$



(v) Graph of $\text{cosec}^{-1}x$



(vi) Graph of $\cot^{-1}x$



PRINCIPAL VALUES FOR INVERSE TRIGONOMETRIC FUNCTIONS

If the domain of inverse trigonometric function is not stated we only consider the principal values which are stated as

Table 26.2

Principal values for x ≥ 0	Principal values of x < 0
$0 \le \sin^{-1} x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \sin^{-1} x < 0$
$0 \le \cos^{-1} x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1} x \le \pi$
$0 \le \tan^{-1} x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1} x < 0$
$0 \le \cot^{-1} x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1} x < \pi$
$0 \le \sec^{-1} x < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1} x \le \pi$
$0 \le \operatorname{cosec}^{-1} x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \operatorname{cosec}^{-1} x < 0$

CAUTION

$\sin^{-1}x$ and $\frac{1}{\sin x}$ i.e., $(\sin x)^{-1}$ have different meanings. They are not equal.

$\therefore \quad \sin^{-1}x \ne (\sin x)^{-1}$

Similarly, for other trigonometric ratios.

SOLVED EXAMPLES

2. If $\sum_{i=1}^{2n} \cos^{-1} x_i = 0$, then $\sum_{i=1}^{2n} x_i$ is

(A) n (B) $2n$

(C) $\frac{n(n+1)}{2}$ (D) none of these

Solution: (B)

Since $0 \le \cos^{-1} x_i \le \pi$, $\therefore \cos^{-1} x_i = 0$ for all i.

$\therefore \quad x_i = 1$ for all $_i$ $\therefore \sum_{i=1}^{2n} x_i = \mathbf{2n}$

3. If $\sum_{i=1}^{20} \sin^{-1} x_i = 10\pi$ then $\sum_{i=1}^{20} x_i$ is equal to

(A) 20 (B) 10

(C) 0 (D) none of these

Solution: (A)

Since $-\frac{\pi}{2} \le \sin{-1}\, x_i \le \frac{\pi}{2}$, $\therefore \sin^{-1} x_i = \frac{\pi}{2}$, $1 \le i \le 20$

$\therefore \quad x_i = 1$, $1 \le i \le 20$. Thus, $\sum_{i=1}^{20} x_i = 20$.

4. $\sec^{-1}(\sin^2 x)$ is well-defined if and only if

(A) $x \in \left[-\frac{\neq}{2}, \frac{\neq}{2}\right]$

(B) $x \in R$

(C) $x \in \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

(D) none of these

Solution: (C)

$\sec^{-1}(\sin^2 x)$ is well defined if and only if

$\sin^2 x \ge 1 \Leftrightarrow \sin^2 x = 1$ ($\therefore \sin^2 x \not> 1$)

$\therefore x = n\pi + \frac{\pi}{2}$, $n \in Z$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

1. (i) $\sin^{-1}(\sin\theta) = \theta$ and $\sin(\sin^{-1} x) = x$, provided $-1 \le x \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

(ii) $\cos^{-1}(\cos\theta) = \theta$ and $\cos(\cos^{-1} x) = x$, provided $-1 \le x \le 1$ and $0 \le \theta \le \pi$.

(iii) $\tan^{-1}(\tan\theta) = \theta$ and $\tan(\tan^{-1} x) = x$, provided $-\infty < x < \infty$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(iv) $\cot^{-1}(\cot\theta) = \theta$ and $\cot(\cot^{-1} x) = x$, provided $-\infty < x < \infty$ and $0 < \theta < \pi$.

(v) $\sec^{-1}(\sec\theta) = \theta$ and $\sec(\sec^{-1} x) = x$.

(vi) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$ and $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$

where θ and x in (v) and (vi) satisfy the corresponding domains and range.

2. (i) $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$ or $\operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$

(ii) $\cos^{-1}x = \sec^{-1}\frac{1}{x}$ or $\sec^{-1}x = \cos^{-1}\frac{1}{x}$

(iii) $\tan^{-1}x = \cot^{-1}\frac{1}{x}$ if $x > 0$ and

$\tan^{-1}x = \cot^{-1}\frac{1}{x} - \pi$ if $x < 0$ and

$\cot^{-1}x = \tan^{-1}\frac{1}{x}$ if $x > 0$ and

$\cot^{-1}x = \tan^{-1}\frac{1}{x} + \pi$ if $x < 0$

3. (i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$

$= \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{x}$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$

$= \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\frac{1}{\sqrt{1-x^2}}$

(iii) $\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$

$= \cot^{-1}\frac{1}{x} = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{1}\frac{\sqrt{1+x^2}}{x}$

4. (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, where $-1 \le x \le 1$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, where $-\infty < x < \infty$

(iii) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$, where $x \le -1$ or $x \ge 1$.

5.(a) $\sin^{-1}x + \sin^{-1}y$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) \\ \text{if } xy \le 0 \text{ or } (xy > 0 \text{ and } x^2 + y^2 \le 1) \\ \neq - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}] \\ \text{if } x > 0,\ y > 0 \text{ and } x^2 + y^2 > 1 \\ -\neq - \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}] \\ \text{if } x < 0,\ y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

(b) $\sin^{-1}x - \sin^{-1}y$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}) \\ \text{if } xy \ge 0 \text{ or } (xy < 0 \text{ and } x^2 + y^2 \le 1) \\ \neq - \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}] \\ \text{if } x > 0,\ y < 0 \text{ and } x^2 + y^2 > 1 \\ -\neq - \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}] \\ \text{if } x < 0,\ y > 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

(c) $\cos^{-1}x + \cos^{-1}y$

$$= \begin{cases} \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) \\ \text{if } |x|,\ |y| \le 1,\ x + y \ge 0 \\ 2\neq - \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) \\ \text{if } |x|,\ |y| \le 1,\ x + y \le 0 \end{cases}$$

(d) $\cos^{-1}x - \cos^{-1}y$

$$= \begin{cases} \cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2}) \\ \text{if } |x|,\ |y| \le 1,\ x \le y \\ -\cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2}) \\ \text{if } |x|,\ |y| \le 1,\ x \ge y. \end{cases}$$

(e) $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy < 1 \\ \neq + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy > 1,\ x > 0. \\ \tan^{-1}\left(\frac{x+y}{1-xy}\right) - \neq \text{ if } xy > 1,\ x < 0. \\ \frac{\neq}{2} \quad \text{if } xy = 1,\ x > 0 \\ -\frac{\neq}{2} \quad \text{if } xy = 1,\ x < 0 \end{cases}$$

(f) $\tan^{-1}x - \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right) \quad \text{if } xy > -1 \\ \neq + \tan^{-1}\left(\frac{x-y}{1+xy}\right) \text{ if } xy < -1,\ x > 0. \\ \tan^{-1}\left(\frac{x-y}{1+xy}\right) - \neq \text{ if } xy < -1,\ x < 0. \end{cases}$$

$$\begin{cases} \dfrac{\neq}{2} & \text{if } xy = -1, x > 0 \\ -\dfrac{\neq}{2} & \text{if } xy = -1, x < 0 \end{cases}$$

(g) $2\sin^{-1}x$

$$= \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}) & \text{if } \dfrac{-1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}) & \text{if } \dfrac{1}{\sqrt{2}} < x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}) & \text{if } -1 \le x < -\dfrac{1}{\sqrt{2}} \end{cases}$$

(h) $2\cos^{-1}x$

$$\begin{cases} \cos^{-1}(2x^2 - 1) & \text{if } 0 \le x \le 1 \\ 2\neq - \cos^{-1}(2x^2 - 1) & \text{if } -1 \le x \le 0 \end{cases}$$

(i) $2\tan^{-1}x =$

$$\begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) - \neq & \text{if } -1 < x < 1 \\ \neq + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) & \text{if } x > 1 \\ \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) - \neq & \text{if } x < -1 \\ \dfrac{\neq}{2} & \text{if } x = 1 \\ -\dfrac{\neq}{2} & \text{if } x = -1 \end{cases}$$

(j) $2\tan^{-1}x =$

$$\begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & \text{if } -1 \le x \le 1 \\ \neq - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & \text{if } x > 1 \\ -\neq - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) & \text{if } x < -1 \end{cases}$$

(k) $2\tan^{-1}x =$

$$\begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & \text{if } -\infty < x \le 0 \end{cases}$$

6. (i) $\sin^{-1}(-x) = -\sin^{-1}x$
(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$
(iii) $\tan^{-1}(-x) = -\tan^{-1}x$
(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

7. (i) $3\sin^{-1}x =$

$$\begin{cases} \sin^{-1}(3x - 4x^3) & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \neq - \sin^{-1}(3x - 4x^3) & \text{if } \dfrac{1}{2} < x \le 1 \\ -\neq - \sin^{-1}(3x - 4x^3) & \text{if } -1 \le x < -\dfrac{1}{2} \end{cases}$$

(ii) $3\cos^{-1}x =$

$$\begin{cases} \cos^{-1}(4x^3 - 3x) & \text{if } \dfrac{1}{2} \le x \le 1 \\ 2\neq - \cos^{-1}(4x^3 - 3x) & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 2\neq + \cos^{-1}(4x^3 - 3x) & \text{if } -1 \le x \le -\dfrac{1}{2} \end{cases}$$

(iii) $3\tan^{-1}x =$

$$\begin{cases} \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) & \text{if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \neq + \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) & \text{if } x > \dfrac{1}{\sqrt{3}} \\ -\neq + \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) & \text{if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$$

8. $\mathbf{\tan^{-1}x + \tan^{-1}y + \tan^{-1}z} = \tan^{-1}\dfrac{x + y + z - xyz}{1 - xy - yz - zx}$

TRICK(S) FOR PROBLEM SOLVING

- $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ are also written as arc sin x, arc cos x and arc tan x, respectively.
- It should be noted that if not otherwise stated, only principal values of inverse circular functions are to be considered.
- If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$, then $xy + yz + zx = 1$.
- If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $x + y + z = xyz$.
- If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{\pi}{2}$,
  then $x^2 + y^2 + z^2 + 2xyz = 1$
- If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$,
  then $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

- If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then $xy + yz + zx = 3$.
- If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then $x^2 + y^2 + z^2 + 2xyz = 1$.
- If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then $xy + yz + zx = 3$.
- If $\sin^{-1}x + \sin^{-1}y = \theta$, then $\cos^{-1}x + \cos^{-1}y = \pi - \theta$.
- If $\cos^{-1}x + \cos^{-1}y = \theta$, then $\sin^{-1}x + \sin^{-1}y = \pi - \theta$.
- If $\tan^{-1}x + \tan^{-1}y = \frac{\neq}{2}$, then $xy = 1$.
- If $\cot^{-1}x + \cot^{-1}y = \frac{\neq}{2}$, then $xy = 1$.
- If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \theta$,

  then $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\theta + \frac{y^2}{b^2} = \sin^2\theta$.

SOLVED EXAMPLES

5. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\right)$

$= \frac{\pi}{2}$ for $0 < |x| < \sqrt{2}$, then x equals

(A) $\frac{1}{2}$ (B) 1

(C) $-\frac{1}{2}$ (D) -1

Solution: (B)

Since $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for $|x| \le 1$.

$\therefore\ x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots = x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots$

$\Rightarrow \frac{x}{1+\frac{x}{2}} = \frac{x^2}{1+\frac{x^2}{2}}$ $(0 < |x| < \sqrt{2})$

$\Rightarrow \frac{x}{2+x} = \frac{x^2}{2+x^2} \Rightarrow 2x + x^3 = 2x^2 + x^3$

$\Rightarrow x^2 = x \Rightarrow x = 0, 1$

But $x \neq 0$, $\therefore\ x = \mathbf{1}$.

6. $\sin^{-1}[\cos(\sin^{-1}x)] + \cos^{-1}[\sin(\cos^{-1}x)]$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) $\frac{3\pi}{4}$ (D) 0

Solution: (B)

We have

$\sin^{-1}(\cos(\sin^{-1}x)) + \cos^{-1}(\sin(\cos^{-1}x))$

$= \sin^{-1}\left[\cos\left(\frac{\pi}{2} - \cos^{-1}x\right)\right] + \cos^{-1}\left[\sin\left(\frac{\pi}{2} - \sin^{-1}x\right)\right]$

$= \sin^{-1}[\sin(\cos^{-1}x)] + \cos^{-1}[\cos(\sin^{-1}x)]$

$= \cos^{-1}x + \sin^{-1}x = \boldsymbol{\frac{\pi}{2}}$

7. The value of $\cos(2\cos^{-1}x + \sin^{-1}x)$ at $x = \frac{1}{5}$ is

(A) $\frac{2\sqrt{6}}{5}$ (B) $-\frac{2\sqrt{6}}{5}$

(C) $\frac{3\sqrt{6}}{5}$ (D) none of these

Solution: (B)

$\cos(2\cos^{-1}x + \sin^{-1}x)$

$= \cos(\cos^{-1}x + \cos^{-1}x + \sin^{-1}x)$

$= \cos\left(\frac{\pi}{2} + \cos^{-1}x\right) = -\sin(\cos^{-1}x)$

$= -\sin(\sin^{-1}\sqrt{1-x^2}) = -\sqrt{1-x^2}$

$= -\sqrt{1-\frac{1}{25}} = -\boldsymbol{\frac{2\sqrt{6}}{5}}$

8. If $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, then $x =$

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$

(C) $\frac{1}{\sqrt{2}}$ (D) $-\frac{1}{\sqrt{2}}$

Solution: (C, D)

We have, $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \tan^{-1}1$

$\Rightarrow \tan^{-1}\frac{x-1}{x+1} = \tan^{-1}1 - \tan^{-1}\frac{x+1}{x+2}$

$\Rightarrow \tan^{-1}\frac{x-1}{x-2} = \tan^{-1}\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}$

$\Rightarrow \tan^{-1}\frac{x-1}{x-2} = \tan^{-1}\frac{1}{2x+3}$

$\Rightarrow \frac{x-1}{x-2} = \frac{1}{2x+3} \Rightarrow (x-1)(2x+3) = x-2$.

$\Rightarrow 2x^2 + x - 3 = x - 2 \Rightarrow 2x^2 = 1$

$\therefore\ x = \pm\boldsymbol{\frac{1}{\sqrt{2}}}$

9. If $a \le \tan^{-1}x + \cot^{-1}x + \sin^{-1}x \le b$, then

(A) $a = 0$ (B) $b = \frac{\pi}{2}$

(C) $a = \frac{\pi}{4}$ (D) $b = \pi$

Solution: (A, D)

We have,

$\tan^{-1}x + \cot^{-1}x + \sin^{-1}x = \frac{\pi}{2} + \sin^{-1}x$ ...(1)

Since $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2} \Rightarrow 0 \le \frac{\pi}{2} + \sin^{-1}x \le \pi$

$\Rightarrow 0 \le \tan^{-1}x + \cot^{-1}x + \sin^{-1}x \le \pi$ [From (1)]

$\therefore a = \mathbf{0}$ and $b = \pi$

10. If $(\sin^{-1}x)^2 + (\cos^{-1}x)^2 = \frac{5\pi^2}{8}$, then x is equal to

(A) 1 (B) -1

(C) $\frac{1}{\sqrt{2}}$ (D) $-\frac{1}{\sqrt{2}}$

Solution: (C, D)

We have, $(\sin^{-1}x)^2 + (\cos^{-1}x)^2 = \frac{5\pi^2}{8}$

$$\Rightarrow (\sin^{-1}x)^2 + \left(\frac{\pi}{2} - \sin^{-1}x\right)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow 2(\sin^{-1}x)^2 = -\pi \sin^{-1}x - \frac{3\pi^2}{8} = 0$$

$$\Rightarrow \sin^{-1}x = \frac{\pi \pm \sqrt{4\pi^2}}{4} \Rightarrow \sin^{-1}x = \frac{3\pi}{4} \text{ or } -\frac{\pi}{4}$$

$$\Rightarrow x = \sin\frac{3\pi}{4} \text{ or } \sin\left(-\frac{\pi}{4}\right)$$

$$\therefore x = \mathbf{\frac{1}{\sqrt{2}}} \text{ or } -\mathbf{\frac{1}{\sqrt{2}}}$$

11. Solution of the equation $\cot^{-1}x + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$ is

(A) $x = 3$ (B) $x = \frac{1}{\sqrt{5}}$

(C) $x = 0$ (D) none of these

Solution: (A)

We have, $\cot^{-1}x + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1/\sqrt{5}}{\sqrt{1-\frac{1}{5}}} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{2} = \tan^{-1}1$$

$$\Rightarrow \tan^{-1}\frac{1}{x} = \tan^{-1}1 - \tan^{-1}\frac{1}{2}$$

$$\Rightarrow \tan^{-1}\frac{1}{x} = \tan^{-1}\left(\frac{1-\frac{1}{2}}{1+1\cdot\frac{1}{2}}\right)$$

$$\Rightarrow \tan^{-1}\frac{1}{x} = \tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \mathbf{x = 3}$$

12. The value of $\sin^2\left(\cos^{-1}\frac{1}{2}\right) + \cos^2\left(\sin^{-1}\frac{1}{3}\right)$ is

(A) $\frac{17}{36}$ (B) $\frac{59}{36}$

(C) $\frac{36}{59}$ (D) none of these

Solution: (B)

$$\sin^2\left(\cos^{-1}\frac{1}{2}\right) + \cos^2\left(\sin^{-1}\frac{1}{3}\right)$$

$$= 1 - \cos^2\left(\cos^{-1}\frac{1}{2}\right) + 1 - \sin^2\left(\sin^{-1}\frac{1}{3}\right)$$

$$= 1 - \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{3}\right)^2 = 2 - \frac{1}{4} - \frac{1}{9} = \mathbf{\frac{59}{36}}$$

13. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$ then x is equal to

(A) 1 (B) 0

(C) $\frac{4}{5}$ (D) $\frac{1}{5}$

Solution: (D)

