CHAPTER 13

Differentiation

Chapter Highlights

Derivative of a function, Derivative at a point, Standard derivatives, Rules for differentiation, Derivative of parametric functions, Derivative of implicit functions, Differentiation of a function with respect to another function, Logarithmic differentiation, Successive differentiation

DERIVATIVE OF A FUNCTION

Let $y = f(x)$ be a function defined on the interval $[a, b]$. Let for a small increment δx in x, the corresponding increment in the value of y be δy. Then

$$y = f(x) \text{ and } y + \delta y = f(x + \delta x)$$

On subtraction, we get

$$\delta y = f(x + \delta x) - f(x)$$

or
$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Taking limit on both sides when $\delta x \to 0$ we have,

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

if this limit exists, is called the *derivative* or *differential coefficient* of y with respect to x and is written as $\frac{dy}{dx}$ or $f'(x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

IMPORTANT POINTS

The instantaneous rate of change of $f(x)$ at x, which is written as $f'(x)$ or $\frac{df}{dx}$ is called the derivative of f at x. Geometrically, it represents the slope of the tangent at the point (x, y) on the curve $y = f(x)$.

DERIVATIVE AT A POINT

The value of $f'(x)$ obtained by putting $x = a$, is called the derivative of $f(x)$ at $x = a$ and it is denoted by $f'(a)$ or $\left\{\frac{dy}{dx}\right\}_{x=a}$.

STANDARD DERIVATIVES

Algebraic Functions

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

   $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \frac{d}{dx} f(x)$

2. $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -nx^{-n-1}$

   $\frac{d}{dx}\left(\frac{1}{f(x)}\right)^n = \frac{-n}{[f(x)]^{n-1}} \frac{d}{dx} f(x)$

3. $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

   $\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \frac{d}{dx} f(x)$

SOLVED EXAMPLES

1. If S_n denotes the sum of n terms of a G.P. whose common ratio is r, then $(r-1)\frac{dS_n}{dr}$ is equal to

(A) $(n-1)\,S_n + n\,S_{n-1}$ (B) $(n-1)\,S_n - n\,S_{n-1}$
(C) $(n-1)\,S_n$ (D) None of these

Solution: (B)

We have,

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$\Rightarrow \quad (r-1)\,S_n = ar^n - a$$

Differentiating both sides with respect to r, we get

$$(r-1)\,\frac{dS_n}{dr} + S_n = nar^{n-1} - 0$$

$$\Rightarrow \quad (r-1)\,\frac{dS_n}{dr} = nar^{n-1} - S_n$$

$$= n\,(n\text{th term of G.P.}) - S_n$$

$$= n\,(S_n - S_{n-1}) - S_n$$

$$= (n-1)\,S_n - n\,S_{n-1}.$$

2. If $y^2 = P(x)$, a polynomial of degree $n \geq 3$, then

$$2\,\frac{d}{dx}\left(y^3\,\frac{d^2y}{dx^2}\right) =$$

(A) $-P(x)\cdot P'''(x)$ (B) $P(x) \times P'''(x)$
(C) $P(x)\cdot P''(x)$ (D) None of these

Solution: (B)

We have,

$$y^2 = P(x) \qquad (1)$$

$$\Rightarrow \quad 2y\,\frac{dy}{dx} = P'(x) \qquad (2)$$

$$\Rightarrow \quad 2\,\frac{dy}{dx}\cdot\frac{dy}{dx} + 2y\cdot\frac{d^2y}{dx^2} = P''(x)$$

$$\Rightarrow \quad 2y^2\left(\frac{dy}{dx}\right)^2 + 2y^3\cdot\frac{d^2y}{dx^2} = y^2\,P''(x)$$

$$\Rightarrow \quad 2y^3\,\frac{d^2y}{dx^2} = y^2\,P''(x) - 2y^2\left(\frac{dy}{dx}\right)^2$$

$$= y^2\,P''(x) - \frac{1}{2}\,[P'(x)]^2$$

[from (2)]

$$\Rightarrow \quad 2\,\frac{d}{dx}\left(y^3\,\frac{d^2y}{dx^2}\right)$$

$$= 2y\,\frac{dy}{dx}\,P''(x) + y^2\,P'''(x) - \frac{1}{2}\,2P'(x) \times P''(x)$$

$$= P'(x)\,P''(x) + y^2\,P'''(x) - P'(x)\,P''(x)$$

$$\left[\because 2y\frac{dy}{dx} = P'(x)\right]$$

$$= y^2\,P'''(x) = P(x)\,P'''(x). \qquad [\because y^2 = P(x)]$$

3. Let f be a twice differentiable function such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x) = [f(x)]^2 + [g(x)]^2$ and $h(5) = 11$ then $h(10) =$

(A) 11 (B) 0
(C) -1 (D) None of these

Solution: (A)

We have,

$$h(x) = [f(x)]^2 + [g(x)]^2$$

Differentiating both sides with respect to x, we get

$$h'(x) = 2f(x) \times f'(x) + 2g(x)\,g'(x)$$

$$= 2f(x)\,g(x) + 2g(x)\,g'(x)$$

$$[\because f'(x) = g(x)]$$

$$= 2\,g(x)\,[f(x) + g'(x)] \qquad (1)$$

But $\quad g(x) = f'(x) \Rightarrow g'(x) = f''(x) = -f(x)$

$$[\because f''(x) = -f(x)]$$

$\therefore$ From (1),

$$h'(x) = 2\,g(x)\,[f(x) - f(x)] = 0$$

$\Rightarrow \quad h(x) =$ constant for all x.

Given $\quad h(5) = 11,$

$\therefore \quad h(10) = 11$

4. A function $f(x)$ is so defined that for all x, $[f(x)]^n = f(nx)$. If $f'(x)$ denotes derivative of $f(x)$ with respect to x, then $f'(x) \times f(nx) =$

(A) $f(x)$ (B) 0
(C) $f(x)\cdot f'(nx)$ (D) None of these

Solution: (C)

We have,

$$[f(x)]^n = f(nx)$$

Differentiating with respect to x, we get

$$n\,[f(x)]^{n-1} \times f'(x) = f'(nx)\cdot n$$

$$\Rightarrow \quad [f(x)]^{n-1}\cdot f'(x) = f'(nx)$$

$\Rightarrow \quad [f(x)]^n \cdot f'(x) = f'(nx) \cdot f(x)$

[Multiplying both sides by $f(x)$]

$\Rightarrow \quad f(nx) \cdot f'(x) = f'(nx) \cdot f(x)$

$[\because\ [f(x)]^n = f(nx)]$

5. If $f(x) = (1 - x)^n$, then the value of

$$f(0) + f'(0) + \frac{f''(0)}{2!} + \dots + \frac{f^n(0)}{n!} \text{ is equal to}$$

(A) 2^n (B) 0
(C) 2^{n-1} (D) None of these

Solution: (B)

We have,

$$f(x) = (1-x)^n, f'(x) = -n(1-x)^{n-1},$$

$$f''(x) = n(n-1)(1-x)^{n-2},$$

$$f'''(x) = -n(n-1)(n-2)(1-x)^{n-3}\dots$$

$$f^n(x) = (-1)^n n(n-1)(n-2)\dots 1$$

$\Rightarrow \quad f(0) = 1, f'(0) = -n, f''(0) = n(n-1),$

$$f'''(0) = -n(n-1)(n-2)\dots f^n(0) = (-1)^n n!$$

Therefore,

$$f(0) + f'(0) + \frac{f''(0)}{2!} + \dots + \frac{f^n(0)}{n!}$$

$$= 1 - n + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \dots + \frac{(-1)^n n!}{n!}$$

$$= {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots + (-1)^n\, {}^nC_n$$

$$= (1-1)^n = 0.$$

6. If $f'(x) = \phi(x)$ and $\phi'(x) = f(x)$ for all x. Also, $f(3) = 5$ and $f'(3) = 4$. Then the value of $[f(10)]^2 - [\phi(10)]^2$ is

(A) 0 (B) 9
(C) 41 (D) None of these

Solution: (B)

$$\frac{d}{dx}\{[f(x)]^2 - [\phi(x)]^2\} = 2[f(x)\cdot f'(x) - \phi(x)\cdot \phi'(x)]$$

$$= 2[f(x)\cdot\phi(x) - \phi(x)\cdot f(x)]$$

$[\because f'(x) = \phi(x) \text{ and } \phi'(x) = f(x)]$

$$= 0$$

$\Rightarrow \quad [f(x)]^2 - [\phi(x)]^2 = \text{constant}$

$\therefore \quad [f(10)]^2 - [\phi(10]^2 = [f(3)]^2 - [\phi(3)]^2$

$$= [f(3)]^2 - [f'(3)]^2$$

$$= 25 - 16 = 9$$

7. If $8f(x) + 6f\left(\frac{1}{x}\right) = x + 5$ and $y = x^2 f(x)$, then $\frac{dy}{dx}$ at $x = -1$ is equal to

(A) 0 (B) $\frac{1}{14}$
(C) $-\frac{1}{14}$ (D) None of these

Solution: (C)

We have,

$$8f(x) + 6f\left(\frac{1}{x}\right) = x + 5 \text{ for all } x \qquad (1)$$

Therefore,

$$8f\left(\frac{1}{x}\right) + 6f(x) = \frac{1}{x} + 5 \qquad (2)$$

$\left(\text{Putting } x = \frac{1}{x}\right)$

From (1) and (2), we have

$$f(x) = \frac{1}{28}\left(8x - \frac{6}{x} + 10\right)$$

$\therefore \quad y = x^2 f(x) = \frac{1}{28}(8x^3 - 6x + 10x^2)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{28}(24x^2 + 20x - 6)$

$\therefore \quad \left.\frac{dy}{dx}\right]_{x=-1} = \frac{1}{28}(24 - 20 - 6) = -\frac{1}{14}$

8. Let $f(x)$ be a polynomial function satisfying $f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$. If $f(4) = 65$ and l_1, l_2, l_3 are in G.P. then $f'(l_1), f'(l_2), f'(l_3)$ are in

(A) A.P. (B) G.P.
(C) H.P. (D) None of these

Solution: (B)

Since $f(x)$ is a polynomial function satisfying

$$f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right),$$

$\therefore \quad f(x) = x^n + 1 \text{ or } f(x) = -x^n + 1$

If $\quad f(x) = -x^n + 1$, then $f(4) = -4^n + 1 \neq 65$

So, $\quad f(x) = x^n + 1.$

Since $\quad f(4) = 65 \therefore 4^n + 1 = 65 \Rightarrow n = 3$

$\therefore \quad f(x) = x^3 + 1 \Rightarrow f'(x) = 3x^2$

$\therefore\ f'(l_1) = 3l_1^2, f'(l_2) = 3l_2^2, f'(l_3) = 3l_3^2$

Since l_1, l_2, l_3 are in G.P.,

$\therefore\quad f'(l_1), f'(l_2), f'(l_3)$ are also in G.P.

9. Let $f(x) = (x^3 + 2)^{30}$. If $f^n(x)$ is a polynomial of degree 20, where $f^n(x)$ denotes the nth derivative of $f(x)$ with respect to x, then the value of n is

(A) 60 (B) 40
(C) 70 (D) None of these

Solution: (C)

$f(x)$ is a polynomial of degree 90. $f'(x)$ reduces the degree of $f(x)$ by one. Thus, in order to get a polynomial of degree 20, we must reduce the degree of $f(x)$ by 70. Hence, $f(x)$ should be differentiated 70 times to get a polynomial of degree 20.

$$\therefore\quad n = 70.$$

10. If $f(x) + f(y) + f(z) + f(x)\cdot f(y)\cdot f(z) = 14$ for all $x, y, z \in R$, then

(A) $f(0) = 2$
(B) $f'(x) = 0$, for all $x \in R$
(C) $f'(x) > 0$, for all $x \in R$
(D) None of these

Solution: (A, B)

We have,

$$f(x) + f(y) + f(z) + f(x)\cdot f(y)\cdot f(z) = 14 \quad (1)$$

for all $x, y, z \in R$

Putting $x = y = z = 0$, we get

$$3f(0) + [f(0)]^3 = 14$$

$$\Rightarrow\quad [f(0)]^3 + 3f(0) - 14 = 0$$

$$\Rightarrow\quad f(0) = 2.$$

Now, putting $y = z = x$ in (1), we get

$$3f(x) + [f(x)]^3 = 14$$

Differentiating with respect to x, we get

$$3f'(x) + 3[f(x)]^2\cdot f'(x) = 0$$

$$\Rightarrow\quad 3f'(x)\{1 + [f(x)]^2\} = 0$$

$$\Rightarrow\quad f'(x) = 0, \text{ for all } x.$$

11. Let $f(x)$ be a polynomial of degree 3 such that $f(3) = 1$, $f'(3) = -1$, $f''(3) = 0$ and $f'''(3) = 12$. Then the value of $f'(1)$ is

(A) 12 (B) 23
(C) – 13 (D) None of these

Solution: (B)

Let $f(x) = a(x-3)^3 + b(x-3)^2 + c(x-3) + d$

Then,

$$f(3) = 1 = d \Rightarrow d = 1$$

$$f'(3) = -1 = c \Rightarrow c = -1$$

$$f''(3) = 0 = 2b \Rightarrow b = 0$$

$$f'''(3) = 12 = 6a \Rightarrow a = 2.$$

$$\therefore\quad f(x) = 2(x-3)^3 - x + 4 \Rightarrow f'(x) = 6(x-3)^2 - 1$$

$$\therefore\quad f'(1) = 6(4) - 1 = 23.$$

12. A triangle has two of its vertices at $P(a, 0)$, $Q(0, b)$ and the third vertex $R(x, y)$ is moving along the straight line $y = x$. If A be the area of the triangle, then $\frac{dA}{dx} =$

(A) $\frac{a-b}{2}$ (B) $\frac{a-b}{4}$

(C) $\frac{a+b}{2}$ (D) $\frac{a+b}{4}$

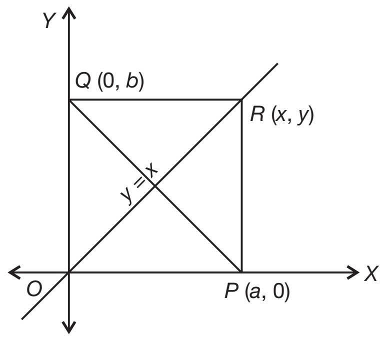
Solution: (C)

Area of $\Delta PQR = A$

$$= \frac{1}{2}[x(b-0) + 0(0-y) + a(y-b)]$$

$$= \frac{1}{2}(bx + ax - ab) \text{ (As } y = x)$$

$$\therefore\quad \frac{dA}{dx} = \frac{1}{2}(a+b)$$



Exponential Functions

1. $\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}e^{[f(x)]} = \{e^{[f(x)]}\}\frac{d}{dx}f(x)$

2. $\frac{d}{dx}(a^x) = a^x \log_e a$

$\frac{d}{dx}a^{[f(x)]} = \{a^{[f(x)]}\log_e a\}\frac{d}{dx}f(x)$

SOLVED EXAMPLE

13. Let $f(x) = 2^{2x-1}$ and $\phi(x) = -2^x + 2x \log 2$. If $f'(x) > \phi'(x)$, then

(A) $0 < x < 1$ (B) $0 \le x < 1$
(C) $x > 0$ (D) $x \ge 0$

Solution: (C)

Since $f'(x) > \phi'(x)$

$\Rightarrow$ $2^{2x-1}\, 2 \log 2 > -2^x \log 2 + 2 \log 2$

$\Rightarrow$ $2^{2x} > -2^x + 2$

$\Rightarrow$ $2^{2x} + 2^x - 2 > 0$

$\Rightarrow$ $(2^x - 1)(2^x + 2) > 0$

$\Rightarrow$ $2^x - 1 > 0$ ($\because$ $2^x + 2 > 0$ for all x)

$\Rightarrow$ $2^x > 1$

$\therefore$ $x > 0$.

Logarithmic Functions

1. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

$\frac{d}{dx}[\log_e f(x)] = \frac{1}{f(x)}\frac{d}{dx}f(x)$

2. $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}$

$\frac{d}{dx}[\log_a f(x)] = \frac{1}{f(x)\log_e a}\frac{d}{dx}f(x)$

Trigonometric Functions

1. $\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}\sin[f(x)] = \cos[f(x)]\frac{d}{dx}f(x)$

2. $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}\cos[f(x)] = -\sin[f(x)]\frac{d}{dx}f(x)$

3. $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}\tan[f(x)] = \sec^2[f(x)]\frac{d}{dx}f(x)$

4. $\frac{d}{dx}(\cot x) = -\text{cosec}^2(x)$

$\frac{d}{dx}\cot[f(x)] = -\text{cosec}^2[f(x)]\frac{d}{dx}f(x)$

5. $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}\sec[f(x)] = \sec[f(x)]\tan[f(x)]\frac{d}{dx}f(x)$

6. $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

$\frac{d}{dx}\text{cosec}[f(x)] = -\text{cosec}[f(x)]\cot[f(x)]\frac{d}{dx}f(x)$

SOLVED EXAMPLES

14. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(A) 1 (B) -1
(C) 0 (D) None of these

Solution: (A)

When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$

$\therefore$ $\cos x - \sin x > 0$

When $\frac{\pi}{4} < x < \frac{\pi}{2}$, $\cos x < \sin x$

$\therefore$ $\cos x - \sin x < 0$

When $\frac{\pi}{2} < x < \pi$, $\cos x - \sin x < 0$

$\therefore$ When $\frac{\pi}{4} < x < \pi$, $\cos x - \sin x < 0$

$\therefore |\cos x - \sin x| = -(\cos x - \sin x)$, when $\frac{\pi}{4} < x < \pi$

$\therefore \quad f(x) = -\cos x + \sin x$, when $\frac{\pi}{4} < x < \pi$

$\Rightarrow \quad f'(x) = \sin x + \cos x$

$\Rightarrow \quad f'\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1$

15. If $f(x) = (ax + b)\sin x + (cx + d)\cos x$, then the values of a, b, c and d such that $f'(x) = x\cos x$ for all x are

(A) $b = c = 0, a = d = 1$ (B) $b = d = 0, a = c = 1$
(C) $c = d = 0, a = b = 1$ (D) None of these

Solution: (A)

We have,

$$f'(x) = a\sin x + (ax + b)\cos x + c\cos x - (cx + d)\sin x$$

But $\quad f'(x) = x\cos x$ for all x (given).

$\therefore x\cos x = (a - d)\sin x + (b + c)\cos x + ax\cos x - cx\sin x$

Equating the coefficients of $\sin x$, $\cos x$, $x\cos x$ and $x\sin x$, we get

$$a - d = 0,\ b + c = 0,\ a = 1,\ c = 0$$

$\therefore \quad b = c = 0$ and $a = d = 1$

16. Let $f(x) = \sin x$, $g(x) = 2x$ and $h(x) = \cos x$. If $\phi(x) = [go(fh)](x)$, then $\phi''\left(\frac{\pi}{4}\right)$ is equal to

(A) 4 (B) 0
(C) – 4 (D) None of these

Solution: (C)

We have,

$$(fh)(x) = f(x)\cdot h(x) = \sin x\cos x$$

$\therefore [go(fh)](x) = g[(fh)(x)] = g[f(x)\cdot h(x)]$

$= g(\sin x\cos x) = 2\sin x\cos x = \sin 2x$

i.e., $\quad \phi(x) = \sin 2x$

$\Rightarrow \quad \phi'(x) = 2\cos 2x$ and $\phi''(x) = -4\sin 2x$

$\therefore \quad \phi''\left(\frac{\pi}{4}\right) = -4\sin\frac{\pi}{2} = -4.$

17. If $y = \sin^{-1}\left[\sqrt{x - ax} - \sqrt{a - ax}\right]$, then $\frac{dy}{dx} =$

(A) $\frac{1}{\sin\sqrt{a - ax}}$ (B) $\sin\sqrt{x}\cdot\sin\sqrt{a}$

(C) $\frac{1}{2\sqrt{x}\sqrt{1-x}}$ (D) Zero

Solution: (C)

Put $\quad x = \sin^2\theta$ and $a = \sin^2\phi$, then

$y = \sin^{-1}(\sin\theta\cos\phi - \cos\theta\sin\phi]$

$= \sin^{-1}[\sin(\theta - \phi)] = \theta - \phi$

$= \sin^{-1}\sqrt{x} - \sin^{-1}\sqrt{a}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$

18. If $y = \sin x$, then $\frac{d^2}{dy^2}(\cos^7 x)$ is equal to

(A) $35\cos^3 x - 42\cos^5 x$
(B) $35\cos^3 x + 42\cos^5 x$
(C) $42\cos^3 x - 35\cos^5 x$
(D) None of these

Solution: (A)

We have, $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x.$

Now, $\frac{d^2}{dy^2}(\cos^7 x) = \frac{d}{dy}\left(\frac{d}{dy}\cos^7 x\right)$

$= \frac{d}{dy}\left(7\cos^6 x\cdot(-\sin x)\frac{dx}{dy}\right)$

$= \frac{d}{dy}(-7\sin x\cos^5 x)$

$= -7[\cos x \times \cos^5 x - 5\cos^4 x \sin^2 x]\frac{dx}{dy}$

$= -42\cos^5 x + 35\cos^3 x$

Inverse Circular Functions

1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$; $|x| < 1$

$\frac{d}{dx}[\sin^{-1} f(x)] = \frac{1}{\sqrt{1 - [f(x)]^2}}\frac{d}{dx}f(x)$; $|f(x)| < 1$

2. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$; $|x| < 1$

$\frac{d}{dx}[\cos^{-1} f(x)] = -\frac{1}{\sqrt{1-[f(x)]^2}}\frac{d}{dx}f(x)$; $|f(x)| < 1$

3. $\frac{d}{dx}\tan^{-1} x = \left(\frac{1}{1+x^2}\right)$; $x = \mathbb{R}$

$\frac{d}{dx}[\tan^{-1} f(x)] = \frac{1}{1+[f(x)]^2}\frac{d}{dx}f(x)$; $f(x) \in \mathbb{R}$

