CHAPTER

2

Functions

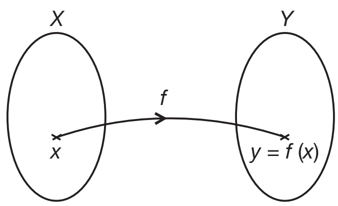
Chapter Highlights

Function or mapping, Features of a mapping $f : X \to Y$, Value of a function, Domain and range of a function, Intervals in R, Method to find the domain of a function, Method to find the range of a function, Types of functions, Method to check whether the function $f : X \to Y$ is one–one or many-one, Onto or surjective function, Method to check whether the function $f : X \to Y$ is onto or Into, Bijective function, Some important functions, Identity function, Modulus function or absolute value function, Greatest integer function/step function/floor function, Least integer function/ceiling function, Fractional-part function, Signum function, Reciprocal function, Exponential function, Logarithmic function, Polynomial function, Rational function, Trigonometric functions, Inverse trigonometric functions, Two ways of defining a function, Explicit and implicit functions, Operations on functions, Composition of functions, Properties of composite functions, Inverse functions, Method to find the inverse of a function, Properties of inverse functions, Odd and even functions, Properties of odd and even functions, Periodic function, Short-cut method to check the periodicity of a function.

FUNCTION OR MAPPING

Let X and Y be any two non-empty sets and there be correspondence or association between the elements of X and Y such that for every element $x \in X$, there exists a unique element $y \in Y$, written as $y = f(x)$. Then we say that f is a mapping or function from X to Y, and is written as

$$f : X \to Y \text{ such that } y = f(x), x \in X, y \in Y$$



IMPORTANT POINTS

- If $f : X \to Y$ be a function from a non-empty set X to another non-empty set Y, where $X, Y \subseteq R$ (set of all real numbers), then we say that f is a **real valued function** or in short a **real function**.
- Throughout this chapter a 'function' will mean a 'real function'.

FEATURES OF A MAPPING $F : X \to Y$

1. For each element $x \in X$, there exist a unique element $y \in Y$.
2. The element $y \in Y$ is called the image of x under the mapping f.
3. If there is an element in X which has more than one image in Y, then $f : X \in Y$ is not a function. But distinct elements of X may be associated to the same element of Y.
4. If there is an element in X which does not have an image in Y, then $f : X \to Y$ is not a function.

CAUTION

- If $f : X \to Y$ is a function, then there may be some elements in Y, which are not images of elements of X.
- But there should not be any x left (element of X) for which there is no elements in set Y.

VALUE OF A FUNCTION

The value of a function $y = f(x)$ at $x = a$ is denoted by $f(a)$. It is obtained by putting $x = a$ in $f(x)$.

NOTE

- If for some value of x say $x = a$, $f(a)$ takes the form $\frac{0}{0}$, we say that $f(a)$ is indeterminate.
- If for some value of x say $x = a$, the denominator vanishes, we say that $f(a)$ is undefined (or does not exist)

SOLVED EXAMPLE

1. If $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real number x, then the minimum value of f

(A) does not exist because f is unbounded
(B) is not attained even though f is bounded
(C) is equal to 1
(D) is equal to –1

Solution: (D)

We have,

$$f(x) = \frac{x^2 - 1}{x^2 + 1} = 1 - \frac{2}{x^2 + 1}.$$

$f(x)$ will attain its minimum value when $\frac{2}{x^2 + 1}$ is maximum, i.e. when $x^2 + 1$ is minimum i.e. at $x = 0$.

$\therefore$ minimum value of $f(x)$ is $f(0) = -1$.

DOMAIN AND RANGE OF A FUNCTION

If $f : X \to Y$ be a function, then the set X is said to be the domain of f and range of f

= set of all image points in Y under the map f.

$= f(X) = \{ f(x) : f(x) \in Y; x \in X\}$

The set Y is also called the co-domain of f. Clearly $f(X) \subseteq Y$ as shown in Fig. 2.1.

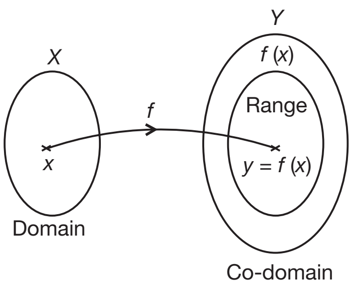


Fig. 2.1

In other words, we can say

Domain = All possible values of x for which $f(x)$ exists.

Range = For all values of x, all possible values of $f(x)$ as shown in Fig. 2.2.

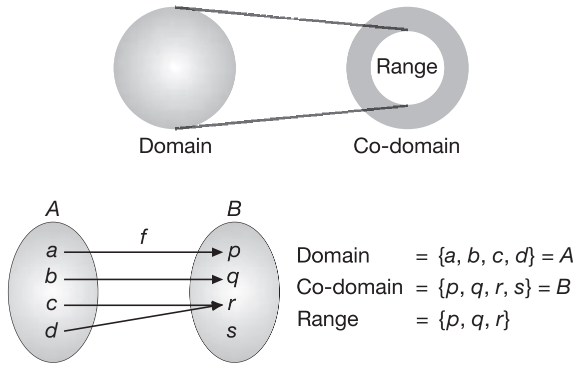


Fig. 2.2

NOTE

- Range is the subset of co-domain
- Range can never exceed co-domain for a given function.
- The projection of the graph of $y = f(x)$ on the x-axis is equal to the domain of f whereas the projection on the y-axis is equal to the range of f.

TRICK(S) FOR PROBLEM SOLVING

The total number of functions from set A to set B containing m and n elements respectively is n^m.

INTERVALS IN *R*

The set of all numbers lying between two given real numbers is called an interval in R.

Let a and b be any two real numbers such that $a < b$, then we define the following types of intervals.

1. Closed interval [*a*, *b*]

= closed interval from a to b

$= \{x : x \in R; a \leq x \leq b\}$

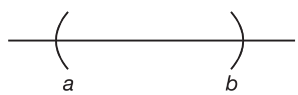
= set of all real numbers lying between a and b including the end points a and b.

2. Open interval (*a*, *b*) or]*a*, *b*[

= open interval from a to b

$= \{x : x \in R; a < x < b\}$

= set of all real numbers lying between a and b, excluding the end points a and b.



3. Closed-open interval [*a*, *b*) and open–closed interval (*a*, *b*]

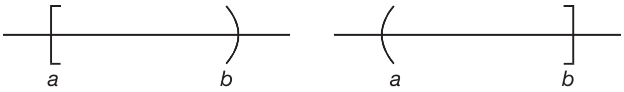


Fig. 2.3

$[a, b) = \{x : x \in R; a \le x < b\}$
$(a, b] = \{x : x \in R; a < x \le b\}$

4. Real number set *R* as an open interval

We introduce two special numbers $-\infty$ and $+\infty$, where
$-\infty = a$ number less than any real number,
$+\infty = a$ number greater than any real number.
$-\infty < x$ for all $x \in R$, and $x < \infty$ for all $x \in R$.
Hence, the set R can be thought of as the open interval $(-\infty, \infty)$, so that

$$R = (-\infty, \infty) = \{x : x \in R; -\infty < x < \infty\}$$

Also, the infinite intervals in R can be given by,

$$(-\infty, a), (a, +\infty), (-\infty, a], [a, +\infty)$$

NOTE

The numbers $+\infty$ and $-\infty$ do not follow the ordinary rules of arithmetic, i.e., $\infty - \infty \ne 0$, $0 \times \infty \ne 0$, $\frac{0}{0} \ne 1$, $\frac{\infty}{\infty} \ne 1$, $\infty + \infty \ne 2\infty$, $1^\infty \ne 1$, $0^0 \ne 1$ etc.

METHOD TO FIND THE DOMAIN OF A FUNCTION

Algebraic Functions

1. Denominator should be non-zero.
2. Expression under the even root should be non-negative.

Trigonometric Functions

1. $\sin x$ and $\cos x$ are defined for all real values of x.
2. $\tan x$ and $\sec x$ are defined for all real values of x except $x = (2n + 1)\,\frac{\pi}{2}$, where $n \in Z$.
3. $\cot x$ and $\operatorname{cosec} x$ are defined for all real values of x except $x = n\pi$, where $n \in Z$.

Inverse Trigonometric Functions

1. $\sin^{-1}x$ and $\cos^{-1}x$ are defined for $-1 \le x \le 1$.
2. $\tan^{-1}x$ and $\cot^{-1}x$ are defined for all real values of x.
3. $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$ are defined for $x \le -1$ or $x \ge 1$.

Logarithmic Functions

1. $\log_b a$ is defined when $a > 0$, $b > 0$ and $b \ne 1$.

Exponential Functions

1. a^x is defined for all real values of x, where $a > 0$.

NOTE

- $(x - a)\ (x - b) > 0 \Rightarrow x < a$ or $x > b$, for $a < b$
- $(x - a)\ (x - b) < 0 \Rightarrow a < x < b$, for $a < b$
- $|x| < a \Rightarrow -a < x < a$
- $|x| > a \Rightarrow x < -a$ or $x > a$
- $\log_b a > k \Rightarrow \begin{cases} a > b^k, & \text{if } b > 1 \\ a < b^k, & \text{if } b < 1 \end{cases}$
- If $\log_a x > \log_a y$, then if $a > 1$ then $x > y$ and if $0 < a < 1$ then $x < y$
- $\sqrt{x^2} = |x|$
- $\sqrt[n]{x^n} = |x|$, if n is even and $\sqrt[n]{x^n} = x$, if n is odd.

SOLVED EXAMPLES

2. The domain of the function $f(x) = \sqrt{x - \sqrt{1 - x^2}}$ is

(A) $\left[-1, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, 1\right]$

(B) $[-1, 1]$

(C) $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{\sqrt{2}}, +\infty\right)$

(D) $\left[\frac{1}{\sqrt{2}}, 1\right]$

Solution: (D)

For $f(x)$ to be defined, we must have

$$x - \sqrt{1 - x^2} \ge 0 \text{ or } x \ge \sqrt{1 - x^2} > 0$$

$$\therefore \quad x^2 \ge 1 - x^2 \text{ or } x^2 \ge \frac{1}{2}$$

Also, $\quad 1 - x^2 \ge 0$ or $x^2 \le 1$

Now, $\quad x^2 \ge \frac{1}{2} \Rightarrow \left(x - \frac{1}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}}\right) \ge 0$

$$\Rightarrow \quad x \le -\frac{1}{\sqrt{2}} \quad \text{or} \quad x \ge \frac{1}{\sqrt{2}}$$

Also, $\quad x^2 \le 1 \Rightarrow (x - 1)(x + 1) \le 0 \Rightarrow -1 \le x \le 1$

Thus, $\quad x > 0, x^2 \ge \frac{1}{2}$ and $x^2 \le 1 \Rightarrow x \in \left[\frac{1}{\sqrt{2}}, 1\right]$

3. The domain of the function

$$f(x) = \sin^{-1}\left\{\log_2\left(\frac{1}{2}x^2\right)\right\} \text{ is}$$

(A) $[-2, -1) \cup [1, 2]$ (B) $(-2, -1] \cup [1, 2]$
(C) $[-2, -1] \cup [1, 2]$ (D) $(-2, -1) \cup (1, 2)$

Solution: (C)

For $f(x)$ to be defined, we must have

$$-1 \le \log_2\left(\frac{1}{2}x^2\right) \le 1 \Rightarrow 2^{-1} \le \frac{1}{2}x^2 \le 2^1$$

[$\because$ the base = 2 > 1]

$\Rightarrow \quad 1 \le x^2 \le 4 \quad (1)$

Now, $1 \le x^2 \Rightarrow x^2 - 1 \ge 0$ i.e. $(x-1)(x+1) \ge 0$

$\Rightarrow \quad x \le -1$ or $x \ge 1 \quad (2)$

Also, $x^2 \le 4 \Rightarrow x^2 - 4 \le 0$ i.e. $(x-2)(x+2) \le 0$

$\Rightarrow \quad -2 \le x \le 2 \quad (3)$

From Eq. (2) and (3), we get the domain of f

$$= ((-\infty, -1] \cup [1, \infty)) \cap [-2, 2]$$

$$= [-2, -1] \cup [1, 2].$$

4. The domain of the function

$$f(x) = \frac{1}{\sqrt{x^{12} - x^9 + x^4 - x + 1}} \text{ is}$$

(A) $(-\infty, -1)$ (B) $(1, \infty)$
(C) $(-1, 1)$ (D) $(-\infty, \infty)$

Solution: (D)

$f(x)$ is defined for

$$x^{12} - x^9 + x^4 - x + 1 > 0$$

$\Rightarrow \quad x^4(x^8 + 1) - x(x^8 + 1) + 1 > 0$

$\Rightarrow \quad (x^8 + 1)\, x\,(x^3 - 1) + 1 > 0$

If $x \ge 1$ or $x \le -1$, then the above expression is positive.
If $-1 < x \le 0$, the above inequality still holds.
If $0 < x < 1$, then $x^{12} - x(x^8 + 1) + (x^4 + 1) > 0$
[$\because x^4 + 1 > x^8 + 1$ and so $x^4 + 1 > x(x^8 + 1)$]
The domain of $f = (-\infty, \infty)$.

5. The domain of the function

$$f(x) = {}^{24-x}C_{3x-1} + {}^{40-6x}C_{8x-10} \text{ is,}$$

(A) $\{2, 3\}$ (B) $\{1, 2, 3\}$
(C) $\{1, 2, 3, 4\}$ (D) None of these

Solution: (A)

${}^{24-x}C_{3x-1}$ is defined if,

$24 - x > 0$, $3x - 1 \ge 0$ and $24 - x \ge 3x - 1$

$\Rightarrow \quad x < 24,\ x \ge \dfrac{1}{3}$ and $x \le \dfrac{25}{4}$

$\Rightarrow \quad \dfrac{1}{3} \le x \le \dfrac{25}{4} \quad (1)$

${}^{40-6x}C_{8x-10}$ is defined if

$40 - 6x > 0$, $8x - 10 \ge 0$ and $40 - 6x \ge 8x - 10$

$\Rightarrow \quad x < \dfrac{20}{3},\ x \ge \dfrac{5}{4}$ and $x \le \dfrac{25}{7}$

$\Rightarrow \quad \dfrac{5}{4} \le x \le \dfrac{25}{7} \quad (2)$

From Eq. (1) and (2), we get

$$\frac{5}{4} \le x \le \frac{25}{7}$$

But, $24 - x \in N$,

$\therefore$ x must be an integer,

$\therefore \quad x = 2, 3.$

Hence, domain $(f) = \{2, 3\}$.

6. The domain of the function $f(x) = \sqrt{\cos^{-1}\left(\dfrac{1-|x|}{2}\right)}$ is

(A) $(-\infty, -3) \cup (3, \infty)$
(B) $[-3, 3]$
(C) $(-\infty, -3] \cup [3, \infty)$
(D) ϕ

Solution: (B)

$\cos^{-1}\left(\dfrac{1-|x|}{2}\right)$ is defined if $-1 \le \dfrac{1-|x|}{2} \le 1$

$\Rightarrow \quad -2 \le 1 - |x| \le 2 \Rightarrow -3 \le -|x| \le 1$

$\Rightarrow \quad -1 \le |x| \le 3$

$|x| \ge -1$ is true for all real values of x.

$$|x| \le 3 \Rightarrow -3 \le x \le 3$$

Also $\sqrt{\cos^{-1}\left(\dfrac{1-|x|}{2}\right)}$ is defined if $\cos^{-1}\left(\dfrac{1-|x|}{2}\right) \ge 0$

$\Rightarrow \quad \dfrac{1-|x|}{2} \ge \cos 0 = 1$

$\Rightarrow \quad 1 - |x| \ge 2$

$\Rightarrow \quad |x| \le -1$ (Absurd)

$\therefore$ Domain $(f) = [-3, 3]$

7. The domain of the function

$$f(x) = \log_{1/2}\left(x - \frac{1}{2}\right) + \log_2\sqrt{4x^2 - 4x + 5} \text{ is}$$

(A) $\left[\dfrac{1}{2}, \infty\right)$ (B) $\left(\dfrac{1}{2}, \infty\right)$

(C) $(-\infty, \infty)$ (D) None of these

Solution: (B)

$\log_{1/2}\left(x-\dfrac{1}{2}\right)$ is defined if $x-\dfrac{1}{2}>0$

i.e., $x>\dfrac{1}{2}$

$\log_2\sqrt{4x^2-4x+5}$ is defined if $4x^2-4x+5>0$

$\Rightarrow 4\left[\left(x-\dfrac{1}{2}\right)^2+1\right]>0$ which is true for all real x.

$\therefore$ Domain of $f=\left(\dfrac{1}{2},\infty\right)$.

8. The domain of the function

$$f(x)=\cos\left[\log\left(\frac{\sqrt{16-x^2}}{3-x}\right)\right] \text{ is}$$

(A) $(-4, 4)$ (B) $(-4, 3)$
(C) $(-\infty, -4)\cup(3,\infty)$ (D) None of these

Solution: (B)

$f(x)$ is defined if $\dfrac{\sqrt{16-x^2}}{3-x}>0$

$\Rightarrow\quad 16-x^2>0$ and $3-x>0$

$\Rightarrow\quad (x-4)(x+4)<0$ and $x<3$

$\Rightarrow\quad -4<x<4$ and $x<3$ or $-4<x<3$

$\therefore$ Domain of $f=(-4, 3)$.

9. The domain of the function $f(x)=\log_2\log_3\log_4 x$ is

(A) $[4,\infty)$ (B) $(4,\infty)$
(C) $(-\infty, 4)$ (D) None of these

Solution: (B)

$f(x)$ is defined if $\log_3\log_4 x>0$, $\log_4 x>0$ and $x>0$

$\Rightarrow\quad \log_4 x>3^\circ=1$, $x>4^\circ$ and $x>0$

$\Rightarrow\quad x>4^1$, $x>1$ and $x>0\Rightarrow x>4$

Domain of $f=(4,\infty)$.

10. The domain of the function

$$f(x)=\cos^{-1}\left(\frac{2-|x|}{4}\right)+[\log(3-x)]^{-1} \text{ is}$$

(A) $[-6, 3)\backslash\{2\}$ (B) $[-6, 2)\cup(2, 3]$
(C) $[-6, 3]$ (D) $[-6, 3)$

Solution: (A)

$\cos^{-1}\left(\dfrac{2-|x|}{4}\right)$ is defined for $-1\le\dfrac{2-|x|}{4}\le 1$

$\Rightarrow -4\le 2-|x|\le 4\Rightarrow -6\le -|x|\le 2$

$\Rightarrow -2\le|x|\le 6\Rightarrow|x|\le 6\Rightarrow -6\le x\le 6$ (1)

$\dfrac{1}{\log(3-x)}$ is defined if $\log(3-x)\ne 0$ and $3-x>0$

$\Rightarrow\ 3-x\ne e^0=1$ and $x<3\Rightarrow x\ne 2$ and $x<3$ (2)

From Eq. (1) and (2), we get domain of f

$=[-6, 6]\cap((-\infty, 3)\setminus\{2\})=[-6, 3)\backslash\{2\}$.

11. The domain of the function

$$f(x)=\cos^{-1}\left(\frac{3}{4+2\sin x}\right) \text{ is}$$

(A) $\left[-\dfrac{\pi}{6}+2n\pi, \dfrac{\pi}{6}+2n\pi\right]$

(B) $\left(-\dfrac{\pi}{6}+2n\pi, \dfrac{\pi}{6}+2n\pi\right)$

(C) $\left(-\dfrac{\pi}{6}+2n\pi, \dfrac{\pi}{6}+2n\pi\right)$

(D) $\left[-\dfrac{\pi}{6}+2n\pi, \dfrac{\pi}{6}+2n\pi\right)$

Solution: (A)

$f(x)$ is defined if $-1\le\dfrac{3}{4+2\sin x}\le 1$

Since, $4+2\sin x>0$ for all real x, therefore

$\dfrac{3}{4+2\sin x}\le 1\Rightarrow 3\le 4+2\sin x\Rightarrow\sin x\ge-\dfrac{1}{2}$

$\Rightarrow\quad -\dfrac{\pi}{6}+2n\pi\le x\le\dfrac{\pi}{6}+2n\pi,\ n\in Z$

Domain of $f=\left[-\dfrac{\pi}{6}+2n\pi, \dfrac{\pi}{6}+2n\pi\right]$.

12. The domain of definition of $f(x)=\sqrt{\dfrac{1-|x|}{2-|x|}}$ is

(A) $(-\infty,\infty)\backslash[-1, 1]$
(B) $(-\infty,\infty)\backslash[-2, 2]$
(C) $[-1, 1]\cup(-\infty, -2)\cup(2,\infty)$
(D) None of these

Solution: (C)

$f(x)$ is defined if $\dfrac{1-|x|}{2-|x|}\ge 0$ and $2-|x|\ne 0$

$\Rightarrow\quad\dfrac{(1-|x|)(2-|x|)}{(2-|x|)^2}\ge 0$ and $x\ne -2, 2$

$\Rightarrow\ (|x|-1)(|x|-2)\ge 0$ and $x\ne -2, 2$

$\Rightarrow\ |x|\le 1$ or $|x|>2$

$\Rightarrow\ -1\le x\le 1$ or $(x<-2$ or $x>2)$

Domain of $f=[-1, 1]\cup(-\infty, -2)\cup(2,\infty)$

13. The domain of the function $f(x) = \sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$ is

(A) $(-\infty, 1)$ (B) $(-1, \infty)$
(C) $[0, 1]$ (D) $[-1, 1]$

Solution: (D)

$f(x)$ is defined if

$1-\sqrt{1-\sqrt{1-x^2}} \geq 0$, $1-\sqrt{1-x^2} \geq 0$ and $1-x^2 \geq 0$

$1-x^2 \geq 0 \Rightarrow (x+1)(x-1) \leq 0 \Rightarrow -1 \leq x \leq 1$

Clearly for these values, the other two inequalities hold.

Thus domain of $f = [-1, 1]$.

14. The domain of the function

$$f(x) = \log_{10}[1-\log_{10}(x^2-5x+16)] \text{ is}$$

(A) $(2, 3)$ (B) $[2, 3]$
(C) $(2, 3]$ (D) $[2, 3)$

Solution: (A)

$f(x)$ is defined if

$1-\log_{10}(x^2-5x+16) > 0$ and $x^2-5x+16 > 0$

$\Rightarrow \log_{10}(x^2-5x+16) < 1$ and $\left(x-\frac{5}{2}\right)^2 + \frac{39}{4} > 0$

$\Rightarrow x^2-5x+16 < 10^1 = 10$

$$\left[\because \left(x-\frac{5}{2}\right)^2 + \frac{39}{4} > 0 \text{ for all real } x\right]$$

$\Rightarrow x^2-5x+6 < 0$

$\Rightarrow (x-3)(x-2) < 0 \Rightarrow 2 < x < 3$

$\therefore$ Domain of $f = (2, 3)$

15. The domain of the function $f(x) = \dfrac{1}{\sqrt{|\sin x| + \sin x}}$ is

(A) $(-2n\pi, 2n\pi)$ (B) $(2n\pi, (2n+1)\pi)$
(C) $\left((4n-1)\frac{\pi}{2}, (4n+1)\frac{\pi}{2}\right)$ (D) None of these

Solution: (B)

$f(x)$ is defined if $|\sin x| + \sin x > 0$

$\Rightarrow \sin x > 0 \Rightarrow 2n\pi < x < 2n\pi + \pi$

$\therefore$ Domain of $f = (2n\pi, (2n+1)\pi)$.

16. The domain of the function

$$f(x) = \log_{\left[x+\frac{1}{2}\right]}|x^2-5x+6| \text{ is}$$

(A) $\left[\frac{3}{2}, 2\right) \cup (2, 3) \cup (3, \infty)$ (B) $\left[\frac{3}{2}, \infty\right)$
(C) $\left[\frac{1}{2}, \infty\right)$ (D) None of these

Solution: (A)

$f(x)$ is defined if

$x^2-5x+6 \neq 0$, $\left[x+\frac{1}{2}\right] > 0$, $\left[x+\frac{1}{2}\right] \neq 1$

$x^2-5x+6 \neq 0 \Rightarrow (x-2)(x-3) \neq 0 \Rightarrow x \neq 2, 3$ (1)

$$\left[x+\frac{1}{2}\right] > 0 \Rightarrow x \geq \frac{1}{2} \quad (2)$$

$$\left[x+\frac{1}{2}\right] \neq 1 \Rightarrow x \notin \left[\frac{1}{2}, \frac{3}{2}\right) \quad (3)$$

From Eq. (1), (2) and (3), we get domain of f

$$= \left[\frac{3}{2}, 2\right) \cup (2, 3) \cup (3, \infty).$$

17. The domain of the function $f(x) = \sqrt{e^{\sin^{-1}(\log_{16} x^2)}}$ is

(A) $\left[\frac{1}{4}, 4\right]$ (B) $\left[-4, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, 4\right]$
(C) $\left[-4, -\frac{1}{4}\right]$ (D) None of these

Solution: (B)

$f(x)$ is defined if

$$-1 \leq \log_{16} x^2 \leq 1 \Rightarrow 16^{-1} \leq x^2 \leq 16^1$$

$\Rightarrow \frac{1}{16} \leq x^2 \leq 16$

$$x^2 \geq \frac{1}{16} \Rightarrow \left(x-\frac{1}{4}\right)\left(x+\frac{1}{4}\right) \geq 0$$

$\Rightarrow x \leq -\frac{1}{4}$ or $x \geq \frac{1}{4}$ (1)

$x^2 \leq 16 \Rightarrow (x-4)(x+4) \leq 0 \Rightarrow -4 \leq x \leq 4$ (2)

From Eq. (1) and (2), we get domain of f

$$= \left[-4, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, 4\right].$$

18. The domain of the function

$$f(x) = \underbrace{\log_2 \log_2 \log_2 \ldots \log_2 x}_{n \text{ times}} \text{ is}$$

(A) $(2^{n-1}, \infty)$ (B) $[2^n, \infty)$
(C) $(2^{n-2}, \infty)$ (D) None of these

Solution: (D)

$f(x)$ is defined if

(1) $x > 0$
(2) $\log_2 x > 0 \Rightarrow x > 2^0 = 1$
(3) $\log_2 \log_2 x > 0 \Rightarrow \log_2 x > 2^0 = 1 \Rightarrow x > 2^1 = 2$
(4) $\log_2 \log_2 \log_2 x > 0 \Rightarrow \log_2 \log_2 x > 2^0 = 1$
$\Rightarrow \log_2 x > 2^1 = 2$
$\Rightarrow x > 2^2$

(5) $\log_2 \log_2 \log_2 \log_2 x > 0 \Rightarrow \log_2 \log_2 \log_2 x > 2^0 = 1$

$\Rightarrow \log_2 \log_2 x > 2^1$

$\Rightarrow \log_2 x > 2^2 \Rightarrow x > 2^{2^2}$

$\vdots \quad \vdots \quad \vdots$

Continuing like this, we get $x > 2^{2^{2\ldots(n-1)\text{ times}}}$

$\therefore$ Domain of $f = (2^{2^{2\ldots(n-1)\text{ times}}}, \infty)$.

19. The domain of the function $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$ is

(A) $(-\infty, -2) \cup [4, \infty)$ (B) $(-\infty, -2] \cup [4, \infty)$
(C) $(-\infty, -2) \cup (4, \infty)$ (D) None of these

Solution: (A)

$f(x)$ is defined for

$[x]^2 - [x] - 6 > 0 \Rightarrow ([x] - 3)([x] + 2) > 0$

$\Rightarrow \quad [x] < -2$ or $[x] > 3$

But $\quad [x] < -2 \Rightarrow [x] = -3, -4, -5, \ldots$

$\therefore \quad x < -2$

Also, $\quad [x] > 3 \Rightarrow [x] = 4, 5, 6, \ldots$

$\therefore \quad x \geq 4$

$\therefore$ Domain of $f = (-\infty, -2) \cup [4, \infty)$.

20. The domain of the function

$$f(x) = \log_3\left[-\log_{1/2}\left(1 + \frac{1}{x^{1/5}}\right) - 1\right] \text{ is}$$

(A) $(-\infty, 1)$ (B) $(0, 1)$
(C) $(1, \infty)$ (D) None of these

Solution: (B)

$f(x)$ is defined if

$-\log_{1/2}\left(1 + \dfrac{1}{x^{1/5}}\right) - 1 > 0,\ 1 + \dfrac{1}{x^{1/5}} > 0,\ x \neq 0$

$\Rightarrow \quad \log_{1/2}\left(1 + \dfrac{1}{x^{1/5}}\right) < -1,\ x^{1/5} + 1 > 0,\ x \neq 0$

$\Rightarrow \quad 1 + \dfrac{1}{x^{1/5}} > \left(\dfrac{1}{2}\right)^{-1},\ x > (-1)^5,\ x \neq 0$

$\Rightarrow \quad \dfrac{1}{x^{1/5}} > 1,\ x > -1$ and $x \neq 0$

$\Rightarrow \quad 0 < x < 1$ and $x > -1 \Rightarrow 0 < x < 1$.

$\therefore \quad$ Domain $(f) = (0, 1)$.

21. The domain of the function

$$f(x) = \log_3[-(\log_3 x)^2 + 5\log_3 x - 6] \text{ is}$$

(A) $(0, 9) \cup (27, \infty)$
(B) $[9, 27]$
(C) $(9, 27)$
(D) None of these

Solution: (C)

$f(x)$ is defined if

$-(\log_3 x)^2 + 5\log_3 x - 6 > 0$ and $x > 0$

$\Rightarrow \quad (\log_3 x - 3)(2 - \log_3 x) > 0$ and $x > 0$

$\Rightarrow \quad (\log_3 x - 2)(\log_3 x - 3) < 0$ and $x > 0$

$\Rightarrow \quad 2 < \log_3 x < 3$ and $x > 0$

$\Rightarrow \quad 3^2 < x < 3^3 \Rightarrow 9 < x < 27$

$\therefore \quad$ Domain of $f = (9, 27)$.

22. The domain of definition of the function $y(x)$ given by the equation $2^x + 2^y = 2$ is

(A) $0 < x \leq 1$ (B) $0 \leq x \leq 1$
(C) $-\infty < x \leq 0$ (D) $-\infty < x < 1$

Solution: (D)

We have, $2^x + 2^y = 2 \Rightarrow 2^y = 2 - 2^x$

$\Rightarrow \quad y = \dfrac{\log(2 - 2^x)}{\log 2}$

For y to be defined, $2 - 2^x > 0 \Rightarrow 2^x < 2$

Since, 2^x is an increasing function, therefore we get $x < 1$.

23. The domain of the function

$$f(x) = \cot^{-1}\left(\frac{x}{\sqrt{x^2 - [x^2]}}\right), x \in R \text{ is}$$

(A) $R - \{\pm\sqrt{n}, n \in N\}$ (B) $R - \{\sqrt{n}, n \geq 0, n \in I\}$
(C) R (D) $R - \{0\}$

Solution: (B)

Domain of $\cot^{-1}x$ is R and $\dfrac{x}{\sqrt{x^2 - [x^2]}}$ is defined if $x^2 \neq [x^2]$

$(\because x^2 \geq [x^2])$

$\Rightarrow \quad x^2 \neq 0$ or +ve integer.

Hence, domain $= R - \{\sqrt{n} : n \geq 0, n \in Z\}$.

METHOD TO FIND THE RANGE OF A FUNCTION

1. Find the domain of the function $y = f(x)$.
2. If the domain is an infinite interval, solve the equation $y = f(x)$ and find x in terms of y to get $x = g(y)$. Find the real values of y for which x is real. The set of values of y so obtained constitutes the range of f. Note that if finite number of values of x are excluded from the domain, find the values of y for these values of x and exclude these values of y from the range of f found earlier.
3. If the domain is a finite interval, find the least and greatest value of y for values of x in the domain. If a is the least value and b the greatest value of y, then range $(f) = [a, b]$.

SOLVED EXAMPLES

24. The range of the function $f(x) = \sqrt{3x^2 - 4x + 5}$ is

(A) $\left[-\infty, \sqrt{\frac{11}{3}}\right]$ (B) $\left(-\infty, \sqrt{\frac{11}{3}}\right)$

(C) $\left[\sqrt{\frac{11}{3}}, \infty\right)$ (D) $\left(\sqrt{\frac{11}{3}}, \infty\right)$

Solution: (C)

$f(x)$ is defined if $3x^2 - 4x + 5 \geq 0$

$\Rightarrow 3\left[x^2 - \frac{4}{3}x + \frac{5}{3}\right] \geq 0 \Rightarrow 3\left[\left(x - \frac{2}{3}\right)^2 + \frac{11}{9}\right] \geq 0$

which is true for all real x

$\therefore$ Domain $(f) = (-\infty, \infty)$

Let $y = \sqrt{3x^2 - 4x + 5}$

$\Rightarrow y^2 = 3x^2 - 4x + 5$ i.e. $3x^2 - 4x + (5 - y^2) = 0$

For x to be real, $16 - 12(5 - y^2) \geq 0 \Rightarrow y \geq \sqrt{\frac{11}{3}}$

$\therefore$ Range of $y = \left[\sqrt{\frac{11}{3}}, \infty\right)$.

25. The range of the function $f(x) = \log_e(3x^2 - 4x + 5)$ is

(A) $\left(-\infty, \log_e \frac{11}{3}\right]$ (B) $\left[\log_e \frac{11}{3}, \infty\right)$

(C) $\left[-\log_e \frac{11}{3}, \log_e \frac{11}{3}\right]$ (D) None of these

Solution: (B)

$f(x)$ is defined if $3x^2 - 4x + 5 > 0$

$\Rightarrow 3\left[x^2 - \frac{4}{3}x + \frac{5}{3}\right] > 0 \Rightarrow 3\left[\left(x - \frac{2}{3}\right)^2 + \frac{11}{9}\right] > 0,$

which is true for all real x.