We have, $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$

$$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}1$$

$$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}\frac{1}{5}$$

$$\Rightarrow \cos^{-1}x = \cos^{-1}\frac{1}{5} \quad \therefore x = \mathbf{\frac{1}{5}}$$

14. $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}x\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}x\right)$, $x \ne 0$ is equal to

(A) x (B) $2x$

(C) $\frac{2}{x}$ (D) none of these

Solution: (C)
Let $\cos^{-1}x = \theta$, then the given expression

$$= \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{2}} + \frac{1+\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}} = \frac{2\left(1+\tan^2\frac{\theta}{2}\right)}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{2}{\cos\theta} = \frac{2}{x}$$

15. The value of

$$\sin^{-1}\left\{\cot\left[\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\left(\frac{\sqrt{12}}{4}\right) + \sec^{-1}\sqrt{2}\right]\right\}$$

is

(A) 0 (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{2}$

Solution: (A)
Given expression

$$= \sin^{-1}\cot\left\{\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right\}$$

$$= \sin^{-1}\cot\left\{\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}\right\}$$

$$= \sin^{-1}\left\{\cot\frac{\pi}{2}\right\} = \sin^{-1}0 = 0$$

16. The value of $\tan\left[\cos^{-1}\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right]$ is

(A) $\frac{2}{3\sqrt{5}}$ (B) $\frac{2}{3}$

(C) $\frac{1}{\sqrt{5}}$ (D) $\frac{4}{\sqrt{5}}$

Solution: (A)

$$\tan\left\{\cos^{-1}\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right\}$$

$$= \tan\left\{\pi - \cos^{-1}\left(\frac{2}{7}\right) - \frac{\pi}{2}\right\}$$

$$= \tan\left\{\frac{\pi}{2} - \cos^{-1}\left(\frac{2}{7}\right)\right\} = \tan\left\{\sin^{-1}\left(\frac{2}{7}\right)\right\}$$

$$= \tan\tan^{-1}\left(\frac{2}{3\sqrt{5}}\right) = \frac{2}{3\sqrt{5}}$$

17. If $x \in \left[\frac{\pi}{2}, \pi\right]$ then $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) =$

(A) $\frac{x-\pi}{2}$ (B) $\frac{\pi-x}{2}$

(C) $\frac{3\pi-x}{2}$ (D) none of these

Solution: (B)

$$\sqrt{1+\sin x} = \sin\frac{x}{2} + \cos\frac{x}{2}$$

$$\sqrt{1-\sin x}\quad x = \sin\frac{x}{2} - \cos\frac{x}{2}$$

$$\left(\text{for } \frac{\pi}{4} \le \frac{x}{2} \le \frac{\pi}{2} \quad \sin\frac{x}{2} \ge \cos\frac{x}{2}\right)$$

$\therefore$ the given expression

$$= \cot^{-1}\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right)$$

$$= \cot^{-1}\left(\tan\frac{x}{2}\right) = \cot^{-1}\cot\left(\frac{\pi}{2}-\frac{x}{2}\right) = \frac{\pi-x}{2}$$

18. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$ then the value of

$$x^{100} + y^{100} + z^{100} - \frac{3}{x^{101}+y^{101}+z^{101}} \text{ is}$$

(A) 0 (B) 1
(C) 2 (D) 3

Solution: (C)
We have $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z =$
$\frac{3\pi}{2}$ it is possible only when

$$\sin^{-1}x = \frac{\pi}{2} \Rightarrow x = 1 \quad \because \quad \sin^{-1}x \le \frac{\pi}{2}$$

$$\sin^{-1}y = \frac{\pi}{2} \Rightarrow y = 1; \quad \sin^{-1}z = \frac{\pi}{2} \Rightarrow z = 1$$

$$\therefore \quad x^{100} + y^{100} + z^{100} - \frac{3}{x^{101}+y^{101}+z^{101}}$$

$$= 1+1+1-\frac{3}{3} = 3 - 1 = 2$$

19. $\cos^{-1}[\cos\{2\cot^{-1}(\sqrt{2}-1)\}]$ is equal to

(A) $\sqrt{2}-1$ (B) $1-\sqrt{2}$

(C) $\frac{\pi}{4}$ (D) $\frac{3\pi}{4}$

Solution: (D)

Let $\cot^{-1}(\sqrt{2}-1) = 0$

$\Rightarrow \cot\theta = \sqrt{2}-1$

Now, $\cos^{-1}(\cos 2\theta) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}\left(\frac{\cot^2\theta - 1}{\cot^2\theta + 1}\right) = \cos^{-1}\left(\frac{2-2\sqrt{2}}{4-2\sqrt{2}}\right)$

$\therefore \quad \cos^{-1}\cos(2\theta) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

20. If $x \in (7\pi, 8\pi)$, then $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} =$

(A) $-\frac{x}{2}$ (B) $\frac{x}{2}$

(C) $4\pi - \frac{x}{2}$ (D) none of these

Solution: (C)

$\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} = \tan^{-1}\left(-\tan\frac{x}{2}\right)$

$= \tan^{-1}\tan\left(4\pi - \frac{x}{2}\right) = 4\pi - \frac{x}{2}$

$\left(\because \; 7\pi < x < 8\pi \Rightarrow \frac{7\pi}{2} < \frac{x}{2} < 4\pi, \text{ so } \tan\frac{x}{2} < 0\right)$

21. If $3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{1}{2}\right)$ then x is equal to

(A) 1 (B) 3

(C) $\sqrt{3}$ (D) none of these

Solution: (B)

Given equation can be written as

$$3\tan^{-1}(2-\sqrt{3}) - \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{3\pi}{12} - \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{2}\right)$$

Taking tangent on both sides, we get

$$\frac{1}{x} = \frac{1-\frac{1}{2}}{1+\frac{1}{2}} \Rightarrow x = 3$$

22. The positive integral solution of

$\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$ is

(A) $x = 1, y = 2;\; x = 2, y = 7$
(B) $x = 1, y = 3;\; x = 2, y = 4$
(C) $x = 0, y = 0;\; x = 3, y = 4$
(D) none of these

Solution: (A)

Converting cos and sin into tan, we have,

$$\tan^{-1}x + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{3}{1}\right)$$

$$\Rightarrow \tan^{-1}\left[\frac{x+\left(\frac{1}{y}\right)}{1-x\left(\frac{1}{y}\right)}\right] = \tan^{-1}\left(\frac{3}{1}\right) \Rightarrow \frac{xy+1}{y-x} = 3$$

$\therefore$ The equality is true for
$x = 1, y = 2$ and for $x = 2, y = 7$

SOLUTIONS OF BASIC INVERSE TRIGONOMETRIC INEQUALITIES

Table 27.3

Inequality	Solution
$\sin^{-1}x > \alpha,\; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\sin\alpha < x < 1$
$\sin^{-1}x < \alpha,\; -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$	$-1 \le x < \sin\alpha$
$\cos^{-1}x > \alpha,\; 0 < \alpha < \pi$	$-1 \le x < \cos\alpha$
$\cos^{-1}x < \alpha,\; -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\cos\alpha < x \le 1$
$\tan^{-1}x > \alpha,\; \sqrt{a^2 - x^2}$	$\tan\alpha < x < \infty$
$\tan^{-1}x < \alpha,\; \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$-\infty < x < \tan\alpha$
$\cot^{-1}x > \alpha,\; 0 < \alpha < \pi$	$-\infty < x < \cot\alpha$
$\cot^{-1}x < \alpha,\; 0 < \alpha < \pi$	$\cot\alpha < x < \infty$

SOLVED EXAMPLES

23. If $\tan^{-1}\frac{x}{\pi} < \frac{\pi}{3}$, $x \in N$, then the maximum value of x is

(A) 2 (B) 5
(C) 7 (D) none of these

Solution: (B)

We have, $\tan^{-1}\frac{x}{\pi} < \frac{\pi}{3}$

$\Rightarrow \tan\left(\tan^{-1}\frac{x}{\pi}\right) < \tan\frac{\pi}{3}$

$\Rightarrow \frac{x}{\pi} < \sqrt{3} \Rightarrow x < \sqrt{3}\,\pi = 5.5$ (approx.)

$\therefore$ The maximum value of x is **5.**

24. If $\alpha = \sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{3}$ and

$\beta = \cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{1}{3}$, then

(A) $\alpha > \beta$ (B) $\alpha = \beta$
(C) $\alpha < \beta$ (D) $\alpha + \beta = 2\pi$

Solution: (C)

From given equations, it can be seen that
$\alpha + \beta = \pi$

Since, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \;\forall\; x$

Also, $\alpha = \frac{\pi}{3} + \sin^{-1}\frac{1}{3} < \frac{\pi}{3} + \sin^{-1}\frac{1}{2}$

as $\sin\theta$ is increasing in $\left[0, \frac{\pi}{2}\right]$

$\therefore \alpha < \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2} \Rightarrow \beta > \frac{\pi}{2} > \alpha \Rightarrow \alpha < \beta$

25. If $[\tan^{-1}x]^2 - 2[\tan^{-1}x] + 1 \le 0$, where [.] denotes greatest integer $\le x$, x belongs to

(A) $[\tan 1, \tan 2]$ (B) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

(C) $[\tan 1, \infty)$ (D) none of these

Solution: (A)

The given inequation is

$[(\tan^{-1}x) - 1]^2 \le 0$

$\Rightarrow (\tan^{-1}x) = 1$ as $[(\tan^{-1}x) - 1]^2 \nless 0$

$\Rightarrow 1 \le \tan^{-1}x < 2$

$\Rightarrow x \in (\tan 1, \tan 2)$

SOME USEFUL SUBSTITUTIONS

Table 27.4

Expression	Substitution
$\sqrt{a^2 + x^2}$	Put $x = a\sin\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\sqrt{x^2 - a^2}$	Put $x = a\tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\sqrt{x^2 - a^2}$	Put $x = a\sec\theta$, $(0, \pi)$
$\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$	Put $x^2 = a^2\cos 2\theta$, $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

SOLVED EXAMPLES

26. $\sin\{\cot^{-1}[\cos(\tan^{-1}x)]\} =$

(A) $\sqrt{\frac{1+x^2}{2+x^2}}$ (B) $\sqrt{\frac{1-x^2}{2+x^2}}$

(C) $\sqrt{\frac{1+x^2}{2-x^2}}$ (D) $\sqrt{\frac{2+x^2}{1+x^2}}$

Solution: (A)

Let $\tan^{-1}x = \alpha \Rightarrow \tan\alpha = x$

So, $\cos\alpha = \frac{1}{\sqrt{(1+x^2)}}$

$\therefore \sin\{\cot^{-1}[\cos(\tan^{-1}x)]\}$

$= \sin\left(\cot^{-1}\frac{1}{\sqrt{(1+x^2)}}\right)$

Let $\cot^{-1}\frac{1}{\sqrt{(1+x^2)}} = \beta \Rightarrow \cot\beta = \frac{1}{\sqrt{(1+x^2)}}$

$\therefore \sin\beta = \frac{\sqrt{(1+x^2)}}{\sqrt{(2+x^2)}}$

$\therefore \sin\beta = \sin\{\cot^{-1}[\cos(\tan^{-1}x)]\} = \sqrt{\frac{\mathbf{1+x^2}}{\mathbf{2+x^2}}}$.

27. If $\tan^{-1}\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} = \alpha$, then $x^2 =$

(A) $\cos 2\alpha$ (B) $\sin 2\alpha$
(B) $\tan 2\alpha$ (D) $\cot 2\alpha$

Solution: (B)

Since, $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)=\alpha$

$\Rightarrow \dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=\dfrac{\tan\alpha}{1}$

Using componendo and dividendo, we get

$\dfrac{2\sqrt{1+x^2}}{2\sqrt{1-x^2}}=\dfrac{1+\tan\alpha}{1-\tan\alpha}=\tan\left(\dfrac{\pi}{4}+\alpha\right)$

$\Rightarrow \dfrac{1+x^2}{1-x^2}=\dfrac{\tan^2\left(\dfrac{\pi}{4}+\alpha\right)}{1}$

Again using componendo and dividendo, we have

$\dfrac{(1+x^2)-(1-x^2)}{(1+x^2)+(1-x^2)}=\dfrac{\tan^2(\pi/4+\alpha)-1}{\tan^2(\pi/4+\alpha)+1}$

$\Rightarrow x^2=-\cos\left(\dfrac{\pi}{2}+2\alpha\right)=\sin 2\alpha$

28. If $(a<0)$ and $x\in(-a,a)$, then $\tan^{-1}\left(\dfrac{x}{\sqrt{a^2-x^2}}\right)=$

(A) $-\sin^{-1}\left(\dfrac{x}{a}\right)$ (B) $\sin^{-1}\left(\dfrac{x}{a}\right)$

(C) $\cos^{-1}\left(\dfrac{x}{a}\right)$ (D) none of these

Solution: (A)

Put $x=a\sin\theta$, then the given expression

$=\tan^{-1}\left(\dfrac{a\sin\theta}{-a\cos\theta}\right)$ $(\because\ a<0)$

$=\tan^{-1}(-\tan\theta)=-\theta=-\sin^{-1}\dfrac{x}{a}$

29. Given $0\le x\le\dfrac{1}{2}$ then the value of

$\tan\left[\sin^{-1}\left(\dfrac{x}{\sqrt{2}}+\dfrac{\sqrt{1-x^2}}{\sqrt{2}}\right)-\sin^{-1}x\right]$ is

(A) -1 (B) 1

(C) $\dfrac{1}{\sqrt{3}}$ (D) $\sqrt{3}$

Solution: (B)

Put $x=\sin\theta$ then $\sin^{-1}\left(\dfrac{x}{\sqrt{2}}+\dfrac{\sqrt{1-x^2}}{\sqrt{2}}\right)$

$=\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\sin\theta+\dfrac{1}{\sqrt{2}}\cos\theta\right)=\sin^{-1}\sin\left(\theta+\dfrac{\pi}{4}\right)$

$=\theta+\dfrac{\pi}{4}$

$\therefore$ given expression $=\tan\left(\theta+\dfrac{\pi}{4}-\theta\right)=\tan\dfrac{\pi}{4}=1$

30. If $-1<x<0$ then $\tan^{-1}x$ equals

(A) $\pi-\cos^{-1}(\sqrt{1-x^2})$

(B) $\sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$

(C) $-\cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

(D) $\operatorname{cosec}^{-1}x$

Solution: (B)

$\because -1<x<0,\quad \therefore\ -\dfrac{\pi}{4}<\tan^{-1}x<0$

Let $\tan^{-1}x=\alpha\Rightarrow -\dfrac{\pi}{4}<\alpha<0$

$\therefore\ \tan\alpha=x,\ -\dfrac{\pi}{4}<\alpha<0$

$\therefore\ \sin\alpha=\dfrac{x}{\sqrt{1+x^2}}\Rightarrow\alpha=\sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$

and $\cos\alpha=\dfrac{1}{\sqrt{1+x^2}}\Rightarrow\cos(-\alpha)=\dfrac{1}{\sqrt{1+x^2}}$,

where $0<-\alpha<\dfrac{\pi}{4}\Rightarrow\alpha=-\cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$

EXERCISES

Single Option Correct Type

1. The sum of the the series

$\cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + \ldots$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{6}$ (D) none of these.

2. If $A = \cot^{-1}\sqrt{\tan\theta} - \tan^{-1}\sqrt{\tan\theta}$, then $\tan\left(\frac{\pi}{4} - \frac{A}{2}\right)$ is equal to

(A) $\sqrt{\cot\theta}\ \theta$ (B) $\tan\theta$

(C) $\sqrt{\tan\theta}$ (D) none of these

3. $\cos\{\tan^{-1}[\sin(\cot^{-1}\sqrt{3})]\}$ is equal to

(A) $\frac{4}{\sqrt{5}}$ (B) $\frac{2}{\sqrt{5}}$

(C) $-\frac{2}{\sqrt{5}}$ (D) none of these

4. The equation $\cos^{-1}\sqrt{\frac{\alpha-\theta}{\alpha-\beta}} = \sin^{-1}\sqrt{\frac{\theta-\beta}{\alpha-\beta}}$ is valid for

(A) $\alpha > \beta$ and θ takes any value

(B) $\alpha = \theta = \beta$

(C) $\alpha > \theta > \beta$ or $\alpha < \theta < \beta$

(D) none of these.

5. The value of $\sin\left(4\tan^{-1}\frac{1}{3}\right) - \cos\left(2\tan^{-1}\frac{1}{7}\right)$ is

(A) $\frac{4}{7}$ (B) 0

(C) $\frac{7}{8}$ (D) none of these

6. If $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}\theta] = 1$, where [.] denotes the greatest integer function, the θ lies in the interval

(A) [tan sin cos1, sin tan cos sin1]

(B) [sin tan cos1, tan sin cos sin1]

(C) [tan sin cos1, tan sin cos sin1]

(D) none of these.

7. If $\sum_{i=1}^{2n}\sin^{-1}x_i = n\pi$, then $\sum_{i=1}^{2n}x_i$ is equal to

(A) n (B) $2n$

(C) $\frac{n(n+1)}{2}$ (D) none of these

8. If $\cos^{-1}\left(\frac{n}{2\pi}\right) > \frac{2\pi}{3}$ then the minimum and the maximum values of integer n are respectively.