4. $\frac{d}{dx}\cot^{-1} x = -\left(\frac{1}{1+x^2}\right)$; $x = \mathbb{R}$

$\frac{d}{dx}[\cot^{-1} f(x)] = -\frac{1}{1+[f(x)]^2}\frac{d}{dx}f(x)$; $f(x) \in \mathbb{R}$

5. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

$\frac{d}{dx}[\sec^{-1} f(x)] = \frac{1}{|f(x)|\sqrt{[f(x)]^2-1}}\frac{d}{dx}f(x)$;

$|f(x)| > 1$

6. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\left(\frac{1}{|x|\sqrt{x^2-1}}\right)$; $|x| > 1$

$\frac{d}{dx}[\operatorname{cosec}^{-1} f(x)] = -\frac{1}{|f(x)|\sqrt{[f(x)]^2-1}}\frac{d}{dx}f(x)$;

$|f(x)| > 1$

SOLVED EXAMPLE

19. If $y = \sum_{r=1}^{x}\tan^{-1}\frac{1}{1+r+r^2}$ then $\frac{dy}{dx}$ is equal to

(A) $\frac{1}{1+x^2}$ (B) $\frac{1}{1+(1+x)^2}$

(C) 0 (D) None of these

Solution: (B)

We have,

$$y = \sum_{r=1}^{x}\tan^{-1}\frac{1}{1+r+r^2} = \sum_{r=1}^{x}\tan^{-1}\left(\frac{(r+1)-r}{1+(r+1)r}\right)$$

$$= \sum_{r=1}^{x}\left[\tan^{-1}(r+1)-\tan^{-1} r\right]$$

$$= [\tan^{-1}2 - \tan^{-1}1 + \tan^{-1}3 - \tan^{-1}2 + ... + \tan^{-1}x - \tan^{-1}(x-1) + \tan^{-1}(x+1) - \tan^{-1}x]$$

$$= [\tan^{-1}(x+1) - \tan^{-1}1]$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+(x+1)^2}$$

RULES FOR DIFFERENTIATION

1. The derivative of a constant function is zero, i.e.,

$\frac{d}{dx}(c) = 0$

2. The derivative of constant times a function is constant times the derivative of the function, i.e.,

$\frac{d}{dx}[c \cdot f(x)] = c\frac{d}{dx}[f(x)]$

3. The derivative of the sum or difference of two function is the sum or difference of their derivatives, i.e.,

$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}[g(x)]$

Product Rule of Differentiation

The derivative of the product of two functions

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\frac{d}{dx}\cdot[g(x)] + g(x)\cdot\frac{d}{dx}[f(x)]$$

= (first function) × (derivative of second function) + (second function) × (derivative of first function)

Quotient Rule of Differentiation

The derivative of the quotient of two functions

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}[f(x)] - f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

$$= \frac{(\text{denom.} \times \text{derivative of num.}) - (\text{num.} \times \text{derivative of denom.})}{(\text{denominator})^2}$$

SOLVED EXAMPLES

20. If $\frac{d}{dx}\left(\frac{1+x^4+x^8}{1+x^2+x^4}\right) = ax^3 + bx$ then

(A) $a = 4, b = 2$ (B) $a = 4, b = -2$

(C) $a = -2, b = 4$ (D) None of these

Solution: (B)

We have,

$$\frac{d}{dx}\left[\frac{(1+x^2+x^4)(1-x^2+x^4)}{(1+x^2+x^4)}\right] = ax^3 + bx$$

$$\Rightarrow \quad \frac{d}{dx}(1 - x^2 + x^4) = ax^3 + bx$$

$$\Rightarrow \quad -2x + 4x^3 = ax^3 + bx$$

$$\Rightarrow \quad a = 4 \text{ and } b = -2.$$

21. If $y = (1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2n})$, then $\frac{dy}{dx}$ at $x = 0$ is

(A) -1 (B) 1
(C) 0 (D) None of these

Solution: (B)

We have,

$$y = (1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2^n})$$

$$= \frac{(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})}{1-x}$$

$$= \frac{1-x^{2^{n+1}}}{1-x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1-x)\cdot -2^{n+1}\cdot x^{2^{n+1}-1} + (1-x^{2^{n+1}})}{(1-x)^2}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=0} = 1.$$

Derivative of a Function of a Function (Chain Rule)

If y is a differentiable function of t and t is a differentiable function of x i.e., $y = f(t)$ and $t = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

Similarly, if $y = f(u)$, where $u = g(v)$ and $v = h(x)$, then,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

SOLVED EXAMPLES

22. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\frac{dy}{dx}$ is equal to

(A) $\sin\left(\frac{2x-1}{x^2+1}\right)^2\cdot\left(\frac{2+2x+x^2}{(x^2+1)^2}\right)$

(B) $\sin\left(\frac{2x-1}{x^2+1}\right)^2\left(\frac{2+2x-2x^2}{(x^2+1)^2}\right)$

(C) $\sin\left(\frac{2x-1}{x^2+1}\right)^2\cdot\left(\frac{2+2x-x^2}{(x^2+1)^2}\right)$

(D) None of these

Solution: (B)

We have, $y = f\left(\frac{2x-1}{x^2+1}\right)$

$$\Rightarrow \quad \frac{dy}{dx} = f'\left(\frac{2x-1}{x^2+1}\right)\cdot\left[\frac{(x^2+1)2-(2x-1)\cdot 2x}{(x^2+1)^2}\right]$$

$$= \sin\left(\frac{2x-1}{x^2+1}\right)^2\cdot\left[\frac{2+2x-2x^2}{(x^2+1)^2}\right]$$

$$\left[\because f'(x) = \sin x^2, \therefore f'\left(\frac{2x-1}{x^2+1}\right) = \sin\left(\frac{2x-1}{x^2+1}\right)^2\right]$$

23. If g is the inverse of f and $f'(x) = \frac{1}{1+x^3}$, then $g'(x)$ is equal to

(A) $1 + [g(x)]^3$ (b) $\frac{1}{1+[g(x)]^3}$

(C) $[g(x)]^3$ (D) None of these

Solution: (A)

We have, $g = \text{inverse of } f = f^{-1}$

$$\Rightarrow \quad g(x) = f^{-1}(x) \Rightarrow f[g(x)] = x$$

Differentiating with respect to x, we get

$$f'[g(x)] \times g'(x) = 1$$

$$\therefore \quad g'(x) = \frac{1}{f'[g(x)]} = 1 + [g(x)]^3$$

$$\left[\because f'(x) = \frac{1}{1+x^3}, \therefore f'[g(x)] = \frac{1}{1+[g(x)]^3}\right]$$

24. If $u = f(x^3)$, $v = g(x^2)$, $f'(x) = \cos x$ and $g'(x) = \sin x$, then $\frac{du}{dv} =$

(A) $\frac{1}{2}x\cos x^3 \operatorname{cosec} x^2$

(B) $\frac{3}{2}x\cos x^3 \operatorname{cosec} x^2$

(C) $\frac{3}{2}x\sec x^3 \operatorname{cosec} x^2$

(D) None of these

Solution: (B)

We have, $u = f(x^3)$

$$\Rightarrow \quad \frac{du}{dx} = f'(x^3)\cdot 3x^2 = 3x^2\cos x^3$$

Also, $\quad v = g(x^2)$

$$\Rightarrow \quad \frac{dv}{dx} = g'(x^2)\cdot 2x = 2x\sin x^2$$

$$\Rightarrow \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{3x^2\cos x^3}{2x\sin x^2}$$

$$= \frac{3}{2}x\cos x^3 \operatorname{cosec} x^2$$

25. Let f be a function defined for all $x \in R$. If f is differentiable and $f(x^3) = x^5$ for all $x \in R$ $(x \neq 0)$, then the value of $f'(27)$ is

(A) 15 (B) 45
(C) 0 (D) None of these

Solution: (A)

We have, for all x $(x \neq 0)$

$$f(x^3) = x^5$$

Differentiating with respect to x, we get

$$f'(x^3)\times 3x^2 = 5x^4 \Rightarrow f'(x^3) = \frac{5}{3}x^2$$

$$\therefore \quad f'(27) = f'(3^3) = \frac{5}{3}(3)^2 = 15$$

26. If $f(x) = \sin\left(\frac{\pi}{2}[x] - x^5\right)$, $1 < x < 2$ and $[x]$ denotes the greatest integer less than or equal to x, then $f'\left(\sqrt[5]{\frac{\pi}{2}}\right)$ is equal to

(A) $5\left(\frac{\pi}{2}\right)^{4/5}$ (B) $-5\left(\frac{\pi}{2}\right)^{4/5}$

(C) 0 (D) None of these

Solution: (B)

Since, $\quad 1 < x < 2.$

$\therefore \quad [x] = 1$

$$\therefore \quad f(x) = \sin\left(\frac{\pi}{2} - x^5\right) = \cos x^5$$

$$\Rightarrow \quad f'(x) = -\sin x^5 \cdot 5x^4$$

$$\Rightarrow \quad f'\left(\sqrt[5]{\frac{\pi}{2}}\right) = -5\left(\frac{\pi}{2}\right)^{4/5}\cdot \sin\frac{\pi}{2} = -5\left(\frac{\pi}{2}\right)^{4/5}.$$

TRICK(S) FOR PROBLEM SOLVING

If $y = u^n$, where u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = nu^{n-1}\times\frac{du}{dx} \quad \left[\because \frac{dy}{du} = nu^{n-1}\right]$$

DERIVATIVE OF PARAMETRIC FUNCTIONS

Sometimes x and y are separately given as functions of a single variable t (called a parameter) i.e., $x = f(t)$ and $y = g(t)$. In this case,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{dt}$$

SOLVED EXAMPLES

27. Let $y = x^3 - 8x + 7$ and $x = f(t)$. If $\frac{dy}{dt} = 2$ and $x = 3$ at $t = 0$, then $\frac{dx}{dt}$ at $t = 0$ is given by

(A) 1 (B) $\frac{19}{2}$

(C) $\frac{2}{19}$ (D) None of these

Solution: (C)

We have,

$$y = x^3 - 8x + 7 \Rightarrow \frac{dy}{dx} = 3x^2 - 8$$

It is given that when $t = 0$, $x = 3$.

$\therefore$ When $\quad t = 0,\ \frac{dy}{dx} = 3\times 3^2 - 8 = 19.$

Also, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad (1)$$

Since, when $\quad t = 0,\ \frac{dy}{dx} = 19$ and $\frac{dy}{dt} = 2,$

$\therefore$ from (1), $\quad 19 = \frac{2}{dx/dt} \Rightarrow \frac{dx}{dt} = \frac{2}{19}$

28. If $x = f(t)\cos t - f'(t)\sin t$, $y = f(t)\sin t + f'(t)\cos t$ then $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ is equal to

(A) $f(t) - f''(t)$ (B) $[f(t) - f''(t)]^2$
(C) $[f(t) + f''(t)]^2$ (D) None of these

Solution: (C)

$$\frac{dx}{dt} = -f(t)\sin t + f'(t)\cos t - f'(t)\cos t - f''(t)\sin t$$
$$= -[f(t) + f''(t)]\sin t$$

and $$\frac{dy}{dt} = f(t)\cos t + f'(t)\sin t - f'(t)\sin t + f''(t)\cos t$$
$$= [f(t) + f''(t)]\cos t$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = [f(t) + f''(t)]^2$$

29. If $x = \sec\theta - \cos\theta$, $y = \sec^n\theta - \cos^n\theta$, then $(x^2+4)\left(\frac{dy}{dx}\right)^2$ is equal to

(A) $n^2(y^2-4)$ (B) $n^2(4-y^2)$
(C) $n^2(y^2+4)$ (D) None of these

Solution: (C)

$$\frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta = \tan\theta(\sec\theta + \cos\theta)$$
$$\frac{dy}{d\theta} = n[\sec^n\theta\tan\theta + \cos^{n-1}\theta\sin\theta]$$
$$= n\tan\theta(\sec^n\theta + \cos^n\theta)$$
$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{n\tan\theta(\sec^n\theta + \cos^n\theta)}{\tan\theta(\sec\theta + \cos\theta)}$$
$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{n^2(\sec^n\theta + \cos^n\theta)^2}{(\sec\theta + \cos\theta)^2}$$
$$= n^2\,\frac{(\sec^n\theta - \cos^n\theta)^2 + 4}{(\sec\theta - \cos\theta)^2 + 4}.$$
$$= \frac{n^2(y^2+4)}{(x^2+4)}$$
$$\Rightarrow (x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$

DERIVATIVE OF IMPLICIT FUNCTIONS

The derivative of an implicit function, given by the relation $f(x, y) = 0$ in which y is not expressible explicitly in terms of x, can be found by the following steps:

Step 1: Differentiate each term of the equation $f(x, y) = 0$ with respect to x, keeping in mind that $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$; $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$ and so on.

Step 2: Collect the terms containing $\frac{dy}{dx}$ on one side and the terms not involving $\frac{dy}{dx}$ on the other side.

Step 3: Divide by coefficient of $\frac{dy}{dx}$ to get $\frac{dy}{dx}$ as a function of x or y or both.

TRICK(S) FOR PROBLEM SOLVING

Shorter Method for Finding the Derivative of an Implicit Function

Step 1: Take all the terms of the function to be differentiated to the left hand side and put left hand side equal to $\phi(x, y)$.

Step 2.
$$\frac{dy}{dx} = -\frac{\text{derivatie of } \phi(x,y) \text{ w.r.t. } x \text{ treating } y \text{ as constant}}{\text{derivatie of } \phi(x,y) \text{ w.r.t. } y \text{ treating } x \text{ as constant}}.$$

DIFFERENTIATION OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

If $y = f(x)$ and $z = g(x)$, then in order to find the derivative of $f(x)$ with respect to $g(x)$, we use the formula
$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$$

SOLVED EXAMPLES

30. The derivative of $f(\tan x)$ with respect to $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'(\sqrt{2}) = 4$, is

(A) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$
(C) 1 (D) None of these

Solution: (A)

Let $y = f(\tan x)$ and $u = g(\sec x)$

$$\Rightarrow \quad \frac{dy}{dx} = f'(\tan x)\sec^2 x$$

and $$\frac{du}{dx} = g'(\sec x)\cdot\sec x\tan x$$

$$\therefore \quad \frac{dy}{du} = \frac{dy}{dx}\Big/\frac{du}{dx} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x}$$

$$\therefore \quad \left(\frac{dy}{du}\right)_{x=\frac{\pi}{4}} = \frac{f'\left(\tan\frac{\pi}{4}\right)}{g'\left(\sec\frac{\pi}{4}\right)\sin\frac{\pi}{4}}$$

$$= \frac{f'(1)}{g'(\sqrt{2})\cdot\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}\times 2}{4} = \frac{1}{\sqrt{2}}$$

31. If $y = x - x^2$, then the derivative of y^2 with respect to x^2 is

(A) $2x^2 + 3x - 1$ (B) $2x^2 - 3x + 1$
(C) $2x^2 + 3x + 1$ (D) None of these

Solution: (B)

Let $u = y^2$ and $v = x^2$

Then, $\frac{du}{dx} = 2y\frac{dy}{dx} = 2y(1-2x) \quad \left(\because \frac{dy}{dx} = 1 - 2x\right)$

and $\frac{dv}{dx} = 2x$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{2y(1-2x)}{2x} = \frac{(x-x^2)(1-2x)}{x} \quad (\because y = x - x^2)$$

$$= 2x^2 - 3x + 1$$

LOGARITHMIC DIFFERENTIATION

If differentiation of an expression or an equation is done after taking log on both sides, then it is called logarithmic differentiation. This method is useful for the function having following forms:

1. $y = [f(x)]^{g(x)}$
2. $y = \frac{f_1(x).f_2(x)...}{g_1(x).g_2(x)...}$ where $g_i(x) \neq 0$

(where i = 1, 2, 3, ...), $f_i(x)$ and $g_i(x)$ both are differentiable.

(i) **Case I:** $y = [f(x)]^{g(x)}$, where $f(x)$ and $g(x)$ are function of x. To find the derivative of this type of functions we proceed as follows: Let $y = [f(x)]^{g(x)}$. Taking logarithm of both the sides, we have $\log y = g(x).\log f(x)$ and then we differentiate with respect to x.

(ii) **Case II:** $y = \frac{f_1(x).f_2(x)}{g_1(x).g_2(x)}$

Taking logarithm of both the sides, we have
$\log y = \log[f_1(x)] + \log[f_2(x)] - \log[g_1(x)] - \log[g_2(x)]$
and differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} - \frac{g_1'(x)}{g_1(x)} - \frac{g_2'(x)}{g_2(x)}$$

Properties of Logarithms

1. $\log_e(mn) = \log_e m + \log_e n$
2. $\log_e\left(\frac{m}{n}\right) = \log_e m - \log_e n$
3. $\log_e(m)^n = n\log_e m$
4. $\log_e e = 1$
5. $\log_n m = \frac{\log_e m}{\log_e n}$
6. $\log_n m \times \log_m n = 1$.

TRICK(S) FOR PROBLEM SOLVING

Shorter Methods of Finding the Derivative of a Function of the from $[f(x)]^{g(x)}$

If $y = [f(x)]^{g(x)}$, then to find $\frac{dy}{dx}$, in addition to the method discussed above, we can also apply any of the following two methods:

Method 1

Step 1: Express $y = [f(x)]^{g(x)} = e^{g(x)\log f(x)}$ $(\because a^x = e^{x\log a})$

Step 2: Differentiate with respect to x to obtain $\frac{dy}{dx}$.

Method 2

Step 1: Evaluate

A = Differential coefficient of y treating $f(x)$ as constant.

Step 2: Evaluate

B = Differential coefficient of y treating $g(x)$ as constant.

Step 3: $\frac{dy}{dx} = A + B$.

SOLVED EXAMPLES

32. If $\phi(x) = \log_5\log_3 x$, then $\phi'(e)$ is equal to

(A) $e\log 5$ (B) $-e\log 5$
(C) $\frac{1}{e\log 5}$ (D) None of these

Solution: (C)

We have,

$$\phi(x) = \log_5\log_3 x = \log_5\left(\frac{\log x}{\log 3}\right)$$

$$= \log_5(\log x) - \log_5(\log 3)$$

$$= \frac{\log(\log x)}{\log 5} - \log_5(\log 3)$$

$$\phi'(x) = \frac{1}{\log 5}\cdot\frac{1}{\log x}\cdot\frac{1}{x} - 0$$

$$\therefore \quad \phi'(e) = \frac{1}{\log 5}\cdot\frac{1}{\log e}\cdot\frac{1}{e} = \frac{1}{e\log 5}.$$

33. If $f(x) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}$, then $f'(0)$ is equal to

(A) $\left(2\log\frac{a}{b} + \frac{a^2-b^2}{ab}\right)\left(\frac{a}{b}\right)^{a+b}$

(B) $\left(2\log\frac{a}{b} + \frac{b^2-a^2}{ab}\right)\left(\frac{a}{b}\right)^{a+b}$

(C) $\left(2\log\frac{a}{b} + \frac{a^2+b^2}{ab}\right)\left(\frac{a}{b}\right)^{a+b}$

(D) None of these

Solution: (B)

We have,

$$\log f(x) = (a + b + 2x)\,[\log (a + x) - \log (b + x)]$$

Differentiating both sides with respect to x, we get

$$\frac{f'(x)}{f(x)} = 2\,[\log (a + x) - \log (b + x)] + (a + b + 2x)\left[\frac{1}{a+x} - \frac{1}{b+x}\right]$$

$$\Rightarrow\ f'(0) = f(0)\left[2(\log a - \log b) + (a+b)\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$

$$= \left(\frac{a}{b}\right)^{a+b}\left[2\log\frac{a}{b} + \frac{b^2-a^2}{ab}\right]$$

Differentiation of Infinite Series

If y is given in the form of an infinite series of x and we have to find out $\frac{dy}{dx}$, then we remove one or more terms, it does not affect the series.

1. If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots \infty}}}$,

then $\quad y = \sqrt{f(x) + y}$

$$\Rightarrow\quad y^2 = f(x) + y \Rightarrow 2y\frac{dy}{dx} = f'(x) + \frac{dy}{dx};$$

$$\therefore\quad \frac{dy}{dx} = \frac{f'(x)}{2y-1}$$

2. If $y = f(x)^{f(x)^{f(x)^{f(x)}}}$ then $y = f(x)^y$

$$\therefore\ \log y = y \log f(x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{y \cdot f'(x)}{f(x)} + \log f(x).\frac{dy}{dx}$$

$$\therefore\quad \frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1 - y\log f(x)]}$$

3. If $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + \ddots\infty}}$,

then $y = f(x) + \frac{1}{y} \Rightarrow \frac{dy}{dx} = \frac{yf'(x)}{2y - f(x)}$

Differentiation of Inverse Trigonometric Functions

For problems involving inverse trigonometric functions, first try for a suitable substitution to simplify it and then differentiate. If no such substitution is found, then differentiate directly.