$\therefore$ Domain $(f) = (-\infty, \infty)$

Let $y = \log_e(3x^2 - 4x + 5) \Rightarrow e^y = 3x^2 - 4x + 5$.

$\Rightarrow 3x^2 - 4x + (5 - e^y) = 0$

For x to be real,

$16 - 12(5 - e^y) \geq 0 \Rightarrow 12e^y \geq 44 \Rightarrow e^y \geq \frac{11}{3}$

$\Rightarrow y \geq \log_e \frac{11}{3}$

$\therefore$ Range of $f = \left[\log_e \frac{11}{3}, \infty\right)$.

26. The value of the function $f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$ lies in the interval

(A) $(-\infty, \infty) \setminus \left\{\frac{1}{5}, 1\right\}$ (B) $(-\infty, \infty)$

(C) $(-\infty, \infty) \setminus \{1\}$ (D) None of these

Solution: (B)

$f(x)$ is defined if $x^2 + x - 6 \neq 0$

i.e., $(x + 3)(x - 2) \neq 0$ i.e. $x \neq -3, 2$

$\therefore$ Domain $(f) = (-\infty, \infty) \setminus \{-3, 2\}$

Let $y = \frac{x^2 - 3x + 2}{x^2 + x - 6}$

$\Rightarrow x^2y + xy - 6y = x^2 - 3x + 2$

$\Rightarrow x^2(y - 1) + x(y + 3) - (6y + 2) = 0$

For x to be real, $(y + 3)^2 + 4(y - 1)(6y + 2) \geq 0$

$\Rightarrow 25y^2 - 10y + 1 \geq 0$ i.e. $(5y - 1)^2 \geq 0$

which is true for all real y.

$\therefore$ Range of $f = (-\infty, \infty)$.

27. The range of the function $y = \sin^{-1}\left(\frac{x^2}{1 + x^2}\right)$ is

(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left[0, \frac{\pi}{2}\right)$

(C) $\left[0, \frac{\pi}{2}\right]$ (D) None of these

Solution: (B)

Clearly, for y to be defined, $\left|\frac{x^2}{1 + x^2}\right| \leq 1$ which is true for all $x \in R$. So, the domain $= (-\infty, \infty)$.

Now, $y = \sin^{-1}\left(\frac{x^2}{1 + x^2}\right) \Rightarrow \frac{x^2}{1 + x^2} = \sin y$

$\Rightarrow x = \frac{x^2}{1 + x^2}$

For x to be real, $\sin y \geq 0$ and $1 - \sin y > 0$

$\Rightarrow 0 \leq \sin y < 1 \Rightarrow y \in \left[0, \frac{\pi}{2}\right)$

$\therefore$ Range $= \left[0, \frac{\pi}{2}\right)$

28. The range of the function $y = 3\sin\sqrt{\frac{\pi^2}{16} - x^2}$ is

(A) $\left[0, \frac{3}{\sqrt{2}}\right]$ (B) $\left[-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right]$

(C) $\left[-\frac{3}{\sqrt{2}}, 0\right]$ (D) None of these

Solution: (A)

For y to be defined, $\frac{\pi^2}{16} - x^2 \geq 0$

$\Rightarrow \left(\frac{\pi}{4}-x\right)\left(\frac{\pi}{4}+x\right) \geq 0 \Rightarrow \left(x-\frac{\pi}{4}\right)\left(x+\frac{\pi}{4}\right) \leq 0$

$\Rightarrow -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

$\therefore$ Domain of $y = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Clearly, for $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, $\sqrt{\frac{\pi^2}{16}-x^2} \in \left[0, \frac{\pi}{4}\right]$.

Since sin x is an increasing function on $\left[0, \frac{\pi}{4}\right]$.

Therefore, $\sin 0 \leq \sin \sqrt{\frac{\pi^2}{16}-x^2} \leq \sin\frac{\pi}{4}$

$\Rightarrow 0 \leq 3\sin\sqrt{\frac{\pi^2}{16}-x^2} \leq \frac{3}{\sqrt{2}} \Rightarrow 0 \leq y \leq \frac{3}{\sqrt{2}}$

$\therefore$ Range of $y = \left[0, \frac{3}{\sqrt{2}}\right]$

29. The range of the function

$$f(x) = \sin\left[\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right] \text{ is}$$

(A) [0, 1] (B) (–1, 0)
(C) [–1, 1] (D) (–1, 1)

Solution: (C)

For $f(x)$ to be defined,

$\frac{\sqrt{4-x^2}}{1-x} > 0$, $4 - x^2 > 0$ and $1 - x \neq 0$

Since $\sqrt{4-x^2} \not< 0$,

$\therefore$ We have $1 - x > 0$ and $4 - x^2 > 0$

$\Rightarrow x < 1$ and $(x - 2)(x + 2) < 0$

$\Rightarrow x < 1$ and $-2 < x < 2$

$\Rightarrow -2 < x < 1$

$\therefore$ Domain of $f = (-2, 1)$.

Since $-\infty < \log\left(\frac{\sqrt{4-x^2}}{1-x}\right) < \infty$

$\Rightarrow -1 \leq \sin\left[\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right] \leq 1$

$\therefore$ Range of $f = [-1, 1]$.

30. If $[2\sin x] + [\cos x] = -3$, then the range of the function $f(x) = \sin x + \sqrt{3}\cos x$ in $[0, 2\pi]$ (where $[\cdot]$ denotes the greatest integer function) is

(A) [–2, –1] (b) $\left(-1, -\frac{1}{2}\right)$
(C) (–2, –1) (D) None of these

Solution: (C)

We have, $[2\sin x] + [\cos x] = -3$

only if $[2\sin x] = -2$ and $[\cos x] = -1$

$\Rightarrow -2 \leq 2\sin x < -1$ and $-1 \leq \cos x < 0$

$\Rightarrow -1 \leq \sin x < -\frac{1}{2}$ and $-1 \leq \cos x < 0$

$\Rightarrow \frac{7\pi}{6} < x < \frac{11\pi}{6}$ and $\frac{\pi}{2} < x < \frac{3\pi}{2}$

$\Rightarrow \frac{7\pi}{6} < x < \frac{3\pi}{2}$

For the above values of x, $\sin x + \sqrt{3}\cos x = 2\sin\left(\frac{\pi}{3}+x\right)$ lies between –2 and –1.

$\therefore$ Rage of $f(x)$ is (–2, –1).

31. The range of the function $y = \frac{e^{-x}}{1+[x]}$ is

(A) $(-\infty, \infty)$ (B) $R - \{0\}$
(C) R (D) None of these

Solution: (B)

We have,

$$y = \frac{e^{-x}}{1+[x]}$$

$\Rightarrow (1 + [x])y = e^{-x} > 0$

$\Rightarrow (1 + [x])y > 0$

$\Rightarrow y > 0$ if $1 + [x] > 0$ and $y < 0$ if $1 + [x] < 0$

$\therefore y \in R - \{0\}$

TYPES OF FUNCTIONS

One–One or Injective Function

A function $f : X \rightarrow Y$ is said to be one-one or injective if distinct elements of X have distinct images in Y, as shown in Fig. 2.4.

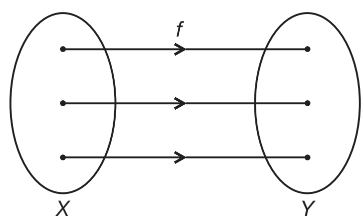


Fig. 2.4

Many–One Function

A function $f: X \to Y$ is said to be many-one if there exists atleast two distinct elements in X whose images are same, as shown in the Fig. 2.5

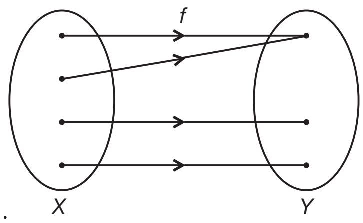


Fig. 2.5

METHOD TO CHECK WHETHER THE FUNCTION $f: X \to Y$ IS ONE–ONE OR MANY-ONE

1. Consider any two points $x, y \in X$.
2. Put $f(x) = f(y)$ and solve the equation.
3. If we get $x = y$ only, then f is one-one, otherwise it is many-one.

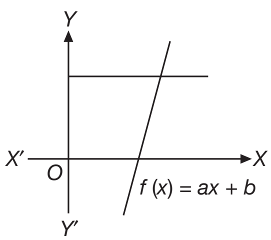
TRICK(S) FOR PROBLEM SOLVING

Derivative Test to Check the Injectivity

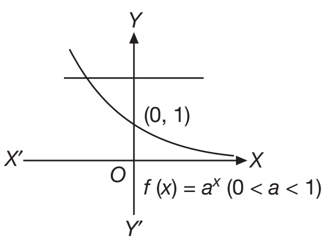
If a function is either strictly increasing or strictly decreasing in the whole domain (or equivalently, $f'(x) > 0$ or $f'(x) < 0$, $\forall\, x \in X$), then it is one-one, otherwise it is many-one.

Graphical Test

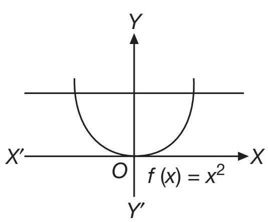
If any straight line parallel to x-axis intersects the graph of the function atmost at one point, then the function is one-one, otherwise it is many-one (i.e., it intersects the graph of the function in atleast two points).



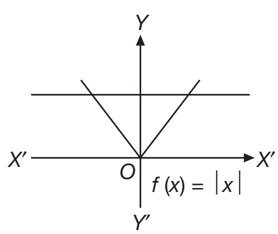
(i) One–One Function



(ii) One–One Function



(i) Many–One Function



(ii) Many–One Function

> **NOTE**
>
> - Any continuous function $f(x)$ which has atleast one local maxima or local minima is many-one.
> - All even functions are many-one.
> - All polynomials of even degree defined on R have atleast one local maxima or minima and hence are many one on the domain R. Polynomials of odd degree can be one-one or many-one.

TRICK(S) FOR PROBLEM SOLVING

Number of One–One Functions (Injections)

If X and Y are any two finite sets having m and n elements respectively, then the number of one-one functions from X to Y would be

$$= \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$$

ONTO OR SURJECTIVE FUNCTION

A function $f: X \to Y$ is said to be onto or surjective if every element of Y is the image of atleast one clement in X under the map f, as shown in the Fig. 2.6

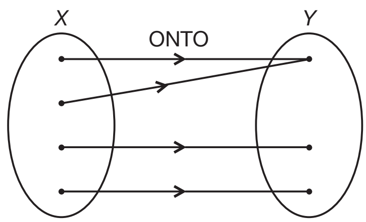


Fig. 2.6

A function $f: X \to Y$ is an into function if it is not an onto function, as shown in the Fig. 2.7

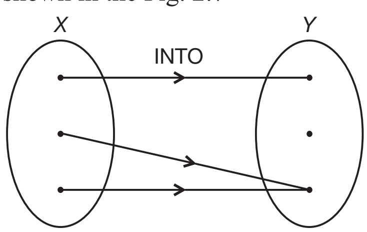


Fig. 2.7

In other words, if the function $f: X \to Y$ is such that there is atleast one element in Y which is not image of any element in X, then we say that f is a function of A into B, i.e., f is called into function if, for some $y \in Y$, there does not exist any $x \in X$ such that $y = f(x)$.

METHOD TO CHECK WHETHER THE FUNCTION $f: X \rightarrow Y$ IS ONTO OR INTO

1. Find the range of the function f.
2. If range of $f = Y$ (the co-domain), then f is onto, otherwise it is into.

NOTE

For an 'onto' function, the range overlaps or equals co-domain, whereas for an into function, the range does not overlap but fits inside the co-domain.

TRICK(S) FOR PROBLEM SOLVING

Number of Onto Functions (Surjections)

If X and Y are any two finite sets having m and n elements respectively, where $1 \leq n \leq m$, then the number of onto functions from X to Y is given by

$$\sum_{r-1}^{n} (-1)^{n-r}\ {}^{n}C_{r} r^{m}$$

Any polynomial function f is onto if degree is odd and into if degree of f is even.

BIJECTIVE FUNCTION

A function $f: X \rightarrow Y$ is said to be bijective, if f is both one-one and onto, as shown in the Fig. 2.8

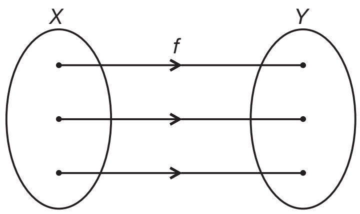


Fig. 2.8

TRICK(S) FOR PROBLEM SOLVING

Number of One–One Onto Functions (Dijections)

If X and Y are any two finite sets having the same number of elements, say n, then the number of bijective functions from X to Y is $n!$.

SOLVED EXAMPLES

32. The function $f: R \rightarrow R$ defined by,

$$f(x) = 4^x + 4^{|x|} \text{ is}$$

(A) one-one and into (B) many-one and into
(C) one-one and onto (D) many-one and onto

Solution: (A)

Since, for different x, 4^x and $4^{|x|}$ are different positive numbers,

$\therefore$ f is one–one. Also, f is not onto as its range $(0, \infty)$ is a proper subset of its co-domain R.

33. Let $f: R \rightarrow R$ be a function defined by,

$$f(x) = x + \sqrt{x^2}, \text{ then } f \text{ is}$$

(A) injective (B) surjective
(C) bijective (D) None of these

Solution: (D)

We have,

$$f(x) = x + \sqrt{x^2} = x + |x|$$

Clearly, f is not one-one as $f(-1) = f(-2) = 0$ but $-1 \neq -2$.

Also, f is not onto as $f(x) \geq 0\ \forall\ x \in R$,

$\therefore$ range of $f = (0, \infty) \subset R$.

SOME IMPORTANT FUNCTIONS

Constant Function

A function $f: R \rightarrow R$ defined as $f(x) = c$, $\forall\ x \in R$, where c is a constant, is called a constant function. Its domain is R and range is singleton set $\{c\}$.

The graph of a constant function is a straight line parallel to x-axis as shown in the Fig. 2.9. It is above or below the x-axis according as c is positive or negative. If $c = 0$, then the straight line coincides with x-axis.

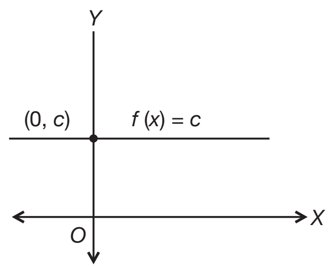


Fig. 2.9

IDENTITY FUNCTION

The function $f: R \rightarrow R$ defined as $f(x) = x$, $\forall\ x \in R$, is called the identity function. Its do-main is R and range is also R.

The graph of the identity function is a straight line passing through origin and inclined at an angle of 45º with x-axis, as shown in the Fig. 2.10.

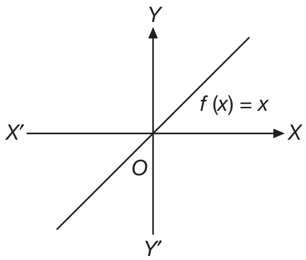


Fig. 2.10

MODULUS FUNCTION OR ABSOLUTE VALUE FUNCTION

The function $f: R \to R$, defined as

$$f(x) = |x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is called the absolute value function or modulus function. Its domain is R and its range is $[0, \infty)$. The graph of the modulus function is as shown in the Fig. 2.11.

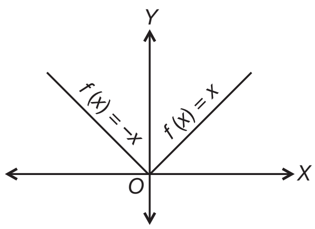


Fig. 2.11

TRICK(S) FOR PROBLEM SOLVING

- For every $x \in R$, $|x| = \max\{x, -x\}$
- For every $x \in R$, $|x| = \sqrt{x^2}$
- $|x + y| \leq |x| + |y|$
- $|x + y| = |x| + |y|$ if x, y have the same sign and $|x + y| < |x| + |y|$ if x, y have opposite signs
- $|x - y| \geq ||x| + |y||$
- $|x - y| = ||x| - |y||$ if x, y have same sign and $|x| \geq |y|$
- $|x - y| > ||x| + |y||$ if x, y have opposite signs
- $|xy| = |x|\ |y|$

GREATEST INTEGER FUNCTION/STEP FUNCTION/FLOOR FUNCTION

The function $f: R \to R$ defined as $f(x) = [x]$ is called the greatest integer function,

where $[x]$ = integral part of x or greatest integer not greater than x or greatest integer less than or equal to x.

i.e. $$f(x) = n,$$

where $n \leq x < n + 1$, $n \in Z$ (the set of integers).

Its domain is R and range is Z. The graph of the greatest integer function is as shown in Fig. 2.12:

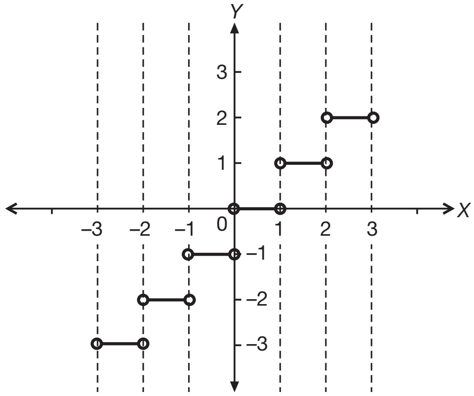


Fig. 2.12

TRICK(S) FOR PROBLEM SOLVING

- $[x] \leq x < [x] + 1$
- $[x + y] = \begin{cases} [x] + [y] & \text{if } \{x\} + \{y\} < 1 \\ [x] + [y] + 1, & \text{if } \{x\} + \{y\} \geq 1 \end{cases}$

  where $\{x\}$ denotes the fractional part of x.
- $n \leq x < n + 1 \Leftrightarrow [x] = n$
- $n_1 \leq [x] \leq n_2 \Rightarrow n_1 \leq x < n_2 + 1$
- $x - 1 < [x] \leq x$
- $[[x]] = [x]$
- $[n + x] = n + [x]$, where n is any integer
- $[x] + [-x] = \begin{cases} 0 & \text{if } x \in Z \\ -1 & \text{if } x \notin Z \end{cases}$
- If $[f(x)] \geq n$, then $f(x) \geq n$

  If $[f(x)] \leq n$, then $f(x) < n + 1$

  where n is any integer.
- $\left[x + \left[y + \left[z + \left[w + [u]\right]\right]\right]\right] = [x] + [y] + [z] + [w] + [u]$

LEAST INTEGER FUNCTION/CEILING FUNCTION

The function $f: R \to R$ defined as $f(x) = (x)$ is called least integer function,

where (x) = least integer greater than or equal to x.

i.e., $$f(x) = n,$$

where $n - 1 < x \leq n$, $n \in Z$ (the set of integers)

Its domain is R and range is Z. The graph of the least integer function is as shown in Fig. 2.13:

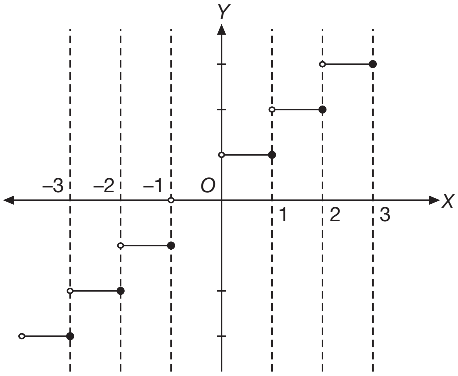


Fig. 2.13

FRACTIONAL-PART FUNCTION

The function $f : R \to R$ defined as $f(x) = x - [x]$ or $f(x) = \{x\}$, where $\{x\}$ denotes the fractional part of x, is called the fractional-part function. Its domain is R and range is $[0, 1)$. The graph of the fractional part function is as shown in Fig. 2.14:

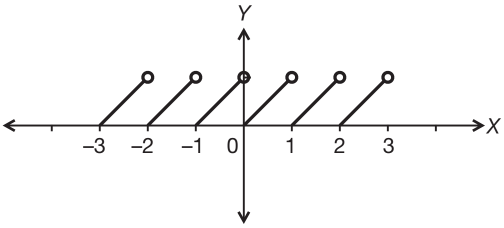


Fig. 2.14

TRICK(S) FOR PROBLEM SOLVING

- If $0 \le x < 1$, then $\{x\} = x$
- If x is an integer, then $x = [x] \Rightarrow \{x\} = 0$
  $\Rightarrow \{[x]\} = 0$
- $[\{x\}] = 0$
- $0 \le \{x\} < 1$
- $\{x\} + \{-x\} = \begin{cases} 0 & \text{if } x \in \text{integer} \\ -1, & \text{if } x \notin \text{integer} \end{cases}$
- If $x \notin Z$ and $x > 0$, then $\{-x\} = 1 - \{x\}$

SOLVED EXAMPLES

34. Which of the following is a function ([.] denotes the greatest integer function, {.} denotes the fractional part function)?

(A) $\dfrac{1}{\log[1-|x|]}$ (B) $\dfrac{x!}{\{x\}}$

(C) $x!\ \{x\}$ (D) $\dfrac{\log(x-1)}{\sqrt{1-x^2}}$

Solution: (C)

Since only $x!\ \{x\}$ is defined so $x!\ \{x\}$ represents a function.

35. Range of the function f defined by $f(x) = \left[\dfrac{1}{\sin\{x\}}\right]$ (where [.] and {.} respectively denote the greatest integer and the fractional part functions) is

(A) Z, the set of integers
(B) N, the set of natural numbers
(C) W, the set of whole numbers
(D) $\{2, 3, 4, ...\}$

Solution: (B)

$\because \quad \{x\} \in [0, 1)$

$\therefore \quad \sin\{x\} \in [0, \sin 1)$ but $f(x)$ is defined if $\sin\{x\} \ne 0$

$\therefore \quad \dfrac{1}{\sin\{x\}} \in \left(\dfrac{1}{\sin 1}, \infty\right).$

$\therefore \quad \left[\dfrac{1}{\sin\{x\}}\right] \in \{1, 2, 3.....\}.$

SIGNUM FUNCTION

The function $f : R \to R$ defined as

$$f(x) = \begin{Bmatrix} \dfrac{|x|}{x} & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{Bmatrix}$$

is called the signum function.

Its domain is R and range is the set $\{-1, 0, 1\}$. The graph of the signum function is as shown in Fig. 2.15:

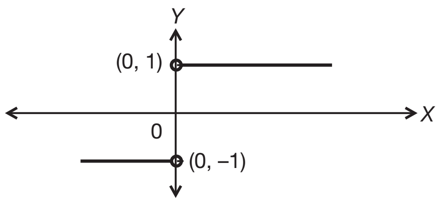


Fig. 2.15

RECIPROCAL FUNCTION

The function $f : R\backslash\{0\} \to R$ defined by $f(x) = \dfrac{1}{x}$, is called the reciprocal function. Its domain as well as range is $R\backslash\{0\}$. The graph of the reciprocal function is as shown in Fig. 2.16:

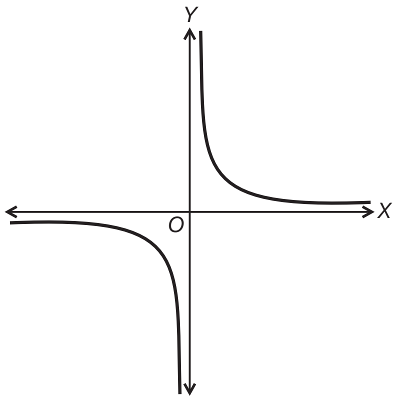


Fig. 2.16

EXPONENTIAL FUNCTION

Let a ($\neq 1$) be a positive real number. Then the function $f : R \to R$, defined by $f(x) = a^x$, is called the exponential function. Its domain is R and range is $(0, \infty)$. The graph of the exponential function is as shown in Fig. 2.17(a) and (b).

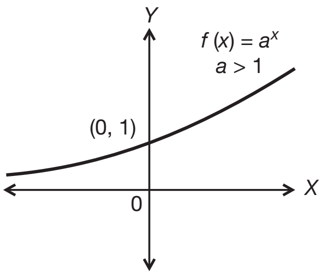


Fig. 2.17(a)

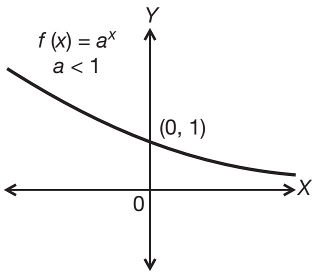


Fig. 2.17(b)

NOTE

- $a^x = e^{x \log_e a}$, $(a > 0)$.
- $a^{\log_a x} = x$, $(a > 0, a \neq 1)$.
- $\log_a b = \dfrac{\log_c b}{\log_c a}$, $a, b, c > 0$ and $a, c \neq 1$
- $\log_b a = \dfrac{1}{\log_a b}$, provided $a \neq 1$, $b \neq 1$ and $a, b > 0$
- If $a^{x_1} > a^{x_2}$ then $x_1 > x_2$ if $a > 1$ and $x_1 < x_2$ if $0 < a < 1$

LOGARITHMIC FUNCTION

Let a ($\neq 1$) be a positive real number. Then the function $f : (0, \infty) \to$ R, defined by $f(x) = \log_a x$, is called the logarithmic function. Its domain is $(0, \infty)$ and range is R. The graph of the logarithmic function is as shown in Fig. 2.18(a) and (b):

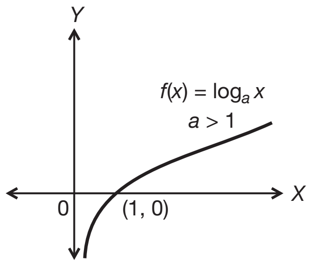


Fig. 2.18(a)

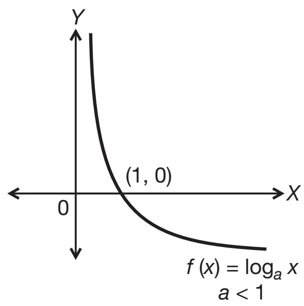


Fig. 2.18(b)

NOTE

- $\log_a a = 1$, $\log_a 1 = 0$, provided $a > 0$, $a \neq 1$
- $\log_a 0 = \begin{cases} -\infty, & \text{if} \quad a > 1 \\ +\infty, & \text{if} \quad 0 < a < 1 \end{cases}$
- $\log_e x$ is also denoted as: $ln\ x$.
- If $\log_a x_1 > \log_a x_2$ then $x_1 > x_2$ if $a > 1$ and $x_1 < x_2$ if $0 < a < 1$.

POLYNOMIAL FUNCTION

A function $f : R \to R$, defined by $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where $n \in N$ and $a_0, a_1, a_2, \ldots, a_n \in R$, is called a polynomial function.

If $a_n \neq 0$, then n is called the degree of the polynomial. The domain of a polynomial function is R.

RATIONAL FUNCTION

A function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials over the set of real numbers and $q(x) \neq 0$, is called a rational function. Its domain is $R \setminus \{x \mid q(x) = 0\}$.

TRIGONOMETRIC FUNCTIONS

Table 2.1

Function	Domain	Range
$y = \sin x$	R	$[-1, 1]$
$y = \cos x$	R	$[-1, 1]$
$y = \tan x$	$R \setminus \left\{(2n+1)\dfrac{\pi}{2} \middle\vert n \in Z\right\}$	R
$y = \cot x$	$R \setminus \{n\pi \mid n \in Z\}$	R
$y = \sec x$	$R \setminus \left\{(2n+1)\dfrac{\pi}{2} \middle\vert n \in Z\right\}$	$(-\infty, -1] \cup [1, \infty)$
$y = \operatorname{cosec} x$	$R \setminus \{n\pi \mid n \in Z\}$	$(-\infty, -1] \cup [1, \infty)$

INVERSE TRIGONOMETRIC FUNCTIONS

Table 2.2

Function	Domain	Range
$y = \sin^{-1}x$	$-1 \le x \le 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}x$	$-1 \le x \le 1$	$[0, \pi]$
$y = \tan^{-1}x$	$-\infty < x < \infty$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1}x$	$-\infty < x < \infty$	$(0, \pi)$
$y = \text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$y = \sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

TWO WAYS OF DEFINING A FUNCTION

1. **Uniform Definition:** If a function is defined as $y = f(x), x \in [a, b]$, we say that it is uniformly defined.
   Illustration
   (i) $y = f(x) = \sin x, x \in R$,
   (ii) $y = f(x) = x^2 + 1, x \in [-1, 1]$.
2. **Piecewise Definition:** If a function $y = f(x), x \in [a, b]$ assumes different forms in different subsets of $[a, b]$, we say that it is piecewise defined, as shown in the Fig. 2.19

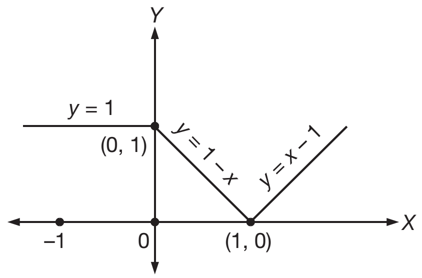


Fig. 2.19

Illustration

$$y = f(x) = \begin{cases} 1, & -1 \le x < 0 \\ 1 - x, & 0 \le x < 1 \\ x - 1, & x \ge 1 \end{cases}$$

EXPLICIT AND IMPLICIT FUNCTIONS

Explicit Function

A function y is said to be an explicit function of x, if the dependent variable y can be expressed totally in terms of the independent variable x.

Illustration

1. $y = \sin^2 x, x \in R$.
2. $y = \log x + x - 2e^x, x > 0$.

Implicit Function

When the variables x and y occur together in an equation $f(x, y) = 0$, in which y cannot be expressed explicitly in terms of x, then y is said to be an implicit function of x.

Illustration

1. $xy = \tan(x + y)$
2. $e^{xy} + xy - 2 = 0$.

OPERATIONS ON FUNCTIONS

Let f and g be two real functions with domain D_1 and D_2 respectively. Then,

1. The sum function $(f + g)$ is defined by,
   $$(f + g)(x) = f(x) + g(x), \forall x \in D_1 \cap D_2$$
   The domain of $f + g$ is $D_1 \cap D_2$
2. The difference function $(f - g)$ is defined by,
   $$(f - g)(x) = f(x) - g(x), \forall x \in D_1 \cap D_2$$
   The domain of $f - g$ is $D_1 \cap D_2$
3. The product function fg is defined by,
   $$(fg)(x) = f(x) \cdot g(x), \forall x \in D_1 \cap D_2$$
   The domain of fg is $D_1 \cap D_2$
4. The quotient function $\left(\frac{f}{g}\right)$ is defined by,
   $$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \forall x \in D_1 \cap D_2 \backslash [x : g(x) = 0]$$
   The domain of $\frac{f}{g}$ is $D_1 \cap D_2 \backslash [x : g(x) = 0]$
5. The scalar multiple function cf is defined by,
   $$(cf)(x) = c \cdot f(x), \forall x \in D_1$$
   The domain of cf is D_1

TRICK(S) FOR PROBLEM SOLVING

In a function $f(x)$ is such that

- $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, n \in R$
- $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$
- $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \log x$ or $f(x) = 0$
- $f(x + y) = f(x) = f(y) \Rightarrow f(x) = k$, where k is constant
- $f(x) \times f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \Rightarrow f(x) = \pm x^n + 1$

COMPOSITION OF FUNCTIONS

Let f and g be two real functions with domain D_1 and D_2 respectively.

If range of $f \subseteq$ domain of g, then composite function (gof) is defined by,

$$(gof)(x) = g[f(x)],\ \forall x \in D_1$$

Also, if range of $g \subseteq$ domain of f, then composite function (fog) is defined by,

$$(fog)(x) = f[g(x)],\ \forall x \in D_2$$

CAUTION

gof exists only if range $f \subseteq$ domain and *fog* exists only if range $g \subseteq$ domain if

PROPERTIES OF COMPOSITE FUNCTIONS

Let $f : X \to Y$ and $g : Y \to Z$ be two functions.

1. If both f and g are one-one, then so is gof.
2. If both f and g are onto, then so is gof.
3. If gof is one-one, then f is one-one but g may not be one-one.
4. If gof is onto, then g is onto but f may not be onto.
5. If f and g are bijective, then so is gof.
6. It may happen that gof may exist and fog may not exist. Moreover, even if both gof and fog exist, they may not be equal.

SOLVED EXAMPLES

36. Let f be a function defined on [0, 1] such that

$$f(x) = \begin{cases} x & x \in Q \\ 1 - x, & x \notin Q \end{cases}$$

Then for all $x \in [0, 1]$, $fof(x)$ is

(A) a constant (B) $1 + x$
(C) x (D) None of these

Solution: (C)

We have,

$$f(x) = \begin{cases} x & x \in Q \\ 1 - x, & x \notin Q \end{cases}$$

$$\therefore\ fof(x) = \begin{cases} f(x), & f(x) \in Q \\ 1 - f(x), & f(x) \notin Q \end{cases}$$

$$= \begin{cases} x, & x \in Q, & x \in Q \\ 1 - x, & x \notin Q, & x \in Q \\ 1 - x, & 1 - x \in Q, & x \notin Q \\ 1 - (1 - x), & 1 - x \notin Q, & x \notin Q \end{cases}$$

$$= \begin{cases} x, & x \in Q \\ x, & x \notin Q \end{cases}$$

$\therefore\ fof(x) = x,\ x \in [0, 1].$

37. If $g[f(x)] = |\sin x|$ and $f[g(x)] = (\sin\sqrt{x})^2$, then

(A) $f(x) = \sin^2 x,\ g(x) = \sqrt{x}$
(B) $f(x) = \sin x,\ g(x) = |x|$
(C) $f(x) = x^2,\ g(x) = \sin\sqrt{x}$
(D) f and g cannot be determined.