(A) – 6 and – 4 (B) 4 and 6

(C) – 6 and – 3 (D) none of these

9. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then $\frac{\sum_{k=1}^{2}(x^{100k} + y^{106k})}{\sum x^{207}\ y^{207}}$ is

(A) $\frac{1}{3}$ (B) $\frac{4}{3}$

(C) $\frac{2}{3}$ (D) none of these

10. If $\tan^{-1}\frac{1}{1+2} + \tan^{-1}\frac{1}{1+2.3} + \tan^{-1}\frac{1}{1+3.4} + \ldots + \tan^{-1}\frac{1}{1+n(n+1)}$ $= \tan^{-1}x$, then x is equal to

(A) $\frac{n}{n+2}$ (B) $\frac{n}{n+1}$

(C) $\frac{n-1}{n+2}$ (D) none of these

11. The value of

$\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$

(C) π (D) 0

12. The number of real solutions of the equations $\tan^{-1}\sqrt{x^2 - 3x + 2} + \cos^{-1}\sqrt{4x - x^2 - 3} = \pi$ is

(A) one (B) two

(C) zero (D) infinite

13. $\sum_{m=1}^{n}\tan^{-1}\frac{2m}{m^4 + m^2 + 2} =$

(A) $\tan^{-1}(n^2+n+1)$ (B) $\tan^{-1}(n^2-n+1)$

(C) $\tan^{-1}\dfrac{n^2+n}{n^2+n+2}$ (D) none of these

14. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\dfrac{3\pi}{2}$ and $f(1)=1, f(p+q)=f(p).f(q)\ \forall\, p, q \in R$

then, $x^{f(1)}+y^{f(2)}+z^{f(3)}-\dfrac{x+y+z}{x^{f(1)}+y^{f(2)}+z^{f(3)}}=$

(A) 0 (B) 1
(C) 2 (D) 3

15. The number of solutions of the equation

$2\sin^{-1}\sqrt{x^2-x+1}+\cos^{-1}(\sqrt{x^2-x})=\dfrac{3\pi}{2}$ is

(A) 0 (B) infinite
(C) 2 (D) 4

16. $\cot^{-1}(\sqrt{\cos\alpha})-\tan^{-1}(\sqrt{\cos\alpha})=x$, then $\sin x=$

(A) $\tan^2\left(\dfrac{\alpha}{2}\right)$ (B) $\cot^2\left(\dfrac{\alpha}{2}\right)$

(C) $\tan\alpha$ (D) $\cot\left(\dfrac{\alpha}{2}\right)$

17. If a, b are positive quantities and if

$a_1=\dfrac{a+b}{2}, b_1=\sqrt{a_1 b}, a_2=\dfrac{a_1+b_1}{2}, b_2=\sqrt{a_2 b_1}$

and so on, then

(A) $a_\infty=\dfrac{\sqrt{b^2-a^2}}{\cos^{-1}\left(\dfrac{a}{b}\right)}$ (B) $b_\infty=\dfrac{\sqrt{b^2-a^2}}{\cos^{-1}\left(\dfrac{a}{b}\right)}$

(C) $b_\infty=\dfrac{\sqrt{a^2-b^2}}{\cos^{-1}\left(\dfrac{b}{a}\right)}$ (D) none of these

18. If $\cos^{-1}\sqrt{p}+\cos^{-1}\sqrt{1-p}+\cos^{-1}\sqrt{1-q}=\dfrac{3\pi}{2}$, then the value of q is

(A) 1 (B) $\dfrac{1}{\sqrt{2}}$

(C) $\dfrac{1}{3}$ (D) $\dfrac{1}{2}$

19. The value of x for which

$\sin[\cot^{-1}(1+x)]=\cos(\tan^{-1}x)$ is

(A) $\dfrac{1}{2}$ (B) 1

(C) 0 (D) $-\dfrac{1}{2}$

20. If $\cos^{-1}\dfrac{x}{2}+\cos^{-1}\dfrac{y}{3}=\theta$, then

$9x^2-12xy\cos\theta+4y^2=$

(A) 36 (B) $-36\sin^2\theta$
(C) $36\sin^2\theta$ (D) $36\cos^2\theta$

21. If $\theta=\tan^{-1}\alpha$, $\varphi=\tan^{-1}b$ and $ab=-1$ then $\theta-\varphi=$

(A) 0 (B) $\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{2}$ (D) none of these

22. The number of real solutions of

$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\dfrac{\pi}{2}$ is

(A) 0 (B) 1
(C) 2 (D) infinite

23. The number of solutions of

$\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$ is

(A) 1 (B) 0
(C) 2 (D) 4

24. The domain of $\sin^{-1}[x]$ is given by

(A) $[-1, 1]$ (B) $[-1, 2)$
(C) $\{-1, 0, 1\}$ (D) none of these

25. If $A=\tan^{-1}\left(\dfrac{x\sqrt{3}}{2k-x}\right)$ and $B=\tan^{-1}\left(\dfrac{2x-k}{k\sqrt{3}}\right)$ then the value of A – B is

(A) 0° (B) 45°
(C) 60° (D) 30°

26. If $\tan^{-1}y=4\tan^{-1}x$, then $1/y$ is zero for

(A) $x=1\pm\sqrt{2}$ (B) $x=\sqrt{2}\pm\sqrt{3}$

(C) $x=3\pm2\sqrt{2}$ (D) all values of x

27. $\cos^{-1}\sqrt{\dfrac{a-x}{a-b}}=\sin^{-1}\sqrt{\dfrac{x-b}{a-b}}$ is possible if

(A) $a>x>b$ or $a<x<b$
(B) $a=x=b$
(C) $a>b$ and x takes any value
(D) $a<b$ and x takes any value

28. The greater of the two angles

$A=2\tan^{-1}(2\sqrt{2}-1)$ and $B=3\sin^{-1}\dfrac{1}{3}+\sin^{-1}\dfrac{3}{5}$ is

(A) B (B) A
(C) C (D) none of these

29. If $x = \sin(2\tan^{-1}2)$, $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$, then

(A) $x = 1 - y$ (B) $x^2 = 1 - y$
(C) $x^2 = 1 + y$ (D) $y^2 = 1 - x$

30. Sum of infinite terms of the series

$\cot^{-1}\left(1^2 + \frac{3}{4}\right) + \cot^{-1}\left(2^2 + \frac{3}{4}\right) + \cot^{-1}\left(3^2 + \frac{3}{4}\right) + \ldots$ is

(A) $\frac{\pi}{4}$ (B) $\tan^{-1}2$
(C) $\tan^{-1}3$ (D) none of these

31. The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is

(A) 6/17 (B) 3/17
(C) 4/17 (D) 5/17

32. Solution of the equation $\tan(\cos^{-1}x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$ is

(A) $x = \pm\frac{\sqrt{7}}{3}$ (B) $x = \pm\frac{\sqrt{5}}{3}$
(C) $x = \pm\frac{3\sqrt{5}}{2}$ (D) none of these

33. $\cos\left[\tan^{-1}\left[\sin(\cot^{-1}x)\right]\right] =$

(A) $\sqrt{\frac{x^2+2}{x^2+3}}$ (B) $\sqrt{\frac{x^2+2}{x^2+1}}$
(C) $\sqrt{\frac{x^2+1}{x^2+2}}$ (D) none of these

34. If $\sum_{i=1}^{2n}\cos^{-1}x_i = 0$, then $\sum_{i=1}^{2n}x_i$ is

(A) n (B) $2n$
(C) $\frac{n(n+1)}{2}$ (D) none of these

35. If $\alpha = \sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{3}$ and

$\beta = \cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{1}{3}$, then

(A) $\alpha > \beta$ (B) $\alpha = \beta$
(C) $\alpha < \beta$ (D) $\alpha + \beta = 2\pi$

36. If $-1 < x < 0$ then $\tan^{-1}x$ equals

(A) $\pi - \cos^{-1}(\sqrt{1-x^2})$

(B) $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

(C) $-\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

(D) $\operatorname{cosec}^{-1}x$

37. The sum of the series

$\cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + \ldots$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$
(C) $\frac{\neq}{6}$ (D) none of these.

38. If $A = \cot^{-1}\sqrt{\tan\theta} - \tan^{-1}\sqrt{\tan\theta}$, then $\tan\left(\frac{\pi}{4} - \frac{A}{2}\right)$ is equal to

(A) $\sqrt{\cot\theta}$ (B) $\tan\theta$
(C) $\sqrt{\tan\theta}$ (D) none of these

39. If $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}\theta] = 1$, where [.] denotes the greatest integer function, the θ lies in the interval

(A) [tan sin cos1, sin tan cos sin1]
(B) [sin tan cos1, tan sin cos sin1]
(C) [tan sin cos1, tan sin cos sin1]
(D) none of these

40. If $\sum_{i=1}^{2n}\sin^{-1}x_i = n\pi$, then $\sum_{i=1}^{2n}x_i$ is equal to

(A) n (B) $2n$
(C) $\frac{n(n+1)}{2}$ (D) none of these

41. If $\cos^{-1}\left(\frac{n}{2\pi}\right) > \frac{2\pi}{3}$ then the minimum and the maximum values of integer n are respectively

(A) – 6 and – 4 (B) 4 and 6
(C) – 6 and – 3 (D) none of these

42. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then

$\frac{\sum_{k=1}^{2}(x^{100k} + y^{106k})}{\sum x^{207}\,y^{207}}$ is

(A) $\frac{1}{3}$ (B) $\frac{4}{3}$
(C) $\frac{2}{3}$ (D) none of these

43. The number of real solutions of the equations

$\tan^{-1}\sqrt{x^2 - 3x + 2} + \cos^{-1}\sqrt{4x - x^2 - 3} = \pi$ is

(A) one (B) two
(C) zero (D) infinite

44. $\sum_{m=1}^{n} \tan^{-1} \frac{2m}{m^4 + m^2 + 2} =$

(A) $\tan^{-1}(n^2 + n + 1)$ (B) $\tan^{-1}(n^2 - n + 1)$

(C) $\tan^{-1} \frac{n^2 + n}{n^2 + n + 2}$ (D) none of these

45. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$ and $f(1) = 1, f(p + q) = f(p).f(q)$ v– $p, q \in R$

then, $x^{f(1)} + y^{f(2)} + z^{f(3)} - \frac{x + y + z}{x^{f(1)} + y^{f(2)} + z^{f(3)}} =$

(A) 0 (B) 1
(C) 2 (D) 3

46. The value of x for which
$\sin(\cot^{-1}(1 + x)) = \cos(\tan^{-1} x)$ is

(A) $\frac{1}{2}$ (B) 1

(C) 0 (D) $-\frac{1}{2}$

47. If $\tan^{-1} y = 4 \tan^{-1}x$, then $1/y$ is zero for

(A) $x = 1 \pm \sqrt{2}$ (B) $x = \sqrt{2} \pm \sqrt{3}$

(C) $x = 3 \pm 2\sqrt{2}$ (D) all values of x

48. The greater of the two angles

$A = 2\tan^{-1}(2\sqrt{2} - 1)$ and $B = 3\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{3}{5}$ is

(A) B (B) A
(C) C (D) none of these

49. If $x = \sin(2\tan^{-1}2)$, $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$, then

(A) $x = 1 - y$ (B) $x^2 = 1 - y$
(C) $x^2 = 1 + y$ (D) $y^2 = 1 - x$

50. Sum of infinite terms of the series

$\cot^{-1}\left(1^2 + \frac{3}{4}\right) + \cot^{-1}\left(2^2 + \frac{3}{4}\right) + \cot^{-1}\left(3^2 + \frac{3}{4}\right) + \ldots$ is

(A) $\frac{\pi}{4}$ (B) $\tan^{-1}2$

(C) $\tan^{-1} 3$ (D) none of these

51. If $r = x + y + z$, then

$\tan^{-1}\sqrt{\frac{xr}{yz}} + \tan^{-1}\sqrt{\frac{yr}{zx}} + \tan^{-1}\sqrt{\frac{zr}{xy}} =$

(A) π (B) 2π

(C) $\frac{\pi}{2}$ (D) none of these

52. If $u = \cot^{-1}\sqrt{\cos 2\theta} - \tan^{-1}\sqrt{\cos 2\theta}$, then $\sin u =$

(A) $\sin^2\theta$ (B) $\cos^2\theta$
(C) $\tan^2\theta$ (D) $\tan^2 2\theta$

53. $2\tan^{-1}\left(\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right) =$

(A) $\cos^{-1}\left(\frac{\cos\theta + \cos\phi}{1 + \cos\theta\cos\phi}\right)$

(B) $\cos^{-1}\left(\frac{\cos\theta - \cos\phi}{1 + \cos\theta\cos\phi}\right)$

(C) $\cos^{-1}\left(\frac{\cos\theta + \cos\phi}{1 - \cos\theta\cos\phi}\right)$

(D) none of these

54. Solution of the equation $\sin^{-1}x + \sin^{-1} 2x = \frac{\pi}{3}$ is

(A) $x = \frac{\sqrt{3}}{2\sqrt{7}}$ (B) $x = -\frac{\sqrt{3}}{2\sqrt{7}}$

(C) $x = \pm\frac{1}{\sqrt{2}}$ (D) none of these

55. If $f(x) = 2\tan^{-1}x + \sin^{-1}\frac{2x}{1 + x^2}$, then for $x \ge 1$, $f(x)$ is equal to

(A) π (B) 2π

(C) $\frac{\pi}{2}$ (D) none of these

56. If θ and φ are the roots of the equation $8x^2 + 22x + 5 = 0$, then

(A) both $\sin^{-1}\theta$ and $\sin^{-1}\varphi$ are real
(B) both $\sec^{-1}\theta$ and $\sec^{-1}\varphi$ are real
(C) both $\tan^{-1}\theta$ and $\tan^{-1}\varphi$ are real
(D) none of these

57. The positive integral solution of the equation

$\tan^{-1}x + \cos^{-1}\left(\frac{y}{\sqrt{1 + y^2}}\right) = \sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ is

(A) $x = 1, y = 2$ (B) $x = 2, y = 1$
(C) $x = 3, y = 2$ (D) $x = -2, y = -1$

58. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{18} + \ldots + \tan^{-1}\left(\frac{1}{n^2 + n + 1}\right) + \ldots$ to ∞ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$

(C) $\frac{2\pi}{3}$ (D) 0.

59. The set of values of x for which the identity $\cos^{-1}x + \cos^{-1}\left(\frac{\pi}{2}+\frac{1}{2}\sqrt{3-3x^2}\right) = \frac{\pi}{3}$ holds good, is

(A) $\left[\frac{1}{2},1\right]$ (B) $\left(\frac{1}{2},1\right)$

(C) (0, 1) (D) [0, 1]

60. If $ax + b\,(\sec(\tan^{-1}x)) = c$ and $ay + b\,(\sec(\tan^{-1}y)) = c$, then $\frac{x+y}{1-xy} =$

(A) $\frac{ac}{a^2-c^2}$ (B) $\frac{2ac}{a^2-c^2}$

(C) $\frac{2ac}{a^2+c^2}$ (D) $\frac{ac}{a^2+c^2}$

61. $\operatorname{cosec}^{-1}\sqrt{5} + \operatorname{cosec}^{-1}\sqrt{65} + \operatorname{cosec}^{-1}\sqrt{325} + \ldots\infty =$

(A) 0 (B) π

(C) $\frac{\pi}{2}$ (D) none of these

62. $\cot^{-1}\left(2^2+\frac{1}{2}\right) + \cot^{-1}\left(2^3+\frac{1}{2^2}\right) + \cot^{-1}\left(2^4+\frac{1}{2^3}\right) + \ldots \infty =$

(A) $\tan^{-1}2$ (B) $\cot^{-1}2$

(C) $2\tan^{-1}2$ (D) $2\tan^{-1}2$

63. The set of values of x satisfying $[\tan^{-1}x] + [\cot^{-1}x] = 2$, where $[x]$ denotes the greatest integer less than or equal to x, is

(A) (cot 3, – tan 1) (B) (cot 3, cot 2)

(C) (cot 3, 0) (D) none of these

64. If $a < \frac{1}{32}$, then the number of solutions of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = a\pi^3$ is

(A) 0 (B) 1

(C) 2 (D) infinite

65. $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right) + \frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$ is equal to

(A) $(\alpha-\beta)(\alpha^2+\beta^2)$ (B) $(\alpha+\beta)(\alpha^2-\beta^2)$

(C) $(\alpha+\beta)(\alpha^2+\beta^2)$ (D) none of these

66. $\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right) + \sin^{-1}\sum_{i=1}^{2n}\cos^{-1}x_i + \ldots + \sin\left(\frac{\sqrt{n}-\sqrt{n-1}}{n\sqrt{n+1}}\right) =$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$ (D) π

67. $\cot^{-1}(2.1^2) + \cot^{-1}(2.2^2) + \cot^{-1}(2.3^2) + \ldots\ldots =$

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{2}$ (D) none of these

More than One Option Correct Type

68. If $(\sin^{-1}x)^2 + (\cos^{-1}x)^2 = \frac{5\pi^2}{8}$, then x is equal to

(A) 1 (B) – 1

(C) $\frac{1}{\sqrt{2}}$ (D) $-\frac{1}{\sqrt{2}}$

69. Solution of the equation

$\sin\left[2\cos^{-1}\{\cot(2\tan^{-1}x)\}\right] = 0$ is

(A) $x = \pm 1$ (B) $1 \pm \sqrt{2}$

(C) $-(1 \pm \sqrt{3})$ (D) $1 \pm \sqrt{2}$

70. If $\tan^{-1}y = 4\tan^{-1}x$, then y is finite if

(A) $x^2 \neq 3 + 2\sqrt{2}$ (B) $x^2 \neq 3 - 2\sqrt{2}$

(C) $x^4 \neq 6x^2 - 1$ (D) $x^4 \neq 6x^2 + 1$

71. If $f(x) = \sin^{-1}x + \cos^{-1}x$, then $\frac{\neq}{2}$ is equal to

(A) $f\left(-\frac{1}{2}\right)$

(B) $f(k^2 - 2k + 3),\ k \in R$

(C) $f\left(\frac{1}{1+k^2}\right),\ k \in R$

(D) $f(-2)$

72. If $x = \text{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ and $y = \sec(\cot^{-1}(\sin(\tan^{-1}(\text{cosec}(\cos^{-1}a)))))$, where $a \in [0, 1]$. Then

(A) $x > y$ (B) $x = y$
(C) $y^2 + a^2 = 3$ (D) $x^2 + a^2 = 3$

73. If $\cos^{-1}x + (\sin^{-1}y)^2 = \frac{p\pi^2}{4}$ and $(\cos^{-1}x)(\sin^{-1}y)^2 = \frac{\pi^2}{16}$, then

(A) $0 \le p \le \frac{4}{\pi} + 1$
(B) $p = 2$ is the only integral value of p
(C) $p = 0, 1, 2$ (integral values).
(D) $p = 1$ is the only integral value of p

Match the Column Type

74.