IMPORTANT POINTS

Substitutions to Reduce the Function to a Simpler Form

Expressions	Substitutions
$\sqrt{a^2 - x^2}$	Put $x = a \sin \theta$ or $x = a \cos \theta$
$\sqrt{x^2 - a^2}$	Put $x = a \sec \theta$ or $x = a \operatorname{cosec} \theta$
$\sqrt{a^2 + x^2}$	Put $x = a \tan \theta$ or $x = a \cot \theta$
$\frac{a-x}{a+x}$ or $\frac{a+x}{a-x}$	Put $x = a \tan \theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	Put $x = a \cos \theta$

Some Useful Trigonometric and Inverse Trigonometric Transformations

1. $1 + \cos mx = 2\cos^2\frac{mx}{2}$

2. $1 - \cos mx = 2\sin^2\frac{mx}{2}$

3. $\sin mx = \dfrac{2\tan\frac{mx}{2}}{1+\tan^2\frac{mx}{2}}$

4. $\cos mx = \dfrac{1-\tan^2\frac{mx}{2}}{1+\tan^2\frac{mx}{2}} = \dfrac{\cot^2\frac{mx}{2}-1}{\cot^2\frac{mx}{2}+1}$

5. $\tan\left(\frac{\pi}{4}+x\right) = \frac{1+\tan x}{1-\tan x}$

6. $\tan\left(\frac{\pi}{4}-x\right) = \frac{1-\tan x}{1+\tan x}$

7. $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$,
provided $x, y > 0$ and $xy < 1$

8. $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}, \left(\frac{x+y}{1-xy}\right)$
provided $x, y > 0$ and $xy > 1$

9. $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$; if $x, y > 0$

10. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} = \tan^{-1}x + \cot^{-1}x$
$= \sec^{-1}x + \text{cosec}^{-1}x$

11. $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$,
provided $x, y \ge 0$ and $x^2+y^2 \le 1$

12. $\sin^{-1}x \pm \sin^{-1}y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$,
if $x, y \ge 0$ and $x^2+y^2 > 1$

13. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right)$,
if $x, y > 0$ and $x^2+y^2 \le 1$

14. $\cos^{-1}x \pm \cos^{-1}y = \pi - \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right)$,
if $x, y > 0$ and $x^2+y^2 > 1$

15. $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$

16. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$

17. $\tan^{-1}\sqrt{\frac{1-x}{1+x}} = \frac{1}{2}\cos^{-1}x$

SUCCESSIVE DIFFERENTIATION

Let $y = f(x)$ be a function of x, then $\frac{dy}{dx}$ is again a function of x and is called the first derivative of y with respect to x. If the first derivative is differentiable, its derivative is called second derivative of the original function and is denoted by $\frac{d^2y}{dx^2}$ or y_2. If the second derivative is differentiable, its derivative is called the third derivative of the original function and is denoted by $\frac{d^3y}{dx^3}$ or y_3 and so on. This process of differentiating a function more than once is called *successive differentiation.*

Differentiation of a function given in the form of a determinant

If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, then

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ p'(x) & q'(x) & r'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Note that differentiation of a determinant can be done in columns also.

TRICK(S) FOR PROBLEM SOLVING

If α is a repeated root of the equation $f(x) = 0$, repeated n times, then $f(x)$ can be written as

$$f(x) = (x-\alpha)^n g(x) \quad (1)$$

where $g(\alpha) \neq 0$.

Clearly, form (1)

$f(\alpha) = 0, f'(x) = 0, f''(\alpha) = 0, \ldots f^{(n-1)}(\alpha) = 0$

SOLVED EXAMPLES

34. If $f_r(x)$, $g_r(x)$, $h_r(x)$, $r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$, $r = 1, 2, 3$ and

$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$, then $F'(a)$ is equal to

(A) a (B) $-a$
(C) 0 (D) None of these

Solution: (C)

We have,

$$F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

$\therefore F'(a) =$

$$\begin{vmatrix} f_1'(a) & f_2'(a) & f_3'(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1'(a) & g_2'(a) & g_3'(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1'(a) & h_2'(a) & h_3'(a) \end{vmatrix}$$

$= 0 + 0 + 0$ [$\because f_r(a) = g_r(a) = h_r(a), r = 1, 2, 3$] $= 0$

35. If $f(x)$, $g(x)$, $h(x)$ are polynomials in x of degree 2 and $F(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$, then $F'(x)$ is equal to

(A) 1 (B) 0
(C) – 1 (D) None of these

Solution: (B)

$F'(x) =$

$$\begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ f'' & g'' & h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f''' & g''' & h''' \end{vmatrix}$$

$= 0 + 0 + 0 = 0.$

[Since $f(x)$, $g(x)$, $h(x)$ are polynomials of degree 2,

$\therefore \quad f'''(x) = g'''(x) = h'''(x) = 0.$]

36. If f, g, h are differentiable functions of x and

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix}$$

then Δ' (the derivative of Δ with respect to x) is given by

(A) $\begin{vmatrix} f' & g' & h' \\ f & g & h \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$

(B) $\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^2 f'')' & (x^2 g'')' & (x^2 h'')' \end{vmatrix}$

(C) $\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$

(D) None of these

Solution: (C)

We have,

$\Delta =$

$$\begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ x^2 f'' + 4xf' + 2f & x^2 g'' + 4xg' + 2g & x^2 h'' + 4xh' + 2h \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' + 4xf' & x^2 g'' + 4xg' & x^2 h'' + 4xh' \end{vmatrix}$$

(Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$)

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$$ (Applying $R_3 \to R_3 - 4R_2$)

$$= x^3 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix}$$

$$\therefore \Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

37. If $f(x) = \begin{vmatrix} x^n & n! & 2 \\ \cos x & \cos\frac{n\pi}{2} & 4 \\ \sin x & \sin\frac{n\pi}{2} & 8 \end{vmatrix}$ then the value of

$\frac{d^n}{dx^n}[f(x)]_{x=0}$ is

(A) 0 (B) 1
(C) – 1 (D) None of these

Solution: (A)

$$\frac{d^n}{dx^n}[f(x)] = \begin{vmatrix} n! & n! & 2 \\ \cos\left(x + \frac{n\pi}{2}\right) & \cos\frac{n\pi}{2} & 4 \\ \sin\left(x + \frac{n\pi}{2}\right) & \sin\frac{n\pi}{2} & 8 \end{vmatrix}$$

$$\therefore \quad \frac{d^n}{dx^n}[f(x)]_{x=0} = \begin{vmatrix} n! & n! & 2 \\ \cos\frac{n\pi}{2} & \cos\frac{n\pi}{2} & 4 \\ \sin\frac{n\pi}{2} & \sin\frac{n\pi}{2} & 8 \end{vmatrix} = 0$$

(C_1 and C_2 are identical)

38. If α is a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degree > 2, then the determinant

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} \text{ is divisible by}$$

(A) $A(x)$ (B) $B(x)$
(C) $C(x)$ (D) $f(x)$

Solution: (D)

Let $\Delta(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

So $\Delta(\alpha) = 0 = \Delta'(\alpha)$, therefore α is a repeated root of $\Delta(x) = 0$ and α is a repeated root of the quadratic equation $f(x) = 0$, so Δ is divisible by $f(x)$.

EXERCISES

Single Correct Options Type

1. If $f(x) = \sqrt{x^2 - 10x + 25}$, then the derivative of $f(x)$ on the interval $[0, 7]$ is

(A) 1 (B) – 1
(C) 0 (D) Does not exist

2. If the capital letters denote the cofactors of the corresponding small letters in the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ then the value of}$$

$$\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \text{ is}$$

(A) 0 (B) 2Δ (C) Δ^2 (D) Δ

3. If $f(x) = \cos^2 x + \cos^2\left(x + \frac{\pi}{3}\right) + \sin x \sin\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 3$ then $(gof)(x)$ is equal to

(A) 1 (B) 2
(C) 3 (D) None of these

4. If $f(x) = \cos x \cos 2x \cos 4x \cos 8x$, then $f'\left(\frac{\pi}{4}\right)$ is

(A) – 1 (B) 2
(C) $\sqrt{2}$ (D) None of these

5. If $y = e^{nx}$, then $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)$ is equal to

(A) ne^{nx} (B) n^2e^{nx}
(C) $-ne^{nx}$ (D) $-ne^{-nx}$

6. If the parametric equation of a curve is given by $x = \cos\theta + \log\tan\frac{\theta}{2}$ and $y = \sin\theta$, then the points for which $\frac{d^2y}{dx^2} = 0$ are given by

(A) $\theta = n\pi, n \in Z$
(B) $\theta = (2n + 1)\pi/2, n \in Z$
(C) $\theta = (2n + 1)\pi, n \in Z$
(D) $\theta = 2n\pi, n \in Z$.

7. Let $f(x) = \sqrt{x-1} + \sqrt{x + 24 - 10\sqrt{x-1}}$; $1 < x < 26$ be real valued function. Then $f'(x)$, for $1 < x < 26$ is

(A) 0 (B) $\frac{1}{\sqrt{x-1}}$
(C) $2\sqrt{x-1} - 5$ (d) None of these

8. If $y = \tan^{-1}\left(\frac{\log(e/x^3)}{\log(ex^3)}\right) + \tan^{-1}\left(\frac{\log(e^4x^3)}{\log(e/x^{12})}\right)$, then $\frac{d^2y}{dx^2}$ is equal to

(A) 1 (B) 0
(C) – 1 (D) None of these

9. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is a constant.

Then $\frac{d^3}{dx^3}[f(x)]$ at $x = 0$ is

(A) p (B) $p + p^2$
(C) $p + p^3$ (D) independent of p

10. The function y defined by the equation $xy - \log y = 1$ satisfies $x(yy'' + y'^2) - y'' + kyy' = 0$. The value k is

(A) – 3 (B) 3
(C) 1 (D) None of these

11. If the function $y(x)$ represented by $x = \sin t$, $y = ae^{t\sqrt{2}} + be^{t\sqrt{2}}$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies the equation $(1 - x^2)y'' - xy' = ky$, then k is equal to

(A) 1 (B) – 2
(C) 2 (D) None of these

12. If $f(x) = \frac{x^2 - x}{x^2 + 2x}$, then $\frac{df^{-1}(x)}{dx}$ is equal to

(A) $-\frac{3}{(1-x)^2}$ (B) $\frac{3}{(1-x)^2}$
(C) $\frac{1}{(1-x)^2}$ (D) None of these

13. Let $F(x) = f(x)\,g(x)\,h(x)$ for all real x, where $f(x)$, $g(x)$ and $h(x)$ are differentiable functions. At some point x_0, if $F'(x_0) = 21\,F(x_0)$, $f'(x_0) = 4f(x_0)$, $g'(x_0) = -7\,g(x_0)$ and $h'(x_0) = kh(x_0)$ then k is equal to

(A) 24 (B) 12
(C) – 12 (D) – 24

14. If $f(x)$ is a polynomial of degree n (> 2) and $f(x) = f(k - x)$, (where k is a fixed real number), then degree of $f'(x)$ is

(A) n (B) $n - 1$
(C) $n - 2$ (D) None of these

15. If $f(x) = |x - 1|$ and $g(x) = f\{f[f(x)]\}$, then for $x > 2$, $g'(x)$ is equal to

(A) – 1 if $2 \le x < 3$ (B) 1 if $2 \le x < 3$
(C) 1 for all $x > 2$ (D) None of these

16. If $f(x) = |(x - 4)(x - 5)|$, then $f'(x)$ is equal to

(A) $-2x + 9$, for all $x \in R$
(B) $2x - 9$ if $4 < x < 5$
(C) $-2x + 9$ if $4 < x < 5$
(D) None of these

17. If $2f(\sin x) + f(\cos x) = x$, then $\frac{d}{dx}f(x)$ is

(A) $\sin x + \cos x$ (B) 2
(C) $\frac{1}{\sqrt{1-x^2}}$ (D) None of these

18. Let $f(x) = |x - a|$; ($a > 0$) and $g(x) = f\{f[f(x)]\}$. Then $g'(\alpha)$; ($\alpha > 3a$)

(A) does not exist (B) equal to 3
(C) equal to 1 (D) None of these

19. Let $\phi(x)$ be the inverse of the function $f(x)$ and $f'(x) = \frac{1}{1+x^5}$ then $\frac{d}{dx}\phi(x)$ is

(A) $\frac{1}{1+[\phi(x)]^5}$ (B) $\frac{1}{1+[f(x)]^5}$
(C) $1 + [\phi(x)]^5$ (D) $1 + [f(x)]^5$

20. If $y = \frac{1}{x}$ then $\frac{dy}{\sqrt{1+y^4}} + \frac{dx}{\sqrt{1+x^4}} =$

(A) 0 (B) 1
(C) $\frac{x}{y}$ (D) $\frac{y}{x}$

21. If $y = \frac{f(x)}{\phi(x)}$ and $z = \frac{f'(x)}{\phi'(x)}$, then $\frac{f''}{f} - \frac{\phi''}{\phi} + \frac{2(y-z)}{f\phi}(\phi')^2 =$

(A) $\frac{d^2y}{dx^2}$ (B) $\frac{1}{y}\frac{d^2y}{dx^2}$
(C) $y\frac{d^2y}{dx^2}$ (D) None of these

22. Let $g(x)$ be the inverse of an invertible function $f(x)$, which is differentiable for all real x, then $g''(f(x))$ equals

(A) $-\frac{f''(x)}{[f'(x)]^3}$

(B) $\frac{f'(x)f''(x) - [f'(x)]^2}{f'(x)}$

(C) $\dfrac{f'(x)f''(x)-[f'(x)]^2}{[f'(x)]^2}$

(D) None of these

23. Let $f(x)=x^n$, n being a non-negative integer. The value of n for which equality
$f'(a+b)=f'(a)+f'(b)$ is valid for all $a, b>0$ is

(A) 5 (B) 1
(C) 2 (D) 4

24. The solution set of $f'(x)>g'(x)$ where $f(x)=(1/2)5^{2x+1}$ and $g(x)=5^x+4x\log 5$ is

(A) $(1,\infty)$ (B) $(0,1)$
(C) $(0,\infty)$ (D) $[0,\infty)$

25. If $I_n=\dfrac{d^n}{dx^n}(x^n\log x)$, then $I_n-nI_{n-1}=$

(A) n (B) $n-1$
(C) $n!$ (D) $(n-1)!$

26. If $u=f(x^3)$, $v=g(x^2)$, $f'(x)=\cos x$ and $g'(x)=\sin x$ then $\dfrac{du}{dv}$ is

(A) $\dfrac{3}{2}x\cdot\cos x^3\cdot\operatorname{cosec} x^2$

(B) $\dfrac{2}{3}\sin x^3\cdot\operatorname{cosec} x^2$

(C) $\tan x$

(D) None of these

27. If $f(x)$ be a differentiable function such that $f(xy)=f(x)+f(y)$ for all x and y, then $f(e)+f(1/e)=$

(A) 1 (B) 0
(C) –1 (D) None of these

28. If for all x, y the function f is defined by $f(x)+f(y)+f(x)\cdot f(y)=1$ and $f(x)>0$ then

(A) $f'(x)$ does not exist
(B) $f'(x)=0$ for all x
(C) $f'(0)<f'(1)$
(D) None of these

29. Let $3f(x)-2f(1/x)=x$, then $f'(2)$ is equal to

(A) $\dfrac{2}{7}$ (B) $\dfrac{1}{2}$ (C) 2 (D) $\dfrac{7}{2}$

30. If $\sqrt{x+y}+\sqrt{y-x}=c$ then $\dfrac{d^2y}{dx^2}$ equals

(A) $\dfrac{2}{c^2}$ (B) $\dfrac{-2}{c^2}$ (C) $\dfrac{2}{c}$ (D) $\dfrac{-2}{c}$

31. $\dfrac{d^2x}{dy^2}$ equals

(A) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$ (B) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$

(C) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$ (D) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$

32. Let y be an implicit function of x defined by,
$x^{2x}-2x^x\cot y-1=0$
Then $y'(1)$ equals

(A) – 1 (B) 1
(C) log 2 (D) – log 2

33. If y and z are the functions of x and if $y^2+z^2=\lambda^2$, then $y\dfrac{d}{dx}\left(\dfrac{y}{\lambda}\right)+\dfrac{d}{dx}\left(\dfrac{z^2}{\lambda}\right)$ is equal to

(A) $\dfrac{z}{\lambda}\dfrac{dz}{dx}$ (B) $\dfrac{z}{\lambda}\dfrac{dx}{dz}$

(C) $\dfrac{\lambda}{z}\dfrac{dz}{dx}$ (D) None of these

34. If S_n denotes the sum of n terms of a G.P. whose common ratio is r, then $(r-1)\dfrac{dS_n}{dr}$ is equal to

(A) $(n-1)S_n+nS_{n-1}$ (B) $(n-1)S_n-nS_{n-1}$
(C) $(n-1)S_n$ (D) None of these

35. Let $f\left(\dfrac{x_1+x_2+\ldots+x_n}{n}\right)=\dfrac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n}$,
where all $x_i\in R$ are independent to each other and $n\in N$. If $f(x)$ is differentiable and $f'(0)=a$, $f(0)=b$ then $f'(x)$ is equal to

(A) a (B) 0
(C) b (D) None of these

36. If $y^2=P(x)$, a polynomial of degree $n\geq 3$, then $2\dfrac{d}{dx}\left(y^3\dfrac{d^2y}{dx^2}\right)=$

(A) $-P(x)\times P'''(x)$ (B) $P(x)\times P'''(x)$
(C) $P(x)\times P''(x)$ (D) None of these

37. A function $f(x)$ is so defined that for all x, $[f(x)]^n=f(nx)$. If $f'(x)$ denotes derivative of $f(x)$ with respect to x, then $f'(x)\times f(nx)=$

(A) $f(x)$ (B) 0
(C) $f(x)\times f'(nx)$ (D) None of these

38. If f, g, h are differentiable functions of x and

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix}$$

then Δ' (the derivative of Δ with respect to x) is given by

(A) $\begin{vmatrix} f' & g' & h' \\ f & g & h \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$

(B) $\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^2 f'')' & (x^2 g'')' & (x^2 h'')' \end{vmatrix}$

(C) $\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$

(D) None of these

39. If α is a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degree > 2, then the determinant $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by

(A) $A(x)$ (B) $B(x)$
(C) $C(x)$ (D) $f(x)$

40. If the capital letters denote the cofactors of the corresponding small letters in the determinant

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then the value of

$\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ is

(A) 0 (B) 2Δ
(C) Δ^2 (D) Δ

41. If $f(x) = \dfrac{x^2 - x}{x^2 + 2x}$, then $\dfrac{df^{-1}(x)}{dx}$ is equal to

(A) $-\dfrac{3}{(1-x)^2}$ (B) $\dfrac{3}{(1-x)^2}$

(C) $\dfrac{1}{(1-x)^2}$ (D) None of these

42. If $f(x)$ is a polynomial of degree n (> 2) and $f(x) = f(k - x)$, (where k is a fixed real number), then degree of $f'(x)$ is

(A) n (B) $n - 1$
(C) $n - 2$ (D) None of these

43. If $y = \dfrac{f(x)}{\phi(x)}$ and $z = \dfrac{f'(x)}{\phi'(x)}$, then

$$\frac{f''}{f} - \frac{\phi''}{\phi} + \frac{2(y-z)}{f\phi}(\phi')^2 =$$

(A) $\dfrac{d^2y}{dx^2}$ (B) $\dfrac{1}{y}\dfrac{d^2y}{dx^2}$

(C) $y\dfrac{d^2y}{dx^2}$ (D) None of these

44. The solution set of $f'(x) > g'(x)$ where $f(x) = (1/2)5^{2x+1}$ and $g(x) = 5^x + 4x \log 5$ is

(A) $(1, \infty)$ (B) $(0, 1)$
(C) $(0, \infty)$ (D) $[0, \infty)$

45. If for all x, y the function f is defined by $f(x) + f(y) + f(x) \cdot f(y) = 1$ and $f(x) > 0$, then

(A) $f'(x)$ does not exist
(B) $f'(x) = 0$ for all x
(C) $f'(0) < f'(1)$
(D) None of these

46. If $\sqrt{x+y} + \sqrt{y-x} = c$ then $\dfrac{d^2y}{dx^2}$ equals

(A) $\dfrac{2}{c^2}$ (B) $\dfrac{-2}{c^2}$

(C) $\dfrac{2}{c}$ (D) $\dfrac{-2}{c}$

47. Let $f(x) = \prod_{k=1}^{n}(\cos(2k-1)x + i\sin(2k-1)x)$, then $(\text{Re}\, f(x))'' + i(\text{Im}\, f(x))''$ is equal to

(A) $n^2 f(x)$ (B) $-n^4 f(x)$
(C) $-n^2 f(x)$ (D) $n^4 f(x)$

48. If $2f(\sin x) + f(\cos x) = x$, then $\dfrac{d}{dx} f(x)$ is

(A) $\sin x + \cos x$ (B) 2

(C) $\dfrac{1}{\sqrt{1-x^2}}$ (D) None of these

49. Let $f(x) = \sqrt{x-1} + \sqrt{x + 24 - 10\sqrt{x-1}}$; $1 < x < 26$ be a real valued function. Then, $f'(x)$, for $1 < x < 26$ is

(A) 0 (B) $\frac{1}{\sqrt{x-1}}$

(C) $2\sqrt{x-1}-5$ (D) None of these

50. If $f(x) = |x-2|$ and $g(x) = f\{f(x)\}$, then $g'(x)$ for $x > 2$ is

(A) – 1 (B) 1
(C) 0 (D) does not exist

51. The derivative of the function represented parametrically as $x = 2t - |t|$, $y = t^3 + t^2|t|$ at $t = 0$ is

(A) 0 (B) 1
(C) –1 (D) does not exist

52. A polynomial $f(x)$ leaves remainder 15 when divided by $(x-3)$ and $(2x+1)$ when divided by $(x-1)^2$. When f is divided by $(x-3)(x-1)^2$, the remainder is

(A) $2x^2+2x+3$ (B) $2x^2-2x-3$
(C) $2x^2-2x+3$ (D) None of these

53. If for a non-zero x, the function $f(x)$ satisfies the equation

$$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \ (a \neq b),$$

then $f'(x)$ is equal to

(A) $\frac{1}{b^2-a^2}\left(\frac{a}{x^2}+b\right)$ (B) $\frac{1}{a^2-b^2}\left(\frac{a}{x^2}+b\right)$

(C) $\frac{1}{a^2-b^2}\left(\frac{a}{x^2}-b\right)$ (D) None of these

54. If $x = \cos^7\theta$ and $y = \sin\theta$, then $\frac{d^3x}{dy^3} =$

(A) $\frac{105}{4}\sin 4\theta$ (B) $\frac{105}{2}\sin 2\theta$

(C) $\frac{105}{4}\cos 4\theta$ (D) None of these

55. If $y = \sec^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$, then $\frac{dy}{dx} =$

(A) $0\ \forall x \in R$ (B) $0\ \forall x \in (0, \infty)$
(A) $0\ \forall x \in R - \{0\}$ (D) None of these