Solution: (A)

When $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$,

$$(fog)(x) = f[g(x)] = f(\sqrt{x}) = (\sin\sqrt{x})^2$$

and $(gof)(x) = g[f(x)] = g(\sin^2 x) = |\sin x|$

When $f(x) = \sin x$ and $g(x) = |x|$

$$(fog)(x) = f(g(x)) = f(|x|) = \sin|x| \neq (\sin\sqrt{x})^2$$

When $f(x) = x^2$ and $g(x) = \sin\sqrt{x}$

$$(fog)(x) = f[g(x)] = f(\sin\sqrt{x}) = (\sin\sqrt{x})^2$$

and $(gof)(x) = g[f(x)] = g(x^2) = \sin\sqrt{x^2}$

$$= \sin|x| \neq |\sin x|$$

38. Let f be a function with domain $[-3, 5]$ and let $g(x) = |3x + 4|$. Then the domain of $(fog)(x)$ is

(A) $\left(-3, \frac{1}{3}\right)$ (B) $\left[-3, \frac{1}{3}\right]$
(C) $\left[-3, \frac{1}{3}\right)$ (D) None of these

Solution: (B)

$$(fog)(x) = f[g(x)] = f(|3x + 4|)$$

Since the domain of f is $[-3, 5]$,

$\therefore\quad -3 \le |3x + 4| \le 5 \Rightarrow |3x + 4| \le 5$

$\Rightarrow\quad -5 \le 3x + 4 \le 5$

$\Rightarrow\quad -9 \le 3x \le 1$

$\Rightarrow\quad -3 \le x \le \frac{1}{3}$

$\therefore$ Domain of fog is $\left[-3, \frac{1}{3}\right]$

INVERSE FUNCTIONS

If $f: X \to Y$ be a one-one onto (bijection) function, then the mapping $f^{-1}: Y \to X$ which associates each element $y \in Y$ with element $x \in X$ such that $f(x) = y$, is called the inverse fucntion of the function $f: X \to Y$.

We define inverse function $f^{-1}: Y \to X$ by the rule

$$y = f(x) \Leftrightarrow f^{-1}(y) = x, \forall x \in X, \forall y \in Y$$

as shown in the Fig. 2.20:

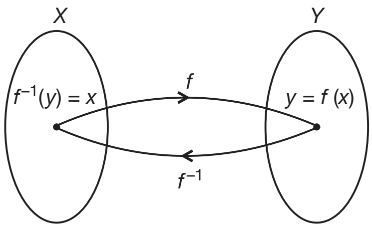


Fig. 2.20

NOTE

For the existence of inverse function, it should be one-one and onto.

METHOD TO FIND THE INVERSE OF A FUNCTION

Let $f: X \to Y$ be a bijective function.

1. Put $f(x) = y$.
2. Solve the equation $y = f(x)$ to obtain x in terms of y. Interchange x and y to obtain the inverse of f.

Aliter: let g be the inverse of f. Simplify the equation $f(g(x)) = x$ to find $g(x)$.

IMPORTANT POINTS

If (x, y) is a point on the graph of an invertible function f, then the corresponding point on the graph of f^{-1} is (y, x). Thus, the graphs of f and f^{-1} are mirror image of each other in the line $y = x$.

PROPERTIES OF INVERSE FUNCTIONS

1. Inverse of a bijection is also a bijection function.
2. Inverse of a bijection is unique.
3. $(f^{-1})^{-1} = f$
4. If f and g are two bijections such that (gof) exists then $(gof)^{-1} = f^{-1}og^{-1}$.

SOLVED EXAMPLES

39. Let $f: (4, 6) \to (6, 8)$ be a function defined by $f(x) = x + \left[\frac{x}{2}\right]$ (where $[\cdot]$ denotes the greatest integer function), then $f^{-1}(x)$ is equal to

(A) $x - \left[\frac{x}{2}\right]$ (B) $-x - 2$

(C) $x - 2$ (D) $\dfrac{1}{x + \left[\frac{x}{2}\right]}$

Solution: (C)

Since $f: (4, 6) \to (6, 8)$,

$\therefore \quad f(x) = x + 2$

$\therefore \quad f^{-1}(x) = x - 2$

40. Let $f: [4, \infty) \to [4, \infty)$ be a function defined by, $f(x) = 5^{x(x-4)}$, then $f^{-1}(x)$ is

(A) $2 - \sqrt{4 + \log_5 x}$ (B) $2 + \sqrt{4 + \log_5 x}$

(C) $\left(\frac{1}{5}\right)^{x(x-4)}$ (D) None of these

Solution: (B)

Let $y = 5^{x(x-4)} \Rightarrow x(x-4) = \log_5 y$

$\Rightarrow \quad x^2 - 4x - \log_5 y = 0$

$$\Rightarrow \quad x = \frac{4 \pm \sqrt{16 + 4\log_5 y}}{2}$$

$$= (2 \pm \sqrt{4 + \log_5 y})$$

But $x \geq 4$, so $x = (2 + \sqrt{4 + \log_5 y})$

$\therefore \quad f^{-1}(y) = 2 + \sqrt{4 + \log_5 y}$

41. The inverse of the function $f(x) = \dfrac{a^x - a^{-x}}{a^x + a^{-x}}$ is

(A) $\frac{1}{2}\log_a\left(\frac{1-x}{1+x}\right)$ (B) $\frac{1}{2}\log_a\left(\frac{1+x}{1-x}\right)$

(C) $\log_a\left(\frac{1+x}{1-x}\right)$ (D) None of these

Solution: (B)

Let $y = \dfrac{a^x - a^{-x}}{a^x + a^{-x}} = \dfrac{a^{2x} - 1}{a^{2x} + 1}$

$$\Rightarrow \quad \frac{y-1}{y+1} = \frac{(a^{2x} - 1) - (a^{2x} + 1)}{(a^{2x} - 1) + (a^{2x} + 1)}$$

(Using componendo and dividendo)

$$\Rightarrow \frac{y-1}{y+1} = \frac{-2}{2a^{2x}} \Rightarrow a^{2x} = \frac{1+y}{1-y}$$

$$\Rightarrow 2x \log_a a = \log_a \left(\frac{1+y}{1-y}\right)$$

$$\Rightarrow x = \frac{1}{2}\log_a \left(\frac{1+y}{1-y}\right)$$

ODD AND EVEN FUNCTIONS

Odd Function

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$ for every real number x in the domain of f.

Illustration

$$y = f(x) = \sin x \text{ is odd.}$$

Even Function

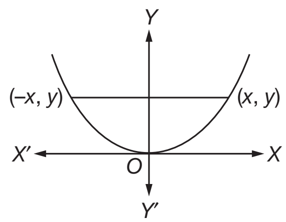
A function $f(x)$ is said to be even if $f(-x) = f(x)$ for every real number x in the domain of f.

Illustration

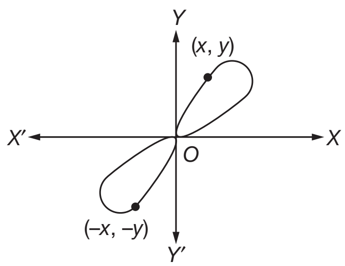
$$y = f(x) = x^2 \text{ is even.}$$

NOTE

- (x, y) is a point on the graph of an even function if an only if $(-x, y)$ is a point on the graph.



- (x, y) is a point on the graph of an odd function if and only if $(-x, -y)$ is a point on the graph.



PROPERTIES OF ODD AND EVEN FUNCTIONS

1. The graph of even function is always symmetric with respect to y-axis. The graph of odd function is always symmetric with respect to origin.
2. The product of two even functions is an even function.
3. The sum and difference of two even functions is an even function.
4. The sum and difference of two odd functions is an odd function.
5. The product of two odd functions is an even function.
6. The product of an even and an odd function is an odd function.
7. It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor odd, e.g. $f(x) = x^2 + x^3$, $f(x) = \log_e x$, $f(x) = e^x$.
8. The sum of even and odd function is neither even nor odd function.
9. Zero function $f(x) = 0$ is the only function which is even and odd both.

TRICK(S) FOR PROBLEM SOLVING

- The graph of an odd function is symmetric about origin and it is placed either in the first and third quadrant or in the second and fourth quadrant.
- The graph of an even function is symmetric about the y-axis.
- To express a given function $f(x)$ as the sum of an even and odd function, we write

  $f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

  where $\frac{1}{2}[f(x) + f(-x)]$ is an even function and $\frac{1}{2}[f(x) - f(-x)]$ is an odd function.
- $f(x) = 0$ is the only function which is both even and odd.
- If $f(x)$ is an odd function, then $f'(x)$ is an even function provided $f(x)$ is differentiable on R.
- If $f(x)$ is an even function, then $f'(x)$ is an odd function provided $f(x)$ is differentiable on R.
- If f and g are even functions, then fog is also an even function, provided fog is defined.
- If f and g are odd functions, then fog is also an odd function, provided fog is defined.
- If f is an even function and g is an odd function, then fog is an even function.
- If f is an odd function and g is an even function, then fog is an even function.
- For a real domain, even functions are not one-one.
- Even functions are many-one functions, because they are symmetric about y-axis.

SOLVED EXAMPLES

42. Let $f(x) = (-1)^{[x]}$ (where [.] denotes the greatest integer function), then

(A) Range of f is $\{-1, 1\}$
(B) f is an even function
(C) f is an odd function
(D) $\lim_{x \to n} f(x)$ exists, for every integer n

Solution: (A)

$$f(x) = (-1)^{[x]} = \{-1, 1\}, \text{ since } [x] \in Z.$$

43. If f is an even function defined on the interval [– 5, 5], then the real values of x satisfying the equation

$$f(x) = f\left(\frac{x+1}{x+2}\right) \text{ are}$$

(A) $\frac{-1 \pm \sqrt{5}}{2}$ (B) $\frac{-3 \pm \sqrt{5}}{2}$
(C) $\frac{-2 \pm \sqrt{5}}{2}$ (D) None of these

Solution: (A, B)

Since $f(x) = f\left(\frac{x+1}{x+2}\right) \Rightarrow x = \frac{x+1}{x+2}$

$\Rightarrow \quad x^2 + x - 1 = 0$

$\Rightarrow \quad x = \frac{-1 \pm \sqrt{5}}{2} \qquad (1)$

Since $f(x)$ is an even function defined on [–5, 5],

$\therefore \quad f(-x) = f(x), \forall x \in [-5, 5]$

$\Rightarrow \quad x = -\left(\frac{x+1}{x+2}\right) \Rightarrow x^2 + 3x + 1 = 0$

$\Rightarrow \quad x = \frac{-3 \pm \sqrt{5}}{2} \qquad (2)$

From Eq. (1) and (2), the values of x are

$$\frac{-1 \pm \sqrt{5}}{2} \text{ and } \frac{-3 \pm \sqrt{5}}{2}$$

44. The function $f(x) = \sec[\log(x + \sqrt{1+x^2})]$ is

(A) even (B) odd
(C) constant (D) None of these

Solution: (A)

Since the function sec x is an even function and $\log(x + \sqrt{1+x^2})$ is an odd function, therefore the function $\sec[\log(x + \sqrt{1+x^2})]$ is an even function.

45. A function whose graph is symmetrical about the origin is given by

(A) $f(x) = (3^x + 3^{-x})$
(B) $f(x) = \cos[\log(x + \sqrt{1+x^2})]$
(C) $f(x + y) = f(x) + f(y) \ \forall\ x, y \in R$
(D) None of these

Solution: (C)

A function whose graph is symmetrical about the origin must be odd.

$$(3^x + 3^{-x}) \text{ is an even function.}$$

Since cos x is an even function and $\log(x + \sqrt{1+x^2})$ is an odd function,

$\therefore \quad \cos(\log(x + \sqrt{1+x^2}))$ is an even function.

If $f(x + y) = f(x) + f(y) \ \forall\ x, y \in R$, then $f(x)$ must be odd.

46. Let the function $f(x) = 3\sin x - 4\cos x + \log(|x| + \sqrt{1+x^2})$ be defined on the interval [0, 1]. The odd extension of $f(x)$ to the interval [– 1, 1] is

(A) $3\sin x - 4\cos x + \log(|x| + \sqrt{1+x^2})$
(B) $3\sin x + 4\cos x - \log(|x| + \sqrt{1+x^2})$
(C) $3\sin x + 4\cos x + \log(|x| + \sqrt{1+x^2})$
(D) None of these

Solution: (B)

To make $f(x)$ an odd function in the interval [–1, 1], we re-define $f(x)$ as follows :

$$f(x) = \begin{cases} f(x), & 0 \le x \le 1 \\ -f(-x), & -1 \le x < 0 \end{cases}$$

$$= \begin{cases} 3\sin x - 4\cos x + \log(|x| + \sqrt{1+x^2}), & 0 \le x \le 1 \\ -[-3\sin x - 4\cos x + \log(|x| + \sqrt{1+x^2})], & -1 \le x < 0 \end{cases}$$

$$= \begin{cases} 3\sin x - 4\cos x + \log(|x| + \sqrt{1+x^2}), & 0 \le x \le 1 \\ 3\sin x + 4\cos x - \log(|x| + \sqrt{1+x^2}), & -1 \le x < 0 \end{cases}$$

Thus, the odd extension of $f(x)$ to the interval [–1, 1] is

$3\sin x + 4\cos x - \log(|x| + \sqrt{1+x^2})$

PERIODIC FUNCTION

A function $f(x)$ is said to be a periodic function of x, provided there exists a real number $T > 0$ such that

$$f(x + T) = f(x), \forall x \in R.$$

The smallest positive real number T, satisfying the above condition is known as the *period or the fundamental period* of $f(x)$.

SOLVED EXAMPLES

47. The period of the function $3^{(\sin^2 \pi x + x - [x] + \sin^4 \pi x)}$, where $[\cdot]$ denotes the greatest integer function, is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) None periodic

Solution: (B)

$$\sin^2 \pi x = \frac{1 - \cos 2\pi x}{2}.$$

Since $\cos 2\pi x$ is a periodic function with period $\frac{2\pi}{2\pi} = 1$, therefore $\sin^2 \pi x$ is periodic with period 1. (1)

$x - [x]$ is a periodic function with period 1. (2)

$$\sin^4 \pi x = (\sin^2 \pi x)^2 = \frac{1}{4}(1 - \cos 2\pi x)^2$$

$$= \frac{1}{4}(1 + \cos^2 2\pi x - 2\cos 2\pi x)$$

$$= \frac{1}{8}(3 + \cos 4\pi x - 4\cos 2\pi x)$$

Since, $\cos 4\pi x$ is a periodic function with period $\frac{2\pi}{4\pi} = \frac{1}{2}$ and $\cos 2\pi x$ is a periodic function with period $\frac{2\pi}{2\pi} = 1$, therefore, period of $\sin^4 \pi x$ is equal to

$$\text{L.C.M.}\left(1, \frac{1}{2}\right) = \frac{\text{L.C.M. }(1, 1)}{\text{H.C.F. }(1, 2)} = \frac{1}{1} = 1 \quad (3)$$

From Eq. (1), (2) and (3), we get

Period of $3^{(\sin^2 \pi x + x - [x] + \sin^4 \pi x)} = 1$

48. The period of the function $f(x) = a \sin kx + b \cos kx$ is

(A) $\frac{2\pi}{k}$ (B) $\frac{2\pi}{|k|}$

(C) $\frac{\pi}{|k|}$ (D) None of these

Solution: (B)

We have,

$$f(x) = a \sin kx + b \cos kx$$

$$= \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}}\sin kx + \frac{b}{\sqrt{a^2 + b^2}}\cos kx\right)$$

$$= \sqrt{a^2 + b^2}\,(\cos\theta \sin kx + \sin\theta \cos kx),$$

where $\cos\theta = \frac{a}{\sqrt{a^2 + b^2}}$

$$= \sqrt{a^2 + b^2}\,\sin(kx + \theta),$$

which is a periodic function of period $\frac{2\pi}{|k|}$.

49. Let f be a real valued function with domain R satisfying $f(x + k) = 1 + [2 - 5f(x) + 10\{f(x)\}^2 - 10\{f(x)\}^3 + 5\{f(x)\}^4 - \{f(x)\}^5]^{1/5}$ for all real x and some positive constant k, then the period of the function $f(x)$ is

(A) k (B) $2k$

(C) Non periodic (D) None of these

Solution: (B)

We have,

$$f(x + k) = 1 + [1 + \{1 - f(x)\}^5]^{1/5}$$

$$\Rightarrow f(x + k) - 1 = [1 - (f(x) - 1)^5]^{1/5}$$

$$\Rightarrow \phi(x + k) = [1 - \{\phi(x)\}^5]^{1/5},$$

where $\phi(x) = f(x) - 1$

$$\Rightarrow \phi(x + 2k) = [1 - \{\phi(x + k)\}^5]^{1/5}$$

$$\Rightarrow \phi(x + 2k) = [1 - \{1 - (\phi(x))^5\}]^{1/5} = \phi(x), \forall x \in R$$

$$\Rightarrow f(x + 2k) - 1 = f(x) - 1$$

$$\Rightarrow f(x + 2k) = f(x), \forall x \in R$$

$\therefore$ $f(x)$ is periodic with period $2k$.

50. The period of the function $f(x) = \sin\sqrt{x}$ is

(A) π (B) 2π

(C) $\frac{\pi}{2}$ (D) None of these

Solution: (D)

Let $f(T + x) = f(x) \Rightarrow \sin\sqrt{T + x} = \sin\sqrt{x}$

$$\Rightarrow \sqrt{T + x} = n\pi + (-1)^n\sqrt{x}$$

Clearly, from here, no positive value of T independent of x is possible because $\sqrt{x}$ on RHS can be cancelled only when $T = 0$.

$\therefore$ $f(x)$ is a non-periodic function.

51. The period of the function $f(x) = \sqrt{\tan x}$ is

(A) π (B) 2π

(C) $\frac{\pi}{2}$ (D) None of these

Solution: (A)

Let $f(T+x)=f(x)$

$\Rightarrow \sqrt{\tan(T+x)}=\sqrt{\tan x} \Rightarrow \tan(T+x)=\tan x$

$\Rightarrow T+x=n\pi+x,\ n\in Z$

Clearly, from here, the least positive value of T independent of x is π. Therefore, $f(x)$ is a periodic function of period π.

52. The period of the function $f(x)=\cos x^2$ is

(A) 2π (B) π
(C) $\frac{\pi}{2}$ (D) None of these

Solution: (D)

Let $f(T+x)=f(x) \Rightarrow \cos(T+x)^2=\cos x^2$

$\Rightarrow (T+x)^2=2n\pi\pm x^2$

Clearly, from here, no positive value of T independent of x is possible because x^2 on R.H.S. can be cancelled only when $T=0$.

$\therefore$ $f(x)$ is a non periodic function.

53. The period of the function

$$f(x)=\cos\left(\frac{\pi x}{n!}\right)-\sin\left(\frac{\pi x}{(n+1)!}\right) \text{ is}$$

(A) $2\,(n+1)!$ (B) $2\,(n!)$
(C) $(n+1)$ (D) Not periodic

Solution: (A)

Since $\sin x$ and $\cos x$ are periodic functions with period 2π.

$\therefore$ Period of $\cos\left(\frac{\pi x}{n!}\right)=\frac{2\pi}{\pi/n!}=2\,(n!)$

and period of $\sin\left(\frac{\pi x}{(n+1)!}\right)=\frac{2\pi}{\pi/(n+1)!}=2\,(n+1)!$

$\therefore$ Period of $f(x)$ = L.C.M. of $\{2\,(n!), 2\,(n+1)!\}=2\,(n+1)!$

54. If the period of the function $f(x)=\sin(\sqrt{[n]}\,x)$, where $[n]$ denotes the greatest integer less than or equal to n, is 2π, then

(A) $1\le n<2$ (B) $1<n<2$
(C) $1\le n\le 2$ (D) None of these

Solution: (A)

Sin x is a periodic function with period 2π, therefore $\sin(\sqrt{[n]}\,x)$ is a periodic function with period $\frac{2\pi}{\sqrt{[n]}}$.

But the period of $f(x)$ is 2π (given).

$\therefore \frac{2\pi}{\sqrt{[n]}}=2\pi \Rightarrow \sqrt{[n]}=1 \Rightarrow [n]=1 \Rightarrow 1\le n<2.$

55. The function $f(x)=k\,|\cos x|+k^2\,|\sin x|+\phi(k)$ has period $\frac{\pi}{2}$ if k is equal to

(A) 1 (B) 2
(C) 3 (D) None of these

Solution: (A)

Since $|\sin x|+|\cos x|$ is a periodic function with period $\frac{\pi}{2}$, therefore period of $f(x)$ will be $\frac{\pi}{2}$ when $k=1$.

56. The period of the function $f(x)=\frac{|\sin x|-|\cos x|}{|\sin x+\cos x|}$ is

(A) $\frac{\pi}{2}$ (B) 2π
(C) π (D) None of these

Solution: (C)

We have,

$$f(\pi+x)=\frac{|\sin(\pi+x)|-|\cos(\pi+x)|}{|\sin(\pi+x)+\cos(\pi+x)|}$$

$$=\frac{|\sin x|-|\cos x|}{|\sin x+\cos x|}=f(x) \text{ for all } x$$

$\therefore$ $f(x)$ is periodic with period π.

57. The period of the function

$$f(x)=\begin{Bmatrix}1, \text{ when } x \text{ is a rational}\\ 0, \text{ when } x \text{ is irrational}\end{Bmatrix} \text{ is}$$

(A) 1 (B) 2
(C) non-periodic (D) None of these

Solution: (D)

For every rational number T, we have

$$f(T+x)=\begin{Bmatrix}1, \text{ when } x \text{ is a rational}\\ 0, \text{ when } x \text{ is irrational}\end{Bmatrix}=f(x),$$

but there is no least positive value of T for which $f(T+x)=f(x)$ because there are infinite number of rational numbers between any two rational numbers. Therefore, $f(x)$ is a periodic function having no fundamental period.

58. Which of the following functions has period π?

(A) $2\cos\left(\frac{2\pi x}{3}\right)+3\sin\left(\frac{\pi x}{3}\right)$

(B) $|\tan x|+\cos 2x$

(C) $4\cos\left(2\pi x+\frac{\pi}{2}\right)+2\sin\left(\pi x+\frac{\pi}{4}\right)$

(D) None of these

Solution: (B)

Period of 2 cos $\left(\frac{2\pi x}{3}\right)$ + 3 sin $\left(\frac{\pi x}{3}\right)$

$$= \text{L.C.M.}\left(\frac{2\pi}{2\pi/3}, \frac{2\pi}{\pi/3}\right) = 6.$$

Period of 4 cos $\left(2\pi x+\frac{\pi}{2}\right)$ + 2 sin $\left(\pi x+\frac{\pi}{4}\right)$

$$= \text{L.C.M.}\left(\frac{2\pi}{2\pi}, \frac{2\pi}{\pi}\right) = 2$$

Period of |tan x| + cos 2x = L.C.M. $\left(\pi, \frac{2\pi}{2}\right) = \pi.$

59. The period of the function $f(x) = x\,[x]$ is

(A) 1 (B) 2
(C) non periodic (D) None of these

Solution: (C)

Let $n \le x < n+1$

Then, $f(x) = x \cdot n$, where n changes with x.

Clearly no constant $k > 0$ is possible for which $f(x) = f(x+k)$ corresponding to all x.

$\therefore$ $f(x)$ is a non periodic function.

60. The period of the function

$$f(x) = 3x + 3 - [3x + 3] + \sin\frac{\pi x}{2},$$

where $[x]$ denotes the greatest integer $\le x$, is

(A) 4 (B) 1
(C) 2 (D) Non-periodic

Solution: (A)

$3x + 3 - [3x + 3]$ has the period 1 and sin $\frac{\pi x}{2}$ has the period $\frac{2\pi}{\pi/2}$ i.e., 4. Therefore, the period of $f(x)$ is L.C.M. (1, 4) = 4.

61. The value of $n \in I$ for which the function

$$f(x) = \frac{\sin nx}{\sin\left(\frac{x}{n}\right)} \text{ has } 4\pi \text{ as its period is}$$

(A) 2 (B) 3 (C) 4 (D) 5

Solution: (A)

For $n = 2$, we have

$$f(x) = \frac{\sin 2x}{\sin\left(\frac{x}{2}\right)} = \frac{2\sin x\cos x}{\sin\left(\frac{x}{2}\right)}$$

$$= \frac{4\sin x/2\cos x/2\cos x}{\sin\left(\frac{x}{2}\right)} = 4\cos\frac{x}{2}\cos x.$$

The period of cos x is 2π and that of cos $\frac{x}{2}$ is $\frac{2\pi}{1/2} = 4\pi$.

Hence, the period of $f(x)$ is 4π.

62. π is the period of the function

(A) |sin x| + |cos x|
(B) $\sin^4 x + \cos^4 x$
(C) sin (sin x) + sin (cos x)
(D) $\frac{1+2\cos x}{\sin x(2+\sec x)}$

Solution: (D)

The period of |sin x| + |cos x| and $\sin^4 x + \cos^4 x$ is $\frac{\pi}{2}$.

sin (sin x) + sin (cos x) has period 2π. The function $\frac{1+2\cos x}{\sin x(2+\sec x)}$ can be written in a simplified form as $\frac{\cos x}{\sin x}$ = cot x, so it has period π.

63. If $f(x) = \frac{x^2+1}{[x]}$, ([·] denotes the greatest integer function), $1 \le x < 4$, then

(A) range of f is $\left[2, \frac{17}{3}\right)$
(B) f is monotonically increasing in [1, 4]
(C) the maximum value of $f(x)$ is $\frac{17}{3}$
(D) the maximum value of $f(x)$ is $\frac{17}{4}$

Solution: (A)

We have, $f(x) = \frac{x^2+1}{[x]}$.

When $x \in [1, 2)$ then $f(x) = x^2 + 1 \Rightarrow R_f = [2, 5)$.

When $x \in [2, 3)$ then $f(x) = \frac{x^2+1}{2} \Rightarrow R_f = \left[\frac{5}{2}, 5\right)$.

When $x \in [3, 4)$ then $f(x) = \frac{x^2+1}{3} \Rightarrow R_f = \left[\frac{10}{3}, \frac{17}{3}\right)$.

$\therefore$ $R_f = [2, 17/3)$.

64. The number of solutions of the equation $a^{f(x)} + g(x) = 0$, $a > 0$, $g(x) \ne 0$ and has minimum value $\frac{1}{2}$ is

(A) One (B) Two
(C) Zero (D) Infinitely many

Solution: (C)

We have

$$a^{f(x)} + g(x) = 0,\ a > 0$$

Since minimum value of $g(x)$ is 1/2

$\therefore \quad g(x) > 0$ and $a^{f(x)} > 0$

$\therefore \quad a^{f(x)} + g(x) > 0, \ \forall \ x$

Hence number of solutions is zero.

65. If $f : R \to R$, $g : R \to R$ be two given functions then $f(x) = 2 \min \{f(x) - g(x), 0\}$ equals

(A) $f(x) + g(x) - |g(x) - f(x)|$
(B) $f(x) + g(x) + |g(x) - f(x)|$
(C) $f(x) - g(x) + |g(x) - f(x)|$
(D) $f(x) - g(x) - |g(x) - f(x)|$

Solution: (D)

We have,

$$f(x) = 2 \min \{f(x) - g(x), 0\}$$

$$= \begin{cases} 0 & f(x) > g(x) \\ 2[f(x) - g(x)], & f(x) \le g(x) \end{cases}$$

$$= \begin{cases} f(x) - g(x) - |f(x) - g(x)|, & f(x) > g(x) \\ f(x) - g(x) - |f(x) - g(x)|, & f(x) \le g(x) \end{cases}$$

$\therefore \quad f(x) = f(x) - g(x) - |g(x) - f(x)|$

66. If $f : R \to R$ is a function satisfying the property $f(2x + 3) + f(2x + 7) = 2, \ \forall \ x \in$ R, then the period of $f(x)$ is

(A) 2 (B) 4 (C) 8 (D) 12

Solution: (B)

We have,

$$f(2x + 3) + f(2x + 7) = 2 \quad (1)$$

Replace x by $x + 1$,

$$f(2x + 5) + f(2x + 9) = 2 \quad (2)$$

Now replace x by,

$$x + 2, f(2x + 7) + f(2x + 11) = 2 \quad (3)$$

From (1) – (3), we get

$$f(2x + 3) - f(2x + 11) = 0$$

i.e., $f(2x + 3) = f(2x + 11) \Rightarrow T = 4$

67. If T_1 is the period of the function $y = e^{3(x-[x])}$ and T_2 is the period of the function $y = e^{3x-[3x]}$ ([·] denotes the greatest integer function), then

(A) $T_1 = T_2$ (B) $T_1 = \frac{T_2}{3}$
(C) $T_1 = 3T_2$ (D) None of these

Solution: (C)

Let $\quad g(x) = e^{3\{x\}} \Rightarrow T_1 = 1$

and $\quad f(x) = e^{\{3x\}} \Rightarrow T_2 = 1/3$

$\therefore \quad T_1 = 3T_2$.

68. If $f(x + y, x - y) = xy$, then the arithmetic mean of $f(x, y)$ and $f(y, x)$ is

(A) y (B) x
(C) 0 (D) None of these

Solution: (C)

Let $\quad x + y = a$ and $x - y = b$

$\Rightarrow \quad x = \frac{a+b}{2}$ and $y = \frac{a-b}{2}$

$\therefore \quad f(x + y, x - y) = xy$

$$\Rightarrow \quad f(a, b) = \frac{a+b}{2} \cdot \frac{a-b}{2} = \frac{a^2 - b^2}{4}$$

$$\therefore \quad f(x, y) = \frac{x^2 - y^2}{4} \text{ and } f(y, x) = \frac{y^2 - x^2}{4}$$

$\therefore$ Arithmetic mean of $f(x, y)$ and $f(y, x)$ is zero.

69. If $3f(x) + 5f\left(\frac{1}{x}\right) = \frac{1}{x} - 3, \forall \ x \ (\ne 0) \in R$, then $f(x) =$

(A) $\frac{1}{14}\left(\frac{3}{x} + 5x - 6\right)$ (B) $\frac{1}{14}\left(-\frac{3}{x} + 5x - 6\right)$

(C) $\frac{1}{14}\left(-\frac{3}{x} + 5x - 6\right)$ (D) None of these

Solution: (B)

We have,

$$3f(x) + 5f\left(\frac{1}{x}\right) = \frac{1}{x} - 3, \forall \ x \ (\ne 0) \in R \quad (1)$$

$$\Rightarrow \quad 3f\left(\frac{1}{x}\right) + 5f(x) = x - 3 \quad (2)$$

$\left[\text{Replacing } x \text{ by } \frac{1}{x}\right]$

Multiplying Eq. (1) by 3 and (2) by 5 and subtracting, we get

$$9f(x) - 25f(x) = \left(\frac{3}{x} - 9\right) - (5x - 15)$$

$$\Rightarrow \quad -14f(x) = \frac{3}{x} - 5x + 6$$

$$\Rightarrow \quad f(x) = \frac{1}{14}\left(-\frac{3}{x} + 5x - 6\right), \forall \ x \ (\ne 0) \in R$$

CAUTION

There are certain function which are periodic but don't have a fundamental period. For example the constant function $f(x) = c$ is periodic as

$$f(x + T) = f(x)$$

is true for every real number T, but it does not have a fundamental period since the least positive value of T cannot be obtained.

SHORT-CUT METHOD TO CHECK THE PERIODICITY OF A FUNCTION

1. Put $f(T + x) = f(x)$ and solve this equation to find the positive values of T independent of x.
2. If no positive value of T independent of x is obtained, then $f(x)$ is a non-periodic function.
3. If positive values of T independent of x are obtained, then $f(x)$ is a periodic function and the least positive value of T is the period of the function $f(x)$.

TRICK(S) FOR PROBLEM SOLVING

- Constant function is periodic with no fundamental period.
- If $f(x)$ is periodic with period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ are also periodic with same period T.
- If $f_1(x)$, $f_2(x)$ and $f_3(x)$ are periodic functions with periods T_1, T_2 and T_3 respectively. Also, if $h(x) = af_1(x) \pm bf_2(x) \pm cf_3(x)$, then, period of $[h(x)]$

$$= \begin{cases} \text{L.C.M. of } \{T_1, T_2, T_3\} & \text{if } h(x) \text{ is an odd function} \\ \frac{1}{2}\text{L.C.M. of } \{T_1, T_2, T_3\} & \text{if } h(x) \text{ is an even function} \end{cases}$$

- If $f(x)$ is periodic with period T, then $kf(ax + b)$ is also periodic with period $\frac{T}{|a|}$, where $a, b, k \in R$ and $a, k \neq 0$.
- $\sin x$, $\cos x$, $\sec x$ and $\text{cosec } x$ are periodic functions with period 2π.
- $\tan x$ and $\cot x$ are periodic functions with period π.
- $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$, $|\sec x|$ and $|\text{cosec } x|$ are periodic functions with period π.
- $\sin^n x$, $\cos^n x$, $\sec^n x$ and $\text{cosec}^n x$ are periodic functions with period 2π when n is odd or π when n is even.
- $\tan^n x$ and $\cot^n x$ are periodic functions with period π.
- If $f(x)$ is a periodic function with period T and $g(x)$ is any function such that range of $f \subset$ domain of g, then gof is also periodic with period T.