I. $\cot^{-1} 9 + \text{cosec}^{-1} \frac{\sqrt{41}}{4} =$	(A) $\frac{3\pi}{4}$
II. $\sin^{-1}(\cos(\sin^{-1}x)) + \cos^{-1}(\sin(\cos^{-1}x)) =$	(B) π
III. $\cos^{-1}\left[\cos\left\{2\cot^{-1}(\sqrt{2}-1)\right\}\right] =$	(C) $\frac{\pi}{4}$
IV. $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} =$	(D) $\frac{\pi}{2}$

75.

I. The value of $\cos(2\cos^{-1}x + \sin^{-1}x)$ at $x = \frac{1}{5}$ is	(A) $\frac{2}{3\sqrt{5}}$
II. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then $x =$	(B) $-\frac{2\sqrt{6}}{5}$
III. The value of $\tan\left\{\cos^{-1}\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right\}$ is	(C) $\frac{3\pi}{4}$
IV. If $\sqrt{p} + \cos^{-1}\sqrt{1-p} + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$, then $q =$	(D) $\frac{1}{5}$

Assertion-Reason Type

Instructions: *In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:*

(A) Assertion(A) is True and Reason(R) is True; Reason(R) ***is*** a correct explanation for Assertion(A)
(B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)
(C) Assertion(A) is True, Reason(R) is False
(D) Assertion(A) is False, Reason(R) is True

76. Assertion: If $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$, then $\sin x = \tan^2\frac{\alpha}{2}$

Reason: $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

77. Assertion: If a, b are positive quantities and if

$$a_1 = \frac{a+b}{2},\ b_1 = \sqrt{a_1 b},\ a_2 = \frac{a_1+b_1}{2},\ b_2 = \sqrt{a_2 b_1}$$

and so on, then $b_\infty = \frac{\sqrt{b^2-a^2}}{\cos^{-1}\left(\frac{a}{b}\right)}$.

Reason: $b_n = b\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\ldots\cos\frac{\theta}{2^n}$

78. Assertion: If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \theta$, then $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

Reason: $\cos^{-1}x + \cos^{-1}y =$

$\cos^{-1}\left(xy - \sqrt{1-x^2}\ \sqrt{1-y^2}\right)$

79. Assertion: $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$

Reason: $\sin^{-1}x + \sin^{-1}y$

$= \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

Previous Year's Questions

80. 10. $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$, then sin x is equal to: **[2002]**

(A) $\tan^2\left(\frac{\alpha}{2}\right)$ (B) $\cot^2\left(\frac{\alpha}{2}\right)$

(C) tan α (D) $\cot\left(\frac{\alpha}{2}\right)$

81. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$ is equal to: **[2002]**

(A) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$

(C) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (D) $\tan^{-1}\left(\frac{1}{2}\right)$

82. The trigonometric equation $\sin^{-1} x = 2\sin^{-1} a$, has a solution for **[2003]**

(A) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$ (B) all real values of a

(C) $|a| < \frac{1}{2}$ (D) $|a| \le \frac{1}{\sqrt{2}}$

83. If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$ then a value of x is

(A) 1 (B) 3

(C) 4 (D) 5

84. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in **[2007]**

(A) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (B) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

(C) $\left(0, \frac{\pi}{2}\right)$ (D) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

85. The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is **[2008]**

(A) $\frac{6}{17}$ (B) $\frac{3}{17}$

(C) $\frac{4}{17}$ (D) $\frac{5}{17}$

86. If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then **[2013]**

(A) $2x = 3y = 6z$ (B) $6x = 3y = 2z$

(C) $6x = 4y = 3z$ (D) $x = y = z$

87. Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$, Then a value of y is : **[2015]**

(A) $\frac{3x+x^3}{1-3x^2}$ (B) $\frac{3x-x^3}{1+3x^2}$

(C) $\frac{3x+x^3}{1+3x^2}$ (D) $\frac{3x-x^3}{1-3x^2}$

ANSWER KEYS

Single Option Correct Type

1. (B)	**2.** (C)	**3.** (B)	**4.** (C)	**5.** (B)	**6.** (C)	**7.** (B)	**8.** (A)	**9.** (B)	**10.** (A)
11. (C,D)	**12.** (C)	**13.** (C)	**14.** (C)	**15.** (C)	**16.** (A)	**17.** (B)	**18.** (D)	**19.** (D)	**20.** (C)
21. (A)	**22.** (C)	**23.** (C)	**24.** (B)	**25.** (D)	**26.** (A)	**27.** (A)	**28.** (B)	**29.** (D)	**30.** (B)
31. (A)	**32.** (B)	**33.** (C)	**34.** (B)	**35.** (C)	**36.** (B)	**37.** (B)	**38.** (C)	**39.** (C)	**40.** (B)
41. (A)	**42.** (B)	**43.** (C)	**44.** (C)	**45.** (C)	**46.** (D)	**47.** (A)	**48.** (B)	**49.** (D)	**50.** (B)
51. (A)	**52.** (C)	**53.** (A)	**54.** (A)	**55.** (A)	**56.** (C)	**57.** (A)	**58.** (B)	**59.** (A)	**60.** (B)
61. (C)	**62.** (B)	**63.** (D)	**64.** (A)	**65.** (C)	**66.** (C)	**67.** (B)			

More than One Option Correct Type

68. (C), and (D) **69.** (A), (B) and (C) **70.** (A), (B) and (C) **71.** (A), and (C)
72. (B),(C) and (D) **73.** (A), and (B)

Match the Column Type

74. (A) → 3; (B) → 4; (C) → 1; (D) → 2; **75.** (A) → 2; (B) → 4; (C) → 1; (D) → 3

Assertion-Reason Type

76. (A) **77.** (A) **78.** (A) **79.** (A)

Previous Year's Questions

80. (A) **81.** (D) **82.** (D) **83.** (B) **84.** (B) **85.** (A) **86.** (D) **87.** (D)

HINTS AND SOLUTIONS

Single Option Correct Type

1. The given series

$= \cot^{-1}2(1)^2 + \cot^{2}2(2)^2 + \cot^{-1}2(3)^2 + \cot^{-1}2(4)^2 + ...$

$\therefore\ T_n = \cot^{-1}(2n^2) = \tan^{-1}\frac{1}{2n^2}$

$= \tan^{-1}\left[\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right]$

$= \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

$\therefore\ T_1 = \tan^{-1}3 - \tan^{-1}1,\quad T_2 = \tan^{-1}5 - \tan^{-1}3,$
$T_3 = \tan^{-1}7 - \tan^{-1}5$
and so on.

$\therefore\ S_n = \tan^{-1}(2n+1) - \tan^{-1}1$

$\therefore\ \lim_{n\to\infty} S_n = \tan^{-1}\infty - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

The correct options is (B)

2. Let $\sqrt{\tan\theta} = \tan\alpha$

$\therefore\ A = \cot^{-1}(\tan\alpha) - \tan^{-1}(\tan\alpha)$

$= \cot^{-1}\left[\cot\left(\frac{\pi}{2}-\alpha\right)\right] - \tan^{-1}(\tan\alpha)$

$= \frac{\pi}{2} - \alpha - \alpha = \frac{\pi}{2} - 2\alpha$

$\Rightarrow 2\alpha = \frac{\pi}{2} - A$ or $\alpha = \frac{\pi}{4} - \frac{A}{2}$

$\therefore\ \sqrt{\tan\theta} = \tan\alpha = \tan\left(\frac{\pi}{4} - \frac{A}{2}\right)$

The correct options is (C)

3. $\cos\{\tan^{-1}[\sin(\cot^{-1}\sqrt{3})]\}$

$= \cos\left[\tan^{-1}\left(\sin\frac{\pi}{6}\right)\right] = \cos\left(\tan^{-1}\frac{1}{2}\right)$

$= \cos\theta$, where $\theta = \tan^{-1}\frac{1}{2}$ $\left(\tan\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{2}{\sqrt{5}}\right)$

$= \frac{2}{\sqrt{5}}$

The correct options is (B)

4. The given equation is valid if $\frac{\alpha-\theta}{\alpha-\beta} > 0$ and $\frac{\theta-\beta}{\alpha-\beta} > 0$

$\Rightarrow$ Either $\alpha > \theta > \beta$ or $\alpha < \theta < \beta$

The correct options is (C)

5. $\sin\left(4\tan^{-1}\frac{1}{3}\right) = 2\sin\left(2\tan^{-1}\frac{1}{3}\right)\cos\left(2\tan^{-1}\frac{1}{3}\right)$

$= 2\sin\left(\tan^{-1}\frac{3}{4}\right)\cos\left(\tan^{-1}\frac{3}{4}\right)$

$\left[\because\ 2\tan^{-1}\frac{1}{3} = \tan^{-1}\left(\frac{2.\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right) = \tan^{-1}\frac{3}{4}\right]$

$= 2\,.\,\frac{3}{5}\,.\,\frac{4}{5} = \frac{24}{25}$

Also, $\cos\left(2\tan^{-1}\frac{1}{7}\right) = \cos\left(\tan^{-1}\frac{7}{24}\right) = \frac{24}{25}$

$\therefore$ The given expression = 0

The correct options is (B)

6. We have, $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}\theta] = 1$

$\Rightarrow\ 1 \le \sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}\theta \le \frac{\pi}{2}$

$\Rightarrow\ \sin 1 \le \cos^{-1}\sin^{-1}\tan^{-1}\theta \le 1$

$\Rightarrow\ \cos\sin 1 \ge \sin^{-1}\tan^{-1}\theta \ge \cos 1$

$\Rightarrow\ \sin\cos\sin 1 \ge \tan^{-1}\theta \ge \sin\cos 1$

$\Rightarrow\ \tan\sin\cos\sin 1 \ge \theta \ge \tan\sin\cos 1$.

The correct options is (C)

7. We have, $\sum_{i=1}^{2n}\sin^{-1}x_i = n\pi$

$\Rightarrow\ \sin^{-1}x_1 + \sin^{-1}x_2 + \ldots + \sin^{-1}x_{2n} = n\pi$

Let $\sin^{-1}x_1 = \alpha_1, \sin^{-1}x_2 = \alpha_2, \ldots, \sin^{-1}x_n = \alpha_n$

$\therefore\ \alpha_1 + \alpha_2 + \ldots + \alpha_{2n} = n\pi$ ....(1)

$\Rightarrow x_1 = \sin\alpha_1, x_2 = \sin\alpha_2, \ldots, x_{2n} = \sin\alpha_{2n}$

$\therefore\ \sum_{i=1}^{2n} x_i = x_1 + x_2 + \ldots + x_{2n}$

$= \sin\alpha_1 + \sin\alpha_2 + \ldots + \sin\alpha_{2n}$

Clearly from (1), $\alpha_1 = \frac{\pi}{2}, \forall\ i = 1, 2, \ldots, 2n$

$\therefore\ \sum_{i=1}^{2n} x_i = \underbrace{1+1+\ldots+1}_{2n \text{ times}} = 2n$

The correct options is (B)

8. We have,

$\frac{2\pi}{3} < \cos^{-1}\left(\frac{n}{2\pi}\right) < \pi\ \Rightarrow\ \cos\frac{2\pi}{3} > \frac{n}{2\pi} > \cos\pi$

$\Rightarrow -1 < \frac{n}{2\pi} < -\frac{1}{2}\ \Rightarrow\ -2\pi < n < -\pi$

$\Rightarrow\ -6 < n < -4$.

The correct options is (A)

9. We have, $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$.

It is possible only when

$\sin^{-1}x = \frac{\pi}{2}$, $\sin^{-1}y = \frac{\pi}{2}$ and $\sin^{-1}z = \frac{\pi}{2}$

$\Rightarrow x = 1, y = 1$ and $z = 1$

$\therefore\ \frac{\sum_{k=1}^{2}(x^{100k} + y^{106k})}{\sum x^{207}y^{207}} = \frac{(x^{100} + y^{106}) + (x^{200} + y^{212})}{x^{207}y^{207} + y^{207}z^{207} + z^{207}x^{207}}$

$= \frac{1+1+1+1}{1+1+1} = \frac{4}{3}$

The correct options is (B)

10. We have,

L.H.S. $= \tan^{-1}\frac{2-1}{1+2.1} + \tan^{-1}\frac{3-2}{1+3.2} + \tan^{-1}\frac{4-3}{1+4.3}$

$+, \ldots, + \tan^{-1}\frac{n+1-n}{1+(n+1)n}$

$= (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + (\tan^{-1}4 - \tan^{-1}3) +, \ldots, + (\tan^{-1}(n+1) - \tan^{-1}n)$

$= \tan^{-1}(n+1) - \tan^{-1}1 = \tan^{-1}\left(\frac{(n+1)-1}{1+(n+1)1}\right)$

$= \tan^{-1}\frac{n}{n+2} = \tan^{-1}x$ (given)

$\therefore\ x = \frac{n}{n+2}$

The correct options is (A)

11. The given expression can be written as

$\tan^{-1}\left(a\sqrt{\frac{a+b+c}{abc}}\right) + \tan^{-1}\left(b\sqrt{\frac{a+b+c}{abc}}\right)$

$+ \tan^{-1}\left(c\sqrt{\frac{a+b+c}{abc}}\right)$

$= \tan^{-1}(ay) + \tan^{-1}(by) + \tan^{-1}(cy)$

where, $y = \sqrt{\frac{a+b+c}{abc}}$

$= \tan^{-1}\left(\frac{ay+by+cy-abcy^3}{1-aby^2-bcy^2-acy^2}\right)$

$= \tan^{-1}\left[y\left(\frac{a+b+c-abcy^2}{1-y^2(ab+bc+ca)}\right)\right]$

$= \tan^{-1}0 = 0$ or π

The correct options is (C,D)

12. Since $\sqrt{x^2-3x+2} \ge 0 \Rightarrow$

$\tan^{-1}\sqrt{x^2-3x+2} < \frac{\pi}{2}$

Since $\sqrt{4x-x^2-3} \ge 0 \Rightarrow 0 < \cos^{-1}\sqrt{4x-x^2-3} \le \frac{\pi}{2}$

$\Rightarrow\ 0 <$ L.H.S. $< \pi$

$\Rightarrow$ The given equation has no solution

The correct options is (C)

13. $\tan^{-1}\frac{2m}{m^4+m^2+2}$

$= \tan^{-1}\frac{2m}{1+(m^2+m+1)(m^2-m+1)}$

$= \tan^{-1}(m^2+m+1) - \tan^{-1}(m^2-m+1)$

so that $\sum_{m=1}^{n}\tan^{-1}\frac{2m}{m^4+m^2+2} = (\tan^{-1}3 - \tan^{-1}1)$

$+ (\tan^{-1}7 - \tan^{-1}3) + \ldots + [\tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)]$

$= \tan^{-1}(n^2+n+1) - \tan^{-1}1$

$= \tan^{-1}\dfrac{n^2+n}{n^2+n+2}$

The correct options is (C)

14. Since, $-\dfrac{\pi}{2} \le \sin^{-1} x \le \dfrac{\pi}{2}$

$\therefore \ \sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{3\pi}{2}$

$\Rightarrow \ \sin^{-1}x = \sin^{-1}y = \sin^{-1}z = \dfrac{\pi}{2}$

$\Rightarrow \ x = y = z = 1$

Also, $f(p+q) = f(p).\,f(q) \ \forall\ p, q \in R$ .... (1)

Given, $f(1) = 1$

From (1),

$f(1+1) = f(1).\,f(1)$

$\Rightarrow f(2) = 1^2 = 1$ ....(2)

From (2), $f(2+1) = f(2)\,.\,f(1)$

$\Rightarrow \ f(3) = 1^2\,.1 = 1^3 = 1$

Now, given expression = $3 - \dfrac{3}{3} = 2$

The correct options is (C)

15. $\sin^{-1}x$, $\cos^{-1}x$ are defined for $x \le 1$ and $x \ge 0$

$\therefore \ \sqrt{x^2 - x + 1} \le 1$ and $\sqrt{x^2 - x} \ge 0$

$\Rightarrow \ x^2 - x \le 0$ and $x^2 - x \ge 0 \Rightarrow x^2 - x = 0$

$\Rightarrow \ x = 1, 0$

$\therefore$ There are two solutons, both satisfy the equation.

The correct options is (C)

16. $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$

$\Rightarrow \ \tan^{-1}\left(\dfrac{1}{\sqrt{\cos\alpha}}\right) - \tan^{-1}(\sqrt{\cos\alpha}) = x$

$\Rightarrow \ \tan^{-1}\dfrac{\dfrac{1}{\sqrt{\cos\alpha}} - \sqrt{\cos\alpha}}{1 + \dfrac{1}{\sqrt{\cos\alpha}},\sqrt{\cos\alpha}} = x$

$\Rightarrow \ \tan^{-1}\dfrac{1-\cos\alpha}{2\sqrt{\cos\alpha}} = x \Rightarrow \tan x = \dfrac{1-\cos\alpha}{2\sqrt{\cos\alpha}}$

or $\cot x = \dfrac{2\sqrt{\cos\alpha}}{1-\cos\alpha}$ or $\operatorname{cosec} x = \dfrac{1+\cos\alpha}{1-\cos\alpha}$

$\therefore \ \sin x = \dfrac{1-\cos\alpha}{1+\cos\alpha} = \dfrac{1-(1-2\sin^2\alpha/2)}{1+2\cos^2\alpha/2-1}$

or $\sin x = \tan^2 \alpha/2$

The correct options is (A)

17. Let $a = b\cos\theta$,

then $a_1 = \dfrac{a+b}{2} = \dfrac{b(1+\cos\theta)}{2} = b\cos^2\dfrac{\theta}{2}$

$\Rightarrow \ b_1 = \sqrt{a_1\, b} = b\cos\dfrac{\theta}{2}$

Now, $a_2 = \dfrac{a_1 + b_1}{2} = b\cos\dfrac{\theta}{2}\cos^2\dfrac{\theta}{4}$

$\therefore \ b_2 = b\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{2^2}$

Similarly, $b_3 = b\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{2^2}\cos\dfrac{\theta}{2^3}$

and so on.