56. If $f(x) = \cos^{-1}\left(\frac{x^{-1}-x}{x^{-1}+x}\right)$, then $f'(x)$ is

(A) odd (B) even
(C) periodic (D) None of these

57. If $f(x) = (1-x)^n$, then the value of

$$f(0) + f'(0) + \frac{f''(0)}{2!} + \ldots + \frac{f^n(0)}{n!}$$ is

(A) n (B) 0
(C) 2^n (D) $2^n - 1$

58. If $y = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$, then $\frac{dy}{dx}$ at $x = 1$, equals

(A) 0 (B) 1
(C) –1 (D) None of these

59. If $y = \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}$, then

(A) $y'\left(\frac{\pi}{4}\right) = y'\left(\frac{3\pi}{4}\right)$

(B) $y'\left(\frac{\pi}{4}\right) \cdot y'\left(\frac{3\pi}{4}\right) = -4$

(C) $y'\left(\frac{\pi}{4}\right)$ and $y'\left(\frac{3\pi}{4}\right)$ do not exist

(D) None of these

60. Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a_1, a_2, a_3 are in A. P., then $f'(a_1), f'(a_2), f'(a_3)$ are in

(A) A.P. (B) G.P.
(C) H.P. (D) None of these

61. If $5f(x) + 3f\left(\frac{1}{x}\right) = x + 2$ and $y = x\,f(x)$ then $\frac{dy}{dx}$ at $x = 1$ is equal to

(A) 1 (B) –1
(C) $\frac{7}{8}$ (D) $-\frac{7}{8}$

62. Let $f(x)$ be a polynomial function of degree 2 and $f(x) > 0$ for all $x \in R$. If $g(x) = f(x) + f'(x) + f''(x)$, then for any x

(A) $g(x) > 0$ (B) $g(x) < 0$
(C) $g(x) = 0$ (D) $g(x) \le 0$

63. Let $f(x+y) = f(x) + f(y) + 2xy - 1\ \forall x, y \in R$. If $f(x)$ is differentiable and $f'(0) = \sin\theta$, then

(A) $f(x) > 0\ \forall x \in R$ (B) $f(x) < 0\ \forall x \in R$
(C) $f(x) = \sin\theta\ \forall x \in R$ (D) None of these

64. If $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all $x \in R$, then which of the following is false?

(A) $f(0) + f(2) = f(1)$ (B) $f(0) + f(3) = 0$
(C) $f(1) + f(3) = f(2)$ (D) $f(1) + f(3) = f(0)$

65. Let $f\left(\frac{x+y}{2}\right) = \frac{1}{2}[f(x) + f(y)]$ for real x and y. If $f'(0)$ exists and equals –1 and $f(0) = 1$ then the value of $f(2)$ is

(A) 1 (B) –1
(C) 0 (D) None of these

66. Let the function f satisfy the equation $f(x+y) = f(x)\,f(y)$ for all x and y and $f(x) = 1 + xg(x)$ where $\lim_{x\to 0} g(x) = \log a$. If $f^n(x) = kf(x)$, then $k =$

(A) $\log a$ (B) $n \log a$
(C) $(\log a)^n$ (D) $n(\log a)^n$

67. Let f be a differentiable function satisfying $f(x + y) = f(x) + f(y) + xy$. If $\lim_{h\to 0} \frac{1}{h} f(h) = 3$, then

(A) $f(x) = 3x$ (B) $f(x) = 3x + x^2$
(C) $f(x) = 3x + \frac{x^2}{2}$ (D) None of these

68. If $f(x) = x + \tan x$ and f is the inverse of g, then $g'(x)$ is equal to

(A) $\frac{1}{1+[g(x)-x]^2}$ (B) $\frac{1}{2-[g(x)-x]^2}$
(C) $\frac{1}{2+[g(x)-x]^2}$ (D) None of these

69. If $y = \sqrt{(a-x)(x-b)} - (a-b)\tan^{-1}\sqrt{\frac{a-x}{x-b}}$, then $\frac{dy}{dx} =$

(A) 1 (B) $\sqrt{\frac{a-x}{x-b}}$
(C) $\sqrt{(a-x)(x-b)}$ (D) $\frac{1}{\sqrt{(a-x)(x-b)}}$

70. If $y^3 - y = 2x$, then $\left(x^2 - \frac{1}{27}\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} =$

(A) y (B) $\frac{y}{3}$
(C) $\frac{y}{9}$ (D) $\frac{y}{27}$

71. If $x < 1$, then $\frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \ldots\infty =$

(A) $\frac{1}{1+x+x^2}$ (B) $\frac{1+2x}{1+x+x^2}$
(C) $\frac{1-x+x^2}{1+x+x^2}$ (D) 1

More than One Option Correct Type

72. If $f(x) = x^m$, m being a non-negative integer, then the value of m for which $f'(\alpha+\beta) = f'(\alpha) + f'(\beta)$, for all $\alpha, \beta > 0$, is

(A) 1 (B) 2
(C) 0 (D) None of these

73. If $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in \mathbb{R}$ where $f(x)$ is a polynomial of degree 3, then

(A) $f(0) + f(2) = f(1)$ (B) $f(0) + f(3) = 0$
(C) $f(1) + f(3) = f(2)$ (D) None of these

74. If $f(x-y)$, $f(x)\cdot f(y)$ and $f(x+y)$ are in A.P. for all x, y and $f(0) \neq 0$, then

(A) $f(2) = f(-2)$ (B) $f(3) + f(-3) = 0$
(C) $f'(2) + f'(-2) = 0$ (D) $f'(3) = f'(-3)$

75. If $f(x) + f(y) + f(z) + f(x)\cdot f(y)\cdot f(z) = 14$ for all $x, y, z \in R$, then

(A) $f(0) = 2$
(B) $f'(x) = 0$, for all $x \in R$
(C) $f'(x) > 0$, for all $x \in R$
(D) None of these

76. If $f(x-y)$, $f(x)\cdot f(y)$ and $f(x+y)$ are in A. P. for all x, y and $f(0) \neq 0$, then

(A) $f'(3) + f'(-3) = 0$ (B) $f(3) + f(-3) = 0$
(C) $f'(2) + f'(-2) = 0$ (D) $f'(3) = f'(-3)$

77. A function $f: (0, \infty) \to R$ satisfies the equation $f(xy) = 2f(x) - f\left(\frac{x}{y}\right)$.

If f is differentiable on R and $f(1) = 0$, $f'(1) = 1$, then

(A) $f(y) = -f\left(\frac{1}{y}\right)$ (B) $f'(x) = \frac{1}{x}$
(C) $f(x) = \ln x$ (D) $f(x) = e^x$

78. If $f: R \to R$ be a function such that $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ is differentiable for every $x \in R$, then

(A) $f'(1) = -5$ (B) $f''(2) = 2$
(C) $f'''(3) = 6$ (D) None of these

79. If $f(x-y) + f(x+y) = 2f(x)f(y)\ \forall x, y \in R$, then

(A) f is even (B) f is odd
(C) f' is even (D) f' is odd

80. If $f(x) = x^2 + xg'(1) + g''(2)$ and $g(x) = f(1) \cdot x^2 + xf'(x) + f''(x)$, then

(A) $f(x) = x^2 - 3x$ (B) $f(x) = x^2 + 3x$
(C) $g(x) = 3x + 2$ (D) $g(x) = -3x + 2$

81. If $\sum_{r=1}^{n} rx^{r-1} = \frac{1}{(1-x)^2} \cdot \{1 + ax^n + bx^{n+1}\}$, then

(A) $a = (n + 1)$ (B) $b = n$
(C) $a = -(n + 1)$ (D) $b = -n$

82. If $y = f(x) = \min \phi(t)$; $-3 \le t \le x$ where $\phi(x) = || x - 1 | - | x + 1||$, then

(A) $f(x)$ is non-differentiable at $x = 0, -1$
(B) $f(x)$ is non-differentiable at $x = 1, -1$
(C) $f''(100) = 0$
(D) $\int_{-3}^{10} f(x)\, dx = 5$

83. If $f(x) = \begin{vmatrix} x^n & \sin x & -\cos x \\ n! & \sin(n\pi/2) & \cos(n\pi/2) \\ a & a^2 & a^3 \end{vmatrix}$, then $f^n(0)$ for $n = 2m + 1$ is

(A) 1 (B) -1
(C) 0 (D) independent of a

84. Let $f(x) = x^3 + 3x^2 - 33x - 33$ for $x > 0$ and g be its inverse, then the value of k such that $kg'(2) = 1$ is equal to

(A) -36 (B) 51
(C) 72 (D) 36

85. If $F(x) = f(x)\, g(x)$ and $f'(x)\, g'(x) = c$, then

(A) $F' = c\left(\frac{f}{f'} + \frac{g}{g'}\right)$

(B) $\frac{F''}{F} = \frac{f''}{f} + \frac{g''}{g} + \frac{2c}{fg}$

(C) $\frac{F'''}{F} = \frac{f'''}{f} + \frac{g'''}{g}$

(D) $\frac{F'''}{F''} = \frac{f'''}{f''} + \frac{g'''}{g''}$

Passage Based Questions

Passage 1

Let f be a function such that $f : (-1, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Let f satisfy the equation $f(x) + f(y) = f\,(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

86. The function $f(x)$ is

(A) even (B) odd
(C) constant (D) None of these

87. If $f(x)$ is differentiable on $(-1, 1)$ and $f'(0) = 1$, then $f'(x)$ is equal to

(A) $\frac{1}{\sqrt{1-x^2}}$ (B) $-\frac{1}{\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1+x^2}}$ (D) $-\frac{1}{\sqrt{1+x^2}}$

88. The function $f(x)$ is equal to

(A) $\cos^{-1} x$
(B) $\sin^{-1} x$
(C) $\tan^{-1} x$
(D) $\sec^{-1} x$

Passage 2

Let $f : R \to R$ be a function satisfying the condition $f\left(\frac{x+y}{k}\right) = \frac{f(x) + f(y)}{k}$, where $k \ne 0, 2$. The function $f(x)$ is differentiable on R and $f'(0) = m$.

89. $f'(x)$ is equal to

(A) m (B) $2m$
(C) $m + 1$ (D) 0

90. The function $f(x)$ is equal is

(A) mx (B) $mx + 1$
(C) $-2m\,x$ (D) None of these

Passage 3

A function $f : R \to [1, \infty)$ satisfies the equation $f(xy) = f(x)\, f(y) - f(x) - f(y) + 2$. The function f is differentiable on R and $f(2) = 5$.

91. $f'(x)$ is equal to

(A) $\frac{f(x)-1}{x} f'(1)$ (B) $\frac{f(x)+1}{x} f'(1)$

(C) $\frac{1-f(x)}{x} f'(1)$ (D) None of these

92. $f(x)$ is equal to

(A) $x^2 - 1$ (B) $1 - x^2$

(C) $x^2 + 1$ (D) None of these

Passage 4

A function $f: R \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies the equation $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$. The function f is differentiable on R and $f'(0) = 2$.

93. The function f is

(A) an even function (B) an odd function

(C) a constant function (D) None of these

94. $f'(x)$ is equal to

(A) $\frac{1}{1+x^2}$ (B) $\frac{1}{1-x^2}$

(C) $\frac{2}{1+x^2}$ (D) $\frac{2}{x^2-1}$

95. $f(x)$ is equal to

(A) $\tan^{-1}x$ (B) $2\tan^{-1}x$

(C) $4\tan^{-1}x$ (D) None of these

Passage 5

Let $z = f(x, y)$ be a function of two variables x and y. The partial derivative with respect to x of the function $z = f(x, y)$ at (x, y) is defined as $\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$, provided the limit exists and is finite. It is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x. Clearly, $\frac{\partial z}{\partial x}$ is the derivative of $z = f(x, y)$ with respect to x, regarding y as a constant. Similarly, we can define $\frac{\partial z}{\partial y}$. $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$, denoted by $\frac{\partial^2 z}{\partial x^2}$ or f_{xx}, $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$, denoted by $\frac{\partial^2 z}{\partial y\, \partial x}$ or f_{yx}, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$, denoted by $\frac{\partial^2 z}{\partial x\, \partial y}$ or f_{xy}, $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$, denoted by $\frac{\partial^2 z}{\partial y^2}$ or f_{yy} are called second order partial derivatives of $z = f(x, y)$.

96. If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$

(A) 0 (B) 1

(C) – 1 (D) None of these

Passage 6

If U and V are two functions of x having derivatives of the nth order, then $(UV)_n = U_nV + {}^nC_1U_{n-1}V_1 + {}^nC_2U_{n-2}V_2 + \ldots + {}^nC_rU_{n-r}V_r + \ldots + {}^nC_n\, UV_n$.

97. If $y = x^2 \sin x$, then $\frac{d^n y}{dx^n} = (x^2 - n^2 + n)\sin\left(x + \frac{n\pi}{2}\right) + k\cos\left(x + \frac{n\pi}{2}\right)$, where $k =$

(A) nx (B) $2nx$

(C) $-nx$ (D) $-2nx$

98. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, then $x^2y_{n+2} + (2n+1)x\, y_{n+1} + ky_n = 0$ where $k =$

(A) n^2 (B) $2n^2$

(C) $-n^2$ (D) $-2n^2$

99. If $f(x) = \tan x$, then $f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0) - \ldots =$

(A) $\sin\frac{n\pi}{2}$ (B) $\cos\frac{n\pi}{2}$

(C) $\tan\frac{n\pi}{2}$ (D) None of these

100. If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, then $I_n = n\, I_{n-1} + k$, where $k =$

(A) $n\,!$ (B) $(n-1)\,!$

(C) $(n-2)\,!$ (D) None of these

Match the Column Type

101.

Column-I	Column-II
I. The derivative of $f(\tan x)$ with respect to $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'(\sqrt{2}) = 4$, is	(A) 3
II. Let $y = x^3 - 8x + 7$ and $x = f(t)$. If $\frac{dy}{dt} = 2$ and $x = 3$ at $t = 0$, then $\frac{dx}{dt}$ at $t = 0$ is	(B) -4
III. Let $f(x) = \sin x$, $g(x) = 2x$ and $h(x) = \cos x$. If $\phi(x) = [go(fh)](x)$, then $\phi''\left(\frac{\pi}{4}\right)$ is equal to	(C) $\frac{2}{19}$
IV. If $f(x) = \cos^2 x + \cos^2\left(x + \frac{\pi}{3}\right) + \sin x \sin\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 3$, then $(g\,o\,f)(x)$ is equal to	(D) $\frac{1}{\sqrt{2}}$

102.

Column-I	Column-II
I. The function y defined by the equation $xy - \log y = 1$ satisfies $x(yy'' + y'^2) - y'' + k\,yy' = 0$. The value of k is	(A) 24
II. If the function $y(x)$ represented by $x = \sin t$, $y = ae^{t\sqrt{2}} + be^{t\sqrt{2}}$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies the equation $(1 - x^2)y'' - xy' = ky$, then k is equal to	(B) 2
III. Let $F(x) = f(x)\,g(x)\,h(x)$ for all real x, where $f(x)$, $g(x)$ and $h(x)$ are differentiable functions. At some point x_0, if $F'(x_0) = 21\,F(x_0)$, $f'(x_0) = 4\,f(x_0)$, $g'(x_0) = -7g(x_0)$ and $h'(x_0) = kh(x_0)$ then k is equal to	(C) 4
IV. Let $f(x) = x^n$, n being a non-negative integer. The number of values of n for which the equality $f'(a + b) = f'(a) + f'(b)$ is valid for all $a, b > 0$, is	(D) 3

Assertion-Reason Type

Instructions: *In the following questions an Assertion (A) is given followed by a Reason (R). Mark your responses from the following options:*

(A) Assertion(A) is True and Reason(R) is True; Reason(R) is a correct explanation for Assertion(A)
(B) Assertion(A) is True, Reason(R) is True; Reason(R) is not a correct explanation for Assertion(A)
(C) Assertion(A) is True, Reason(R) is False
(D) Assertion(A) is False, Reason(R) is True

103. **Assertion:** Let $f(x)$ be a polynomial function satisfying

$f(x)\,.\,f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$. If $f(4) = 65$ and l_1, l_2, l_3 are in G.P., then $f'(l_1), f'(l_2), f'(l_3)$, are also in G.P.

Reason: $f(x) = \pm x^n + 1$

104. **Assertion:** If $y = (1 + x)(1 + x^2)(1 + x^4)\ldots(1 + x^{2n})$, then $\frac{dy}{dx}$ at $x = 0$ is 1.

Reason: $y = \frac{1 - x^{2^{n+1}}}{1 - x}$

105. **Assertion:** If $f(x) = (\cos x + i \sin x)(\cos 2x + i \sin 2x)(\cos 3x + i \sin 3x) \ldots (\cos nx + i \sin nx)$ and $f(1) = 1$ then $f''(1)$ is equal to $-\left(\frac{n(n+1)}{2}\right)^2$.

Reason: $f(x) = \cos\frac{n(n-1)}{2}x + i\sin\frac{n(n-1)}{2}x$

Previous Year's Questions

106. If $y = (x + \sqrt{1+x^2})^n$, then $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}$ is : **[2002]**

(A) n^2y (B) $-n^2y$
(C) $-y$ (D) $2x^2y$

107. If $\sin y = x \sin(\alpha + y)$, then $\dfrac{dy}{dx}$ is : **[2002]**

(A) $\dfrac{\sin\alpha}{\sin^2(\alpha+y)}$ (B) $\dfrac{\sin^2(\alpha+y)}{\sin\alpha}$

(C) $\sin\alpha\sin^2(\alpha+y)$ (D) $\dfrac{\sin^2(\alpha-y)}{\sin\alpha}$

108. If $x^y = e^{x-y}$, then $\dfrac{dy}{dx}$ is: **[2002]**

(A) $\dfrac{1+x}{1+\log x}$ (B) $\dfrac{1-\log x}{1+\log x}$

(C) not defined (D) $\dfrac{\log x}{(1+\log x)^2}$

109. Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in A. P., then $f'(a)$, $f'(B)$ and $f'(C)$ are in **[2003]**

(A) A.P.
(B) G.P.
(C) H. P.
(D) arithmetic-geometric progression

110. If $f(x) = x$, then the value of

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + \frac{(-1)^n f^n(1)}{n!} \text{ is}$$

[2003]

(A) 2^n (B) 2^{n-1}
(C) 0 (D) 1

111. If $x = e^{y+e^{y+\ldots \text{to} \infty}}$, $x > 0$, then $\dfrac{dy}{dx}$ is **[2004]**

(A) $\dfrac{x}{1+x}$ (B) $\dfrac{1}{x}$

(C) $\dfrac{1-x}{x}$ (D) $\dfrac{1+x}{x}$

112. Suppose $f(x)$ is differentiable $x = 1$ and

$\lim\limits_{h\to 0}\dfrac{1}{h}f(1+h) = 5$, then $f'(1)$ equals **[2005]**

(A) 3 (B) 4
(C) 5 (D) 6

113. If f is a real-valued differentiable function satisfying $|f(x) - f(y)| \le (x-y)^2$, $x, y \in R$ and $f(0) = 0$, then $f(1)$ equals **[2005]**

(A) -1 (B) 0
(C) 2 (D) 1

114. The set of points where $f(x) = \dfrac{x}{1+|x|}$ is differentiable is **[2006]**

(A) $(-\infty, 0) \cup (0, \infty)$
(B) $(-\infty, -1) \cup (-1, \infty)$
(C) $(-\infty, \infty)$
(D) $(0, \infty)$

115. If $x^m \cdot y^m = (x + y)^{m+n}$, then $\dfrac{dy}{dx}$ is **[2006]**

(A) $\dfrac{y}{x}$ (B) $\dfrac{x+y}{xy}$

(C) xy (D) $\dfrac{x}{y}$

116. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals **[2009]**

(A) -1 (B) 1
(C) log 2 (D) $-\log 2$

117. Let $f : (-1, 1) \to R$ be a differentiable function such that $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0) =$ **[2010]**

(A) -4 (B) 0
(C) -2 (D) 4

118. $\dfrac{d^2x}{dy^2}$ is equal to **[2011]**

(A) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$

(B) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$

(C) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

(D) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$

119. If $y = \sec(\tan^{-1} x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to **[2013]**

(A) $\frac{1}{2}$ (B) 1
(C) $\sqrt{2}$ (D) $\frac{1}{\sqrt{2}}$

120. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to **[2014]**

(A) $1 + x^5$ (B) $5x^4$
(C) $\frac{1}{1+\{g(x)\}^5}$ (D) $1+\{g(x)\}^5$

ANSWER KEYS

Single Option Correct Type

1. (D)	**2.** (C)	**3.** (C)	**4.** (C)	**5.** (D)	**6.** (A)	**7.** (A)	**8.** (B)	**9.** (D)	**10.** (B)
11. (C)	**12.** (B)	**13.** (A)	**14.** (B)	**15.** (A)	**16.** (C)	**17.** (C)	**18.** (C)	**19.** (C)	**20.** (A)
21. (B)	**22.** (A)	**23.** (C)	**24.** (C)	**25.** (D)	**26.** (A)	**27.** (B)	**28.** (B)	**29.** (B)	**30.** (A)
31. (A)	**32.** (A)	**33.** (A)	**34.** (B)	**35.** (A)	**36.** (B)	**37.** (C)	**38.** (C)	**39.** (D)	**40.** (C)
41. (B)	**42.** (B)	**43.** (B)	**44.** (C)	**45.** (B)	**46.** (A)	**47.** (B)	**48.** (C)	**49.** (A)	**50.** (D)
51. (A)	**52.** (C)	**53.** (A)	**54.** (A)	**55.** (B)	**56.** (A)	**57.** (B)	**58.** (C)	**59.** (B)	**60.** (A)
61. (C)	**62.** (A)	**63.** (A)	**64.** (D)	**65.** (B)	**66.** (C)	**67.** (C)	**68.** (C)	**69.** (B)	**70.** (C)
71. (B)									

More than One Option Correct Type

72. (B) and (C)	**73.** (A), (B) and (C)	**74.** (A) and (C)	**75.** (A) and (B)	**76.** (A) and (C)
77. (A), (B) and (C)	**78.** (A), (B) and (C)	**79.** (A) and (D)	**80.** (A) and (D)	**81.** (B) and (C)
82. (A), (C) and (D)	**83.** (C) and (D)	**84.** (A), (B) and (C)	**85.** (A), (B) and (C)	

Passage Based Questions

86. (B)	**87.** (A)	**88.** (B)	**89.** (A)	**90.** (A)
91. (A)	**92.** (C)	**93.** (B)	**94.** (C)	**95.** (B)
96. (A)	**97.** (D)	**98.** (B)	**99.** (A)	**100.** (B)

Match the Column Type

101. I $\rightarrow$ (D), II $\rightarrow$ (C), III $\rightarrow$ (B), IV $\rightarrow$ (A) **102.** I $\rightarrow$ (D), II $\rightarrow$ (B), III $\rightarrow$ (A), IV $\rightarrow$ (B)

Assertion-Reason Type

103. (A) **104.** (A) **105.** (C)

Previous Year's Questions

106. (A)	**107.** (B)	**108.** (D)	**109.** (A)	**110.** (C)
111. (C)	**112.** (C)	**113.** (B)	**114.** (C)	**115.** (A)
116. (A)	**117.** (A)	**118.** (C)	**119.** (D)	**120.** (D)

HINTS AND SOLUTIONS

Single Option Correct Type

1. We have,

$$f(x) = \sqrt{x^2 - 10x + 25} = \sqrt{(x-5)^2} = |x-5|$$

$$= \begin{cases} x-5, & x \ge 5 \\ 5-x, & x < 5 \end{cases}$$

Clearly, $f(x)$ is differentiable at all points on the interval [0, 7] except at $x = 5$.