EXERCISES

Single Option Correct Type

1. Let $f(x) = x^3 + x^2 + 100x + 7\sin x$, then the equation $\frac{1}{y - f(1)} + \frac{2}{y - f(2)} + \frac{3}{y - f(3)} = 0$ has

(A) one real root (B) two real roots
(C) more than two real roots (D) no real root

2. If $b^2 - 4ac = 0$ and $a > 0$, then the domain of the function $f(x) = \log(ax^3 + (2a + b)x^2 + (2b + c)x + 2c)$ is

(A) $(-2, \infty)\backslash\left\{-\frac{b}{2a}\right\}$ (B) $[-2, \infty)\backslash\left\{-\frac{b}{2a}\right\}$
(C) $(-\infty, -2)\backslash\left\{-\frac{b}{2a}\right\}$ (D) None of these

3. If $e^x + e^{f(x)} = e$, then range of the function f is

(A) $(-\infty, 1]$ (B) $(-\infty, 1)$
(C) $(1, \infty)$ (D) $[1, \infty)$

4. Which of the following functions is are injective map(s)?

(A) $f(x) = x^2 + 2, x \in (-\infty, \infty)$
(B) $f(x) = |x + 2|, x \in [-2, \infty)$
(C) $f(x) = (x - 4)(x - 5), x \in (-\infty, \infty)$
(D) $f(x) = \frac{4x^2 + 3x - 5}{4 + 3x - 5x^2}, x \in (-\infty, \infty)$

5. The graph of the function $\cos x \cos(x + 2) - \cos^2(x + 1)$ is

(A) a straight line passing through $(0, -\sin^2 1)$ with slope 2
(B) a straight line passing through $(0, 0)$
(C) a parabola with vertex $(1, -\sin^2 1)$
(D) a straight line parallel to x-axis passing through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$

6. Let $f: R \to R$ be a function defined by, $f(x) = \frac{x^2 - 8}{x^2 + 2}$, then f is

(A) one-one but not onto
(B) one-one and onto
(C) onto but not one-one
(D) neither one-one nor onto

7. If $f(x) = 64x^3 + \frac{1}{x^3}$ and a, b are the roots of $4x + \frac{1}{x} = 3$, then

(A) $f(a) = 12$ (B) $f(b) = 11$
(C) $f(a) = f(b)$ (D) None of these

8. If the functions f, g, h are defined from the set of real numbers R to R such that

$f(x) = x^2 - 1,\ g(x) = \sqrt{x^2+1},\ h(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } x \ge 0 \end{cases}$

then the composite function $(hofog)(x) =$

(A) $\begin{cases} 0, & x = 0 \\ x^2, & x > 0 \\ -x^2, & x < 0 \end{cases}$ (B) $\begin{cases} 0, & x = 0 \\ x^2, & x \ne 0 \end{cases}$

(C) $\begin{cases} 0, & x \le 0 \\ x^2, & x > 0 \end{cases}$ (D) None of these

9. If S is the set of all real x and such that $\dfrac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains

(A) $\left(-\infty, -\dfrac{3}{2}\right)$ (B) $\left(-\dfrac{3}{2}, -\dfrac{1}{4}\right)$

(C) $\left(-\dfrac{1}{4}, \dfrac{1}{2}\right)$ (D) $\left(\dfrac{1}{2}, 3\right)$

10. The number of values of x, where the function $f(x) = \cos x + \cos(\sqrt{2}\,x)$ attains its maximum, is

(A) 0 (B) 1
(C) 2 (D) infinite

11. The distinct linear function (s) which map (s) [–1, 1] onto [0, 2] is (are)

(A) $x + 1, -x + 1$ (B) $x - 1, x + 1$
(C) $-x + 1$ (D) None of these

12. Let $f(x) = \max.\{(1-x), (1+x), 2\}, \forall x \in R$. Then

(A) $f(x) = \begin{cases} 1+x, & x \le -1 \\ 2, & -1 < x < 1 \\ 1-x, & x \ge 1 \end{cases}$

(B) $f(x) = \begin{cases} 1-x, & x \le -1 \\ 1, & -1 < x < 1 \\ 1+x, & x \ge 1 \end{cases}$

(C) $f(x) = \begin{cases} 1-x, & x \le -1 \\ 2, & -1 < x < 1 \\ 1+x, & x \ge 1 \end{cases}$

(D) None of these

13. If $f(x) = \sin[\pi^2]x + \sin[-\pi^2]x$, where $[\cdot]$ denotes the greatest integer function, then

(A) $f\left(\dfrac{\pi}{2}\right) = 1$ (B) $f(\pi) = 2$

(C) $f\left(\dfrac{\pi}{4}\right) = -1$ (D) None of these

14. The image of the interval [1, 3] under the mapping $f: R \to R$, given by $f(x) = 2x^3 - 24x + 107$ is

(A) [0, 89] (B) [75, 89]
(C) [0, 75] (D) None of these

15. If $2f(x) - 3f\left(\dfrac{1}{x}\right) = x^2$, x is not equal to zero, then $f(2)$ is equal to

(A) $-\dfrac{7}{4}$ (B) $\dfrac{5}{2}$

(C) -1 (D) None of these

16. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and $m(b)$ the minimum value of $f(x)$ for a given b. As b varies, the range of $m(b)$ is

(A) [0, 1] (B) $\left(0, \dfrac{1}{2}\right]$

(C) $\left[\dfrac{1}{2}, 1\right]$ (D) (0, 1]

17. Let $f: R \to R$, $g: R \to R$ be two functions given by $f(x) = 2x - 3$, $g(x) = x^3 + 5$. Then $(fog)^{-1}(x)$ is equal to

(A) $\left(\dfrac{x-7}{2}\right)^{1/3}$ (B) $\left(\dfrac{x+7}{2}\right)^{1/3}$

(C) $\left(x - \dfrac{7}{2}\right)^{1/3}$ (D) $\left(\dfrac{x-2}{7}\right)^{1/3}$

18. The functions $f(x) = \log(x-1) - \log(x-2)$ and $g(x) = \log\left(\dfrac{x-1}{x-2}\right)$ are identical when x lies in the interval

(A) [1, 2] (B) $[2, \infty]$
(C) $(2, \infty)$ (D) $(-\infty, \infty)$

19. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$. Then, for all x, $f[g(x)]$ is equal to

(A) x (B) 1 (C) $f(x)$ (D) $g(x)$

20. The domain of the function $y = \sqrt{\log \dfrac{1}{|\sin x|}}$ is

(A) $R \setminus \{n\pi : n \in Z\}$ (B) $R \setminus (-\pi, \pi)$
(C) $R \setminus \{2n\pi : n \in Z\}$ (D) $(-\infty, \infty)$

21. If x is real, then the expression $\dfrac{x^2+34x-71}{x^2+2x-7}$

(A) cannot lie between 5 and 9
(B) always lies between 5 and 9
(C) is not real
(D) None of these

22. If $f(x)$ is an odd periodic function with period 2, then $f(4)$ equals

(A) -4 (B) 4 (C) 2 (D) 0

23. The function

$f(x) = \cot^{-1}[\sqrt{(x+3)x}] + \cos^{-1}(\sqrt{x^2+3x+1})$ is defined on the set S, where S is equal to

(A) $\{-3, 0\}$ (B) $[-3, 0]$
(C) $[0, 3]$ (D) $(-3, 0)$

24. If $f(x) = a^{\cos x}$ and $g(x) = (\sin x)^a$, $a \in$ N, then

(A) $f(x) > g(x)$, $\forall x$
(B) $f(x) < g(x)$, $\forall x$
(C) $f(x) = g(x)$, for infinitely many values of x
(D) $f(x) \neq g(x)$, for any x

25. Let f be a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If $f(1) = 3$, then $\sum_{r=1}^{n} f(r)$ is equal to

(A) $\frac{3}{2}(3^n - 1)$ (B) $\frac{3}{2}n(n+1)$
(C) $3^{n+1} - 3$ (D) None of these

26. Let $f(x)$ be defined for all $x > 0$ and be continuous. Let $f(x)$ satisfy $f\left(\frac{x}{y}\right) = f(x) - f(y)$ for all x, y and $f(e) = 1$. Then

(A) $f(x)$ is bounded (B) $f\left(\frac{1}{x}\right) \to 0$ as $x \to 0$
(C) $x f(x) \to 1$ as $x \to 0$ (D) $f(x) = \log x$

27. If $g(x) = 1 + \sqrt{x}$ and $f[g(x)] = 3 + 2\sqrt{x} + x$, then $f(x) =$

(A) $1 + 2x^2$ (B) $2 + x^2$
(C) $1 + x$ (D) $2 + x$

28. Range of values of $f(x) = 1 + \sin x + \sin^3 x + \sin^5 x \ldots$, $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is

(A) $(0, 1)$ (B) $(-\infty, \infty)$
(C) $(-2, 2)$ (D) None of these

29. The function $f: (-\infty, -1] \to (0, e^5]$ defined by,

$$f(x) = e^{x^3 - 3x + 2} \text{ is}$$

(A) Many one and onto
(B) Many one and into
(C) One-one and onto
(D) One-one and into

30. The domain of the function

$$f(x) = \log_2\left(-\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right) \text{ is}$$

(A) $0 < x < 1$ (B) $0 < x \leq 1$
(C) $x \geq 1$ (D) $x > 1$

31. The range of the function

$f(x) = \frac{\sin(\pi[x^2+1])}{x^4+1}$ where, [ ] is greatest integer function, is

(A) $[0, 1]$ (B) $[-1, 1]$
(C) $\{0\}$ (D) None of these

32. If $a f(x) + bf\left(\frac{1}{x}\right) = x + \frac{5}{x}$, $(a \neq b)$, then $f(x)$ is equal to

(A) $\frac{1}{a^2 - b^2}\left(x + \frac{1}{x}\right)$

(B) $\frac{1}{a^2 - b^2}\left[x(5a - b) + \frac{1}{x}(5b - a)\right]$

(C) $\frac{1}{a^2 - b^2}\left[x(a - 5b) + \frac{1}{x}(5a - b)\right]$

(D) None of the above

33. Let $f: R \to R$ defined by,

$f(x) = x^3 + x^2 + 100x + 5\sin x$, then f is

(A) many–one onto (B) many–one into
(C) one–one onto (D) one–one into

34. Let f be a real valued function with domain R satisfying $0 \leq f(x) \leq \frac{1}{2}$ and for some fixed $a > 0$

$$f(x+a) = \frac{1}{2} - \sqrt{f(x) - (f(x))^2} \ \forall x \in R,$$

then the period of the function $f(x)$ is

(A) a (B) $2a$
(C) non-periodic (D) None of these

35. Let $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$. Then $g(f(x))$ is invertible for $x \in$

(A) $\left[-\frac{\pi}{2}, 0\right]$ (B) $\left[-\frac{\pi}{2}, \pi\right]$

(C) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ (D) $\left[0, \frac{\pi}{2}\right]$

36. If $f\left(2x + \frac{y}{8}, 2x - \frac{y}{8}\right) = xy$, then $f(m, n) + f(n, m) = 0$

(A) only when $m = n$
(B) only when $m \neq n$
(C) only when $m = -n$
(D) for all m and n.

37. If $f(x)$ is defined on $(0, 1)$, then the domain of definition of $f(e^x) + f(\ln|x|)$ is

(A) $(-e, -1)$
(B) $(-e, -1) \cup (1, e)$
(C) $(-\infty, -1) \cup (1, \infty)$
(D) $(-e, e)$

38. The value of $\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{100}\right]+\left[\frac{1}{2}+\frac{2}{100}\right]+\cdots+\left[\frac{1}{2}+\frac{99}{100}\right]$ is

(A) 49 (B) 50 (C) 51 (D) 98

39. The domain of definition of

$$f(x)=\sqrt{\frac{\log_{0.3}|x-2|}{|x|}} \text{ is}$$

(A) $[1, 2) \cup (2, 3]$ (B) $[1, 3]$
(C) $\mathbb{R} - (1, 3]$ (D) None of these

40. Let $f : R \to R$ be a function defined by, $f(x) = -\frac{|x|^3+|x|}{1+x^2}$, then the graph of $f(x)$ lies in which quadrant(s)?

(A) I and II (B) I and III
(C) II and III (D) III and IV

41. The domain of definition of the function

$$f(x)=\frac{\sqrt{\sin^{-1}x+\sqrt{x^2+1}}+\sqrt{x-[x]+\log x}}{e^{\sqrt{\sin x+\cos x}}+\log\left|\sin\left(\frac{1}{\sqrt{-x^2}}\right)\right|} \text{ is}$$

(A) $(-1, 1)$ (B) $(0, 1)$
(C) $(1, 0)$ (D) None of these

42. If $f : R \to R$ and $g : R \to R$ are given by $f(x) = |x|$ and $g(x) = [x]$ for each $x \in R$, then $\{x \in R : g[f(x)] \le f[g(x)]\} =$

(A) $Z \cup (-\infty, 0)$ (B) $(-\infty, 0)$
(C) Z (D) R

43. The function

$$f(x)=\sin^{-1}(x-x^2)+\sqrt{1-\frac{1}{|x|}}+\frac{1}{[x^2-1]}$$

is defined in the interval (where $[\cdot]$ is the greatest integer)

(A) $x \in \left(\sqrt{2}, \frac{1+\sqrt{5}}{2}\right)$

(B) $x \in \left(1, \frac{1+\sqrt{5}}{2}\right)$

(C) $x \in \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$

(D) $x \in \left(-\sqrt{2}, \frac{1+\sqrt{5}}{2}\right)$

44. If the graph of $y = ax^3 + bx^2 + cx + d$ is symmetric about the line $x = k$, then the value of $a + k$ is

(A) $-\frac{c}{2b}$ (B) c
(C) $c - bd$ (D) None of these

45. If function $f(x) = \frac{1}{2} - \tan\left(\frac{\pi x}{2}\right)$; $(-1 < x < 1)$ and $g(x)=\sqrt{3+4x-4x^2}$, then the domain of gof is

(A) $(-1, 1)$ (B) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(C) $\left[-1, \frac{1}{2}\right]$ (d) $\left[-\frac{1}{2}, -1\right]$

46. If $f : R \to R$ is a function such that $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in \mathbb{R}$, then $f(2) - f(1) =$

(A) $f(0)$ (B) $-f(0)$ (C) $f'(0)$ (D) $-f'(0)$

47. Let $f : \mathbb{R} \to A = \left\{y : 0 \le y < \frac{\pi}{2}\right\}$ be a function such that $f(x) = \tan^{-1}(x^2 + x + k)$, where k is a constant. The minimum value of k for which f is an onto function, is

(A) 1 (B) 0
(C) $\frac{1}{4}$ (D) None of these

48. Suppose $f : [2, 2] \to R$ is defined by,

$$f(x)=\begin{cases}-1 & \text{for } -2 \le x \le 0\\ x-1 & \text{for } 0 \le x \le 2\end{cases},$$

then $\{x \in (-2, 2) : x \le 0 \text{ and } f(|x|) = x\} =$

(A) $\{-1\}$ (B) $\{0\}$
(C) $\{-1/2\}$ (D) ϕ

49. Let $f(x) = [x]^2 + [x + 1] - 3$, where $[x]$ is greatest integer less than or equal to x, then

(A) $f(x)$ is a many one and into function
(B) $f(x) = 0$ for infinite number of values of x
(C) $f(x) = 0$ for only two real values
(D) None of these

50. If $q^2 - 4pr = 0$, $p > 0$, then the domain of the function $f(x) = \log[px^3 + (p + q)x^2 + (q + r)x + r]$ is

(A) $R - \left\{-\frac{q}{2p}\right\}$

(B) $R - \left[(-\infty, -1] \cup \left\{-\frac{q}{2p}\right\}\right]$

(C) $R - \left[(-\infty, -1) \cap \left\{-\frac{q}{2p}\right\}\right]$

(D) None of these

51. If $f:(3,6)\rightarrow(1,3)$ is a function defined by $f(x) = x-\left[\frac{x}{3}\right]$ (where [ ] dentoes the greatest integer function), then $f^{-1}(x) =$

(A) $x-1$ (B) $x+1$
(C) x (D) None of these

52. If $[x]$ denotes the integral part of x, then the domain of the function $f(x) = \sin^{-1}[2x^2-3] + \log_2[\log_{1/2}(x^2-5x+5)]$ is

(a) $\left(-\sqrt{\frac{5}{2}}, -1\right]$ (B) $\left[1, \sqrt{\frac{5}{2}}\right)$

(C) $\left(-\sqrt{\frac{5}{2}}, -1\right] \cup \left[1, \sqrt{\frac{5}{2}}\right)$ (D) None of these

53. If $f(n+1) = \dfrac{2f(n)+1}{2}$, $n = 1, 2, \ldots$ and $f(1) = 2$, then $f(101)$ equals

(A) 52 (B) 49 (C) 48 (D) 51

54. If $f(x) = \dfrac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}$ for $x \in R$, then $f(2002) =$

(A) 1 (B) 2 (C) 3 (D) 4

55. Let $f(x) = x + 1$ and $\phi(x) = x - 2$, then the values of x satisfying$|f(x) + \phi(x)| = |f(x)| + |\phi(x)|$ are

(A) $(-\infty, 1)$ (B) $(2, \infty)$
(C) $(-\infty, -2)$ (D) $(1, \infty)$

56. A function f from the set of natural numbers to integers defined by,

$$f(n) = \begin{cases} \dfrac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\dfrac{n}{2}, & \text{when } n \text{ is even} \end{cases} \quad \text{is}$$

(A) neighter one–one nor onto
(B) one–one but not onto
(C) onto but not one–one
(D) one–one and onto both

57. If $\alpha \in \left(0, \frac{\pi}{2}\right)$ then $\sqrt{x^2+x} + \dfrac{\tan^2\alpha}{\sqrt{x^2+x}}$ is always greater than or equal to

(A) $2\tan\alpha$ (B) 1
(C) 2 (D) $\sec^2\alpha$

58. The function $f(x)$ is defined in [0, 1], then the domain of definition of the function $f[\log(1-x^2)]$, is

(A) $x \in \{0\}$
(B) $x \in [-\sqrt{1+e}, -1] \cup [1, +\sqrt{1+e}]$
(C) $x \in (-\infty, \infty)$
(D) None of these

59. If $f: R \rightarrow S$, defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$, is onto then the interval of S is

(A) [– 1, 3] (B) [– 1, 1]
(C) [0, 1] (D) [0, 3]

60. If a and b are natural numbers and

$$f(x) = \sin(\sqrt{a^2-3})\,x + \cos(\sqrt{b^2+7})\,x$$

is periodic with finite fundamental period, then period of $f(x)$ is

(A) π (B) 2π
(C) $2\pi(\sqrt{a^2-3} + \sqrt{b^2+7})$
(D) $\pi(\sqrt{a^2-3} + \sqrt{b^2+7})$

61. If $2f(x) + 3f\left(\frac{1}{x}\right) = x^2 - 1$, then $f(x)$ is

(A) a periodic function (B) an even function
(C) an odd function (D) None of these

62. If $f(x_1) - f(x_2) = f\left(\dfrac{x_1 - x_2}{1 - x_1x_2}\right)$ for $x_1, x_2 \in [-1, 1]$ then $f(x)$ is

(A) $\log\left(\dfrac{1-x}{1+x}\right)$ (B) $\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$

(C) $\log\left(\dfrac{1+x}{1-x}\right)$ (D) $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

63. Let $f: R \rightarrow R$ be a periodic function such that

$$f(T+x) = 1 + \{1 - 3f(x) + 3[f(x)]^2 - [f(x)]^3\}^{1/3},$$

where T is a fixed positive number, then period of $f(x)$ is

(A) T (B) $2T$
(C) $3T$ (D) None of these

64. The domain of the function

$$f(x) = \sqrt{\frac{-\log_{0.3}(x-1)}{-x^2+3x+18}} \text{ is}$$

(A) [2, 6] (B) (2, 6)
(C) [2, 6) (D) None of these

65. Suppose $f(x) = (x + 1)^2$ for $x \geq -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals

(A) $-\sqrt{x}-1, x \geq 0$ (B) $\dfrac{1}{(x+1)^2}, x > -1$

(C) $\sqrt{x+1}, x \geq -1$ (D) $\sqrt{x}-1, x \geq 0$

66. The function $f(x) = \dfrac{\sin^{101} x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$, where $[x]$ denotes the integral part of x is

(A) an odd function
(B) an even function
(C) neither odd nor even function
(D) both odd and even function

67. The domain of the function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is

(A) $(-\infty, \infty) - \{0\}$ (B) $(-\infty, \infty)$
(C) $(0, \infty)$ (D) $(-\infty, 0)$

68. The total number of injective mappings from a set with n element to a set with n elements, for $m > n$, is

(A) $\dfrac{m!}{n!(m-n)!}$ (B) $\dfrac{m!}{(m-n)!}$
(C) n^m (D) zero

69. Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in N : y = 4x + 3$ for some $x \in N\}$. Show that f is invertible and its inverse is

(A) $g(y) = \dfrac{3y+4}{3}$ (B) $g(y) = 4 + \dfrac{y+3}{4}$
(C) $g(y) = \dfrac{y+3}{4}$ (D) $g(y) = \dfrac{y-3}{4}$

70. For real x, let $f(x) = x^3 + 5x + 1$, then
(A) f is one–one but not onto R
(B) f is onto R but not one–one
(C) f is one–one and onto R
(D) f is neither one–one nor onto R

71. Let $f(x) = (x + 1)^2 - 1, x \geq -1$

Statement 1: The set $\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$
Statement 2: f is a bijection.
(A) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1
(B) Statement 1 is true, Statement 2 is true; Statement 2 is not a correct explanation for Statement 1
(C) Statement 1 is true, Statement 2 is false
(D) Statement 1 is false, Statement 2 is true

72. The period of the function

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is a rational} \\ 0, & \text{when } x \text{ is irrational} \end{cases} \text{ is}$$

(A) 1 (B) 2
(C) non-periodic (D) None of these

73. Let $f_1(n) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}$, then $f_1(1) + f_1(2) + f_1(3) + \ldots + f_1(n)$ is equal to

(A) $nf_1(n) - 1$ (B) $(n + 1)f_1(n) + n$
(C) $(n + 1)f_1(n) - n$ (D) $nf_1(n) + n$

74. Range of the function f defined by $f(x) = \left[\dfrac{1}{\sin\{x\}}\right]$ (where [·] and { · } respectively denote the greatest integer and the fractional part functions is:)
(A) Z, the set of integers
(B) N, the set of natural numbers
(C) W, the set of whole numbers
(D) $\{2, 3, 4, \ldots\}$

75. If $f : R \to R$, $g : R \to R$ be two given functions then $f(x) = 2 \min \{|f(x) - g(x)|, 0\}$ equals
(A) $f(x) + g(x) - |g(x) - f(x)|$
(B) $f(x) + g(x) + |g(x) - f(x)|$
(C) $f(x) - g(x) + |g(x) - f(x)|$
(D) $f(x) - g(x) - |g(x) - f(x)|$

76. Let $f(x)$ be a function defined on [0, 1] such that

$$f(x) = \begin{cases} x & x \in Q \\ 1 - x & x \notin Q \end{cases}$$

Then, for all $x \in [0, 1]$, $fof(x)$ is
(A) a constant (B) $1 + x$
(C) x (D) None of these

77. If a function $f : R \to R$ be such that $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1, \forall x, y \in R$, then $f(x) =$

(A) $\dfrac{x^2}{2} - 1$ (B) $\dfrac{x^2}{2} + 1$
(C) $1 - \dfrac{x^2}{2}$ (D) None of these

78. A function whose graph is symmetrical about the origin is given by
(A) $f(x) = (3^x + 3^{-x})$
(B) $f(x) = \cos[\log(x + \sqrt{1 + x^2})]$
(C) $f(x + y) = f(x) + f(y)\ \forall x, y \in$ R
(D) None of these

79. Which of the following functions is (are) injective map(*s*)?
(A) $f(x) = x^2 + 2, x \in (-\infty, \infty)$
(B) $f(x) = |x + 2|, x \in [-2, \infty)$
(C) $f(x) = (x - 4)(x - 5), x \in (-\infty, \infty)$
(D) $f(x) = \dfrac{4x^2 + 3x - 5}{4 + 3x - 5x^2}, x \in (-\infty, \infty)$

80. Let f be a function with domain [–3, 5] and let $g(x) = |3x + 4|$. Then, the domain of $(fog)(x)$ is

(A) $\left(-3, \dfrac{1}{3}\right)$ (B) $\left[-3, \dfrac{1}{3}\right]$
(C) $\left[-3, \dfrac{1}{3}\right)$ (D) None of these

81. If for a real number x, $[x]$ denotes the greatest integer less than or equal to x, then for any $n \in N$

$$\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\left[\frac{n+8}{16}\right]+\ldots=$$

(A) n (B) $n-1$
(C) $n+1$ (D) $n+2$

82. Let $g(x) = 1 + x - [x]$ and $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$. Then, for all x, $f[g(x)]$ is equal to

(A) x (B) 1 (C) $f(x)$ (D) $g(x)$.

83. If $g(x)=1+\sqrt{x}$ and $f[g(x)]=3+2\sqrt{x}+x$, then $f(x)=$

(A) $1+2x^2$ (B) $2+x^2$ (C) $1+x$ (D) $2+x$

84. The domain of the function $f(x) =$

$$\log_2\left(-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right) \text{ is}$$

(A) $0 < x < 1$ (B) $0 < x \leq 1$
(C) $x \geq 1$ (D) $x > 1$

85. Let f be a real valued function with domain R satisfying $0 \leq f(x) \leq \frac{1}{2}$ and for some fixed $a > 0$

$$f(x+a)=\frac{1}{2}-\sqrt{f(x)-(f(x))^2} \quad \forall x \in R,$$

then the period of the function $f(x)$ is

(A) a (B) $2a$
(C) non-periodic (D) None of these

86. If $f(x)$ is defined on $(0, 1)$, then the domain of definition of $f(e^x) + f(\log |x|)$ is

(A) $(-e, -1)$ (B) $(-e, -1) \cup (1, e)$
(C) $(-\infty, -1)$ (D) $(-e, e)$

87. The function $f(x)=\sin^{-1}(x-x^2)+\sqrt{1-\frac{1}{|x|}}+\frac{1}{[x^2-1]}$ is defined in the interval (where $[\cdot]$ is the greatest integer):

(A) $x \in \left(\sqrt{2}, \frac{1+\sqrt{5}}{2}\right)$

(B) $x \in \left(1, \frac{1+\sqrt{5}}{2}\right)$

(C) $x \in \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$

(D) $x \in \left(-\sqrt{2}, \frac{1+\sqrt{5}}{2}\right)$

88. If $f: R \to R$ is a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all $x \in R$, then $f(2) - f(1) =$

(A) $f(0)$ (B) $-f(0)$
(C) $f'(0)$ (D) $-f'(0)$

89. If $q^2 - 4pr = 0$, $p > 0$, then the domain of the function $f(x) = \log \{px^3 + (p+q)x^2 + (q+r)x + r\}$ is

(A) $R-\left\{-\frac{q}{2p}\right\}$

(B) $R-\left[(-\infty,-1]\cup\left\{-\frac{q}{2p}\right\}\right]$

(C) $R-\left[(-\infty,-1)\cap\left\{-\frac{q}{2p}\right\}\right]$

(D) None of these

90. If a and b are natural numbers and

$$f(x)=\sin\left(\sqrt{a^2-3}\right)x+\cos\left(\sqrt{b^2+7}\right)x$$

is periodic with finite fundamental period, then period of $f(x)$ is:

(A) π (B) 2π

(C) $2\pi\left(\sqrt{a^2-3}+\sqrt{b^2+7}\right)$

(D) $\pi\left(\sqrt{a^2-3}+\sqrt{b^2+7}\right)$

91. The function $f(x)=\dfrac{\sin^{101}x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}$, where $[x]$ denotes the integral part of x, is

(A) an odd function
(B) an even function
(C) neither odd nor even
(D) both odd and even functon

92. If the graph of $y = ax^3 + bx^2 + cx + d$ is symmetric about the line $x = k$, then the value of $a + k$ is

(A) $-\frac{c}{2b}$ (B) c
(C) $c - bd$ (D) None of these

93. The domain of definition of the function $f(x) = \ln\{x\}+\sqrt{x-2\{x\}}$, where $\{\ \}$ denotes the fractional part, is

(A) $\{0\} \cup [1, \infty)$ (B) $(1, \infty)$
(C) $(1, \infty) - I^+$ (D) None of these

94. The domain of the function $f(x) = \ln(1 - 2|\cos x|) + e^{\cos-1(2x/\pi)}$ is

(A) $\left(-\frac{\pi}{2}, \frac{-\pi}{3}\right) \cup \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

(B) $\left[-\frac{\pi}{2}, \frac{-\pi}{3}\right) \cup \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

(C) $\left[-\frac{\pi}{2}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{3}, \frac{\pi}{2}\right]$

(D) None of these

95. The domain of the function $f(x) = \ln\{\text{sgn}(9 - x^2)\} + \sqrt{[x]^3 - 4[x]}$, where [·] denotes integral part, is

(A) $(-2, 1) \cup (2, 3)$ (B) $[-2, 1) \cup [2, 3)$
(C) $[-2, 1] \cup [2, 3)$ (D) $[-2, 1) \cup [2, 3]$

96. The domain of the function

$$f(x) = \sqrt{(\log_{0.2} x)^3 + (\log_{0.2} x^3)(\log_{0.2} 0.0016x) + 36} \text{ is}$$

(A) (0 , 125) (B) [0, 125]
(C) (0, 125] (D) None of these

97. The range of the function $y = [x^2] - [x]^2$, $x \in [0, 2]$ where [·] denotes the integral part, is

(A) {0} (B) {0, 1}
(C) {1, 2} (D) {0, 1, 2}

98. The range of the function

$$y = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right], \text{ where } [\cdot]$$

denotes the integral part, is

(A) $(0, \pi)$ (B) $[0, \pi]$
(C) $\{\pi\}$ (D) $\{0, \pi\}$

99. Let $f(x) = \frac{ax^2 + 2x + 1}{2x^2 - 2x + 1}$. If $f : R \to [-1, 2]$ is onto, then the values of a are

(A) $(-\infty, 2)$ (B) $[2, \infty)$
(C) $(-\infty, -7] \cup [-2, \infty)$ (D) None of these

100. Let $f : N \to N$, where $f(x) = x + (-1)^{x-1}$. Then,

(A) $f^{-1}(x) = x + (-1)^{x-1}$ (B) $f^{-1}(x) = x$
(C) $f^{-1}(x) = x - (-1)^{x-1}$ (D) None of these

101. Let $f : R \to R$, where $f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. The values of α, for which f to be onto, are

(A) (2, 14) (B) [2, 14)
(C) (2, 14] (D) [2, 14]

102. Domain of definition of the function

$$f(x) = \sqrt{\sin x} + \sin^{-1}\left(\frac{2|x|}{1 + x^2}\right) \text{ is}$$

(A) $(2n\pi, (2n + 1)\pi)$, $n \in I$
(B) $[2n\pi, (2n + 1)\pi]$, $n \in I$
(C) R
(D) None of these

103. The domain of the function

$$f(x) = [\sin x] \cos\left(\frac{\pi}{[x - 1]}\right) \text{ is}$$

(A) (1, 2) (B) $R - [1, 2)$
(C) $R - (1, 2)$ (D) None of these

104. The range of the function $y = \frac{x - [x]}{1 - [x] + x}$ is

(A) $\left(0, \frac{1}{2}\right)$ (B) $\left[0, \frac{1}{2}\right]$
(C) $\left[0, \frac{1}{2}\right)$ (D) $\left(0, \frac{1}{2}\right]$

105. If the domain for $y = f(x)$ is $[-3, 2]$, then the domain of $g(x) = f\{|[x]|\}$ is

(A) $(-2, 3)$ (B) $[-2, 3]$
(C) $[-2, 3)$ (D) $(-2, 3]$

106. If $\{x\}$ and $[x]$ represent fractional and integral part of x, then the value of $[x] + \sum_{r=1}^{2000} \frac{\{x + r\}}{2000}$ is

(A) x (B) $2000\,x$
(C) 0 (D) None of these

107. If $f : (0, \pi) \to R$ be defined by $f(x) = \sum_{k=1}^{n} ([1 + \sin kx])$, where $[x]$ denotes the integral part of x, then the range of $f(x)$ is

(A) $\{n - 1, n + 1\}$ (B) $\{n - 1, n, n + 1\}$
(C) $\{n, n + 1\}$ (D) None of these

108. Consider a function $f(n)$ defined for all $n \in N$. The function satisfies the following two conditions

(i) $f(1) + f(2) + f(3) + \ldots$ to $\infty = 1$
(ii) $f(n) = \{(1 - p)p^{-1}\}\{f(n + 1) + f(n + 2) + \ldots \text{ to } \infty\}$
where $0 < p < 1$. Then, $f(2)$ is equal to

(A) $p(1 - p)$ (B) $1 - p$
(C) $1 + p$ (D) None of these

109. If $f(x) = \lim_{n\to\infty}\left(\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \cdots \text{ to } n \text{ terms}\right)$ then range of $f(x)$ is

(A) {0, 1} (B) {– 1, 0}
(C) {– 1, 1} (D) [0, 1]

110. If p and q are positive integers, f is a function defined for positive numbers and attains only positive values such that $f(xf(y)) = x^p y^q$, then

(A) $p = q$ (B) $p = q^2$

(C) $p^2 = q$ (D) $|p| = |q|$

111. If the function f satisfies the relation $f(x + y) + f(x - y) = 2f(x)f(y)\ \forall\ x, y \in R$ and $f(0) \neq 0$, then $f(x)$ is

(A) an even function

(B) an odd function

(C) odd if $f(x) > 0$

(D) neither even nor odd

112. Let $f: R - \{2\} \to R$ be a function satisfying $2f(x) + 3f\left(\dfrac{2x+29}{x-2}\right) = 100x + 80\ \forall\ x \in R - \{2\}$, then $f(x) =$

(A) $16 - 40x - \dfrac{60(2x+29)}{x-2}$

(B) $100x + 80 - \dfrac{3(2x+29)}{x-2}$

(C) $40 - 16x + \dfrac{30(2x+29)}{x-2}$

(D) None of these

113. Let $g: R \to R$ be given by $g(x) = 3 + 4x$. If $g^n(x) = gogo \ldots og(x)$, then $g^{-n}(x)$ (where $g^{-n}(x)$ denotes inverse of $g^n(x)$) is equal to

(A) $(4^n - 1) + 4^n x$ (B) $(x + 1)4^{-n} - 1$

(C) $(x + 1)4^n - 1$ (D) $(4^{-n} - 1)x + 4^n$

114. Let $f(x + p) = 1 + [2 - 3f(x) + 3(f(x))^2 - (f(x))^3]^{1/3}$, $\forall\ x \in R$, where $p > 0$. Then, $f(x)$ is periodic with period.