Now, $b_n = b\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{2^2}\ldots\cos\dfrac{\theta}{2^n}$

$\therefore \ b_\infty = \lim\limits_{n\to\infty}\dfrac{b\sin\theta}{2^n\sin\left(\dfrac{\theta}{2^n}\right)} = \lim\limits_{n\to\infty}\dfrac{b\sin\theta}{\theta\left(\dfrac{\sin(\theta/2^n)}{\theta/2^n}\right)}$

$= \dfrac{b\sin\theta}{\theta} = \dfrac{b\sqrt{1-\cos^2\theta}}{\cos^{-1}(a/b)}$

$\therefore \ b_\infty = \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}$

The correct options is (B)

18. Let $\alpha = \cos^{-1}\sqrt{p}$, $\beta = \cos^{-1}\sqrt{1-p}$

and $\gamma = \cos^{-1}\sqrt{1-q}$ or $\cos\alpha = \sqrt{p}$, $\cos\beta = \sqrt{1-p}$

and $\cos\gamma = \sqrt{1-q}$

Therefore, $\sin\alpha = \sqrt{1-p}$, $\sin\beta = \sqrt{p}$, $\sin\gamma = \sqrt{q}$

The given equation may be written as $\alpha + \beta + \gamma = \dfrac{3\pi}{4}$

or $\alpha + \beta = \dfrac{3\pi}{4} - \gamma$ or $\cos(\alpha+\beta) = \cos\left(\dfrac{3\pi}{4} - \gamma\right)$

$\Rightarrow \ \cos\alpha\cos\beta - \sin\gamma\sin\beta = \cos[\pi - (\pi/4 + \gamma)]$

$= -\cos\left(\dfrac{\pi}{4} + \gamma\right)$

$\Rightarrow \ \sqrt{p}\sqrt{1-p} - \sqrt{1-p}\sqrt{p}$

$= -\left(\dfrac{1}{\sqrt{2}}\sqrt{1-q} - \dfrac{1}{\sqrt{2}}\sqrt{q}\right)$

$\Rightarrow \ 0 = \sqrt{1-q} - \sqrt{q} \Rightarrow 1 - q = q \Rightarrow q = \dfrac{1}{2}$

The correct options is (D)

19. We have, $\sin(\cot^{-1}(x+1)) = \cos(\tan^{-1}x)$

$\Rightarrow \ \cos\left(\dfrac{\pi}{2} - \cot^{-1}(x+1)\right) = \cos(\tan^{-1}x)$

$\Rightarrow \dfrac{\pi}{2} - \cot^{-1}(x+1) = 2n\pi \pm \tan^{-1}x$

Put $n = 0 \Rightarrow \dfrac{\pi}{2} - \cot^{-1}(x+1) = \pm \tan^{-1}x = \tan^{-1}(\pm x)$

$\Rightarrow \dfrac{\pi}{2} = \tan^{-1}(\pm x) + \cot^{-1}(x+1)$

$\Rightarrow x + 1 = \pm x \Rightarrow 2x + 1 = 0$

$\therefore\ x = -\dfrac{1}{2}$

The correct options is (D)

20. Given, $\theta = \cos^{-1}\left(\dfrac{x}{2}\cdot\dfrac{y}{3} - \sqrt{1-\dfrac{x^2}{4}}\sqrt{1-\dfrac{y^2}{9}}\right)$

$\Rightarrow \cos\theta = \dfrac{xy}{6} - \dfrac{\sqrt{4-x^2}\sqrt{9-y^2}}{6}$

$\Rightarrow (xy - 6\cos\theta)^2 = (4-x^2)(9-y^2)$

$= 36 - 9x^2 - 4y^2 + x^2y^2$

$\Rightarrow 36\cos^2\theta - 12xy\cos\theta + 4y^2 + 9x^2 = 36$

$\therefore\ 9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

The correct options is (C)

21. Let $\sin^{-1}\dfrac{5}{13} = x \Rightarrow \sin x = \dfrac{5}{13}$

$\Rightarrow \cos x = \sqrt{1-\dfrac{25}{169}} = \dfrac{12}{13}$

$\Rightarrow \cos\left(\sin^{-1}\dfrac{5}{13}\right) = \cos\left(\cos^{-1}\dfrac{12}{13}\right) = \dfrac{12}{13}$

The correct options is (A)

22. $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \dfrac{\pi}{2}$

$\Rightarrow \cos^{-1}\dfrac{1}{\sqrt{1+(x^2+x)^2}} + \sin^{-1}\sqrt{x^2+x+1} = \dfrac{\pi}{2}$

$\Rightarrow \cos^{-1}\dfrac{1}{\sqrt{1+(x^2+x)^2}} = \dfrac{\pi}{2} - \sin^{-1}\sqrt{x^2+x+1}$

$= \cos^{-1}\sqrt{x^2+x+1}$

$\Rightarrow \dfrac{1}{\sqrt{1+(x^2+x)^2}} = \sqrt{x^2+x+1}$

$\Rightarrow (x^2+x+1)\,[(x^2+x)^2+1] = 1$

$\Rightarrow (x^2+x)^3 + (x^2+x)^2 + (x^2+x) = 0$

$\Rightarrow x^2 + x = 0 \Rightarrow x = -1, 0$

The correct options is (C)

23. $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$

$\Rightarrow \sin^{-1}(1-x) = \cos^{-1}x - \sin^{-1}x = \dfrac{\pi}{2} - 2\sin^{-1}x$

$\Rightarrow 1 - x = \cos(2\sin^{-1}x) = 1 - 2(\sin\sin^{-1}x)^2 = 1 - 2x^2$

$\Rightarrow 2x^2 - x = 0 \Rightarrow x = 0$ or $\dfrac{1}{2}$

The correct options is (C)

24. Domain of $\sin^{-1}x$ is $[-1, 1]$

and domain of $\sin^{-1}[x]$ is $\{x : -1 \le [x] \le 1\}$

$= \{x : [x] = -1, 0, 1\}$

But $[x] = \begin{cases} -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \end{cases}$

$\therefore\ x \in [-1, 2)$

The correct options is (B)

25. $\tan(A - B) = \dfrac{\tan A + \tan B}{1 + \tan A\tan B}$

$= \dfrac{\dfrac{x\sqrt{3}}{2k-x} - \dfrac{2x-k}{k\sqrt{3}}}{1 + \dfrac{x\sqrt{3}}{2k-x}\cdot\dfrac{2x-k}{k\sqrt{3}}} = \dfrac{1}{\sqrt{3}}$

$\therefore\ A - B = 30°$

The correct options is (D)

26. If we put $x = \tan\theta$, the given equality becomes $\tan^{-1}y = 4\theta$

$\Rightarrow y = \tan 4\theta = \dfrac{2\tan 2\theta}{1-\tan^2 2\theta} = \dfrac{2\left[\dfrac{2\tan\theta}{1-\tan^2\theta}\right]}{1-\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)^2}$

$= \dfrac{2\times 2x(1-x^2)}{(1-x^2)^2 - 4x^2} = \dfrac{4x(1-x^2)}{1-6x^2+x^4}$

$\therefore\ \dfrac{1}{y}$ is zero if $x^4 - 6x^2 + 1 = 0$

$\Rightarrow x^2 = \dfrac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2} = (1 \pm \sqrt{2})^2$

The correct options is (A)

27. LHS and RHS of the given equation are defined if $\dfrac{x-b}{a-b} > 0$

and $\dfrac{x-b}{a-b} > 0$

$\therefore$ either $a > x > b$ or $a < x < b$

The correct options is (A)

28. $A = 2\tan^{-1}(2\sqrt{2} - 1) = 2\tan^{-1}(1.828)$

$\therefore A > 2\tan^{-1}\sqrt{3}\ (\sqrt{3} = 1.732 < 1.828)$

$\Rightarrow A > \dfrac{2\pi}{3}$ .... (1)

We have $\sin^{-1}\frac{1}{3} < \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\Rightarrow 3\sin^{-1}\frac{1}{3} < \frac{\pi}{2}$

Using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$,

we have, $\sin^{-1}\frac{1}{3} = \sin^{-1}\left(3\times\frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right)$

$= \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852)$

$\therefore 3\sin^{-1}\frac{1}{3} < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \quad \left(\because \frac{\sqrt{3}}{2} = 0.868 > 0.852\right)$

i.e., $3\sin^{-1}\frac{1}{3} < \frac{\pi}{3}$ .... (2)

Also, $\sin^{-1}\frac{3}{5} = \sin^{-1}(0.6) < \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$\therefore \sin^{-1}\frac{3}{5} < \frac{\pi}{3}$

$\therefore B = 3\sin^{-1}\frac{1}{3} + \sin\frac{3}{5} < \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}\ \frac{2\pi}{3}$

$\therefore B < \frac{2\pi}{3}$ ... (3)

From (1) and (3), $A > B$

The correct options is (B)

29. $x = \sin 2\theta$, where $\tan\theta = 2$

$\Rightarrow x = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{4}{1+4} = \frac{4}{5}$

If $\alpha = \tan^{-1}\frac{4}{3}$, $y = \sin\left(\frac{\alpha}{2}\right) = \frac{1}{\sqrt{2}}\sqrt{1-\cos\alpha}$

$= \frac{1}{\sqrt{2}}\sqrt{1-\frac{3}{5}} = \frac{1}{\sqrt{5}}$

$\therefore y^2 = \frac{1}{5} \Rightarrow y^2 = 1 - x$

The correct options is (D)

30. Let $T_n = \cot^{-1}\left(n^2 + \frac{3}{4}\right) \Rightarrow T_n = \cot^{-1}\left(\frac{4n^2+3}{4}\right)$

$\Rightarrow T_n = \tan^{-1}\left(\frac{4}{4n^2+3}\right) = \tan^{-1}\left(\frac{1}{n^2+\frac{3}{4}}\right)$

$\Rightarrow T_n = \tan^{-1}\left(\frac{1}{1+\left(n^2-\frac{1}{4}\right)}\right)$

$\Rightarrow T_n = \tan^{-1}\left[\frac{1}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right]$

$\Rightarrow T_n = \tan^{-1}\left[\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right]$

$\Rightarrow T_n = \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(n-\frac{1}{2}\right)$

Now $S_n = T_1 + T_2 + T_3 + \ldots + T_n$

$\Rightarrow S_n = \tan^{-1}\left(\frac{3}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{5}{2}\right) - \tan^{-1}\left(\frac{3}{2}\right)$

$+ \tan^{-1}\left(\frac{7}{2}\right) - \tan^{-1}\left(\frac{5}{2}\right) + \ldots.$

$+ \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(n-\frac{1}{2}\right)$

$\therefore S_\infty = \tan^{-1}\infty - \tan^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right) = \cot^{-1}\left(\frac{1}{2}\right)$

$\therefore S_\infty = \tan^{-1} 2$

The correct options is (B)

31. Let $E = \cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$

$\Rightarrow E = \cot\left(\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$

$\Rightarrow E = \cot\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right)$

$\Rightarrow E = \cot\left(\tan^{-1}\frac{17}{6}\right) = \frac{6}{17}$

32. We have, $\tan(\cos^{-1}x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$

$$\Rightarrow \tan\left[\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right] = \sin(\tan^{-1} 2)$$

$$\Rightarrow \tan\left[\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right] = \sin\left(\sin^{-1}\frac{2}{\sqrt{1+4}}\right)$$

$$\Rightarrow \tan\left[\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right] = \sin\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}} \Rightarrow 5(1-x^2) = 4x^2$$

$$\Rightarrow 9x^2 = 5, \therefore \boldsymbol{x = \pm\left(\frac{\sqrt{5}}{3}\right)}.$$

The correct option is (B)

33. Let $\cot^{-1}x = \theta \Rightarrow \cot\theta = x$

$$\therefore \sin(\cot^{-1}x) = \sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+x^2}}$$

Thus, $\cos\left[\tan^{-1}\left[\sin(\cot^{-1}x)\right]\right]$

$$= \cos\left[\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right] = \cos\varphi$$

$$\left(\text{Put } \tan^{-1}\left\{\frac{1}{\sqrt{1+x^2}}\right\} = \phi \Rightarrow \tan\phi = \frac{1}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{\sec\phi} = \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+\frac{1}{1+x^2}}} = \sqrt{\frac{\boldsymbol{1+x^2}}{\boldsymbol{2+x^2}}}.$$

The correct option is (C)

34. Since $0 \le \cos^{-1}x_i \le \pi$, $\therefore \cos^{-1}x_i = 0$ for all i.

$\therefore x_i = 1$ for all i $\therefore \sum_{i=1}^{2n} x_i = \boldsymbol{2n}$.

The correct option is (B)

35. From given equations, it can be seen that

$\alpha + \beta = \pi$

Since, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \ \forall\ x$

Also, $\alpha = \frac{\pi}{3} + \sin^{-1}\frac{1}{3} < \frac{\pi}{3} + \sin^{-1}\frac{1}{2}$

as $\sin\theta$ is increasing in $\left[0, \frac{\pi}{2}\right]$

$$\therefore \alpha < \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2} \Rightarrow \beta > \frac{\pi}{2} > \alpha \Rightarrow \alpha < \beta$$

The correct option is (C)

36. $\because -1 < x < 0$, $\therefore -\frac{\pi}{4} < \tan^{-1}x < 0$

Let $\tan^{-1}x = \alpha \Rightarrow -\frac{\pi}{4} < \alpha < 0$

$\therefore \tan\alpha = x$, $-\frac{\pi}{4} < \alpha < 0$

$$\therefore \sin\alpha = \frac{x}{\sqrt{1+x^2}} \Rightarrow \alpha = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$$

and, $\cos\alpha = \frac{1}{\sqrt{1+x^2}} \Rightarrow \cos(-\alpha) = \frac{1}{\sqrt{1+x^2}}$,

where, $0 < -\alpha < \frac{\pi}{4} \Rightarrow \alpha = -\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

The correct option is (B)

37. The given series

$= \cot^{-1}2(1)^2 + \cot^2 2(2)^2 + \cot^{-1}2(3)^2 + \cot^{-1}2(4)^2 + \ldots.$

$$\therefore T_n = \cot^{-1}(2n^2) = \tan^{-1}\frac{1}{2n^2}$$

$$= \tan^{-1}\left[\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right]$$

$= \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

$\therefore T_1 = \tan^{-1}3 - \tan^{-1}1$

$T_2 = \tan^{-1}5 - \tan^{-1}3$

$T_3 = \tan^{-1}7 - \tan^{-1}5$

and so on.

$\therefore S_n = \tan^{-1}(2n+1) - \tan^{-1}1$

$$\therefore \lim_{n\to\infty} S_n = \tan^{-1}\infty - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

The correct option is (B)

38. Let $\sqrt{\tan\theta} = \tan\alpha$

$\therefore A = \cot^{-1}(\tan\alpha) - \tan^{-1}(\tan\alpha)$

$$= \cot^{-1}\left(\cot\left(\frac{\pi}{2}-\alpha\right)\right) - \tan^{-1}(\tan\alpha)$$

$$= \cot^{-1}\left(\cot\left(\frac{\pi}{2}-\alpha\right)\right) - \tan^{-1}(\tan\alpha) = \frac{\pi}{2} - \alpha - \alpha$$

$\Rightarrow\ 2\alpha = \frac{\pi}{2} - 2\alpha$ or $\alpha = \frac{\pi}{2} - A$.

$\therefore\ \sqrt{\tan\theta} = \tan\alpha = \tan\left(\frac{\pi}{4} - \frac{A}{2}\right)$.

The correct option is (C)

39. We have, $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}\theta] = 1$

$\Rightarrow\ 1 \le \sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}\theta \le \frac{\pi}{2}$

$\Rightarrow\ \sin 1 \le \cos^{-1}\sin^{-1}\tan^{-1}\theta \le 1$

$\Rightarrow\ \cos\sin 1 \ge \sin^{-1}\tan^{-1}\theta \ge \cos 1$

$\Rightarrow\ \sin\cos\sin 1 \ge \tan^{-1}\theta \ge \sin\cos 1$

$\Rightarrow\ \tan\sin\cos\sin 1 \ge \theta \ge \tan\sin\cos 1$.

The correct option is (C)

40. We have, $\sum_{i=1}^{2n}\sin^{-1}x_i = n\pi$

$\Rightarrow\ \sin^{-1}x_1 + \sin^{-1}x_2 + ... + \sin^{-1}x_{2n} = n\pi$

Let $\sin^{-1}x_1 = \alpha_1$, $\sin^{-1}x_2 = \alpha_2$, ..., $\sin^{-1}x_n = \alpha_n$

$\therefore\ \alpha_1 + \alpha_2 + ... + \alpha_{2n} = n\pi$ ....(1)

$\Rightarrow\ x_1 = \sin\alpha_1, x_2 = \sin\alpha_2, ..., x_{2n} = \sin\alpha_{2n}$

$\therefore\ \sum_{i=1}^{2n} x_i = x_1 + x_2 + ... + x_{2n}$

$= \sin\alpha_1 + \sin\alpha_2 + ... + \sin\alpha_{2n}$

Clearly, from (1), $\alpha_1 = \frac{\pi}{2}$, $\forall\ i = 1, 2, ..., 2n$

$\therefore\ \sum_{i=1}^{2n} x_i = \underbrace{1+1+...+1}_{2n\text{ times}} = \mathbf{2n}$

The correct option is (B)

41. We have,

$\frac{2\pi}{3} < \cos^{-1}\left(\frac{n}{2\pi}\right) < \pi \Rightarrow \cos\frac{2\pi}{3} > \frac{n}{2\pi} > \cos\pi$

$\Rightarrow -1 < \frac{n}{2\pi} < -\frac{1}{2} \Rightarrow\ -2\pi < n < -\pi\ \Rightarrow$

$-6 < n < -4$.