$\therefore$ The derivative of $f(x)$ on the interval [0, 7] does not exist.

The correct option is (D)

2. We have, $\Delta\Delta' = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$

$\therefore \quad \Delta' = \Delta^2$

The correct option is (C)

3. $f'(x) = -2\cos x \sin x - 2\cos\left(x + \frac{\pi}{3}\right)\sin\left(x + \frac{\pi}{3}\right)$

$$+ \cos x \sin\left(x + \frac{\pi}{3}\right) + \sin x \cos\left(x + \frac{\pi}{3}\right)$$

$$= -\sin 2x - \sin\left(2x + \frac{2\pi}{3}\right) + \sin\left(x + x + \frac{\pi}{3}\right)$$

$$= -2\sin\left(2x + \frac{\pi}{3}\right)\cos\frac{\pi}{3} + \sin\left(2x + \frac{\pi}{3}\right)$$

$$= -\sin\left(2x + \frac{\pi}{3}\right) + \sin\left(2x + \frac{\pi}{3}\right) = 0.$$

$\Rightarrow \quad f(x) = \text{constant for all } x.$

But, $f(0) = \cos^2 0 + \cos^2\frac{\pi}{3} + \sin 0 \cdot \sin\frac{\pi}{3} = \frac{5}{4}$

$\therefore \quad f(x) = \frac{5}{4}$ for all x.

Thus, $(g \, o \, f)(x) = g[f(x)] = g\left(\frac{5}{4}\right) = 3$

The correct option is (C)

4. $f(x) = \dfrac{2\sin x \cos x \cos 2x \cos 4x \cos 8x}{2\sin x} = \dfrac{\sin 16x}{2^4 \sin x}$

$$\Rightarrow \quad f'(x) = \frac{1}{16}\left[\frac{\sin x \cdot \cos 16x \cdot 16 - \sin 16x \cdot \cos x}{\sin^2 x}\right]$$

$$\therefore \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{16}\left[\frac{\frac{1}{\sqrt{2}} \cdot 1 \cdot 16 - \frac{1}{\sqrt{2}} \cdot 0}{\left(\frac{1}{\sqrt{2}}\right)^2}\right] = \sqrt{2}$$

The correct option is (C)

5. $y = e^{nx} \Rightarrow \dfrac{dy}{dx} = n \cdot e^{nx}, \ \dfrac{d^2y}{dx^2} = n^2 \times e^{nx}$

Now, $\quad nx = \log y \Rightarrow n \cdot \dfrac{dx}{dy} = \dfrac{1}{y}$

$$\Rightarrow \quad n\frac{d^2x}{dy^2} = -\frac{1}{y^2} \Rightarrow n\frac{d^2x}{dy^2} = -e^{-2nx}$$

Now, $\quad \dfrac{d^2y}{dx^2} \cdot \dfrac{d^2x}{dy^2} = n^2 \cdot e^{nx}\left(\dfrac{-e^{-2nx}}{n}\right) = -ne^{-nx}$

The correct option is (D)

6. We have,

$$\frac{dx}{d\theta} = -\sin\theta + \frac{\sec^2\theta/2}{2\tan\theta/2} = -\sin\theta + \frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= -\sin\theta + \frac{1}{\sin\theta} = \frac{1 - \sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta}$$

and $\dfrac{dy}{d\theta} = \cos\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta \sin\theta}{\cos^2\theta} = \tan\theta$$

Also, $\dfrac{d^2y}{dx^2} = \sec^2\theta \cdot \dfrac{d\theta}{dx} = \dfrac{\sin\theta}{\cos^4\theta}$

$$\therefore \quad \frac{d^2y}{dx^2} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = n\pi, n \in Z$$

The correct option is (A)

7. $f(x) = \sqrt{x-1} + \sqrt{25 + (x-1) - 10\sqrt{x-1}}$

$$= \sqrt{x-1} + \sqrt{(5 - \sqrt{x-1})^2}$$

$$= \sqrt{x-1} + |5 - \sqrt{x-1}| = 5$$

$(\because \sqrt{x-1} < 5 \text{ for } 1 < x < 26)$

$\therefore f'(x) = 0$

The correct option is (A)

8. We have,

$$y = \tan^{-1}\left(\frac{1 - 3\log x}{1 + 3\log x}\right) + \tan^{-1}\left(\frac{4 + 3\log x}{1 - 12\log x}\right)$$

$$= \tan^{-1}1 - \tan^{-1}(3\log x) + \tan^{-1}4 + \tan^{-1}(3\log x)$$

$$\left[\begin{array}{l} \text{Using } \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\dfrac{A-B}{1+AB}\right) \\ \text{and } \tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\dfrac{A+B}{1-AB}\right) \end{array}\right]$$

$= \tan^{-1}1 + \tan^{-1}(4)$

$\therefore \dfrac{dy}{dx} = 0$ and hence $\dfrac{d^2y}{dx^2} = 0.$

The correct option is (B)

9. We have,

$$f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\text{and } f'''(x) = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\therefore \quad f'''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$$

($\because R_1$ and R_2 are identical)

The correct option is (D)

10. Differentiating the given equation with respect to x, we get

$$xy' + y \cdot 1 - \frac{1}{y} y' = 0$$

$\Rightarrow \; xyy' - y' + y^2 = 0$ i.e.$(xy - 1)\, y' + y^2 = 0$

Differentiating again with respect to x, we get

$(xy - 1)\, y'' + y'\,(xy' + y \cdot 1) + 2yy' = 0$

$\Rightarrow \quad x\,(yy'' + y'^2) - y'' + 3yy' = 0$

$\therefore \quad k = 3$

The correct option is (B)

11. We have,

$$\frac{dx}{dt} = \cos t$$

$$\text{and } \frac{dy}{dt} = \sqrt{2}\,(a+b)\,e^{t\sqrt{2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sqrt{2}(a+b)e^{t\sqrt{2}}}{\cos t} = \frac{\sqrt{2}y}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x^2)\left(\frac{dy}{dx}\right)^2 = 2y^2$$

Differentiating with respect to x, we get

$(1 - x^2)\, 2y'\, y'' - 2x\,(y')^2 = 4yy'$

$\Rightarrow \; (1 - x^2)\, y'' - xy' = 2y$ (dividing by $2y'$)

$\therefore \; k = 2$

The correct option is (C)

12. We have, $f(x) = \dfrac{x^2 - x}{x^2 + 2x}$

Clearly, $f(0)$ and $f(-2)$ are not defined.

So domain of $f = R\backslash\{0, -2\}$.

Then, in this domain, we have

$$y = f(x) = \frac{x-1}{x+2} \Rightarrow yx + 2y = x - 1$$

$$\text{or} \quad x = \frac{2y+1}{1-y}, \text{ i.e., } f^{-1}(x) = \frac{2x+1}{1-x}$$

$$\therefore \quad \frac{df^{-1}(x)}{dx} = \frac{2(1-x)+2x+1}{(1-x)^2} = \frac{3}{(1-x)^2}$$

The correct option is (B)

13. We have,

$F(x) = f(x)\, g(x)\, h(x)$

$\Rightarrow \; \log F(x) = \log f(x) + \log g(x) + \log h(x)$

Differentiating both the sides with respect to x, we get

$$\frac{F'(x)}{F(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} + \frac{h'(x)}{h(x)}$$

$$\Rightarrow \quad \frac{F'(x_0)}{F(x_0)} = \frac{f'(x_0)}{f(x_0)} + \frac{g'(x_0)}{g(x_0)} + \frac{h'(x_0)}{h(x_0)}$$

$\Rightarrow \quad 21 = 4 - 7 + k \Rightarrow k = 24$

The correct option is (A)

14. Clearly, $f(x)$ must be of the form

$f(x) = a_0\,[x^n + (k - x)^n] + a_1\,[x^{n-1} + (k - x)^{n-1}]$
$+ \ldots + a_{n-1}\,[x + (k - x)] + a_n$

It may be noted that n must be even for otherwise $f(x)$ will become a polynomial of degree $n - 1$.

Clearly, $f'(x)$ is a polynomial of degree $\boldsymbol{n-1}$.

The correct option is (B)

15. We have,

$f(x) = x - 1$ $\quad (\because x > 2)$

$f[f(x)] = f(x - 1) = |(x - 1) - 1| = |x - 2|$

$= (x - 2)$ $\quad (\because x > 2)$

$\therefore \quad g(x) = f\{f[f(x)]\} = f(x - 2)$

$= |(x - 2) - 1| = |x - 3|$

$$= \begin{cases} x-3, & \text{if } x \ge 3 \\ -x+3, & \text{if } 2 \le x < 3 \end{cases}$$

$$\therefore \quad g'(x) = \begin{cases} 1, & \text{if } x \ge 3 \\ \mathbf{-1}, & \textbf{if } \mathbf{2 \le x < 3} \end{cases}$$

The correct option is (A)

16. $f(x) = |(x - 4)(x - 5)|$

$$= \begin{cases} (x-4)(x-5) & \text{if } (x-4)(x-5) \ge 0 \\ -(x-4)(x-5) & \text{if } (x-4)(x-5) < 0 \end{cases}$$

$$= \begin{cases} x^2 - 9x + 20 & \text{if } x \le 4 \text{ or } x \ge 5 \\ -(x^2 - 9x + 20) & \text{if } 4 < x < 5 \end{cases}$$

$$\therefore \quad f'(x) = \begin{cases} 2x - 9 & \text{if } x \le 4 \text{ or } x \ge 5 \\ -2x + 9 & \text{if } 4 < x < 5 \end{cases}$$

The correct option is (C)

17. $2f(\sin x) + f(\cos x) = x$ (1)

Replace x by $\frac{\pi}{2} - x$

$2f(\cos x) + f(\sin x) = \frac{\pi}{2} - x$ (2)

Solving we get, $3f(\sin x) = \frac{\pi}{2} + 3x$

$\therefore f(x) = \frac{\pi}{6} + \sin^{-1} x \ \therefore \ \frac{d}{dx} f(x) = \frac{1}{\sqrt{1-x^2}}$

The correct option is (C)

18. $f(x) = |x - a| = x - a$ (if $x > a$)

$f[f(x)] = f(x - a) = |x - 2a| = x - 2a$ (if $x > 2a$)

$f\{f[f(x)]\} = f(|x - 2a|) = |x - 3a|$

$= x - 3a$ (if $x > 3a$)

$\Rightarrow \quad g'(\alpha) = 1, \forall \alpha > 3a.$

The correct option is (C)

19. $\phi(x) = f^{-1}(x) \Rightarrow x = f[\phi(x)]$

$\Rightarrow \ 1 = \{f'[\phi(x)]\} \cdot \phi'(x)$

$\Rightarrow \ \phi'(x) = \frac{1}{f'[\phi(x)]}$

Now $f'(x) = \frac{1}{1+x^5} \Rightarrow f'[\phi(x)] = \frac{1}{1+[\phi(x)]^5}$

$\therefore \ \phi'(x) = \frac{1}{f'[\phi(x)]} = 1 + [\phi(x)]^5$

The correct option is (C)

20. $y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2}$

Now, $\frac{\sqrt{1+y^4}}{\sqrt{1+x^4}} = \frac{\sqrt{1+\frac{1}{x^4}}}{\sqrt{1+x^4}} = \frac{\frac{\sqrt{x^4+1}}{x^2}}{\sqrt{x^4+1}} = \frac{1}{x^2} = -\frac{dy}{dx}$

Writing in the form of differentials, we have

$\frac{dx}{\sqrt{1+x^4}} = -\frac{dy}{\sqrt{1+y^4}} \Rightarrow \frac{dx}{\sqrt{1+x^4}} + \frac{dy}{\sqrt{1+y^4}} = 0.$

The correct option is (A)

21. $\frac{dy}{dx} = \frac{f'\phi - f\phi'}{\phi^2}$

$\frac{d^2y}{dx^2} = \frac{(f''\phi - f\phi'')\phi^2 - (f'\phi - f\phi')2\phi\phi'}{\phi^4}$

$\therefore \ \frac{1}{y}\frac{d^2y}{dx^2} = \frac{f''\phi - f\phi''}{f\phi} - \frac{2(f'\phi - f\phi')}{f\phi^2 \cdot \phi'}(\phi')^2$

$= \frac{f''}{f} - \frac{\phi''}{\phi} + \frac{2(y-z)}{f\phi}(\phi')^2$

The correct option is (B)

22. Given that $g^{-1}(x) = f(x)$

$\Rightarrow \ x = g[f(x)]$ or $g'[f(x)]f'(x) = 1$

$\Rightarrow \ g'[f(x)] = \frac{1}{f'(x)} \Rightarrow g''[f(x)] \times f'(x) = -\frac{f''(x)}{[f'(x)]^2}$

$\Rightarrow \ g''[f(x)] = -\frac{f''(x)}{[f'(x)]^3}$

The correct option is (A)

23. $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

$f'(a+b) = f'(a) + f'(b)$

$\Rightarrow \ n(a+b)^{n-1} = na^{n-1} + nb^{n-1}$

$\Rightarrow \ (a+b)^{n-1} = a^{n-1} + b^{n-1}$

Which is true for $n = 2$ and false for $n = 1$ and $n = 4$.

Also, for $n = 0, f(x) = 1$,

So, $f'(x) = 0; f'(a+b) = 0$

So, $f'(a+b) = f'(a) + f'(b)$

Hence, there are two values of n.

The correct option is (C)

24. $f'(x) = (1/2)5^{2x+1}(\log 5)(2) = \log 5 \cdot (5^{2x+1})$

Also, $g'(x) = 5^x \log 5 + 4 \log 5$

So $\{x : f'(x) > g'(x)\}$

$= \{x : \log 5 \times 5^{2x+1} > \log 5 \ 5^x + 4 \log 5\}$

$= \{x : 5^{2x+1} > 5^x + 4\}$

$= \{t = 5^x : 5t^2 - t - 4 > 0\}$

$= \{t = 5^x : (5t+4)(t-1) > 0\}$

$= \{t = 5^x : t > 1 \text{ or } t < -4/5\}$

$= \{t = 5^x : t > 1\} = (0, \infty).$

The correct option is (C)

25. $I_n = \frac{d^{n-1}}{dx^{n-1}}(x^{n-1} + nx^{n-1} \log x)$

$\Rightarrow \ I_n = (n-1)! + nI_{n-1} \Rightarrow I_n - nI_{n-1} = (n-1)!.$

The correct option is (D)

26. $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{f'(x^3) \cdot 3x^2}{g'(x^2) \cdot 2x} = \frac{\cos x^3 \cdot 3x^2}{\sin x^2 \cdot 2x}$

$\therefore \ \frac{du}{dv} = \frac{3}{2} x \cos x^3 \cdot \operatorname{cosec} x^2$

The correct option is (A)

27. Put $x = y = 1$, we get $f(1) = 0$

Put $y = \frac{1}{x}$, we get $f(x) + f\left(\frac{1}{x}\right) = f(1) = 0$

$\therefore \ f(e) + f\left(\frac{1}{e}\right) = 0$

The correct option is (B)

28. Putting $x = 0, y = 0$, we get

$2f(0) + [f(0)]^2 = 1 \Rightarrow f(0) = \sqrt{2} - 1$ $[\because f(x) > 0]$

Putting $y = x$, $2f(x) + [f(x)]^2 = 1$

Differentiating with respect to x, we get

$2f'(x) + 2f(x) \cdot f'(x) = 0$ or $f'(x)\,[1 + f(x)] = 0$

$\Rightarrow\ f'(x) = 0$, because $f(x) > 0$

The correct option is (B)

29. $3f(x) - 2f\left(\dfrac{1}{x}\right) = x$ (1)

Put $x = \dfrac{1}{x}$, then $3f\left(\dfrac{1}{x}\right) - 2f(x) = \dfrac{1}{x}$ (2)

Solving (1) and (2), we get

$$5f(x) = 3x + \frac{2}{x} \Rightarrow f'(x) = \frac{3}{5} - \frac{2}{5x^2}$$

$$\therefore\ f'(2) = \frac{3}{5} - \frac{2}{20} = \frac{1}{2}$$

The correct option is (B)

30. Given: $\sqrt{x+y} + \sqrt{y-x} = c$ (1)

$$\Rightarrow \frac{(y+x)-(y-x)}{\sqrt{x+y}-\sqrt{y-x}} = c$$

$$\Rightarrow \sqrt{y+x} - \sqrt{y-x} = \frac{2x}{c} \quad (2)$$

By adding Eq. (1) and (2), we get

$$2\sqrt{y+x} = c + \frac{2x}{c} \Rightarrow 4(y+x) = c^2 + \frac{4x^2}{c^2} + 4x$$

$$\therefore\ 4\frac{dy}{dx} = \frac{8x}{c^2}$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{2}{c^2}$$

The correct option is (A)

31. We have

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{1}{dy/dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{dy/dx}\right)\cdot\frac{dx}{dy}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\cdot\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)^3}\cdot\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^{-3}\left(\frac{d^2y}{dx^2}\right)$$

The correct option is (A)

32. $x^{2x} - 2x^x \cot y - 1 = 0$ (1)

Now $x = 1$,

$1 - 2\cot y - 1 = 0$

$\Rightarrow\ \cot y = 0 \Rightarrow y = \dfrac{\pi}{2}$

Now differentiating eq., (1) with respect to 'x'

$$2x^{2x}(1+\log x) - 2\left[x^x(\text{cosec}^2 y)\frac{dy}{dx} + \cot y x^x(1+\log x)\right] = 0$$

Now at $\left(1, \dfrac{\pi}{2}\right)$

$$2(1+\log 1) - 2\left(1(-1)\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} + 0\right) = 0$$

$$\Rightarrow 2 + 2\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = -1$$

The correct option is (A)

33. We have, $\lambda = \sqrt{y^2+z^2}$

Now, $y\dfrac{d}{dx}\left(\dfrac{y}{\lambda}\right) + \dfrac{d}{dx}\left(\dfrac{z^2}{\lambda}\right)$

$$= y\frac{d}{dx}\left(\frac{y}{\sqrt{y^2+z^2}}\right) + \frac{d}{dx}\left(\frac{z^2}{\sqrt{y^2+z^2}}\right)$$

$$= y\left[\frac{\sqrt{y^2+z^2}\,\frac{dy}{dx} - \frac{y\left(y\frac{dy}{dx}+z\frac{dz}{dx}\right)}{\sqrt{y^2+z^2}}}{y^2+z^2}\right] + \left[\frac{(\sqrt{y^2+z^2})\,2z\frac{dz}{dx} - \frac{z^2\left(y\frac{dy}{dx}+z\frac{dz}{dx}\right)}{\sqrt{y^2+z^2}}}{y^2+z^2}\right]$$

$$= \frac{1}{(y^2+z^2)^{3/2}}\left[y\frac{dy}{dx}(y^2+z^2) - y^2\left(y\frac{dy}{dx}+z\frac{dz}{dx}\right) + 2z\frac{dz}{dx}(y^2+z^2) - z^2\left(y\frac{dy}{dx}+z\frac{dz}{dx}\right)\right]$$

$$= \frac{1}{(y^2+z^2)^{3/2}}\left[z\,(y^2+z^2)\frac{dz}{dx}\right]$$

$$= \frac{z}{\sqrt{y^2+z^2}}\frac{dz}{dx} = \frac{z}{\lambda}\frac{dz}{dx}.$$

The correct option is (A)

34. We have, $S_n = \dfrac{a(r^n-1)}{r-1}$

$\Rightarrow\ (r-1)\,S_n = ar^n - a$

Differentiating both sides with respect to r, we get

$$(r-1)\frac{dS_n}{dr} + S_n = nar^{n-1} - 0$$

$$\Rightarrow\ (r-1)\frac{dS_n}{dr} = nar^{n-1} - S_n$$

$$= n\,[n\text{th term of G.P.}] - S_n$$

$= n(S_n - S_{n-1}) - S_n$

$= (n-1)S_n - nS_{n-1}$.

The correct option is (B)

35. Differentiating the given equation with respect to x_1, we get

$$\frac{1}{n} f'\left(\frac{x_1 + x_2 + .. + x_n}{n}\right) = \frac{f'(x_1)}{n}$$

[Since all x_i's are independent to each other, $\therefore \frac{dx_i}{dx_j} = 0$ if $i \neq j$ and $\frac{dx_i}{dx_j} = 1$ if $(i = j)$]

On putting $x_1 = x_2 = ... = x_{n-1} = 0$

and $x_n = x$, we get $f'\left(\frac{x}{n}\right) = f'(0) = a$.