(A) p (B) $2p$

(C) $4p$ (D) None of these

115. If $\sum_{k=0}^{n} f(x + ka) = 0$, where $a > 0$, then the period of $f(x)$ is

(A) a (B) $(n + 1)a$

(C) $\dfrac{a}{n+1}$ (D) None of these

116. If $y = \log_3 x$ and $S = (3, 27)$, the set onto which the set S is mapped is

(A) $(0, 3)$ (B) $(1, 4)$

(C) $(1, 3)$ (D) $(0, 2)$

117. The values of x for which the functions $f(x) = x - 3$ and $\phi(x) = 4 - x$ satisfy the inequality $|f(x) + \phi(x)| < |f(x)| + |\phi(x)|$ are

(A) $[3, 4]$ (B) $(-\infty, \infty)$

(C) $(-\infty, \infty) - [3, 4]$ (D) None of these

More Than One Option Correct Type

118. If f is an even function defined on the interval $[-5, 5]$, then the real values of x satisfying the equation

$$f(x) = f\left(\frac{x+1}{x+2}\right) \text{ are}$$

(A) $\dfrac{-1 \pm \sqrt{5}}{2}$ (B) $\dfrac{-3 \pm \sqrt{5}}{2}$

(C) $\dfrac{-2 \pm \sqrt{5}}{2}$ (D) None of these

119. The distinct linear function which maps $[-1, 1]$ onto $[0, 2]$ is

(A) $x - 1$ (B) $x + 1$

(C) $-x + 1$ (D) $-x - 1$

120. Let $f(x)$ be defined for all $x > 0$ and be continuous. Let $f(x)$ satisfy $f\left(\dfrac{x}{y}\right) = f(x) - f(y)$ for all x, y and $f(e) = 1$. Then

(A) $f(x)$ is bounded

(B) $f\left(\dfrac{1}{x}\right) \to 0$ as $x \to 0$

(C) $x f(x) \to 0$ as $x \to 0$

(D) $f(x) = \log x$

121. Let $f: R \to R$ be a function defined by $f(x) = \dfrac{|x|^3 + |x|}{1 + x^2}$, then the graph of $f(x)$ lies in which quadrant

(A) I (B) II

(C) III (D) IV

122. If $f(x_1) - f(x_2) = f\left(\dfrac{x_1 - x_2}{1 - x_1 x_2}\right)$ for $x_1, x_2 \in [-1, 1]$, then $f(x)$ is

(A) $\log\left(\dfrac{1-x}{1+x}\right)$ (B) $\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$

(C) $\log\left(\dfrac{1+x}{1-x}\right)$ (D) $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

123. Which of the following functions have period 2?

(A) $\{x\} + \cos \pi x$ (B) $\tan\left(\dfrac{\pi}{2}[x]\right)$

(C) $\sin x + \{x\}$ (D) $\sin(\cos x)$

124. The values of x for which the domain of definition of the function, $f(x) = \dfrac{1}{[|x-1|] + |7-x| - 6}$, where [.] denotes the greatest integer part, is not defined are

(A) (0, 1] (B) [7, 8)
(C) {2, 3, 4, 5, 6} (D) $[0, 1] \cup [7, 8]$

125. If $f(x) = \dfrac{x(\sin x + \tan x)}{\left[\dfrac{x+\pi}{\pi}\right] - \dfrac{1}{2}}$, where [ ] denotes greatest integer function, then

(A) $f(x)$ is an odd function if $x = n\pi$
(B) $f(x)$ is an even function if $x \neq n\pi$
(C) $f(x)$ is an odd function if $x \neq n\pi$
(D) $f(x)$ is an even function if $x = n\pi$

126. If $f: R \to R$ be defined by $f(x) = \dfrac{e^x - e^{-x}}{2}$, then

(A) f is one-one
(B) f is onto
(C) $f^{-1}(x) = \log(x - \sqrt{x^2 + 1})$
(D) $f^{-1}(x) = \log(x + \sqrt{x^2 + 1})$

127. Let $f(x) = \sin^{-1}(\log[x]) + \log(\sin^{-1}[x])$, where [ ] denotes the greatest integer function. Then,

(A) domain of f is [1, 2)
(B) domain of f is [1, 3)
(C) range of f is $\left\{\log\dfrac{\pi}{2}\right\}$
(D) range of f is {0}

128. If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then

(A) f is one-one (B) f is onto

(C) $f^{-1}(x) = \dfrac{1 + \sqrt{1 + 4\log_2 x}}{2}$

(D) $f^{-1}(x) = \dfrac{1 - \sqrt{1 + 4\log_2 x}}{2}$

129. Let $f(x) = \dfrac{9^x}{9^x + 3}$. Then,

(A) $f(x) + f(1-x) = 1$
(B) $f(x) + f(1-x) = -1$

(C) $f\left(\dfrac{1}{1996}\right) + f\left(\dfrac{2}{1996}\right) + f\left(\dfrac{3}{1996}\right) + \ldots + f\left(\dfrac{1995}{1996}\right) = 998$

(D) $f\left(\dfrac{1}{1996}\right) + f\left(\dfrac{2}{1996}\right) + f\left(\dfrac{3}{1996}\right) + \ldots + f\left(\dfrac{1995}{1996}\right) = 997\dfrac{1}{2}$

130. Let n be a positive integer with $f(n) = 1! + 2! + 3! + \ldots + n!$ and $P(x)$ and $Q(x)$ be polynomials in x such that $f(n+2) = P(n) f(n+1) + Q(n) f(n)$ for all $n \geq 1$, then

(A) $P(x) = x + 3$
(B) $Q(x) = -x - 2$
(C) $P(x) = -x - 2$
(D) $Q(x) = x + 3$

Passage Based Questions

Passage 1

A function $f(x)$ is called **periodic** if there exists a positive real number T independent of x such that $f(x + T) = f(x)$ for all x in the domain of x. The smallest such value of T is called the **fundamental period of f**.

If $f(x)$ is periodic having period T, then $k f(ax + b)$ is also periodic having period $T/|a|$

131. If the period of the function $f(x) = \sin\left(\sqrt{[n]}\, x\right)$, where $[n]$ denotes the greatest integer less than or equal to n, is 2π, then

(A) $1 \leq n < 2$ (B) $1 < n < 2$
(C) $1 \leq n \leq 2$ (D) None of these

132. The period of the function $f(x) = \cos(\sin x) + \cos(\cos x)$ is

(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{4}$

(C) $\dfrac{\pi}{6}$ (D) None of these

Passage 2

For composite functions, if T_1, T_2, … be the fundamental periods of the various functions involved, then the period of the composite function is the L.C.M. of (T_1, T_2, …). But in the case of functions where modulus is involved, the L.C.M. rule gives the period of the function but it may not be the fundamental period.

For example, according to the L.C.M. rule,

Period of $|\sin x| + |\cos x|$ = L.C.M. of $(\pi, \pi) = \pi$, but it is not the fundamental period since

$$\left|\sin\left(x+\frac{\pi}{2}\right)\right| + \left|\cos\left(x+\frac{\pi}{2}\right)\right| = |\cos x| + |\sin x|$$

which shows that the fundamental period is $\frac{\pi}{2}$.

Thus, the period of $|\sin px| + |\cos qx|$

$$= \text{L.C.M. of } \left(\frac{\pi}{p}, \frac{\pi}{q}\right) \text{ if } p \neq q$$

$$= \frac{1}{2} \text{ L.C.M. of } \left(\frac{\pi}{p}, \frac{\pi}{q}\right) \text{ if } p = q$$

133. The function $f(x) = k|\cos x| + k^2|\sin x| + \phi(k)$ has period $\frac{\pi}{2}$ if k is equal to

(A) 1 (B) 2
(C) 3 (D) None of these

134. The period of the function $f(x) = 3x + 3 - [3x + 3] + \sin\frac{\pi x}{2}$,
where $[x]$ denotes the greatest integer $\leq x$, is

(A) 4 (B) 1
(C) 2 (D) None of these

135. π is the period of the function

(A) $|\sin x| + |\cos x|$
(B) $\sin^4 x + \cos^4 x$
(C) $\sin(\sin x) + \sin(\cos x)$
(D) $\dfrac{1+2\cos x}{\sin x(2+\sec x)}$

136. The period of the function $f(x) = \sin 5x + \cos\sqrt{3}\,x$ is

(A) $\sqrt{3}\,\pi$ (B) π
(C) non-periodic (D) None of these

Passage 3

The range of a function $y = f(x)$ is the set of all possible output values $f(x)$ corresponding to every input x in the domain of f and is denoted as $f(A)$ if A is the domain. For finding the range of a function $y = f(x)$, first of all, find the domain of f.

If the domain consists of finite number of points, then the range consists of set of corresponding $f(x)$ values.

If the domain consists of whole real line or real line minus some finite points, then express x in terms of y as $x = g(y)$. The values of y for which g is defined is the required range.

If the domain is a finite interval, find the intervals in which $f(x)$ increases/decreases and then find the extreme values of the function in those intervals. The union of those intervals is the required range.

137. If $e^x + e^{f(x)} = e$, then range of the function f is

(A) $(-\infty, 1]$ (B) $(-\infty, 1)$
(C) $(1, \infty)$ (D) $[1, \infty)$

138. The range of the function $f(x) = \dfrac{\sin(\pi[x^2+1])}{x^4+1}$,
where $[\cdot]$ denotes the greatest integer function, is

(A) $[0, 1]$ (B) $[-1, 1]$
(C) $\{0\}$ (D) None of these

139. Range of values of $f(x) = 1 + \sin x + \sin^3 x + \sin^5 x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is

(A) $(0, 1)$
(B) $(-\infty, \infty)$
(C) $(-2, 2)$
(D) None of these

Passage 4

The domain of a function $y = f(x)$ is the set of input values x for which the operation f is being defined (real).

For finding the domain of a function, the denominator should not be equal to 0. Also, expression under the even root should be greater than or equal to 0.

If domain of the functions, $y = f(x)$ and $y = g(x)$ are D_1 and D_2, respectively, then the domain of $f(x) \pm g(x)$ or $f(x)\cdot g(x)$ is $D_1 \cap D_2$, whereas the domain of is $\frac{f(x)}{g(x)}$ $D_1 \cap D_2 - \{g(x) = 0\}$.

$\log_a x$ is defined if x, $a > 0$ and $a \neq 1$.

$\text{Sin}^{-1}x$ and $\cos^{-1}x$ are defined for $-1 \leq x \leq 1$

140. If $[x]$ denotes the integral part of x, then the domain of the function $f(x) = \sin^{-1}[2x^2 - 3] + \log_2[\log_{1/2}(x^2 - 5x + 5)]$ is

(A) $\left(-\sqrt{\frac{5}{2}}, -1\right]$ (B) $\left[1, \sqrt{\frac{5}{2}}\right)$

(C) $\left(-\sqrt{\frac{5}{2}}, -1\right] \cup \left[1, \sqrt{\frac{5}{2}}\right)$ (D) None of these

Passage 5

Consider a one-one onto function $y = f(x)$ having domain A and range B, i.e. $f : A \to B$. Then, there exists a function g such that $x = g(y)$ which therefore has domain B and range A. i.e. $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x$, $\forall\, x \in A$ and $y \in B$. g is said to be the inverse of f.

141. If $f(x)$ satisfies $x + |f(x)| = 2f(x)$, then $f^{-1}(x)$ satisfies

(A) $3x + |f^{-1}(x)| = 2f^{-1}(x)$

(B) $x + f^{-1}(x) = 2f^{-1}(x)$

(C) $f^{-1}(x) - |x| = 2x$

(D) $3x - |f^{-1}(x)| = 2f^{-1}(x)$

Match the Column Type

142.

Column-I	Column-II
I. Domain of the function $f(x) = \dfrac{1}{\sqrt{\lvert\sin x\rvert + \sin x}}$	(A) $R - \{\sqrt{n},\ n \ge 0, n \in I\}$
II. Domain of the function $f(x) = \log_3\left[-\log_{1/2}\left(1 + \dfrac{1}{x^{1/5}}\right) - 1\right]$	(B) $(2n\pi, (2n+1)\pi)$
III. Domain of the function $f(x) = \log_3\left[-(\log_3 x)^2 + 5\log_3 x - 6\right]$	(C) $(0, 1)$
IV. Domain of the function $f(x) = \cot^{-1}\left(\dfrac{x}{\sqrt{x^2 - [x^2]}}\right)$, $x \in R$	(D) $(9, 27)$

143.

Column-I	Column-II
I. Range of the function $f(x) = \sqrt{3x^2 - 4x + 5}$	(A) $\left[\log_e \dfrac{11}{3}, \infty\right)$
II. Range of the function $f(x) = \log_e(3x^2 - 4x + 5)$	(B) $(-\infty, \infty)$
III. The value of the function $f(x) = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$ lies in the interval	(C) $\left[\sqrt{\dfrac{11}{3}}, \infty\right)$
IV. The range of the function $f(x) = \sin\left[\log\left(\dfrac{\sqrt{4 - x^2}}{1 - x}\right)\right]$	(D) $[-1, 1]$

144. The range of the function

Column-I	Column-II
I. $y = \log_{\sqrt{5}}\{\sqrt{2}(\sin x - \cos x) + 3\}$	(A) $[0, 1)$
II. $y = \log_2\{2 - \log_{\sqrt{5}}(16\sin^2 x + 1)\}$	(B) $(-\infty, 1]$
III. $y = \dfrac{e^x - e^{-\lvert x\rvert}}{e^x + e^{\lvert x\rvert}}$	(C) $[0, 2]$
IV. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, $x \ge 0$	(D) $\left[0, \dfrac{1}{2}\right)$

Assertion-Reason Type

Instructions: In the following questions an **Assertion (A)** is given followed by a **Reason (R)**. Mark your responses from the following options.

(A) Assertion(A) is True and Reason(R) is True; Reason(R) is a correct explanation for Assertion(A)

(B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)

(C) Assertion(A) is True, Reason(R) is False

(D) Assertion(A) is False, Reason(R) is True

145. **Assertion:** If $3f(x) - f(1/x) = \ln x^4$, $x > 0$, then the area of the region bounded by $f(e^x)$, x-axis, the lines $x = 1$ and $x = 1$ is 0.

Reason: $f(x) = \ln x$, $x > 0$.

146. **Assertion:** If $f(x) = \dfrac{a^x}{a^x + \sqrt{a}}\ (a > 0)$, then $\sum_{r=1}^{2n-1} 2f\left(\dfrac{r}{2n}\right) = 2n - 1$

Reason: $f(x) + f(1 - x) = 1\ \forall\, x$

147. Assertion: The range of the function $f(x) = g(x) + h(x)$ where $g(x) = \sqrt{-x^2+4x-3}$ and $h(x) = \sqrt{\sin\left(\frac{\pi}{2}\left(\sin\frac{\pi}{2}(x-1)\right)\right)}$ is [0, 2]

Reason: Maximum and minimum values of both g and h are attained at 2 and 1, respectively.

148. Assertion: If $[x]$ denotes the integral part of x, then domain of the function $f(x) = g(x) + h(x)$, where $g(x) = \frac{\sqrt{3-x}}{(x-1)(x-2)(x-3)}$ and $h(x) = \sin^{-1}\left[\frac{3x-2}{2}\right]$ is $[0, 2) - \{1\}$

Reason: Domain of $h(x)$ is [0, 2)

149. Assertion: Suppose, $f(x) = (x+1)^2$ for $x \ge -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x) = \sqrt{x} - 1, x \ge 0$.

Reason: $g(x)$ is the inverse of $f(x)$

Previous Year's Questions

150. The period of $\sin^2\theta$ is : **[2002]**

(A) π^2 (B) π
(C) 2π (D) $\pi/2$

151. The domain of $\sin^{-1}[\log_3(x/3)]$ is : **[2002]**

(A) [1, 9] (B) [–1, 9]
(C) [–9, 1] (D) [– 9, –1]

152. The period of the function $f(x) = \sin^4 x + \cos^4 x$ is : **[2002]**

(A) π (B) $\frac{\pi}{2}$
(C) 2π (D) None of these

153. A function f from the set of natural numbers to integers defined by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$$ is **[2003]**

(A) one-one but not onto
(B) onto but not one-one
(C) one-one and onto both
(D) neither one-one nor onto

154. Domain of definition of the function

$f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$, is **[2003]**

(A) (1, 2)
(B) $(-1, 0) \cup (1, 2)$
(C) $(1, 2) \cup (2, \infty)$
(D) $(-1, 0) \cup (1, 2) \cup (2, \infty)$

155. The range of the function $f(x) = {}^{7-x}P_{x-3}$ is **[2004]**

(A) {1, 2, 3} (B) {1, 2, 3, 4, 5}
(C) {1, 2, 3, 4} (D) {1, 2, 3, 4, 5, 6}

156. If $f: R \to S$, defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$, is onto, then the interval of S is **[2004]**

(A) [0, 3] (B) [–1, 1]
(C) [0, 1] (D) [–1, 3]

157. The graph of the function $y = f(x)$ is symmetrical about the line $x = 2$, then **[2004]**

(A) $f(x+2) = f(x-2)$
(B) $f(2+x) = f(2-x)$
(C) $f(x) = f(-x)$
(D) $f(x) = -f(-x)$

158. The domain of the function $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is **[2004]**

(A) [2, 3] (B) [2, 3)
(C) [1, 2] (D) [1, 2)

159. Let $f: (-1, 1) \to B$, be a function defined by $f(x) = \tan^{-1}\frac{2x}{1-x^2}$, then f is both one-one and onto when B is the interval **[2005]**

(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left[0, \frac{\pi}{2}\right)$
(C) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (D) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

160. A real valued function $f(x)$ satisfies the functional equation $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$ where a is a given constant and $f(0) = 1$, $f(2a-x)$ is equal to **[2005]**

(A) $-f(x)$ (B) $f(x)$
(C) $f(A) + f(a-x)$ (D) $f(x)$

161. The largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which the function $f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$ is defined, is **[2007]**

(A) $[0, \pi]$ (B) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(C) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (D) $\left[0, \frac{\pi}{2}\right)$

162. Let $f: N \to Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in N: y = 4x + 3$ for some $x \in N\}$. Show that f is invertible and its inverse is **[2008]**

(A) $g(y) = \frac{3y+4}{3}$ (B) $g(y) = 4 + \frac{y+3}{4}$

(C) $g(y) = \frac{y+3}{4}$ (D) $g(y) = \frac{y-3}{4}$

163. For real x, let $f(x) = x^3 + 5x + 1$, then **[2009]**

(A) f is one-one but not onto R
(B) f is onto R but not one-one
(C) f is one-one and onto R
(D) f is neither one-one nor onto R

164. The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is **[2011]**

(A) $(0, \infty)$ (B) $(-\infty, 0)$
(C) $(-\infty, \infty) - \{0\}$ (D) $(-\infty, \infty)$

165. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x, x \neq 0$, and $S = \{x \in R : f(x) = f(-x)\}$; then S: **[2016]**

(A) contains more than two elements.
(B) is an empty set.
(C) contains exactly one element
(D) contains exactly two elements

ANSWER KEYS

Single Option Correct Type

1. (B)	**2.** (A)	**3.** (B)	**4.** (B)	**5.** (B)	**6.** (D)	**7.** (C)	**8.** (B)	**9.** (A, D)	**10.** (B)
11. (A)	**12.** (C)	**13.** (A)	**14.** (B)	**15.** (A)	**16.** (D)	**17.** (A)	**18.** (C)	**19.** (B)	**20.** (A)
21. (A)	**22.** (D)	**23.** (A)	**24.** (C)	**25.** (A)	**26.** (D)	**27.** (B)	**28.** (B)	**29.** (D)	**30.** (A)
31. (C)	**32.** (C)	**33.** (C)	**34.** (B)	**35.** (C)	**36.** (D)	**37.** (A)	**38.** (B)	**39.** (A)	**40.** (D)
41. (D)	**42.** (D)	**43.** (A)	**44.** (A)	**45.** (A)	**46.** (B)	**47.** (C)	**48.** (C)	**49.** (A, B)	**50.** (B)
51. (B)	**52.** (D)	**53.** (A)	**54.** (A)	**55.** (B)	**56.** (D)	**57.** (A)	**58.** (A)	**59.** (A)	**60.** (B)
61. (B)	**62.** (A, B, C)		**63.** (B)	**64.** (C)	**65.** (D)	**66.** (B)	**67.** (D)	**68.** (D)	**69.** (D)
70. (C)	**71.** (C)	**72.** (D)	**73.** (C)	**74.** (B)	**75.** (D)	**76.** (C)	**77.** (C)	**78.** (C)	**79.** (B)
80. (B)	**81.** (A)	**82.** (B)	**83.** (B)	**84.** (A)	**85.** (B)	**86.** (A)	**87.** (A)	**88.** (B)	**89.** (B)
90. (B)	**91.** (B)	**92.** (A)	**93.** (C)	**94.** (C)	**95.** (B)	**96.** (C)	**97.** (D)	**98.** (C)	**99.** (C)
100. (A)	**101.** (D)	**102.** (B)	**103.** (B)	**104.** (C)	**105.** (C)	**106.** (A)	**107.** (C)	**108.** (A)	**109.** (A)
110. (C)	**111.** (A)	**112.** (A)	**113.** (B)	**114.** (B)	**115.** (B)	**116.** (C)	**117.** (C)		

More than One Option Correct Type

118. (A) and (B)	**119.** (B) and (C)	**120.** (C) and (D)	**121.** (C) and (D)	**122.** (A), (B) and (C)
123. (A) and (B)	**124.** (A), (B) and (C)	**125.** (C) and (D)	**126.** (A), (B) and (D)	**127.** (A) and (C)
128. (A), (B) and (C)	**129.** (A) and (D)	**130.** (A) and (B)		

Passage Based Questions

131. (A)	**132.** (A)	**133.** (A)	**134.** (A)	**135.** (D)	**136.** (C)	**137.** (B)	**138.** (C)	**139.** (B)	**140.** (D)
141. (D)									

Match the Column Type

142. I. → (B); II. → (C); III. → (D); IV. → (A)
143. I. → (C); II. → (A); III. → (B); IV. → (D)
144. I. → (C); II. → (B); III. → (D); IV. → (A)

Assertion-Reason Type

145. (A) **146.** (A) **147.** (A) **148.** (A) **149.** (A)

Previous Year's Questions

150. (B) **151.** (A) **152.** (B) **153.** (C) **154.** (D) **155.** (A) **156.** (D) **157.** (B) **158.** (B) **159.** (D)
160. (A) **161.** (D) **162.** (D) **163.** (C) **164.** (B) **165.** (D)

HINTS AND SOLUTIONS

Single Option Correct Type

1. We have, $f(x) = x^3 + x^2 + 100x + 7\sin x$

$\Rightarrow f'(x) = 3x^2 + 2x + 100 + 7\cos x > 0, \forall x \in R$

$\therefore$ $f(x)$ is an increasing function

$\Rightarrow f(1) < f(2) < f(3)$

Let $a = f(1)$, $b = f(2)$ and $c = f(3)$, then $a < b < c$ (1)

Then given equation is

$$\frac{1}{y-a} + \frac{2}{y-b} + \frac{3}{y-c} = 0$$

$\Rightarrow (y-b)(y-c) + 2(y-a)(y-c) + 3(y-a)(y-b) = 0$ (2)

Let $\phi(y) = (y-b)(y-c) + 2(y-a)(y-c) + 3(y-a)(y-b)$,

Then, $\phi(a) = (a-b)(a-c) > 0$

and $\phi(b) = 2(b) = 2(b-a)(b-c) < 0$

and $\phi(c) = 3(c-a)(c-b) > 0$

So the equation (2), i.e. $\phi(y) = 0$ has one real root between a and b and other between b and c.

The correct option is (B)

2. $f(x) = \log(ax^3 + (2a+b)x^2 + (2b+c)x + 2c)$

$= \log((ax^2 + bx + c)(x+2))$

$= \log(ax^2 + bx + c) + \log(x+2)$

Since $a > 0$ and $D = 0$,

$\therefore$ $ax^2 + bx + c \geq 0 \; \forall x \in R$

$\therefore$ $\log(ax^2 + bx + c)$ is defined if $ax^2 + bx + c \neq 0$

$$\Rightarrow a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \neq 0$$

$$\Rightarrow a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right] \neq 0$$

$$\Rightarrow a\left(x + \frac{b}{2a}\right)^2 \neq 0 \qquad [\because b^2 - 4ac = 0]$$

$$\Rightarrow x \neq -\frac{b}{2a}$$

$\therefore$ $f(x)$ is defined for $x \neq -\dfrac{b}{2a}$ and $x + 2 > 0$

$\therefore$ Domain of $f = (-2, \infty)\setminus\left\{-\dfrac{b}{2a}\right\}$.

The correct option is (A)

3. We have,

$e^x + e^{f(x)} = e \Rightarrow e^{f(x)} = e - e^x \Rightarrow f(x) = \log(e - e^x)$.

For $f(x)$ to be defined, $e - e^x > 0 \Rightarrow e^1 > e^x \Rightarrow x < 1$

$\therefore$ Domain of $f = (-\infty, 1)$

Let $y = \log(e - e^x) \Rightarrow e^y = e - e^x$

$\Rightarrow e^x = e - e^y$

$\Rightarrow x = \log(e - e^y)$

For x to be real, $e - e^y > 0 \Rightarrow e^1 > e^y \Rightarrow y < 1$

$\therefore$ Range of $f = (-\infty, 1)$.

The correct option is (B)

4. The function $f(x) = x^2 + 2$, $x \in (-\infty, \infty)$ is not injective as $f(1) = f(-1)$ but $1 \neq -1$.

The function $f(x) = (x-4)(x-5)$, $x \in (-\infty, \infty)$ is not one-one as $f(4) = f(5)$ but $4 \neq 5$.

The function, $f(x) = \dfrac{4x^2 + 3x - 5}{4 + 3x - 5x^2}$, $x \in (-\infty, \infty)$ is also not injective as $f(1) = f(-1)$ but $1 \neq -1$.

For the function, $f(x) = |x+2|$, $x \in [-2, \infty)$.

Let $f(x) = f(y)$, $x, y \in [-2, \infty) \Rightarrow |x+2| = |y+2|$

$\Rightarrow \quad x + 2 = y + 2$

$\Rightarrow \quad x = y.$

So, f is an injection.

The correct option is (B)

5. $y = \frac{1}{2}\{\cos(2x+2) + \cos 2 - [1 + \cos(2x+2)]\}$

or $\quad y = -\frac{1}{2}(1 - \cos 2) = -\sin^2 1$ i.e., constant

$\therefore$ graph is a line parallel to x-axis. Also, when $x = \frac{\pi}{2}$,

$y = -\cos^2\left(\frac{\pi}{2} + 1\right) = -\sin^2 1$ and hence it passes through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$.

The correct option is (B)

6. Since $f(x) = f(-x)$

$\therefore$ f is not one–one.

Let $y \in R$. Then $f(x) = y \Rightarrow y = \frac{x^2 - 8}{x^2 + 2}$

$\Rightarrow \quad x^2 = \frac{8 + 2y}{1 - y}$

For x to be real, $(8 + 2y)(1 - y) \geq 0$

and $1 - y \neq 0 \Rightarrow (y + 4)(y - 1) \leq 0$ and $y \neq 1$

$\Rightarrow \quad -4 \leq y < 1$

$\therefore$ Range of $f = [-4, 1) \subset R$

$\therefore$ f is not onto.

The correct option is (D)

7. We have,

$f(a) = 64\, a^3 + \frac{1}{a^3} = (4a)^3 + \frac{1}{a^3}$

$= \left(4a + \frac{1}{a}\right)^3 - 3 \cdot 4a \cdot \frac{1}{a}\left(4a + \frac{1}{a}\right)$

$= (3)^3 - 12 \cdot 3 = 27 - 36 = -9.$

$\left[\begin{array}{l} \text{Since } a, b \text{ are roots of } 4x + \frac{1}{x} = 3 \\ \therefore 4a + \frac{1}{a} = 3 \end{array}\right]$

Similarly, $\quad f(b) = -9$

$\therefore \quad f(a) = f(b) = -9$

The correct option is (C)

8. We have,

$(fog)(x) = f[g(x)] = f\left(\sqrt{x^2+1}\right)$

$= \left(\sqrt{x^2+1}\right)^2 - 1 = x^2$

$\therefore \quad (hofog)(x) = h[(fog)(x)] = h(x^2) = \begin{Bmatrix} 0 & \text{if } x = 0 \\ x^2 & \text{if } x \neq 0 \end{Bmatrix}.$

The correct option is (B)

9. Let $y = \frac{2x-1}{2x^3 + 3x^2 + x} = \frac{2x-1}{x(2x+1)(x+1)}$

$= \frac{(2x-1)^2}{(2x+1)(x+1)x(2x-1)}$

For y to be positive,

$(x + 1)(2x + 1)\, x\, (2x - 1) > 0$

$\Rightarrow \quad x \in (-\infty, -1) \cup \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right).$

Thus, (a) and (d) are the correct answers.

The correct option is (A) and (D)

10. The maximum value of $f(x) = \cos x + \cos(\sqrt{2}\, x)$ is 2 which occurs at $x = 0$. Also, there is no value of x for which this value will be attained again.

The correct option is (B)

11. Let the required function be $f(x) = ax + b$.

If $a > 0$, then $f(-1) = 0$ and $f(1) = 2$.

$\Rightarrow \quad -a + b = 0$ and $a + b = 2$

$\Rightarrow \quad a = 1$ and $b = 1$.

If $a < 0$, then $f(-1) = 2$ and $f(1) = 0$

$\Rightarrow \quad -a + b = 2$ and $a + b = 0$

$\Rightarrow \quad a = -1$ and $b = 1$.

Hence, $f(x) = x + 1$ or $f(x) = -x + 1$.

The correct option is (A)

12. For $x \leq -1$, $1 - x \geq 2$ and $1 - x \geq 1 + x$

$\therefore$ max $[(1 - x), 2, (1 + x)] = 1 - x$

For $-1 < x < 1$, $0 < 1 - x < 2$ and $0 < 1 + x < 2$.

$\therefore$ max $[(1 - x), 2, (1 + x)] = 2$.

For $x \geq 1$, $1 + x \geq 2$, $1 + x > 1 - x$

$\therefore$ max $[(1 - x), 2, (1 + x)] = 1 + x$

Hence, $f(x) = \begin{cases} 1 - x, & x \leq -1 \\ 2, & -1 < x < 1 \\ 1 + x, & x \geq 1 \end{cases}.$

The correct option is (C)

13. We have,

$f(x) = \sin[\pi^2]\, x + \sin[-\pi^2]\, x$

$= \sin 9x + \sin(-10)\, x$

$= \sin 9x - \sin 10x$

$\therefore \quad f\left(\frac{\pi}{2}\right) = \sin\frac{9\pi}{2} - \sin 5\pi = 1 - 0 = 1$

$f(\pi) = \sin 9\pi - \sin 10\pi = 0$

$f\left(\frac{\pi}{4}\right) = \sin\frac{9\pi}{4} - \sin\frac{10\pi}{4} = \frac{1}{\sqrt{2}} - 1.$

The correct option is (A)

14. Since the given function has minimum value 75 which is attained at $x = 2$ and maximum value 89 which is attained at $x = 3$. Hence, the range of f is [75, 89].

The correct option is (B)

15. We have, $2f(x) - 3f\left(\frac{1}{x}\right) = x^2, x \neq 0.$

Putting $x = 2$, we get

$$2f(2) - 3f\left(\frac{1}{2}\right) = 4 \qquad (1)$$

Now, putting $x = \frac{1}{2}$, we get

$$2f\left(\frac{1}{2}\right) - 3f(2) = \frac{1}{4} \quad (2)$$

Solving (1) and (2), we get $f(2) = -\frac{7}{4}$.

The correct option is (A)

16. $f(x) = (1 + b^2)$

$$\left\{x^2 + \frac{2b}{1+b^2}x + \frac{b^2}{(1+b^2)^2}\right\} - \frac{b^2}{1+b^2} + 1$$

$$= (1 + b^2)\left(x + \frac{b}{1+b^2}\right)^2 + \frac{1}{1+b^2} \geq \frac{1}{1+b^2}$$

$\therefore \quad m(b) = \frac{1}{1+b^2}$. So, range of $m(b) = (0, 1]$.

The correct option is (D)

17. Let $y = (fog)(x) = f[g(x)] = f(x^3 + 5)$

$$= 2(x^3 + 5) - 3 = 2x^3 + 7.$$

$$\therefore \quad x^3 = \frac{y-7}{2} \Rightarrow x = \left(\frac{y-7}{2}\right)^{1/3}$$

$$\Rightarrow \quad (fog)^{-1}(y) = \left(\frac{y-7}{2}\right)^{1/3}$$

$$\Rightarrow \quad (fog)^{-1}(x) = \left(\frac{x-7}{2}\right)^{1/3}$$

The correct option is (A)

18. Since $f(x) = \log(x - 1) - \log(x - 2)$.

Domain of $f(x)$ is $x > 2$ or $x \in (2, \infty)$ (1)

$g(x) = \log\left(\frac{x-1}{x-2}\right)$ is defined if $\frac{x-1}{x-2} > 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (2, \infty)$ (2)

From Eq. (1) and (2), $x \in (2, \infty)$.

The correct option is (C)

19. We have,

$$g(x) = \begin{cases} 1 + n - n = 1, \; x = n \in Z \\ 1 + n + k - n = 1 + k, \;\; x = n + k \end{cases}$$

where $n \in Z$, $0 < k < 1$.

$$\text{Now,} \quad f[g(x)] = \begin{cases} -1, & g(x) < 0 \\ 0, & g(x) = 0 \\ 1, & g(x) > 0 \end{cases}$$

Clearly, $g(x) > 0$ for all x. So, $f[g(x)] = 1$, for all x.