The correct option is (A)

42. We have, $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$.

It is possible only when

$\sin^{-1}x = \frac{\pi}{2}$, $\sin^{-1}y = \frac{\pi}{2}$ and $\sin^{-1}z = \frac{\pi}{2}$

$\Rightarrow\ x = 1, y = 1$ and $z = 1$

$\therefore\ \frac{\sum_{k=1}^{2}(x^{100k} + y^{106k})}{\sum x^{207}y^{207}} = \frac{(x^{100} + y^{106}) + (x^{200} + y^{212})}{x^{207}y^{207} + y^{207}z^{207} + z^{207}x^{207}}$

$= \frac{1+1+1+1}{1+1+1} = \mathbf{\frac{4}{3}}$.

The correct option is (B)

43. Since $\sqrt{x^2 - 3x + 2} \ge 0 \Rightarrow$

$\tan^{-1}\sqrt{x^2 - 3x + 2} < \frac{\pi}{2}$

Since $\sqrt{4x - x^2 - 3} \ge 0 \Rightarrow 0 < \cos^{-1}\sqrt{4x - x^2 - 3} \le \frac{\pi}{2}$

$\Rightarrow\ 0 <$ L.H.S. $< \pi$

$\Rightarrow$ The given equation has **no solution**

The correct option is (C)

44. $\tan^{-1}\frac{2m}{m^4 + m^2 + 2}$

$= \tan^{-1}\frac{2m}{1 + (m^2 + m + 1)(m^2 - m + 1)}$

$= \tan^{-1}(m^2 + m + 1) - \tan^{-1}(m^2 - m + 1)$

so that $\tan^{-1}\frac{n^2 + n}{n^2 + n + 2} = (\tan^{-1}3 - \tan^{-1}1) +$

$(\tan^{-1}7 - \tan^{-1}3) + ... + (\tan^{-1}(n^2 + n + 1) - \tan^{-1}(n^2 - n + 1))$

$= \tan^{-1}(n^2 + n + 1) - \tan^{-1}1$

$= -\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

The correct option is (C)

45. Since, $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

$\therefore\ \sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$

$\Rightarrow\ \sin^{-1}x = \sin^{-1}y = \sin^{-1}z = \frac{\pi}{2}$

$\Rightarrow\ x = y = z = 1$

Also, $f(p + q) = f(p).f(q)\ \forall\ p, q \in R$ ....(1)

Given, $f(1) = 1$

From (1),

$f(1 + 1) = f(1).f(1)$

$\Rightarrow\ f(2) = 1^2 = 1$ ....(2)

From (2), $f(2 + 1) = f(2).f(1)$

$\Rightarrow\ f(3) = 1^2.1 = 1^3 = 1$

Now, given expression $= 3 - \frac{3}{3} = \mathbf{2}$

The correct option is (C)

46. We have, $\sin(\cot^{-1}(x + 1)) = \cos(\tan^{-1}x)$

$\Rightarrow\ \cos\left(\frac{\pi}{2} - \cot^{-1}(x + 1)\right) = \cos\left(\tan^{-1}x\right)$

$\Rightarrow \frac{\pi}{2} - \cot^{-1}(x+1) = 2n\pi \pm \tan^{-1}x$

Put $n = 0 \Rightarrow \frac{\pi}{2} - \cot^{-1}(x+1) = \pm \tan^{-1}x = \tan^{-1}(\pm x)$

$\Rightarrow \frac{\pi}{2} = \tan^{-1}(\pm x) + \cot^{-1}(x+1)$

$\Rightarrow x + 1 = \pm x \Rightarrow 2x + 1 = 0$

$\therefore x = -\frac{1}{2}$

The correct option is (D)

47. If we put $x = \tan\theta$, the given equality becomes $\tan^{-1}y = 4\theta$

$$\Rightarrow y = \tan 4\theta = \frac{2\tan 2\theta}{1-\tan^2 2\theta} = \frac{2\left[\frac{2\tan\theta}{1-\tan^2\theta}\right]}{1-\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)^2}$$

$$= \frac{2\times 2x(1-x^2)}{(1-x^2)^2 - 4x^2} = \frac{4x(1-x^2)}{1-6x^2+x^4}$$

$\therefore \frac{1}{y}$ is zero if $x^4 - 6x^2 + 1 = 0$

$$\Rightarrow x^2 = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2} = (1\pm\sqrt{2})^2$$

The correct option is (A)

48. $A = 2\tan^{-1}(2\sqrt{2}-1) = 2\tan^{-1}(1.828)$

$\therefore A > 2\tan^{-1}\sqrt{3}$ $[\sqrt{3} = 1.732 < 1.828]$

$\Rightarrow A > \frac{2\pi}{3}$ ....(1)

We have, $\sin^{-1}\frac{1}{3} < \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\Rightarrow 3\sin^{-1}\frac{1}{3} < \frac{\pi}{2}$

Using $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$,

we have, $\sin^{-1}\frac{1}{3} = \sin^{-1}\left(3\times\frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right)$

$= \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852)$

$\therefore 3\sin^{-1}\frac{1}{3} < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ $\left(\because \frac{\sqrt{3}}{2} = 0.868 > 0.852\right)$

i.e., $3\sin^{-1}\frac{1}{3} < \frac{\pi}{3}$ ....(2)

Also, $\sin^{-1}\frac{3}{5} = \sin^{-1}(0.6) < \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

$\therefore \sin^{-1}\frac{3}{5} < \frac{\pi}{3}$

$\therefore B = 3\sin^{-1}\frac{1}{3} + \sin\frac{3}{5} < \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore B < \frac{2\pi}{3}$ ... (3)

From (1) and (3), $\boldsymbol{A > B}$

The correct option is (B)

49. $x = \sin 2\theta$, where $\tan\theta = 2$

$$\Rightarrow x = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{4}{1+4} = \frac{4}{5}$$

If $\alpha = \tan^{-1}\frac{4}{3}$, $y = \sin\left(\frac{\alpha}{2}\right) = \frac{1}{\sqrt{2}}\sqrt{1-\cos\alpha}$

$$= \frac{1}{\sqrt{2}}\sqrt{1-\frac{3}{5}} = \frac{1}{\sqrt{5}}$$

$\therefore y^2 = \frac{1}{5} \Rightarrow \boldsymbol{y2 = 1 - x}$

The correct option is (D)

50. Let $T_n =$

$\cot^{-1}\left(n^2 + \frac{3}{4}\right) \Rightarrow T_n = \cot^{-1}\left(\frac{4n^2+3}{4}\right)$

$$\Rightarrow T_n = \tan^{-1}\left(\frac{4}{4n^2+3}\right) = \tan^{-1}\left(\frac{1}{n^2+\frac{3}{4}}\right)$$

$$\Rightarrow T_n = \tan^{-1}\left[\frac{1}{1+\left(n^2-\frac{1}{4}\right)}\right]$$

$$\Rightarrow T_n = \tan^{-1}\left[\frac{1}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right]$$

$$\Rightarrow T_n = \tan^{-1}\left[\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right]$$

$$\Rightarrow T_n = \tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(n-\frac{1}{2}\right)$$

Now, $S_n = T_1 + T_2 + T_3 + \ldots + T_n$

$$\Rightarrow S_n = \tan^{-1}\left(\cancel{\frac{3}{2}}\right) - \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\cancel{\frac{5}{2}}\right) - \tan^{-1}\left(\cancel{\frac{3}{2}}\right)$$

$$+\tan^{-1}\left(\frac{7}{2}\right)-\tan^{-1}\left(\frac{5}{2}\right)+\ldots.$$

$$+\tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\left(n-\frac{1}{2}\right)$$

$$\therefore\ S_\infty=\tan^{-1}\infty-\tan^{-1}\left(\frac{1}{2}\right)$$

$$=\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{2}\right)=\cot^{-1}\left(\frac{1}{2}\right)$$

$\therefore\ S_\infty=$ **tan–1 2**

The correct option is (B)

51. Let $\sqrt{\frac{xr}{yz}}=\alpha,\ \sqrt{\frac{yr}{zx}}=\beta$ and $\sqrt{\frac{zr}{xy}}=\gamma$

Then, the given expression $=\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma$

$$=\tan^{-1}\left[\frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-\alpha\beta-\beta\gamma-\gamma\alpha}\right]$$

Now, $\alpha+\beta+\gamma-\alpha\beta\gamma$

$$=\sqrt{\frac{xr}{yz}}+\sqrt{\frac{yr}{zx}}+\sqrt{\frac{zr}{xy}}-\sqrt{\frac{xr}{yz}}\cdot\sqrt{\frac{yr}{zx}}\cdot\sqrt{\frac{zr}{xy}}$$

$$=\frac{x\sqrt{r}+y\sqrt{r}+z\sqrt{r}}{\sqrt{xyz}}-\frac{r\sqrt{r}}{\sqrt{xyz}}$$

$$=\frac{x\sqrt{r}+y\sqrt{r}+z\sqrt{r}}{\sqrt{xyz}}-\frac{r\sqrt{r}}{\sqrt{xyz}}$$

$$=\frac{r\sqrt{r}}{\sqrt{xyz}}-\frac{r\sqrt{r}}{\sqrt{xyz}}\quad = x+y+z=r]$$

Also, $1-\alpha\beta-\beta\gamma-\gamma\alpha$

$$=1-\sqrt{\frac{xr}{yz}}\cdot\sqrt{\frac{yr}{zx}}-\sqrt{\frac{yr}{zx}}\cdot\sqrt{\frac{zr}{zy}}-\sqrt{\frac{zr}{xy}}\cdot\sqrt{\frac{xr}{yz}}$$

$$=1-\frac{r}{z}-\frac{r}{x}-\frac{r}{y}=1-r\left[\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right]\neq 0.$$

$\therefore$ The given expression $=\tan^{-1}0=n\pi$

$=-\pi, 0, \pi$

$$\left[\begin{array}{l}\because -\frac{\pi}{2}<\tan^{-1}\sqrt{\frac{xr}{yz}}<\frac{\pi}{2},\ -\frac{\pi}{2}<\tan^{-1}\sqrt{\frac{yr}{zx}}<\frac{\pi}{2},\\ -\frac{\pi}{2}<\tan^{-1}\sqrt{\frac{zr}{xy}}<\frac{\pi}{2}\ \therefore -\frac{3\pi}{2}<\tan^{-1}\sqrt{\frac{xr}{yz}}\\ +\tan^{-1}\sqrt{\frac{yr}{zx}}+\tan^{-1}\sqrt{\frac{zr}{xy}}<\frac{3\pi}{2}\end{array}\right]$$

$$=\pi.\left[\begin{array}{l}\because \sqrt{\frac{xr}{yz}},\sqrt{\frac{yr}{zx}},\sqrt{\frac{zr}{xy}}\text{ are positive }\therefore \tan^{-1}\sqrt{\frac{xr}{yz}},\\ \tan^{-1}\sqrt{\frac{yr}{zx}},\ \tan^{-1}\sqrt{\frac{zr}{xy}}\text{ are positive angles, and}\\ \text{sum of three positive angles is positive}\end{array}\right]$$

The correct option is (A)

52. We have, $u=\tan^{-1}\frac{1}{\sqrt{\cos 2\theta}}-\tan^{-1}\sqrt{\cos 2\theta}$

$$=\tan^{-1}\left[\frac{\frac{1}{\sqrt{\cos 2\theta}}-\sqrt{\cos 2\theta}}{1+\frac{1}{\sqrt{\cos 2\theta}}\cdot\sqrt{\cos 2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{1-\cos 2\theta}{2\sqrt{\cos 2\theta}}\right]$$

$$=\tan^{-1}\frac{\sin^2\theta}{\sqrt{\cos 2\theta}}$$

[$\because\ 1-\cos 2\theta=2\sin^2\theta$]

$$\therefore\ \tan u=\frac{\sin^2\theta}{\sqrt{\cos 2\theta}}=\frac{AB}{BC}\ \text{(say)}$$

Then, $AC=\sqrt{\sin^4\theta+\cos 2\theta}$

$$=\sqrt{\sin^4\theta+1-2\sin^2\theta}=\sqrt{(1-\sin^2\theta)^2}$$

$=|1-\sin^2\theta|=\cos^2\theta$

$$\therefore\ \sin u=\frac{AB}{AC}=\frac{\sin^2\theta}{\cos^2\theta}=\mathbf{\tan 2\theta.}$$

The correct option is (C)

53. We have, $2\tan^{-1}\left(\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)$

$$=\cos^{-1}\left[\frac{1-\tan^2\frac{\theta}{2}\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\theta}{2}\tan^2\frac{\phi}{2}}\right]=\cos^{-1}\left[\frac{1-\frac{\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}}{\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}}}{1+\frac{\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}}{\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}}}\right]$$

$$=\cos^{-1}\left[\frac{\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}-\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}}{\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}+\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}}\right]$$

$=\cos^{-1}$

$$\left[\frac{\left(\cos\frac{\theta}{2}\cos\frac{\phi}{2}+\sin\frac{\theta}{2}\sin\frac{\phi}{2}\right)\left(\cos\frac{\theta}{2}\cos\frac{\phi}{2}-\sin\frac{\theta}{2}\sin\frac{\phi}{2}\right)}{\left(\cos\frac{\theta}{2}\cos\frac{\phi}{2}+\sin\frac{\theta}{2}\sin\frac{\phi}{2}\right)^2-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\phi}{2}}\right]$$

$$= \cos^{-1}\left[\frac{\cos\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)}{\cos^2\frac{\theta-\phi}{2}-\frac{\sin\theta\sin\phi}{2}}\right]$$

$$= \cos^{-1}\left[\frac{2\cos\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)}{2\cos^2\frac{\theta-\phi}{2}-\sin\theta\sin\phi}\right]$$

$$= \cos^{-1}\left[\frac{\cos\theta+\cos\phi}{1+\cos(\theta-\phi)-\sin\theta\sin\phi}\right]$$

$$= \cos^{-1}\left[\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi+\sin\theta\sin\phi-\sin\theta\sin\phi}\right]$$

$$= \cos^{-1}\left[\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right].$$

The correct option is (A)

54. We have, $\sin^{-1}x + \sin^{-1}2x = \sin^{-1}\frac{\sqrt{3}}{2}$

$\Rightarrow \sin^{-1}x - \sin^{-1}\frac{\sqrt{3}}{2} = -\sin^{-1}2x$ ...(1)

Let $\sin^{-1}x = \alpha$ and $\sin^{-1}\frac{\sqrt{3}}{2} = \beta$

$\Rightarrow \sin\alpha = x$ and $\sin\beta = \frac{\sqrt{3}}{2}$

$\therefore \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= x\sqrt{1-\frac{3}{4}} - \sqrt{1-x^2}\,\frac{\sqrt{3}}{2} = \frac{x}{2} - \frac{\sqrt{3}\sqrt{1-x^2}}{2}$

$\therefore \alpha-\beta = \sin^{-1}\left[\frac{x}{2}-\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right]$

i.e., $\sin^{-1}x - \sin^{-1}\frac{\sqrt{3}}{2} = \sin^{-1}\left[\frac{x}{2}-\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right]$

Hence, from (1),

$\sin\sin^{-1}\left[\frac{x}{2}-\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right] = \sin[-\sin^{-1}2x]$

$\Rightarrow \frac{x}{2}-\frac{\sqrt{3}}{2}\sqrt{1-x^2} = -2x$ [$\because \sin(-\theta) = -\sin\theta$]

$\Rightarrow 5x = \sqrt{3}\sqrt{1-x^2}$ ...(2)

$\Rightarrow 25x^2 = 3(1-x^2) \Rightarrow 28x^2 = 3 \therefore x = \pm\frac{\sqrt{3}}{2\sqrt{7}}$

But $x = -\frac{\sqrt{3}}{2\sqrt{7}}$ does not satisfy the equation as negative value of x makes L.H.S. of the equation (*2*) negative where as R.H.S. is positive.

$\therefore x = \frac{\sqrt{3}}{2\sqrt{7}} = \frac{1}{2}\sqrt{\frac{3}{7}}$.

The correct option is (A)

55. Let $x = \tan\theta$

Then, $\sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} = \sin^{-1}(\sin 2\theta)$

$\therefore 2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2} = 2\theta + \sin^{-1}(\sin 2\theta)$

If $-\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$,

$2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2} = 2\theta + 2\theta$

$= 4\tan^{-1}x \ne$ independent of x.

If $-\frac{\pi}{2} \le \pi - 2\theta \le \frac{\pi}{2}$, $2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2}$

$= 2\theta + \sin^{-1}[\sin(\pi-2\theta)] = 2\theta + \pi - 2\theta$

$= \pi =$ independent of x.

$\therefore \theta \notin \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ but $\theta \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ and from the principal value of $\tan^{-1}x$,

$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Hence, $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

$\therefore \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \Rightarrow 2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2} = \pi$

Also, at $\theta = \frac{\pi}{4}$, $2\tan^{-1}x + \sin^{-}\frac{2x}{1+x^2} = 2\cdot\frac{\pi}{4} + \sin^{-1}1$

$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

$\therefore$ The given function $= \pi =$ constant if $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$,

i.e., $x \in [1, +\infty)$.

The correct option is (A)

56. $8x^2 + 22x + 5 = 0 \Rightarrow x = -\frac{1}{4}, -\frac{5}{2}$.