On integrating, we get $nf\left(\frac{x}{n}\right) = ax + c$

Since $f(0) = b$, we have $c = nb$

$\therefore \quad nf\left(\frac{x}{n}\right) = ax + nb \Rightarrow nf(x) = nax + nb$

$\Rightarrow \; f(x) = ax + b$.

$\therefore \; f'(x) = a, \forall x \in R$

The correct option is (A)

36. We have,

$y^2 = P(x)$ (1)

$\Rightarrow \; 2y\frac{dy}{dx} = P'(x)$ (2)

$\Rightarrow \; 2\frac{dy}{dx}\cdot\frac{dy}{dx} + 2y\cdot\frac{d^2y}{dx^2} = P''(x)$

$\Rightarrow \; 2y^2\left(\frac{dy}{dx}\right)^2 + 2y^3\cdot\frac{d^2y}{dx^2} = y^2P''(x)$

$\Rightarrow 2y^3\frac{d^2y}{dx^2} = y^2P''(x) - 2y^2\left(\frac{dy}{dx}\right)^2$

$= y^2P''(x) - \frac{1}{2}[P'(x)]^2$ [from (2)]

$\Rightarrow 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$

$= 2y\frac{dy}{dx}P''(x) + y^2P'''(x) - \frac{1}{2}2P'(x)\cdot P''(x)$

$= P'(x)P''(x) + y^2P'''(x) - P'(x)P''(x)$

$\left[\because 2y\frac{dy}{dx} = P'(x)\right]$

$= y^2P'''(x)$

$= P(x)P'''(x)$ $[\because y^2 = P(x)]$

The correct option is (B)

37. We have, $[f(x)]^n = f(nx)$

Differentiating with respect to x, we get

$n[f(x)]^{n-1}\cdot f'(x) = f'(nx)\cdot n$

$\Rightarrow \; [f(x)]^{n-1}\cdot f'(x) = f'(nx) \Rightarrow [f(x)]^n \cdot f'(x) = f'(nx)\cdot f(x)$

[Multiplying both sides by $f(x)$]

$\Rightarrow \; f(nx)\cdot f'(x) = f'(nx)\cdot f(x)$ $[\because [f(x)]^n = f(nx)]$

The correct option is (C)

38. We have,

$$\Delta = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ x^2f'' + 4xf' + 2f & x^2g'' + 4xg' + 2g & x^2h'' + 4xh' + 2h \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' + 4xf' & x^2g'' + 4xg' & x^2h'' + 4xh' \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$]

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix}$$ [Applying $R_3 \to R_3 - 4R_2$]

$$= x^3\begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$\therefore \Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

The correct option is (C)

39. Let $\Delta(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

So, $\Delta(\alpha) = 0 = \Delta'(\alpha)$, therefore α is a repeated root of $\Delta(x)$ and α is a repeated root of the quadratic equation $f(x) = 0$, so Δ is divisible by $f(x)$.

The correct option is (D)

40. We have, $\Delta\Delta' = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$

$\therefore \quad \Delta' = \Delta^2$.

The correct option is (C)

41. We have, $f(x) = \frac{x^2 - x}{x^2 + 2x}$

Clearly, $f(0)$ and $f(-2)$ are not defined.

So, domain of $f = R\backslash\{0, -2\}$

Then, in this domain, we have

$$y = f(x) = \frac{x-1}{x+2} \Rightarrow yx + 2y = x - 1$$

or, $x = \dfrac{2y+1}{1-y}$, i.e., $f^{-1}(x) = \dfrac{2x+1}{1-x}$

$$\therefore \frac{df^{-1}(x)}{dx} = \frac{2(1-x)+2x+1}{(1-x)^2} = \frac{3}{(1-x)^2}$$

The correct option is (B)

42. Clearly, $f(x)$ must be of the form

$f(x) = a_0\,[x^n + (k-x)^n] + a_1\,[x^{n-1} + (k-x)^{n-1}]$
$+ \ldots + a_{n-1}\,[x + (k-x)] + a_n.$

It may be noted that n must be even for otherwise $f(x)$ will become a polynomial of degree $n - 1$.

Clearly, $f'(x)$ is a polynomial of degree $n - 1$.

The correct option is (B)

43. $\dfrac{dy}{dx} = \dfrac{f'\phi - f\phi'}{\phi^2}$

$$\frac{d^2y}{dx^2} = \frac{(f''\phi - f\phi'')\phi^2 - (f'\phi - f\phi')2\phi\phi'}{\phi^4}$$

$$\therefore \frac{1}{y}\frac{d^2y}{dx^2} = \frac{f''\phi - f\phi''}{f\phi} - \frac{2(f'\phi - f\phi')}{f\phi^2\cdot\phi'}(\phi')^2$$

$$= \frac{f''}{f} - \frac{\phi''}{\phi} + \frac{2(y-z)}{f\phi}\cdot(\phi')^2$$

The correct option is (B)

44. $f'(x) = (1/2)5^{2x+1}\,(\log 5)(2) = \log 5 \cdot (5^{2x+1})$

Also, $g'(x) = 5^x \log 5 + 4 \log 5$

So, $\{x: f'(x) > g'(x)\}$

$= \{x: \log 5 \times 5^{2x+1} > \log 5\; 5^x + 4 \log 5\}$

$= \{x: 5^{2x+1} > 5^x + 4\}$

$= \{t = 5^x : 5t^2 - t - 4 > 0\}$

$= \{t = 5^x : (5t + 4)(t - 1) > 0\}$

$= \{t = 5^x : t > 1 \text{ or } t < -4/5\}$

$= \{t = 5^x : t > 1\} = (0, \infty).$

The correct option is (C)

45. Putting $x = 0$, $y = 0$, we get

$2f(0) + \{f(0)\}^2 = 1 \Rightarrow f(0) = \sqrt{2} - 1$ $\quad(\because f(x) > 0)$

Putting $y = x$, $2f(x) + \{f(x)\}^2 = 1$

Differentiating with respect to x, we get

$2f'(x) + 2f(x) \times f'(x) = 0$ or $f'(x)\,\{1 + f(x)\} = 0$

$\Rightarrow f'(x) = 0$, because $f(x) > 0$

The correct option is (B)

46. Given: $\sqrt{x+y} + \sqrt{y-x} = c$ $\quad(1)$

$$\Rightarrow \frac{(y+x)-(y-x)}{\sqrt{x+y} - \sqrt{y-x}} = \frac{2x}{c}$$

$$\Rightarrow \sqrt{y+x} - \sqrt{y-x} = \frac{2x}{c} \qquad (2)$$

By adding (1) and (2) we get

$$2\sqrt{y+x} = c + \frac{2x}{c} \Rightarrow 4(y+x) = c^2 + \frac{4x^2}{c^2} + 4x$$

$$\therefore 4\frac{dy}{dx} = \frac{8x}{c^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2}{c^2}$$

The correct option is (A)

47. $f(x) = \displaystyle\prod_{k=1}^{n} (\cos(2k-1)x + i\sin(2k-1)x)$

$$= \cos\sum_{k=1}^{n}(2k-1)x + i\sin\sum_{k=1}^{n}(2k-1)x$$

$$= \cos n^2x + i\sin n^2x$$

(Using De Moivre's Theorem)

$\therefore$ $(\text{Re}\, f(x))'' = -n^4 \cos n^2x$

and, $(\text{Im} f(x))'' = -n^4 \sin n^2x$. Thus,

$(\text{Re}\, f(x))'' + i(\text{Im} f(x))'' = -n^4\,[\cos n^2x + i\sin n^2x]$
$= -n^4 f(x).$

The correct option is (B)

48. $2f(\sin x) + f(\cos x) = x$ $\quad(1)$

Replace x by $\dfrac{\pi}{2} - x$

$2f(\cos x) + f(\sin x) = \dfrac{\pi}{2} - x$ $\quad(2)$

Solving we get, $3f(\sin x) = \dfrac{\pi}{2} + 3x$

$$\therefore f(x) = \frac{\pi}{6} + \sin^{-1}x \;\; \therefore \frac{d}{dx}f(x) = \frac{1}{\sqrt{1-x^2}}$$

The correct option is (C)

49. $f(x) = \sqrt{x-1} + \sqrt{25 + (x-1) - 10\sqrt{x-1}}$

$$= \sqrt{x-1} + \sqrt{(5 - \sqrt{x-1})^2}$$

$$= \sqrt{x-1} + |5 - \sqrt{x-1}| = 5$$

$[\because \sqrt{x-1} < 5 \text{ for } 1 < x < 26]$

$\therefore f'(x) = 0$

The correct option is (A)

50. We have,

$f(x) = 2 - x, x < 2$
$= x - 2, x \geq 2$

Thus, we have,

$$g(x) = f\{f(x)\} = 2 - f, f < 2$$
$$= f - 2, f \geq 2$$

i.e., $$\left.\begin{aligned} g(x) &= 2-(2-x), & 2-x<2 \\ &= (2-x)-2, & 2-x\geq 2 \end{aligned}\right\} x<2$$

$$\left.\begin{aligned} &= 2-(x-2), & x-2<2 \\ &= (x-2)-2, & x-2\geq 2 \end{aligned}\right\} x\geq 2$$

i.e., $g(x) = x,\ 0 < x < 2$
$= -x,\ x \leq 0$
$= 4 - x,\ 2 \leq x < 4$
$= x - 4,\ 4 \leq x$

i.e., $g(x) = -x,\ x \leq 0$
$= x,\ 0 < x < 2$
$= 4 - x,\ 2 \leq x < 4$
$= x - 4,\ 4 \leq x$

Hence, we have for $x > 2$

$g'(x) = -1,\ 2 < x < 4$
$= 1,\ 4 \leq x$

The derivative of $g(x)$ does not exist at $x = 4$.

The correct option is (D)

51. We have,

$x = 2t - |t|,\ y = t^3 + t^2\,|t|$

$\Rightarrow\ x = 3t,\ y = 0$ when $t < 0$

$x = t,\ y = 2t^3$ when $t \geq 0$

Eliminating the parameter t, we get

$$y = \begin{cases} 0, & x<0 \\ 2x^3, & x\geq 0 \end{cases}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \begin{cases} 0, & x<0 \\ 6x^2, & x\geq 0 \end{cases}$$

Hence, the function is differentiable everywhere and its derivative at $x = 0$ $(t = 0)$ is 0

The correct option is (A)

52. Since function $f(x)$ leaves remainder 15 when divided by $x - 3$, therefore $f(x)$can be written as

$$f(x) = (x-3)\, l(x) + 15 \qquad (1)$$

Also, $f(x)$ leaves remainder $2x + 1$ when divided by $(x - 1)^2$. Thus, $f(x)$ can also be written as

$$f(x) = (x-1)^2 m(x) + 2x + 1 \qquad (2)$$

It $R(x)$ be the remainder when $f(x)$ is divided by $(x - 3)(x - 1)^2$, then we may write

$$f(x) = (x-3)(x-1)^2 n(x) + R(x) \qquad (3)$$

Since $(x - 3)(x - 1)^2$ is a polynomial of degree three, the remainder has to be a polynomial of degree less than or equal to two. Thus, let

$R(x) = ax^2 + bx + c$

From equations (1) and (3), we have

$f(3) = 15 = R(3)$

i.e., $9a + 3b + c = 15$ (4)

From equations (2) and (3), we have

$f(1) = 3 = R(1)$

i.e., $a + b + c = 3$ (5)

From equations (2) and (3), we have

$f'(1) = 2 = R'(1)$

i.e., $2a + b = 2$ (6)

Solving equations (4), (5) and (6), we get

$a = 2,\ b = -2,\ c = 3$

The correct option is (C)

53. We have,

$$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \qquad (1)$$

Substituting $\frac{1}{x}$ in place of x, we have

$$a f\left(\frac{1}{x}\right) + bf(x) = x - 5 \qquad (2)$$

Eliminating $f\left(\frac{1}{x}\right)$ from equations (1) and (2), we have

$$(a^2 - b^2) f(x) = a\left(\frac{1}{x} - 5\right) - b(x-5)$$

$$\Rightarrow\ f(x) = \frac{1}{a^2-b^2}\left[\frac{a}{x} - bx + 5(b-a)\right]$$

$$\therefore\ f'(x) = \frac{1}{b^2-a^2}\left[\frac{a}{x^2} + b\right]$$

The correct option is (A)

54. We have, $x = \cos^7\theta$ and $y = \sin\theta$

Differentiating with respect to θ, we get

$$\frac{dx}{d\theta} = -7\cos^6\theta\sin\theta$$

and, $$\frac{dy}{d\theta} = \cos\theta$$

Thus, we have,

$$\frac{dx}{dy} = -7\cos^5\theta\sin\theta$$

and, $$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{d\theta}\left(\frac{dx}{dy}\right)\cdot\frac{d\theta}{dy}$$

$$= (35\cos^4\theta\sin^2\theta - 7\cos^6\theta)\frac{1}{\cos\theta}$$
$$= 35\cos^3\theta\sin^2\theta - 7\cos^5\theta$$
$$= 35\cos^3\theta(1-\cos^2\theta) - 7\cos^5\theta$$
$$= 35\cos^3\theta - 42\cos^5\theta$$

and, $$\frac{d^3x}{dy^3} = \frac{d}{dy}\left(\frac{d^2x}{dy^2}\right) = \frac{d}{d\theta}\left(\frac{d^2x}{dy^2}\right)\cdot\frac{d\theta}{dy}$$

$= (-105 \cos^2\theta \sin\theta + 210 \cos^4 q\text{sn } \theta)\dfrac{1}{\cos\theta}$

$= 105 \sin q\cos\theta (2\cos^2\theta - 1)$

$= \dfrac{105}{2} \sin 2q\cos 2\theta = \dfrac{105}{4} \sin 4\theta$

The correct option is (A)

55. We have,

$$y = \sec^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$$

$$= \cos^{-1}\left(\frac{x-1}{x+1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{2}$$

However, the above function is defined only for values of x, given by

$$-1 \le \frac{x-1}{x+1} \le 1$$

i.e., $\dfrac{x-1}{x+1} + 1 \ge 0$ and $\dfrac{x-1}{x+1} - 1 \le 0$

i.e., $\dfrac{2x}{x+1} \ge 0$ and $\dfrac{2}{x+1} \ge 0$

i.e., $x < -1$ or ≥ 0 and $x > -1$

i.e., $x \ge 0$

Hence, we have,

$y = \dfrac{\pi}{2}, x \ge 0$

and, $\dfrac{dy}{dx} = 0, x > 0$

The correct option is (B)

56. We have,

$$f(x) = \cos^{-1}\left(\frac{x^{-1} - x}{x^{-1} + x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

and,

$$f'(x) = \frac{-1}{\sqrt{1 - \left(\dfrac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{-1}{\sqrt{4x^2}} \cdot \frac{-4x}{(1+x^2)^2} = \frac{2x}{|x|(1+x^2)^2}$$

which is an odd function, since

$f'(-x) = -f'(x)$

The correct option is (A)

57. We have,

$f(x) = (1-x)^n$

$f'(x) = -n(1-x)^{n-1}$

$f''(x) = (-1)^2 n(n-1)(1-x)^{n-2}$

. . .

. . .

. . .

$f^n(x) = (-1)^n n!$

Thus, $f(0) = 1, f'(0) = -n, f''(0) = n(n-1), \ldots,$

$f^n(0) = (-1)^n n!$

Hence, $f(0) + f'(0) + \dfrac{f''(0)}{2!} + \ldots + \dfrac{f^{(n)}(0)}{n!}$

$= 1 - n + \dfrac{n(n-1)}{2!} + \ldots + (-1)^n \dfrac{n!}{n!}$

$= {}^nC_0 - {}^nC_1 + {}^nC_2 + \ldots + (-1)^{nn}C_n$

$= 0$ [Putting $x = 1$ in the expansion of $(1-x)^n$]

The correct option is (B)

58. We have,

$$y = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right) = \cot^{-1}\left(\frac{x^{2x} - 1}{2x^x}\right)$$

$$= -\tan^{-1}\left(\frac{2x^x}{1 - x^{2x}}\right)$$

$$= -2\tan^{-1}(x^x)$$

and, $y' = \dfrac{-2}{1 + x^{2x}} \cdot x^x (1 + \ln x)$

Hence, $y'(1) = \dfrac{-2}{1+1} \cdot 1 = -1$

The correct option is (C)

59. We have,

$$y = \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}} = \sqrt{\frac{\cos^2\theta}{\sin^2\theta}} = |\cot\theta|$$

In the neighbourhood of $\theta = \pi/4$, we have

$y = \cot\theta$

and, $y' = -\text{cosec}^2\theta \Rightarrow y'\left(\dfrac{\pi}{4}\right) = -\text{cosec}^2\left(\dfrac{\pi}{4}\right) = -2$

In the neighbourhood of $\theta = \dfrac{3\pi}{4}$, we have

$y = -\cot\theta$

and, $y' = \text{cosec}^2\theta$

$\Rightarrow y'\left(\dfrac{3\pi}{4}\right) = \text{cosec}2\left(\dfrac{3\pi}{4}\right) = 2$

Hence, $y'\left(\dfrac{\pi}{4}\right) y'\left(\dfrac{3\pi}{4}\right) = -4$

The correct option is (B)

60. Let $f(x) = ax^2 + bx + c$

Then, $f'(x) = 2ax + b$

Also, $f(1) = f(-1)$

$\Rightarrow a + b + c = a - b + c \Rightarrow 2b = 0 \Rightarrow b = 0$

$\therefore f'(x) = 2ax$

$\therefore f'(a_1) = 2aa_1, f'(a_2) = 2aa_2$ and $f'(a_3) = 3aa_3$

As a_1, a_2, a_3 are in A.P., we have

$f'(a_1), f'(a_2), f'(a_3)$ are in A. P.

The correct option is (A)

61. We have, $5f(x) + 3f\left(\frac{1}{x}\right) = x + 2$ (1)

Put $x = \frac{1}{x}$, we get

$$5f\left(\frac{1}{x}\right) + 3f(x) = \frac{1}{x} + 2 \quad (2)$$

Solving (1) and (2), we get

$$16f(x) = 5x - \frac{3}{x} + 4 \quad (3)$$

$\therefore$ $y = x\,f(x)$

$$\Rightarrow y = x \cdot \frac{1}{16}\left\{5x - \frac{3}{x} + 4\right\} = \frac{1}{16}\left[5x^2 - 3 + 4x\right]$$

or, $\frac{dy}{dx} = \frac{1}{16}[10x + 4]$

Therefore, $\left(\frac{dy}{dx}\right)_{\text{at } x=1} = \frac{10+4}{16} = \frac{7}{8}$

The correct option is (C)

62. Let, $f(x) = ax^2 + bx + c$

Then, $g(x) = f(x) + f'(x) + f''(x)$

$= (ax^2 + bx + c) + (2ax + b) + 2a$

$= ax^2 + (b + 2a)x + (c + b + 2a)$ (1)

As $f(x) > 0\ \forall\ x$, we have

$a > 0$ and $D < 0$

i.e., $a > 0$ and $b^2 - 4ac < 0$ (2)

Now, $D = (b + 2a)^2 - 4a\,(c + b + 2a)$

$= b^2 + 4a^2 + 4ab - 4ac - 4ab - 8a^2$

$= b^2 - 4a^2 - 4ac$

$= b^2 - 4ac - 4a^2$

$= (b^2 - 4ac) - (4a^2)$,

where $b^2 - 4ac < 0$ from (2)

$\therefore$ $D = -$ve as $b^2 - 4ac - (4a^2) < 0$

Hence, $g(x)$ is a quadratic in which $a > 0$ and $D < 0$

$\Rightarrow$ $g(x) > 0$ for all x

The correct option is (A)

63. We have,

$f(x + y) = f(x) + f(y) + 2xy - 1$

Put $x = y = 0 \Rightarrow f(0) = 2f(0) - 1 \Rightarrow f(0) = 1$

Also, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{f(x) + f(h) + 2xh - 1 - f(x)}{h}$

$= 2x + \lim_{h\to 0} \frac{f(h) - 1}{h}$

$= 2x + f'(0) = 2x + \sin\theta$

Integrating, we get $f(x) = x^2 + x\sin\theta + c$

$f(0) = 1 \Rightarrow 1 = c$

$\therefore f(x) = x^2 + x\sin\theta + 1 > 0\ \forall\ x \in R$

$[\because D = \sin^2\theta - 4 < 0]$

The correct option is (A)

64. $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$

$\therefore$ $f'(x) = 3x^2 + 2xf'(1) + f''(2)$

$\Rightarrow$ $f''(x) = 6x + 2f'(1)$

$\Rightarrow$ $f'''(x) = 6$

Now, $f'(1) = 3 + 2f'(1) + f''(2)$

$\Rightarrow$ $f''(2) + f'(1) + 3 = 0$ (1)

Again, $f''(2) = 12 + 2f'(1)$

$\Rightarrow$ $f''(2) - 2f'(1) - 12 = 0$ (2)

Again, $f'''(3) = 6$ (3)

From (1) and (2)

$2f''(2) + f''(2) - 6 = 0 \Rightarrow f''(2) = 2$ (4)

$\therefore$ (1) gives $f'(1) + 2 + 3 = 0$

$\Rightarrow$ $f'(1) = -5$ (5)

$\therefore$ $f(x) = x^3 - 5x^2 + 2x + 6$

$\therefore$ $f(0) = 6$,

$f(1) = 1 - 5 + 2 + 6 = 4$

$f(2) = 8 - 20 + 4 + 6 = -2$

$f(3) = 27 - 45 + 6 + 6 = -6$

$f(0) + f(2) = 6 - 2 = f(1)$

$\therefore$ (a) is true

$f(0) + f(3) = 6 - 6 = 0$

$f(1) + f(3) = (1 - 5 + 2 + 6) - 6$

$= -2 = f(2)$

$f(1) + f(3) = -2 \neq f(0)$ $[\because f(0) = 6]$

$\therefore$ (d) is false

The correct option is (D)

65. Putting $y = 0$ in the given functional equation,

we get, $f(x/2) = \frac{f(x) + f(0)}{2} = \frac{1}{2}[1 + f(x)]$ $(\because f(0) = 1)$