The correct option is (B)

20. Clearly, y is defined for all $x \in R$ except when $\sin x = 0$ i.e., $x = n\pi : n \in Z$

$\therefore \quad$ Domain of $f = R \setminus \{n\pi : n \in Z\}$

The correct option is (A)

21. Let $y = \frac{x^2 + 34x - 71}{x^2 + 2x - 7}$

$\Rightarrow \quad (y - 1)x^2 + (2y - 34)x - 7y + 71 = 0$

For x to be real,

$(2y - 34)^2 \geq 4(y - 1)(71 - 7y)$ $\quad$ ($\because$ Discriminent ≥ 0)

$\Rightarrow \quad y^2 + 289 - 34y \geq -7y^2 - 71 + 78y$

$\Rightarrow \quad 8y^2 - 112y + 360 \geq 0$

$\Rightarrow \quad y^2 - 14y + 45 \geq 0$

$\Rightarrow \quad (y - 9)(y - 5) \geq 0$

$\Rightarrow \quad y \leq 5$ or $y \geq 9$

$\therefore \quad y$ cannot lie between 5 and 9.

The correct option is (A)

22. Since $f(x)$ is an odd periodic function with period 2

$\therefore \quad f(-x) = -f(x)$ and $f(x + 2) = f(x)$

$\therefore \quad f(2) = f(0 + 2) = f(0)$

and $\quad f(-2) = f(-2 + 2) = f(0)$

Now, $\quad f(0) = f(-2) = -f(2) = -f(0)$

$\Rightarrow \quad 2f(0) = 0$ *i.e.* $f(0) = 0$

$\therefore \quad f(4) = f(2 + 2) = f(2) = f(0) = 0$

Thus, $f(4) = 0$

The correct option is (D)

23. For the two components to be meaningful, we must have $x(x + 3) \geq 0$ and $0 \leq x^2 + 3x + 1 \leq 1$.

Hence, $(x + 3)x = 0$ i.e., $x = 0, -3$.

$\therefore \quad S = \{-3, 0\}$

The correct option is (A)

24. Since $f(x) = g(x)$, $\forall\, x \in (4n + 1)\frac{\pi}{2}$, $n \in Z$

$\therefore \quad f(x) = g(x)$ for infinitely many values of x.

The correct option is (C)

25. Since, $f(x + y) = f(x)f(y)$

$\Rightarrow \quad f(x) = a^x$

Also, given that $f(1) = 3$

$\Rightarrow \quad a^1 = 3,$

$\therefore \quad a = 3$

Hence, $f(x) = 3^x$

$$\text{Now, } S = \sum_{r=1}^{n} f(r) \Rightarrow S = 3^1 + 3^2 + \ldots + 3^n$$

$$= 3\left(\frac{1-3^n}{1-3}\right)$$

$$\therefore \quad S = \frac{3}{2}(3^n - 1).$$

The correct option is (A)

26. Taking $f(x) = \log x$, we see that

$$f\left(\frac{x}{y}\right) = f(x) - f(y)$$

Clearly $f(x)$ is not bounded

and $f\left(\frac{1}{x}\right) = -\log x \to \infty$ as $x \to 0$.

Also, $x f(x) = x \log x \to 0$ as $x \to 0$.

The correct option is (D)

27. We have, $g(x) = 1 + \sqrt{x}$

and $f[g(x)] = 3 + 2\sqrt{x} + x$ (1)

Also, $f[g(x)] = f(1 + \sqrt{x})$ (2)

From Eq. (1) and (2), we get

$f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$.

Let $1 + \sqrt{x} = y$ or $x = (y - 1)^2$.

$\therefore$ $f(y) = 3 + 2(y - 1) + (y - 1)^2$

$= 3 + 2y - 2 + y^2 - 2y + 1 = 2 + y^2$

$\therefore$ $f(x) = 2 + x^2$

The correct option is (B)

28. We have, $f(x) = 1 + \dfrac{\sin x}{\cos^2 x}$

$\Rightarrow$ $f'(x) = \dfrac{\cos^2 x(\cos x) + \sin x(2\cos x \sin x)}{\cos^4 x}$

$= \dfrac{\cos x(\cos^2 x + 2\sin^2 x)}{\cos^4 x} = \dfrac{1 + \sin^2 x}{\cos^3 x}$

$\Rightarrow$ $f'(x) > 0$

$\therefore$ $f(x)$ is increasing function.

$\lim\limits_{x \to \frac{-\pi}{2}} \left(1 + \dfrac{\sin x}{\cos^2 x}\right) = -\infty$

and $\lim\limits_{x \to \frac{\pi}{2}} \left(1 + \dfrac{\sin x}{\cos^2 x}\right) = \infty$

$\therefore$ Range $= (-\infty, \infty)$

The correct option is (B)

29. We have, $f(x) = e^{x^3 - 3x + 2}$

Let $g(x) = x^3 - 3x + 2$

$\Rightarrow$ $g'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$\Rightarrow$ $g'(x) \geq 0$, for $x \in (-\infty, -1]$

$\therefore$ $g(x)$ is increasing function

$\therefore$ $f(x)$ is one-one.

Now, Range of $f(x)$ is $(0, e^4]$, but co-domain is $(0, e^5]$.

$\therefore$ $f(x)$ is into function.

The correct option is (D)

30. For f to be defined, we must have

$\log_{1/2}\left(1 + \dfrac{1}{\sqrt[4]{x}}\right) < -1 \Rightarrow 1 + \dfrac{1}{\sqrt[4]{x}} > (2^{-1})^{-1} = 2$ which is possible only if $\dfrac{1}{\sqrt[4]{x}} > 1$ i.e. $0 < x < 1$.

Hence, the domain of the given function is $\{x : 0 < x < 1\}$.

The correct option is (A)

31. Since $[x^2 + 1]$ is an integer

$\therefore$ $\sin(\pi[x^2 + 1]) = 0$

$\Rightarrow$ $f(x) = \dfrac{\sin(\pi[x^2 + 1])}{x^4 + 1} = 0$

Hence, Range of $f = R_f = \{0\}$

The correct option is (C)

32. $af(x) + bf\left(\dfrac{1}{x}\right) = x + \dfrac{5}{x}$ (1)

$af\left(\dfrac{1}{x}\right) + bf(x) = \dfrac{1}{x} + 5x$ (2)

Multiply Eq. (1) by a and equation (2) by b, then subtract

$(a^2 - b^2) f(x) = ax + \dfrac{5a}{x} - \dfrac{b}{x} - 5bx$

$\therefore$ $f(x) = \dfrac{1}{(a^2 - b^2)}\left(x(a - 5b) + \dfrac{1}{x}(5a - b)\right)$

The correct option is (C)

33. $f(x) = x^3 + x^2 + 100x + 5\sin x$

$\therefore$ $f'(x) = 3x^2 + 2x + 100 + 5\cos x = 3x^2 + 2x + 94 + (6 + 5\cos x) > 0$

$\therefore$ f is an increasig function and consequently a one-one function.

Clearly, $f(-\infty) = -\infty$, $f(\infty) = \infty$ and $f(x)$ is continuous, therefore range $f = R =$ codomain f. Hence, f is onto.

The correct option is (C)

34. We have, $f(x + 2a) = f((x + a) + a)$

$= \dfrac{1}{2} - \sqrt{f(x + a) - (f(x + a))^2}$

$= \dfrac{1}{2} - \sqrt{\dfrac{1}{2} - \sqrt{f(x) - (f(x))^2} - \left(\dfrac{1}{2} - \sqrt{f(x) - (f(x))^2}\right)^2}$

$= \dfrac{1}{2} - \sqrt{\dfrac{1}{4} - f(x) + (f(x))^2}$

$= \dfrac{1}{2} - \left(\dfrac{1}{2} - f(x)\right) = f(x)$

Hence, $f(x)$ is periodic with period $2a$.

The correct option is (B)

35. $g[f(x)] = (\sin x + \cos x)^2 - 1$

$\Rightarrow$ $g[f(x)] = \sin 2x$

We know that $\sin x$ is bijective only when $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

Thus, $f(x)$ is bijective if $-\dfrac{\pi}{2} \leq 2x \leq \dfrac{\pi}{2}$

$\Rightarrow$ $-\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{4}$.

The correct option is (C)

36. Let $2x + \dfrac{y}{8} = \alpha$ and $2x - \dfrac{y}{8} = \beta$,

then $x = \dfrac{\alpha + \beta}{4}$ and $y = 4(\alpha - \beta)$

Given, $f\left(2x + \dfrac{y}{8}, 2x - \dfrac{y}{8}\right) = xy$

$\Rightarrow$ $f(\alpha, \beta) = \alpha^2 - \beta^2$

$\Rightarrow$ $f(m, n) + f(n, m) = m^2 - n^2 + n^2 - m^2 = 0$ for all m, n.

The correct option is (D)

37. Since the domain of f is (0, 1),

$\therefore$ $0 < e^x < 1$ and $0 < \ln |x| < 1$

$\Rightarrow$ $\log 0 < x < \log 1$ and $e^0 < |x| < e^1$

$\Rightarrow$ $-\infty < x < 0$ and $1 < |x| < e$

$\Rightarrow$ $x \in (-\infty, 0)$ and $x \in ((-\infty, -1) \cup (1, \infty)) \cap (-e, e)$

$\Rightarrow$ $x \in (-\infty, 0)$ and $x \in (-e, -1) \cup (1, e) \Rightarrow x \in (-e, -1)$

The correct option is (A)

38. We have $\frac{1}{2} + \frac{n}{100} < 1$ if $n \leq 49$

$\therefore$ $\left[\frac{1}{2} + \frac{n}{100}\right] = 0$ if $n \leq 49$

Again $\frac{1}{2} + \frac{n}{100} \geq 1$ if $n \geq 50$

$\therefore$ $\left[\frac{1}{2} + \frac{n}{100}\right] = 1$ if $50 \leq n \leq 99$

$\therefore$ The desired sum = 50.

The correct option is (D)

39. We have, $f(x) = \sqrt{\frac{\log_{0.3}|x - x|}{|x|}}$

Now, by definition, the domain of f is defined as

$\frac{\log_{0.3}|x-2|}{|x|} \geq 0, |x| \neq 0$

Since denominator i.e. $|x|$ is always positive

$\therefore$ $\log_{0.3}|x - 2| \geq 0 \Rightarrow \log_{0.3}|x - 2| \geq \log_{0.3}$ (1)

$\Rightarrow$ $|x - 2| \leq 1$

$\Rightarrow$ $-1 \leq x - 2 \leq 1$

$\therefore$ $1 \leq x \leq 3$ (1)

and $|x - 2| > 0$, which is always true, but $x \neq 2$ (2)

Hence, domain of $f = D_f = [1, 2) \cup (2, 3]$

The correct option is (A)

40. Since, $f(-x) = f(x)$

$\Rightarrow$ $f(x)$ is an even function, its graph will be symmetrical about y-axis.

Also, $f(x) = -\left(\frac{|x|^3 + |x|}{1 + x^2}\right) \Rightarrow f(x) = -(+ive) = -ive$

i.e., the graph of $f(x)$ completely lies below the x-axis, and is also symmetric about y-axis (as discussed above).

$\therefore$ The graph of $f(x)$ lies in III and IV quadrants.

The correct option is (D)

41. Since, $\sqrt{-x^2}$ is not defined for any real x,

$\therefore$ domain of the given real fucntion is null set.

The correct option is (D)

42. $g[f(x)] \leq f[g(x)] \Rightarrow g(|x|) \leq f([x]) \Rightarrow [|x|] \leq |[x]|$

This is true for each $x \in R$.

The correct option is (D)

43. $\sin^{-1}(x - x^2)$ is defined when $-1 \leq x - x^2 \leq 1$

$\Rightarrow$ $\frac{1-\sqrt{5}}{2} \leq x \leq \frac{1+\sqrt{5}}{2}$ (1)

$\sqrt{1 - \frac{1}{|x|}}$ is defined when $1 - \frac{1}{|x|} \geq 0$

$\Rightarrow$ $x \leq -1, x \geq 1$ (2)

and $\frac{1}{[x^2 - 1]}$ is defined when $x^2 - 1 < 0, x^2 - 1 \geq 1$

i.e., $x \in (-\infty, -\sqrt{2}) \cup (-1, 1) \cup (-\sqrt{2}, \infty)$ (3)

From Eq. (1), (2) and (3), we get

$x \in \left(\sqrt{2}, \frac{1+\sqrt{5}}{2}\right)$

The correct option is (A)

44. Graph is symmetric about $x = k$

if $f(k - x) = f(k + x)$

$\Rightarrow$ $a(k - x)^3 + b(k - x)^2 + c(k - x) + d$

$= a(k + x)^3 + b(k + x) + c(k + x) + d$

$\Rightarrow$ $2ax^3 - (6ak^2 + 4bk + 2c)x = 0$

It is true for all x if $a = 0$ and $6ak^2 + 4bk + 2c = 0$

i.e. $a = 0$ and $k = -\frac{c}{2b} \Rightarrow a + k = -\frac{c}{2b}$.

The correct option is (A)

45. Domain of f and domain of composite function gof are same.

The correct option is (A)

46. Since,

$f(x) = x^3 + x^2 f'(1) + x f''(2) + f'(3)$ (1)

$\Rightarrow$ $f'(x) = 3x^2 + 2x f'(1) + f''(2)$ (2)

$\Rightarrow$ $f''(x) = 6x + 2f'(1)$ (3)

$\Rightarrow$ $f'''(x) = 6$ i.e., a constant function

Hence, $f'(3) = 6$

Using Eq. (3), we have

$f''(2) = 12 + 2f'(1)$ (5)

Substituting $x = 1$, in equation (2), we have

$f'(1) = 3 + 2f'(1) + f''(2)$

$\Rightarrow$ $f'(1) = 3 + 2f'(1) + 12 + 2f'(1)$ [using (5)]

$\Rightarrow$ $3f'(1) = -15$

$\therefore$ $f'(1) = -5$ (6)

Substituting respective value in Eq. (5), we have

$f''(2) = 2$

Hence, the polynomial $f(x)$ can be written as

$f(x) = x^3 - 5x^2 + 2x + 6$

Therefore, $f(2) - f(1)$

$= (8 - 20 + 4 + 6) - (1 - 5 + 2 + 6)$

$= -2 - 4 = -6$

$\therefore$ $f(2) - f(1) = -6 = -f(0)$.

The correct option is (B)

47. $f : \mathbb{R} \to A$, where $A = \left\{y : 0 \leq y < \frac{\pi}{2}\right\}$

Also, the function f is defined as

$f(x) = \tan^{-1}(x^2 + x + k)$

For the function f to be onto, the range must overlap completely on the codomain or range must be equal to codomain such that

$R_f = A$

$\Rightarrow$ $R_f = \left[0, \frac{\pi}{2}\right)$ i.e. 1st Quadrant which is possible if and only if

$x^2 + x + k \geq 0 \Rightarrow \Delta \leq 0$

$\Rightarrow$ $1^2 - 4 \cdot 1 \cdot (k) \leq 0$

$\therefore$ $k \geq \frac{1}{4}$

The correct option is (C)

48. By verification, $f\left(\left|-\frac{1}{2}\right|\right) = f\left(\frac{1}{2}\right) = \frac{1}{2} - 1 = -\frac{1}{2}$

Hence, $f(|x|) = x$

The correct option is (C)

49. Given that

$f(x) = [x]^2 + [x+1] - 3$

$\Rightarrow$ $f(x) = [x]^2 + [x] + 1 - 3$

$\Rightarrow$ $f(x) = [x]^2 + [x] - 2$

$\Rightarrow$ $f(x) = ([x] + 2)([x] - 1)$

Now, $f(x) = 0$

$\Rightarrow$ $([x] + 2)([x] - 1) = 0$

$\Rightarrow$ either $[x] + 2 = 0$ or $[x] - 1 = 0$

$\Rightarrow$ either $[x] = -2$ or $[x] = 1$

$\Rightarrow$ either $-2 \leq x < -1$ or $1 \leq x < 2$

i.e., $f(x) = 0$ for infinite number of values of x.

Also, $f(x)$ will not be one-to-one

Since, $f(x) = \underbrace{\underbrace{(|x| + 2)}_{\text{Integer}}\underbrace{(|x| - 1)}_{\text{Integer}}}_{\text{Integer}}$

Hence, $f(x)$ is into on the set of real numbers (R) as co-domain.

The correct option is (A) and (B)

50. Given, $q^2 - 4pr = 0$ and $p > 0$

For $f(x)$ to be defined,

$px^3 + (p+q)x^2 + (q+r)x + r > 0$

$\Rightarrow$ $px^2(x+1) + qx(x+1) + r(x+1) > 0$

$\Rightarrow$ $(x+1)(px^2 + qx + r) > 0$

$\Rightarrow$ $x > -1$ and $x \neq -\frac{q}{2p}$

[Since $q^2 - 4pr = 0$

$\therefore$ at $x = -\frac{q}{2p}$, $px^2 + qx + r = 0$ and at $x \neq -\frac{q}{2p}$, $px^2 + qx + r > 0$]

$\therefore$ Domain $= R - \left[(-\infty, -1] \cup \left\{-\frac{q}{2p}\right\}\right]$

The correct option is (B)

51. Given that $f(x) = x - \left[\frac{x}{3}\right]$

where $f : (3, 6) \to (1, 3)$

Now, inside the given domain, we will always have

$\left[\frac{x}{3}\right] = 1$

$\therefore$ $f(x) = x - 1$ throughout its domain

$\Rightarrow$ $y = x - 1$

$\Rightarrow$ $x = y + 1 = f^{-1}(y)$

$\therefore$ $f^{-1}(x) = x + 1$

The correct option is (B)

52. For $f(x)$ to be defined,

(i) $[2x^2 - 3] = -1, 0, 1$

$\Rightarrow$ $-1 \leq 2x^2 - 3 < 2 \Rightarrow 2 \leq 2x^2 < 5$

$\Rightarrow$ $1 \leq x^2 < \frac{5}{2}$

$\Rightarrow$ $\begin{cases} 1 \leq x^2 \Rightarrow x \leq -1 \text{ or } x \geq 1 \\ x^2 < \frac{5}{2} \Rightarrow -\sqrt{\frac{5}{2}} < x < \sqrt{\frac{5}{2}} \end{cases}$

$\Rightarrow$ $-\sqrt{\frac{5}{2}} < x \leq -1$ or $1 \leq x < \sqrt{\frac{5}{2}}$ (1)

(ii) $x^2 - 5x + 5 > 0$

$\Rightarrow$ $x < \frac{5 - \sqrt{5}}{2}$ or $x > \frac{5 + \sqrt{5}}{2}$ (2)

(iii) $\log_{1/2}(x^2 - 5x + 5) > 0$

$\Rightarrow$ $x^2 - 5x + 5 < \left(\frac{1}{2}\right)^0$

$\Rightarrow$ $x^2 - 5x + 5 < 1$

$\Rightarrow$ $x^2 - 5x + 4 < 0$

$\Rightarrow$ $1 < x < 4$ (3)

From Eq. (1), (2) and (3), $1 \leq x < \frac{5 - \sqrt{5}}{2}$

The correct option is (D)

53. We have, $f(n+1) = \frac{2f(n) + 2}{2}$ and $f(1) = 2$

$\therefore$ $f(101) = f(1) + 100 \times 1/2 = 2 + 50$

$\therefore$ $f(101) = 52$.

The correct option is (A)

54. $f(x) = \frac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}$

$\Rightarrow$ $f(x) = \frac{\cos^2 x + \sin^2 x(1 - \cos^2 x)}{\sin^2 x + \cos^2 x(1 - \sin^2 x)}$

$\Rightarrow$ $f(x) = \frac{\sin^2 x + \cos^2 x - \sin^2 x \cos^2 x}{\sin^2 x + \cos^2 x - \sin^2 x \cos^2 x}$

$\Rightarrow$ $f(x) = 1 \Rightarrow f(2002) = 1$

The correct option is (A)

55. We know that

$|x + y| = |x| + |y|$

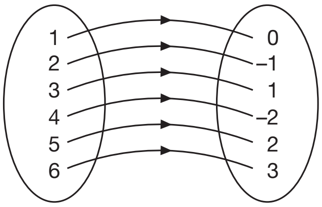
if x and y are of same sign i.e.,

either x and y are > 0 or x and y are < 0

Since, $|f(x) + \phi(x)| = |f(x)| + |\phi(x)|$

i.e., $|x+1+x-2| = |x+1| + |x-2|$
therefore, either $x + 1 > 0$ and $x - 2 > 0$
$\Rightarrow$ $x > -1$ and $x > 2$
$\therefore$ $x > 2$
or $x + 1 < 0$ and $x - 2 < 0$
$\Rightarrow$ $x < -1$ and $x < 2$
$\therefore$ $x < -1$
Hence, for the given equality to be valid, we have
$x \in (-\infty, -1) \cup (2, \infty)$
Since, option (B) is the subset of the solution set, Hence, option (B) is true.
The correct option is (B)

56. $f: N \rightarrow Z$
$f(1) = 0, f(2) = -1, f(3) = 1, f(4) = -2,$
$f(5) = 2$, and $f(6) = -3$ so on.



In this type of function every element of set A has unique image in set B and there is no element left in set B.
Hence, f is one-one and onto function.
The correct option is (D)

57. Since A.M. ≥ G.M.

$\therefore \quad \sqrt{x^2+x} + \dfrac{\tan^2\alpha}{\sqrt{x^2+x}} \geq 2\sqrt{\tan^2\alpha}$

$= 2\tan\alpha \qquad \left(\tan\alpha > 0 \text{ as } 0 < \alpha < \dfrac{\pi}{2}\right)$

The correct option is (A)

58. $0 \leq \log(1-x^2) \leq 1 \Rightarrow 1 \leq 1 - x^2 \leq e$
Now, $1 - x^2 \leq e \ \forall\, x \in R$
But $1 - x^2 \geq 1$ is possible only when $x = 0$.
The correct option is (A)

59. $f(x)$ is onto,
$\therefore$ S = range of $f(x)$
Now $f(x) = \sin x - \sqrt{3}\cos x + 1 = 2\sin\left(x - \dfrac{\pi}{3}\right) + 1$

$\because \quad -1 \leq \sin\left(x - \dfrac{\pi}{3}\right) \leq 1, -1 \leq 2\sin\left(x - \dfrac{\pi}{3}\right) + 1 \leq 3$

$\therefore$ $f(x) \in [-1, 3] = S$
The correct option is (A)

60. Since the function is periodic, $a^2 - 3$ and $b^2 + 7$ should be perfect squares, which is possible only if $a = 2$, $b = 3$ in which case $f(x) = \sin x + \cos 4x$, whose period is 2π.
The correct option is (B)

61. We have, $2f(x) + 3f\left(\dfrac{1}{x}\right) = x^2 - 1$ (1)

Putting $\dfrac{1}{x}$ in place of x, we get

$3f(x) + 2f\left(\dfrac{1}{x}\right) = \dfrac{1}{x^2} - 1$ (2)

Solving Eq. (1) and (2), we get $f(x) = \dfrac{(2x^2+3)(1-x^2)}{5x^2}$,
which is a non-periodic even function.
The correct option is (B)

62. When $x_1 = -1$ and $x_2 = 1$,

then $f(-1) - f(1) = f\left(\dfrac{-1-1}{1+1(1)}\right) = f(-1)$

$\Rightarrow$ $f(1) = 0$,
which is satisfied when $f(x) = \tan^{-1}\left(\dfrac{1-x}{1+x}\right)$
When $x_1 = x_2 = 0$, then

$f(0) - f(0) = f\left(\dfrac{0-0}{1-0}\right) = f(0) \Rightarrow f(0) = 0$

When $x_1 = -1$ and $x_2 = 0$, then

$f(-1) - f(0) = f\left(\dfrac{-1-0}{1-0}\right) = f(-1) \Rightarrow f(0) = 0,$

which is satisfied when $f(x) = \log\left(\dfrac{1-x}{1+x}\right)$

and $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$.

The correct option is (A), (B) and (C)

63. Given: $f(T+x) = 1 + [(1-f(x))^3]^{1/3}$
$= 1 + (1 - f(x))$
$\Rightarrow$ $f(T+x) + f(x) = 2$ (1)
$\Rightarrow$ $f(2T+x) + f(T+x) = 2$ (2)
(2) – (1) $\Rightarrow f(2T+x) - f(x) = 0$
$\Rightarrow$ $f(2T+x) = f(x)$
Also, T is positive and least therefore period of $f(x) = 2T$.
The correct option is (B)

64. Since, $f(x) = \sqrt{\dfrac{-\log_{0.3}(x-1)}{-x^2+3x+18}}$

$\Rightarrow$ $f(x) = \sqrt{\dfrac{\log_{0.3}(x-1)}{x^2-3x-18}}$

By definition, $f(x)$ is defined, if

$\dfrac{\log_{0.3}(x-1)}{x^2-3x-18} \geq 0$

$\therefore$ either $\log_{0.3}(x-1) \geq 0$ and $x^2 - 3x - 18 > 0$
$\Rightarrow$ $1 < x \leq 2$ and $x < -3$ or $x > 6$
No solution
or $\log_{0.3}(x-1) \leq 0$ and $x^2 - 3x - 18 < 0$
$\Rightarrow$ $x \geq 2$ and $-3 < x < 6 \Rightarrow x \in [2, 6)$
Hence, domain of $f(x) = [2, 6)$.
The correct option is (C)

65. Just interchange x and y and in first case $y = 0$, now $x \geq 0$

$x = (y+1)^2\ x \geq 0, y \geq -1$

or $\quad y = \sqrt{x} - 1\ \ x \geq 0$

The correct option is (D)

66. We have, $f(x) = \dfrac{\sin^{101} x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$

$$\Rightarrow \quad f(-x) = \frac{\sin^{101}(-x)}{\left[-\frac{x}{\pi}\right] + \frac{1}{2}}$$

Case I: when $x = n\pi\ (n \in Z)$

$$f(-x) = \frac{\sin^{101}(-n\pi)}{\left[\frac{-n\pi}{\pi}\right] + \frac{1}{2}} = 0$$

Case II: When $x \neq n\pi, n \in Z$

$$f(-x) = \frac{-\sin^{101}(x)}{\left[-\frac{x}{\pi}\right] + \frac{1}{2}}$$

$$\Rightarrow \quad f(-x) = \frac{-\sin^{101}(x)}{-1 - \left[\frac{x}{\pi}\right] + \frac{1}{2}} \qquad \left[\because \left[\frac{x}{\pi}\right] + \left[-\frac{x}{\pi}\right] = -1\right]$$

$$\Rightarrow \quad f(-x) = \frac{\sin^{101}(x)}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} = f(x)$$

$\therefore \quad f(x)$ is an even function.

The correct option is (B)

67. The given function f is well defined only when $|x| - x > 0$

$\Rightarrow \quad x < 0$

Required domain is $(-\infty, 0)$

The correct option is (D)

68. Number of one–one functions $= \begin{cases} {}^nP_m & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases}$

The correct option is (D)

69. Function is increasing

$x = \dfrac{y-3}{4} = g(y)$

The correct option is (D)

70. Given $f(x) = x^3 + 5x + 1$

Now $f'(x) = 3x^2 + 5 > 0, \forall x \in R$

$\therefore \quad f(x)$ is strictly increasing function

$\therefore \quad$ It is one–one

Clearly, $f(x)$ is a continuous function and also increasing on R,

$\underset{x \to -\infty}{\text{Lt}} f(x) = -\infty$ and $\underset{x \to \infty}{\text{Lt}} f(x) = \infty$

$\therefore \quad f(x)$ takes every value between $-\infty$ and ∞.

Thus, $f(x)$ is onto function.

The correct option is (C)

71. There is no information about co-domain therefore $f(x)$ is not necessarily onto.

The correct option is (C)

72. For every rational number T, we have,

$$f(T+x) = \begin{Bmatrix} 1, \text{ when } x \text{ is a rational} \\ 0, \text{ when } x \text{ is irrational} \end{Bmatrix},$$

but there is no least positive value of T for which $f(T+x) = f(x)$ because there are infinite number of rational numbers between any two rational numbers. Therefore, $f(x)$ is a periodic function having no fundamental period.

The correct option is (D)

73. In the sum, $f_1(1) + f_1(2) + f_1(3) + \ldots + f_1(n)$, 1 occurs n times, $\frac{1}{2}$ occurs $(n-1)$ times, $\frac{1}{3}$ occurs $(n-2)$ times and so on.

$\therefore \quad f_1(1) + f_1(2) + f_1(3) + \ldots + f_1(n)$

$$= n \cdot 1 + (n-1) \cdot \frac{1}{2} + (n-2) \cdot \frac{1}{3} + \ldots + 1 \cdot \frac{1}{n}$$

$$= n\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{n-1}{n}\right)$$

$$= nf_1(n) - \left[\left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{4}\right) + \ldots + \left(1 - \frac{1}{n}\right)\right]$$

$$= nf_1(n) - [n - f_1(n)] = (n+1) f_1(n) - n$$

The correct option is (C)

74. $\because \{x\} \in [0, 1)$

$\therefore \quad \sin\{x\} \in [0, \sin 1)$ but $f(x)$ is defined if $\sin\{x\} \neq 0$

$$\therefore \quad \frac{1}{\sin\{x\}} \in \left(\frac{1}{\sin 1}, \infty\right)$$

$$\therefore \quad \left[\frac{1}{\sin\{x\}}\right] \in \{1, 2, 3.....\}$$

The correct option is (B)

75. We have,

$f(x) = 2 \min\{|f(x) - g(x)|, 0\}$

$$= \begin{cases} 0 & f(x) > g(x) \\ 2(f(x) - g(x)), & f(x) \leq g(x) \end{cases}$$

$$= \begin{cases} f(x) - g(x) - |f(x) - g(x)|, & f(x) > g(x) \\ f(x) - g(x) - |f(x) - g(x)|, & f(x) \leq g(x) \end{cases}$$

$\therefore \quad f(x) = f(x) - g(x) - |g(x) - f(x)|$

The correct option is (D)

76. We have,

$$f(x) = \begin{cases} x & x \in Q \\ 1 - x, & x \notin Q \end{cases}$$

$$\therefore \quad fof(x) = \begin{cases} f(x) & , f(x) \in Q \\ 1 - f(x) & , f(x) \notin Q \end{cases}$$

$$= \begin{cases} x, & x \in Q, & x \in Q \\ 1 - x, & x \notin Q, & x \in Q \\ 1 - x, & 1 - x \in Q, & x \notin Q \\ 1 - (1 - x), & 1 - x \notin Q, & x \notin Q \end{cases}$$

$$= \begin{cases} x, & x \in Q \\ x, & x \notin Q \end{cases}$$

$\therefore \quad fof(x) = x, x \in [0, 1].$

The correct option is (C)

77. We have,

$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1 \qquad (1)$

Put $\quad x = f(y) = 0$

then $\quad f(0) = f(0) + 0 + f(0) - 1$

$\therefore \quad f(0) = 1 \qquad (2)$

Again, put $x = f(y) = \lambda$ in (1)

Then, $\quad f(o) = f(\lambda) + \lambda^2 + f(\lambda) - 1$

$\Rightarrow \quad 1 = 2f(\lambda) + \lambda^2 - 1$

$\therefore \quad f(\lambda) = \dfrac{2 - \lambda^2}{2} = 1 - \dfrac{\lambda^2}{2}$

Hence, $f(x) = 1 - \dfrac{x^2}{2}$

The correct option is (C)

78. A function whose graph is symmetrical about the origin must be odd.

$(3^x + 3^{-x})$ is an even function.

Since $\cos x$ is an even function and $\log(x + \sqrt{1+x^2})$ is an odd function,

$\therefore \quad \cos(\log(x + \sqrt{1+x^2}))$ is an even function.

If $f(x + y) = f(x) + f(y)\ \forall\ x, y \in R$, then $f(x)$ must be odd.

The correct option is (C)

79. The function $f(x) = x^2 + 2, x \in (-\infty, \infty)$ is not injective as $f(1) = f(-1)$ but $1 \neq -1$

The function $f(x) = (x - 4)(x - 5), x \in (-\infty, \infty)$ is not one-one as $f(4) = f(5)$ but $4 \neq 5$

The function, $f(x) = \dfrac{4x^2 + 3x - 5}{4 + 3x - 5x^2}$, $x \in (-\infty, \infty)$ is also not injective as $f(1) = f(-1)$ but $1 \neq -1$

For the function, $f(x) = |x + 2|, x \in [-2, \infty)$

Let $f(x) = f(y), x, y \in [-2, \infty) \Rightarrow |x + 2| = |y + 2|$

$\Rightarrow \quad x + 2 = y + 2$

$\Rightarrow \quad x = y$

So, f is an injection.