$\because -1 < -\dfrac{1}{4} < 1$ and $-\dfrac{5}{2} < -1$,

$\therefore\ \sin^{-1}\left(-\dfrac{1}{4}\right)$ exists but $\sin^{-1}\left(-\dfrac{5}{2}\right)$ does not exist,

$\sec^{-1}\left(-\dfrac{5}{2}\right)$ exists but $\sec^{-1}\left(-\dfrac{1}{4}\right)$ does not exist,

$\tan^{-1}\left(-\dfrac{1}{4}\right)$ and $\tan^{-1}\left(-\dfrac{5}{2}\right)$ both exist.

The correct option is (C)

57. First we take $\cos^{-1}\left(\dfrac{y}{\sqrt{1+y^2}}\right) = \theta$

$\Rightarrow\ \cos\theta = \dfrac{y}{\sqrt{1+y^2}}$

then, $\tan\theta = \dfrac{1}{y}$, so $\theta = \tan^{-1}\dfrac{1}{y}$

Hence, $\cos^{-1}\left(\dfrac{y}{\sqrt{1+y^2}}\right) = \theta = \tan^{-1}\dfrac{1}{y}$

Now, we take $\sin^{-1}\left(\dfrac{3}{\sqrt{10}}\right) = \varphi$ or $\sin\varphi = \left(\dfrac{3}{\sqrt{10}}\right)$

Then, $\tan\varphi = 3$ or $\varphi = \tan^{-1} 3$

Substituting the value of θ and φ in the given equation, we get

$$\tan^{-1} x + \tan^{-1}\frac{1}{y} = \tan^{-1} 3 \Rightarrow \tan^{-1}\frac{x+\frac{1}{y}}{1-x\cdot\frac{1}{y}} = \tan^{-1} 3$$

$$\left(\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}\right)$$

or, $\dfrac{x+\frac{1}{y}}{1-\frac{x}{y}} = 3$ or $\dfrac{xy+1}{y-x} = 3$ or $(xy+1) = 3(y-x)$

i.e., $x = 1, y = 2$ or $x = -2, y = -1$

Clearly, $\boldsymbol{x = 1, y = 2}$ **is** the only positive integral solution of the given equation.

The correct option is (A)

58. Let $t_n = \tan^{-1}\left(\dfrac{1}{n^2+n+1}\right)$

$= \tan^{-1}\left(\dfrac{(n+1)-n}{n^2+n+1}\right) = \tan^{-1}\left(\dfrac{(n+1)-n}{1+(n+1)n}\right)$

$= \tan^{-1}(n+1) - \tan^{-1}(n),\ n = 1, 2, 3, \ldots n$

$\Rightarrow t_1 = \tan^{-1} 2 - \tan^{-1} 1$

$t_2 = \tan^{-1} 3 - t_2$

. .

. .

. .

$t_n = \tan^{-1}(n+1) - \tan^{-1} n.$

On adding, we get

$t_1 + t_2 + \ldots t_n = \tan^{-1}(n+1) - \tan^{-1} 1$

$$= \tan^{-1}\left(\frac{(n+1)-1}{1+n+1}\right) = \tan^{-1}\frac{n}{2+n}$$

As $n \to \infty$, it becomes $\tan^{-1}(1) = \dfrac{\pi}{4}$.

Hence,

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \ldots + \tan^{-1}\frac{1}{n^2+n+1} + \ldots \text{upto}\ \infty = \frac{\pi}{4}.$$

The correct option is (B)

59. **Case 1:** If $0 \le x\ \dfrac{1}{2}$,

then $\cos^{-1}\left(\dfrac{x}{2} + \dfrac{1}{2}\sqrt{3-3x^2}\right) = \cos^{-1} x - \cos^{-}\dfrac{1}{2}$

$\therefore$ Equation is $\cos^{-1} x + \cos^{-1} x - \cos^{-1}\dfrac{1}{2}\ \dfrac{\pi}{3} \Rightarrow x = \dfrac{1}{2}$

Case 2: If $\dfrac{1}{2} \le x \le 1$, then

$\cos^{-}\left(\dfrac{x}{2} + \dfrac{1}{2}\sqrt{3-3x^2}\right) = \cos^{-1}\dfrac{1}{2} - \cos^{-1} x$

$\therefore$ Equation is $\cos^{-1} x + \cos^{-1}\dfrac{1}{2} - \cos^{-1} x = \dfrac{\pi}{3}$, which is identity

Hence, the identity holds good for $x \in \left[\dfrac{1}{2}, 1\right]$.

The correct option is (A)

60. Let $\tan^{-1} x = \alpha$ and $\tan^{-1} y = \beta$

$\Rightarrow\ \tan\alpha = x, \tan\beta = y$

The given system of equations is

$a\tan\alpha + b\sec\alpha = c$ and $a\tan\beta + b\sec\beta = c$

$\therefore\ \alpha$ and β are roots of $a\tan\theta + b\sec\theta = c$

$\Rightarrow (b\sec\theta)^2 = (c - a\tan\theta)^2$

$\Rightarrow (a^2 - b^2)\tan^2\theta - 2ac\tan\theta + c^2 - b^2 = 0$

$\therefore\ \tan\alpha + \tan\beta = \dfrac{2ac}{a^2-b^2}$ and $\tan\alpha\tan\beta = \dfrac{c^2b^2}{a^2-b^2}$

$\therefore\ x + y = \dfrac{c^2-b^2}{a^2-b^2}$ and $xy = \dfrac{c^2-b^2}{a^2-b^2}$

$\Rightarrow\ 1 - xy = \dfrac{c^2-b^2}{a^2-b^2}$

$\therefore\ \dfrac{x+y}{1-xy} = \dfrac{2ac}{a^2-c^2}$

The correct option is (B)

61. $\text{cosec}^{-}\sqrt{5}+\text{cosec}^{-}\sqrt{65}+\text{cosec}^{-1}\sqrt{325}+\ldots$

$$=\tan^{-}\frac{1}{2}+\tan^{-}\frac{1}{8}+\tan^{-}\frac{1}{18}+\ldots.+\tan^{-}\frac{1}{2n^2}+\ldots.$$

$$=\underset{n\to\infty}{Lt}\sum_{r=1}^{n}\tan^{-1}\frac{1}{2n^2}$$

$$\underset{n\to\infty}{Lt}\sum_{r=1}^{n}\tan^{-1}\left(\frac{(2n+1)-(2n-1)}{1+(2n-1)(2n+1)}\right)$$

$$\underset{n\to\infty}{Lt}\sum_{r=1}^{n}[\tan^{-1}(2n+1)-\tan^{-1}(2n-1)]$$

$$=\underset{n\to\infty}{Lt}[\tan^{-1}(2n+1)-\tan^{-1}1]$$

$$=\tan^{-1}\infty\ \frac{\pi}{4}=\frac{\pi}{2}-\frac{\pi}{4}\ \frac{\pi}{2}\ \frac{\pi}{4}\ \frac{\pi}{2}$$

The correct option is (C)

62. Given series $=\sum_{r=1}^{n}\cot^{-1}\left(\frac{2^{2r+1}+1}{2^r}\right)$

$$=\sum_{r=1}^{n}\cot^{-1}\left(\frac{2^{2r+1}+1}{2^r}\right)\ \sum_{r=1}^{n}\tan^{-1}\left(\frac{2^r}{1+2^r.2^{r+1}}\right)$$

$$=\sum_{r=1}^{n}\tan^{-1}\left(\frac{2^r}{1+2^r.2^{r+1}}\right)$$

$$=\sum_{r=1}^{n}(\tan^{-1}2^{r+1}-\tan^{-1}2^r)$$

$=\tan^{-1}2^2-\tan^{-1}2+\tan^{-1}2^3$
$-\tan 2^2+\ldots.+\tan^{-1}2^{n+1}-\tan^{-1}2^n$
$=\tan^{-1}2^{n+1}-\tan^{-1}2$

$\therefore$ reqd. sum $=\underset{n\to\infty}{Lt}\tan^{-1}2^{n+1}-\tan^{-1}2$

$$=\frac{\pi}{2}-\tan^{-1}2=\mathbf{cot}^{\ 1}\mathbf{2}$$

The correct option is (B)

63. Let $[\cot^{-1}x]=1$, $[\tan^{-1}x]=1$
$\Rightarrow 1\le\cot^{-1}x<2$, $1\le\tan^{-1}x<2$
$\Rightarrow x\in(\cot 2,\cot 1]$, $x\in[\tan 1,\infty)$
No such x exists. [$\because$ cot 1 < tan 1]
Again let, $[\cot^{-1}x]=2$, $[\tan^{-1}x]=-1$
$\Rightarrow x\in(\cot 3,\cot 2]$, $x\in[0,\tan 1)$
No such x exists. ($\because$ cot 2 < 0)
Again let $[\cot^{-1}x]=3$, $[\tan^{-1}x]=-1$
$\Rightarrow x\in(-\infty,\cot 3]$, $x\in[-\tan 1,0)$
No such x exists. [$\because$ cot 3 < –tan1]
Thus, $[\cot^{-1}x]+[\tan^{-1}x]$ can not be equal to 2 for any x.
The correct option is (D)

64. **(a).** We know that

$$\sin^{-1}x+\cos^{-1}x=\frac{\neq}{2}$$

$\therefore\quad(\sin^{-1}x)^3+(\cos^{-1}x)^3=(\sin^{-1}x+\cos^{-1}x)^3$
$-3\sin^{-1}x\cos^{-1}x(\sin^{-1}x+\cos^{-1}x)$

$$=\left(\frac{\pi}{2}\right)^3\ \frac{3\pi}{2}\ \sin^{-1}x\cos^{-1}x$$

$$=\frac{\pi^3}{8}-\frac{3\pi}{2}\sin^{-1}x\cos^{-1}x$$

$$\therefore\ \frac{3\pi}{2}\sin^{-1}x\cos^{-1}x=\frac{\pi^3}{8}-a\pi^3=\pi^3\left(\frac{1}{8}-a\right)$$

$$\Rightarrow\ \sin^{-1}\left(\frac{\pi}{2}-\sin^{-1}x\right)=\frac{\pi^2}{12}(1-8\alpha)$$

$$\Rightarrow\ (\sin^{-1}x)^2\ \frac{\pi}{2}\ \sin^{-1}x+\frac{\pi^2}{12}(1-8\alpha)=0$$

$$\text{Disc.}=\frac{\pi^2}{4}-\frac{\pi^2}{3}(1-8\alpha)=\frac{\pi^2}{12}(3-4+32\alpha)$$

$$=\frac{\pi^2}{12}(32\alpha-1)<\mathbf{0}$$

$\therefore$ given equation has no solution.
The correct option is (A)

65. We have, $\frac{\alpha^3}{2}\text{cosec}\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)+\frac{\beta^3}{2}\sec\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$

$$=\alpha^3\frac{1}{1-\cos\left(\tan^{-1}\frac{\alpha}{\beta}\right)}+\beta\frac{1}{1+\cos\left(\tan^{-1}\frac{\beta}{\alpha}\right)}$$

$$=\ \alpha\frac{1}{1-\cos\left(\cos^{-1}\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}$$

$$+\beta^3\frac{1}{1+\cos\left(\cos^{-1}\frac{1}{\sqrt{\alpha^2+\beta^2}}\right)}$$

$$=\alpha^3\frac{1}{1-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}+\beta^3.\frac{1}{1+\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}}$$

$$=\sqrt{\alpha^2+\beta^2}\left[\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}-\beta}+\frac{\beta^3}{\sqrt{\alpha^2+\beta^2}+\beta}\right]$$

$$=\sqrt{\alpha^2+\beta^2}\left[\frac{\alpha^3(\sqrt{\alpha^2+\beta^2}+\beta)}{\alpha^2}+\frac{\beta^3(\sqrt{\alpha^2+\beta^2}-\alpha)}{\beta^2}\right]$$

$$\sqrt{\alpha^2+\beta^2}\left[\alpha(\sqrt{\alpha^2+\beta^2}+\beta)+\beta(\sqrt{\alpha^2+\beta^2}-\alpha)\right]$$

$$=\sqrt{\alpha^2+\beta^2}\left[(\alpha+\beta)(\sqrt{\alpha^2+\beta^2})\right]=(\alpha+\beta)\ (\alpha2+\beta2)$$

The correct option is (C)

66. **(c).** Let $S=\sin^{-}\frac{1}{\sqrt{2}}+\sin^{-}\frac{\sqrt{2}-1}{\sqrt{6}}+\sin\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}$

$$+......+\sin^{-}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$$

Now, $T_n=\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$

$$=\sin^{-}\left[\frac{1}{\sqrt{n}}\sqrt{1-\left(\frac{1}{\sqrt{n+1}}\right)^2}-\frac{1}{\sqrt{n+1}}\sqrt{1-\left(\frac{1}{\sqrt{n}}\right)^2}\right]$$

$$=\sin^{-}\frac{1}{\sqrt{n}}-\sin^{-}\frac{1}{\sqrt{n+1}}$$

$$\therefore\ S=\sin^{-}\frac{1}{\sqrt{2}}+\left(\sin^{-1}\frac{1}{\sqrt{2}}-\sin\frac{1}{\sqrt{3}}\right)$$

$$+\left(\sin^{-1}\frac{1}{\sqrt{3}}-\sin\frac{1}{\sqrt{4}}\right)+....+\infty$$

$$=2\sin^{-}\frac{1}{\sqrt{2}}=\left(\frac{\pi}{4}\right)=\frac{\pi}{2}$$

The correct option is (C)

67. $S=\cot^{-1}2.1^2+\cot^{-1}2.2^2+\cot^{-1}2.3^2+\cot^{-1}2.4^2$ $+......$

$$T_n=\cot^{-1}2n^2=\tan^{-}\frac{1}{2n^2}=\tan^{-}\left(\frac{2}{4n^2}\right)$$

$$=\tan^{-}\left[\frac{(2n+1)-(2n-1)}{1+4(4n^2-1)}\right]$$

$$=\tan^{-1}(2n+1)-\tan^{-1}(2n-1)$$

$$\therefore\ S=\tan^{-1}3-\tan^{-1}1+\tan^{-1}5-\tan^{-1}3$$

$$+\tan^{-1}7-\tan^{-1}5+.....\infty$$

$$=\tan^{-1}\infty-\tan^{-1}1=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

The correct option is (B)

More than One Option Correct Type

68. We have, $(\sin^{-1}x)^2+(\cos^{-1}x)^2=\frac{5\pi^2}{8}$

$$\Rightarrow(\sin^{-1}x)^2+\left(\frac{\pi}{2}-\sin^{-1}x\right)^2=\frac{5\pi^2}{8}$$

$$\Rightarrow 2(\sin^{-1}x)^2=-\pi\sin^{-1}x-\frac{3\pi^2}{8}=0$$

$$\Rightarrow\sin^{-1}x=\frac{\pi\pm\sqrt{4\pi^2}}{4}\ \Rightarrow\sin^{-1}x=\frac{3\pi}{4}\ \text{or}-\frac{\pi}{4}$$

$$\Rightarrow x=\sin\frac{3\pi}{4}\ \text{or}\ \sin\left(-\frac{\pi}{4}\right)$$

$$\therefore\ x=\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\ \text{or}\ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}.$$

69. $\sin\left[2\cos^{-1}\{\cot(2\tan^{-1}x)\}\right]=0$

$$\Rightarrow\sin\left[2\cos^{-1}\left\{\cot\tan^{-1}\frac{2x}{1-x^2}\right\}\right]=0$$

$$\Rightarrow\sin\left[2\cos^{-1}\left\{\cot\cot^{-1}\left(\frac{1-x^2}{2x}\right)\right\}\right]=0$$

$$\Rightarrow\sin\left[2\cos^{-1}\frac{1-x^2}{2x}\right]=0$$

Let $\cos^{-1}\left(\frac{1-x^2}{2x}\right)=\theta$

$$\Rightarrow\cos\theta=\left(\frac{1-x^2}{2x}\right)$$

$$\therefore\ \cos^{-1}\left(\frac{1-x^2}{2x}\right)=\theta$$

$$\Rightarrow\sin\theta=\frac{\sqrt{[(4x^2-(1-x^2)^2]}}{2x}.$$

$$\therefore\ \sin\left[2\cos^{-1}\frac{1-x^2}{2x}\right]=0$$

$$\Rightarrow\sin(2\cos^{-1}\cos\theta)=0\Rightarrow\sin2\theta=0$$

$$\Rightarrow 2\sin\theta\cos\theta=0$$

$$\Rightarrow 2\times\sqrt{\left(\frac{4x^2-(1-x^2)^2}{2x}\right)}\times\frac{1-x^2}{2x}=0$$

If $\left(\frac{1-x^2}{2x}\right)=0\Rightarrow 1-x^2=0\ \therefore\ x=\pm 1$

If $\sqrt{\left(\frac{4x^2-(1-x^2)^2}{2x}\right)} = 0$

$\Rightarrow 4x^2 - (1 - x^2)^2 = 0$

$\Rightarrow 4x^2 = (1 - x^2)^2 \Rightarrow 2x = \pm (1 - x^2)$

From positive sign

$\Rightarrow 2x = 1 - x^2 \Rightarrow x^2 + 2x - 1 = 0$

$x = \frac{-2\pm\sqrt{[4-4(1)(-1)]}}{2\times 1} = \frac{-2\pm\sqrt{8}}{2} = -1 \pm \sqrt{2}$

From negative sign

$2x = -1 + x^2 \Rightarrow x^2 - 2x - 1 = 0$

$x = \frac{-2\pm\sqrt{[4-4(1)(-1)]}}{2\times 1} = \frac{-2\pm\sqrt{8}}{2} = 1 \pm \sqrt{2}$

$\therefore$ **$x = \pm 1, \pm(1 \pm \sqrt{2})$.**

70. Put $x = \tan\theta$, given equality becomes $\tan^{-1} y = 4\theta$

$\Rightarrow y = \tan 4\theta \quad \frac{2\tan 2\theta}{1-\tan^2 2\theta} \quad \frac{2\left[\frac{2\tan\theta}{1-\tan^2\theta}\right]}{1-\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)}$

$\frac{2\times 2x(1-x^2)}{(1-x^2)^2-4x^2} \quad \frac{4x(1-x^2)}{1-6x^2+x^4}$

Therefore, y is finite if $x^4 - 6x^2 + 1 \neq 0$

$\Rightarrow x2 \neq \frac{6\pm\sqrt{36-4}}{2} \neq 3\pm 2\sqrt{2}$

71. $\sin^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ if $-1 \le x \le 1$

Since $\frac{1}{2} \in [-1, 1]$, $\therefore \left(-\frac{1}{2}\right) = \frac{\pi}{2}$

Again, $\frac{1}{1+k^2} \le 1 \qquad \forall k \in R$

$\therefore \left(\frac{1}{1+k^2}\right) = \frac{\pi}{2}$

$k^2 - 2k + 3 = (k-1)^2 + 2 \ge 2$

$\therefore$ (b) does not hold. Also (d) does not hold clearly.