$\Rightarrow$ $f(x) = 2f(x/2) - 1$ (1)

Since $f'(0) = -1$, we get $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = -1$

$\Rightarrow$ $\lim_{h\to 0} \frac{f(h) - 1}{h} = -1$

Now,

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \dfrac{\dfrac{f(2x) + f(2h)}{2} - f(x)}{h}$ [using (1)]

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{2}\{2f(x) - 1 + 2f(h) - 1\} - f(x)\right]$

$$\lim_{h\to 0}\frac{1}{h}\left[f(x)-1+f(h)-f(x)\right] = \lim_{h\to 0}\frac{f(h)-1}{h} = -1$$

Thus, $f'(x) = -1$, so we get $f(x) = -x + c$

But $f(0) = 1$, therefore, $1 = f(0) = -0 + c$

$\Rightarrow$ $c = 1$

Thus, $f(x) = 1 - x$

$\therefore$ $f(2) = 1 - 2 = -1$

The correct option is (B)

66. $f'(x = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)\{f(h)-1\}}{h}$$

$$= \lim_{h\to 0}\frac{f(x)\{1+hg(h)-1\}}{h}$$

$$= \lim_{h\to 0} f(x)\,g(h) = \log a\, f(x)$$

Therefore, $f^n(x) = (\log a)^n f(x)$, so $k = (\log a)^n$

The correct option is (C)

67. We have, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)+f(h)+xh-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}f(h)+x$$

$$= 3 + x$$

Integrating, we get $f(x) = 3x + \frac{x^2}{2} + c$

Putting $x = y = 0$ in the given equation,

we get, $f(0) = 0 \Rightarrow c = 0$

$\therefore$ $f(x) = 3x + \frac{x^2}{2}$

The correct option is (C)

68. We have,

$f(x) = x + \tan x$

$\Rightarrow$ $f(f^{-1}(y)) = f^{-1}(y) + \tan f^{-1}(y)$

$\Rightarrow$ $y = g(y) + \tan g(y)$ or $x = g(x) + \tan g(x)$

On differentiating, we get $1 = g'(x) + \sec^2 g(x)\, g'(x)$

$$\Rightarrow g'(x) = \frac{1}{1+\sec^2 g(x)}$$

$$= \frac{1}{2+[g(x)-x]^2}$$

The correct option is (C)

69. Let $x = a\cos^2\theta + b\sin^2\theta$

$\therefore$ $a - x = a - a\cos^2\theta - b\sin^2\theta = (a-b)\sin^2\theta$

and, $x - b = a\cos^2\theta + b\sin^2\theta - b = (a-b)\cos^2\theta$

$\therefore$ $y = (a-b)\sin q\cos\theta - (a-b)\tan^{-1}\tan\theta$

$$= \frac{a-b}{2}\sin 2\theta - (a-b)\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(a-b)\cos 2\theta-(a-b)}{(b-a)\sin 2\theta}$$

$$= \frac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta = \sqrt{\frac{a-x}{x-b}}$$

The correct option is (B)

70. We have,

$y^3 - y = 2x$

Differentiating both sides with respect to x, we get

$$(3y^2-1)\frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} = \frac{2}{3y^2-1} \quad (1)$$

Again, differentiating both sides with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{-2\cdot 6y\frac{dy}{dx}}{\left(3y^2-1\right)^2}$$

using (1), we get

$$\frac{d^2y}{dx^2} = \frac{-24y}{\left(3y^2-1\right)^3} \quad (2)$$

Now,

$$\left(x^2-\frac{1}{27}\right)\frac{d^2y}{dx^2}+\frac{xdy}{dx}$$

$$= \left(x^2-\frac{1}{27}\right)\left(\frac{-24y}{(3y^2-1)^3}\right)+\frac{2x}{(3y^2-1)}$$

[From (1) and (2)]

$$= \left(\frac{y^2(y^2-1)^2}{4}-\frac{1}{27}\right)\left(\frac{-24y}{(3y^2-1)^3}\right)+\frac{y(y^2-1)}{3y^2-1}$$

$(\because y^3 - y = 2x)$

$$= \frac{\{27y^2(y^2-1)^2-4\}}{108}\frac{(-24y)}{\left(3y^2-1\right)^3}+\frac{y(y^2-1)}{3y^2-1}$$

$$= \frac{y}{9}\left\{\frac{-54y^2(y^2-1)^2+8}{(3y^2-1)^3}+\frac{9(y^2-1)}{3y^2-1}\right\}$$

$$= \frac{y}{9}\left\{\frac{-2(1+\alpha)(\alpha-2)^2+8}{\alpha^3}\right\}+\frac{3(\alpha-2)}{\alpha}$$

where $\alpha = 3y^2 - 1$

$$= \frac{y}{9}$$

The correct option is (C)

71. We have,

$(1 + x + x^2)(1 - x + x^2) = (1 - x^2)^2 - x^2 = 1 + x^2 + x^4$

Now, $(1 + x + x^2)(1 - x + x^2)(1 - x^2 + x^4)$

$= (1 + x^2 + x^4)(1 - x^2 + x^4)$

$= 1 + x^4 + x^8$

Continuing in this way, we have

$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)$

$\ldots\left(1-x^{2^{n-1}}+x^{2^n}\right)$

$=\left(1+x^{2^n}+x^{2^{n+1}}\right)$

Now, for $x<1$, $x^\infty=0$.

Taking limits as $n\to\infty$ in (1), we get

$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\ldots=1$

Taking logarithm of both sides, we get

$\Rightarrow \ln(1+x+x^2)+\ln(1-x+x^2)+\ln(1-x^2+x^4)$
$+\ln(1-x^4+x^8)+\ldots=0$

Differentiating both sides with respect to x, we get

$$\Rightarrow \frac{1+2x}{1+x+x^2}+\frac{-1+2x}{1-x+x^2}+\frac{-2x+4x^3}{1-x^2+x^4}+\frac{-4x^3+8x^7}{1-x^4+x^8}+\ldots=0$$

Hence, $$\frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1-x^2+x^4}+\frac{4x^3-8x^7}{1-x^4+x^8}+\ldots=\frac{1+2x}{1+x+x^2}$$

The correct option is (B)

More than One Option Correct Type

72. We have, $f'(\alpha+\beta)=f'(\alpha)+f'(\beta)$

$\Rightarrow\ (\alpha+\beta)^{m-1}=a^{m-1}+b^{m-1}$

Since, for $m>2$, the above equality is not valid

$\therefore$ we must have $m=2$.

Also, for $m=0$, $f'(x)=0$ for all x. So the equality is trivially true.

The correct option is (B) and (C)

73. Since $f(x)$ is a polynomial of degree 3,

$\therefore\quad f(x)=x^3+ax^2+bx+c$

Comparing with the given equation, we have

$a=f'(1),\ b=f''(2),\ c=f'''(3)$

Now, $f'(x)=3x^2+2ax+b$

$f''(x)=6x+2a$

$f'''(x)=6$

$\therefore\quad a=f'(1)=3+2a+b$

$b=f''(2)=12+2a$

$c=f'''(3)=6$

Solving the above system of equations

$3+a+b=0$ and $b=12+2a$

$\therefore\quad a=-5,\ b=2$

Now, $f(x)=x^3-5x^2+2x+6$

Thus, $f(0)=6, f(1)=4, f(2)=-2, f(3)=-6.$

The correct option is (A), (B) and (C)

74. Since $f(x-y), f(x)\cdot f(y)$ and $f(x+y)$ are in A.P.

$\Rightarrow\ f(x+y)+f(x-y)=2f(x)\cdot f(y)$ (1)

Putting $x=0, y=0$ in (1), we get

$f(0)+f(0)=2f(0)\cdot f(0)\Rightarrow f(0)=1$ $[\because f(0)\neq 0]$

Putting $x=0, y=x$ in (1), we get

$f(x)+f(-x)=2f(0)\cdot f(x)\Rightarrow f(2)=f(-2), f(3)=f(-3)$

Differentiating $f(x)=f(-x)$ with respect to x,

$f'(x)+f'(-x)=0$

$\therefore f'(2)+f'(-2)=0, f'(3)+f'(-3)=0$

The correct option is (A) and (C)

75. We have,

$f(x)+f(y)+f(z)+f(x)\times f(y)\times f(z)=14$ (1)

for all $x, y, z\in R$

Putting $x=y=z=0$, we get

$3f(0)+[f(0)]^3=14\Rightarrow[f(0)]^3+3f(0)-14=0$

$\Rightarrow\ f(0)=2$

Now, putting $y=z=x$ in (1), we get

$3f(x)+[f(x)]^3=14$

Differentiating with respect to x, we get

$3f'(x)+3[f(x)]^2\times f'(x)=0$

$\Rightarrow\quad 3f'(x)[1+\{f(x)\}^2]=0$

$\Rightarrow\quad f'(x)=0$, for all x.

The correct option is (A) and (B)

76. Since $f(x-y), f(x)\cdot f(y)$ and $f(x+y)$ are in A.P.

$\Rightarrow f(x+y)+f(x-y)=2f(x)\cdot f(y)$ (1)

Putting $x=0, y=0$ in (1), we get

$f(0)+f(0)=2f(0)\cdot f(0)\Rightarrow f(0)=1\{\because f(0)\neq 0\}$

Putting $x=0, y=x$ in (1), we get

$f(x)+f(-x)=2f(0)\cdot f(x)\Rightarrow f(2)=f(-2), f(3)=f(-3)$

Differentiating $f(x)=f(-x)$ with respect to x, we have

$f'(x)+f'(-x)=0$

$\therefore f'(2)+f'(-2)=0, f'(3)+f'(-3)=0$

The correct option is (A) and (C)

77. We have,

$$f(xy)=2f(x)-f\left(\frac{x}{y}\right)\quad(1)$$

Putting $x=1$ in equation (1), we get

$$2f(1)=f(y)-\frac{1}{y}$$

$$\Rightarrow\quad f(y)=-f\left(\frac{1}{y}\right)\ [f(1)=0;\text{ given}]\quad(2)$$

Now, $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left\{x\left(1+\frac{h}{x}\right)\right\} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2}{x} \frac{f\left(1+\frac{h}{x}\right) - f(1)}{\frac{h}{x}} - \frac{1}{x^2} \cdot \frac{f\left(\frac{1}{x}+\frac{h}{x^2}\right) + f(x)}{\frac{h}{x^2}}$$

[using (2)]

$$= \frac{2}{x} - \frac{1}{x^2} f'\left(\frac{1}{x}\right)$$

Differentiating (2) with respect to y, we get

$$f'(y) = f'\left(\frac{1}{y}\right)\frac{1}{y^2}$$

$\therefore \quad f'(x) = \frac{2}{x} - f'(x)$ i.e. $f'(x) = \frac{1}{x}$

$\Rightarrow \quad df = \frac{1}{x}\, dx$

On integrating, we get

$f(x) = \ln x + c$, where c is a constant

Since, $f(1) = 0 \Rightarrow c = 0$. Thus, we have

$f(x) = ln x$

The correct option is (A) , (B) and (C)

78. We have,

$f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ (1)

Differentiating with respect to x, we have

$f'(x) = 3x^2 + 2x f'(1) + f''(2)$ (2)

Differentiating with respect to x, we get

$f''(x) = 6x + 2f'(1)$ (3)

Differentiating with respect to x, we get

$f'''(x) = 6$ (4)

Putting $x = 3$ is equation (4), we get

$f'''(3) = 6$

Putting $x = 1$ in equation (2) and in equation (3), we get

$f'(1) = 3 + 2f'(1) + f''(2)$

i.e., $f'(1) + f''(2) = -3$ (5)

and, $f''(2) = 12 + 2f'(1)$ (6)

Solving equation (5) and equation (6), we have

$f'(1) = -5$ and $f''(2) = 2$

The correct option is (A), (B) and (C)

79. We have,

$f(x - y) + f(x + y) = 2 f(x) f(y)$ (1)

Putting $y = 0$ in equation (1), we have

$f(x) \{f(0) - 1\} = 0$

$\Rightarrow f(0) = 1$

Putting $x = 0$ in equation (1), we have

$f(-y) + f(y) = 2f(0) + f(y) = 2f(y)$

$\Rightarrow f(-y) = f(y)$ (2)

$\Rightarrow f$ is even

Differentiating equation (2) with respect to y, we get

$-f'(-y) = f'(y)$

$\Rightarrow f'$ is odd.

The correct option is (A) and (D)

80. Let $g'(1) = a$ and $g'(2) = b$ (1)

Then, $\quad f(x) = x^2 + ax + b, f(1) = 1 + a + b$

$f'(x) = 2x + a, f''(x) = 2$

$\therefore \quad g(x) = (1 + a + b)x^2 + (2x + a) \cdot x + 2$

$= x^2(3 + a + b) + ax + 2$

$\Rightarrow \quad g'(x) = 2x(3 + a + b) + a$

Hence, $\quad g'(1) = 2(3 + a + b) + a$ (2)

$g'(2) = 4(3 + a + b) + a$ (3)

From (1), (2) and (3), we have,

$a = 2(3 + a + b) + a$ and $b = 2(3 + a + b)$

i.e., $3 + a + b = 0$ and $b + 2a + 6 = 0$

Hence, $\quad b = 0$ and $a = -3$

So $\quad f(x) = x^2 - 3x$ and $g(x) = -3x + 2$

The correct option is (A) and (D)

81. Let $S = 1 + x + x^2 + x^3 + x^4 + ... + x^n$

which is a geometric progression

$\therefore \quad S = 1 + x + x^2 + x^3 + ... + x^n = \frac{1(1 - x^{n+1})}{1 - x}$

On differentiating both sides, we get,

$0 + 1 + 2x + 3x^2 + 4x^3 + ... + nx^{n-1}$

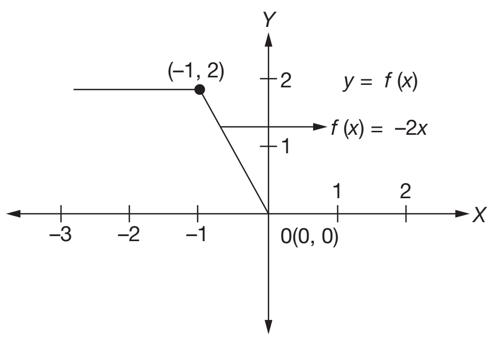
$$= \frac{(1-x)\cdot[-(n+1)x^n] - \left(1 - x^{n+1}\right)\cdot(-1)}{(1-x)^2}$$

$$\therefore \quad \sum_{r=1}^{n} r\, x^{r-1} = \frac{1}{(1-x)^2}\{1 - (n+1)x^n + n \bullet x^{n+1}\}$$

Thus, $a = -(n + 1)$ and $b = n$

The correct option is (B) and (C)

82. From the graph it is clear that $f(x)$ is non-differentiable at $x = 0, -1$



Also, $f''(100) = 0$

$$\left[\because \text{on } x\text{-axis, } y = 0 \Rightarrow \frac{dy}{dx} = 0 \Rightarrow f'(x) = 0\right]$$

$$\int_{-3}^{10} f(x)\,dx = \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{10} f(x)\,dx$$

$$= \int_{-3}^{-1} 2\,dx + \int_{-1}^{0} (-2x)\,dx + \int_{0}^{10} 0\,dx$$

$$= |2x|_{-3}^{-1} + |-x^2|_{-1}^{0}$$

$$= (-2-(-6)) + (-0+(-1)^2) = 4 + 1 = 5$$

The correct option is (A), (C) and (D)

83. $f^n(x) = \begin{vmatrix} n! & \sin\left(x + \frac{n\pi}{2}\right) & -\cos\left(x + {}^{n\pi}/_{2}\right) \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left({}^{n\pi}/_{2}\right) \\ a & a^2 & a^3 \end{vmatrix}$

$\therefore f^n(0) = \begin{vmatrix} n! & \sin\left({}^{n\pi}/_{2}\right) & -\cos\left({}^{n\pi}/_{2}\right) \\ n! & \sin\left({}^{n\pi}/_{2}\right) & \cos\left({}^{n\pi}/_{2}\right) \\ a & a^2 & a^3 \end{vmatrix}$

$= \begin{vmatrix} n! & \sin\left({}^{n\pi}/_{2}\right) & 0 \\ n! & \sin\left({}^{n\pi}/_{2}\right) & 0 \\ a & a^2 & a^3 \end{vmatrix} = 0$

$(\because n = (2m+1))$

The correct option is (C) and (D)

84. We have,

$g(f(x)) = x \Rightarrow g'(f(x))\,f'(x) = 1 \Rightarrow g'(f(x)) = \dfrac{1}{(f'(x))}$

Now, $f(x) = 2 \Rightarrow x^3 + 3x^2 - 33x - 33 = 2$

$\Rightarrow \quad x^3 + 3x^2 - 33x - 35 = 0$

$\Rightarrow \quad x^3 - 5x^2 + 8x^2 - 40x + 7x - 35 = 0$

$\Rightarrow \quad (x-5)(x^2 + 8x + 7) = 0$

$\Rightarrow \quad (x-5)(x+1)(x+7) = 0$

$\therefore x = -7, -1, 5$

Thus, we have,

$k = f'(-1) = 3(-1)^2 + 6(-1) - 33 = 3 - 6 - 33 = -36$

$k = f'(-7) = 3(-7)^2 + 6(-7) - 33 = 147 - 63 - 33 = 51$

$k = f'(5) = 3(5)^2 + 6(5) - 33 = 75 + 30 - 33 = 72$

The correct option is (A), (B) and (C)

85. Given $F(x) = f(x) \cdot g(x)$ (1)

Differentiating both sides with respect to x,we get

$F'(x) = f'(x)\,g(x) + g'(x)\,f(x)$

$\Rightarrow \quad F'(x) = f'(x)\,g'(x)\left[\dfrac{f(x)}{f'(x)} + \dfrac{g(x)}{g'(x)}\right]$

$\Rightarrow \quad F' = c\left[\dfrac{f}{f'} + \dfrac{g}{g'}\right] \Rightarrow$ (a) is correct

Again, differentiating both sides w.r.t, x we get

$F''(x) = f''(x)\cdot g(x) + g''(x)\cdot f(x) + 2f'(x)\cdot g'(x)$

$\Rightarrow \quad F''(x) = f''(x)\cdot g(x) + g''(x)\cdot f(x) + 2c$ (2)

Dividing both sides by $F(x) = f(x)\cdot g(x)$

$\{\because f'(x)\,g'(x) = c\}$

Then, $\quad \dfrac{F''(x)}{F(x)} = \dfrac{f''(x)}{f(x)} + \dfrac{g''(x)}{g(x)} + \dfrac{2c}{f(x)\,g(x)}$

or, $\quad \dfrac{F''}{F} = \dfrac{f''}{f} + \dfrac{g''}{g} + \dfrac{2c}{fg}$

$\Rightarrow$ (b) is correct

Again, given $f'(x)\,g'(x) = c$

Differentiating both sides with respect to x, we get

$f'(x)\,g''(x) + g'(x)\,f''(x) = 0$ (3)

From (2), $F''(x) = f''(x) + g(x) + g'''(x)\,f(x) + 2c$

Differentiating both sides with respect to x, we get

$F'''(x) = f''(x)\,g'(x) + f'''(x)\,g(x) + g''(x)\,f'(x) + g''(x)\,f(x) + 0$

$= f'''(x)\,g(x) + g'''(x)\,f(x) + 0$ [from (3)]

Now, dividing both sides by $F(x) = f(x)\,g(x)$, we get

$\dfrac{F'''(x)}{F(x)} = \dfrac{f'''(x)}{f(x)} + \dfrac{g'''(x)}{g(x)}$

or, $\quad \dfrac{\boldsymbol{F'''}}{\boldsymbol{F}} = \dfrac{\boldsymbol{f'''}}{\boldsymbol{f}} + \dfrac{\boldsymbol{g'''}}{\boldsymbol{g}} \Rightarrow$ (c) is correct

The correct option is (A), (B) and (C)

Passage Based Questions

86. We have,

$f(x) + f(y) = f\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ (1)

Putting $y = -x$ in equation (1), we have

$f(x) + f(-x) = f\left(x\sqrt{1-x^2} - x\sqrt{1-x^2}\right)\cos^{-1} = f(0)$

Putting $x = 0$ and $y = 0$ in equation (1), we get

$f(0) + f(0) = f(0)$

i.e., $f(0) = 0$

So, by (1)

$f(x) + f(-x) \quad = 0$ (2)

Therefore, the function $f(x)$ is odd.

The correct option is (B)

87. We have,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + f(-x)}{h} \quad \text{[using (2)]}$$

$$= \lim_{h\to 0} \frac{f\left\{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}\right\}}{h}$$

[using (1)]

$$= \lim_{y\to 0} \frac{f(y)}{y} \cdot \lim_{h\to 0} \frac{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}}{h}$$

$$\left[\text{Putting } (x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2} = y\right]$$

Now, we have,

$$\lim_{y\to 0} \frac{f(y)}{y} = \lim_{y\to 0} \frac{f(y) - f(0)}{y} = f'(0) = 1 \quad [\because f(0) = 0]$$

and,
$$\lim_{h\to 0} \frac{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}}{h}$$

$$= \sqrt{1-x^2} + \lim_{h\to 0} \frac{x\left(\sqrt{1-x^2} - \sqrt{1-(x+h)^2}\right)}{h}$$

$$= \sqrt{1-x^2} + \lim_{h\to 0} \frac{x(1-x^2-1+(x+h)^2)}{h\left(\sqrt{1-x^2} + \sqrt{1-(x+h)^2}\right)}$$

$$= \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

Hence, $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$ (3)

The correct option is (A)

88. Integrating (3), we get

$f(x) = \sin^{-1}x + C$, where C is a contant.

Since $f(0) = 0$, we get $c = 0$.