The correct option is (B)

80. $(fog)(x) = f[g(x)] = f(|3x + 4|)$

Since the domain of f is $[-3, 5]$,

$\therefore \quad -3 \leq |3x + 4| \leq 5 \Rightarrow |3x + 4| \leq 5$

$\Rightarrow \quad -5 \leq 3x + 4 \leq 5$

$\Rightarrow \quad -9 \leq 3x \leq 1 \Rightarrow -3 \leq x \leq \dfrac{1}{3}$

$\therefore \quad$ Domain of fog is $\left[-3, \dfrac{1}{3}\right]$

The correct option is (B)

81. For any $x \in R$, we have

$[x] = \left[\dfrac{x}{2}\right] + \left[\dfrac{x+1}{2}\right] \qquad (1)$

$\Rightarrow \quad [n] = \left[\dfrac{n+1}{2}\right] + \left[\dfrac{n}{2}\right]$

$\Rightarrow \quad n = \left[\dfrac{n+1}{2}\right] + \left[\dfrac{n}{2}\right]$

$= \left[\dfrac{n+1}{2}\right] + \left[\dfrac{n}{4}\right] + \left[\dfrac{\frac{n}{2}+1}{2}\right] \qquad$ [Using (1)]

$= \left[\dfrac{n+1}{2}\right] + \left[\dfrac{n+2}{4}\right] + \left[\dfrac{n}{4}\right]$

$= \left[\dfrac{n+1}{2}\right] + \left[\dfrac{n+2}{4}\right] + \left[\dfrac{n}{8}\right] + \left[\dfrac{\frac{n}{4}+1}{2}\right]$

[Using (1)]

$= \left[\dfrac{n+1}{2}\right] + \left[\dfrac{n+2}{4}\right] + \left[\dfrac{n+4}{8}\right] + \left[\dfrac{n}{8}\right]$

Continuing in this manner, we have,

$\left[\dfrac{n+1}{2}\right] + \left[\dfrac{n+2}{4}\right] + \left[\dfrac{n+4}{8}\right] + \ldots = n$

The correct option is (A)

82. We have,

$$g(x) = \begin{cases} 1 + n - n = 1, & x = n \in I \\ 1 + n + k - n = 1 + k, & x = n + k, \end{cases}$$

where $n \in I, 0 < k < 1$

Now, $f[g(x)] = \begin{cases} -1, & g(x) < 0 \\ 0, & g(x) = 0 \\ 1, & g(x) > 0 \end{cases}$

Clearly, $g(x) > 0$ for all x. So, $f[g(x)] = 1$, for all x.

The correct option is (B)

83. We have, $g(x) = 1 + \sqrt{x}$

and $f[g(x)] = 3 + 2\sqrt{x} + x \qquad (1)$

Also, $f[g(x)] = f(1 + \sqrt{x}) \qquad (2)$

From (1) and (2), we get

$f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x.$

Let $1 + \sqrt{x} = y$ or $x = (y - 1)^2$

$\therefore \quad f(y) = 3 + 2(y - 1) + (y - 1)^2$

$= 3 + 2y - 2 + y^2 - 2y + 1 = 2 + y^2$

$\therefore \quad f(x) = 2 + x^2$

The correct option is (B)

84. For f to be defined, we must have

$\log_{1/2}\left(1 + \dfrac{1}{\sqrt[4]{x}}\right) < -1 \Rightarrow 1 + \dfrac{1}{\sqrt[4]{x}} > (2^{-1})^{-1} = 2$ which is possible only if $\dfrac{1}{\sqrt[4]{x}} > 1$, i.e., $0 < x < 1$

Hence, the domain of the given function is $\{x: 0 < x < 1\}$

The correct option is (A)

85. We have, $f(x+2a) = f((x+a)+a)$

$$= \frac{1}{2} - \sqrt{f(x+a) - (f(x+a))^2}$$

$$= \frac{1}{2} - \sqrt{\frac{1}{2} - \sqrt{f(x) - (f(x))^2} - \left(\frac{1}{2} - \sqrt{f(x) - (f(x))^2}\right)^2}$$

$$= \frac{1}{2} - \sqrt{\frac{1}{4} - f(x) + (f(x))^2}$$

$$= \frac{1}{2} - \left(\frac{1}{2} - f(x)\right) = f(x)$$

Hence, $f(x)$ is periodic with period $2a$.

The correct option is (B)

86. Since the domain of f is (0, 1),

$\therefore\ 0 < e^x < 1$ and $0 < \ln|x| < 1$

$\Rightarrow\ \log 0 < x < \log 1$ and $e^0 < |x| < e^1$

$\Rightarrow\ -\infty < x < 0$ and $1 < |x| < e$

$\Rightarrow\ x \in (-\infty, 0)$ and $x \in ((-\infty, -1) \cup (1, \infty)) \cap (-e, e)$

$\Rightarrow\ x \in (-\infty, 0)$ and $x \in (-e, -1) \cup (1, e)$

$\Rightarrow\ x \in (-e, -1)$

The correct option is (A)

87. $\sin^{-1}(x - x^2)$ is defined when $-1 \le x - x^2 \le 1$

$$\Rightarrow\ \frac{1-\sqrt{5}}{2} \le x \le \frac{1+\sqrt{5}}{2} \qquad (1)$$

$\sqrt{1 - \frac{1}{|x|}}$ is defined when $1 - \frac{1}{|x|} \ge 0$

$\Rightarrow\ x \le -1$ or $x \ge 1$ (2)

and $\frac{1}{[x^2-1]}$ is defined when $x^2 - 1 < 0$ or $x^2 - 1 \ge 1$

i.e., $x \in (-\infty, -\sqrt{2}) \cup (-1, 1) \cup (-\sqrt{2}, \infty)$ (3)

From (1), (2) and (3), we get

$$x \in \left(\sqrt{2}, \frac{1+\sqrt{5}}{2}\right)$$

The correct option is (A)

88. Since,

$f(x) = x^3 + x^2 f'(1) + x f''(2) + f'(3)$ (1)

$\Rightarrow\ f'(x) = 3x^2 + 2x f'(1) + f''(2)$ (2)

$\Rightarrow\ f''(x) = 6x + 2f'(1)$ (3)

$\Rightarrow\ f'''(x) = 6$ i.e., a constant function

Hence, $f'(3) = 6$

Using equation (3), we have

$f''(2) = 12 + 2f'(1)$ (5)

Substituting $x = 1$, in equation (2), we have

$f'(1) = 3 + 2f'(1) + f''(2)$

$\Rightarrow\ f'(1) = 3 + 2f'(1) + 12 + 2f'(1)$ {using (5)}

$\Rightarrow\ 3f'(1) = -15$

$\therefore\ f'(1) = -5$ (6)

Substituting respective value in (5), we have

$f''(2) = 2$

Hence, the polynomial $f(x)$ can be written as

$f(x) = x^3 - 5x^2 + 2x + 6$

Therefore, $f(2) - f(1)$

$= (8 - 20 + 4 + 6) - (1 - 5 + 2 + 6)$

$= -2 - 4 = -6$

$\therefore\ f(2) - f(1) = -6 = -f(0)$

The correct option is (B)

89. Given, $q^2 - 4pr = 0$ and $p > 0$

For $f(x)$ to be defined

$px^3 + (p+q)x^2 + (q+r)x + r > 0$

$\Rightarrow\ px^2(x+1) + qx(x+1) + r(x+1) > 0$

$\Rightarrow\ (x+1)(px^2 + qx + r) > 0$

$\Rightarrow\ x > -1$ and $x \ne -\frac{q}{2p}$ [Since $q^2 - 4pr = 0$

$\therefore$ at $x = -\frac{q}{2p}$, $px^2 + qx + r = 0$ and at $x \ne -\frac{q}{2p}$, $px^2 + qx + r > 0$]

$$\therefore\ \text{Domain} = R - \left[(-\infty, -1] \cup \left\{-\frac{q}{2p}\right\}\right]$$

The correct option is (B)

90. Since the function is periodic, $a^2 - 3$ and $b^2 + 7$ should be perfect squares, which is possible only if $a = 2$, $b = 3$ in which case $f(x) = \sin x + \cos 4x$, whose period is 2π.

The correct option is (B)

91. We have, $f(x) = \dfrac{\sin^{101} x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$

$$\Rightarrow\ f(-x) = \frac{\sin^{101}(-x)}{\left[-\frac{x}{\pi}\right] + \frac{1}{2}}$$

Case I: when $x = n\pi\ (n \in Z)$

$$f(-x) = \frac{\sin^{101}(-n\pi)}{\left[\frac{-n\pi}{\pi}\right] + \frac{1}{2}} = 0$$

Case II: When $x \ne n\pi$, $n \in Z$

$$f(-x) = \frac{-\sin^{101}(x)}{\left[-\frac{x}{\pi}\right] + \frac{1}{2}}$$

$$\Rightarrow\ f(-x) = \frac{-\sin^{101}(x)}{-1 - \left[\frac{x}{\pi}\right] + \frac{1}{2}} \qquad \left[\because \left[\frac{x}{\pi}\right] + \left[-\frac{x}{\pi}\right] = -1\right]$$

$$\Rightarrow\ f(-x) = \frac{\sin^{101}(x)}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} = f(x)$$

$\therefore\ f(x)$ is an even function.

The correct option is (B)

92. Graph is symmetric about $x = k$ if $f(k - x) = f(k + x)$

$\Rightarrow \quad a(k - x)^3 + b(k - x)^2 + c(k - x) + d$

$= a(k + x)^3 + b(k + x) + c(k + x) + d$

$\Rightarrow \quad 2ax^3 - (6ak^2 + 4bk + 2c)x = 0$

It is true for all x if $a = 0$ and $6ak^2 + 4bk + 2c = 0$

i.e., $a = 0$ and $k = -\dfrac{c}{2b} \Rightarrow a + k = -\dfrac{c}{2b}$.

The correct option is (A)

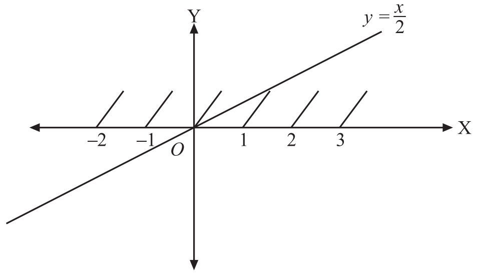
93. For the *ln* operation to be defined, we have $\{x\} > 0$

Plotting the curves $y = \{x\}$ and $y = 0$, we can see that the values of x satisfying the above inequality, are

$x \in R - I$ (1)

For the square root operation to be defined, we have

$\{x\} \le x/2$



Plotting the curves $y = \{x\}$ and $y = x/2$ as shown above, we can see that the values of x satisfying the above inequality, are

$x \in \{0\} \cup [1, \infty)$ (2)

The domain of definition is the intersection of (1) and (2) which gives $x \in (1, \infty) - I^+$

The correct option is (C)

94. For the *ln* operation to be defined, we have

$1 - 2\,|\cos x| > 0$

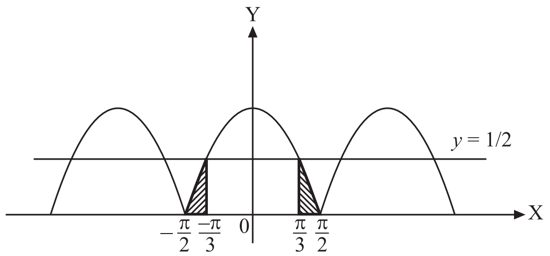
i.e., $|\cos x| < \dfrac{1}{2}$ (1)

For the inverse cos operation to be defined, we have

$-1 \le \dfrac{2x}{\pi} \le 1$

i.e., $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ (2)

To find the values of x satisfying both (1) and (2), let us plot the curves $y = |\cos x|$ and $y = \dfrac{1}{2}$ in the interval $[-\pi/2, \pi/2]$ as shown below



The domain of definition is the darkened portion on the X-axis

i.e. $x \in \left[-\dfrac{\pi}{2}, -\dfrac{\pi}{3}\right) \cup \left(\dfrac{\pi}{3}, \dfrac{\pi}{2}\right]$

The correct option is (C)

95. For the sgn operation to be defined, we have

$9 - x^2 > 0$ i.e., $-3 < x < 3$ (1)

For the square root operation to be defined, we have

$[x]^3 - 4[x] \ge 0$

$\Rightarrow \quad [x]\,([x] + 2)\,([x] - 2) \ge 0$

$\Rightarrow \quad -2 \le [x] \le 0$ or $[x] \ge 2$

i.e., $-2 \le x < 1$ or $x \ge 2$ (2)

The union of the intervals (1) and (2) gives the domain as $[-2, 1) \cup [2, 3)$

The correct option is (B)

96. we have

$$f(x) = \sqrt{\left(\log_{0.2} x\right)^3 + \left(\log_{0.2} x^3\right)\left(\log_{0.2} 0.0016x\right) + 36}$$

$$= \sqrt{\left(\log_{0.2} x\right)^3 + \left(3\log_{0.2} x\right)\left\{\log_{0.2}{}^{x} + \log_{0.2}(0.2)^4\right\} + 36}$$

$$= \sqrt{\lambda^3 + 3\lambda(\lambda + 4) + 36} \qquad [\text{putting } \log_{0.2} x = \lambda]$$

$$= \sqrt{\lambda^3 + 3\lambda^2 + 12\lambda + 36} = \sqrt{(\lambda + 3)\,(\lambda^2 + 12)}$$

For the square root operation to be defined, we have

$(\lambda + 3)\,(\lambda^2 + 12) \ge 0$

i.e., $\lambda + 3 \ge 0$ $\quad (\because \lambda^2 + 12$ is *a* positive quantity)

i.e., $\log_{0.2} x \ge -3$ i.e., $x \le (0.2)^{-3} = 5^3 = 125$

Also, $x > 0$ for log to be defined

Hence, the required domain of definition is $(0, 125]$

The correct option is (C)

97. We have,

$y = [x^2] - [x]^2,$	$x \in [0, 2]$
i.e., $y = [x^2],$	$0 \le x < 1$
$y = [x^2] - 1,$	$1 \le x < 2$
$= [x^2] - 1,$	$x = 2$
$= 0,$	$x = 2$
i.e., $y = 0,$	$0 \le x < 1$
$= 1 - 1 = 0,$	$1 \le x < \sqrt{2}$
$= 2 - 1 = 1,$	$\sqrt{2} \le x < \sqrt{3}$
$= 3 - 1 = 2,$	$\sqrt{3} \le x < 2$
$= 0,$	$x = 2$

Hence, the range is $\{0, 1, 2\}$

The correct option is (D)

98. For the inverse sin operation to be defined, we have

$$-1 \le \left[x^2 + \frac{1}{2}\right] \le 1$$

i.e., $-1 \le x^2 + \dfrac{1}{2} < 2$ $\quad$ i.e., $-\dfrac{3}{2} \le x^2 < \dfrac{3}{2}$ (1)

For the inverse cos operation to be defined, we have

$$-1 \le \left[x^2 - \frac{1}{2}\right] \le 1$$

i.e., $-1 \le x^2 - \frac{1}{2} < 2$

i.e., $\frac{-1}{2} \le x^2 < \frac{5}{2}$ (2)

Intersection of inequalities (1) and (2) gives

$\frac{-1}{2} \le x^2 < \frac{3}{2}$ i.e., $0 \le x^2 < \frac{3}{2}$ [$\because x^2$ cannot be negative]

Now, we have

$y = \sin^{-1}(0) + \cos^{-1}(-1) = \pi$, $0 \le x^2 < \frac{1}{2}$

$y = \sin^{-1}(1) + \cos^{-1}(0) = \pi$, $\frac{1}{2} \le x^2 < \frac{3}{2}$

Hence, the range is $\{\pi\}$

The correct option is (C)

99. We have,

$$f(x) = \frac{ax^2 + 2x + 1}{2x^2 - 2x + 1}$$

which is defined $\forall\, x \in R$, since

$2x^2 - 2x + 1 = 2\left(x - \frac{1}{2}\right)^2 + \frac{1}{2} \ne 0$ for any real x.

Now, if $f: R \to [-1, 2]$ is onto, then

$$-1 \le \frac{ax^2 + 2x + 1}{2x^2 - 2x + 1} \le 2, \forall\, x \in R$$

Solving the left-hand inequality, we have

$(a + 2)x^2 + 2 \ge 0$

which is true $\forall\, x \in R$ if $a \ge -2$ (1)

Solving the right-hand inequality, we have

$(a - 2)x^2 + 6x - 1 \le 0$

which is true for all $x \in$ R if coefficient of $x^2 < 0$ and D ≤ 0

i.e., $a < 2$ and $36 + 4\,(a - 2) \le 0$

i.e., $a < 2$ and $a \le -7$

i.e., $a \le -7$ (2)

Hence, from (1) and (2) the permissible values of a are given by

$(-\infty, -7] \cup [-2, \infty)$

The correct option is (C)

100. We have, $f(x) = x + (-1)^{x-1}, x \in N$

Thus, we have

$f(1) = 1 + 1 = 2$, $f(2) = 2 - 1 = 1$

$f(3) = 3 + 1 = 4$, $f(4) = 4 - 1 = 3$

$f(5) = 5 + 1 = 6$, $f(6) = 6 - 1 = 5$ and so on.

The graph of $f(x)$ consists of the points (1, 2), (2, 1), (3, 4), (4, 3), (5, 6), (6, 5)… Thus, if (a, b) is a point on the graph, then (b, a) is also a point on the graph. Hence the inverse of f is f itself

i.e., $f^{-1}(x) = x + (-1)^{x-1}, x \in N$

The correct option is (A)

101. Let $y = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

i.e., $(\alpha + 8y)x^2 + 6(1 - y)x - (\alpha y + 8) = 0$

According to the given condition, y takes all real values, for real x. In other words, the above quadratic equation in x should have real roots for every real y

i.e., $D \ge 0, \forall\, y \in R$

i.e., $36(1 - y)^2 + 4(\alpha y + 8)\,(\alpha + 8y) \ge 0, \forall\, y \in R$

i.e., $(9 + 8\alpha)\,y^2 + (\alpha^2 + 46)y + (9 + 8\alpha) \ge 0, \forall\, y \in R$

i.e., $D \le 0$ and coefficient of $y^2 > 0$

i.e., $(\alpha^2 + 46)^2 \le 4(9 + 8\alpha)^2$ and $9 + 8\alpha > 0$

i.e., $\alpha^2 + 46 \le 2(9 + 8\alpha)$ and $\alpha > -9/8$

i.e., $\alpha^2 - 16\alpha + 28 \le 0$ and $\alpha > -9/8$

i.e., $2 \le \alpha \le 14$ and $\alpha > -9/8$

i.e., $2 \le \alpha \le 14$

Hence, $f: R \to R$ is onto for $2 \le \alpha \le 14$

The correct option is (D)

102. The function $f(x) = \sin^{-1}\left(\frac{2|x|}{1+x^2}\right)$ is defined for

$\left|\frac{2|x|}{1+x^2}\right| \le 1$ i.e., $\frac{2|x|}{1+x^2} \le 1$

$\left[\because \frac{2|x|}{1+x^2} \text{ is a positive quantity}\right]$

i.e., $x^2 - 2\,|x| + 1 \ge 0$

i.e., $(|x| - 1)^2 \ge 0$, which is true $\forall\, x \in R$ (1)

The function $\sqrt{\sin x}$ is defined for, $\sin x \ge 0$

i.e., $2n\pi \le x \le (2n + 1)\pi, n \in I$ (2)

Intersection of inequalities (1) and (2) gives the domain as $[2n\pi, (2n + 1)\pi], n \in I$

The correct option is (B)

103. $[\sin x]$ is always defined.

$\cos\left(\frac{\pi}{[x-1]}\right)$ is also defined everywhere except when $[x - 1] = 0$

$\Rightarrow\ 0 \le x - 1 < 1$

$\Rightarrow\ 1 \le x < 2$

Hence, domain $\in R - [1, 2)$

The correct option is (B)

104. Here, $y = \frac{x - [x]}{1 + x - [x]} = \frac{\{x\}}{1 + \{x\}}$

Thus, domain $= (-\infty, \infty)$

So, from $y = \frac{\{x\}}{1 + \{x\}}$ we have,

$y + y\,\{x\} = \{x\}$

$\Rightarrow\ \{x\} = \frac{y}{1 - y}$

Here, $0 \le \{x\} < 1$ so, $0 \le \frac{y}{1-y} < 1$

$\Rightarrow \quad 0 \le y < \frac{1}{2}$

Hence, range $= \left[0, \frac{1}{2}\right)$

The correct option is (C)

105. For $g(x) = f|[x]|$ to be defined, we must have

$-3 \le |[x]| \le 2$

$\Rightarrow \quad 0 \le |[x]| \le 2 \qquad$ [as $|x| \ge 0 \ \forall \ x$]

$\Rightarrow \quad -2 \le [x] \le 2 \qquad$ [as $|x| \le a \Rightarrow -a \le x \le a$]

$\Rightarrow \quad -2 \le x < 3 \quad$ [by definition of greatest integer function]

Hence, domain of $g(x)$ is $[-2, 3)$.

The correct option is (C)

106. We know that $\{x + r\} = \{x\}$ as $r \in$ Integer

$\therefore \quad [x] + \sum_{r=1}^{2000} \frac{\{x+r\}}{2000} = [x] + \sum_{r=1}^{2000} \frac{\{x\}}{2000}$

$= [x] + \left[\frac{\{x\}}{2000} + \frac{\{x\}}{2000} + \ldots + \text{up to 2000 times}\right]$

$= [x] + \frac{2000\{x\}}{2000} = [x] + \{x\} = x$

The correct option is (A)

107. $f(x) = \sum_{k=1}^{n} (1 + [\sin kx])$

$= n + [\sin x] + [\sin 2x] + \ldots + [\sin nx] \qquad (1)$

Case I: When $kx \ne \frac{\pi}{2}$ for $k = 1, 2, 3, \ldots, n$

Since $0 < kx < \pi$ and $kx \ne \frac{\pi}{2}$

$\therefore \quad 0 < \sin kx < 1$, for $k = 1, 2, \ldots, n$

$\therefore \quad [\sin kx] = 0$, for $k = 1, 2, 3, \ldots, n$

$\therefore \quad$ From (1), $f(x) = n$

Case II: When exactly one of $x, 2x, 3x, \ldots, nx$ is $\frac{\pi}{2}$

Here, not more than one of $x, 2x, 3x, \ldots nx$ can be $\frac{\pi}{2}$

In this case, one of $\sin x, \sin 2x, \ldots, \sin nx$ is 1 and others lie between 0 and 1

$\therefore \quad$ From (1), $f(x) = n + 1$

Hence, range of $f = \{n, n + 1\}$

The correct option is (C)

108. We have,

$f(n) = \{(1-p)p^{-1}\}\{f(n+1) + f(n+2) + \ldots \text{ to } \infty\}$

Put $n = 1$

$f(1) = \{(1-p)p^{-1}\}\{f(2) + f(3) + \ldots \text{ to } \infty\}$

$= \{(1-p)p^{-1}\}\{1 - f(1)\}$

$= (1-p)p^{-1} - (1-p)p^{-1} f(1)$

$\Rightarrow \quad f(1)\{1 + (1-p)p^{-1}\} = (1-p)p^{-1}$

$\Rightarrow \quad f(1) \times p^{-1} = (1-p)p^{-1}$

$\Rightarrow \quad f(1) = 1 - p$

Put $n = 2$,

$f(2) = \{(1-p)p^{-1}\}\{f(3) + f(4) + \ldots\}$

$\Rightarrow \quad f(2) = \{(1-p)p^{-1}\}\{1 - 1 + p - f(2)\}$

$= [(1-p)p^{-1}][p - f(2)]$

$= (1-p) - (1-p)p^{-1}f(2)$

$\Rightarrow \quad f(2)\{1 + (1-p)p^{-1}\} = 1 - p$

$\Rightarrow \quad f(2) = p(1-p)$

The correct option is (A)

109. Let $S_n = \frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \ldots + \frac{x}{((n-1)x+1)(nx+1)}$

$= \left(1 - \frac{1}{1+x}\right) + \left(\frac{1}{1+x} - \frac{1}{1+2x}\right) + \left(\frac{1}{1+2x} - \frac{1}{1+3x}\right) + \ldots + \left(\frac{1}{1+(n-1)x} - \frac{1}{1+nx}\right)$

$= 1 - \frac{1}{1+nx}$

$\therefore \quad f(x) = \lim_{n\to\infty} S_n = \begin{cases} 1 & \text{when } x \ne 0 \\ 0 & \text{when } x = 0 \end{cases}$

$\therefore \quad$ Range $f = \{0, 1\}$

The correct option is (A)

110. For $x = \frac{1}{f(y)}$, we have

$f\left(x \cdot \frac{1}{x}\right) = \frac{1}{f(y)^p} \cdot y^q \Rightarrow f(1) = \frac{y^q}{\{f(y)\}^p}$

$\Rightarrow \quad f(y) = \frac{y^{q/p}}{\{f(1)\}^{1/p}}$

For $y = 1$, we have $f(1) = 1$

$\therefore \quad f(y) = y^{q/p} \quad$ or $\quad f(x) = x^{q/p} \qquad (1)$

Hence, $f(x \cdot y^{q/p}) = x^p \cdot y^q$

Let $y^{q/p} = z \Rightarrow y = z^{p/q}$

$\Rightarrow \quad f(x \cdot z) = x^p \cdot z^p \quad$ or $\quad f(x) = x^p \qquad (2)$

From (1) and (2), we have

$x^{q/p} = x^p$

$\Rightarrow \quad \frac{q}{p} = p \quad$ or $\quad q = p^2$

The correct option is (C)

111. Given, $f(x + y) + f(x - y) = 2f(x)f(y) \qquad (1)$

Interchange x and y in (1), we get

$f(y + x) + f(y - x) = 2f(y)f(x) \qquad (2)$

From (1) and (2), $f(x - y) = f(y - x)$

Putting $y = 2x$, we get $f(x) = f(-x)$

The correct option is (A)

112. We have,

$f(x) = -\frac{3}{2} f\left(\frac{2x+29}{x-2}\right) + 50x + 40 \qquad (1)$

Replacing x by $\frac{2x+29}{x-2}$ in the given functional equation we get

$$f\left(\frac{2x+29}{x-2}\right) = \frac{-3}{2} f\left[\frac{2\left(\frac{2x+29}{x-2}\right)+29}{\left(\frac{2x+29}{x-2}\right)-2}\right] + 50\left(\frac{2x+29}{x-2}\right) + 40$$

$$\Rightarrow f\left(\frac{2x+29}{x-2}\right) = -\frac{3}{2} f(x) - 50\left(\frac{2x+29}{x-2}\right) + 40 \quad (2)$$

Using (2) in (1), we get

$$f(x) = \frac{9}{4} f(x) + 75\left(\frac{2x+29}{x-2}\right) - 60 + 50x + 40$$

$$\Rightarrow \frac{9}{4} f(x) - f(x) = 20 - 50x - 75\left(\frac{2x+29}{x-2}\right)$$

$$\Rightarrow \frac{5}{4} f(x) = 20 - 50x - 75\left(\frac{2x+29}{x-2}\right)$$

$$\Rightarrow f(x) = 16 - 40x - 60\left(\frac{2x+29}{x-2}\right)$$

The correct option is (A)

113. Here,

$g^2(x) = (gog)(x) = g\{g(x)\} = g(3 + 4x)$

$\Rightarrow g^2(x) = 3 + 4(3 + 4x)$

$\Rightarrow g^2(x) = 15 + 4^2x$

$\Rightarrow g^2(x) = (4^2 - 1) + (4^2)x$

On generalizing, we have

$g^n(x) = (4^n - 1) + (4^n)x$

Then, for finding inverse, $g^n(x) = y$

$\Rightarrow y = (4^n - 1) + (4^n)x$

$\Rightarrow x = (y + 1 - 4^n)4^{-n}$

$\Rightarrow g^{-n}(y) = (y + 1 - 4^n)4^{-n}$

$\Rightarrow g^{-n}(x) = (x + 1 - 4^n)\, 4^{-n}$

The correct option is (B)

114. We have,

$f(x + p) = 1 + [2 - 3f(x) + 3\{f(x)\}^2 - \{f(x)\}^3]^{1/3}$

$\Rightarrow f(x + p) = 1 + [1 + \{1 - f(x)\}^3]^{1/3}$

$\Rightarrow f(x + p) - 1 = [1 - \{f(x) - 1\}^3]^{1/3}$

$\Rightarrow g(x + p) = [1 - \{g(x)\}^3]^{1/3}$ (1)

where $g(x) = f(x) - 1$ and $g(x + p) = f(x + p) - 1$

$\Rightarrow g(x + 2p) = [1 - \{g(x + p)\}^3]^{1/3}$ (2)

$\Rightarrow g(x + 2p) = [1 - \{\{1 - \{g(x)\}^3\}^{1/3}\}^3]^{1/3}$

from (1) and (2), we have

$\Rightarrow g(x + 2p) = [1 - \{1 - \{g(x)\}^3\}]^{1/3}$

$\Rightarrow g(x + 2p) = [1 - 1 + \{g(x)\}^3]^{1/3}$

$\Rightarrow g(x + 2p) = [\{g(x)\}^3]^{1/3}$

$\Rightarrow g(x + 2p) = g(x)$

which shows $f(x + 2p) - 1 = f(x) - 1$

or, $f(x + 2p) = f(x)$

Hence, $f(x)$ is periodic with period $2p$.

The correct option is (B)

115. Given $f(x) + f(x + a) + \ldots + f(x + an) = 0$ (1)

Replace x by $x + a$, we get

$f(x + a) + f(x + 2a) + \ldots + f\{x + a(n + 1)\} = 0$ (2)

Subtracting (2) from (1), we get

$f(x) - f\{x + a(n + 1)\} = 0$

$\Rightarrow f(x)$ is periodic with period $a(n + 1)$

The correct option is (B)

116. Since $y = \log_3 x$ is an increasing function, so S is mapped onto the set $(\log_3 3, \log_3 27) = (1, 3)$

The correct option is (C)

117. We have $f(x) + \phi(x) = x - 3 + 4 - x = 1$, so that

$|f(x)| + |\phi(x)| = 1$

Furthermore,

$$|f(x)| = \begin{cases} x - 3 & \text{if } x \ge 3 \\ 3 - x & \text{if } x < 3 \end{cases};$$

$$\text{and } |\phi(x)| = \begin{cases} 4 - x & \text{if } x \le 4 \\ x - 4 & \text{if } x > 4 \end{cases}$$

$$\Rightarrow |f(x)| + |\phi(x)| = \begin{cases} 7 - 2x & \text{if } x < 3 \\ 1 & \text{if } 3 \le x \le 4 \\ 2x - 7 & \text{if } x > 4 \end{cases}$$

We need those points for which the L.H.S. is greater than 1. Clearly, we can exclude values of x between 3 and 4. Now, for values of x less than 3, $7 - 2x$ is greater than 1, and for values of x greater than 4, $2x - 7$ is greater than 1.

Therefore, the given inequality is true for values of x given by $(-\infty, \infty) - [3, 4]$.

The correct option is (C)

More Than One Option Correct Type

118. Since $f(x) = f\left(\frac{x+1}{x+2}\right) \Rightarrow x = \frac{x+1}{x+2}$

$\Rightarrow x^2 + x - 1 = 0$

$$\Rightarrow x = \frac{-1 \pm \sqrt{5}}{2} \quad (1)$$

Since $f(x)$ is an even function defined on $[-5, 5]$,

$\therefore f(-x) = f(x), \forall x \in [-5, 5]$

$$\Rightarrow x = -\left(\frac{x+1}{x+2}\right) \Rightarrow x^2 + 3x + 1 = 0$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{5}}{2} \quad (2)$$

From (1) and (2), the values of x are

$$\frac{-1\pm\sqrt{5}}{2} \text{ and } \frac{-3\pm\sqrt{5}}{2}$$

The correct option is (A) and (B)

119. Let the required function be $f(x) = ax + b$

If $a > 0$, then $f(-1) = 0$ and $f(1) = 2$

$\Rightarrow$ $-a + b = 0$ and $a + b = 2$

$\Rightarrow$ $a = 1$ and $b = 1$

If $a < 0$, then $f(-1) = 2$ and $f(1) = 0$

$\Rightarrow$ $-a + b = 2$ and $a + b = 0$

$\Rightarrow$ $a = -1$ and $b = 1$

Hence, $f(x) = x + 1$ or $f(x) = -x + 1$

The correct option is (B) and (C)

120. Taking $f(x) = \log x$, we see that

$$f\left(\frac{x}{y}\right) = f(x) - f(y)$$

Clearly, $f(x)$ is not bounded

and $f\left(\frac{1}{x}\right) = -\log x \to \infty$ as $x \to 0$

Also, $x\,f(x) = x \log x \to 0$ as $x \to 0$

The correct option is (C) and (D)

121. Since, $f(-x) = f(x)$

$\Rightarrow$ $f(x)$ is an even function, its graph will be symmetrical about y-axis.

Also, $f(x) = -\left(\frac{|x|^3 + |x|}{1 + x^2}\right)$

$\Rightarrow$ $f(x) = -$(positive) = negative

i.e., the graph of $f(x)$ completely lies below the x-axis, and is also symmetric about y-axis (as discussed above).

$\therefore$ The graph of $f(x)$ lies in III and IV quadrants.

The correct option is (C) and (D)

122. When $x_1 = -1$ and $x_2 = 1$,

then $f(-1) - f(1) = f\left(\frac{-1-1}{1+1(1)}\right) = f(-1)$

$\Rightarrow$ $f(1) = 0$,

which is satisfied when $f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

When $x_1 = x_2 = 0$, then

$$f(0) - f(0) = f\left(\frac{0-0}{1-0}\right) = f(0) \Rightarrow f(0) = 0$$

When $x_1 = -1$ and $x_2 = 0$, then

$$f(-1) - f(0) = f\left(\frac{-1-0}{1-0}\right) = f(-1) \Rightarrow f(0) = 0,$$

which is satisfied when $f(x) = \log\left(\frac{1-x}{1+x}\right)$

and, $f(x) = \log\left(\frac{1+x}{1-x}\right)$

The correct option is (A), (B) and (C)

123. (A) The period of $\cos \pi x$ is $\frac{2\pi}{\pi} = 2$, and period of $\{x\}$ is 1

Hence, period of the given function is L.C.M. of$(1, 2) = 2$

(B) Solving $\tan\left(\frac{\pi}{2}[x+T]\right) = \tan\left(\frac{\pi}{2}[x]\right)$

i.e., $[x + T] - [x] = 2n$

gives a value of T independent of x only if T is an integer. In that case, the above equation reduces to

$[x] + T - [x] = 2n$

i.e., $T = 2n$

Hence, period of $f(x)$, is the smallest positive value of T, i.e., 2.

(C) We have period of $\sin x = 2\pi$ and period of $\{x\} = 1$

Hence, period of the given functions is L.C.M. of$(2\pi, 1)$ which does not exist since 2π is an irrational number. Hence, the function is not periodic

(D) Let us solve

$\sin\{\cos (x + T)\} = \sin\{\cos x\}$

i.e., $\cos (x + T) = n\pi + (-1)^n \cos x,\ n \in I$

Putting $n = 0$, gives $\cos (x + T) = \cos x$,

which gives $T = 2\pi$ as the smallest positive value. For no other value of n can a value of T be found independent of x.