72. We have

$x = \text{cosec}\,(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a)))))$

$= \text{cose}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\sec\left(\sec^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)\right)$

$= \text{cose}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)$

$= \text{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\frac{1}{\sqrt{2-a^2}}\right)\right)\right)$

$= \text{cosec}\left(\tan^{-1}\frac{1}{\sqrt{2-a^2}}\right)$

$= \text{cosec}\,(\text{cosec}^{-1}\sqrt{3-a^2})) \quad \sqrt{3-a^2}$

$\therefore x^2 + a^2 = 3$

Also, $y = \sec\,(\cot^{-1}(\sin(\tan^{-1}(\text{cosec}(\cos^{-1}a)))))$

$= \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\text{cosec}\left(\text{cosec}^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)\right)$

$= \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)$

$= \sec\left(\cot^{-1}\left(\sin\left(\sin^{-1}\frac{1}{\sqrt{2-a^2}}\right)\right)\right)$

$= \sec\left(\cot^{-1}\frac{1}{\sqrt{2-a^2}}\right) = \sec\left(\sec^{-1}\sqrt{3-a^2}\right)$

$= \sqrt{3-a^2}$

$\therefore$ **$x2 + a2 = 3$.** Also, clearly **$x = y$**

73. $\cos^{-1}x + (\sin^{-1}y)^2 = \frac{p\pi^2}{4}$ ....(1)

and, $(\cot^{-1}x)(\sin^{-1}y)^2 \quad \frac{\pi^2}{16}$ .....(2)

Let $\cos^{-1}x = a \;\therefore a \in [0, \pi]$

and, $\sin^{-1}y = b \;\therefore b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, $a + b^2 = \frac{p\pi^2}{4}$ ....(3)

and, $ab^2 = \frac{\pi^2}{16}$ ....(4)

Since $b^2 \left[0, \frac{\pi^2}{4}\right]$, $\therefore a + b^2 \in \left[0, \pi + \frac{\pi^2}{4}\right]$

$\therefore \quad 0 \le \frac{p\pi^2}{4} \le \pi + \frac{\pi^2}{4}$

i. e., $\mathbf{0 \le p \le} \frac{4}{\pi} + 1 \therefore$ (a) holds.

Again, if p is an integer, then $p = 0$, 1 or 2.

But if $p = 0$, $a = b = 0 \therefore$ (4) is not satisfied.

Again, putting the value of b^2 from (3) in (4), we get $a\left(\frac{p\pi^2}{4}-a\right)=\frac{\pi^2}{16}$

$\Rightarrow 16a^2 - 4p\pi^2 a + \pi^2 = 0$

Since $a \in R$, i.e., a is real $\therefore$ Disc. ≥ 0 ...(5)

$\therefore\ 16p^2\pi^4 - 64\pi^2 \geq 0$

$\Rightarrow p^2\pi^2 \geq 4 \Rightarrow p \geq \frac{2}{\pi}$

$\therefore$ **only integral value of p which satisfies all the conditions is $p = 2$** $\therefore$ (b) holds.

Match the Column Type

74. **I** $\cot^{-1} 9 + \text{cosec}^{-1}\frac{\sqrt{41}}{4} = \cot^{-1} 9 + \cot^{-1}\sqrt{\frac{41}{16}-1}$

$\left[\because \text{cosec}^{-1}x = \cot^{-1}\sqrt{x^2-1}\right]$

$= \cot^{-1} 9 + \cot^{-1}\frac{5}{4} = \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{4}{5}$

$= \tan^{-1}\left(\frac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}\cdot\frac{4}{5}}\right) = \tan^{-1}\left(\frac{41}{41}\right) = \tan^{-1} 1 = \frac{\boldsymbol{\pi}}{\mathbf{4}}$.

II We have, $\sin^{-1}(\cos(\sin^{-1}x)) + \cos^{-1}(\sin(\cos^{-1}x))$

$= \sin^{-1}\left[\cos\left(\frac{\pi}{2}-\cos^{-1}x\right)\right]$

$+ \cos^{-1}\left[\sin\left(\frac{\pi}{2}-\sin^{-1}x\right)\right]$

$= \sin^{-1}[\sin(\cos^{-1}x)] + \cos^{-1}[\cos(\sin^{-1}x)]$

$= \cos^{-1}x + \sin^{-1}x = \frac{\boldsymbol{\pi}}{\mathbf{2}}$.

III Let $\cot^{-1}(\sqrt{2}-1) = \theta$

$\Rightarrow \cot\theta = \sqrt{2}-1$

Now, $\cos^{-1}(\cos 2\theta) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}\left(\frac{\cot^2\theta - 1}{\cot^2\theta + 1}\right) = \cos^{-1}\left(\frac{2-2\sqrt{2}}{4-2\sqrt{2}}\right)$

$\therefore\ \cos^{-1}\cos(2\theta) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$.

IV $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16}$

$= \tan^{-1}\frac{12}{5} + \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{63}{16}$

$\left[\because \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} \text{ and } \cos^{-1}x = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right]$

$= \pi + \tan^{-1}\left(\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\cdot\frac{3}{4}}\right) + \tan^{-1}\frac{63}{16}$

$\left[\because \text{ if } xy > 1 \text{ then } \tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= \pi + \tan^{-1}\left(-\frac{63}{16}\right) + \tan^{-1}\frac{63}{16}$

$= \pi - \tan^{-1}\left(\frac{63}{16}\right) + \tan^{-1}\frac{63}{16}$

$\because \tan^{-1}(-x) = -\tan^{-1}x]$

$= \pi$.

75. **I** $\cos(2\cos^{-1}x + \sin^{-1}x)$

$= \cos(\cos^{-1}x + \cos^{-1}x + \sin^{-1}x)$

$= \cos\left(\frac{\pi}{2} + \cos^{-1}x\right) = -\sin(\cos^{-1}x)$

$= -\sin(\sin^{-1}\sqrt{1-x^2}) = -\sqrt{1-x^2}$

$= -\sqrt{1-\frac{1}{25}} = \frac{\mathbf{2\sqrt{6}}}{\mathbf{5}}$.

II We have, $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$

$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1} 1$

$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}\frac{1}{5}$

$\Rightarrow \cos^{-1}x = \cos^{-1}\frac{1}{5} \quad \therefore \quad x = \frac{1}{5}.$

III $\tan\left\{\cos^{-1}\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right\}$

$= \tan\left\{\pi - \cos^{-1}\left(\frac{2}{7}\right) - \frac{\pi}{2}\right\}$

$= \tan\left\{\frac{\pi}{2} - \cos^{-1}\left(\frac{2}{7}\right)\right\} = \tan\left\{\sin^{-1}\left(\frac{2}{7}\right)\right\}$

$= \tan\tan^{-1}\left(\frac{2}{3\sqrt{5}}\right) = \frac{2}{3\sqrt{5}}$

IV Let $\alpha = \cos^{-1}\sqrt{p}$, $\beta = \cos^{-1}\sqrt{1-p}$

and, $\gamma = \cos^{-1}\sqrt{1-q}$ or $\cos\alpha = \sqrt{p}$, $\cos\beta = \sqrt{1-p}$

and, $\cos\gamma = \sqrt{1-q}$

Therefore, $\sin\alpha = \sqrt{1-p}$, $\sin\beta = \sqrt{p}$, $\sin\gamma = \sqrt{q}$

The given equation may be written as $\alpha + \beta + \gamma = \frac{3\pi}{4}$

or, $\alpha + \beta = \frac{3\pi}{4} - \gamma$ or $\cos(\alpha+\beta) = \cos\{\pi - (\pi/4 + \gamma)\}$

$\Rightarrow \cos\alpha\cos\beta - \sin\gamma\sin\beta \quad = \cos\{\pi - (\pi/4 + \gamma)\}$

$= -\cos\left(\frac{\pi}{4} + \gamma\right)$

$\Rightarrow \sqrt{p}\sqrt{1-p} - \sqrt{1-p}\sqrt{p}$

$= -\left(\frac{1}{\sqrt{2}}\sqrt{1-q} - \frac{1}{\sqrt{2}}\sqrt{q}\right)$

$\Rightarrow \quad 0 = \sqrt{1-q} - \sqrt{q} \Rightarrow 1 - q = q \Rightarrow q = \frac{1}{2}$

Assertion-Reason Type

76. $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$

$\Rightarrow \tan^{-1}\left(\frac{1}{\sqrt{\cos\alpha}}\right) - \tan^{-1}(\sqrt{\cos\alpha}) = x$

$\Rightarrow \tan^{-1}\frac{\frac{1}{\sqrt{\cos\alpha}} - \sqrt{\cos\alpha}}{1 + \frac{1}{\sqrt{\cos\alpha}}\cdot\sqrt{\cos\alpha}} = x$

$\Rightarrow \tan^{-1}\frac{1-\cos\alpha}{2\sqrt{\cos\alpha}} = x \Rightarrow \tan x = \frac{1-\cos\alpha}{2\sqrt{\cos\alpha}}$

or, $\cot x = \frac{2\sqrt{\cos\alpha}}{1-\cos\alpha}$ or $\operatorname{cosec} x = \frac{1+\cos\alpha}{1-\cos\alpha}$

$\therefore \sin x = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{1-(1-2\sin^2\alpha/2)}{1+2\cos^2\alpha/2-1}$

or, $\sin x = \tan^2\alpha/2$

77. Let $a = b\cos\theta$,

then, $a_1 = \frac{a+b}{2} = \frac{b(1+\cos\theta)}{2} = b\cos^2\frac{\theta}{2}$

$\Rightarrow b_1 = \sqrt{a_1 b} = b\cos\frac{\theta}{2}$

Now, $a_2 = \frac{a_1 + b_1}{2} \quad b\cos\frac{\theta}{2}\cos^2\frac{\theta}{4}$

$\therefore b_2 = b\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}$

Similarly, $b_3 = b\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}$

and so on.

Now, $b_n = b\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\ldots\cos\frac{\theta}{2^n}$

$\therefore b_\infty = \lim_{n\to\infty}\frac{b\sin\theta}{2^n\sin\left(\frac{\theta}{2^n}\right)} = \lim_{n\to\infty}\frac{b\sin\theta}{\theta\left(\frac{\sin(\theta/2^n)}{\theta/2^n}\right)}$

$= \frac{b\sin\theta}{\theta} = \frac{b\sqrt{1-\cos^2\theta}}{\cos^{-1}(a/b)}$

$\therefore \quad b_\infty = \frac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}$

78. Given, $\theta = \cos^{-1}\left(\frac{x}{2}\cdot\frac{y}{3} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right)$

$\Rightarrow \cos\theta = \frac{xy}{6} - \frac{\sqrt{4-x^2}\sqrt{9-y^2}}{6}$

$\Rightarrow (xy - 6\cos\theta)^2 = (4-x^2)(9-y^2)$

$= 36 - 9x^2 - 4y^2 + x^2y^2$

$\Rightarrow 36\cos^2\theta - 12xy\cos\theta + 4y^2 + 9x^2 = 36$

$\therefore 9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

79. The given expression

$= \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13}\right) + \sin^{-1}\frac{16}{65}$

$= \sin^{-1}\left[\frac{4}{5}\sqrt{1-\frac{25}{169}} + \frac{5}{13}\sqrt{1-\frac{16}{25}}\right] + \sin^{-1}\frac{16}{25}$

$\left[\because \sin^{-1}x+\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right]\right]$

$=\sin^{-1}\left(\frac{4}{5}\cdot\frac{12}{13}+\frac{5}{13}\cdot\frac{3}{5}\right)+\sin^{-1}\frac{16}{25}$

$=\sin^{-1}\frac{63}{65}+\sin^{-1}\frac{16}{65}=\sin^{-1}\frac{63}{65}+\cos^{-1}\sqrt{1-\left(\frac{16}{65}\right)^2}$

$\left[\because \sin^{-1}x=\cos^{-1}\sqrt{1-x^2}\right]$

$=\sin^{-1}\frac{63}{65}+\cos^{-1}\frac{63}{65}=\frac{\pi}{2}.$

Previous Year's Questions

80. We know that

$\cot^{-1}(\sqrt{\cos\alpha})+\tan^{-1}(\sqrt{\cos\alpha})=\frac{\pi}{2}$ ... (i)

and given that

$\cot^{-1}(\sqrt{\cos\alpha})-\tan^{-1}(\sqrt{\cos\alpha})=x$ ...(ii)

Adding above eqs. (i) and (ii), we obtain

$2\cot^{-1}(\sqrt{\cos\alpha})=\frac{\pi}{2}+x$

$\Rightarrow \sqrt{\cos\alpha}=\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$\Rightarrow \sqrt{\cos\alpha}=\dfrac{\cot\frac{x}{2}-1}{1+\cot\frac{x}{2}}$

$\Rightarrow \sqrt{\cos\alpha}=\dfrac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}$

$\Rightarrow \cos\alpha=\dfrac{1-\sin x}{1+\sin x}$

$\Rightarrow \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\dfrac{1-\sin x}{1+\sin x}$

$\Rightarrow \sin x=\tan^2\frac{\alpha}{2}$

Alternate Solution

$\because \cot^{-1}(\sqrt{\cos\alpha})-\tan^{-1}(\sqrt{\cos\alpha})=x$

$\Rightarrow \tan^{-1}\left(\frac{1}{\sqrt{\cos\alpha}}\right)-\tan^{-1}(\sqrt{\cos\alpha})=x$

$\Rightarrow \tan^{-1}\left(\dfrac{\frac{1}{\sqrt{\cos\alpha}}-\sqrt{\cos\alpha}}{1+\frac{1}{\sqrt{\cos\alpha}}\cdot\sqrt{\cos\alpha}}\right)=x$

(Using $\tan^{-1}(A)-\tan^{-1}(B)=\tan^{-1}(\frac{A-B}{1+AB})$)

$\Rightarrow \dfrac{1-\cos\alpha}{2\sqrt{\cos\alpha}}-\tan x$

$\Rightarrow \cot x=\dfrac{2\sqrt{\cos\alpha}}{1-\cos\alpha}$

$\because \operatorname{cosec} x=\sqrt{1+\cot^2 x}$

$\therefore \operatorname{cosec} x=\dfrac{1+\cos\alpha}{1-\cos\alpha}$

$\Rightarrow \sin x=\dfrac{1-\cos\alpha}{1+\cos\alpha}=\tan^2\frac{\alpha}{2}$

81. Since, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\therefore \tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\tan^{-1}\left(\dfrac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right)$

$=\tan^{-1}\left(\frac{1}{2}\right)$

82. $-\frac{\pi}{4}\le\frac{\sin^{-1}x}{2}\le\frac{\pi}{4}$

$\Rightarrow -\frac{\pi}{4}\le\sin^{-1}(a)\le\frac{\pi}{4}$

$\Rightarrow |a|\le\frac{1}{\sqrt{2}}$

Hence, (D) is the correct answer

83.

Given equation implies

$$\frac{x}{5}\sqrt{1-\frac{16}{25}}+\frac{4}{5}\sqrt{1-\frac{x^2}{25}}=1$$

$$\Rightarrow \frac{4}{5}\sqrt{1-\frac{x^2}{25}}=1-\frac{3x}{25}$$

$$\Rightarrow \frac{4}{5}\sqrt{\frac{25-x^2}{25}}=\frac{25-3x}{25}$$

$$\Rightarrow 4\sqrt{25-x^2}=25-3x$$

$$\Rightarrow 16(25-x^2)=(25-3x)^2$$

$$\Rightarrow x=3$$

84.

Sol: For given function , the derivative is given by

$$f'(x)=\frac{1}{1+(\sin x+\cos x)^2}(\cos x-\sin x)$$

$$=\frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+(\sin x+\cos x)^2}$$

Now, f(x) is increasing if $-\frac{\pi}{2}<x+\frac{\pi}{4}<\frac{\pi}{2}$

i.e. when x $\in \left(-\frac{\pi}{2},\frac{\pi}{4}\right)$

85.

Let E = $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right)$, then

$$E=\cot\left(\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$\Rightarrow E=\cot\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{2}\cdot\frac{2}{3}}\right)\right)$$

$$\Rightarrow E=\cot\left(\tan^{-1}\frac{17}{6}\right)=\frac{6}{17}$$

86.

If x, y, z **are in A.P., then**

$2y=x+z$

Also $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are in A.P. which implies

$$2\tan^{-1}y=\tan^{-1}x+\tan^{-1}z\Rightarrow x=y=z.$$

Note: If y = 0, then none of the options is appropriate.

87. $\tan^{-1}y=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$=\tan^{-1}\left(\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right)$$

$$=\tan^{-1}\left(\frac{x-x^3+2x}{1-x^2-2x^2}\right)$$

$$\therefore\ \tan^{-1}y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\therefore\ y=\frac{3x-x^3}{1-3x^2}.$$