$\therefore f(x) = \sin^{-1}x$

The correct option is (B)

89. Given: $f\left(\dfrac{x+y}{k}\right) = \dfrac{f(x) + f(y)}{k}$ (1)

We have,

$$f'(x) = \lim_{h\to 0} \frac{f\left(x + \frac{h}{k}\right) - f(x)}{h/k}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{kx+h}{k}\right) - f(x)}{h/k}$$

$$= \lim_{h\to 0} \frac{f(kx) + f(h) - k\,f(x)}{h} \quad (2)$$

[using (1)]

Putting kx in place x and 0 in place of y in equation (1), we get

$$f\left(\frac{kx+0}{k}\right) = \frac{f(kx) + f(0)}{k}$$

i.e., $f(kx) - kf(x) = -f(0)$

Putting the above result in equation (2), we get

$$f'(x) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = f'(0)$$

i.e., $f'(x) = m$ $[f'(0) = m \text{ (given)}]$

The correct option is (A)

90. We have, $f'(x) = m$

$$\Rightarrow \frac{df}{dx} = m$$

i.e., $f(x) = mx + c$, where c is a constant (3)

Putting $x = 0$ and $y = 0$ in equation (1), we get

$$f(0) = \frac{2\,f(0)}{k}$$

$$\Rightarrow f(0)\left(1 - \frac{2}{k}\right) = 0 \text{ or } f(0) = 0$$

Therefore, from (3), we get $c = 0$

Hence, we have,

$f(x) = mx$

The correct option is (A)

91. Given:

$$f(xy) = f(x)f(y) - f(x) - f(y) + 2 \quad (1)$$

Therefore,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left\{x\left(1 + \frac{h}{x}\right)\right\} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)\,f\left(1+\frac{h}{x}\right) - f(x) - f\left(1+\frac{h}{x}\right) + 2 - f(x)}{h}$$

[using (1)]

$$= \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - 2}{\frac{h}{x}} \cdot \frac{f(x) - 1}{x} \quad (2)$$

Putting $x = 1$ and $y = 2$ in equation (1), we have

$$f(2) = f(1) + (2) - f(1) - f(2) + 2$$

$$\Rightarrow \quad 5 = 5f(1) - f(1) - 5 + 2$$

$[\because f(2) = 5; \text{ given}]$

i.e., $f(1) = 2$

Equation (2), thus reduces to

$$f'(x) = \frac{f(x)-1}{x} \cdot \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - f(1)}{\frac{h}{x}}$$

$$= \frac{f(x)-1}{x} \cdot f'(1) \qquad (3)$$

The correct option is (A)

92. From (3), we have,

$$\frac{df}{f-1} = f'(1)\frac{dx}{x}$$

On integrating, we get

$\ln(f-1) = f'(1)\ln x + C$, where C is a constant

On using the condition $f(1) = 2$, we get

$\ln(2-1) = f'(1)\ln 1 + C$

$\Rightarrow \quad C = 0$

Using the condition $f(2) = 5$, we get

$\ln(5-1) = f'(1)\ln 2$

$$\Rightarrow \quad f'(1) = \frac{\ln 4}{\ln 2} = 2$$

Hence, we have,

$\ln[f(x)-1] = 2\ln x = \ln x^2$

$\therefore \quad f(x) = 1 + x^2$

The correct option is (C)

93. Given $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right) \qquad (1)$

Putting $y = -x$ in equation (1), we get

$f(x) + f(-x) = f(0)$

Putting $x = 0$ and $y = 0$ in equation (1), we get

$f(0) + f(0) = f(0)$

$\Rightarrow \quad f(0) = 0$

Thus, we have,

$f(x) + f(-x) = 0$

which proves that **$f(x)$ is an odd function.**

The correct option is (B)

94. We have,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)+f(-x)}{h} \qquad [\because f \text{ is odd}]$$

$$= \lim_{h\to 0}\frac{f\left(\frac{h}{1+x^2+xh}\right)}{h} \qquad [\text{using equation (1)}]$$

$$= \lim_{h\to 0}\frac{f\left(\frac{h}{1+x^2+xh}\right)-f(0)}{\frac{h}{1+x^2+xh}} \cdot \frac{1}{1+x^2+xh}$$

$[\because f(0) = 0]$

$$= \frac{1}{1+x^2}\lim_{\delta\to 0}\frac{f(\delta)-f(0)}{\delta} = \frac{f'(0)}{1+x^2}$$

$$= \frac{2}{1+x^2} \qquad [f'(0) = 2;\ \text{given}]$$

The correct option is (C)

95. We have, $f'(x) = \dfrac{2}{1+x^2}$

$$\Rightarrow \quad df = \left(\frac{2}{1+x^2}\right)dx$$

On integrating, we get

$f(x) = 2\tan^{-1}x + C$, where C is a constant.

Now, using the condition $f(0) = 0$, we get $C = 0$.

Hence, we have,

$f(x) = 2\tan^{-1}x$

The correct option is (B)

96. We have

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x/y)^2}}\cdot\frac{1}{y} + \frac{1}{\{1+(y/x)^2\}}\left(\frac{-y}{x^2}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \qquad (1)$$

$$\text{Now, } \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-x^2/y^2}}\left(\frac{-x}{y}\right) + \frac{1}{\{1+(y^2/x^2)\}}\cdot\frac{1}{x}$$

$$\therefore \quad y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \qquad (2)$$

Adding (1) and (2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

The correct option is (A)

97. Let $U = \sin x$ and $V = x^2$ so that $y = UV$

We have, $U_n = \sin\left(x + \frac{n\pi}{2}\right)$

$$\therefore \frac{d^n y}{dx^n} = \sin\left(x+\frac{n\pi}{2}\right)\cdot x^2 + {}^nC_1 \sin\left[x+(n-1)\frac{\pi}{2}\right]\cdot 2x + {}^nC_2 \sin\left[x+(n-1)\frac{\pi}{2}\right]\cdot 2$$

$$= x^2\sin\left(x+\frac{n\pi}{2}\right) - 2nx\sin\left[\frac{\pi}{2}-\left(x+\frac{n\pi}{2}\right)\right] - n(n-1)\sin\left[n-\left(x+\frac{n\pi}{2}\right)\right]$$

$$= (x^2-n^2+n)\sin\left(x+\frac{n\pi}{2}\right) - 2nx\cos\left(x+\frac{n\pi}{2}\right)$$

$\therefore\ k = -2nx$

The correct option is (D)

98. **Step 1:** We have $\cos^{-1}\left(\frac{y}{b}\right) = n(\log x - \log n)$

$\Rightarrow\ y = b\cos[n(\log x - \log n)]$ (1)

Differentiating, we get

$$y_1 = -b\sin[n(\log x - \log n)]\cdot\frac{n}{x}$$

Squaring both sides, we get

$$x^2 y_1^2 = n^2b^2\sin^2[n(\log x - \log n)]$$
$$= n^2b^2[1 - \cos^2\{n(\log x - \log n)\}]$$

Using (1), we obtain

$$x^2y_1^2 = n^2b^2 - n^2y^2$$

Differentiating again, we get

$$2xy_1^2 + 2x^2y_1y_2 = -2n^2y\,y_1$$

Cancelling out $2y_1$, we get $x^2y_2 + xy_1 + n^2y = 0$ (2)

Step 2 : Differentiating (2) n times by the theorem, we get

$[x^2y_{n+2} + {}^nC_1 y_{n+1}\,.\,2x + {}^nC_2 y_n.\,2]$

$+ [xy_{n+1} + {}^nC_1 y_n.\,1] + n^2y_n = 0$

or, $x^2y_{n+2} + 2n\,x\,y_{n+1} + n(n-1)\,y_n + x\,y_{n+1} + n\,y_n + n^2y_n = 0$

or, $x^2y_{n+2} + (2n+1)\,x\,y_{n+1} + 2n^2\,y_n = 0\ \therefore k = 2n^2$

The correct option is (B)

99. Apply the theorem to $f(x)\cos x = \sin x$.

The correct option is (A)

100. We have,

$$I_n = \frac{d^{n-1}}{dx^{n-1}}\left\{\frac{d}{dx}\left(x^n\log x\right)\right\}$$

$$I_n = \frac{d^{n-1}}{dx^{n-1}}\left\{n\,x^{n-1}\log x + x^{n-1}\right\}$$

$$= n\frac{d^{n-1}}{dx^{n-1}}\left\{x^{n-1}\log x\right\} + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$\therefore\ I_n = n\,I_{n-1} + (n-1)\,!$

$\therefore\ k = (n-1)\,!$

The correct option is (B)

Match the Column Type

101. **I.** Let $y = f(\tan x)$ and $u = g(\sec x)$

$$\Rightarrow\ \frac{dy}{dx} = f'(\tan x)\sec^2 x$$

and $$\frac{du}{dx} = g'(\sec x)\cdot\sec x\tan x$$

$$\therefore\ \frac{dy}{du} = \frac{dy}{dx}\Big/\frac{du}{dx} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x}$$

$$\therefore\ \left.\frac{dy}{du}\right]_{x=\frac{\pi}{4}} = \frac{f'\left(\tan\frac{\pi}{4}\right)}{g'\left(\sec\frac{\pi}{4}\right)\sin\frac{\pi}{4}}$$

$$= \frac{f'(1)}{g'(\sqrt{2})\cdot\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}\times 2}{4} = \frac{1}{\sqrt{2}}$$

The correct option is (D)

II. We have, $y = x^3 - 8x + 7 \Rightarrow \frac{dy}{dx} = 3x^2 - 8$

It is given that when $t = 0$, $x = 3$

$\therefore$ When $t = 0$, $\frac{dy}{dx} = 3\cdot 3^2 - 8 = 19$

Also, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ (1)

Since, when $t = 0$, $\frac{dy}{dx} = 19$ and $\frac{dy}{dt} = 2$,

$\therefore$ from (1), $19 = \frac{2}{dx/dt} \Rightarrow \frac{dx}{dt} = \frac{2}{19}$

The correct option is (C)

III. We have, $(fh)(x) = f(x)\cdot h(x) = \sin x\cos x$

$\therefore\ [go(fh)](x) = g[(fh)(x)] = g[f(x)\cdot h(x)]$

$= g(\sin x\cos x) = 2\sin x\cos x = \sin 2x$

i.e., $\phi(x) = \sin 2x$

$\Rightarrow\ \phi'(x) = 2\cos 2x$ and $\phi''(x) = -4\sin 2x$

$\therefore\ \phi''\left(\frac{\pi}{4}\right) = -4\sin\frac{\pi}{2} = -4$

The correct option is (B)

IV. $f'(x) = -2\cos x\sin x - 2\cos\left(x + \frac{\pi}{3}\right)\sin\left(x + \frac{\pi}{3}\right)$

$$+ \cos x\sin\left(x + \frac{\pi}{3}\right) + \sin x\cos\left(x + \frac{\pi}{3}\right)$$

$$= -\sin 2x - \sin\left(2x + \frac{2\pi}{3}\right) + \sin\left(x + x + \frac{\pi}{3}\right)$$

$$= -2\sin\left(2x + \frac{\pi}{3}\right)\cos\frac{\pi}{3} + \sin\left(2x + \frac{\pi}{3}\right)$$

$$= -\sin\left(2x + \frac{\pi}{3}\right) + \sin\left(2x + \frac{\pi}{3}\right) = 0$$

$\Rightarrow f(x) =$ constant for all x.

But, $f(0) = \cos^2 0 + \cos^2\frac{\pi}{3} + \sin 0\cdot\sin\frac{\pi}{3} = \frac{5}{4}$

$\therefore\ f(x) = \frac{5}{4}$ for all x.

Thus, $(gof)(x) = g[f(x)] = g\left(\frac{5}{4}\right) = 3$

The correct option is (A)

102. **I.** Differentiating the given equation with respect to x, we get

$$xy' + y\cdot 1 - \frac{1}{y}y' = 0$$

$\Rightarrow xyy' - y' + y^2 = 0$ i.e., $(xy - 1)y' + y^2 = 0$

Differentiating again with respect to x, we get

$(xy - 1)y'' + y'(xy' + y\cdot 1) + 2yy' = 0$

$\Rightarrow x(yy'' + y'^2) - y'' + 3yy' = 0$

$\therefore k = 3$

The correct option is (D)

II. We have,

$$\frac{dx}{dt} = \cos t$$

and, $\dfrac{dy}{dt} = \sqrt{2}\,(a+b)\,e^{t\sqrt{2}}$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sqrt{2}(a+b)e^{t\sqrt{2}}}{\cos t} = \frac{\sqrt{2}y}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = 2y^2$$

Differentiating with respect to x, we get

$(1 - x^2)\,2y'y'' - 2x(y')^2 = 4yy'$

$\Rightarrow (1 - x^2)y'' - xy' = 2y$ [dividing by $2y'$]

$\therefore k = 2$

The correct option is (B)

III. We have,

$F(x) = f(x)\,g(x)\,h(x)$

$\Rightarrow \log F(x) = \log f(x) + \log g(x) + \log h(x)$

Differentiating both the sides with respect to x, we get

$$\frac{F'(x)}{F(x)} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} + \frac{h'(x)}{h(x)}$$

$$\Rightarrow \frac{F'(x_0)}{F(x_0)} = \frac{f'(x_0)}{f(x_0)} + \frac{g'(x_0)}{g(x_0)} + \frac{h'(x_0)}{h(x_0)}$$

$\Rightarrow 21 = 4 - 7 + k \Rightarrow k = 24$

The correct option is (A)

IV. $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

$f'(a+b) = f'(a) + f'(b)$

$\Rightarrow n(a+b)^{n-1} = na^{n-1} + nb^{n-1}$

$\Rightarrow (a+b)^{n-1} = a^{n-1} + b^{n-1}$

Which is true for $n = 2$ and false for $n = 1$ and $n = 4$.

Also, for $n = 0, f(x) = 1$,

So, $f'(x) = 0; f'(a+b) = 0$

So, $f'(a+b) = f'(a) + f'(b)$

Hence, there are two values of n.

The correct option is (B)

Assertion-Reason Type

103. Since $f(x)$ is a polynomial function satisfying

$$f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right),$$

$\therefore\ f(x) = x^n + 1$ or $f(x) = -x^n + 1$

If $f(x) = -x^n + 1$, then $f(4) = -4^n + 1 \neq 65$

So, $f(x) = x^n + 1$

Since $f(4) = 65 \therefore 4^n + 1 = 65 \Rightarrow n = 3$

$\therefore\ f(x) = x^3 + 1 \Rightarrow f'(x) = 3x^2$

$\therefore\ f'(l_1) = 3l_1^2, f'(l_2) = 3l_2^2, f'(l_3) = 3l_3^2$

Since l_1, l_2, l_3 are in G.P.,

$\therefore\ f'(l_1), f'(l_2), f'(l_3)$ are also in **G.P.**

The correct option is (A)

104. We have,

$$y = (1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$$

$$= \frac{(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})}{1-x}$$

$$= \frac{1-x^{2^{n+1}}}{1-x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1-x)\cdot -2^{n+1}\cdot x^{2^{n+1}-1} + (1-x^{2^{n+1}})}{(1-x)^2}$$

$$\therefore \left.\frac{dy}{dx}\right]_{x=0} = 1$$

The correct option is (A)

105. $f(x) = (\cos x + i\sin x)(\cos 2x + i\sin 2x)$

$(\cos 3x + i\sin 3x)\ldots(\cos nx + i\sin nx)$

$= \cos(x + 2x + 3x + \ldots + nx) + i\sin(x + 2x + 3x + \ldots + nx)$

$$= \cos\frac{n(n+1)}{2}x + i\sin\frac{n(n+1)}{2}x$$

$$\Rightarrow f'(x) = \frac{n(n+1)}{2}\left[-\sin\frac{n(n+1)}{2}x + i\cos\frac{n(n+1)}{2}x\right]$$

$$\Rightarrow f''(x) = -\left(\frac{n(n+1)}{2}\right)^2\left(\cos\frac{n(n+1)}{2}x + i\sin\frac{n(n+1)}{2}x\right)$$

$$= -\left(\frac{n(n+1)}{2}\right)^2 f(x)$$

$$\therefore f''(1) = -\left(\frac{n(n+1)}{2}\right)^2 f(1) = -\left(\frac{n(n+1)}{2}\right)^2.$$

The correct option is (C)

Previous Year's Questions

106. Given $y=(x+\sqrt{1+x^2})^n$

$\therefore$ On differentiating with respect to x, we get

$$\frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$=\frac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2)\left(\frac{dy}{dx}\right)^2=n^2y^2$$

On differentiating again with respect to x, we get

$$(1+x^2)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)^2=n^2 2y\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$$

The correct option is (A)

107. Since, $\sin y = x\sin(a+y)$ we have that

$$x=\frac{\sin y}{\sin(a+y)}$$

On differentiating with respect to y, we get

$$\frac{dx}{dy}=\frac{\sin(a+y)\cos y-\sin y\cos(a+y)}{\sin^2(a+y)}$$

$$=\frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy}=\frac{\sin a}{\sin^2(a+y)}$$

The correct option is (B)

108. $\because x^y=e^{x-y}$

Taking log on both sides, we obtain

$y\log x=(x-y)\log_e e$

$$\Rightarrow y=\frac{x}{1+\log x}$$

On differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{(1+\log x)\cdot 1-x\cdot\frac{1}{x}}{(1+\log x)^2}$$

$$=\frac{\log x}{(1+\log x)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$$

The correct option is (D)

109. $f(x)=ax^2+bx+c$

So that, $f(1)=a+b+c$

And, $f(-1)=a\text{-}b+c$

$\Rightarrow a+b+c=a-b+c$ also $2b=a+c$

$f'(x)=2ax+b=2ax$

$f'(a)=2a^2$

$f'(b)=2ab$

$f'(c)=2ac$

So they are in AP.

The correct option is (A)

110. We have

$f'(x)=1$

$f''(x)=0$

And all other derivatives are 0.

So the expression $=1-1+0+0+0+\ldots$

$=0$

The correct option is (C)

111. Given that $x=e^{y+e^{y+e^{y+\ldots\infty}}}$ $\Rightarrow x=e^{y+x}$

$$\Rightarrow \ln x-x=y\Rightarrow\frac{dy}{dx}=\frac{1}{x}-1=\frac{1-x}{x}.$$

The correct option is (C)

112. The value of $f'(1)=\lim\limits_{h\to0}\dfrac{f(1+h)-f(1)}{h}$.

As function is differentiable so it is continuous as it is given that $\lim\limits_{h\to0}\dfrac{f(1+h)}{h}=5$ and hence $f(1)=0$

Hence $f'(1)=\lim\limits_{h\to0}\dfrac{f(1+h)}{h}=5$

The correct option is (C)

113. $f'(x)=\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h}$ implies

$$|f'(x)|=\lim_{h\to0}\left|\frac{f(x+h)-f(x)}{h}\right|\le\lim_{h\to0}\left|\frac{(h)^2}{h}\right|$$

$\Rightarrow$ $|f'(x)|\le 0\Rightarrow f'(x)=0\Rightarrow f(x)=$ constant

As $f(0)=0$ we have $f(1)=0$.

This correct option is (B)

114. Representing the given function as

$$f(x)=\begin{cases}\dfrac{x}{1-x}, & x<0\\[2mm] \dfrac{x}{1+x}, & x\ge0\end{cases}\Rightarrow f'(x)=\begin{cases}\dfrac{x}{(1-x)^2}, & x<0\\[2mm] \dfrac{x}{(1+x)^2}, & x\ge0\end{cases}$$

$\therefore f'(x)$ exist at everywhere.

The correct option is (C)

115. Given equation $x^m\cdot y^n=(x+y)^{m+n}$

$\Rightarrow m\ln x+n\ln y=(m+n)\ln(x+y)$

$$\therefore \frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=\frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{m}{x}-\frac{m+n}{x+y}\right)=\left(\frac{m+n}{x+y}-\frac{n}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{my-mx}{x(x+y)}=\left(\frac{my-mx}{y(x+y)}\right)\frac{dy}{dx}\Rightarrow \frac{dy}{dx}=\frac{y}{x}$$

The correct option is (A)

116. Given equation $x^{2x} - 2x^x \cot y - 1 = 0$ (1)

Now at $x = 1$,

$1 - 2\cot y - 1 = 0 \Rightarrow \cot y = 0 \Rightarrow y = \frac{\pi}{2}$

Now, differentiating eq. (1) with respect to 'x', we get

$2x^{2x}(1 + \log x) - 2$

$$\left[x^x(-\operatorname{cosec}^2 y)\frac{dy}{dx}+\cot y x^x(1+\log x)\right]=0$$

Now at $\left(1, \frac{\pi}{2}\right)$,

$$2(1+\log 1)-2\left(1(-1)\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)}+0\right)=0$$

$$\Rightarrow 2+2\left(\frac{dy}{dx}\right)_{\left(1,\frac{x}{2}\right)}=0\Rightarrow \left(\frac{dy}{dx}\right)_{\left(1,\frac{x}{2}\right)}=-1$$

The correct option is (A)

117. $$g'(x)=2(f(2f(x)+2))\left(\frac{d}{dx}(f(2f(x)+2))\right)$$

$$=2f(2f(x)+2)f'(2f(x)+2)\cdot(2f'(x))$$

$$\Rightarrow g'(0) = 2f(2f(0)+2)\,.\,f'(2f(0)+2)\,.\,2(f'(0)$$

$$= 4f(0)f'(0)$$

$$= 4(-1)(1) = -4$$

The correct option is (A)

118. $$\frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dy}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right)=-\frac{1}{\left(\frac{dy}{dx}\right)^2}\frac{d}{dy}\left(\frac{dy}{dx}\right)$$

$$=-\left(\frac{dy}{dx}\right)^{-2}\frac{1}{\left(\frac{dy}{dx}\right)}\frac{d}{dx}\left(\frac{dy}{dx}\right)=-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

The correct option is (C)

119. Given, $y = \sec(\tan^{-1} x)$

$$\therefore \frac{dy}{dx}=\sec\left(\tan^{-1}x\right)\tan\left(\tan^{-1}x\right)\cdot\frac{1}{1+x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1}=\sqrt{2}\times 1\times\frac{1}{2}=\frac{1}{\sqrt{2}}.$$

The correct option is (D)

120. $f(g(x)) = x$

$f'(g(x))\, g'(x) = 1$

$g'(x) = 1 + (g(x))^5$

The correct option is (D)