Hence, the required fundamental period is 2π.

The correct option is (A) and (B)

124. The function is defined for all real values of x except those which satisfy the equation

$[|x-1|] + [|7-x|] - 6 = 0$ (1)

Case I: $(1 < x < 7)$

Equation (1) reduces to

$[x - 1] + [7 - x] - 6 = 0$

i.e., $[x] - 1 + [-x] + 7 - 6 = 0$ or $[x] + [-x] = 0$

which is true $\forall\ x \in I$

Thus, every integer in (1, 7) satisfies equation (1).

Case II: $(x \le 1)$

Equation (1) reduces to

$[1 - x] + [7 - x] - 6 = 0$

i.e., $1 + [-x] + 7 + [-x] - 6 = 0$

i.e., $[-x] = -1$ i.e., $-1 \le -x < 0$ or $0 < x \le 1$

Thus, equation (1) is satisfied $\forall\ 0 < x \le 1$

Case III: $(x \ge 7)$

Equation (1) reduces to

$[x - 1] + [x - 7] - 6 = 0$

i.e., $[x] - 1 + [x] - 7 - 6 = 0$ or $[x] = 7$

$\therefore$ $7 \le x < 8$

Thus, equation (1) is satisfied $\forall\ 7 \le x < 8$. The union of the intervals obtained in the above three cases gives the domain of definition as

$R - (0, 1] - [7, 8) - \{2, 3, 4, 5, 6\}$

The correct option is (A), (B) and (C)

125. $f(x) = \dfrac{x(\sin x + \tan x)}{\left[\dfrac{x+\pi}{\pi}\right] - \dfrac{1}{2}} = \dfrac{x(\sin x + \tan x)}{\left[\dfrac{x}{\pi}\right] + 1 - \dfrac{1}{2}}$

$= \dfrac{x(\sin x + \tan x)}{\left[\dfrac{x}{\pi}\right] + 0.5}$

$\Rightarrow f(-x) = \dfrac{-x(\sin(-x) + \tan(-x))}{\left[\dfrac{-x}{\pi}\right] + 0.5}$

$\Rightarrow f(-x) = \begin{cases} \dfrac{x(\sin x + \tan x)}{-1 - \left[\dfrac{x}{\pi}\right] + 0.5} & , x \neq n\pi \\ 0 & , x = n\pi \end{cases}$

$\Rightarrow f(-x) = -\left(\dfrac{x(\sin x + \tan x)}{\left[\dfrac{x}{\pi}\right] + 0.5}\right)$, when $x \neq n\pi$

and $f(-x) = 0$, when $x = n\pi$

Hence, $f(x)$ is an odd function (if $x \neq n\pi$) and $f(x)$ is an even function (if $x = n\pi$)

The correct option is (C) and (D)

126. Let us check for invertibility of $f(x)$

(A) one-one: we have, $f(x) = \dfrac{e^x + e^{-x}}{2}$

$\Rightarrow f'(x) = \dfrac{e^{2x} + 1}{2e^x}$, which is strictly increasing as $e^{2x} > 0$ for all x.

Thus, f is one-one

(B) Onto; Let $y = f(x)$

$\Rightarrow y = \dfrac{e^x + e^{-x}}{2}$, where y is strictly monotonic

Hence, range of $f(x) = (f(-\infty), f(\infty))$

$\Rightarrow$ range of $f(x) = (-\infty, \infty)$

So, range of $f(x)$ = co-domain

Hence, $f(x)$ is one-one and onto

(C) To find $f^{-1}: y = \dfrac{e^{2x} - 1}{2e^x}$

$\Rightarrow e^{2x} - 2e^x y - 1 = 0$

$\Rightarrow e^x = \dfrac{2y \pm \sqrt{4y^2 + 4}}{2}$

$\Rightarrow x = \log(y \pm \sqrt{y^2 + 1})$

$\Rightarrow f^{-1}(y) = \log(y \pm \sqrt{y^2 + 1})$

Since, $e^{f^{-1}(x)}$ is always positive, so, neglecting negative sign.

Hence, $f^{-1}(x) = \log(x + \sqrt{x^2 + 1})$

The correct option is (A), (B) and (D)

127. We have,

$f(x) = \sin^{-1}(\log [x]) + \log(\sin^{-1}[x])$ (1)

Let $g(x) = \sin^{-1}(\log [x])$ (2)

and, $h(x) = \log(\sin^{-1}[x])$ (3)

Now for $g(x)$;

$-1 \le \log [x] \le 1$ {as $\sin^{-1}\theta$ exists when $-1 \le \theta \le 1$}

and, $[x] > 0$ {as $\log [x]$ exists when $[x] > 0$}

$\Rightarrow \dfrac{1}{e} \le [x] \le e$ and $[x] > 0$

$\Rightarrow [x] = 1, 2$

$\Rightarrow x \in [1, 3)$ (4)

Again, from (3), we have

$h(x) = \log(\sin^{-1}[x])$ exists when;

$\sin^{-1}[x] > 0$ and $-1 \le [x] \le 1$

$\Rightarrow [x] > 0$ and $-1 \le [x] \le 1$

$\Rightarrow 0 < [x] \le 1 \Rightarrow [x] = 1$

$\Rightarrow x \in [1, 2)$ (5)

$\Rightarrow$ Domain of $f(x)$ is $[1, 2)$

Now, for range,

we know, $f(x) = \sin^{-1}(\log[x]) + \log(\sin^{-1}[x])$

where $x \in [1, 2) \Rightarrow [x] = 1$

$\therefore$ Range of $f(x) = \sin^{-1}(\log 1) + \log(\sin^{-1} 1)$

$= \sin^{-1}(0) + \log\left(\dfrac{\pi}{2}\right)$

$= \log(\pi/2)$

$\Rightarrow$ Range of $f(x) = \left\{\log \dfrac{\pi}{2}\right\}$

The correct option is (A) and (C)

128. (A) one-one:

$f(x) = 2^{x^2 - x}$

$\Rightarrow f'(x) = 2^{x^2 - x}(2x - 1) \cdot \log_2$

For $f(x)$ to be one-one, it should be strictly increasing or strictly decreasing.

So, $f'(x) > 0$

$\Rightarrow 2^{x^2 - x}(2x - 1) > 0$, where $2x^{2-x} > 0$ for all x

$\Rightarrow 2x - 1 > 0$ or $x > \dfrac{1}{2}$

Thus, for given domain $[1, \infty)$, $f(x)$ is always increasing. Hence, f is one-one

(B) onto: As $f(x)$ is strictly increasing

$\Rightarrow$ Range $f(x) \in [f(1), f(\infty))$

$\Rightarrow$ Range $f(x) \in [1, \infty)$

$\Rightarrow$ Range of $f(x)$ = Co-domain of $f(x)$, thus, f is onto.

(C) Inverse:

As f is one-one and onto, f^{-1} can be obtained.

Let $y = f(x)$

$\Rightarrow y = 2x^{2-x}$

$\Rightarrow x^2 - x = \log_2 y$

$\Rightarrow x^2 - x - \log_2 y = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{1+4\log_2 y}}{2}$

$\Rightarrow \quad f^{-1}(y) = \dfrac{1 + \sqrt{1+4\log_2 y}}{2}$ [as $y > 0, \forall\, x \in D$]

The correct option is (A), (B) and (C)

129. $f(x) = \dfrac{9^x}{9^x + 3}$ (1)

and, $f(1-x) = \dfrac{9^{1-x}}{9^{1-x}+3}$

$\Rightarrow \quad f(1-x) = \dfrac{\frac{9}{9^x}}{\frac{9}{9^x}+3} = \dfrac{9}{9+3.9^x}$

$f(1-x) = \dfrac{9}{3(3+9^x)}$ (2)

Adding (1) and (2), we get

$f(x) + f(1-x) = \dfrac{9^x}{9^x+3} + \dfrac{9}{3(3+9^x)}$

$= \dfrac{3.\,9^x + 9}{3(9^x+3)} = \dfrac{3(9^x+3)}{3(9^x+3)}$

$\therefore \quad f(x) + f(1-x) = 1$ (3)

Now, putting $x = \dfrac{1}{1996}, \dfrac{2}{1996}, \dfrac{3}{1996}, \ldots \dfrac{998}{1996}$, in (3), we get

$f\left(\dfrac{1}{1996}\right) + f\left(\dfrac{1995}{1996}\right) = 1$

$\Rightarrow \quad f\left(\dfrac{2}{1996}\right) + f\left(\dfrac{1994}{1996}\right) = 1$

$\Rightarrow \quad f\left(\dfrac{3}{1996}\right) + f\left(\dfrac{1993}{1996}\right) = 1$

… … …

… … …

$\Rightarrow \quad f\left(\dfrac{997}{1996}\right) + f\left(\dfrac{999}{1996}\right) = 1$

$\Rightarrow \quad f\left(\dfrac{998}{1996}\right) + f\left(\dfrac{998}{1996}\right) = 1$ or $f\left(\dfrac{998}{1996}\right) = \dfrac{1}{2}$

Adding all the above expressions, we get

$f\left(\dfrac{1}{1996}\right) + f\left(\dfrac{2}{1996}\right) + \ldots + f\left(\dfrac{1995}{1996}\right)$

$= (1+1+1+\ldots+997) + \dfrac{1}{2}$

$= 997 + \dfrac{1}{2} = 997\dfrac{1}{2}$

The correct option is (A) and (D)

130. We have $f(n+2) - f(n+1)$

$= (n+2)\,! = (n+2)\,(n+1)\,!$

$= (n+2)\,[f(n+1) - f(n)]$

$\Rightarrow \quad f(n+2) = (n+3)\,f(n+1) - (n+2)\,f(n)$

$\therefore \quad P(x) = x + 3$ and $Q(x) = -x - 2$

The correct option is (A) and (B)

Passage Based Questions

131. $\sin x$ is a periodic function with period 2π, therefore $\sin(\sqrt{[n]}\,x)$ is a periodic function with period $\dfrac{2\pi}{\sqrt{[n]}}$.

But the period of $f(x)$ is 2π (given).

$\therefore \quad \dfrac{2\pi}{\sqrt{[n]}} = 2\pi \Rightarrow \sqrt{[n]} = 1 \Rightarrow [n] = 1 \Rightarrow 1 \le n < 2$

The correct option is (A)

132. $f(x+T) = f(x)$

$\Rightarrow \quad \cos(\sin(x+T) + \cos(\cos(x+T)$

$= \cos(\sin x) + \cos(\cos x)$

If $x = 0$, then $\cos(\sin T) + \cos(\cos T)$

$= \cos(0) + \cos(1) = \cos\left(\cos\dfrac{\pi}{2}\right) + \cos\left(\sin\dfrac{\pi}{2}\right)$

On comparing, we get $T = \dfrac{\pi}{2}$.

The correct option is (A)

133. Since $|\sin x| + |\cos x|$ is a periodic function with period $\dfrac{\pi}{2}$, therefore period of $f(x)$ will be $\dfrac{\pi}{2}$ when $k = 1$.

The correct option is (A)

134. $3x + 3 - [3x + 3]$ has the period 1 and $\sin\dfrac{\pi x}{2}$ has the period $\dfrac{2\pi}{\pi/2}$ i.e., 4. Therefore, the period of $f(x)$ is L.C.M. (1, 4) = 4.

The correct option is (A)

135. The period of $|\sin x| + |\cos x|$ and $\sin^4 x + \cos^4 x$ is $\dfrac{\pi}{2}$ $\sin(\sin x)$ + $\sin(\cos x)$ has period 2π. The function $\dfrac{1+2\cos x}{\sin x(2+\sec x)}$ can be written in a simplified form as $\dfrac{\cos x}{\sin x} = \cot x$, so it has period π.

The correct option is (D)

136. The period of sin $5x$ is $\frac{2\pi}{5}$ and that of cos $\sqrt{3}x$ is $\frac{2\pi}{\sqrt{3}}$. As $\frac{2\pi}{5}$ and $\frac{2\pi}{\sqrt{3}}$ do not have a common multiple, $f(x)$ is non-periodic.

The correct option is (C)

137. We have,

$e^x + e^{f(x)} = e \Rightarrow e^{f(x)} = e - e^x$

$\Rightarrow f(x) = \log(e - e^x)$

For $f(x)$ to be defined, $e - e^x > 0$

$\Rightarrow e^1 > e^x \Rightarrow x < 1$

$\therefore$ Domain of $f = (-\infty, 1)$

Let $y = \log(e - e^x) \quad \Rightarrow e^y = e - e^x$

$\Rightarrow e^x = e - e^y$

$\Rightarrow x = \log(e - e^y)$

For x to be real, $e - e^y > 0$

$\Rightarrow e^1 > e^y \Rightarrow y < 1$

$\therefore$ Range of $f = (-\infty, 1)$

The correct option is (B)

138. Since $[x^2 + 1]$ is an integer,

$\therefore \sin(\pi[x^2 + 1]) = 0$

$\Rightarrow f(x) = \frac{\sin(\pi[x^2+1])}{x^4+1} = 0$

Hence, Range of $f = R_f = \{0\}$

The correct option is (C)

139. We have, $f(x) = 1 + \frac{\sin x}{\cos^2 x}$

$\Rightarrow f'(x) = \frac{\cos^2 x(\cos x) + \sin x(2\cos x \sin x)}{\cos^4 x}$

$= \frac{\cos x(\cos^2 x + 2\sin^2 x)}{\cos^4 x} = \frac{1+\sin^2 x}{\cos^3 x}$

$\Rightarrow f'(x) > 0.$

$\therefore f(x)$ is increasing function.

$\lim_{x \to \frac{-\pi}{2}} \left(1 + \frac{\sin x}{\cos^2 x}\right) = -\infty$

and, $\lim_{x \to \frac{\pi}{2}} \left(1 + \frac{\sin x}{\cos^2 x}\right) = \infty$

$\therefore$ Range $= (-\infty, \infty)$

The correct option is (B)

140. For $f(x)$ to be defined,

(1) $[2x^2 - 3] = -1, 0, 1$

$\Rightarrow -1 \le 2x^2 - 3 < 2 \Rightarrow 2 \le 2x^2 < 5$

$\Rightarrow 1 \le x^2 < \frac{5}{2}$

$\Rightarrow \begin{cases} 1 \le x^2 \Rightarrow x \le -1 \text{ or } x \ge 1 \\ x^2 < \frac{5}{2} \Rightarrow -\sqrt{\frac{5}{2}} < x < \sqrt{\frac{5}{2}} \end{cases}$

$\Rightarrow -\sqrt{\frac{5}{2}} < x \le -1 \text{ or } 1 \le x < \sqrt{\frac{5}{2}}$ (1)

(2) $x^2 - 5x + 5 > 0$

$\Rightarrow x < \frac{5-\sqrt{5}}{2} \text{ or } x > \frac{5+\sqrt{5}}{2}$ (2)

(3) $\log_{1/2}(x^2 - 5x + 5) > 0$

$\Rightarrow x^2 - 5x + 5 < \left(\frac{1}{2}\right)^0 \Rightarrow x^2 - 5x + 5 < 1$

$\Rightarrow x^2 - 5x + 4 < 0$

$\Rightarrow 1 < x < 4$ (3)

From Eqs. (1), (2) and (3), $1 \le x < \frac{5-\sqrt{5}}{2}$

The correct option is (D)

141. If $f(x) \ge 0$, then $x + f(x) = 2f(x)$

or, $f(x) = x$

$\therefore f^{-1}(x) = x$, when $f^{-1}(x) \ge 0$ (1)

Also, when $f(x) \le 0$, $x - f(x) = 2f(x)$

or, $f(x) = \frac{x}{3}$

$\therefore f^{-1}(x) = 3x$, when $f^{-1}(x) \le 0$ (2)

Clearly, option (d) satisfies both (1) and (2)

The correct option is (D)

Match the Column Type

142. **I.** $f(x)$ is defined if $|\sin x| + \sin x > 0$

$\Rightarrow \sin x > 0 \Rightarrow 2n\pi < x < 2n\pi + \pi$

$\therefore$ Domain of $f = (2n\pi, (2n+1)\pi)$

The correct option is (B)

II. $f(x)$ is defined if

$-\log_{1/2}\left(1 + \frac{1}{x^{1/5}}\right) - 1 > 0,\ 1 + \frac{1}{x^{1/5}} > 0, x \ne 0$

$\Rightarrow \log_{1/2}\left(1 + \frac{1}{x^{1/5}}\right) < -1,\ x^{1/5} + 1 > 0, x \ne 0$

$\Rightarrow 1 + \frac{1}{x^{1/5}} > \left(\frac{1}{2}\right)^{-1},\ x > (-1)^5, x \ne 0$

$\Rightarrow \frac{1}{x^{1/5}} > 1,\ x > -1 \text{ and } x \ne 0$

$\Rightarrow 0 < x < 1 \text{ and } x > -1 \Rightarrow 0 < x < 1$

$\therefore$ Domain $(f) = (0, 1)$

The correct option is (C)

III. $f(x)$ is defined if

$-(\log_3 x)^2 + 5\log_3 x - 6 > 0$ and $x > 0$

$\Rightarrow (\log_3 x - 3)(2 - \log_3 x) > 0$ and $x > 0$

$\Rightarrow (\log_3 x - 2)(\log_3 x - 3) < 0$ and $x > 0$

$\Rightarrow 2 < \log_3 x < 3$ and $x > 0$

$\Rightarrow 3^2 < x < 3^3 \Rightarrow 9 < x < 27$

Domain of $f = (9, 27)$.

The correct option is (D)

IV. Domain of $\cot^{-1}x$ is R and $\dfrac{x}{\sqrt{x^2 - [x^2]}}$ is defined if $x^2 \neq [x^2]$ $\quad (\because x^2 \geq [x^2])$

$\Rightarrow x^2 \neq 0$ or positive integer.

Hence, domain $= R - \{\sqrt{n} : n \geq 0, n \in I\}$.

The correct option is (A)

143. I. $f(x)$ is defined if $3x^2 - 4x + 5 \geq 0$

$$\Rightarrow 3\left[x^2 - \frac{4}{3}x + \frac{5}{3}\right] \geq 0 \Rightarrow 3\left[\left(x - \frac{2}{3}\right)^2 + \frac{11}{9}\right] \geq 0$$

which is true for all real x

$\therefore$ Domain $(f) = (-\infty, \infty)$

Let $y = \sqrt{3x^2 - 4x + 5}$

$\Rightarrow y^2 = 3x^2 - 4x + 5$ i.e., $3x^2 - 4x + (5 - y^2) = 0$

For x to be real, $16 - 12(5 - y^2) \geq 0 \Rightarrow y \geq \sqrt{\dfrac{11}{3}}$

$\therefore$ Range of $y = \left[\sqrt{\dfrac{11}{3}}, \infty\right)$

The correct option is (C)

II. $f(x)$ is defined if $3x^2 - 4x + 5 > 0$

$$\Rightarrow 3\left[x^2 - \frac{4}{3}x + \frac{5}{3}\right] > 0 \Rightarrow 3\left[\left(x - \frac{2}{3}\right)^2 + \frac{11}{9}\right] > 0,$$

which is true for all real x.

$\therefore$ Domain $(f) = (-\infty, \infty)$

Let, $y = \log_e(3x^2 - 4x + 5) \quad \Rightarrow e^y = 3x^2 - 4x + 5$

$\Rightarrow 3x^2 - 4x + (5 - e^y) = 0$

For x to be real,

$16 - 12(5 - e^y) \geq 0 \Rightarrow 12e^y \geq 44 \Rightarrow e^y \geq \dfrac{11}{3}$

$\Rightarrow y \geq \log_e \dfrac{11}{3}$

Range of $f = \left[\log_e \dfrac{11}{3}, \infty\right)$

The correct option is (A)

III. $f(x)$ is defined if $x^2 + x - 6 \neq 0$

i.e., $(x + 3)(x - 2) \neq 0$ i.e., $x \neq -3, 2$

$\therefore$ Domain $(f) = (-\infty, \infty)\backslash\{-3, 2\}$

Let $y = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$

$\Rightarrow x^2y + xy - 6y = x^2 - 3x + 2$

$\Rightarrow x^2(y - 1) + x(y + 3) - (6y + 2) = 0$

For x to be real, $(y + 3)^2 + 4(y - 1)(6y + 2) \geq 0$

$\Rightarrow 25y^2 - 10y + 1 \geq 0$ i.e., $(5y - 1)^2 \geq 0$

which is true for all real y.

$\therefore$ Range of $f = (-\infty, \infty)$.

The correct option is (B)

IV. For $f(x)$ to be defined,

$\dfrac{\sqrt{4 - x^2}}{1 - x} > 0$, $4 - x^2 > 0$ and $1 - x \neq 0$

Since $\sqrt{4 - x^2} \not< 0$,

$\therefore$ we have $1 - x > 0$ and $4 - x^2 > 0$

$\Rightarrow x < 1$ and $(x - 2)(x + 2) < 0$

$\Rightarrow x < 1$ and $-2 < x < 2$

$\Rightarrow -2 < x < 1$

$\therefore$ Domain of $f = (-2, 1)$

Since $-\infty < \log\left(\dfrac{\sqrt{4 - x^2}}{1 - x}\right) < \infty$

$$\Rightarrow -1 \leq \sin\left[\log\left(\frac{\sqrt{4 - x^2}}{1 - x}\right)\right] \leq 1$$

$\therefore$ Range of $f = [-1, 1]$.

The correct option is (D)

144. I. We have,

$y = \log_{\sqrt{5}}\left\{\sqrt{2}(\sin x - \cos x) + 3\right\}$

$= \log_{\sqrt{5}}\left\{2\sin\left(x - \dfrac{\pi}{4}\right) + 3\right\}$

which is defined for values of x such that

$2\sin\left(\pi - \dfrac{\pi}{4}\right) + 3 > 0$

which is true $\forall\, x \in R$

Now, we have

$-2 \leq 2\sin\left(x - \dfrac{\pi}{4}\right) \leq 2$

i.e., $\quad 1 \leq 2\sin\left(x - \dfrac{\pi}{4}\right) + 3 \leq 5$

i.e., $\quad 0 \leq \log_{\sqrt{5}}\left\{2\sin\left(x - \dfrac{\pi}{4}\right) + 3\right\} \leq \log_{\sqrt{5}} 5$

i.e., $\quad 0 \leq y \leq 2$

Hence, the range is $y \in [0, 2]$

The correct option is (C)

II. The function is defined for values of x such that

$2 - \log_{\sqrt{5}}(16\sin^2 x + 1) > 0$

Also, we have

$2 - \log_{\sqrt{5}}(16\sin^2 x + 1) \leq 2$

$[\because \log_{\sqrt{5}}(16\sin^2 x + 1) \geq 0]$

Together, we have

$0 < 2 - \log_{\sqrt{5}}(16\sin^2 x + 1) \leq 2$

i.e., $-\infty < \log_2\{2 - \log_{\sqrt{5}}(16\sin^2 x + 1)\} \le \log_2 2$

i.e., $-\infty < y \le 1$

Hence, the range is $y \in (-\infty, 1]$

The correct option is (B)

III. We have,

$$y = \frac{e^x - e^x}{e^x + e^{-x}} = 0, x < 0$$

$$= \frac{e^x - e^{-x}}{e^x + e^x} = \frac{1 - e^{-2x}}{2}, x \ge 0$$

Now, we have $\forall\, x \ge 0,\ 0 < e^{-2x} \le 1$

i.e., $-1 \le -e^{-2x} < 0$ i.e., $0 \le 1 - e^{-2x} < 1$

i.e., $0 \le \frac{1 - e^{-2x}}{2} < \frac{1}{2}$ i.e., $0 \le y < \frac{1}{2}$

Hence, the range is $y \in \left[0, \frac{1}{2}\right)$

The correct option is (D)

IV. We have,

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} + 1 - 2}{e^{2x} + 1} = 1 - \frac{2}{1 + e^{2x}}$$

Now, we have $\forall\, x \ge 0,\ 2 \le 1 + e^{2x} < \infty$

i.e., $\frac{1}{2} \ge \frac{1}{1 + e^{2x}} > 0$

i.e., $-1 \le \frac{-2}{1 + e^{2x}} < 0$

i.e., $-1 + 1 \le 1 - \frac{2}{1 + e^{2x}} < 0 + 1$

Hence, the range is $y \in [0, 1)$

The correct option is (A)

Assertion-Reasoning Type

145. We have,

$$3f(x) - f\left(\frac{1}{x}\right) = 4\ln x$$

Putting $1/x$ in place of x, we have

$3f(1/x) - f(x) = -4\ln x$

Solving the above equations, we have

$f(x) = \ln x \Rightarrow f(e^x) = x$

Hence, required area is

$$\int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

The correct option is (A)

146. Given, $f(x) = \frac{a^x}{a^x + \sqrt{a}}$ (1)

Now, $f(1 - x) = \frac{a^{1-x}}{a^{1-x} + \sqrt{a}} = \frac{\sqrt{a}}{\sqrt{a} + a^x}$ (2)

From (1) and (2), we have $f(x) + f(1 - x) = 1$ (3)

$$\Rightarrow f\left(\frac{r}{2n}\right) + f\left(\frac{2n - r}{2n}\right) = 1$$

$$\Rightarrow \sum_{r=1}^{2n-1} f\left(\frac{r}{2n}\right) + \sum_{r=1}^{2n-1} f\left(\frac{2n - r}{2n}\right) = 2n - 1$$

$$\Rightarrow \sum_{r=1}^{2n-1} f\left(\frac{r}{2n}\right) + \sum_{r=1}^{2n-1} f\left(\frac{t}{2n}\right) = 2n - 1$$

(Putting $2n - r = t$)

Hence, $2\sum_{r=1}^{2n-1} f\left(\frac{r}{2n}\right) = 2n - 1$

The correct option is (A)

147. We have,

$g(x) = \sqrt{-x^2 + 4x - 3}$ and

$$h(x) = \sqrt{\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}(x-1)\right)\right)}$$

Since $-x^2 + 4x - 3 = 1 - (x - 2)^2$, maximum value of $g = g(2) = 1$.

Also, $g(1) = 0$

Therefore, minimum value of $g = g(1) = 0$

Now, $h(2) = 1$ and $h(1) = 0$

Hence, maximum and minimum values of both g and h are attained at 2 and 1, respectively. Further, g and h are both continous in [1, 2] Hence, Range of $f = [f(1), f(2)] = [0, 2]$

The correct option is (A)

148. Domain of $g(x)$: $g(x)$ is defined if

$3 - x \ge 0$ and $(x - 1)(x - 2)(x - 3) \ne 0$

$\Rightarrow x \le 3$ and $x \ne 1, 2, 3$

$\therefore$ Domain of $g(x) = (-\infty, 3) - \{1, 2, 3\}$

Domain of $h(x)$:

$$h(x) = \sin^{-1}\left[\frac{3x - 2}{2}\right] \Rightarrow -1 \le \left[\frac{3x - 2}{2}\right] \le 1$$

Case I:

If $\left[\frac{3x - 2}{2}\right] = -1 \Rightarrow -1 \le \frac{3x - 2}{2} < 0$

$\Rightarrow -2 \le 3x - 2 < 0$

$\therefore$ $\mathbf{0 \le x < \frac{2}{3}}$ (1)

Case II:

If $\left[\frac{3x-2}{2}\right]=0$

$\Rightarrow\ 0\le\frac{3x-2}{2}<1\Rightarrow 0\le 3x-2<2$

$\Rightarrow\ 2\le 3x<4$

$\therefore \frac{2}{3}\le x<\frac{4}{3}$ (2)

Case III:

If $\left[\frac{3x-2}{2}\right]=1\Rightarrow 1\le\frac{3x-2}{2}<2\Rightarrow 2\le 3x-2<4$

$\therefore\ \frac{4}{3}\le x<2$ (3)

Thus, from (1), (2) and (3), we have

Domain of $h(x)=[0,2)$

$\therefore$ Domain of $f=[0,2)-\{1\}$

The correct option is (A)

149. Since, $g(x)$ is a function whose graph is the reflection of the graph of $f(x)$ in the line $y=x$

$\Rightarrow$ $g(x)$ is the inverse of $f(x)$ by definition

i.e., $g(x)=f^{-1}(x)$

Now, let $y=f(x)=(x+1)^2,\ \forall\ x\ge -1$ {given}

$\Rightarrow\ (x+1)=\pm\sqrt{y},\ y\ge 0$

$\Rightarrow\ x=-1\pm\sqrt{y}$

$\Rightarrow$ either $x=-1+\sqrt{y}$ or $x=-1-\sqrt{y}$

$\Rightarrow\ f^{1}(y)=\sqrt{y}-1$ Not possible, $\because\ x\ge -1$ (given)

$\therefore\ \boldsymbol{f^{-1}(x)=\sqrt{x}-1=g(x)},\ x\ge 0$

The correct option is (A)

Previous Year's Questions

150. Key Idea : Period of the functions $\sin\theta$ and $\cos\theta$ is 2π.

Since, $\sin^2\theta=\frac{1-\cos 2\theta}{2}=\frac{1}{2}-\frac{1}{2}\cos 2\theta$

$\therefore$ Period of $\sin^2\theta=\frac{2\pi}{2}=\pi$

The correct option is (B)

151. Key Idea : Domain of inverse function $\sin^{-1}x=[-1,1]$ and range of $\sin^{-1}x=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Since, domain of $\sin^{-1}x=[-1,1]$

$\therefore -1\le\log_3\left(\frac{x}{3}\right)\le 1$

$\Rightarrow 3^{-1}\le\frac{x}{3}\le 3$

$\Rightarrow 1\le x\le 9$

$\therefore$ Domain of $\sin^{-1}\left[\log_3\left(\frac{x}{3}\right)\right]$ is [1, 9].

The correct option is (A)

152. $\because\ f(x)=\sin^4x+\cos^4x$

$=(\sin^2x+\cos^2x)^2-2\sin^2x\cos^2x$

$=1-\frac{1}{2}(2\sin x\cos x)^2$

$=1-\frac{1}{2}(\sin 2x)^2=\frac{3}{4}+\frac{1}{4}\cos 4x$

$\because$ $\cos x$ is periodic with period 2π.

$\therefore$ The period of function $f(x)=\frac{2\pi}{4}=\frac{\pi}{2}$.

The correct option is (B)

153. Clearly the function is both one-to-one and onto

Because if n is odd, values are set of all non-negative integers and if n is an even, values are set of all negative integers.

The correct option is (C)

154. $4-x^2\ne 0$

$\Rightarrow x\ne\pm 2$

And, $x^3-x>0$

$\Rightarrow x(x+1)(x-1)>0$.

The correct option is (D)

155. $f(x)={}^{7-x}P_{x-3}$

Now, $7-x\ge 0\Rightarrow x\le 7$

And, $x-3\ge 0\Rightarrow x\ge 3$,

Again, $7-x\ge x-3\Rightarrow x\le 5$

$\Rightarrow 3\le x\le 5\Rightarrow x=3,4,5$

$\Rightarrow$ Range is $\{1,2,3\}$.

The correct option is (A)

156. $-2\le\sin x-\sqrt{3}\cos x\le 2\ \Rightarrow -1\le\sin x-\sqrt{3}\cos x+1\le 3$

$\Rightarrow$ range of $f(x)$ is $[-1,3]$.

Hence the range set S is $[-1,3]$.

The correct option is (D)

157. If the curve $y=f(x)$ is symmetric about the line $x=2$ then $f(2+x)=f(2-x)$.

The correct option is (B)

158. Since $9-x^2>0$ and $-1\le x-3\le 1\Rightarrow x\in[2,3)$

The correct option is (B)

159. Given $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ for $x \in (-1, 1)$

Clearly range of $f(x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore$ co-domain of function $= B = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The correct option is (D)

160. $f(2a - x) = f(a - (x - a)) = f(A)\ f(x-a) - f(0)\,f(x)$
$= -f(x)$ [$\because x = 0, y = 0$ in the given functional equation $f(0)$
$= f^2(0) - f^2(a)$
$\Rightarrow f^2(A) = 0 \Rightarrow f(A) = 0$].
The correct option is (A)

161. $f(x)$ is defined if $-1 \le \frac{x}{2} - 1 \le 1$ and $\cos x > 0$

i.e. if $0 \le x \le 4$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\therefore x \in \left[0, \frac{\pi}{2}\right)$

The correct option is (D)

162. Function is increasing

So, $x = \frac{y-3}{4} = g(y)$

The correct option is (D)

163. Given $f(x) = x^3 + 5x + 1$.
The differential $f'(x) = 3x^2 + 5 > 0, \forall x \in R$
$\therefore f(x)$ is strictly increasing function
$\therefore$ It is one-one
Clearly, $f(x)$ is a continuous function and also increasing on R,

$\underset{x\to-\infty}{Lt} f(x) = -\infty$ and $\underset{x\to\infty}{Lt} f(x) = \infty$

$\therefore f(x)$ takes every value between $-\infty$ and ∞.
Thus, $f(x)$ is onto function.
The correct option is (C)

164. $\frac{1}{\sqrt{|x|-x}} \Rightarrow |x| - x > 0 \Rightarrow |x| > x \Rightarrow x$ is negative
$x \in (-\infty, 0)$
The correct option is (B)

165. $f(x) + 2f(1/x) = 3x$ (1)

$x \to \frac{1}{x} \Rightarrow f(1/x) + 2f(x) = 3/x$ (2)

$f(x) + 2\left(\frac{3}{x} - 2f(x)\right) = 3x$

$\Rightarrow 3f(x) = \frac{6}{x} - 3x$

$\Rightarrow f(x) = \frac{2}{x} - x$

For S, $f(x) = f(-x) \Rightarrow \frac{2}{x} - x = 0$

$\Rightarrow x = \pm\sqrt{2}$
The correct option is (D)